Comparison of GL_N and Divisionalgebra representations II

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Introduction

This is the second part of a paper which is aimed on comparing the irreducible representations of $G := GL_N(F)$ and of a central division algebra D/F of index N over a p-adic number field $F|\mathbb{Q}_p$ by purely local methods. In Part I ([Zi$_{92}$]) we have seen that the explicit constructions of [BK] and of [Zi$_{90}$] enable us to introduce a parameter system which works for $(D^*)^\wedge$ and for the cuspidal representations in $\widehat{G}$ as well. Now we will see that our parameters can be used for the discrete series representations in $\widehat{G}$ too.

We start out from the remark that a simple type in the sense of [BK] uniquely determines an unramified twist class of discrete series representations of G and moreover that there is a natural bijection between conjugacy classes of simple types and unramified twist classes of discrete series representations. Then it is easily seen that this bijection can be lifted to a natural bijection between conjugacy classes of *extended simple types* (EST) and discrete series representations of G (Theorem 2.9). An EST is by definition the extension of a simple type onto its normalizer.

By the results of Bushnell and Howe, Moy, resp., the *level* of a discrete series representation coincides with the level of a corresponding EST (Corollary 2.5) which is given in terms of the principal unit filtration in a maximal compact modulo center subgroup of G. In section 3 we use [BFr] and [GJ] to relate this level to the exponential Artin conductor of the representation which is given in terms of its ε-factor, and we remark that the same result holds for division algebra representations (Theorem 3.1). Moreover, we reduce the computation of the ε-factor to that of the root number W of the corresponding extended simple type (Theorem 3.6).

The next step is to produce extended simple types out of our parameters, which is done in Section 4. The way to do that is basically the same as to construct simple types ([BK], §5) but there is one detail which we will explain more carefully (Main Lemma 4.1). What is immediately clear from our construction is that different parameters give rise to extended simple types which are not conjugated (Theorem 4.6) such that an injection of our parameter set $\mathcal{T}_N^-$ into the set $\widehat{G}_2$ of discrete series representations of G is induced (Theorem 4.7).

First properties of that injection are compatibility with tame character twist, an expression of the level and the exponential Artin conductor resp. of a representation in terms of its parameter and that it is possible to say in these terms if a representation is cuspidal or a character twist of the Steinberg representation, respectively.

In Section 5 we use the methods of [CMS] to compute the formal degree of a discrete series representation in terms of its parameter $t \in \mathcal{T}_N^-$ (Theorem 5.1). And in Theorem 6.1 we see that precisely the same formula gives us the dimension of the division algebra representation which is assigned to t. Combining all these results, in Theorem 6.8 we obtain an injection $D^{*\frown} \to \widehat{G}_2$ which is compatible with tame character twist, takes the dimension into the formal degree and preserves the exponential Artin conductor of representations. Using the local Matching Theorem of [BDKV] and a counting argument, in 6.9 we prove that any injection $D^{*\frown} \to \widehat{G}_2$ which is compatible with unramified character twist and preserves the exponential Artin conductor is in fact surjective too. Especially we see that the injection constructed in 6.8 (hence that in 4.7 too) is surjective, which by the results of [BK] should be possible to prove purely locally. Finally, following an idea of C. Bushnell, we show that any bijection $\Pi' \in D^{*\frown} \mapsto \Pi \in \widehat{G}_2$ which is compatible with unramified character twist and takes the dimension into the formal degree, has the property that Π is cuspidal iff Π' has the Galois dimension N (see Definition 6.3), i.e. Π' can be induced from $K^* \cdot U^1$ where U^1 are the principal units in D and K/F is a subfield of degree N. Especially this applies to the Matching Theorem.

In section 7 we take up the ε-factor computations of section 3. Using the parametrization of the extended simple types we make a first computation of their root numbrs. The aim is to show that the root number $W(\Lambda)$ of an EST Λ is essentially determined by just one character value of Λ (Theorem 7.1). This fact is well known for the cases $p \nmid N$ and $p = N$.

