Quantum groupoids

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Abstract

We introduce a general notion of quantum universal enveloping algebroids (QUE algebroids), or quantum groupoids, as a unification of quantum groups and star-products. Some basic properties are studied including the twist construction and the classical limits. In particular, it is shown that a quantum groupoid naturally gives rise to a Lie bialgebroid as a classical limit. Conversely, we prove that any regular triangular Lie bialgebroid is quantizable. As an application, we study the dynamical quantum groupoid $\mathcal{D}\otimes_\hbar U_\hbar\mathfrak{g}$, which gives an interpretation of quantum dynamical Yang-Baxter equation in terms of Hopf algebroids.

1 Introduction

Poisson tensors in many aspects resemble classical triangular r-matrices in quantum group theory. A notion unifying both Poisson structures and Lie bialgebras was introduced in [33], called Lie bialgebroids. Integration theorem for Lie bialgebroids encomposes both Drinfeld theorem of integration of Lie bialgebras on the one hand, and Karasev-Weinstein theorem of existence of local symplectic groupoids for Poisson manifolds on the other hand. Quantization of Lie bialgebras leads to quantum groups, while quantization of Poisson manifolds is the so called star-products. It is therefore natural to expect that there exists some intrinsic connection between these two quantum objects. The purpose of this paper is to fill in this gap by introducing the notion of quantum universal enveloping algebriods (QUE algebriods), or quantum groupoids, as a general framework unifying these two concepts. Part of the results in this paper has been announced in [44] [45].

The general notion of Hopf algebroids was introduced by Lu [31], where the axioms were obtained essentially by translating those of Poisson groupoids to their quantum counterparts. The special case that base algebras are commutative was studied earlier by Maltsiniotis [35] in 1992 based

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on the work of Deligne on Tannakian category [8]. Hopf algebroids also appeared in Etingof and Varchenko's work on dynamical quantum groups [14] [15]. In this paper, we will mainly follow Lu's definition, but some axioms will be modified.

As we know, many important examples of Hopf algebras arise as deformations of the universal enveloping algebras of Lie algebras. Given a Lie algebroid A, its universal enveloping algebra UA (see the definition in Section 2) carries a natural cocommutative Hopf algebroid structure. For example, when A is the tangent bundle Lie algebroid TP, one obtains a cocommutative Hopf algebroid structure on $\mathcal{D}(P)$, the algebra of differential operators on P. It is natural to expect that deformations of UA, called quantum universal enveloping algebroids or quantum groupoids in this paper, would give us some non-trivial Hopf algebroids. This is the main subject we will undertake the study in the present paper. Examples include the quantum groupoid $\mathcal{D}_\hbar(P)$ corresponding to a star-product on a Poisson manifold P.

Another important class of quantum groupoids is connected with the so called quantum universal dynamical Yang-Baxter equation, also known as Geravais-Neveu-Felder equation:

$$R^{12}(\lambda+\hbar h^{(3)})R^{13}(\lambda)R^{23}(\lambda+\hbar h^{(1)}) = R^{23}(\lambda)R^{13}(\lambda+\hbar h^{(2)})R^{12}(\lambda). \quad (1)$$

Here $R(\lambda)$ is a meromorphic function from η^* to $U_\hbar\mathfrak{g}\otimes U_\hbar\mathfrak{g}$, $U_\hbar\mathfrak{g}$ is a quasi-triangular quantum group, and $\eta\subset\mathfrak{g}$ is an Abelian Lie subalgebra. This equation arises naturally from various contexts in mathematical physics, including quantum Liouville theory, quantum Knizhnik-Zamolodchikov-Bernard equation, and quantum Caloger-Moser model [2] [3] [18]. One approach to this equation due to Babelon et. al. [4] is to use Drinfeld's theory of quais-Hopf algebras [11]. Consider a meromorphic function $F:\eta^*\longrightarrow U_\hbar\mathfrak{g}\otimes U_\hbar\mathfrak{g}$ such that $F(\lambda)$ is invertible for all λ. Set $R(\lambda)=F^{21}(\lambda)^{-1}RF^{12}(\lambda)$, where $R\in U_\hbar\mathfrak{g}\otimes U_\hbar\mathfrak{g}$ is the standard universal R-matrix for the quantum group $U_\hbar\mathfrak{g}$. One can check that $R(\lambda)$ satisfies Equation (1) if $F(\lambda)$ is of zero weight, and satisfies the following shifted cocycle condition:

$$(\Delta_0\otimes id)F(\lambda)F^{12}(\lambda+\hbar h^{(3)}) = (id\otimes\Delta_0)F(\lambda)F^{23}(\lambda), \quad (2)$$

where Δ_0 is the coproduct of $U_\hbar\mathfrak{g}$. If moreover we assume that

$$(\epsilon_0\otimes id)F(\lambda)=1;\quad (id\otimes\epsilon_0)F(\lambda)=1, \quad (3)$$

where ϵ_0 is the counit map, one can form an elliptic quantum group, which is a family of quasi-Hopf algebras $(U_\hbar\mathfrak{g},\Delta_\lambda)$ parameterized by $\lambda\in\eta^*$: $\Delta_\lambda=F(\lambda)^{-1}\Delta F(\lambda)$. For $\mathfrak{g}=\mathfrak{sl}_2(\mathbb{C})$, a solution to Equations (2) and (3) was obtained by Babelon [3] in 1991. For general simple Lie algebras, solutions are recently found independently by Arnaudon et. al. [1] and Jimbo et. al. [24] based on the approach of Fronsdal [19].

As we will see in Section 7, Equation (2) arises naturally from the "twistor" equation of a quantum groupoid. This leads to another interpretation of an elliptic quantum group, namely as a quantum groupoid. Roughly speaking, our construction goes as follows. Instead of $U_\hbar\mathfrak{g}$, we start with the algebra $H=\mathcal{D}\otimes U_\hbar\mathfrak{g}$, where $\mathcal{D}$ denotes the algebra of meromorphic differential operators on η^*. H is no longer a Hopf algebra. Instead it is a QUE algebroid considered as the Hopf algebroid tensor product of $\mathcal{D}$ and $U_\hbar\mathfrak{g}$. Then the shifted cocycle condition is shown to be equivalent to the equation defining a twistor of this Hopf algebroid. Using this twistor, we obtain a new QUE algebroid $\mathcal{D}\otimes_\hbar U_\hbar\mathfrak{g}$ (or a quantum groupoid). We note that $\mathcal{D}\otimes_q U_\hbar\mathfrak{g}$ is co-associative as a Hopf algebroid, although $(U_\hbar\mathfrak{g},\Delta_\lambda)$ is not co-associative. The construction of $\mathcal{D}\otimes_q U_\hbar\mathfrak{g}$ is in some sense to restore the co-associativity by enlarging the algebra $U_\hbar\mathfrak{g}$ by tensoring the dynamical part $\mathcal{D}$. The relation between this quantum groupoid and quasi-Hopf algebras $(U_\hbar\mathfrak{g}, \Delta_\lambda)$ is, in a certain sense, similar to that between a fiber bundle and its fibers. We expect that this quantum groupoid will be useful in understanding elliptic quantum groups, especially their representation theory. The physical meaning of it, however, still needs to be explored.

This paper is organized as follows. In Section 2, we recall some basic definitions and results concerning Lie bialgebroids. Section 3 is devoted to the definition and basic properties of Hopf algebroids. In particular, for Hopf algebroids with anchor, it is proved that the category of left modules is a monoidal category. As a fundamental construction, in Section 4 we study the twist construction of Hopf algebroids, which generalizes the usual twist construction of Hopf algebras. Despite its complexity comparing to Hopf algebras, the fundamentals are in analogue to those of Hopf algebras. In particular, the monoidal categories of left modules of the twisted and untwisted Hopf algebroids are always equivalent. Section 5 is devoted to the introduction of quantum universal enveloping algebroids. The main part is to show that Lie bialgebroids indeed appear as the classical limit of QUE algebroids, as what is expected. However, unlike the quantum group case, the proof is not trivial and is in fact rather involved. On the other hand, the inverse question: the quantization problem, which would encompose both quantization of Lie bialgebras and deformation quantization of Poisson manifolds as special cases, remains widely open. As a very special case, in Section 6, we show that any regular triangular Lie bialgebroids are always quantizable. The discussion on quantum groupoids associated to universal dynamical R-matrices (i.e. solutions to the quantum dynamical Yang-Baxter equation) occupies Section 7. The last section, Section 8, consists of an appendix, as well as a list of open questions.

We would like to mention the recent work by Etingof and Varchenko [14] [15], where a different approach to the quantum dynamical Yang-Baxter equation in the framework of Hopf algebroid was given.

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2 Preliminary on Lie bialgebroids

It is well known that the classical objects corresponding to quantum groups are Lie bialgebras. Therefore, it is not surprising to expect that the classical counterparts of quantum groupoids are Lie bialgebroids. However, unlike Lie bialgebras, Lie bialgebroids were introduced and studied before the invention of quantum groupoids. In fact, they were used mainly in the study of Poisson geometry in connection with symplectic and Poisson groupoids (see [39] [42]).

The purpose of this section is to recall some basic facts concerning Lie bialgebroids. We start with recalling some definitions.

Definition 2.1 *A Lie algebroid is a vector bundle A over P together with a Lie algebra structure* *on the space $\Gamma(A)$ of smooth sections of A, and a bundle map $\rho : A \to TP$ (called the anchor), extended to a map between sections of these bundles, such that*

(i) $\rho([X,Y]) = [\rho(X),\ \rho(Y)]$; *and*

(ii) $[X, fY] = f[X,Y] + (\rho(X)f)Y$

for any smooth sections X and Y of A and any smooth function f on P.

Among many examples of Lie algebroids are usual Lie algebras, the tangent bundle of a manifold, and an integrable distribution over a manifold (see [32]). Another interesting example is connected with Poisson manifolds. Let P be a Poisson manifold with Poisson tensor π. Then T^*P carries a natural Lie algebroid structure, called the cotangent bundle Lie algebroid of the Poisson manifold P [7]. The anchor map $\pi^\# : T^*P \to TP$ is defined by

$$\pi^\# :\ T_p^*P \longrightarrow T_pP :\ \ \pi^\#(\xi)(\eta) = \pi(\xi,\eta), \quad \forall \xi,\eta \in T_p^*P \tag{4}$$

and the Lie bracket of 1-forms α and β is given by

$$[\alpha,\beta] \ = \ L_{\pi^\#(\alpha)}\beta \ - \ L_{\pi^\#(\beta)}\alpha \ - \ d\pi(\alpha,\beta). \tag{5}$$

Given a Lie algebroid A, it is known that $\oplus_k \Gamma(\wedge^k A^*)$ admits a differential d that makes it into a differential graded algebra [26]. Here, $d : \Gamma(\wedge^k A^*) \longrightarrow \Gamma(\wedge^{k+1} A^*)$ is defined by ([32] [33] [41]):

$$\begin{aligned} d\omega(X_1,\ldots,X_{k+1}) \ &= \ \sum_{i=1}^{k+1}(-1)^{i+1}\rho(X_i)(\omega(X_1,\ldots\hat{}\ldots,X_{k+1})) \\ &\quad + \sum_{i\ <j}(-1)^{i+j}\omega([X_i,X_j],X_1,\ldots\hat{}\ldots\hat{}\ldots,X_{k+1}), \end{aligned} \tag{6}$$

for $\omega \in \Gamma(\wedge^k A^*)$, $X_i \in \Gamma A$, $1 \le i \le k+1$. Then $d^2 = 0$ and one obtains a cochain complex whose cohomology is called the Lie algebroid cohomology. On the other hand, the Lie bracket on $\Gamma(A)$ extends naturally to a graded Lie bracket on $\oplus_k \Gamma(\wedge^k A)$ called Schouten bracket, which, together with the usual wedge product $\wedge$, makes it into a Gerstenhaber algebra [46].

As in the case of Lie algebras, associated to any Lie algebroid, there is an associative algebra called the *universal enveloping algebra* of the Lie algebroid A [22], a concept whose definition we now recall.

Let $A \to P$ be a Lie algebroid with anchor ρ. Then the $C^\infty(P)$-module direct sum $C^\infty(P) \oplus \Gamma(A)$ is a Lie algebra over $\mathbb{R}$ with the Lie bracket:

$$[f+X,\ g+Y] = (\rho(X)g - \rho(Y)f) + [X,Y].$$

Let $U = U(C^\infty(P) \oplus \Gamma(A))$ be its standard universal enveloping algebra. For any $f \in C^\infty(P)$ and $X \in \Gamma(A)$, denote by f' and X' their canonical image in U. Denote by I the two-sided ideal of U generated by all elements of the form $(fg)' - f'g'$ and $(fX)' - f'X'$. Define $U(A) = U/I$, which is called the universal enveloping algebra of the Lie algebroid A. When A is a Lie algebra, this definition reduces to the definition of usual universal enveloping algebras. On the other hand, for the tangent bundle Lie algebroid TP, its universal enveloping algebra is $\mathcal{D}(P)$, the algebra of differential operators over P. In between, if $A = TP \times \mathfrak{g}$ as the Lie algebroid direct product, then $U(A)$ is isomorphic to $\mathcal{D}(P)\otimes U\mathfrak{g}$. Note that the maps $f \mapsto f'$ and $X \mapsto X'$ considered above descend to linear embedings $i_1 : C^\infty(P) \mapsto U(A)$, and $i_2 : \Gamma(A) \mapsto U(A)$; the first map i_1 is an algebra morphism. These maps have the following properties:

$$i_1(f)i_2(X) = i_2(fX), \quad [i_2(X), i_1(f)] = i_1(\rho(X)f)), \quad [i_2(X), i_2(Y)] = i_2([X,Y]). \tag{7}$$

In fact, $U(A)$ is universal among all triples $(B, \varphi_1, \varphi_2)$ having such properties (see [22] for a proof of this simple fact). Sometimes, it is also useful to think of UA as the algebra of left invariant differential operators on a local Lie groupoid G, which integrates the Lie algebroid A.

The notion of Lie bialgebroids is a natural generalization of that of Lie bialgebras. Roughly speaking, a Lie bialgebroid is a pair of Lie algebroids $(A,\ A^*)$ satisfying a certain compatibility condition. Such a condition, providing a definition of *Lie bialgebroid*, was given in [33]. We quote here an equivalent formulation from [23].

Definition 2.2 *A* Lie bialgebroid *is a dual pair (A, A^*) of vector bundles equipped with Lie algebroid structures such that the differential d_* on $\Gamma(\wedge^* A)$ coming from the structure on A^* is a derivation of the Schouten bracket on $\oplus\Gamma(\wedge^* A)$. Equivalently, d_* is a derivation for sections of A, i.e.,*

$$d_*[X,Y] = [d_*X, Y] + [X, d_*Y], \quad \forall X, Y \in \Gamma(A). \tag{8}$$

In other words, $(\oplus_k\Gamma(\wedge^k A), \wedge, [\cdot,\cdot], d_*)$ is a strong differential Gerstenhaber algebra [46].

In fact, a Lie bialgebroid is equivalent to a strong differential Gerstenhaber algebra structure on $\oplus_k\Gamma(\wedge^k A)$ (see Proposition 2.3 in [46]).

For a Lie bialgebroid (A, A^*), the base P inherits a natural Poisson structure:

$$\{f,\ g\} =< df,\ d_*g >, \quad \forall f, g \in C^\infty(P), \tag{9}$$

which satisfies the identity: $[df,\ dg] = d\{f,\ g\}$.

As in the case of Lie bialgebras, a useful method of constructing Lie bialgebroids is via some kind of "r-matrices". More precisely, by an *r-matroid*, we mean a section Λ of $\Gamma(\wedge^2 A)$ satisfying

$$L_X[\Lambda, \Lambda] = [X, [\Lambda, \Lambda]] = 0, \quad \forall X \in \Gamma(A). \tag{10}$$

An r-matroid Λ defines a Lie bialgebroid, where the differential $d_* : \Gamma(\wedge^* A) \longrightarrow \Gamma(\wedge^{*+1} A)$ is simply given by $d_* = [\Lambda, \cdot]$. In this case, the bracket on $\Gamma(A^*)$ is given by

$$[\xi, \eta] = L_{\Lambda^\# \xi}\eta - L_{\Lambda^\# \eta}\xi - d[\Lambda(\xi, \eta)], \tag{11}$$

and the anchor is the composition $\rho \circ \Lambda^\# : A^* \longrightarrow TP$, where $\Lambda^\#$ denotes the bundle map $A^* \longrightarrow A$ defined by $\Lambda^\#(\xi)(\eta) = \Lambda(\xi, \eta), \forall \xi, \eta \in \Gamma(A^*)$. Such a Lie bialgebroid is called a *coboundary Lie bialgebroid*, in analogue with the Lie algebra case [29] [30]. It is called a *triangular Lie bialgebroid* if $[\Lambda, \Lambda] = 0$. In particular it is called a *regular triangular Lie bialgebroid* if Λ is of constant rank.

When P reduces to a point, i.e., A is a Lie algebra, Equation (10) is equivalent to that $[\Lambda, \Lambda]$ is *ad*-invariant, i.e, Λ is an r-matrix in the ordinary sense. On the other hand, when A is the tangent bundle TP with the standard Lie algebroid structure, Equation (10) is equivalent to that $[\Lambda, \Lambda] = 0$, i.e., Λ is a Poisson tensor.

Another interesting class of coboundary Lie bialgebroids is connected with the so called dynamical r-matrices.

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$ (or $\mathbb{C}$) and $\eta \subset \mathfrak{g}$ an Abelian Lie subalgebra. A dynamical r-matrix [13] is a smooth function (or meromorphic function in complex case) $r : \eta^* \longrightarrow \wedge^2\mathfrak{g}$ such that

(i). $r(\lambda)$ is η-invariant, i.e., $ad_h r(\lambda) = 0, \ \forall h \in \eta$;

(ii). $\mathrm{Alt}dr - \frac{1}{2}[r, r]$ is constant over η^* with value in $(\wedge^2\mathfrak{g})^{\mathfrak{g}}$.

Here dr is considered as a $\eta \otimes \wedge^2\mathfrak{g}$-valued function over η^* and Alt denotes the standard skew-symmetrization operator. In particular, if $\mathrm{Alt}dr - \frac{1}{2}[r, r] = 0$, it is called a *triangular dynamical r-matrix*. It is called a *regular triangular dynamical r-matrix* if $r(\lambda)$ is of constant rank. The following is a simple example of a dynamical r-matrix.

Example 2.1 Let $\mathfrak{g}$ be a simple Lie algebra with root decomposition $\mathfrak{g} = \eta \oplus \sum_{\alpha\in\Delta_+}(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$, where η is a Cartan subalgebra, and Δ_+ is the set of positive roots. Then

$$r(\lambda) = \frac{1}{2}\sum_{\alpha\in\Delta_+} \coth(\frac{1}{2} \ll \alpha, \lambda \gg) E_\alpha \wedge E_{-\alpha},$$

is a dynamical r-matrix, where $\ll , \gg$ is the Killing form of $\mathfrak{g}$, the E_α and $E'_{-\alpha}s$ are standard root vectors, and $\coth(x) = \frac{e^x+e^{-x}}{e^x-e^{-x}}$ is the hyperbolic cotangent function.

A dynamical r-matrix naturally gives rise to a Lie bialgebroid.

Proposition 2.3 *[5] [30] Let $r : \eta^* \longrightarrow \wedge^2\mathfrak{g}$ be a dynamical r-matrix. Then $A = T\eta^* \times \mathfrak{g}$ equipped with the standard Lie algebroid structure, together with $\Lambda = \sum_{i=1}^k(\frac{\partial}{\partial\lambda_i} \wedge h_i) + r(\lambda) \in \Gamma(\wedge^2 A)$ defines a coboundary Lie bialgebroid. Here $\{h_1, h_2, \cdots, h_k\}$ is a basis of η, and $(\lambda_1, \cdots, \lambda_k)$ its induced coordinate system on η^*.*

We end this section by recalling the definition of Hamiltonian operators, which will be needed later on. Given a Lie bialgebroid (A, A^*) with associated strong differential Gerstanhaber algebra $(\oplus_k\Gamma(\wedge^k A), \wedge, [\cdot,\cdot], d_*)$, one may construct a new Lie bialgebroid via a twist. For that, simply let $\tilde{d_*} = d_* + [H, \cdot]$ for some $H \in \Gamma(\wedge^2 A)$. It is easy to check [27] that this still defines a strong differential Gerstanhaber algebra (therefore a Lie bialgebroid), iff

$$d_* H + \frac{1}{2}[H, \ H] = 0. \tag{12}$$

Such an H is called a Hamiltonian operator of the Lie bialgebroid (A, A^*).

