On Poisson pairs associated to modified R-matrices

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Abstract

For any complex simple Lie algebra $\mathfrak{g}$ we give a complete classification of orbits in $\mathfrak{g}^*$ (with respect of the Ad^*-action of the corresponding group) such that the bracket defined by a modified R-matrix $R \in \wedge^2\mathfrak{g}$ is a Poisson one. We consider the family of Poisson brackets generated by the above bracket and the Kirillov-Kostant-Souriau bracket. These two brackets are compatible.

1 Introduction

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbf{C}$, G be its adjoint group. Consider a homogeneous G-manifold $M = G/H$. Any element $X \in \mathfrak{g}$ defines a holomorphic vector field $\rho(X)$ on the manifold M in the following way $\rho(X)f(m) = f(e^{-tX}m)|_{t=0}$, $f \in Fun(M)$. The correspondence $X \mapsto \rho(X)$ is a representation of $\mathfrak{g}$ into the space $Vect(M)$ of all holomorphic vector fields on M. Let us fix an element $R \in \wedge^2\mathfrak{g}$ and associate the following operator to it

$$f \otimes g \to \{f,g\}_R = \mu < (\rho \otimes \rho)R, df \otimes dg >, \quad f,g \in Fun(M). \qquad (1)$$

Hereafter μ is the usual commutative multiplication in the space of holomorphic functions $Fun(M)$

$$\mu : Fun(M)^{\otimes 2} \to Fun(M)$$

and $<,>$ stands for the pairing between vector fields and differential forms.

Let us consider two conditions

(i) R satisfies the classical Yang-Baxter equation i.e. $[[R,R]]=0$ where

$$[[R,R]] = [R^{12},R^{13}] + [R^{12},R^{23}] + [R^{13},R^{23}]$$

(it is clear that $[[R,R]] \in \wedge^3\mathfrak{g}$ for any $R \in \wedge^2\mathfrak{g}$).

(ii) The operator (1) defines a Poisson bracket i.e. it satisfies the Jacobi identity (since the antisymmetricity and the Leibnitz identity are fulfilled automatically).

It is obvious that the implication (i)$\Rightarrow$(ii) is true. However (ii) could be fulfilled even if the condition (i) fails. In the present paper we investigate the following

PROBLEM. Let $\mathfrak{g}$ be a simple Lie algebra. Describe all orbits $\mathcal{O}$ in $\mathfrak{g}^*$, such that the condition (ii) is fulfilled, where R is a modified R-matrix.

These orbits are said to be the *orbits of R-matrix type.*

In fact the problem under consideration may be formulated without any R-matrix and therefore the property of an orbit to be of the R-matrix type does not depend on a particular choice of R-matrix. More exactly, there exists a unique (up to a scalar multiple) G-invariant 3-form on $\mathfrak{g}^*$ and an orbit $\mathcal{O}$ is of R-matrix type *iff* the restriction of this 3-form on $\mathcal{O}$ is identically zero (cf. Section 2).

Two families of R-matrix type orbits have been described in [DGM], [DG1], the orbit of the highest weight vectors and symmetric spaces. In the present paper we give a complete solution of the Problem. The answer appears to be rather pretty. Namely, if an orbit $\mathcal{O}$ is of R-matrix type, then it is either semisimple or nilpotent and a semisimple one must be a symmetric space. For nilpotent orbits we give a criterion, formulated in terms of the height of an orbit (cf. Theorem 1). In the case of classical simple Lie algebras this condition may be reformulated in a very simple form (cf. Theorem 2).

The classification of all R-matrix type orbits enables us to construct the new families of compatible Poisson brackets. Recall that two Poisson brackets are called *compatible* if any linear combination of them is again a Poisson bracket. It is well-known that there exists a symplectic structure and therefore a non-degenerated Poisson bracket on any orbit $\mathcal{O} \in \mathfrak{g}^*$, the so-called "Kirillov-Kostant-Souriau bracket" (we denote it $\{,\}_{KKS}$). It is easy to see that the brackets $\{,\}_{KKS}$ and $\{,\}_R$ (assuming the orbit to be of R-matrix type) are always compatible. Therefore on any R-matrix type orbit there exists the following family of Poisson brackets

$$\{,\}_{a,b} = a\{,\}_{KKS} + b\{,\}_R \tag{2}$$

The question of compatibility of the Kirillov-Kostant-Souriau bracket and the reduced Sklyanin bracket was investigated in the paper [KRR] for orbits equipped with a hermitian structure. In Section 4 we describe the relation between the reduced Sklyanin bracket and the R-matrix one and deduce (partially) the result of [KRR] from ours.