Section 1 is separate from the other sections. It is to give an improved definition of our set $\mathfrak{T}$ of parameters which we will denote as R-polynomials and to put $\mathfrak{T}$ into perspective by comparing it with two other sets $\mathfrak{S}$ and $\mathfrak{P}$. $\mathfrak{S}$ seems to be more adapted to parametrize Galois representations which conjecturally should be possible, while $\mathfrak{P}$ can be useful for counting arguments (see the proof of 6.9). Moreover, $\mathfrak{P}$ is helpful to understand the metric on $\mathfrak{T}$, $\mathfrak{S}$, respectively and the definition of $\mathfrak{T}^-$. For more details see [Zi_{89}]. Note that the notation $\mathfrak{T}$ of Part 1, 2.1 is replaced here by $\mathfrak{T}^-$ while $\mathfrak{T}$ is the set of all R-polynomials.

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Beginning from Section 2, we will use the following standard notations:

$A = M_N(F)$	the matrix algebra over F
$A_E =$	centralizer of a subfield E/F in A
$G = A^* = GL_N(F)$	
$GL_E = A_E^*$	is the centralizer of E^* in G
$\mathfrak{A} =$	principal $\mathfrak{o}_F$-order in A
$\mathfrak{A}_{L/F} =$	principal $\mathfrak{o}_F$-order in $\text{End}_F(L)$ which corresponds. to the $\mathfrak{o}_F$-lattice chain $\{\mathfrak{p}_L^i\}_{i\in\mathbb{Z}}$ of prime ideal powers in the extension field L/F
$\mathfrak{A}_E = \mathfrak{A} \cap A_E$	
$\mathfrak{K} = N_G(\mathfrak{A})$	normalizer of $\mathfrak{A}$ in G
$\mathfrak{K}_E = \mathfrak{K} \cap GL_E$	
$P = \text{Jac}(\mathfrak{A})$	the Jacobson radical of $\mathfrak{A}$
$\mathfrak{A}^* =$	group of units in $\mathfrak{A}$

$U^i = 1 + P^i$	the principal units in $\mathfrak{A}$
D/F	a central division algebra of index N
$\mathcal{O}_D$	the integers in D
$U_D^i = 1 + P_D^i$	the principal units in D
$\widehat{G}_2 =$	equivalence classes of irreducible admissible representations of G which are in the discrete series
$D^{*\wedge} =$	irreducible admissible representations of the multiplicative group D^* modulo equivalence

If $M \subseteq N$ are groups and π is a representation of M we write $\mathrm{Ind}_{M\uparrow N}(\pi)$ or $\mathrm{Ind}_N(\pi)$ for the induction of π from M to N. If the factor N/M is not compact, then $\mathrm{ind}_N(\pi)$ denotes the induction with compact support.

1. R-polynomials

Let $F|\mathbb{Q}_p$ be a p-adic number field. Throughout this paper $F[T]$ will denote the set of irreducible polynomials with highest coefficient 1 and degree ≥ 1. If $\bar{F}/F$ is a fixed algebraic closure and $G_F = \mathrm{Gal}(\bar{F}/F)$, the corresponding Galois group, we get a natural bijection

$$(1) \qquad G_F \backslash \bar{F} \longleftrightarrow F[T]$$

between Galois orbits of F-algebraic numbers and irreducible polynomials. Let $\tilde{F}|F$ be the maximal tamely ramified subextension of $\bar{F}/F$. If $\beta \in \bar{F}$, then $\tilde{F}(\beta)$ is the maximal tamely ramified extension of $F(\beta)$. Let $\Gamma_\beta \subset \mathrm{Gal}(\tilde{F}(\beta/F(\beta))$ be the tame Weil group, i.e. we restrict to automorphisms which on the maximal unramified extension of $F(\beta)$ induce an integral power of the Frobenius. As proposed by (1) we say:

1.1 Definition: An *R-polynomial* is a Galois orbit of pairs (R, β), where $\beta \in \bar{F}$ and $R \in \Gamma_\beta\hat{}$ is an irreducible admissible representation of Γ_β and where the Galois action is given by

$$(2) \qquad \sigma \cdot (R, \beta) := (\sigma R \sigma^{-1},\, \sigma(\beta))$$

for $\sigma \in G_F$. (Note that $\Gamma_{\sigma(\beta)} = \sigma \Gamma_\beta \sigma^{-1}$ and $(\sigma R \sigma^{-1})(\tau) := R(\sigma^{-1} \tau \sigma))$

With $t = [R, \beta]$ we will denote the R-polynomial which is generated by (R, β), and the set of all R-polynomials is denoted by $\mathcal{T}$. then we have a natural projection

$$(3) \qquad \mathcal{T} \longrightarrow F[T], \quad t \longmapsto f_t(T)$$

which assigns the minimal polynomial of β over F to an R-polynomial $t = [R, \beta]$.