Finally we note that even though we are mainly dealing with real vector bundles and real Lie algebroids in this paper, one may also consider (as already suggested by the early example of dynamical r-matrices) complex Lie algebroids and complex Lie bialgebroids. In this context, one may have to use sheaves of holomorphic sections etc. instead of global ones. Most results in this section are expected to hold with certain suitable modifications.

3 Hopf algebroids

Definition 3.1 *A Hopf algebroid* $(H, R, \alpha, \beta, m, \Delta, \epsilon)$ *consists of the following data:*

1) a **total algebra** H *with product* m*, a base algebra* R*, a* **source map***: an algebra homomorphism* $\alpha: R \longrightarrow H$*, and a* **target map***: an algebra anti-homomorphism* $\beta: R \longrightarrow H$ *such that the images of* α *and* β *commute in* H*, i.e.,* $\forall a, b \in R$, $\alpha(a)\beta(b) = \beta(b)\alpha(a)$*. There is then a natural* (R, R)*-bimodule structure on* H *given by* $a \cdot h = \alpha(a)h$ *and* $h \cdot a = \beta(a)h$*. Thus, we can form the* (R, R)*-bimodule product* $H \otimes_R H$*. It is easy to see that* $H \otimes_R H$ *again admits an* (R, R)*-bimodule structure. This will allow us to form the triple product* $H \otimes_R H \otimes_R H$ *and etc.*

2) a **co-product***: an* (R, R)*-bimodule map* $\Delta: H \longrightarrow H \otimes_R H$ *with* $\Delta(1) = 1 \otimes 1$ *satisfying the co-associativity:*

$$(\Delta \otimes_R id_H)\Delta = (id_H \otimes_R \Delta)\Delta : H \longrightarrow H \otimes_R H \otimes_R H; \tag{13}$$

3) the product and the co-product are **compatible** *in the following sense:*

$$\Delta(h)(\beta(a) \otimes 1 - 1 \otimes \alpha(a)) = 0, \quad \text{in } H \otimes_R H, \ \forall a \in R \text{ and } h \in H, \text{ and} \tag{14}$$

$$\Delta(h_1 h_2) = \Delta(h_1)\Delta(h_2), \quad \forall h_1, h_2 \in H, \qquad \text{(see the remark below)}; \tag{15}$$

4) a co-unit map: an (R, R)-bimodule map $\epsilon: H \longrightarrow R$ *satisfying* $\epsilon(1_H) = 1_R$ *(it follows then that* $\epsilon\beta = \epsilon\alpha = id_R$*) and*

$$(\epsilon \otimes_R id_H)\Delta = (id_H \otimes_R \epsilon)\Delta = id_H : H \longrightarrow H. \tag{16}$$

Here we have used the identification: $R \otimes_R H \cong H \otimes_R R \cong H$ *(note that both maps on the left hand sides of Equation (16) are well-defined).*

Remark It is clear that any left H-module is automatically an (R, R)-bimodule. Now given any left H-modules M_1 and M_2, define,

$$h \cdot (m_1 \otimes_R m_2) = \Delta(h)(m_1 \otimes m_2), \quad \forall h \in H, \ m_1 \in M_1, \ m_2 \in M_2. \tag{17}$$

The RHS is a well-defined element in $M_1 \otimes_R M_2$ due to Equation (14). In particular, when taking $M_1 = M_2 = H$, we see that the RHS of Equation (15) makes sense. In fact, Equation (15) implies that $M_1 \otimes_R M_2$ is again a left H-module under the action defined by Equation (17). Left H-modules are also called *representations* of the Hopf algebroid H (as an associative algebra). The category of representations of H is denoted by Rep$H$.

There is an equivalent version for the compatibility condition 3), which was used by Lu [31]:

Proposition 3.2 *The compatibility condition 3) (Equations (14) and (15)) is equivalent to that the kernel of the map*

$$\Psi : H \otimes H \otimes H \longrightarrow H \otimes_R H : \quad \sum h_1 \otimes h_2 \otimes h_3 \longmapsto \sum (\Delta h_1)(h_2 \otimes h_3) \tag{18}$$

is a left ideal of $H \otimes H^{op} \otimes H^{op}$*, where* H^{op} *denotes* H *with the opposite product.*

Proof. Assume that $Ker\Psi$ is a left ideal. It is clear that for any $a \in H$, $1\otimes\beta(a)\otimes 1 - 1\otimes 1\otimes\alpha(a) \in Ker\Psi$. Hence $h\otimes\beta(a)\otimes 1 - h\otimes 1\otimes\alpha(a) = (h\otimes 1\otimes 1)(1\otimes\beta(a)\otimes 1 - 1\otimes 1\otimes\alpha(a))$ belongs to $Ker\Psi$. That is, $\Delta(h)(\beta(a)\otimes 1 - 1\otimes\alpha(a)) = 0$. To prove Equation (15), we assume that $\Delta h_2 = \sum_{ij} g_i\otimes_R g_j$ for some $g_i, g_j \in H$. Then $h_2\otimes 1\otimes 1 - \sum_{ij} 1\otimes g_i\otimes g_j \in Ker\Psi$. This implies that $h_1 h_2\otimes 1\otimes 1 - \sum_{ij} h_1\otimes g_i\otimes g_j \in Ker\Psi$ since $Ker\Psi$ is a left ideal. Hence $\Delta(h_1 h_2) - \sum_{ij}(\Delta h_1)(g_i\otimes g_j) = 0$ in $H\otimes_R H$. I.e., $\Delta(h_1 h_2) = \Delta(h_1)\Delta(h_2)$.

Conversely, assume that Equations (14) and (15) hold. Suppose that $\sum h_1\otimes h_2\otimes h_3 \in Ker\Psi$. Then for any $x, y, z \in H$, note that in $H\otimes H^{op}\otimes H^{op}$, we have $(x\otimes y\otimes z)\sum(h_1\otimes h_2\otimes h_3) = \sum xh_1\otimes h_2 y\otimes h_3 z$. Then

$$\begin{aligned} & \Psi((x\otimes y\otimes z)\sum(h_1\otimes h_2\otimes h_3)) \\ = & \sum \Delta(xh_1)(h_2 y\otimes h_3 z) \\ = & \Delta x \sum(\Delta h_1)(h_2\otimes h_3)(y\otimes z) \\ = & 0. \end{aligned}$$

That is, $Ker\Psi$ is a left ideal.

□

Remark (1) In [31], objects satisfying the above axioms are called bi-algebroids, while the term of Hopf algebroids is reserved to those admitting an antipode. In our definition above, we choose not to require the existence of antipode because many interesting examples, as shown below, often do not admit an antipode.

(2) In the classical case, the compatibility between the Poisson structure and the groupoid structure implies that the base manifold is a coisotropic submanifold of the Poisson groupoid [39]. Being considered as a quantum counterpart, for a Hopf algebroid, it would be natural to expect that the quantum analogue should hold as well, which means that the kernel of ϵ is a **left ideal** of H. However, we are not able to prove this at the moment (note that this extra condition was required in the definition in [31]).

In most situation, Hopf algebroids are equipped with an additional structure, called an *anchor* map. Let $(H, R, \alpha, \beta, m, \Delta, \epsilon)$ be a Hopf algebroid (over the ground field k of characteristic zero). By $End_k R$, we denote the algebra of linear endmorphisms of R over k. It is clear that $End_k R$ is an (R, R)-bimodule, where R acts on it from the left by left multiplication and acts from the right by right multiplication. Assume that R is a left H-module and moreover the representation $\mu : H \longrightarrow End_k R$ is an (R, R)-bimodule map. For any $x \in H$ and $a \in R$, we denote by $x(a)$ the element $\mu(x)(a)$ in R. Define

$$\begin{aligned} & \varphi_\alpha, \varphi_\beta : (H\otimes_R H)\otimes R \longrightarrow H, \\ & \varphi_\alpha(x\otimes_R y\otimes a) = x(a)\cdot y, \quad \text{and } \varphi_\beta(x\otimes_R y\otimes a) = x\cdot y(a). \end{aligned} \tag{19}$$

Here $x, y \in H$, $a \in R$, and the dot $\cdot$ denotes the (R, R)-bimodule structure on H. Note that φ_α and φ_β are well defined since μ is an (R, R)-bimodule map.

Proposition 3.3 *Under the above assumption, and moreover assume that*

$$\varphi_\alpha(\Delta x \otimes a) = x\alpha(a), \quad \textit{and} \;\; \varphi_\beta(\Delta x \otimes a) = x\beta(a), \quad \forall x \in H, a \in R. \tag{20}$$

Then the map $\tilde{\epsilon} : H \longrightarrow R$, $\tilde{\epsilon}x = x(1_R)$, satisfies the co-unit property. I.e., it is an (R, R)-bimodule map, $\tilde{\epsilon}(1_H) = 1_R$, and satisfies Equation (16).

Proof. That $\tilde{\epsilon}$ is an (R, R)-bimodule map follows from the assumption that the representation μ is an (R, R)-bimodule map. It is clear that $\tilde{\epsilon}(1_H) = 1_R$. To prove Equation (16), assume that $\Delta x = \sum_i x_i^{(1)} \otimes_R x_i^{(2)}$. Then

$$\begin{aligned} (\tilde{\epsilon}\otimes_R id_H)\Delta x &= \sum \tilde{\epsilon}(x_i^{(1)})\otimes_R x_i^{(2)} \\ &= \sum x_i^{(1)}(1_R)\otimes_R x_i^{(2)} \\ &= \varphi_\alpha(\Delta x \otimes 1_R) \\ &= x\alpha(1_R) \\ &= x. \end{aligned}$$

Similarly, we have $(id_H \otimes_R \tilde{\epsilon})\Delta x = x$.

□

It is thus natural to expect that $\tilde{\epsilon}$ coincides with the co-unit map.

Definition 3.4 *Given a Hopf algebroid $(H, R, \alpha, \beta, m, \Delta, \epsilon)$, an anchor map is a representation $\mu : H \longrightarrow End_k R$, which is also an (R, R)-bimodule map satisfying*

(i). $\varphi_\alpha(\Delta x \otimes a) = x\alpha(a)$ *and* $\varphi_\beta(\Delta x \otimes a) = x\beta(a), \quad \forall x \in H, a \in R;$

(ii). $x(1_R) = \epsilon x, \quad \forall x \in H.$

Remark For a Hopf algebra, since $R = k$ and $End_k R \cong k$, one can simply take μ as the counit. In this case, μ is in fact equivalent to the counit map. However, for a Hopf algebroid, the existence of an anchor map is a stronger assumtion than the existence of a counit. In fact, we can require axioms 1)-3) in Definition (3.1) together with the existence of an anchor map to define a Hopf algebroid with anchor. Then the existence of the counit would be a consequence according to Proposition (3.3).

Given any $x = \sum x_1 \otimes_R x_2 \cdots \otimes_R x_n \in \otimes_R^n H$ and k elements ($k \le n$) $a_{i_1}, a_{i_2}, \cdots, a_{i_k} \in R$, we denote by $x(\cdot, \cdots, a_{i_1}, \cdots, \cdot, a_{i_k}, \cdots, \cdot)$ the element $\sum x_1 \otimes_R \cdots \otimes_R x_{i_1}(a_{i_1}) \otimes_R \cdots \otimes_R x_{i_k}(a_{i_k}) \otimes_R \cdots \otimes_R x_n$ in $\otimes_R^{n-k} H$. The anchor map assumption gurantees that this is a well-defined element.

Proposition 3.5 *For any $a, b \in R, x \in H$,*

$$\alpha(a)(b) = ab, \quad \beta(a)(b) = ba; \tag{21}$$

$$\Delta(x)(a, b) = x(ab); \tag{22}$$

$$\epsilon(xy) = x(\epsilon(y)). \tag{23}$$

Proof. To prove Equation (21), we note that $\alpha(a)(b) = (a \cdot 1_H)(b) = a(1_H(b)) = ab$, where we used the fact that μ is an (R,R)-bimodule map. Similarly, we have $\beta(a)(b) = ba$.

For Equation (22), we have

$$\begin{aligned} \Delta(x)(a,b) &= \varphi_\alpha(\Delta x \otimes a)(b) \\ &= (x\alpha(a))(b) \\ &= x(\alpha(a)(b)) \\ &= x(ab). \end{aligned}$$

Here the last step used Equation (21).

Finally, using (ii) in Definition (3.4), we have

$$\epsilon(xy) = (xy)(1_R) = x(y(1_R)) = x(\epsilon(y)).$$

□

Remark Equation (21) implies that the induced (R,R)-bimodule structure on R, when R being considered as a left H-module, coincides with the usual one by (left and right) multiplications. In fact, this condition is equivalent to requiring that μ is an (R,R)-bimodule map in Definition (3.4). Equation (22) simply means that $R \otimes_R R \cong R$ as left H-modules. And the last equation, Equation (23), amounts to saying that $\epsilon : H \longrightarrow R$ is a morphism of left H-modules, where H acts on H by left multiplication.

The following result follows immediately from definition.

Theorem 3.6 *Let $(H, R, \alpha, \beta, m, \Delta, \epsilon)$ be a Hopf algebroid with anchor μ. Then the category RepH of left H-modules equipped with the tensor product $\otimes_R$ as defined by Equation (17), the unit object (R, μ), and the trivial associativity isomorphisms: $(M_1 \otimes_R M_2) \otimes_R M_3 \longrightarrow M_1 \otimes_R (M_2 \otimes_R M_3)$ is a monoidal category.*

Example 3.1 Let $\mathcal{D}$ denote the algebra of all differential operators on a smooth manifold M, and R the algebra of smooth functions on M. Then $\mathcal{D}$ is a Hopf algebroid over R. Here, $\alpha = \beta$ is the embedding $R \longrightarrow \mathcal{D}$, while the coproduct $\Delta : \mathcal{D} \longrightarrow \mathcal{D} \otimes_R \mathcal{D}$ is defined as

$$\Delta(D)(f,g) = D(fg), \quad \forall D \in \mathcal{D}, \text{ and } f,\ g \in R. \tag{24}$$

Note that $\mathcal{D} \otimes_R \mathcal{D}$ is simply the space of bidifferential operators. Clearly, Δ is co-commutative, i.e., $\Delta^{op} = \Delta$. The usual action of differential operators on $C^\infty(P)$ defines an anchor $\mu : H \longrightarrow End_k R$. In this case, the co-unit $\epsilon : \mathcal{D} \longrightarrow R$ is the natural projection from a differential operator to its $0th$-order part. It is easy to see that left $\mathcal{D}$-modules are D-modules in the usual sense, and the tensor product is the usual tensor product of D-modules over R. We note, however, that this Hopf algebroid does not admit an antipode in any natural sense. Given a differential operator D, its antipode, if it exists, would be the dual operator D^*. However, the latter is a differential operator on 1-densities, which does not possess any canonical identification with a differential operator on R.

The construction above can be generalized to show that the universal enveloping algebra UA of a Lie algebroid A admits a co-commutative Hopf algebroid structure.

Again we take $R = C^\infty(P)$, and let $\alpha = \beta : R \longrightarrow UA$ be the natural embedding. For the co-product, we set

$$\begin{aligned} \Delta(f) &= f \otimes_R 1, \quad \forall f \in R; \\ \Delta(X) &= X \otimes_R 1 + 1 \otimes_R X, \quad \forall X \in \Gamma(A). \end{aligned}$$

This formula extends to a co-product $\Delta : UA \longrightarrow UA \otimes_R UA$ by the compatibility condition: Equations (14) and (15). Alternatively, we may identify UA as the subalgebra of $\mathcal{D}(G)$ consisting of left invariant differential operators on a (local) Lie groupoid G integrating A, and then restrict the co-product Δ_G on $\mathcal{D}(G)$ to this subalgebra. This is well defined since Δ_G maps left invariant differential operators to left invariant bidifferential operators. Finally, the map $(\mu x)(f) = (\rho x)(f), \quad \forall x \in UA, f \in R$ defines an anchor, and the co-unit map is then the projection $\epsilon : UA \longrightarrow R$, where $\rho : UA \longrightarrow \mathcal{D}(P)$ denotes the algebra homomorphism extending the anchor of the Lie algebroid anchor (denoted by the same symbol ρ).

Theorem 3.7 *$(UA, R, \alpha, \beta, m, \Delta, \epsilon)$ is a co-commutative Hopf algebroid with anchor μ.*

4 Twist construction

As in the Hopf algebra case, twist construction is an important method of producing new Hopf algebroids. This section is devoted to the study on this useful construction. We start with the following

Proposition 4.1 *Let $(H, R, \alpha, \beta, m, \Delta, \epsilon)$ be a Hopf algebroid with anchor μ, and let φ_α and φ_β be the maps defined by Equation (19). Then for any $x, y, z \in H$ and $a \in R$,*

$$\varphi_\alpha((\Delta x)(y \otimes_R z) \otimes a) = x\alpha(y(a))z; \tag{25}$$

$$\varphi_\beta((\Delta x)(y \otimes_R z) \otimes a) = x\beta(z(a))y. \tag{26}$$

Proof. Assume that $\Delta x = \sum_i x_i^{(1)} \otimes_R x_i^{(2)}$. Then

$$\begin{aligned} \varphi_\alpha((\Delta x)(y \otimes_R z) \otimes a) &= \varphi_\alpha(\sum_i (x_i^{(1)} y \otimes_R x_i^{(2)} z) \otimes a) \\ &= \sum_i \alpha((x_i^{(1)} y)(a)) x_i^{(2)} z \\ &= \sum_i \alpha(x_i^{(1)}(y(a))) x_i^{(2)} z. \end{aligned}$$

On the other hand, using (i) in Definition 3.4,

$$\begin{aligned} x\alpha(y(a))z &= \varphi_\alpha(\Delta x \otimes y(a))z \\ &= \varphi_\alpha(\sum_i x_i^{(1)} \otimes_R x_i^{(2)} \otimes y(a))z \\ &= \sum_i \alpha(x_i^{(1)}(y(a)))x_i^{(2)} z. \end{aligned}$$

Hence, $\varphi_\alpha((\Delta x)(y\otimes_R z) \otimes a) = x\alpha(y(a))z$. Equation (26) can be proved similarly.

□

Now let $\mathcal{F}$ be an element in $H\otimes_R H$. Define $\alpha_\mathcal{F}, \beta_\mathcal{F} : R \longrightarrow H$, respectively, by

$$\alpha_\mathcal{F}(a) = \varphi_\alpha(\mathcal{F}\otimes a), \quad \beta_\mathcal{F}(a) = \varphi_\beta(\mathcal{F}\otimes a), \ \forall a \in R. \tag{27}$$

And for any $a, b \in R$, set

$$a *_\mathcal{F} b = \alpha_\mathcal{F}(a)(b). \tag{28}$$

More explicitly, if $\mathcal{F} = \sum_i x_i \otimes_R y_i$ for $x_i, y_i \in H$, then $\forall a, b \in R$,

$$\alpha_\mathcal{F}(a) = \sum_i x_i(a) \cdot y_i = \sum_i \alpha(x_i(a))y_i, \tag{29}$$

$$\beta_\mathcal{F}(a) = \sum_i x_i \cdot y_i(a) = \sum_i \beta(y_i(a))x_i, \quad \text{and} \tag{30}$$

$$a *_\mathcal{F} b = \sum_i x_i(a)y_i(b). \tag{31}$$

Proposition 4.2 *Assume that $\mathcal{F} \in H\otimes_R H$ satisfies:*

$$(\Delta\otimes_R id)\mathcal{F}\mathcal{F}^{12} = (id\otimes_R \Delta)\mathcal{F}\mathcal{F}^{23} \quad in \quad H\otimes_R H\otimes_R H; \quad and \tag{32}$$

$$(\epsilon\otimes_R id)\mathcal{F} \ = \ 1_H; \quad (id\otimes_R \epsilon)\mathcal{F} \ = \ 1_H. \tag{33}$$

Here $\mathcal{F}^{12} = \mathcal{F}\otimes 1 \in (H\otimes_R H)\otimes H$, $\mathcal{F}^{23} = 1\otimes\mathcal{F} \in H\otimes(H\otimes_R H)$, and in Equation (33) we have used the identification: $R\otimes_R H \cong H\otimes_R R \cong H$ (note that both maps on the left hand sides of Equation (33) are well-defined). Then

(i). *$(R, *_\mathcal{F})$ is an associative algebra, and $1_R *_\mathcal{F} a = a *_\mathcal{F} 1_R = a, \forall a \in R$.*

(ii). *$\alpha_\mathcal{F} : R_\mathcal{F} \longrightarrow H$ is an algebra homomorphism, and $\beta_\mathcal{F} : R_\mathcal{F} \longrightarrow H$ is anti-homomorphism. Here $R_\mathcal{F}$ stands for the algebra $(R, *_\mathcal{F})$.*

(iii). *$\alpha_\mathcal{F}(a)\beta_\mathcal{F}(b) = \beta_\mathcal{F}(b)\alpha_\mathcal{F}(a), \quad \forall a, b \in R$*

Proof. As a first step, we prove that for any $a, b \in R$,

$$\alpha_\mathcal{F}(a *_\mathcal{F} b) \ = \ (\alpha_\mathcal{F} a)(\alpha_\mathcal{F} b), \tag{34}$$

$$\beta_\mathcal{F}(a *_\mathcal{F} b) \ = \ (\beta_\mathcal{F} b)(\beta_\mathcal{F} a). \tag{35}$$

Assume that $\mathcal{F} = \sum_i x_i \otimes_R y_i$ for $x_i, y_i \in H$. Then

$$(\Delta \otimes_R id)\mathcal{F}\mathcal{F}^{12} = \sum_{ij} \Delta x_i (x_j \otimes_R y_j) \otimes_R y_i, \tag{36}$$

$$(id \otimes_R \Delta)\mathcal{F}\mathcal{F}^{23} = \sum_{ij} x_i \otimes_R \Delta y_i (x_j \otimes_R y_j). \tag{37}$$

Thus

$$\begin{aligned}
[(\Delta \otimes_R id)\mathcal{F}\mathcal{F}^{12}](a, b, \cdot) &= \sum_{ij} \Delta x_i (x_j \otimes y_j)(a, b) \otimes_R y_i \\
&= \sum_{ij} x_i (x_j(a) y_j(b)) \otimes_R y_i \\
&= \sum_{ij} \alpha[x_i (x_j(a) y_j(b))] y_i \\
&= \alpha_{\mathcal{F}} (\sum_j x_j(a) y_j(b)) \\
&= \alpha_{\mathcal{F}} (a *_{\mathcal{F}} b),
\end{aligned}$$

where the second equality used Equation (22).