In the paper [GRZ] a simultaneous quantization of the family $\{,\}_{a,b}$ was constructed assuming R to be a classical R-matrix. The result of the quantization is a two-parameter family of associative algebras which is a flat deformation of the commutative algebra of functions on $\mathfrak{g}^*$. Our next intention is to construct an analogous quantization of the family (2) on orbits of R-matrix type.

2 Algebraization of the Problem

Let $\mathfrak{g}$ be a simple Lie algebra, G be its adjoint group, and $\mathrm{rk}\mathfrak{g} = l$. It is well-known, that $(\wedge^*\mathfrak{g})^G$ is the exterior algebra of l generators of degrees $2m_i+1$, $i=1,\ldots,l$, where $m_1,\ldots,m_l$ are the exponents of $\mathfrak{g}$. In particular $\dim(\wedge^3\mathfrak{g})^G = 1$ since $m_1 = 1$ and $m_i \geq 2, i \geq 2$. Fix some $\varphi \in (\wedge^3\mathfrak{g})^G \setminus \{0\}$. Clearly φ may be regarded as a (G-invariant) 3-form on $\mathfrak{g}^*$.

Recall that an element $R \in \wedge^2\mathfrak{g}$ is called a *modified R-matrix* iff $[[R,R]]$ is G-invariant and therefore $[[R,R]] = c\varphi$ for some $c \in \mathbf{C}^*$.

Remark that all modified R-matrices were classified in [BD]. The most popular solution is

$$R = \frac{1}{2}\sum \frac{X_\alpha \wedge X_{-\alpha}}{(X_\alpha, X_{-\alpha})} \tag{3}$$

where α runs over all positive roots of $\mathfrak{g}$ (this one depends on the choice of a triangular decomposition of $\mathfrak{g}$). This R-matrix is related to some "canonical" Manin triple. However all statements below embrace any modified R-matrix, since they are formulated in terms of the element φ only.

For a fixed homogeneous G-manifold M we shall consider a map

$$\varphi_\rho : (f \otimes g \otimes h) \to \mu < (\rho \otimes \rho \otimes \rho)\varphi, df \otimes dg \otimes dh >,\ f,g,h \in Fun(M).$$

It is obvious that for any modified R-matrix the bracket $\{,\}_R$ is Poisson one iff $\mathrm{Im}\varphi_\rho = 0$. For the sake of brevity we shall write $\varphi\ |_M \equiv 0$ in this case. Therefore now our Problem is transformed into the following one:

Describe all orbits $\mathcal{O}$ in $\mathfrak{g}^*$ such that $\varphi\ |_{\mathcal{O}} \equiv 0$.

Let us introduce some notation. Suppose $x \in \mathfrak{g}^*$. By $\mathfrak{g}_x$ denote the stationary subalgebra of x in $\mathfrak{g}$ and by G_x denote the stabilizer of x in G (relative to the coadjoint representation). Let $\mathcal{O} = \mathcal{O}(x)$ be the G-orbit of a point $x \in \mathfrak{g}^*$.

One may consider φ as the G-invariant map $\varphi : \mathbf{C} \to \wedge^3\mathfrak{g}$. By virtue of the G-invariance of the element φ it suffices to check the condition $\varphi\ |_{\mathcal{O}} \equiv 0$ for a single point x, where this is equivalent to the following one: the composition

$$k \xrightarrow{\varphi} \wedge^3\mathfrak{g} \to \wedge^3(\mathfrak{g}/\mathfrak{g}_x) \tag{4}$$

is equal to 0.