1.2 Definition: The basic numerical invariants of an R-polynomial $t = [R, \beta]$ are the *degree* $\deg(t) := \dim(R) \cdot [F(\beta){:}\, F]$, the *inertial degree* $f_t := \dim(R) \cdot f_{F(\beta)/F}$ and the *ramification exponent* $e_t := e_{F(\beta)/F}$.

Note that $R \in \Gamma_\beta\hat{}$ is induced by a character of the unramified extension of degree $\dim(R)$ over $F(\beta)$.

Sometimes it is useful to look at R-polynomials from the following slightly different point of view:

1.3 Definition (ρ-polynomials)**:** Let $\tilde{F}[T]$ be the set of irreducible monic polynomials over $\tilde{F}$. The coefficient field F_φ of a polynomial $\varphi(T) \in \tilde{F}[T]$ is the tame extension of F which is generated by the coefficients of $\varphi(T)$. Let $\Gamma_\varphi \subseteq \mathrm{Gal}(\tilde{F}/F_\varphi)$ be the Weil subgroup. A *ρ-polynomial* is a Galois orbit of pairs $(\rho, \varphi(T))$ where $\varphi(T) \in \tilde{F}[T]$ and $\rho \in \Gamma_\varphi\hat{}$ is an irreducible admissible representation of Γ_φ. On the pairs $(\rho, \varphi(T))$ only the tame Galois group $G_{\tilde{F}/F}$ acts, namely

$$(4) \qquad \sigma \cdot (\rho, \varphi(T)) = \left(\sigma \rho \sigma^{-1},\, {}^\sigma\varphi(T)\right) \quad \text{for} \quad \sigma \in G_{\tilde{F}/F},$$

where the polynomial ${}^{\sigma}\varphi(T)$ is obtained by applying σ to the coefficients of $\varphi(T)$. $s = [\rho, \varphi(T)]$ denotes the ρ-polynomial which is generated by the pair $(\rho, \varphi(T))$, and the set of all ρ-polynomials is denoted by $\mathcal{S}$. Again, we have a natural projection

$$(5) \qquad \mathcal{S} \longrightarrow F[T], \quad s \longmapsto f_s(T)$$

sending $s = [\rho, \varphi(T)]$ to the F-irreducible polynomial which is generated by $\varphi(T)$. Note that $\deg f_s(T) = [F_\varphi : F] \cdot \deg \varphi(T)$.

1.4 Proposition: *There is a natural bijection*

$$\mathcal{T} \longleftrightarrow \mathcal{S}, \quad t = [R, \beta] \longmapsto s = [\rho, \varphi(T)]$$

where $\varphi(T)$ *is the minimal polynomial of* β *over* $\tilde{F}$ *and* $\rho \in \Gamma_\varphi{}^{\wedge}$ *is obtained from* R *by using the natural isomorphism* $\Gamma_\beta \tilde{\to} \Gamma_\varphi$ *which comes from* $F_\varphi = F(\beta) \cap \tilde{F}$ *and* $Gal(\tilde{F}(\beta)/F(\beta)) \tilde{\to} Gal(\tilde{F}/F(\beta) \cap \tilde{F})$. *Moreover this bijection has the property* $f_s(T) = f_t(T) \in F[T]$ *if* s *corresponds to* t.