On the other hand,

$$\begin{aligned}
[(id \otimes_R \Delta)\mathcal{F}\mathcal{F}^{23}](a, b, \cdot) &= \sum_{ij} x_i(a) \otimes_R \varphi_\alpha (\Delta y_i (x_j \otimes_R y_j) \otimes b) \\
&= \sum_{ij} x_i(a) \otimes_R y_i \alpha(x_j(b)) y_j \\
&= \sum_{ij} \alpha(x_i(a)) y_i \alpha(x_j(b)) y_j \\
&= (\alpha_{\mathcal{F}} a)(\alpha_{\mathcal{F}} b).
\end{aligned}$$

Thus Equation (34) follows from Equation (32). The equation $\beta_{\mathcal{F}}(a *_{\mathcal{F}} b) = (\beta_{\mathcal{F}} b)(\beta_{\mathcal{F}} a)$ can be proved similarly.

Now for any $a, b, c \in R$, $[(\alpha_{\mathcal{F}} a)(\alpha_{\mathcal{F}} b)](c) = (\alpha_{\mathcal{F}} a)((\alpha_{\mathcal{F}} b)(c)) = a *_{\mathcal{F}} (b *_{\mathcal{F}} c)$. On the other hand, $\alpha_{\mathcal{F}}(a *_{\mathcal{F}} b)(c) = (a *_{\mathcal{F}} b) *_{\mathcal{F}} c$. The associativity of $R_{\mathcal{F}}$ thus follows from Equation (34).

Finally, we have $\alpha_{\mathcal{F}}(1_R) = \sum_i x_i(1_R) \cdot y_i = \sum_i \epsilon(x_i) \cdot y_i = (\epsilon \otimes_R id)\mathcal{F} = 1_H$. Similarly, $\beta_{\mathcal{F}}(1_R) = 1_H$. It thus follows that $1_R *_{\mathcal{F}} a = \alpha_{\mathcal{F}}(1_R)(a) = 1_H(a) = a$. Similarly, $a *_{\mathcal{F}} 1_R = a$.

For the last statement, a similar computation leads to

$$[(\Delta \otimes_R id)\mathcal{F}\mathcal{F}^{12}](a, \cdot, b) = \beta_{\mathcal{F}} b *_{\mathcal{F}} \alpha_{\mathcal{F}} a, \quad \text{and}$$
$$[(id \otimes_R \Delta)\mathcal{F}\mathcal{F}^{23}](a, \cdot, b) = \alpha_{\mathcal{F}} a *_{\mathcal{F}} \beta_{\mathcal{F}} b.$$

Thus (iii) follows immediately. This concludes the proof.

□

Proposition 4.3 *Under the same hypothesis as in Proposition 4.2, we have*

$$\mathcal{F}(\beta_{\mathcal{F}}(a)\otimes 1 - 1\otimes\alpha_{\mathcal{F}}(a)) = 0 \quad \text{in } H\otimes_R H, \quad \forall a\in R. \tag{38}$$

Proof.

$$\begin{aligned}
& \mathcal{F}(\beta_{\mathcal{F}}(a)\otimes 1 - 1\otimes\alpha_{\mathcal{F}}(a)) \\
= & (\sum_i x_i\otimes_R y_i)(\sum_j \beta(y_j(a))x_j\otimes 1 - \sum_j 1\otimes\alpha(x_j(a))y_j) \\
= & \sum_{ij}[x_i\beta(y_j(a))x_j\otimes_R y_i - x_i\otimes_R y_i\alpha(x_j(a))y_j].
\end{aligned}$$

Now using Equations (36) (37) and (25) (26), we obtain

$$\begin{aligned}
& [(\Delta\otimes_R id)\mathcal{F}\mathcal{F}^{12}](\cdot, a, \cdot) \\
= & \sum_{ij}\varphi_\beta(\Delta x_i(x_j\otimes_R y_j)\otimes a)\otimes_R y_i \\
= & \sum_{ij} x_i\beta(y_j(a))x_j\otimes_R y_i,
\end{aligned}$$

and

$$\begin{aligned}
& [(id\otimes_R\Delta)\mathcal{F}\mathcal{F}^{23}](\cdot, a, \cdot) \\
= & \sum_{ij} x_i\otimes_R\varphi_\alpha(\Delta y_i(x_j\otimes_R y_j)\otimes a) \\
= & \sum_{ij} x_i\otimes_R y_i\alpha(x_j(a))y_j.
\end{aligned}$$

Thus the conclusion follows immediately from Equation (32).

□

As an immediate consequence, we have

Corollary 4.4 *Let M_1 and M_2 be any left H-modules. Then*

$$\begin{aligned}
\mathcal{F}^{\#}: M_1\otimes_{R_{\mathcal{F}}} M_2 &\longrightarrow M_1\otimes_R M_2 \\
(m_1\otimes_{R_{\mathcal{F}}} m_2) &\longrightarrow \mathcal{F}\cdot(m_1\otimes m_2), \quad m_1\in M_1, \quad \text{and} \quad m_2\in M_2,
\end{aligned} \tag{39}$$

is a well defined linear map.

Note that $M_1\otimes_R M_2$ is automatically an (R,R)-bimodule since both M_1 and M_2 are (R,R)-bimodules. Similarly, $M_1\otimes_{R_{\mathcal{F}}} M_2$ is an $(R_{\mathcal{F}}, R_{\mathcal{F}})$-bimodule. In the meantime, $M_1\otimes_R M_2$ is also a left H-module. The next lemma indicates how these module structures are related.

Lemma 4.5 *For any $a \in R$ and $m \in M_1 \otimes_{R_\mathcal{F}} M_2$,*

$$\mathcal{F}^{\#}(a \cdot_\mathcal{F} m) = \alpha_\mathcal{F}(a) \cdot \mathcal{F}^{\#}(m); \tag{40}$$
$$\mathcal{F}^{\#}(m \cdot_\mathcal{F} a) = \beta_\mathcal{F}(a) \cdot \mathcal{F}^{\#}(m), \tag{41}$$

where the dot on the RHS means the left H-action on $M_1 \otimes_R M_2$, and the dot $\cdot_\mathcal{F}$ on the LHS refers to the both left and right $R_\mathcal{F}$-actions on $M_1 \otimes_{R_\mathcal{F}} M_2$.

Proof. For simplicity, let us assume that $m = m_1 \otimes_{R_\mathcal{F}} m_2$ for $m_1 \in M_1$ and $m_2 \in M_2$. Then

$$\begin{aligned}
& \mathcal{F}^{\#}(a \cdot_\mathcal{F} (m_1 \otimes_{R_\mathcal{F}} m_2)) \\
= \; & \mathcal{F}^{\#}((\alpha_\mathcal{F}(a) m_1) \otimes_{R_\mathcal{F}} m_2) \\
= \; & \mathcal{F} \cdot (\sum_i \alpha(x_i(a)) y_i m_1 \otimes_{R_\mathcal{F}} m_2) \\
= \; & \sum_{ij} x_j \alpha(x_i(a)) y_i m_1 \otimes_R y_j m_2 \\
= \; & (\Delta \otimes_R id) \mathcal{F}\mathcal{F}^{12}(a, \cdot, \cdot) \cdot (m_1 \otimes m_2).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& \alpha_\mathcal{F}(a) \cdot \mathcal{F}^{\#}(m_1 \otimes_{R_\mathcal{F}} m_2) \\
= \; & \alpha_\mathcal{F}(a) \cdot (\sum_j x_j m_1 \otimes_R y_j m_2) \\
= \; & \sum_j \Delta(\alpha_\mathcal{F}(a))(x_j m_1 \otimes_R y_j m_2) \\
= \; & \sum_{ij} \Delta(\alpha(x_i(a)) y_i)(x_j m_1 \otimes_R y_j m_2) \\
= \; & \sum_{ij} ((x_i(a) \cdot \Delta y_i)(x_j m_1 \otimes_R y_j m_2) \\
= \; & \sum_{ij} [(x_i(a) \otimes_R \Delta y_i (x_j \otimes_R y_j)](m_1 \otimes m_2) \\
= \; & (id \otimes_R \Delta) \mathcal{F}\mathcal{F}^{23}(a, \cdot, \cdot) \cdot (m_1 \otimes m_2).
\end{aligned}$$

The conclusion thus follows again from Equation (32).

□

We say that $\mathcal{F}$ is *invertible* if $\mathcal{F}^{\#}$ defined by Equation (39) is a vector space isomorphism for any left H-modules M_1 and M_2. In this case, in particular we can take $M_1 = M_2 = H$ so that we have an isomorphism

$$\mathcal{F}^{\#} : H \otimes_{R_\mathcal{F}} H \longrightarrow H \otimes_R H. \tag{42}$$

An immediate consequence of Lemma 4.5 is the following

Corollary 4.6 *If $\mathcal{F}$ is invertible, then for any $a \in R_{\mathcal{F}}$ and $n \in M_1 \otimes_R M_2$,*

$$\mathcal{F}^{\#^{-1}}(\alpha_{\mathcal{F}}(a) \cdot n) = a \cdot_{\mathcal{F}} \mathcal{F}^{\#^{-1}}(n);$$
$$\mathcal{F}^{\#^{-1}}(\beta_{\mathcal{F}}(a) \cdot n) = \mathcal{F}^{\#^{-1}}(n) \cdot_{\mathcal{F}} a.$$

Definition 4.7 *An element $\mathcal{F} \in H \otimes_R H$ is called a twistor if it is invertible and satisfies Equations (32) and (33).*

Now assume that $\mathcal{F}$ is a twistor. Define a new coproduct $\Delta_{\mathcal{F}} : H \longrightarrow H \otimes_{R_{\mathcal{F}}} H$ by

$$\Delta_{\mathcal{F}} = \mathcal{F}^{-1} \Delta \mathcal{F}, \tag{43}$$

where Equation (43) means that $\Delta_{\mathcal{F}}(x) = \mathcal{F}^{\#^{-1}}(\Delta(x)\mathcal{F})$, $\forall x \in H$. In what follows, we will prove that $\Delta_{\mathcal{F}}$ is indeed a Hopf algebroid co-product.

Lemma 4.8 *For any $x \in H$ and $a \in R_{\mathcal{F}}$,*

$$\Delta(a \cdot_{\mathcal{F}} x) = \alpha_{\mathcal{F}}(a) \cdot \Delta x; \tag{44}$$
$$\Delta(x \cdot_{\mathcal{F}} a) = \beta_{\mathcal{F}}(a) \cdot \Delta x, \tag{45}$$

where $\cdot_{\mathcal{F}}$ refers to the $R_{\mathcal{F}}$ (both the left and the right) actions on H.

Proof. We have

$$\begin{aligned} & \Delta(a \cdot_{\mathcal{F}} x) \\ = & \Delta(\alpha_{\mathcal{F}}(a)x) \\ = & \Delta(\alpha_{\mathcal{F}}(a))\Delta x \\ = & \alpha_{\mathcal{F}}(a) \cdot \Delta x. \end{aligned}$$

Equation (45) can be proved similarly.

□

Proposition 4.9 $\Delta_{\mathcal{F}} : H \longrightarrow H \otimes_{R_{\mathcal{F}}} H$ *is an $(R_{\mathcal{F}}, R_{\mathcal{F}})$-bimodule map.*

Proof. For any $a \in R_{\mathcal{F}}$ and $x \in H$, using Lemma 4.8 and Corollary 4.6, we have

$$\begin{aligned} & \Delta_{\mathcal{F}}(a \cdot_{\mathcal{F}} x) \\ = & \mathcal{F}^{\#^{-1}}[\Delta(a \cdot_{\mathcal{F}} x)\mathcal{F}] \\ = & \mathcal{F}^{\#^{-1}}[\alpha_{\mathcal{F}}(a) \cdot \Delta x \mathcal{F}] \\ = & a \cdot_{\mathcal{F}} \mathcal{F}^{\#^{-1}}(\Delta x \mathcal{F}) \\ = & a \cdot_{\mathcal{F}} \Delta_{\mathcal{F}}(x). \end{aligned}$$

Similarly, we can show that $\Delta_{\mathcal{F}}(x \cdot_{\mathcal{F}} a) = \Delta_{\mathcal{F}} x \cdot_{\mathcal{F}} a$.

$\square$

Proposition 4.10 *The comultiplication $\Delta_{\mathcal{F}} : H \longrightarrow H \otimes_{R_{\mathcal{F}}} H$ is compatible with the multiplication in H.*

Proof. Consider the maps

$$\Psi : \ H \otimes H \otimes H \longrightarrow H \otimes_R H : \quad \sum h_1 \otimes h_2 \otimes h_3 \longmapsto \sum (\Delta h_1)(h_2 \otimes h_3), \tag{46}$$

and

$$\Psi_{\mathcal{F}} : \ H \otimes H \otimes H \longrightarrow H \otimes_{R_{\mathcal{F}}} H : \quad \sum h_1 \otimes h_2 \otimes h_3 \longmapsto \sum (\Delta_{\mathcal{F}} h_1)(h_2 \otimes h_3). \tag{47}$$

We first prove that

$$\mathcal{F}^{\#} \circ \Psi_{\mathcal{F}} = \Psi \circ \mathcal{F}^{\#}_{23}. \tag{48}$$

$$\begin{aligned}
(\mathcal{F}^{\#} \circ \Psi_{\mathcal{F}})(\sum h_1 \otimes h_2 \otimes h_3) &= \sum \mathcal{F}^{\#}((\Delta_{\mathcal{F}} h_1)(h_2 \otimes h_3)) \\
&= \sum \mathcal{F}^{\#}[\mathcal{F}^{\#^{-1}}((\Delta h_1)\mathcal{F})(h_2 \otimes h_3)] \\
&= \sum \Delta h_1 \mathcal{F}(h_2 \otimes h_3) \\
&= (\Psi \circ \mathcal{F}^{\#}_{23})(\sum h_1 \otimes h_2 \otimes h_3).
\end{aligned}$$

Equation (48) implies that $Ker \Psi_{\mathcal{F}} = Ker \Psi$. To see this, note that $x \in Ker \Psi_{\mathcal{F}}$, i.e., $\Psi_{\mathcal{F}}(x) = 0$ is equivalent to $(\mathcal{F}^{\#} \circ \Psi_{\mathcal{F}})(x) = 0$, which is equivalent to $(\Psi \circ \mathcal{F}^{\#}_{23})(x) = 0$, or $\mathcal{F}^{\#}_{23}(x) \in Ker \Psi$. Since $Ker \Psi$ is a left ideal in $H \otimes H^{op} \otimes H^{op}$, the latter is equivalent to that $x \in Ker \Psi$. The final conclusion thus follows from Proposition 3.2.

$\square$

Proposition 4.10 implies that $M_1 \otimes_{R_{\mathcal{F}}} M_2$ is still a left H-module for any left H-modules M_1, M_2. Moreover, it is easy to see that $\mathcal{F}^{\#} : M_1 \otimes_{R_{\mathcal{F}}} M_2 \longrightarrow M_1 \otimes_R M_2$ is an isomorphism of left H-modules. Now we are ready to prove that $\Delta_{\mathcal{F}}$ is coassociative.

Proposition 4.11 *$\Delta_{\mathcal{F}} : H \longrightarrow H \otimes_{R_{\mathcal{F}}} H$ is coassociative.*

Proof. Suppose that M_1, M_2, M_3 are any left H-modules. It suffices to prove that the natural identification $\varphi_{\mathcal{F}} : (M_1 \otimes_{R_{\mathcal{F}}} M_2) \otimes_{R_{\mathcal{F}}} M_3 \longrightarrow M_1 \otimes_{R_{\mathcal{F}}} (M_2 \otimes_{R_{\mathcal{F}}} M_3)$ is an isomorphism of left H-modules.

Consider the following diagram:

$$\begin{array}{ccccc}
(M_1 \otimes_{R_{\mathcal{F}}} M_2) \otimes_{R_{\mathcal{F}}} M_3 & \stackrel{(\mathcal{F}^{12})^{\#}}{\longrightarrow} & (M_1 \otimes_R M_2) \otimes_{R_{\mathcal{F}}} M_3 & \stackrel{[(\Delta \otimes_R id)\mathcal{F}]^{\#}}{\longrightarrow} & (M_1 \otimes_R M_2) \otimes_R M_3 \\
\varphi_{\mathcal{F}} \downarrow & & & & \downarrow \varphi \\
M_1 \otimes_{R_{\mathcal{F}}} (M_2 \otimes_{R_{\mathcal{F}}} M_3) & \stackrel{(\mathcal{F}^{23})^{\#}}{\longrightarrow} & M_1 \otimes_{R_{\mathcal{F}}} (M_2 \otimes_R M_3) & \stackrel{[(id \otimes_R \Delta)\mathcal{F}]^{\#}}{\longrightarrow} & M_1 \otimes_R (M_2 \otimes_R M_3)
\end{array}$$

Equation (32) implies that the above diagram commutes. Since all the other maps involved in the diagram above are isomorphisms of left H-modules, $\varphi_{\mathcal{F}}$ is an H-module isomorphism as well. This concludes the proof.

$\square$

By now, we have actually proved all the Hopf algebroid axioms for $(H, R_{\mathcal{F}}, \alpha_{\mathcal{F}}, \beta_{\mathcal{F}}, m, \Delta_{\mathcal{F}}, \epsilon)$ except for the condition on counit ϵ. Instead of proving this last condition directly, here we show that μ is still an anchor after the twist, and therefore axiom 4) in Definition 3.1 would be a consequence according to the remark following Definition 3.4. Note that $R_{\mathcal{F}}$ can still be considered as a left H-module under the representation $\mu : H \longrightarrow End R_{\mathcal{F}}$ (here only the underlying vector space structure on $R_{\mathcal{F}}$ is involved). We prove that μ still satisfies the anchor axioms.

Lemma 4.12 *For any $x, y \in H$ and $a \in R$,*

$$\begin{aligned} \varphi_\alpha^{\mathcal{F}}((x \otimes_{R_{\mathcal{F}}} y) \otimes a) &= \varphi_\alpha(\mathcal{F}^{\#}(x \otimes_{R_{\mathcal{F}}} y) \otimes a); & (49)\\ \varphi_\beta^{\mathcal{F}}((x \otimes_{R_{\mathcal{F}}} y) \otimes a) &= \varphi_\beta(\mathcal{F}^{\#}(x \otimes_{R_{\mathcal{F}}} y) \otimes a). & (50) \end{aligned}$$

Proof.

$$\begin{aligned} \varphi_\alpha^{\mathcal{F}}((x \otimes_{R_{\mathcal{F}}} y) \otimes a) &= x(a) \cdot_{\mathcal{F}} y \\ &= \alpha_{\mathcal{F}}(x(a))y \\ &= \sum_i \alpha(x_i(x(a)))y_i y \\ &= \sum_i \alpha(x_i x(a))y_i y. \end{aligned}$$

On the other hand,

$$\begin{aligned} & \varphi_\alpha(\mathcal{F}^{\#}(x \otimes_{R_{\mathcal{F}}} y) \otimes a) \\ = \; & \varphi_\alpha(\sum_i x_i x \otimes_R y_i y \otimes a) \\ = \; & \sum_i \alpha(x_i x(a))y_i y. \end{aligned}$$

Hence, $\varphi_\alpha^{\mathcal{F}}((x \otimes_{R_{\mathcal{F}}} y) \otimes a) = \varphi_\alpha(\mathcal{F}^{\#}(x \otimes_{R_{\mathcal{F}}} y) \otimes a)$. Similarly, one can prove that $\varphi_\beta^{\mathcal{F}}((x \otimes_{R_{\mathcal{F}}} y) \otimes a) = \varphi_\beta(\mathcal{F}^{\#}(x \otimes_{R_{\mathcal{F}}} y) \otimes a)$.