Dualizing (4) we get the sequence

$$\wedge^3(\mathfrak{g}/\mathfrak{g}_x)^* \to \wedge^3\mathfrak{g}^* \xrightarrow{\varphi^*} k \tag{5}$$

Clearly, (4) is a complex *iff* (5) is a complex. Let us remark that $(\mathfrak{g}/\mathfrak{g}_x)^*$ is naturally isomorphic to the annihilator subspace $\mathrm{Ann}\,\mathfrak{g}_x \subset \mathfrak{g}^*$. We shall identify $\mathfrak{g}$ and $\mathfrak{g}^*$ via the Killing form (,). Then $\mathrm{Ann}\,\mathfrak{g}_x \cong \mathfrak{g}_x^\perp =: \mathfrak{m}_x \subset \mathfrak{g}$, where $\mathfrak{g}_x^\perp$ is the orthogonal complement to $\mathfrak{g}_x$ relative to the Killing form, and $\varphi^* : \wedge^3\mathfrak{g} \to k$ is defined by the formulae:

$$\varphi^*(x,y,z) = \alpha([x,y],z), \quad \alpha \in \mathbf{C}^*. \tag{6}$$

Therefore our final reformulation looks as follows:

Let $x \in \mathfrak{g}$ be a non-zero element and $\mathfrak{m}_x = (\mathfrak{g}_x)^\perp \subset \mathfrak{g}$. Find all x such that

$$[\mathfrak{m}_x, \mathfrak{m}_x] \subset \mathfrak{g}_x \tag{7}$$

The latter condition means that $[x,y] \in \mathfrak{g}_x$ for any $x, y \in \mathfrak{m}_x$.

Thus we have reduced the problem of describing all R-matrix type orbits to the Lie algebra condition (7).

3 R-matrix type orbits

Keep the previous notation. All assertions on orbits of the adjoint representation with no references to their proofs may be found e.g. in [SS].

Suppose $x \in \mathfrak{g}$. There exists a unique decomposition (*Jordan decomposition*) $x = x_s + x_n$ such that

(1) x_s is semisimple and x_n is nilpotent,

(2) $[x_s, x_n] = 0$,

(3) $\mathfrak{g}_x = \mathfrak{g}_{x_s} \cap \mathfrak{g}_{x_n}$.

In order to formulate our results we need some preliminaries. Take an arbitrary non-zero nilpotent element $e \in \mathfrak{g}$. By the Morozov's theorem there exists a $\mathfrak{sl}_2$-triple $\{e, h, f\}$, containing e (i.e. $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$). The semisimple element h defines a natural grading on $\mathfrak{g}$. Put $\mathfrak{g}(i) = \{x \in \mathfrak{g} \mid [h, x] = ix\}$. Then $\mathfrak{g}(0)$ is a reductive subalgebra of $\mathfrak{g}$ of the maximal rank and $\mathfrak{g} = \bigoplus_{i \in \mathbf{Z}} \mathfrak{g}(i)$ is a $\mathbf{Z}$-grading of $\mathfrak{g}$. Given e, all $\mathfrak{sl}_2$-triples, containing e, form a single G_e-orbit. Therefore properties of the $\mathbf{Z}$-grading under consideration reflect really properties of the orbit $\mathcal{O}(e)$ only.

Definition. The integer $\max\{i \mid \mathfrak{g}(i) \neq 0\}$ is said to be the *height* of e (or the orbit $\mathcal{O}(e)$) and will be denoted by $\mathrm{ht}(e)$.

Since $e \in \mathfrak{g}(2)$, we have $\mathrm{ht}(e) \geq 2$ for any nilpotent $e \neq 0$.
We shall say the orbit $\mathcal{O}$ is *semisimple* (resp., *nilpotent*), if it consists of semisimple (resp., nilpotent) elements.

The following is our solution of the Problem.

Theorem 1. *Let $\mathcal{O}$ be a non-zero orbit in $\mathfrak{g}$.*

1. Suppose $\varphi|_{\mathcal{O}} \equiv 0$, then $\mathcal{O}$ is either semisimple or nilpotent.

2. If $\mathcal{O}$ is semisimple, then $\varphi|_{\mathcal{O}} \equiv 0$ iff $\mathcal{O}$ is a symmetric space.

3. If $\mathcal{O} = \mathcal{O}(e)$ is nilpotent, then $\varphi|_{\mathcal{O}} \equiv 0$ iff $\mathrm{ht}(e) = 2$.

The condition on the height of nilpotent elements seems to be somewhat vague. But for the classical Lie algebras this admits a nice reformulation.