The proof is almost obvious. The inverse map is $s = [\rho, \varphi(T)] \mapsto t = [R, \beta]$ where $\beta \in \bar{F}$ is a zero of $\varphi(T)$ and R is the pullback of ρ with respect to $\Gamma_\beta \tilde{\to} \Gamma_\varphi$. If β_1 is another zero of $\varphi(T)$ then $\beta_1 = \sigma(\beta)$ for some $\sigma \in G_{\bar{F}/\tilde{F}}$, $\Gamma_{\beta_1} = \sigma \Gamma_\beta \sigma^{-1}$, and the pullback of ρ with respect to $\Gamma_{\beta_1} \tilde{\to} \Gamma_\varphi$ is $\sigma R \sigma^{-1}$. □

Now we come to a third interpretation of R-polynomials which is not canonical but useful for counting arguments. Namely, let $C_F \subseteq F^*$ be a complementary subgroup with respect to the principal units $U_F^1 = 1 + \mathfrak{p}_F$, i.e. $F^* = C_F \cdot U_F^1$ is a direct product. C_F contains the roots of unity of order prime to p in F, a group which we can identify with k_F^* (= multiplicative group of the residue field). C_F is fixed by choosing a prime element π_F. $C_F = \langle \pi_F \rangle\, k_F^*$. Let ν_F be the normalized exponent of F, $\nu_F(\pi_F) = 1$. It extends in a unique way to an exponent of $\bar{F}$ with values in $\mathbb{Q}$ and $\nu_F(0) = \infty$. $\mathfrak{p}_{\bar{F}} = \{x \in \bar{F};\ \nu_F(x) > 0\}$ is the valuation ideal in $\bar{F}$ and $U_{\bar{F}}^1 = 1 + \mathfrak{p}_{\bar{F}}$ are the principal units. By [Zi89] we can extend C_F to a complementary group C of $\bar{F}^*$, and C is unique up to applying field automorphisms $s \in G_F$. We fix such a group $C \supseteq C_F$, $\bar{F}^* = C \cdot U_{\bar{F}}^1$, and we consider the field $\mathcal{K} = F(C)$ which is obtained by adjoining all elements from C. Note that C consists of all roots of unity of order prime to p and of a "string" of roots π_F^a, $a \in \mathbb{Q}$, (i.e. $C \cong \mathbb{Q}^+ \times k_{\bar{F}}^*$) and that $\mathcal{K}$ consists of all $x \in \bar{F}$ which expand into a converging series $x = \sum_{\nu \in \mathbb{Q}} x_\nu$ with terms $x_\nu \in C \cup \{0\}$, where $\nu_F(x_\nu) = \nu$ if $x_\nu \neq 0$ and where $x_\nu = 0$ if $\nu < \nu_F(x)$. K/F is said to be a C-field if $K \subseteq \mathcal{K}$ with the property that $C_K := C \cap K$ is a complementary group in K^*. Let $x = \sum_{\nu \in \mathbb{Q}} x_\nu$ be the C-expansion of $x \in \mathcal{K}$. There is a smallest C-field K_x containing x. It is a finite extension of F and is obtained by adjoining the terms of the expansion: $K_x = F(x_\nu;\ \nu \in \mathbb{Q})$.

Moreover, we get $\mathcal{K} = \tilde{F} \cdot K_\infty$ as a composite field, where K_∞/F is generated by all p-power roots of π_F in C. This yields a natural isomorphism of Galois groups:

$$(6) \qquad G(\mathcal{K}/K_\infty) \tilde{\to} G(\tilde{F}/F).$$

Because of $\bar{F} = C \cdot U_{\bar{F}}^1$ we have a projection $y \in \bar{F}^* \mapsto [y] \in C$ which is called the symbol of y. From ramification theory follows: $[s(y)] = [y]$ for all $s \in G(\bar{F}/\tilde{F})$.