$\square$

Proposition 4.13 *The map $\mu : H \longrightarrow End R_{\mathcal{F}}$ satisfies the anchor axioms in Definition 3.4 for $(H, R_{\mathcal{F}}, \alpha_{\mathcal{F}}, \beta_{\mathcal{F}}, m, \Delta_{\mathcal{F}}, \epsilon)$.*

Proof. First we need to show that μ is an $(R_{\mathcal{F}}, R_{\mathcal{F}})$-bimodule map. This can be checked easily since $(a \cdot_{\mathcal{F}} x)(b) = (\alpha_{\mathcal{F}}(a)x)(b) = \alpha_{\mathcal{F}}(a)(x(b)) = a *_{\mathcal{F}} x(b)$. Similarly, $(x \cdot_{\mathcal{F}} a)(b) = x(b) *_{\mathcal{F}} a$.

The axioms (i) and (ii) in Definition 3.4 hold automatically since τ is an anchor for $(H, R, \alpha, \beta, m, \Delta, \epsilon)$. Now according to Lemma 4.12

$$\begin{aligned}
& \varphi^{\mathcal{F}}_{\alpha}(\Delta_{\mathcal{F}} x \otimes a) \\
= & \ \varphi_{\alpha}(\Delta x \mathcal{F} \otimes a) \\
= & \ \sum_i \varphi_{\alpha}(\Delta x (x_i \otimes_R y_i) \otimes a) \\
= & \ \sum_i x \alpha(x_i(a)) y_i \\
= & \ x \alpha_{\mathcal{F}}(a).
\end{aligned}$$

Here the second from the last equality used Equation (25). Similarly, $\varphi^{\mathcal{F}}_{\beta}(\Delta_{\mathcal{F}} x \otimes a) = x \beta_{\mathcal{F}}(a)$. This concludes the proof.

$\square$

In summary, we have proved

Theorem 4.14 *Assume that $(H, R, \alpha, \beta, m, \Delta, \epsilon)$ is a Hopf algebroid with anchor μ, and $\mathcal{F} \in H \otimes_R H$ a twistor. Then $(H, R_{\mathcal{F}}, \alpha_{\mathcal{F}}, \beta_{\mathcal{F}}, m, \Delta_{\mathcal{F}}, \epsilon)$ is a Hopf algebroid, which still admits μ as an anchor. Moreover, its corresponding monoidal category of left H-modules is equivalent to that of $(H, R, \alpha, \beta, m, \Delta, \epsilon)$.*

We say that $(H, R_{\mathcal{F}}, \alpha_{\mathcal{F}}, \beta_{\mathcal{F}}, m, \Delta_{\mathcal{F}}, \epsilon)$ is obtained from $(H, R, \alpha, \beta, m, \Delta, \epsilon)$ by twisting via $\mathcal{F}$.

The following theorem generalizes a standard result in Hopf algebras [9].

Theorem 4.15 *If $\mathcal{F}_1 \in H \otimes_R H$ is a twistor to the Hopf algebroid H, and $\mathcal{F}_2 \in H \otimes_{R_{\mathcal{F}_1}} H$ a twistor for the twisted Hopf algebroid $H_{\mathcal{F}_1}$, then the Hopf algebroid obtained by twisting H via $\mathcal{F}_1$ then via $\mathcal{F}_2$ is equivalent to that obtained by twisting via $\mathcal{F}_1 \mathcal{F}_2$. Here $\mathcal{F}_1 \mathcal{F}_2 \in H \otimes_R H$ is understood as $\mathcal{F}_1^{\#}(\mathcal{F}_2)$, where $\mathcal{F}_1^{\#} : H \otimes_{R_{\mathcal{F}_1}} H \longrightarrow H \otimes_R H$ is the map as defined in Equation (42).*

Proof. Clearly, $\mathcal{F} = \mathcal{F}_1 \mathcal{F}_2 = \mathcal{F}_1^{\#}(\mathcal{F}_2)$ is a well defined element in $H \otimes_R H$. We only need to verify that $\mathcal{F}$ is still a twistor. The rest of the result is a routine verification. For this purpose, it suffices to show that $\mathcal{F}$ satisfies both Equations (32) and (33). To check these equations, in fact we may think of $\mathcal{F}_1$ and $\mathcal{F}_2$ as elements in $H \otimes H$ by taking some representatives. Now

$$\begin{aligned}
& (\Delta \otimes_R id) \mathcal{F} \mathcal{F}^{12} \\
= & \ (\Delta \otimes_R id) \mathcal{F}_1 (\Delta \otimes_R id) \mathcal{F}_2 \mathcal{F}_1^{12} \mathcal{F}_2^{12} \\
= & \ (\Delta \otimes_R id) \mathcal{F}_1 \mathcal{F}_1^{12} [(\mathcal{F}_1^{12})^{-1} (\Delta \otimes_R id) \mathcal{F}_2 \mathcal{F}_1^{12}] \mathcal{F}_2^{12} \\
= & \ [(\Delta \otimes_R id) \mathcal{F}_1 \mathcal{F}_1^{12}][(\Delta_{\mathcal{F}_1} \otimes_{R_{\mathcal{F}}} id) \mathcal{F}_2 \mathcal{F}_2^{12}] \\
= & \ [(id \otimes_R \Delta) \mathcal{F}_1 \mathcal{F}_1^{23}][(id \otimes_{R_{\mathcal{F}}} \Delta_{\mathcal{F}_1}) \mathcal{F}_2 \mathcal{F}_2^{23}] \\
= & \ (id \otimes_R \Delta) \mathcal{F}_1 (id \otimes_R \Delta) \mathcal{F}_2 \mathcal{F}_1^{23} \mathcal{F}_2^{23} \\
= & \ (id \otimes_R \Delta) \mathcal{F} \mathcal{F}^{23}.
\end{aligned}$$

To prove Equation (33), assume that $\mathcal{F}_1 = \sum_i x_i^{(1)} \otimes_R y_i^{(1)}$, and $\mathcal{F}_2 = \sum_i x_i^{(2)} \otimes_{R_{\mathcal{F}_1}} y_i^{(2)}$. Then $\mathcal{F}_1\mathcal{F}_2 = \sum_{ij} x_i^{(1)} x_j^{(2)} \otimes_R y_i^{(1)} y_j^{(2)}$. And

$$
\begin{aligned}
& (\epsilon \otimes_R id)\mathcal{F} \\
= & \sum_{ij} \epsilon(x_i^{(1)} x_i^{(2)}) \otimes_R y_i^{(1)} y_j^{(2)} \\
= & \sum_{ij} \epsilon(x_i^{(1)} x_i^{(2)}) \cdot y_i^{(1)} y_j^{(2)} \quad \text{(using Equation (23))} \\
= & \sum_{ij} x_i^{(1)} (\epsilon x_i^{(2)}) \cdot y_i^{(1)} y_j^{(2)} \\
= & \sum_{ij} \varphi_\alpha (x_i^{(1)} \otimes_R y_i^{(1)} \otimes \epsilon x_j^{(2)}) y_j^{(2)} \\
= & \sum_{j} \varphi_\alpha (\mathcal{F}_1 \otimes \epsilon x_j^{(2)}) y_j^{(2)} \\
= & \sum_{j} \alpha_{\mathcal{F}_1} (\epsilon x_j^{(2)}) y_j^{(2)} \\
= & (\epsilon \otimes_{R_{\mathcal{F}_1}} id)\mathcal{F}_2 \\
= & 1_H.
\end{aligned}
$$

Similarly, we prove that $(id \otimes_R \epsilon)\mathcal{F} = 1_H$.

□

We end this section by the following:

Example 4.1 Let P be a smooth manifold, $\mathcal{D}$ the algebra of differential operators on P, and $R = C^\infty(P)$. Let $\mathcal{D}[[\hbar]]$ denote the space of formal power series in $\hbar$ with coefficients in $\mathcal{D}$. The Hopf algebroid structure on $\mathcal{D}$ naturally extends to a Hopf algebroid structure on $\mathcal{D}[[\hbar]]$ over the base $R[[\hbar]]$, which also admits a natural anchor map.

Let $\mathcal{F} = 1 \otimes_R 1 + \hbar B_1 + \cdots \in \mathcal{D} \otimes_R \mathcal{D}[[\hbar]] (\cong \mathcal{D}[[\hbar]] \otimes_{R[[\hbar]]} \mathcal{D}[[\hbar]])$ be a formal power series of bidifferential operators. It is easy to see that $\mathcal{F}$ is a twistor iff the multiplication on $R[[\hbar]]$ defined by:

$$f *_\hbar g = \mathcal{F}(f, g), \quad \forall f, g \in R[[\hbar]] \tag{51}$$

is associative with identity being the constant function 1, i.e., $*_\hbar$ is a star product on P. In this case, the bracket $\{f, g\} = B_1(f, g) - B_1(g, f), \quad \forall f,\ g \in C^\infty(P)$ defines a Poisson bracket on P, and $f *_\hbar g = \mathcal{F}(f, g)$ is simply a deformation quantization of such a Poisson structure [6].

The twisted Hopf algebroid can be easily described. Here $\mathcal{D}_\hbar = \mathcal{D}[[\hbar]]$ is equipped with the usual multiplication, $R_\hbar = R[[\hbar]]$ is the $*$-product defined by Equation (51), $\alpha : R_\hbar \longrightarrow \mathcal{D}_\hbar$ and $\beta : R_\hbar \longrightarrow \mathcal{D}_\hbar$ are given, respectively, by

$$\alpha_\hbar(f) g = f *_\hbar g, \quad \beta(f) g = g *_\hbar f, \quad \forall f, g \in R.$$

The co-product $\Delta_\hbar : \mathcal{D}_\hbar \longrightarrow \mathcal{D}_\hbar \otimes_{R_\hbar} \mathcal{D}_\hbar$ is

$$\Delta_\hbar = \mathcal{F}^{-1} \Delta \mathcal{F},$$

and the co-unit ϵ remains the same, i.e., the projection $\mathcal{D}_\hbar \longrightarrow R_\hbar$. This twisted Hopf algebroid $(\mathcal{D}_\hbar, R_\hbar, \alpha, \beta, m, \Delta_\hbar, \epsilon)$ is called the quantum groupoid associated to the star product $*_\hbar$.

5 Quantum groupoids and their classical limits

The main purpose of this section is to introduce quantum universal enveloping algebroids (QUE algebroids), also called quantum groupoids in the paper, as a deformation of the standard Hopf algebroid UA.

Definition 5.1 *A deformation of a Hopf algebroid $(H, R, \alpha, \beta, m, \Delta, \epsilon)$ over a field k is a topological Hopf algebroid $(H_\hbar, R_\hbar, \alpha_\hbar, \beta_\hbar, m_\hbar, \Delta_\hbar, \epsilon_\hbar)$ over the ring $k[[\hbar]]$ of formal power series in $\hbar$ such that*

(i). *$H_\hbar$ is isomorphic to $H[[\hbar]]$ as $k[[\hbar]]$ module with identity 1_H, and $R_\hbar$ is isomorphic to $R[[\hbar]]$ as $k[[\hbar]]$ module with identity 1_R;*

(ii). *$\alpha_\hbar = \alpha(mod\ \hbar), \beta_\hbar = \beta(mod\ \hbar), \quad m_\hbar = m(mod\ \hbar), \epsilon_\hbar = \epsilon(mod\ \hbar)$;*

(iii). *$\Delta_\hbar = \Delta(mod\ \hbar)$.*

In this case, we simply say that the quotient $H_\hbar/\hbar H_\hbar$ is isomorphic to H as a Hopf algebroid.

Here the meaning of (i) and (ii) is clear. However, for Condition (iii), we need the following simple fact:

Lemma 5.2 *Under the hypothesis (i) and (ii) as in Definition 5.1, $H_\hbar \otimes_{R_\hbar} H_\hbar/\hbar(H_\hbar \otimes_{R_\hbar} H_\hbar)$ is isomorphic to $H \otimes_R H$ as a vector space.*

Proof. Define $\tau : H_\hbar \otimes H_\hbar \longrightarrow H \otimes_R H$ by

$$(\sum_i x_i \hbar^i) \otimes (\sum_i y_i \hbar^i) \longrightarrow x_0 \otimes_R y_0.$$

For any $a \in R$ and $x, y \in H$, since

$$\begin{aligned} & (\beta_\hbar a \otimes 1 - 1 \otimes \alpha_\hbar a)(x \otimes y) \\ = \ & (\beta a) x \otimes y - x \otimes (\alpha a) y + O(\hbar), \end{aligned}$$

then $\tau[(\beta_\hbar a \otimes 1 - 1 \otimes \alpha_\hbar a)(x \otimes y)] = 0$. In other words, τ descends to a well defined map from $H_\hbar \otimes_{R_\hbar} H_\hbar$ to $H \otimes_R H$. It is easy to see that τ is surjective and $Ker\tau = \hbar(H_\hbar \otimes_{R_\hbar} H_\hbar)$. The conclusion thus follows immediately.

□

By abuse of notation, we will still use τ to denote the induced map $H_\hbar \otimes_{R_\hbar} H_\hbar \longrightarrow H \otimes_R H$. We shall also use the notation $\hbar \mapsto 0$ to denote this map whenever the meaning is clear from the context. Then, Condition (iii) means that $\lim_{\hbar \mapsto 0} \Delta_\hbar(x) = \Delta(x)$ for any $x \in H$.

Definition 5.3 *A quantum universal enveloping algebroid (or a QUE algebroid), also called a quantum groupoid, is a deformation of the standard Hopf algebroid $(UA, R, \alpha, \beta, m, \Delta, \epsilon)$ of a Lie algebroid A.*

Let $U_\hbar A = UA[[\hbar]]$ and $R_\hbar = R[[\hbar]]$. Assume that $(U_\hbar A, R_\hbar, \alpha_\hbar, \beta_\hbar, m_\hbar, \Delta_\hbar, \epsilon_\hbar)$ is a quantum groupoid. Then $R_\hbar$ defines a star product on P so that the equation

$$\{f, g\} = \lim_{\hbar \mapsto 0} \frac{1}{\hbar}(f *_\hbar g - g *_\hbar f), \quad \forall f, g \in R$$

defines a Poisson structure on the base space P.

Now define

$$\begin{aligned}
\delta f &= \lim_{\hbar \mapsto 0} \frac{1}{\hbar}(\alpha_\hbar f - \beta_\hbar f) \in UA, \quad \forall f \in R, \\
\Delta^1 X &= \lim_{\hbar \mapsto 0} \frac{1}{\hbar}(\Delta_\hbar X - (1 \otimes_{R_\hbar} X + X \otimes_{R_\hbar} 1)) \in UA \otimes_R UA, \quad \forall X \in \Gamma(A), \quad \text{and} \\
\delta X &= \Delta^1 X - (\Delta^1 X)_{21} \in UA \otimes_R UA.
\end{aligned}$$

Here for $T = \sum x \otimes_R y \in UA \otimes_R UA$, T_{21} is defined to be $\sum y \otimes_R x$. For the convenience of notations, we introduce

$$\mathrm{Alt} T = T - T_{21}, \quad \forall T \in UA \otimes_R UA$$

so that $\delta X = \mathrm{Alt} \Delta^1 X$.

Below we will use $*_\hbar$ to denote both the multiplications in $U_\hbar A$ and in $R_\hbar$ provided there is no confusion. For any $f, g \in R$, $x, y \in UA$, write

$$\begin{aligned}
\alpha_\hbar f &= f + \hbar \alpha_1 f + \hbar^2 \alpha_2 f + O(\hbar^3); \\
\beta_\hbar f &= f + \hbar \beta_1 f + \hbar^2 \beta_2 f + O(\hbar^3); \\
f *_\hbar g &= fg + \hbar B_1(f, g) + O(\hbar^2); \\
x *_\hbar y &= xy + \hbar m_1(x, y) + O(\hbar^2),
\end{aligned}$$

where $\alpha_1 f, \beta_1 f, \alpha_2 f, \beta_2 f$ and $m_1(x, y)$ are elements in UA. Hence,

$$\begin{aligned}
\{f, g\} &= B_1(f, g) - B_1(g, f), \quad \text{and} \\
\delta f &= \alpha_1 f - \beta_1 f.
\end{aligned}$$

Lemma 5.4 *For any $f, g \in R$,*

(i). $\alpha_1(fg) = g(\alpha_1 f) + f(\alpha_1 g) + [\alpha_1 f, g] + m_1(f, g) - B_1(f, g)$;

(ii). $\beta_1(fg) = g(\beta_1 f) + f(\beta_1 g) + [\beta_1 f, g] + m_1(f, g) - B_1(g, f)$;

(iii). $[\alpha_1 f, g] - [\beta_1 g, f] = m_1(g, f) - m_1(f, g)$.

Proof. From the identity $\alpha_\hbar(f *_\hbar g) = \alpha_\hbar f *_\hbar \alpha_\hbar g$, it follows, by considering its $\hbar$-terms, that

$$\alpha_1(fg) + B_1(f,g) = m_1(f,g) + (\alpha_1 f)g + f(\alpha_1 g).$$

Thus (i) follows immediately. And (ii) can be proved similarly.

On the other hand, we know, from the definition of Hopf algebroids, that $(\alpha_\hbar f) *_\hbar (\beta_\hbar g) = (\beta_\hbar g) *_\hbar (\alpha_\hbar f)$. By considering its $\hbar$-terms, we obtain

$$(\alpha_1 f)g + f(\beta_1 g) + m_1(f,g) = g(\alpha_1 f) + (\beta_1 g)f + m_1(g,f).$$

This proves (iii).

$\square$

Corollary 5.5 *For any $f, g \in R$,*

(i). $\delta(fg) = f\delta g + g\delta f$;

(ii). $[\delta f, g] = \{f, g\}$.

Proof. By symmetrizing the third identity in Lemma 5.4, we obtain that

$$[\alpha_1 f - \beta_1 f, g] - [\beta_1 g - \alpha_1 g, f] = 0,$$

i.e.,

$$[\delta f, g] = -[\delta g, f].$$

Now subtracting Equation (ii) from Equation (i) in Lemma 5.4, one obtains that

$$\delta(fg) = g\delta f + f\delta g + [\delta f, g] - \{f, g\}.$$

I.e.,

$$\delta(fg) - (g\delta f + f\delta g) = [\delta f, g] - \{f, g\}.$$

Note that the LHS of this equation is symmetric with respect to f and g, whereas the RHS is skew-symmetric, so both sides must vanish. The conclusion thus follows immediately.

$\square$

Lemma 5.6 *For any $f \in R$,*

(i). $\Delta_\hbar f = f \otimes_{R_\hbar} 1 + \hbar(\alpha_1 f \otimes_{R_\hbar} 1 - \Delta_\hbar \alpha_1 f) + \hbar^2(\alpha_2 f \otimes_{R_\hbar} 1 - \Delta_\hbar \alpha_2 f) + O(\hbar^3)$;

(ii). $\Delta_\hbar f = 1 \otimes_{R_\hbar} f + \hbar(1 \otimes_{R_\hbar} \beta_1 f - \Delta_\hbar \beta_1 f) + \hbar^2(1 \otimes_{R_\hbar} \beta_2 f - \Delta_\hbar \beta_2 f) + O(\hbar^3)$;

(iii). $f \otimes_{R_\hbar} 1 - 1 \otimes_{R_\hbar} f = \hbar(1 \otimes_{R_\hbar} \alpha_1 f - \beta_1 f \otimes_{R_\hbar} 1) + \hbar^2(1 \otimes_{R_\hbar} \alpha_2 f - \beta_2 f \otimes_{R_\hbar} 1) + O(\hbar^3)$.

Proof. Since $\Delta_\hbar : U_\hbar A \longrightarrow U_\hbar A \otimes_{R_\hbar} U_\hbar A$ is an $(R_\hbar, R_\hbar)$-bimodule map, it follows that

$$\Delta_\hbar(f \cdot_\hbar 1) = f \cdot_\hbar \Delta_\hbar 1.$$

Here, as well as in the sequel, $\cdot_\hbar$ denotes the $R_\hbar$ (both left and right) actions on $U_\hbar A$, or on its appropriate tensor powers.