Theorem 2. *1. Suppose e is a nilpotent matrix in $\mathfrak{sl}(V)$ or $\mathfrak{sp}(V)$. Then $\varphi|_{\mathcal{O}(e)} \equiv 0$ iff $e^2 = 0$.*

2. Suppose e is a nilpotent matrix in $\mathfrak{so}(V)$, then $\varphi|_{\mathcal{O}(e)} \equiv 0$ iff $e^2 = 0$ or $\mathrm{rank}(e) = 2$ *and* $\mathrm{rank}(e^2) = 1$.

Proofs of these theorems will be given in a series of propositions below. Before doing this let us present some useful observations.

(a) It immediately follows from our description that only "small enough" orbits may be of R-matrix type. It is also easy to give the explicit presentation of these orbits in classical Lie algebras via the Jordan normal form.

(b) All R-matrix type orbits appears to be *spherical* or *multiplicity free.* (This important property may be formulated in a various way. The simplest one is that an G-orbit $\mathcal{O}$ is spherical *iff* a Borel subgroup of G has an open orbit on $\mathcal{O}$.) This is well-known for symmetric spaces, and for nilpotent orbits of the height 2 this is proved in [P]. However there exist spherical nilpotent orbits which are not of R-matrix type, namely the ones of the height 3 (cf. [P]).

Recall that any simple Lie algebra contains finitely many nilpotent orbits. In the following tables we indicate the numbers of nilpotent R-matrix type orbits for each simple Lie algebras. For classical Lie algebras these integers may be computed by using Theorem 2 and for exceptional ones one should look through the classification tables of nilpotent orbits.

Table

$\mathbf{A}_n$	$\mathbf{B}_n$	$\mathbf{C}_n$	$\mathbf{D}_n$	$\mathbf{E}_6$	$\mathbf{E}_7$	$\mathbf{E}_8$	$\mathbf{F}_4$	$\mathbf{G}_2$
$[(n+1)/2]$	$[n/2]+1$	n	$[n/2]+1$	2	3	2	2	1

Now let us return to the proofs of Theorems.

Proposition 3. *Suppose $x \in \mathfrak{g}$ is a semisimple element, then $[\mathfrak{m}_x, \mathfrak{m}_x] \subset \mathfrak{g}_x$ iff Gx is a symmetric space.*

Proof. If x is semisimple, then $\mathfrak{g}_x$ is reductive and the restriction of the Killing form on $\mathfrak{g}_x$ is non-degenerate. This gives us the direct sum decomposition $\mathfrak{g} = \mathfrak{g}_x \oplus \mathfrak{m}_x$. Hence, (7) is equivalent to saying that this decomposition is a $\mathbf{Z}_2$-grading of $\mathfrak{g}$. That is, $\mathfrak{g}_x$ is the fixed subspace of an involutive automorphism of $\mathfrak{g}$. □

Proposition 4. *Suppose $e \in \mathfrak{g}$ is a nilpotent element, then $[\mathfrak{m}_e, \mathfrak{m}_e] \subset \mathfrak{g}_e$ iff* $\mathrm{ht}(e) = 2$.

Proof. Denote by $\mathfrak{s}$ the 3-dimensional subalgebra of $\mathfrak{g}$ with the base $\{e, h, f\}$. Obviously $\mathfrak{s} \cong \mathfrak{sl}_2$. Consider $\mathfrak{g}$ as the $\mathfrak{s}$-module (by restricting the adjoint representation of $\mathfrak{g}$ on $\mathfrak{s}$). Then $\mathfrak{g} = \bigoplus_i V(d_i)$, where $V(d_i)$ is the unique irreducible $\mathfrak{s}$-module of dimension $d_i + 1$. The condition $\mathrm{ht}(e) \leq 2$ is equivalent to the following: $d_i \leq 2$ for every i.