1.5 Theorem ([Zi_{89}] 2.1): *There exists a well defined bijection*

$$\mathcal{K} \longleftrightarrow \tilde{F}[T], \quad x \longmapsto \varphi_x(T) \tag{7}$$

such that for all zeros α of $\varphi_x(T)$:

(i) $[x] = [\alpha]$

(ii) $K_x \cap \tilde{F} = F(\alpha) \cap \tilde{F}$ *and*
$\deg \varphi_x(T) = [K_x \cap K_\infty : F] = [K_x : K_x \cap \tilde{F}] = [F(\alpha) : F(\alpha) \cap \tilde{F}]$

(iii) *via (6) the bijeciton (7) is compatible with the natural Galois action on both sides.*

As a consequence, (7) induces a bijection

$$G(\mathcal{K}/K_\infty)\backslash\mathcal{K} \longleftrightarrow F[T]. \tag{7$_*$}$$

For $x \in \mathcal{K}$ let $\Gamma_x \subset G(\tilde{F}/\tilde{F} \cap K_x)$ be the Weil subgroup. Consider pairs (ρ, x) where $x \in \mathcal{K}$ and $\rho \in \Gamma_x\hat{}$. $G(\tilde{F}/F)$ acts on these pairs by

$$s \circ (\rho, x) = \left(s\rho s^{-1},\, s_{\mathcal{K}}(x)\right) \tag{8}$$

where $s_{\mathcal{K}} \in G(\mathcal{K}/K_\infty)$ corresponds to s under (6).

1.6 Definition: Let $\mathcal{P}$ be the set of equivalence classes $[\rho, x]$ of pairs (ρ, x) with respect to the Galois action (8).
Then $(7)_*$ extends to a bijection

$$\mathcal{P} \longleftrightarrow \mathcal{S}, \quad [\rho, x] \longmapsto [\rho, \varphi_x(T)] \tag{9}$$

The most important property of (7) is that the exponential distance on the field $\mathcal{K}$ is transported into a well defined exponential distance w_F on the set $\tilde{F}[T]$ of irreducible polynomials:

$$\nu_F(x_1 - x_2) = w_F\left(\varphi_1(T), \varphi_2(T)\right) \tag{10}$$

where $\varphi_i := \varphi_{x_i}$ and $w_F(\varphi_1, \varphi_2) := \Psi_{\varphi_1}(\nu_F(\varphi_1, \varphi_2)) = \Psi_{\varphi_2}(\nu_F(\varphi_1, \varphi_2))$ is a Herbrand transformation of the exponential mean distance of polynomials (see [Zi_{89}], 3.11 (ii); the metric w_F has been reviewed in [Zi_{90}], 0.4 and Part 1, 1.8 too). Moreover, with respect to the induced map $(7)_*$, we have:

$$\nu_F(\underline{x}_1, \underline{x}_2) = w_F\left(f_1(T), f_2(T)\right) \tag{10$_*$}$$

where $f_i(T) \in F[T]$ is the irreducible polynomial corresponding to the Galois orbit $\underline{x}_i \in G(\mathcal{K}/K_\infty)\backslash\mathcal{K}$, $\nu_F(\underline{x}_1, \underline{x}_2) := \max\{\nu_F(x_1 - x_2);\ x_1 \in \underline{x}_1,\, x_2 \in \underline{x}_2\}$ and where $w_F(f_1, f_2) := \Psi_{f_1}(\nu_F(f_1, f_2)) = \Psi_{f_2}(\nu_F(f_1, f_2))$ is again a Herbrand transformation of the exponential mean distance of polynomials (see [Zi_{89}], 4.3 and the other references as before). As a consequence of (10), $(10)_*$, one gets: $w_F(f, g) = \max\{w_F(\varphi, \gamma)\}$, where $\varphi, \gamma \in \tilde{F}[T]$

are running over the $\tilde{F}$-irreducible factors of $f, g \in F[T]$. Further, on $\mathcal{K}$ one has an *approximation procedure* (see 1.9 of Part I)

$$\mathcal{K} \times \mathbb{Q} \longrightarrow \mathcal{K}, \quad (x, j) \longmapsto x(j) := \sum_{\nu < j} x_\nu \tag{11}$$

which is given by truncating the C-expansion of $x \in \mathcal{K}$ (see Part I, 1.10). Moreover, the action of $G(\mathcal{K}/K_\infty)$ on $\mathcal{K} = F(C)$ takes C into itself. Hence (11) induces an approximation procedure

$$\begin{aligned} G(\mathcal{K}/K_\infty)\backslash\mathcal{K} \times \mathbb{Q} &\longrightarrow G(\mathcal{K}/K_\infty)\backslash\mathcal{K} \\ (\underline{x}, j) &\longmapsto \underline{x}(j) \end{aligned} \tag{11$_*$}$$