Now

$$f \cdot_\hbar 1 = \alpha_\hbar f = f + \hbar\alpha_1 f + \hbar^2\alpha_2 f + O(\hbar^3),$$

while

$$f \cdot_\hbar \Delta_\hbar 1 = f \cdot_\hbar (1 \otimes_{R_\hbar} 1) = \alpha_\hbar f \otimes_{R_\hbar} 1 = (f + \hbar\alpha_1 f + \hbar^2\alpha_2 f) \otimes_{R_\hbar} 1 + O(\hbar^3).$$

Thus it follows that

$$\Delta_\hbar f = f \otimes_{R_\hbar} 1 + \hbar(\alpha_1 f \otimes_{R_\hbar} 1 - \Delta_\hbar \alpha_1 f) + \hbar^2(\alpha_2 f \otimes_{R_\hbar} 1 - \Delta_\hbar \alpha_2 f) + O(\hbar^3).$$

Similarly, one can prove (ii).

Finally, since $1 \otimes_{R_\hbar} \alpha_\hbar f = \beta_\hbar f \otimes_{R_\hbar} 1$, we have

$$1 \otimes_{R_\hbar} (f + \hbar\alpha_1 f + \hbar^2\alpha_2 f + O(\hbar^3)) = (f + \hbar\beta_1 f + \hbar^2\beta_2 f + O(\hbar^3)) \otimes_{R_\hbar} 1.$$

This implies (iii).

$\square$

Corollary 5.7 *For any $f, g \in R$,*

(i). $\Delta(\delta f) = \delta f \otimes_R 1 + 1 \otimes_R \delta f$;

(ii). $\Delta^1(\delta f) = \nu_2 f \otimes_R 1 + 1 \otimes_R \nu_2 f - \Delta\nu_2 f$, *where* $\nu_2 f = \alpha_2 f - \beta_2 f$.

Proof. Combining the three identities in Lemma 5.6 ((i)-(ii)+(iii)), we obtain that

$$\begin{aligned}&\delta f \otimes_{R_\hbar} 1 + 1 \otimes_{R_\hbar} \delta f - \Delta_\hbar \delta f \\ &\quad + \hbar[\alpha_2 f \otimes_{R_\hbar} 1 + 1 \otimes_{R_\hbar} \alpha_2 f - \beta_2 f \otimes_{R_\hbar} 1 - 1 \otimes_{R_\hbar} \beta_2 f - \Delta_\hbar(\alpha_2 f - \beta_2 f)] + O(\hbar^2) = 0.\end{aligned}$$

I.e.,

$$\delta f \otimes_{R_\hbar} 1 + 1 \otimes_{R_\hbar} \delta f - \Delta_\hbar \delta f + \hbar[\nu_2 f \otimes_{R_\hbar} 1 + 1 \otimes_{R_\hbar} \nu_2 f - \Delta_\hbar \nu_2 f] + O(\hbar^2). \tag{52}$$

By letting $\hbar \mapsto 0$, this implies that $\delta f \otimes_R 1 + 1 \otimes_R \delta f - \Delta(\delta f) = 0$. This concludes the proof of (i).

Now writing $\Delta_\hbar \delta f = \delta f \otimes_{R_\hbar} 1 + 1 \otimes_{R_\hbar} \delta f + \hbar\Delta_\hbar^1(\delta f)$, and substituting it back into Equation (52), we obtain

$$\Delta_\hbar^1(\delta f) = \nu_2 f \otimes_{R_\hbar} 1 + 1 \otimes_{R_\hbar} \nu_2 f - \Delta_\hbar \nu_2 f + O(\hbar).$$

(ii) thus follows immediately by letting $\hbar \mapsto 0$.

□

An immediate consequence is

Corollary 5.8 *For any $f \in R$, $\delta f \in \Gamma(A)$ and $\delta^2 f = 0$.*

Proof. From Corollary 5.7 (i), it follows that δf is primitive, i.e., $\delta f \in \Gamma(A)$. According to Corollary 5.7 (ii), $\Delta^1(\delta f)$ is symmetric. Therefore, $\delta^2 f = \delta(\delta f)$, being the skew-symmetric part of $\Delta^1(\delta f)$, equals zero.

□

Lemma 5.9 *For any $X \in \Gamma(A)$,*

$$\Delta^1 X \otimes_R 1 + (\Delta \otimes_R id)\Delta^1 X = 1 \otimes_R \Delta^1 X + (id \otimes_R \Delta)\Delta^1 X. \tag{53}$$

Proof. For any $X \in \Gamma(A)$, denote

$$\Delta^1_\hbar X = \frac{1}{\hbar}[\Delta_\hbar X - (1 \otimes_{R_\hbar} X + X \otimes_{R_\hbar} 1)].$$

Thus $\Delta^1 X = \lim_{\hbar \mapsto 0} \Delta^1_\hbar X$ and

$$\Delta_\hbar X = 1 \otimes_{R_\hbar} X + X \otimes_{R_\hbar} 1 + \hbar \Delta^1_\hbar X.$$

Then

$$\begin{aligned}
(\Delta_\hbar \otimes_{R_\hbar} id)\Delta_\hbar X &= 1 \otimes_{R_\hbar} 1 \otimes_{R_\hbar} X + \Delta_\hbar X \otimes_{R_\hbar} 1 + \hbar(\Delta_\hbar \otimes_{R_\hbar} id)\Delta^1_\hbar X \\
&= 1 \otimes_{R_\hbar} 1 \otimes_{R_\hbar} X + 1 \otimes_{R_\hbar} X \otimes_{R_\hbar} 1 + X \otimes_{R_\hbar} 1 \otimes_{R_\hbar} 1 + \hbar[\Delta^1_\hbar X \otimes_{R_\hbar} 1 + (\Delta_\hbar \otimes_{R_\hbar} id)\Delta^1_\hbar X)];
\end{aligned}$$

and

$$\begin{aligned}
(id \otimes_{R_\hbar} \Delta_\hbar)\Delta_\hbar X &= 1 \otimes_{R_\hbar} \Delta_\hbar X + X \otimes_{R_\hbar} 1 \otimes_{R_\hbar} 1 + \hbar(id \otimes_{R_\hbar} \Delta_\hbar)\Delta^1_\hbar X \\
&= 1 \otimes_{R_\hbar} 1 \otimes_{R_\hbar} X + 1 \otimes_{R_\hbar} X \otimes_{R_\hbar} 1 + X \otimes_{R_\hbar} 1 \otimes_{R_\hbar} 1 + \hbar[1 \otimes_{R_\hbar} \Delta^1_\hbar X + (id \otimes_{R_\hbar} \Delta_\hbar)\Delta^1_\hbar X].
\end{aligned}$$

It thus follows that

$$\Delta^1_\hbar X \otimes_{R_\hbar} 1 + (\Delta_\hbar \otimes_{R_\hbar} id)\Delta^1_\hbar X = 1 \otimes_{R_\hbar} \Delta^1_\hbar X + (id \otimes_{R_\hbar} \Delta_\hbar)\Delta^1_\hbar X. \tag{54}$$

The conclusion thus follows immediately by letting $\hbar \mapsto 0$.

□

According to Proposition 8.1 in Appendix, immediately we have the following

Corollary 5.10 *For any $X \in \Gamma(A)$, $\delta X \in \Gamma(\wedge^2 A)$.*

Lemma 5.11 *For any $f \in R$ and $X \in \Gamma(A)$,*

$$\delta(fX) = f\delta X + \delta f \wedge X.$$

Proof. For any $X \in \Gamma(A)$, again we let

$$\Delta_\hbar^1 X = \frac{1}{\hbar}[\Delta_\hbar X - (1 \otimes_{R_\hbar} X + X \otimes_{R_\hbar} 1)].$$

Now

$$f \cdot_\hbar X = \alpha_\hbar f *_\hbar X = fX + \hbar[(\alpha_1 f)X + m_1(f, X)] + O(\hbar^2).$$

Hence

$$\begin{aligned} & \Delta_\hbar(f \cdot_\hbar X) \\ = \; & \Delta_\hbar(fX) + \hbar[\Delta_\hbar((\alpha_1 f)X) + \Delta_\hbar m_1(f, X)] + O(\hbar^2) \\ = \; & 1 \otimes_{R_\hbar} fX + fX \otimes_{R_\hbar} 1 + \hbar[\Delta_\hbar^1(fX) + \Delta_\hbar((\alpha_1 f)X) + \Delta_\hbar m_1(f, X)] + O(\hbar^2). \end{aligned} \tag{55}$$

On the other hand,

$$\begin{aligned} & f \cdot_\hbar \Delta_\hbar X \\ = \; & f \cdot_\hbar (X \otimes_{R_\hbar} 1 + 1 \otimes_{R_\hbar} X + \hbar \Delta_\hbar^1 X) \\ = \; & [fX + \hbar((\alpha_1 f)X + m_1(f, X))] \otimes_{R_\hbar} 1 + (f + \hbar\alpha_1 f) \otimes_{R_\hbar} X + \hbar f \cdot_\hbar \Delta_\hbar^1 X + O(\hbar^2). \end{aligned} \tag{56}$$

From Equations (55) and (56), it follows that

$$\begin{aligned} 1 \otimes_{R_\hbar} fX - f \otimes_{R_\hbar} X \;=\; & \hbar[(\alpha_1 f)X \otimes_{R_\hbar} 1 + m_1(f, X) \otimes_{R_\hbar} 1 + \alpha_1 f \otimes_{R_\hbar} X + f \cdot_\hbar \Delta_\hbar^1 X \\ & -\Delta_\hbar^1(fX) - \Delta_\hbar((\alpha_1 f)X) - \Delta_\hbar m_1(f, X)] + O(\hbar^2). \end{aligned} \tag{57}$$

From the identity $1 \otimes_{R_\hbar} f \cdot_\hbar X = (1 \cdot_\hbar f) \otimes_{R_\hbar} X$, it follows that

$$1 \otimes_{R_\hbar} (fX + \hbar((\alpha_1 f)X + m_1(f, X))) = (f + \hbar\beta_1 f) \otimes_{R_\hbar} X + O(\hbar^2).$$

That is,

$$1 \otimes_{R_\hbar} fX - f \otimes_{R_\hbar} X = \hbar[\beta_1 f \otimes_{R_\hbar} X - 1 \otimes_{R_\hbar} (\alpha_1 f)X - 1 \otimes_{R_\hbar} m_1(f, X)] + O(\hbar^2). \tag{58}$$

By comparing Equations (57) and (58), one obtains that

$$\begin{aligned} \Delta_\hbar^1(fX) \;=\; & f \cdot_\hbar \Delta_\hbar^1 X + (\alpha_1 f - \beta_1 f) \otimes_{R_\hbar} X + (\alpha_1 f)X \otimes_{R_\hbar} 1 + 1 \otimes_{R_\hbar} (\alpha_1 f)X - \Delta_\hbar((\alpha_1 f)X) \\ & + m_1(f, X) \otimes_{R_\hbar} 1 + 1 \otimes_{R_\hbar} m_1(f, X) - \Delta_\hbar m_1(f, X) + O(\hbar). \end{aligned}$$

Taking the limit by letting $\hbar \mapsto 0$, we obtain

$$\begin{aligned} \Delta^1(fX) \;=\; & f\Delta^1 X + \delta f \otimes_R X + (\alpha_1 f)X \otimes_R 1 + 1 \otimes_R (\alpha_1 f)X - \Delta((\alpha_1 f)X) \\ & + m_1(f, X) \otimes_R 1 + 1 \otimes_R m_1(f, X) - \Delta m_1(f, X). \end{aligned}$$

The conclusion thus follows by taking the skew-symmetrization.

□

In summary, we have proved the following

Proposition 5.12 *For any $f, g \in R$ and $X \in \Gamma(A)$,*

(i). $\delta f \in \Gamma(A)$ *and* $\delta X \in \Gamma(\wedge^2 A)$*;*

(ii). $\delta(fg) = f\delta g + g\delta f$*;*

(iii). $\delta(fX) = f\delta X + \delta f \wedge X$*;*

(iv). $[\delta f, g] = \{f, g\}$*;*

(v). $\delta^2 f = 0$.

Properties (i)-(iii) above allow us to extend δ to a well-defined degree 1 derivation $\delta : \Gamma(\wedge^* A) \longrightarrow \Gamma(\wedge^{*+1} A)$. Below we will show that $(\oplus \Gamma(\wedge^* A), \wedge, [\cdot, \cdot], \delta)$ is a strong differential Gersterhaber algebra. For this purpose, it suffices to show that δ is a derivation with respect to $[\cdot, \cdot]$, and $\delta^2 = 0$. We will prove these facts in two separate propositions below.

Proposition 5.13 *For any $X, Y \in \Gamma(A)$,*

$$\delta[X, Y] = [\delta X, Y] + [X, \delta Y]. \tag{59}$$

Proof. $\forall X, Y \in \Gamma(A)$,

$$\Delta_\hbar(X *_\hbar Y) = \Delta_\hbar X *_\hbar \Delta_\hbar Y = (1 \otimes_{R_\hbar} X + X \otimes_{R_\hbar} 1 + \hbar \Delta_\hbar^1 X) *_\hbar (1 \otimes_{R_\hbar} Y + Y \otimes_{R_\hbar} 1 + \hbar \Delta_\hbar^1 Y).$$

It thus follows that

$$\begin{aligned} \Delta_\hbar [X, Y]_\hbar &= 1 \otimes_{R_\hbar} [X, Y]_\hbar + [X, Y]_\hbar \otimes_{R_\hbar} 1 + \hbar[(\Delta_\hbar^1 X)(1 \otimes Y + Y \otimes 1) + (1 \otimes X + X \otimes 1)\Delta_\hbar^1 Y \\ &\quad -(\Delta_\hbar^1 Y)(1 \otimes X + X \otimes 1) - (1 \otimes Y + Y \otimes 1)\Delta_\hbar^1 X] + O(\hbar^2). \end{aligned} \tag{60}$$

Here $[X, Y]_\hbar = X *_\hbar Y - Y *_\hbar X$. Then $[X, Y]_\hbar = [X, Y] + \hbar l_1(X, Y) + O(\hbar^2)$, where $l_1(X, Y) = m_1(X, Y) - m_1(Y, X)$. Hence

$$\begin{aligned} \Delta_\hbar [X, Y]_\hbar &= \Delta_\hbar [X, Y] + \hbar \Delta_\hbar l_1(X, Y) + O(\hbar^2) \\ &= 1 \otimes_{R_\hbar} [X, Y] + [X, Y] \otimes_{R_\hbar} 1 + \hbar(\Delta_\hbar^1 [X, Y] + \Delta_\hbar l_1(X, Y)) + O(\hbar^2). \end{aligned} \tag{61}$$

Comparing Equation (61) with Equation (60), we obtain

$$\begin{aligned} \Delta_\hbar^1 [X, Y] &= -\Delta_\hbar l_1(X, Y) + 1 \otimes_{R_\hbar} l_1(X, Y) + l_1(X, Y) \otimes_{R_\hbar} 1 \\ &\quad +(\Delta_\hbar^1 X)(1 \otimes Y + Y \otimes 1) + (1 \otimes X + X \otimes 1)\Delta_\hbar^1 Y \\ &\quad -(\Delta_\hbar^1 Y)(1 \otimes X + X \otimes 1) - (1 \otimes Y + Y \otimes 1)\Delta_\hbar^1 X + O(\hbar). \end{aligned}$$

This implies, by letting $h \mapsto 0$, that

$$\begin{aligned}\Delta^1[X,Y] &= -\Delta l_1(X,Y) + 1\otimes_R l_1(X,Y) + l_1(X,Y)\otimes_R 1\\ &\quad +(\Delta^1 X)(1\otimes Y + Y\otimes 1) + (1\otimes X + X\otimes 1)\Delta^1 Y\\ &\quad -(\Delta^1 Y)(1\otimes X + X\otimes 1) - (1\otimes Y + Y\otimes 1)\Delta^1 X.\end{aligned}$$

Equation (59) thus follows immediately by taking the skew-symmetrization.

□

Proposition 5.14 *For any* $X \in \Gamma(A)$,

$$\delta^2 X = 0.$$

Proof. Let

$$J_\hbar = (\Delta_\hbar \otimes_{R_\hbar} id)\Delta^1_\hbar X - (id\otimes_{R_\hbar}\Delta_\hbar)\Delta^1_\hbar X - 1\otimes_{R_\hbar}\Delta^1_\hbar X + \Delta^1_\hbar X\otimes_{R_\hbar} 1. \tag{62}$$

From Equation (54) we know that $J_\hbar = 0$. Let $\{e_i \in UA\}$ $(e_0 = 1)$ be a local basis of UA over the left module R. Assume that $\delta X = \sum Y_i \wedge Z_i$ with $Y_i, Z_i \in \Gamma(A)$, and

$$\Delta^1 X = \sum f^{ij} e_i \otimes_R e_j + \sum (Y_i \otimes_R Z_i - Z_i \otimes_R Y_i),$$

where $f^{ij} \in R$ are symmetric: $f^{ij} = f^{ji}$. We also assume that $\Delta e_i = \sum g_i^{kl} e_k \otimes_R e_l$ with $g_i^{kl} = g_i^{lk} \in R$ since Δ is co-commutative. Let us write

$$\Delta^1_\hbar X = \sum f^{ij} \cdot_\hbar e_i \otimes_{R_\hbar} e_j + \sum (Y_i \otimes_{R_\hbar} Z_i - Z_i \otimes_{R_\hbar} Y_i) + \hbar \Delta^2_\hbar X; \quad \text{and} \tag{63}$$

$$\Delta_\hbar e_i = \sum g_i^{kl} \cdot_\hbar e_k \otimes_{R_\hbar} e_l + \hbar \Delta^1_\hbar e_i, \tag{64}$$

for some $\Delta^2_\hbar X$ and $\Delta^1_\hbar e_i \in U_\hbar A \otimes_{R_\hbar} U_\hbar A$.

Then

$$\begin{aligned}(id\otimes_{R_\hbar}\Delta_\hbar)\Delta^1_\hbar X &= \sum (\alpha_\hbar f^{ij} *_\hbar \beta_\hbar g_j^{kl} *_\hbar e_i)\otimes_{R_\hbar} e_k \otimes_{R_\hbar} e_l + \sum (Y_i \otimes_{R_\hbar} Z_i \otimes_{R_\hbar} 1 + Y_i \otimes_{R_\hbar} 1 \otimes_{R_\hbar} Z_i\\ &\quad - Z_i \otimes_{R_\hbar} Y_i \otimes_{R_\hbar} 1 - Z_i \otimes_{R_\hbar} 1 \otimes_{R_\hbar} Y_i) + \hbar[\sum f^{ij} \cdot_\hbar e_i \otimes_{R_\hbar} \Delta^1_\hbar e_j\\ &\quad + \sum Y_i \otimes_{R_\hbar} \Delta^1_\hbar Z_i - \sum Z_i \otimes_{R_\hbar} \Delta^1_\hbar Y_i + (id\otimes_{R_\hbar}\Delta_\hbar)\Delta^2_\hbar X],\end{aligned}$$

and

$$1\otimes_{R_\hbar}\Delta^1_\hbar X = \sum \beta_\hbar f^{ij} \otimes_{R_\hbar} e_i \otimes_{R_\hbar} e_j + \sum (1\otimes_{R_\hbar} Y_i \otimes_{R_\hbar} Z_i - 1\otimes_{R_\hbar} Z_i \otimes_{R_\hbar} Y_i) + \hbar(1\otimes_{R_\hbar}\Delta^2_\hbar X).$$

Similarly, we may write

$$\Delta^1_\hbar X = \sum e_i \otimes_{R_\hbar} e_j \cdot_\hbar f^{ij} + \sum (Y_i \otimes_{R_\hbar} Z_i - Z_i \otimes_{R_\hbar} Y_i) + \hbar \tilde{\Delta}^2_\hbar X; \quad \text{and} \tag{65}$$

$$\Delta_\hbar e_i = \sum e_k \otimes_{R_\hbar} e_l \cdot_\hbar g_i^{kl} + \hbar \tilde{\Delta}^1_\hbar e_i, \tag{66}$$

for some $\tilde{\Delta}_\hbar^2 X$ and $\tilde{\Delta}_\hbar^1 e_i \in U_\hbar A \otimes_{R_\hbar} U_\hbar A$.