(a) First we prove that $[\mathfrak{m}_e, \mathfrak{m}_e] \not\subset \mathfrak{g}_e$, if $\mathrm{ht}(e) > 2$. Assume that there exist $V(d_i) \subset \mathfrak{g}$ with $d_i \geq 3$. Let $v_0 \in V(d_i)$ be the lowest weight vector, i.e. $[f, v_0] = 0$ and $[h, v_0] = -d_i v_0$. By definition put $v_j = (\mathrm{ad} e)^j v_0$. Then $v_3 \neq 0$, i.e. $v_2 \notin \mathfrak{g}_e$. On the other hand, we have $v_1, e \in \mathrm{Im}(\mathrm{ad} e) = \mathfrak{m}_e$ and $[e, v_1] = v_2$.

(b) Assume $\mathrm{ht}(e) = 2$ and let $\mathfrak{g} = \bigoplus_{i=-2}^{2} \mathfrak{g}(i)$ be the corresponding grading. The structure of $\mathfrak{g}_e$ is as follows. This is a positively graded algebra, $\mathfrak{g}_e = \oplus_{i=0}^{2} (\mathfrak{g}_e)_i$, $(\mathfrak{g}_e)_i = \mathfrak{g}(i)$ for $i = 1, 2$ and $(\mathfrak{g}_e)_0 \subset \mathfrak{g}(0)$. Let $\mathfrak{c}$ be the orthogonal complement to $(\mathfrak{g}_e)_0$ in $\mathfrak{g}(0)$ (we know the restriction of the Killing form on $\mathfrak{g}(0)$ is non-degenerate). Then

$$\mathfrak{c} \oplus (\mathfrak{g}_e)_0 = \mathfrak{g}(0)$$

and $\mathfrak{m}_e = \mathfrak{c} \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2)$. Hence in order to prove the assertion one have to establish that $[\mathfrak{c}, \mathfrak{c}] \subset (\mathfrak{g}_e)_0$. But this has been proved in [P, ch.3]. □

Proposition 5. *Assume x is neither semisimple, nor nilpotent. Then (7) is not satisfied.*

Proof. Let $x = s + n$ be the Jordan decomposition, $s \neq 0$, $n \neq 0$. Then $\mathfrak{m}_x = \mathfrak{m}_s + \mathfrak{m}_n$ (the sum is not direct!). Putting $\mathfrak{m}_{sn} = \mathfrak{m}_x \cap \mathfrak{g}_s$, one get already the direct sum $\mathfrak{m}_x = \mathfrak{m}_s \oplus \mathfrak{m}_{sn}$. The reductivity of $\mathfrak{g}_s$ is used at this point. The single relation (7) for $\mathfrak{m}_x$ is inverted into 3 relations for $\mathfrak{m}_s$ and $\mathfrak{m}_{sn}$. Namely,

$$[\mathfrak{m}_x, \mathfrak{m}_x] \subset \mathfrak{g}_x \Leftrightarrow \begin{cases} [\mathfrak{m}_s, \mathfrak{m}_s] \subset \mathfrak{g}_x \\ [\mathfrak{m}_s, \mathfrak{m}_{sn}] \subset \mathfrak{g}_x \\ [\mathfrak{m}_{sn}, \mathfrak{m}_{sn}] \subset \mathfrak{g}_x \end{cases}$$

Since $\mathfrak{g}_x \subset \mathfrak{g}_s$, the first one give us that $\mathfrak{g} = \mathfrak{g}_s \oplus \mathfrak{m}_s$ is a $\mathbf{Z}_2$-grading of $\mathfrak{g}$. The second one give us $[\mathfrak{m}_s, \mathfrak{m}_{sn}] \subset \mathfrak{g}_s = \mathfrak{m}_s^{\perp}$, i.e.

$$0 = ([\mathfrak{m}_s, \mathfrak{m}_{sn}], \mathfrak{m}_s) = ([\mathfrak{m}_s, \mathfrak{m}_s], \mathfrak{m}_{sn})$$

Since $\mathfrak{m}_{sn} \subset \mathfrak{g}_s$, the last equality implies by the induction that $\mathfrak{m}_{sn}$ is orthogonal to the subalgebra of $\mathfrak{g}$, generated by $\mathfrak{m}_s$. But the latter coincides with $\mathfrak{g}$ [K, lemma 4.1], i.e. $\mathfrak{m}_{sn} = 0$. Hence $\mathfrak{m}_x = \mathfrak{m}_s$ and $x = s$. The contradiction obtained proves the proposition. □

Combining Propositions 3-5 one obtains the proof of Theorem 1. Theorem 2 is a direct consequence of the description of nilpotent elements of the small height in classical Lie algebras given in [P].