Hence,

$$\begin{aligned}(\Delta_\hbar \otimes_{R_\hbar} id)\Delta_\hbar^1 X &= \sum e_k \otimes_{R_\hbar} e_l \otimes_{R_\hbar} (\alpha_\hbar g_i^{kl} *_\hbar \beta_\hbar f^{ij} *_\hbar e_j) + \sum (Y_i \otimes_{R_\hbar} 1 \otimes_{R_\hbar} Z_i + 1 \otimes_{R_\hbar} Y_i \otimes_{R_\hbar} Z_i \\ &\quad - Z_i \otimes_{R_\hbar} 1 \otimes_{R_\hbar} Y_i - 1 \otimes_{R_\hbar} Z_i \otimes_{R_\hbar} Y_i) + \hbar[\sum \tilde{\Delta}_\hbar^1 e_i \otimes_{R_\hbar} e_j \cdot_\hbar f^{ij} \\ &\quad + \sum \Delta_\hbar^1 Y_i \otimes_{R_\hbar} Z_i - \sum \Delta_\hbar^1 Z_i \otimes_{R_\hbar} Y_i + (\Delta_\hbar \otimes_{R_\hbar} id)\tilde{\Delta}_\hbar^2 X],\end{aligned}$$

and

$$\Delta_\hbar^1 X \otimes_{R_\hbar} 1 = \sum e_i \otimes_{R_\hbar} e_j \otimes_{R_\hbar} \alpha_\hbar f^{ij} + \sum_i (Y_i \otimes_{R_\hbar} Z_i \otimes_{R_\hbar} 1 - Z_i \otimes_{R_\hbar} Y_i \otimes_{R_\hbar} 1) + \hbar \tilde{\Delta}_\hbar^2 X \otimes_{R_\hbar} 1.$$

Thus we have $J_\hbar = I_\hbar + \hbar K_\hbar$, where

$$\begin{aligned}I_\hbar &= \sum e_i \otimes_{R_\hbar} e_j \otimes_{R_\hbar} (\alpha_\hbar g_l^{ij} *_\hbar \beta_\hbar f^{lk} *_\hbar e_k) - \sum (\alpha_\hbar f^{il} *_\hbar \beta_\hbar g_l^{jk} *_\hbar e_i) \otimes_{R_\hbar} e_j \otimes_{R_\hbar} e_k \\ &\quad - \sum \beta_\hbar f^{ij} \otimes_{R_\hbar} e_i \otimes_{R_\hbar} e_j + \sum e_i \otimes_{R_\hbar} e_j \otimes_{R_\hbar} \alpha_\hbar f^{ij},\end{aligned}$$

and

$$\begin{aligned}K_\hbar &= \sum \tilde{\Delta}_\hbar^1 e_i \otimes_{R_\hbar} e_j \cdot_\hbar f^{ij} + \sum \Delta_\hbar^1 Y_i \otimes_{R_\hbar} Z_i - \sum \Delta_\hbar^1 Z_i \otimes_{R_\hbar} Y_i + (\Delta_\hbar \otimes_{R_\hbar} id)\tilde{\Delta}_\hbar^2 X + \tilde{\Delta}_\hbar^2 X \otimes_{R_\hbar} 1 \\ &\quad -[\sum f^{ij} \cdot_\hbar e_i \otimes_{R_\hbar} \Delta_\hbar^1 e_j + \sum Y_i \otimes_{R_\hbar} \Delta_\hbar^1 Z_i - \sum Z_i \otimes_{R_\hbar} \Delta_\hbar^1 Y_i + (id \otimes_{R_\hbar} \Delta_\hbar)\Delta_\hbar^2 X + 1 \otimes_{R_\hbar} \Delta_\hbar^2 X].\end{aligned}$$

By Alt, we denote the standard skew-symmetrization operator on $UA \otimes_R UA \otimes_R UA$ defined by

$$\mathrm{Alt}(x_1 \otimes_R x_2 \otimes_R x_3) = \sum_{\sigma \in S_3} \frac{1}{3!}(sign\sigma) x_{\sigma(1)} \otimes_R x_{\sigma(2)} \otimes_R x_{\sigma(3)},$$

where $x_1, x_2, x_3 \in UA$. It is tedious but straightforward to verify that $\mathrm{Alt}(\lim_{\hbar \mapsto 0} \frac{1}{\hbar} I_\hbar) = 0$. Therefore, $\mathrm{Alt}(\lim_{\hbar \mapsto 0} K_\hbar) = 0$, i.e.,

$$\begin{aligned}&\mathrm{Alt}[\sum f^{ij} \tilde{\Delta}^1 e_i \otimes_R e_j + \sum \Delta^1 Y_i \otimes_R Z_i - \sum \Delta^1 Z_i \otimes_R Y_i + (\Delta \otimes_R id)\tilde{\Delta}^2 X + \tilde{\Delta}^2 X \otimes_R 1 \\ &-(\sum f^{ij} e_i \otimes_R \Delta^1 e_j + \sum Y_i \otimes_R \Delta^1 Z_i - \sum Z_i \otimes_R \Delta^1 Y_i + (id \otimes_R \Delta)\Delta^2 X + 1 \otimes_R \Delta^2 X)] = 0.\end{aligned}$$

The final conclusion thus follows immediately by applying the skew-symmetrization operator Alt to the equation above and using the following simple fact:

Lemma 5.15 (i). *$Alt(\tilde{\Delta}^2 X \otimes_R 1 - 1 \otimes_R \Delta^2 X) = 0$.*

(ii). *$Alt \sum (f^{ij} \tilde{\Delta}^1 e_i \otimes_R e_j - f^{ij} e_i \otimes_R \Delta^1 e_j) = 0$.*

(iii). *$Alt((id \otimes_R \Delta)\Delta^2 X) = Alt((\Delta \otimes_R id)\tilde{\Delta}^2 X) = 0$.*

Proof. It follows from Equations (63) and (65) that

$$\begin{aligned}
\hbar\Delta_\hbar^2 X - \hbar\tilde{\Delta}_\hbar^2 X &= \sum (f^{ij}\cdot_\hbar e_i \otimes_{R_\hbar} e_j - e_i\otimes_{R_\hbar} e_j \cdot_\hbar f^{ij}) \\
&= \sum (f^{ij}\cdot_\hbar e_i \otimes_{R_\hbar} e_j - e_i\otimes_{R_\hbar} f^{ij}\cdot_\hbar e_j + e_i\otimes_{R_\hbar} f^{ij}\cdot_\hbar e_j - e_i\otimes_{R_\hbar} e_j \cdot_\hbar f^{ij}) \\
&= \hbar\sum \delta f^{ij} *_\hbar e_i\otimes_{R_\hbar} e_j + \hbar\sum e_i\otimes_{R_\hbar}\delta f^{ij} *_\hbar e_j + O(\hbar^2).
\end{aligned}$$

Hence,

$$\begin{aligned}
& \Delta^2 X - \tilde{\Delta}^2 X \\
= & \sum (\delta f^{ij}) e_i \otimes_R e_j + \sum e_i\otimes_R (\delta f^{ij}) e_j \\
= & \sum \Delta(\delta f^{ij}) e_i\otimes_R e_j,
\end{aligned}$$

which is symmetric. It thus follows that

$$\begin{aligned}
& \mathrm{Alt}(\tilde{\Delta}^2 X\otimes_R 1 - 1\otimes_R \Delta^2 X) \\
= & \mathrm{Alt}(\Delta^2 X\otimes_R 1 - 1\otimes_R\Delta^2 X - \sum \Delta(\delta f^{ij}) e_i\otimes_R e_j\otimes_R 1) \\
= & 0.
\end{aligned}$$

Similarly, one can show that

$$\Delta^1 e_i - \tilde{\Delta}^1 e_i = \sum_{kl} \Delta(\delta g_i^{kl})(e_k\otimes_R e_l).$$

Thus, (ii) follows immediately. Finally, (iii) is obvious since Δ is co-commutative.

□

Combining Propositions 5.12, 5.13 and 5.14, we conclude that $(\oplus\Gamma(\wedge^* A), \wedge, [\cdot,\cdot], \delta)$ is indeed a strong differential Gersterhaber algebra. Hence (A, A^*) is a Lie bialgebroid, which will be called the classical limit of the quantum groupoid $U_\hbar A$. In summary, we have proved

Theorem 5.16 *A quantum groupoid $(U_\hbar A, R_\hbar, \alpha_\hbar, \beta_\hbar, m_\hbar, \Delta_\hbar, \epsilon_\hbar)$ naturally induces a Lie bialgebroid (A, A^*) as a classical limit. The induced Poisson structure of this Lie bialgebroid on the base space P coincides with the one obtained as the classical limit of the base $*$-algebra $R_\hbar$.*

As an example, in what follows, we will examine the classical limit of those quantum groupoids obtained from the standard Hopf algebroid $UA[[\hbar]]$ by twists. Consider $(UA[[\hbar]], R[[\hbar]], \alpha, \beta, m, \Delta, \epsilon)$ equipped with the standard Hopf algebroid structure induced from that on UA. Assume that

$$\mathcal{F}_\hbar = 1\otimes_R 1 + \hbar\bar{\Lambda} + O(\hbar^2) \in UA\otimes_R UA[[\hbar]], \tag{67}$$

with $\bar{\Lambda}\in UA\otimes_R UA$, is a twistor, and let $(U_\hbar A, R_\hbar, \alpha_\hbar, \beta_\hbar, m_\hbar, \Delta_\hbar, \epsilon_\hbar)$ be its resulting twisted QUE algebroid.

Lemma 5.17 *Assume that $\mathcal{F}_\hbar \in UA\otimes_R UA[[\hbar]]$ given by Equation (67) is a twistor. Then*

$$\bar{\Lambda}\otimes_R 1 + (\Delta\otimes_R id)\bar{\Lambda} = 1\otimes_R\bar{\Lambda} + (id\otimes_R\Delta)\bar{\Lambda}. \tag{68}$$

Proof. This follows immediately from the $\hbar^1$-expansion of Equation (32).

$\square$

Using Proposition 8.1 in Appendix, we have

Corollary 5.18 *Under the same hypothesis as in Lemma 5.17, then* $\Lambda = Alt\bar{\Lambda}(\stackrel{def}{=} \bar{\Lambda} - \bar{\Lambda}_{21})$ *is a section of* $\wedge^2 A$.

It is not surprising to expect the following:

Theorem 5.19 *Let* $(U_\hbar A, R_\hbar, \alpha_\hbar, \beta_\hbar, m_\hbar, \Delta_\hbar, \epsilon_\hbar)$ *be the quantum groupoid obtained from* $(UA[[\hbar]], R[[\hbar]], \alpha, \beta, m, \Delta, \epsilon)$ *by twisting via* $\mathcal{F}_\hbar$ *as given by Equation (67). Then its classical limit is a coboundary Lie bialgebroid* (A, A^*, Λ), *where* $\Lambda = Alt\bar{\Lambda}$. *In particular, its induced Poisson structure on the base space is* $\rho\Lambda$, *which admits* $R_\hbar$ *as a deformation quantization.*

Proof. It suffices to prove that $\delta f = [f, \Lambda]$ and $\delta X = [X, \Lambda], \forall f \in R$ and $X \in \Gamma(A)$.

Write

$$\bar{\Lambda} = \sum_i d_i \otimes_R e_i \in UA \otimes_R UA.$$

It is easy to see that for any $f \in R$,

$$\begin{aligned} \alpha_\hbar f &= f + \hbar \sum_i ((\rho d_i) f) e_i + O(\hbar^2), \quad \text{and} \\ \beta_\hbar f &= f + \hbar \sum_i ((\rho e_i) f) d_i + O(\hbar^2). \end{aligned}$$

It thus follows that

$$\delta f = \sum_i [((\rho d_i) f) e_i - ((\rho e_i) f) d_i] = [f, \Lambda].$$

Now we will prove the second identity $\delta X = [X, \Lambda]$. From the definition of $\mathcal{F}_\hbar^\#$, it follows that

$$\begin{aligned} &\mathcal{F}_\hbar^\# [1 \otimes_{R_\hbar} X + X \otimes_{R_\hbar} 1 + \hbar \sum_i (X d_i \otimes_{R_\hbar} e_i + d_i \otimes_{R_\hbar} X e_i - d_i X \otimes_{R_\hbar} e_i - d_i \otimes_{R_\hbar} e_i X)] \\ = \;& \mathcal{F}_\hbar [1 \otimes X + X \otimes 1 + \hbar \sum_i (X d_i \otimes e_i + d_i \otimes X e_i - d_i X \otimes e_i - d_i \otimes e_i X)] \\ = \;& 1 \otimes_R X + X \otimes_R 1 + \hbar \sum_i (X d_i \otimes_R e_i + d_i \otimes_R e_i X) + O(\hbar^2) \\ = \;& (\Delta X) \mathcal{F}_\hbar + O(\hbar^2). \end{aligned}$$

It thus follows that

$$\begin{aligned} &\Delta_\hbar X \\ = \;& \mathcal{F}_\hbar^{\#^{-1}} ((\Delta X) \mathcal{F}_\hbar) \\ = \;& 1 \otimes_{R_\hbar} X + X \otimes_{R_\hbar} 1 + \hbar \sum_i (X d_i \otimes_{R_\hbar} e_i + d_i \otimes_{R_\hbar} X e_i - d_i X \otimes_{R_\hbar} e_i - d_i \otimes_{R_\hbar} e_i X) + O(\hbar^2). \end{aligned}$$

Therefore

$$\Delta^1 X = (1 \otimes X + X \otimes 1)\bar{\Lambda} - \bar{\Lambda}(1 \otimes X + X \otimes 1).$$

This immediately implies that

$$\begin{aligned}
\delta X &= \Delta^1 X - (\Delta^1 X)_{21} \\
&= [(1\otimes X + X\otimes 1)\bar{\Lambda} - \bar{\Lambda}(1\otimes X + X\otimes 1)] - [(1\otimes X + X\otimes 1)\bar{\Lambda}_{21} - \bar{\Lambda}_{21}(1\otimes X + X\otimes 1)] \\
&= (1\otimes X + X\otimes 1)\Lambda - \Lambda(1\otimes X + X\otimes 1) \\
&= [X, \Lambda].
\end{aligned}$$

This concludes the proof.

□

More generally, we have

Theorem 5.20 *Assume that* $(U_\hbar A, R_\hbar, \alpha_\hbar, \beta_\hbar, m_\hbar, \Delta_\hbar, \epsilon_\hbar)$ *is a quantum groupoid with classical limit* (A, A^*). *Let* $\mathcal{F}_\hbar \in U_\hbar A \otimes_{R_\hbar} U_\hbar A$ *be a twistor such that* $\mathcal{F}_\hbar = 1 \otimes_{R_\hbar} 1 (mod \hbar)$. *Then* $\Lambda = Alt(lim_{\hbar \mapsto 0} \hbar^{-1}(\mathcal{F}_\hbar - 1 \otimes_{R_\hbar} 1))$ *is a section of* $\Gamma(\wedge^2 A)$, *and is a Hamiltonian operator of the Lie bialgebroid* (A, A^*). *If* $(U_\hbar A, \tilde{R}_\hbar, \tilde{\alpha}_\hbar, \tilde{\beta}_\hbar, m_\hbar, \tilde{\Delta}_\hbar, \epsilon_\hbar)$ *is obtained from* $(U_\hbar A, R_\hbar, \alpha_\hbar, \beta_\hbar, m_\hbar, \Delta_\hbar, \epsilon_\hbar)$ *by twisting via* $\mathcal{F}_\hbar$, *its corresponding Lie bialgebroid is obtained from* (A, A^*) *by twisting via* Λ. *In particular, if* (A, A^*) *is a coboundary Lie bialgebroid with r-matroid* Λ_0, *the latter is still a coboundary Lie bialgebroid and its r-matroid is* $\Lambda_0 + \Lambda$.

Proof. The proof is similar to that of Theorem 5.19, even though it is a little bit more complicated. We omit it here.

□

Remark. It is easy to see that the classical limit of the quantum groupoid $(\mathcal{D}_\hbar, R_\hbar, \alpha, \beta, m, \Delta_\hbar, \epsilon)$ in Example 4.1 is the standard Lie bialgebroid (TP, T^*P) associated to the Poisson manifold P. It, however, would be interesting to explore its "dual" quantum groupoid, namely the one with the Lie bialgebroid (T^*P, TP).

6 Quantization of Lie bialgebroids

Definition 6.1 *A quantization of a Lie bialgebroid* (A, A^*) *is a quantum groupoid* $(U_\hbar A, R_\hbar, \alpha_\hbar, \beta_\hbar, m_\hbar, \Delta_\hbar, \epsilon_\hbar)$ *whose classical limit is* (A, A^*).

It is a deep theorem of Etingof and Kazhdan [12] that every Lie bialgebra is quantizable. On the other hand, the existence of $*$-products for arbitrary Poisson manifolds was recently proved by Kontsevich [25]. In terms of Hopf algebroids, this amounts to saying that the Lie bialgebroid (TP, T^*P) associated to a Poisson manifold P is always quantizable. It is therefore natural to expect:

Conjecture Every Lie bialgebroid is quantizable.

Below we will prove a very special case of this conjecture by using Fedosov quantization [16] [43].

Theorem 6.2 *Any regular triangular Lie bialgebroid is quantizable.*

We need some preparation first. Recall that given a Lie algebroid $A \longrightarrow P$ with anchor ρ, an A-connection on a vector bundle $E \longrightarrow P$ is an $\mathbb{R}$-linear map:

$$\Gamma(A) \otimes \Gamma(E) \longrightarrow \Gamma(E)$$

$$X \otimes s \longrightarrow \nabla_X s,$$

satisfying the axioms resembling those of the usual linear connections, i.e., $\forall f \in C^\infty(P),\ X \in \Gamma(A), s \in \Gamma(E)$,

$$\begin{aligned}&\nabla_{fX} s = f\nabla_X s;\\ &\nabla_X(fs) = (\rho(X)f)s + f\nabla_X s.\end{aligned}$$

In particular, if $E = A$, an A-connection is called torsion-free if

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

A torsion-free connection always exists for any Lie algebroid.

Let $\omega \in \Gamma(\wedge^2 A^*)$ be a closed two-form, i.e., $d\omega = 0$. An A-connection on A is said to be compatible with ω if $\nabla_X \omega = 0$. If ω is non-degenerate, a compatible torsion-free connection always exists.

Lemma 6.3 *If $\omega \in \Gamma(\wedge^2 A^*)$ is a closed non-degenerate two-form, there exists a compatible torsion-free A-connection on A.*

Proof. This result is standard (see [36] [40]). The proof is simply just a repetition of that of the existence of a symplectic connection for a symplectic manifold. For completeness, we sketch a proof here.

First, take any torsion-free A-connection ∇. Then any other A-connection can be written as

$$\tilde{\nabla}_X Y = \nabla_X Y + S(X, Y), \quad \forall X, Y \in \Gamma(A), \tag{69}$$

where S is a $(2,1)$-tensor. Clearly, $\tilde{\nabla}$ is torsion-free iff S is symmetric, i.e., $S(X, Y) = S(Y, X)$ for any $X, Y \in \Gamma(A)$.

$\tilde{\nabla}$ is compatible with $\omega \in \Gamma(\wedge^2 A^*)$ iff $\tilde{\nabla}_X \omega = 0$. The latter is equivalent to

$$\omega(S(X, Y), Z) - \omega(S(X, Z), Y) = (\nabla_X \omega)(Y, Z). \tag{70}$$

Let S be the $(2,1)$-tensor defined by the equation:

$$\omega(S(X,Y),Z) = \frac{1}{3}[(\nabla_X\omega)(Y,Z) + (\nabla_Y\omega)(X,Z)]. \tag{71}$$

Clearly, $S(X,Y)$, defined in this way, is symmetric with respect to X and Y. Now

$$\begin{aligned}
& \omega(S(X,Y),Z) - \omega(S(X,Z),Y) \\
= & \frac{1}{3}[(\nabla_X\omega)(Y,Z) + (\nabla_Y\omega)(X,Z)] - \frac{1}{3}[(\nabla_X\omega)(Z,Y) + (\nabla_Z\omega)(X,Y)] \\
= & \frac{1}{3}[(\nabla_X\omega)(Y,Z) + (\nabla_Y\omega)(X,Z) + (\nabla_X\omega)(Y,Z) + (\nabla_Z\omega)(Y,X)] \\
= & (\nabla_X\omega)(Y,Z).
\end{aligned}$$

Here the last step follows from the identity:

$$(\nabla_X\omega)(Y,Z) + (\nabla_Y\omega)(Z,X) + (\nabla_Z\omega)(X,Y) = 0,$$

which is equivalent to that $d\omega = 0$. This implies that $\tilde{\nabla}$ is a torsion-free symplectic connection.

□

Proof of Theorem 6.2 Let (A, A^*, Λ) be a regular triangular Lie bialgebroid. Then $\Lambda^\# : A^* \longrightarrow A$ is a Lie algebroid morphism [33]. Therefore its image $\Lambda^\# A^*$ is a Lie subalgebroid of A, and Λ can be considered as a section of $\wedge^2(\Lambda^\# A^*)$. Hence, by restricting to $\Lambda^\# A^*$ if necessary, one may always assume that Λ is nondegenerate. Let $\omega = \Lambda^{-1} \in \Gamma(\wedge^2 A^*)$. Then ω is closed: $d\omega = 0$. Let ∇ be a compatible torsion-free A-connection on A, which always exists according to Lemma 6.3. Let $(G \rightrightarrows P, \alpha, \beta)$ be a local groupoid corresponding to the Lie algebroid A. Let Λ^l denote the left translation of Λ, so Λ^l defines a left invariant Poisson structure on G. This is a regular Poisson structure, whose symplectic leaves are simply α-fibers. The A-connection ∇ induces a leafwise linear connection $\tilde{\nabla}$ for the α-fibration. To see this, simply define for any $X, Y \in \Gamma(A)$,

$$\tilde{\nabla}_{X^l} Y^l = (\nabla_X Y)^l, \tag{72}$$

where X^l, Y^l and $(\nabla_X Y)^l$ denote their corresponding left invariant vector fields on G. Since left invariant vector fields span the tangent space of α-fibers, this indeed defines a linear connection on each α-fiber $\alpha^{-1}(u), \forall u \in P$. Such a connection will be denoted by $\tilde{\nabla}_u$. Clearly, $\tilde{\nabla}_u$ is torsion-free since ∇ is torsion-free. Moreover, $\tilde{\nabla}$ preserves the Poisson structure Λ^l, and is left-invariant in the sense that

$$L_x^* \tilde{\nabla}_u = \tilde{\nabla}_v, \quad \forall x \in G \ \text{ such that } \beta(x) = u, \alpha(x) = v. \tag{73}$$

Applying Fedosov quantization to this situation, one obtains a $*$-product on G

$$f *_\hbar g = fg + \frac{1}{2}\hbar\Lambda^l(f,g) + \cdots + \hbar^k B_k(f,g) + \cdots \tag{74}$$

quantizing the Poisson structure Λ^l. In fact, this $*$-product is given by a family of leafwise $*$-products indexed by $u \in P$ quantizing the leafwise symplectic structures on α-fibers. Since the Poisson structure Λ^l is left invariant, which means that the leafwise symplectic structures are invariant by left translations L_x, and the symplectic connections $\tilde{\nabla}_u$ are left-invariant, the resulting

Fedosov $*$-products are invariant under left translations. In other words, the bidifferential operators $B_k(\cdot,\cdot)$ are all left invariant, and therefore belong to $UA\otimes_R UA$. In this way, we obtain an element $\mathcal{F}_\hbar = 1 + \hbar\Lambda + O(\hbar^2) \in UA\otimes_R UA[[\hbar]]$ so that the $*$-product on G is

$$f *_\hbar g = \mathcal{F}_\hbar(f,g), \quad \forall f,g \in C^\infty(G).$$

The associativity of $*_\hbar$ implies that $\mathcal{F}_\hbar$ satisfies Equation (32):

$$(\Delta\otimes_R id)\mathcal{F}_\hbar\mathcal{F}_\hbar^{12} \;=\; (id\otimes_R\Delta)\mathcal{F}_\hbar\mathcal{F}_\hbar^{23}. \tag{75}$$

The identity $1 *_\hbar f = f *_\hbar 1 = f$ implies that

$$(\epsilon\otimes_R id)\mathcal{F}_\hbar \;=\; 1_H; \quad (id\otimes_R\epsilon)\mathcal{F}_\hbar \;=\; 1_H. \tag{76}$$

Thus $\mathcal{F}_\hbar \in UA\otimes_R UA[[\hbar]]$ is a twistor, and the resulting twisted Hopf algebroid $(U_\hbar A, R_\hbar, \alpha_\hbar, \beta_\hbar, m_\hbar, \Delta_\hbar, \epsilon_\hbar)$ is a quantization of the triangular Lie bialgebroid (A, A^*, Λ). This concludes the proof of the theorem.

□

In particular, when the base P reduces to a point, Theorem 6.2 implies that every finite dimensional triangular r-matrix is quantizable. Of course, there is no need to use Fedosov method in this special case. There is a very nice short proof due to Drinfeld [10].

Remark. We note that the induced Poisson structure on the base P of a non-degenerate triangular Lie bialgebroid was studied by Nest-Tsygan [36] and Weinstein [40], for which a $*$-product was constructed. In fact our algebra $R_\hbar$ provides a $*$-product for such a Poisson structure, so here we obtain an alternative proof of their result.

7 Universal dynamical R-matrices and dynamical quantum groupoids

Let $U_\hbar\mathfrak{g}$ be a quasi-triangular quantum universal enveloping algebra over $\mathbb{C}$ with R-matrix $R \in U_\hbar\mathfrak{g}\otimes U_\hbar\mathfrak{g}$, $\eta \subset \mathfrak{g}$ a finite dimensional Abelian Lie subalgebra such that $U\eta[[\hbar]]$ is a commutative subalgebra of $U_\hbar\mathfrak{g}$. By $\mathcal{M}(\eta^*)$, we denote the space of meromorphic functions on η^*, and by $\mathcal{D}$ the algebra of meromorphic differential operators on η^*. Let us consider $H = \mathcal{D}\otimes U_\hbar\mathfrak{g}$. Then H is a Hopf algebroid over $\mathbb{C}$ with base algebra $R = \mathcal{M}(\eta^*)[[\hbar]]$, whose coproduct and counit are denoted respectively by Δ and ϵ. Moreover the map

$$\mu(D\otimes u)(f) = (\epsilon_0 u)D(f), \;\; \forall D \in \mathcal{D}, \; u \in U_\hbar\mathfrak{g}, f \in \mathcal{M}(\eta^*) \tag{77}$$

is an anchor map. Here ϵ_0 is the counit of the Hopf algebra $U_\hbar\mathfrak{g}$. Let us fix a basis in η, say $\{h_1, \cdots, h_k\}$, and let $\{\xi_1, \cdots, \xi_k\}$ be its dual basis, which in turn defines a coordinate system $(\lambda_1, \cdots, \lambda_k)$ on η^*.

Set

$$\theta = \sum_{i=1}^{k}(\frac{\partial}{\partial\lambda_i}\otimes h_i) \in H\otimes H, \quad \text{and } \Theta = \exp\hbar\theta \in H\otimes H. \tag{78}$$

Note that θ, and hence Θ, is independent of the choice of the basis in η. The following fact can be easily verified.

Lemma 7.1 *Θ satisfies Equations (32) and (33).*

Proof. Consider $A = \mathcal{D}^{inv} \otimes U\eta[[\hbar]]$, where $\mathcal{D}^{inv}$ consists of holomorphic differential operators on η^* invariant under the translations. Then A is a Hopf subalgebroid of H, which is in fact a Hopf algebra. Clearly, $\theta \in A \otimes A$, so $\Theta \in A \otimes A$. It thus suffices to prove that

$$(\Delta \otimes id)\Theta\Theta^{12} = (id \otimes \Delta)\Theta\Theta^{23} \quad \text{in } A \otimes A \otimes A.$$

Now

$$\begin{aligned} & (\Delta \otimes id)\Theta\Theta^{12} \\ = \; & ((\Delta \otimes id) \exp \hbar\theta) \exp \hbar\theta^{12} \\ = \; & \exp \hbar((\Delta \otimes id)\theta + \theta^{12}) \\ = \; & \exp \hbar \sum_{i=1}^{k} (\frac{\partial}{\partial \lambda_i} \otimes 1 \otimes h_i + 1 \otimes \frac{\partial}{\partial \lambda_i} \otimes h_i + \frac{\partial}{\partial \lambda_i} \otimes h_i \otimes 1). \end{aligned}$$

Here in the second equality we used the fact that $(\Delta \otimes id)\theta$ and θ^{12} commute in $A \otimes A \otimes A$.

Similarly, we have

$$(id \otimes \Delta)\Theta\Theta^{23} = \exp \hbar \sum_{i=1}^{k} (\frac{\partial}{\partial \lambda_i} \otimes 1 \otimes h_i + 1 \otimes \frac{\partial}{\partial \lambda_i} \otimes h_i + \frac{\partial}{\partial \lambda_i} \otimes h_i \otimes 1).$$

This proves Equation (32). Finally, Equation (33) follows from a straightforward verification.

□

Remark There is a more intrinsic way of proving this fact in terms of deformation quantization. Consider $T^*\eta^*$ with the standard cotangent bundle symplectic structure $\sum_{i=1}^{k} d\lambda_i \wedge dp_i$. It is well-known that

$$f *_\hbar g = f e^{\hbar(\sum_{i=1}^{k} \overleftarrow{\frac{\partial}{\partial \lambda_i}} \otimes \overrightarrow{\frac{\partial}{\partial p_i}})} g \tag{79}$$

defines a $*$-product on $T^*\eta^*$, called the Wick type $*$-product corresponding to the normal order quantization. Hence $\exp \hbar(\sum_{i=1}^{k} \overleftarrow{\frac{\partial}{\partial \lambda_i}} \otimes \overrightarrow{\frac{\partial}{\partial p_i}})$, as a (formal) bidifferential operator on $T^*\eta^*$ (i.e. as an element in $\mathcal{D}(T^*\eta^*) \otimes_{\mathcal{M}(T^*\eta^*)} \mathcal{D}(T^*\eta^*)[[\hbar]]$ satisfies Equations (32) and (33) according to Example 4.1. Note that elements in $(\mathcal{D}\eta^* \otimes U\eta) \otimes_{\mathcal{M}(\eta^*)} (\mathcal{D}\eta^* \otimes U\eta)[[\hbar]]$ can be considered as (formal) bidifferential operators on $T^*\eta^*$ invariant under the p-translations, so $(\mathcal{D}\eta^* \otimes U\eta) \otimes_{\mathcal{M}(\eta^*)} (\mathcal{D}\eta^* \otimes U\eta)[[\hbar]]$ is naturally a subspace of $\mathcal{D}(T^*\eta^*) \otimes_{\mathcal{M}(T^*\eta^*)} \mathcal{D}(T^*\eta^*)[[\hbar]]$. Clearly $\exp \hbar(\sum_{i=1}^{k} \overleftarrow{\frac{\partial}{\partial \lambda_i}} \otimes \overrightarrow{\frac{\partial}{\partial p_i}})$ is a p-invariant bidifferential operator on $T^*\eta^*$, and is equal to Θ under the above identification. Equations (32) and (33) thus follows immediately.

In other words, Θ is a twistor of the Hopf algebroid H. As we shall see below, it is this Θ that links a shifted cocycle $F(\lambda)$ and a Hopf algebroid twistor.

Now let $F \in \mathcal{M}(\eta^*, U_\hbar\mathfrak{g}\otimes U_\hbar\mathfrak{g})$, define $F^{12}(\lambda+\hbar h^{(3)}) \in \mathcal{M}(\eta^*, U_\hbar\mathfrak{g}\otimes U_\hbar\mathfrak{g}\otimes U_\hbar\mathfrak{g})$ by

$$\begin{aligned} F^{12}(\lambda+\hbar h^{(3)}) &= F(\lambda)\otimes 1 + \hbar\sum_i \frac{\partial F}{\partial\lambda_i}\otimes h_i + \frac{1}{2!}\hbar^2\sum_{i_1 i_2}\frac{\partial^2 F}{\partial\lambda_{i_1}\partial\lambda_{i_2}}\otimes h_{i_1}h_{i_2} \\ &\quad + \cdots + \frac{\hbar^k}{k!}\sum \frac{\partial^k F}{\partial\lambda^{i_1}\cdots\partial\lambda^{i_k}}\otimes h_{i_1}\cdots h_{i_k} + \cdots, \end{aligned} \tag{80}$$

similarly for $F^{23}(\lambda+\hbar h^{(1)})$ etc.

Lemma 7.2 *Let X be a meomorphic vector field on η^*, and $F \in \mathcal{M}(\eta^*, U_\hbar\mathfrak{g}\otimes U_\hbar\mathfrak{g})$. Then*

$$(\Delta X)F = F(\Delta X) + X(F), \qquad \textit{in} \;\; H\otimes_R H. \tag{81}$$

Proof. Note that F, being considered as an element in $H\otimes_R H$, clearly satisfies the condition that $\forall f \in R$,

$$F(f\otimes 1 - 1\otimes f) = 0, \qquad \text{in } H\otimes_R H.$$

So both $(\Delta X)F$ and $F(\Delta X)$ are well defined elements in $H\otimes_R H$. For any $f, g \in R$, considering both sides of Equation (81) as $U_\hbar\mathfrak{g}\otimes U_\hbar\mathfrak{g}$-valued bidifferential operators and applying them to $f\otimes g$, one obtains that

$$\begin{aligned} & [(\Delta X)F - F(\Delta X)](f\otimes g) \\ =\; & X(Ffg) - FX(fg) \\ =\; & X(F)fg. \end{aligned}$$

Thus Equation (81) follows.

□

An element $F \in \mathcal{M}(\eta^*, U_\hbar\mathfrak{g}\otimes U_\hbar\mathfrak{g})$ is said to be of zero weight if

$$[F(\lambda), 1\otimes h + h\otimes 1] = 0, \; \forall\lambda \in \eta^*, h \in \eta. \tag{82}$$

Lemma 7.3 *Assume that $F \in \mathcal{M}(\eta^*, U_\hbar\mathfrak{g}\otimes U_\hbar\mathfrak{g})$ is of zero weight. Then $\forall n \in \mathbb{N}$,*

$$[(\Delta\otimes_R id)\theta^n]F^{12} = \sum_{k=0}^{n}\sum_{0\le i_1\cdots i_k\le n} C_n^k\Big(\frac{\partial^k F}{\partial\lambda_{i_1}\cdots\partial\lambda_{i_k}}\otimes h_{i_1}\cdots h_{i_k}\Big)(\Delta\otimes_R id)\theta^{n-k}; \tag{83}$$

$$[(id\otimes_R\Delta)\theta]F^{23} = F^{23}(id\otimes_R\Delta)\theta. \tag{84}$$

Proof. To prove Equation (83), let's first consider $n = 1$. Then

$$\begin{aligned}
& [(\Delta \otimes_R id)\theta]F^{12} \\
= \; & \sum_i (\Delta \frac{\partial}{\partial \lambda_i} \otimes_R h_i) F^{12} \\
= \; & \sum_i (\Delta \frac{\partial}{\partial \lambda_i}) F \otimes_R h_i \\
= \; & \sum_i (F \Delta \frac{\partial}{\partial \lambda_i} + \frac{\partial F}{\partial \lambda_i}) \otimes_R h_i \\
= \; & F^{12}(\Delta \otimes_R id)\theta + \sum_i \frac{\partial F}{\partial \lambda_i} \otimes_R h_i .
\end{aligned}$$

The general conclusion follows from induction by using the equation above repeatedly.

For Equation (84), we have

$$\begin{aligned}
& [(id \otimes_R \Delta)\theta]F^{23} \\
= \; & \sum_i (\frac{\partial}{\partial \lambda_i} \otimes_R \Delta h_i) F^{23} \\
= \; & \sum_i \frac{\partial}{\partial \lambda_i} \otimes_R (\Delta h_i) F \\
= \; & \sum_i \frac{\partial}{\partial \lambda_i} \otimes_R (F \Delta h_i) \\
= \; & F^{23}(id \otimes_R \Delta)\theta,
\end{aligned}$$

where the second from the last equality follows from the fact that F is of zero weight. Equation (84) thus follows.

□

Proposition 7.4 *Assume that $F \in \mathcal{M}(\eta^*, U_\hbar \mathfrak{g} \otimes U_\hbar \mathfrak{g})$ is of zero weight. Then*

$$[(\Delta \otimes_R id)\Theta]F^{12}(\lambda) = F^{12}(\lambda + \hbar h^{(3)})(\Delta \otimes_R id)\Theta; \tag{85}$$

$$[(id \otimes_R \Delta)\Theta]F^{23}(\lambda) = F^{23}(\lambda)(id \otimes_R \Delta)\Theta. \tag{86}$$

Proof. Note that $(\Delta \otimes_R id)\Theta = \exp \hbar((\Delta \otimes_R id)\theta)$. Equations (85) and (86) thus follow immediately from Lemma 7.3.

□

Remark One may rewrite Equation (85) as

$$\begin{aligned}
F^{12}(\lambda + \hbar h^{(3)}) &= [(\Delta \otimes_R id)\Theta]F^{12}(\lambda)[(\Delta \otimes_R id)\Theta]^{-1} \\
&= e^{\hbar \sum_{i=1}^k (\Delta \frac{\partial}{\partial \lambda_i} \otimes h_i)} F^{12}(\lambda) e^{-\hbar \sum_{i=1}^k (\Delta \frac{\partial}{\partial \lambda_i} \otimes h_i)} .
\end{aligned}$$

This is essentially the definition of $F^{12}(\lambda+\hbar h^{(3)})$ used in [2], where the operator $\sum_{i=1}^{k}(\Delta\frac{\partial}{\partial\lambda_i}\otimes h_i)$ was denoted by $\mathcal{D}=\sum_{i=1}^{k}\frac{\partial}{\partial\lambda_i}h_i^{(3)}$.

Now set

$$\mathcal{F}=F(\lambda)\Theta\in H\otimes_R H. \tag{87}$$

Theorem 7.5 *Assume that $F\in\mathcal{M}(\eta^*,U_\hbar\mathfrak{g}\otimes U_\hbar\mathfrak{g})$ is of zero weight. Then $\mathcal{F}$ is a twistor (i.e. satisfies Equations (32) and (33)) iff*

$$[(\Delta_0\otimes id)F(\lambda)]F^{12}(\lambda+\hbar h^{(3)})=[(id\otimes\Delta_0)F(\lambda)]F^{23}(\lambda), \tag{88}$$
$$(\epsilon_0\otimes id)F(\lambda)=1;\quad (id\otimes\epsilon_0)F(\lambda)=1, \tag{89}$$

where Δ_0 is the coproduct of $U_\hbar\mathfrak{g}$, and ϵ_0 is the counit map.

Proof. Using Proposition 7.4, we have

$$\begin{aligned}
& (\Delta\otimes_R id)\mathcal{F}\mathcal{F}^{12}\\
= & [(\Delta\otimes_R id)F(\lambda)\Theta]F^{12}(\lambda)\Theta^{12}\\
= & [(\Delta_0\otimes id)F(\lambda)][(\Delta\otimes_R id)\Theta]F^{12}(\lambda)\Theta^{12}\\
= & [(\Delta_0\otimes id)F(\lambda)]F^{12}(\lambda+\hbar h^{(3)})(\Delta\otimes_R id)\Theta\Theta^{12},
\end{aligned}$$

and

$$\begin{aligned}
& (id\otimes_R\Delta)\mathcal{F}\mathcal{F}^{23}\\
= & [(id\otimes_R\Delta)F(\lambda)\Theta]F^{23}(\lambda)\Theta^{23}\\
= & [(id\otimes\Delta_0)F(\lambda)](id\otimes_R\Delta)\Theta F^{23}(\lambda)\Theta^{23}\\
= & [(id\otimes\Delta_0)F(\lambda)]F^{23}(\lambda)(id\otimes_R\Delta)\Theta\Theta^{23}.
\end{aligned}$$

Thus it follows from Lemma 7.1 that Equation (32) and Equation (88) are equivalent.

For Equation (33), we note that $\mathcal{F}=F(\lambda)\Theta=\sum_{k=0}^{\infty}\frac{\hbar^k}{k!}F(\lambda)\theta^k$. It is easy to see that for $k\geq 1$, $(\epsilon\otimes_R id)(F(\lambda)\theta^k)=(id\otimes_R\epsilon)(F(\lambda)\theta^k)=0$ since $\epsilon(\frac{\partial^k}{\partial\lambda_{i_1}\cdots\partial\lambda_{i_k}})=0$ and $\epsilon(h_{i_1}\cdots h_{i_k})=0$. Thus it is immediate that Equations (33) and (89) are equivalent. This concludes the proof of the theorem.

□

Now assume that $F(\lambda)$ is a solution to Equations (88) and (89) so that we can form a quantum groupoid by twisting $\mathcal{D}\otimes U_\hbar\mathfrak{g}$ via $\mathcal{F}$. The resulting quantum groupoid is denoted by $\mathcal{D}\otimes_\hbar U_\hbar\mathfrak{g}$, and is called a *dynamical quantum groupoid.*

As an immediate consequence of Theorem 4.14, we have

Theorem 7.6 *As a monoidal category, the category of $\mathcal{D}\otimes_\hbar U_\hbar\mathfrak{g}$-modules is equivalent to that of $\mathcal{D}\otimes U_\hbar\mathfrak{g}$-modules, and therefore is a braided monoidal category.*

Remark. Representations of a quantum dynamical R-matrix can be understood using the representations of the dynamical quantum groupoid $\mathcal{D}\otimes_{\hbar}U_{\hbar}\mathfrak{g}$. The discussion on this topic will appear somewhere else.