4 Discussion

Thus on any R-matrix type orbit one can construct the family (2) of Poisson brackets, generated by the K-K-S bracket and a fixed R-matrix bracket. We leave to the reader to verify that these brackets are compatible (cf. also [DGM]).

Let us describe now another way to construct Poisson brackets arising from modified R-matrices. All constructions of this Section are still valid for real manifolds. In this case $Fun(M)$ (resp., $Vect(M)$) denotes the space of smooth functions (resp., smooth vector fields) on M.

Consider a bracket (introduced by E.Sklyanin)

$$\{f,g\}_S = \{f,g\}_l - \{f,g\}_r,\ \{f,g\}_i = \mu < (\rho_i \otimes \rho_i)R, df \otimes dg >, f,g \in Fun(G),$$

defined on a group G. Here $\rho_i : \mathfrak{g} \to Vect(G)$ $(i = l; r)$ are representations of $\mathfrak{g}$ in right-(left-)invariant vector fields.

Remark that it is natural to consider a representation

$$\rho = \rho_l \oplus \rho_r : \mathfrak{g} \oplus \mathfrak{g} \to Vect(G)$$

of the algebra $\mathfrak{g} \oplus \mathfrak{g}$ and to describe all elements $\mathbf{R} \in \wedge^2(\mathfrak{g} \oplus \mathfrak{g})$ defining Poisson brackets on G with respect to the scheme above. In particular $\mathbf{R} = R_1 - R_2,\ R_i = R \in \wedge^2 \mathfrak{g}$ defines the Sklyanin bracket if R is a modified R-matrix.

Let us fix a homogeneous space $M = G/H$. Suppose that the bracket $\{,\}_S$ can be reduced onto the space M. It means that $\{f,g\}_S$ is H-invariant if $f,g \in Fun(G)$ are H-invariant. Consider the reduced brackets $\{,\}_S^M$, $\{,\}_l^M$, $\{,\}_r^M$. In general the brackets $\{,\}_l^M$, $\{,\}_r^M$ are not Poisson ones. However they become the Poisson ones simultaneously.

If M is a symmetric space then $\{,\}_l^M$ coincides with a R-matrix bracket and $\{,\}_r^M$ is a Poisson bracket as well. Assuming R to be of the form (3) and $M = \mathcal{O}$ to be an orbit in $\mathfrak{g}^*$ equipped with an hermitian structure it is easy to show that the bracket $\{,\}_r^M$ is equal to the Kirillov-Kostant-Souriau bracket up to a factor (cf. [DG1]). Thus $\{,\}_S^M = \{,\}_R + a\{,\}_{KKS}$ for some a and therefore the bracket $\{,\}_S^M$ is compatible with the bracket $\{,\}_{KKS}$ on such a manifold as this follows from our result (more precisely from its real counterpart). This generalizes the "positive" half of a result from [KRR] which states that on a hermitian orbit in $\mathfrak{g}^*$ the brackets $\{,\}_S^M$ and $\{,\}_{KKS}$ are compatible iff M is a symmetric space, and R is of the form (3).

Using the cited result we can state as well that on any hermitian orbit the brackets $\{,\}_S^M$ and $\{,\}_{KKS}$ are compatible iff the brackets $\{,\}_l^M$ and $\{,\}_r^M$ are the Poisson ones (assuming R to be as above).

Concluding the paper we would like to remark that we consider the problem of a simultaneous quantization of the family (2) in the spirit of [GRZ] as a problem of a great interest. Up to now the quantization of the whole family is only done for the case $\mathfrak{g} = \mathfrak{sl}_2$ (cf. [DG2]). In the last case all orbits in $\mathfrak{g}^*$ are of R-matrix type and therefore R-matrix bracket is the Poisson one on the whole $\mathfrak{g}^*$. As a result of the quantization of the family (2) there arises a "braiding" of the enveloping algebra $U(\mathfrak{sl}_2)$. In general situation it is reasonable to expect that a braided deformation of some quotient algebras of the enveloping algebras will arise. However a construction of this quantization is an open problem.

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