In what follows, we describe various structures of $\mathcal{D}\otimes_{\hbar}U_{\hbar}\mathfrak{g}$ more explicitly.

Proposition 7.7 (i). *$f *_{\mathcal{F}} g = fg$, $\forall f,g \in \mathcal{M}(\eta^*)$, i.e., $R_{\mathcal{F}}$ is isomorphic to $\mathcal{M}(\eta^*)[[\hbar]]$ with pointwise multiplication;*

(ii). *$\alpha_{\mathcal{F}} f = \exp(\hbar\sum_{i=1}^{k} h_i \frac{\partial}{\partial\lambda_i})f = \sum_{1\leq i_1,\cdots,i_n\leq k} \frac{\hbar^n}{n!}\frac{\partial^n f}{\partial\lambda_{i_1}\cdots\partial\lambda_{i_n}} h_{i_1}\cdots h_{i_n}$, $\forall f \in R$;*

(iii). *$\beta_{\mathcal{F}} f = f$, $\forall f \in R$.*

Proof. Assume that $F(\lambda) = \sum_i F_i(\lambda) u_i \otimes v_i$, with $u_i, v_i \in U_{\hbar}\mathfrak{g}$. Let

$$\mathcal{F}_n = F(\lambda)\theta^n = \sum_i \sum_{1\leq i_1,\cdots,i_n\leq k} F_i(\lambda) u_i \frac{\partial^n}{\partial\lambda_{i_1}\cdots\partial\lambda_{i_n}} \otimes_R v_i h_{i_1}\cdots h_{i_n}.$$

Then

$$\begin{aligned}
\alpha_{\mathcal{F}_n} f &= \sum_i \sum_{1\leq i_1,\cdots,i_n\leq k} F_i(\lambda)(\epsilon_0 u_i)\frac{\partial^n f}{\partial\lambda_{i_1}\cdots\partial\lambda_{i_n}} v_i h_{i_1}\cdots h_{i_n} \\
&= \sum_{1\leq i_1,\cdots,i_n\leq k} \frac{\partial^n f}{\partial\lambda_{i_1}\cdots\partial\lambda_{i_n}} [\sum_i F_i(\lambda)(\epsilon_0 u_i) v_i] h_{i_1}\cdots h_{i_n} \\
&= \sum_{1\leq i_1,\cdots,i_n\leq k} \frac{\partial^n f}{\partial\lambda_{i_1}\cdots\partial\lambda_{i_n}} h_{i_1}\cdots h_{i_n},
\end{aligned}$$

where the last equality used the fact that $\sum_i F_i(\lambda)(\epsilon_0 u_i) v_i = (\epsilon_0 \otimes id)F(\lambda) = 1$.

Similarly, we have $\beta_{\mathcal{F}_n} f = f$ if $n = 0$, and otherwise $\beta_{\mathcal{F}_n} f = 0$.

Combining these equations, one obtains immediately that

$$\begin{aligned}
&\alpha_{\mathcal{F}} f = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!}\alpha_{\mathcal{F}_n} f = \sum_{1\leq i_1,\cdots,i_n\leq k} \frac{\hbar^n}{n!}\frac{\partial^n f}{\partial\lambda_{i_1}\cdots\partial\lambda_{i_n}} h_{i_1}\cdots h_{i_n}, \\
&\beta_{\mathcal{F}} f = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!}\beta_{\mathcal{F}_n} f = f, \quad \text{and} \\
&f *_{\mathcal{F}} g = (\alpha_{\mathcal{F}} f)(g) = fg.
\end{aligned}$$

$\square$

To describe the relation between $\Delta_{\mathcal{F}}$ and the quasi-Hopf coproducts Δ_λ, we need a kind of "projection" map from $H \otimes_{R_{\mathcal{F}}} H$ to $\mathcal{M}(\eta^*, U_{\hbar}\mathfrak{g}\otimes U_{\hbar}\mathfrak{g})$. This can be defined as follows. Let $Ad_\Theta :$ $H\otimes H \longrightarrow H\otimes H$ be the adjoint operator: $Ad_\Theta w = \Theta w \Theta^{-1}$, $\forall w \in H\otimes H$. Composing with the natural projection, one obtains a map, denoted by the same symbol Ad_Θ, from $H\otimes H$ to $H\otimes_R H$. Since $\alpha_\Theta = \alpha_{\mathcal{F}}$, $\beta_\Theta = \beta_{\mathcal{F}}$ and $\Theta(\beta_\Theta f\otimes 1 - 1\otimes\alpha_\Theta f) = 0$ in $H\otimes_R H$, then $\Theta(\beta_{\mathcal{F}} f\otimes 1 - 1\otimes\alpha_{\mathcal{F}} f) = 0$

in $H\otimes_R H$. This implies that Ad_Θ descends to a map from $H\otimes_{R_\mathcal{F}} H$ to $H\otimes_R H$. On the other hand, there exists an obvious projection Pr from $H\otimes_R H$ to $\mathcal{M}(\eta^*, U_\hbar\mathfrak{g}\otimes U_\hbar\mathfrak{g})$, which is just to take the 0th-order component. Now composing with this projection, one obtains a map from $H\otimes_{R_\mathcal{F}} H$ to $\mathcal{M}(\eta^*, U_\hbar\mathfrak{g}\otimes U_\hbar\mathfrak{g})$, which is denoted by T. The following proposition gives an explicit description of this map T.

An element $x = D\otimes_\hbar u \in H$ is said to be of order k if $D\in\mathcal{D}[[\hbar]]$ is of order k.

Proposition 7.8 (i). *$T(x\otimes_{R_\mathcal{F}} y) = 0$ if either x or y is of order greater than zero.*

(ii). *$T(fu\otimes_{R_\mathcal{F}} gv) = u\otimes g(\alpha_\mathcal{F} f)v$, $\forall f, g\in\mathcal{M}(\eta^*, U_\hbar\mathfrak{g}\otimes U_\hbar\mathfrak{g})$ and $u, v\in U_\hbar\mathfrak{g}$.*

Proof. (i) is obvious. We prove (ii) below.

$$\begin{aligned}
& T(fu\otimes_{R_\mathcal{F}} gv) \\
= \ & Pr(e^{\hbar\sum_{i=1}^k(\frac{\partial}{\partial\lambda_i}\otimes h_i)}(fu\otimes gv)e^{-\hbar\sum_{i=1}^k(\frac{\partial}{\partial\lambda_i}\otimes h_i)}) \\
= \ & Pr(e^{\hbar\sum_{i=1}^k(\frac{\partial}{\partial\lambda_i}\otimes h_i)}(fu\otimes gv)) \\
= \ & Pr\sum_{1\le i_1,\cdots,i_n\le k}\frac{\hbar^n}{n!}(\frac{\partial^n fu}{\partial\lambda_{i_1}\cdots\partial\lambda_{i_n}})\otimes gh_{i_1}\cdots h_{i_n}v \\
= \ & \sum_{1\le i_1,\cdots,i_n\le k}\frac{\hbar^n}{n!}\frac{\partial^n f}{\partial\lambda_{i_1}\cdots\partial\lambda_{i_n}}u\otimes gh_{i_1}\cdots h_{i_n}v \\
= \ & u\otimes g(\alpha_\mathcal{F} f)v.
\end{aligned}$$

□

Proposition 7.9 *The following diagram:*

$$\begin{tikzcd}
H \arrow[r, "\Delta_\mathcal{F}"] & H\otimes_{R_\mathcal{F}} H \arrow[d, "T"] \\
U_\hbar\mathfrak{g} \arrow[u, "i"] \arrow[r, "\Delta_\lambda"'] & \mathcal{M}(\eta^*, U_\hbar\mathfrak{g}\otimes U_\hbar\mathfrak{g})
\end{tikzcd} \tag{90}$$

commutes, where $i: U_\hbar\mathfrak{g}\longrightarrow H$ is the natural embedding. I.e., $\Delta_\lambda = T\circ\Delta_\mathcal{F}\circ i$.

Proof. For any $u\in U_\hbar\mathfrak{g}$, $(\Delta_\mathcal{F}\circ i)(u) = \Delta_\mathcal{F}(u) = \mathcal{F}^{-1}(\Delta_0 u)\mathcal{F} = \Theta^{-1}F(\lambda)^{-1}(\Delta_0 u)F(\lambda)\Theta$. Then $(T\circ\Delta_\mathcal{F}\circ i)(u) = T[(\Delta_\mathcal{F}\circ i)(u)] = Pr[F(\lambda)^{-1}(\Delta_0 u)F(\lambda)] = F(\lambda)^{-1}(\Delta_0 u)F(\lambda) = \Delta_\lambda u$. The conclusion thus follows.

□

The following theorem describes the classical limit of this quantum groupoid $\mathcal{D}\otimes_\hbar U_\hbar\mathfrak{g}$.

Theorem 7.10 *Let $(U_\hbar\mathfrak{g}, R)$ be a quasitriangular quantum universal enveloping algebra, and $R = 1 + \hbar r_0 (mod \hbar)$. Assume that $F(\lambda) \in U_\hbar\mathfrak{g} \otimes U_\hbar\mathfrak{g}$ is a shifted cocycle that $F(\lambda) = 1 + \hbar f(\lambda)(mod\hbar)$. Then the classical limit of the corresponding dynamical quantum groupoid $\mathcal{D}\otimes_\hbar U_\hbar\mathfrak{g}$ is a coboundary Lie bialgebroid (A, A^*, Λ), where $A = T\eta^* \times \mathfrak{g}$ and $\Lambda = \sum_{i=1}^k \frac{\partial}{\partial\lambda_i} \wedge h_i + \frac{1}{2}Alt(r_0) + Alt f(\lambda)$.*

Proof. It is well known [9] that $r_0 \in \mathfrak{g}\otimes\mathfrak{g}$, and the operator $\delta : \mathfrak{g} \longrightarrow \wedge^2\mathfrak{g}$, $\delta a = [1\otimes a + a\otimes 1, r_0]$, $\forall a \in \mathfrak{g}$, defines the cobracket of the corresponding Lie bialgebra of $(U_\hbar\mathfrak{g}, R)$. Thus the Lie bialgbroid corresponding to $\mathcal{D}\otimes U_\hbar\mathfrak{g}$ is a coboundary Lie bialgbroid $(T\eta^*\times\mathfrak{g}, T^*\eta^*\times\mathfrak{g}^*)$ with r-matroid $\frac{1}{2}\mathrm{Alt}(r_0)$. On the other hand, it is obvious that $\mathrm{Alt}\lim_{\hbar\mapsto 0}\hbar^{-1}(\mathcal{F}-1) = \mathrm{Alt}F(\lambda)+\sum_{i=1}^k \frac{\partial}{\partial\lambda_i}\wedge h_i$. The conclusion follows from Theorem 5.20.

□

As a consequence, we have

Corollary 7.11 *Under the same hypothesis as in Theorem 7.10, $r(\lambda) = \frac{1}{2}Alt(r_0) + Alt f(\lambda)$ is a dynamical r-matrix.*

We refer the interested reader to [1] [19] [24] for the construction of shifted cocycles $F(\lambda)$ for simple Lie algebras. In this case, the corresponding dynamical r-matrix is the standard one $r(\lambda) = \frac{1}{2}\sum_{\alpha\in\Delta_+} \coth(\frac{1}{2} \ll \alpha, \lambda \gg) E_\alpha \wedge E_{-\alpha}$.

Remark We may replace θ in Equation (78) by $\tilde{\theta} = \sum_{i=1}^k \frac{1}{2}(\frac{\partial}{\partial\lambda_i}\otimes h_i - h_i\otimes\frac{\partial}{\partial\lambda_i}) \in H\otimes H$ and set $\tilde{\Theta} = \exp\hbar\tilde{\theta} \in H\otimes H$. In this case, one can show that $\tilde{\mathcal{F}} = F(\lambda)\tilde{\Theta}$ satisfies Equation (32) is equivalent to the following condition for $F(\lambda)$:

$$[(\Delta_0\otimes id)F(\lambda)]F^{12}(\lambda + \frac{1}{2}\hbar h^{(3)}) = [(id\otimes\Delta_0)F(\lambda)]F^{23}(\lambda - \frac{1}{2}\hbar h^{(1)}).$$

Then $R(\lambda) = F^{21}(\lambda)^{-1}RF^{12}(\lambda)$ satisfies

$$R^{12}(\lambda+\frac{1}{2}\hbar h^{(3)})R^{13}(\lambda-\frac{1}{2}\hbar h^{(2)})R^{23}(\lambda+\frac{1}{2}\hbar h^{(1)}) = R^{23}(\lambda-\frac{1}{2}\hbar h^{(1)})R^{13}(\lambda+\frac{1}{2}\hbar h^{(2)})R^{12}(\lambda-\frac{1}{2}\hbar h^{(3)}). \tag{91}$$

In fact, both Θ and $\tilde{\Theta}$ can be obtained from the quantization of the cotangent bundle symplectic structure $T^*\eta^*$, using the normal order and the Weyl order respectively, so they are equivalent. This indicates that solutions to Equation (1) and Equation (91) should be equivalent as well.

8 Appendix and open questions

Given any permutation $(i_1i_2i_3)$ of (123), by $\sigma_{i_1i_2i_3}$ we denote the permutation operator on $UA\otimes_R UA\otimes_R UA$ given by

$$\sigma_{i_i i_2 i_3}(\sum x_1\otimes_R x_2\otimes_R x_3) = \sum x_{i_1}\otimes_R x_{i_2}\otimes_R x_{i_3}.$$

Proposition 8.1 *Assume that* $T\in UA\otimes_R UA$ *satisfies*

$$T\otimes_R 1+(\Delta\otimes_R id)T=1\otimes_R T+(id\otimes_R\Delta)T. \tag{92}$$

Then $AltT \stackrel{def}{=} T-T_{21}$ *is a section in* $\Gamma(\wedge^2 A)$.

Proof. First we show that

$$(\Delta\otimes_R id)\mathrm{Alt}T=1\otimes_R\mathrm{Alt}T+\sigma_{132}(\mathrm{Alt}T\otimes_R 1). \tag{93}$$

To see this, write $T=\sum_i u_i\otimes_R v_i$, where $u_i, v_i\in UA$. Then Equation (92) becomes

$$\sum_i u_i\otimes_R v_i\otimes_R 1+\sum_i\Delta u_i\otimes_R v_i=\sum_i 1\otimes_R u_i\otimes_R v_i+\sum_i u_i\otimes_R\Delta v_i. \tag{94}$$

Applying the permutation operators σ_{231} and σ_{132}, respectively, to both sides of the above equation, one leads to

$$\sum_i v_i\otimes_R 1\otimes_R u_i+\sum_i\sigma_{231}(\Delta u_i\otimes_R v_i)=\sum_i u_i\otimes_R v_i\otimes_R 1+\sum_i\Delta v_i\otimes_R u_i; \tag{95}$$

$$\sum_i u_i\otimes_R 1\otimes_R v_i+\sum_i\sigma_{132}(\Delta u_i\otimes_R v_i)=\sum_i 1\otimes_R v_i\otimes_R u_i+\sum_i u_i\otimes_R\Delta v_i. \tag{96}$$

Combining Equations (94), (95) and (96) ((94+(95)-(96)), we obtain:

$$\sum(\Delta u_i\otimes_R v_i-\Delta v_i\otimes_R u_i)=\sum_i(1\otimes_R u_i\otimes_R v_i-1\otimes_R v_i\otimes_R u_i+u_i\otimes_R 1\otimes_R v_i-v_i\otimes_R 1\otimes_R u_i),$$

which is equivalent to Equation (93). Here we have used the identity: $\sigma_{231}(\Delta u_i\otimes_R v_i)=\sigma_{132}(\Delta u_i\otimes_R v_i)$, which can be easily verified using the fact that Δu_i is symmetric.

The conclusion essentially follows from Equation (93). To see this, let us write $\mathrm{Alt}T=\sum_i u_i\otimes_R v_i$, where $\{v_i\in UA\}$ are assumed to be R-linearly independent.

From Equation (93), it follows that

$$\sum_i\Delta u_i\otimes_R v_i=\sum_i(1\otimes_R u_i\otimes_R v_i+u_i\otimes_R 1\otimes_R v_i).$$

I.e., $\sum_i(\Delta u_i-1\otimes_R u_i-u_i\otimes_R 1)\otimes_R v_i=0$. Hence $\Delta u_i=1\otimes_R u_i+u_i\otimes_R 1$, which implies that $u_i\in\Gamma(A)$. Since $\mathrm{Alt}T$ is skew symmetric, we conclude that $\mathrm{Alt}T\in\Gamma(\wedge^2 A)$.

$\square$

Remark It might be useful to consider the following cochain complex

$$0\to R\stackrel{\partial}{\to}UA\stackrel{\partial}{\to}UA\otimes_R UA\stackrel{\partial}{\to}UA\otimes_R UA\otimes_R UA\stackrel{\partial}{\to} \tag{97}$$

where $\partial : \otimes_R^n UA \longrightarrow \otimes_R^{n+1} UA$, $\partial = \partial^0 - \partial^1 + \cdots + (-1)^{n+1}\partial^{n+1}$, $\partial^i(x_1 \otimes_R \cdots \otimes_R x_n) = x_1 \otimes_R \cdots \otimes_R x_{i-1} \otimes_R \Delta x_i \otimes_R x_{i+1} \otimes_R \cdots \otimes_R x_n$ for $1 \leq i \leq n$, and $\partial^0 x = 1 \otimes_R x$, $\partial^{n+1} x = x \otimes_R 1$. It is simple to check that $\partial^2 = 0$. In fact, this is the subcomplex of the Hochschild cochain complex of the algebra $C^\infty(G)$ (G is a local groupoid integrating the Lie algebroid A) by restricting to the left invariant muti-differential operators. It is natural to expect that the cohomology of this complex is isomorphic to $\Gamma(\wedge^* A)$, where the isomorphism from H^* (more precisely, cocycles) to $\Gamma(\wedge^* A)$ is the usual skew-symmetrization map. This is known to be true for Lie algebras [11] and the tangent bundle Lie algebroid. However we could not find such a general result in the literature. In terms of this cochain complex, it is simple to describe what we have proved in Proposition 8.1. It simply means that $\mathrm{Alt} : UA \otimes_R UA \longrightarrow UA \otimes_R UA$ maps 2-cocycles into $\Gamma(\wedge^2 A)$.

We end this paper by a list of open questions.

Question 1: We believe that techniques in [12] would be useful to prove the conjecture in Section 6. While the proof of Etingof and Kazhdan relies heavily on the double of a Lie bialgebra, the double of a Lie bialgeboid is no longer a Lie algeboid. Instead it is a Courant algebroid [27], where certain anomalies are inevitable. As a first step, it is natural to ask: what is the universal enveloping algebra of a Courant algebroid? Roytenberg and Weinstein proved that Courant algebroids give rise to homotopy Lie algebras [37]. It is expected that these homotopy Lie algebras are useful to understand this question as well as the quantization problem.

Question 2: One can form a Kontsevich's formality-type conjecture for Lie algebroids, where one simply replace in Kontsevich formality theorem [25] polyvector fields by sections of $\wedge^* A$ and multi-differential operators by $UA \otimes_R \cdots \otimes_R UA$ for a Lie algebroid A. Does this conjecture hold? It is not clear if Kontsevich's proof in [25] can be generalized to the general context of Lie algebroids. If this conjecture holds, it would imply that any triangular Lie bialgebroid is quantizable.

Question 3: Given a solution $r : \eta^* \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ of the classical dynamical Yang-Baxter equation:

$$\mathrm{Alt} dr + [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0 \tag{98}$$

(in this case $\mathrm{Alt} r$ satisfies Condition (ii) of a dynamical r-matrix (see Introduction) if $r + r^{21}$ is ad-invariant), a quantization of r is a quantum dynamical R-matrix $R : \eta^* \longrightarrow U_\hbar \mathfrak{g} \otimes U_\hbar \mathfrak{g}$ such that $R(\lambda) = 1 + hr(mod \hbar^2)$, where $U_\hbar \mathfrak{g}$ is a quantum universal enveloping algebra. Is every dynamical r-matrix quantizable? Our result in Section 6 should imply that every regular triangular dynamical r-matrix is quantizable. However, it is not even clear whether a general triangular dynamical r-matrix is quantizable.

Question 4: According to the general spirit of deformation theory, any deformation corresponds to a certain cohomology. In particular, the deformation of Hopf algebras is controlled by the cohomology of a certain double complex arising from the Hopf algebra structure [20] [21]. It is natural to ask what is the proper cohomology theory controlling the deformation of a Hopf algebroid, and in particular what is the premier obstruction to the quantization problem.

Question 5: What is the connection between dynamical quantum groupoids and quantum Virasoro algebra or quantum W-algebras [17] [38]?

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