where $\underline{x}(j)$ is the orbit of $x(j)$ for any representative $x \in \underline{x}$. And by means of (7), (10) and $(7)_*$, $(10)_*$ the approximation procedures (11), $(11)_*$ resp. are transported into approximation procedures on $\tilde{F}[T]$ and $F[T]$ resp. These approxiamtion procedures refer to the exponential distance w_F of polynomials. So we have a well defined approximation procedure

$$F[T] \times \mathbb{Q} \longrightarrow F[T], \quad (f(T), j) \longmapsto f^j(T),$$

and we put:

$$\begin{aligned} F[T]^- &:= \{f(T) \in F[T],\ f^0(T) = f(T)\} \\ \mathcal{T}^- &:= \{t \in \mathcal{T};\ f_t(T) \in F[T]^-\} \\ \mathcal{S}^- &:= \{s \in \mathcal{S};\ f_s(T) \in F[T]^-\} \\ \mathcal{P}^- &:= \{[\rho, x] \in \mathcal{P};\ \underline{x}(0) = \underline{x} \in G(\mathcal{K}/K_\infty)\backslash\mathcal{K}\} \end{aligned} \tag{12}$$

Let be $\mathcal{K}^- := \{x \in \mathcal{K};\ x(0) = x\}$. This set is stable under the action of $G(\mathcal{K}/K_\infty)$ and we get: $\mathcal{P}^- = \{[\rho, x] \in \mathcal{P};\ x \in \mathcal{K}^-\}$.

The sets $\mathcal{T}^-$, $\mathcal{S}^-$, $\mathcal{P}^-$ are in bijective correspondence to each other, and the set $\mathcal{P}^-$ is appropriate for counting its elements. Consider $\mathcal{T}_N^- := \{t \in \mathcal{T}^-;\ \deg(t)|N\}$ and the sets $\mathcal{S}_N^-$, $\mathcal{P}_N^-$ which are defined in a similar way. The point is that each of these sets can be used to parametrize the discrete series representations of $GL_N(F)$.

2. Extended simple types

In this section we are going to explain the notion of an extended simple type (EST) which is a slight modification of the notion of a simple type introduced in [BK]. The main result is a natural bijection between conjugacy classes of extended simple types and irreducible discrete series representations of $GL_N(F)$.

Let us remember first some basic facts on simple types. We consider a principal order $\mathfrak{A}$ in $A = M_N(F)$ with Jacobson radical $P = \mathrm{Jac}(\mathfrak{A})$. $\mathfrak{K}$ denotes the normalizer of $\mathfrak{A}$ in $A^* = G$ which is a maximal compact modulo center subgroup in G and $U^i = 1 + P^i$ for $i \geq 1$, $i \in \mathbb{Z}$ are the principal units in $\mathfrak{K}$.

A simple type (J^0, λ) consists of a compact subgroup J^0 in G and an irreducible representation $\lambda \in (J^0)^\frown$. The simple type is associated to a simple stratum $\alpha + \mathfrak{A}$, i.e. α is an $\mathfrak{A}$-pure element in A such that $\alpha \in \alpha + \mathfrak{A}$ is a representative of minimal degree over F. We note the following properties (which are in [BK], but sometimes we give a reformulation in terms of [Zi$_{90}$], Sections 1, 2):

2.1 Let J be the $\mathfrak{K}$-normalizer of the set $\mathfrak{F} = \{\mathrm{Ad}\, U^{i+1} \cdot (\alpha + P^{-i})\}_{i \geq 0}$ of orbits of residue classes derived from $\alpha + \mathfrak{A}$:

$$J := N_{\mathfrak{K}}(\mathfrak{F}) = \bigcap_{i \geq 0} N_{\mathfrak{K}}\left(\mathrm{Ad}\, U^{i+1} \cdot (\alpha + P^{-i})\right) \tag{1}$$

The group J^0 of the simple stratum is given as

$$J^0 = \mathfrak{A}^* \cap J = \mathcal{O}^* \tag{2}$$

where $\mathcal{O} = \sum_{v \geq 0} P^v \cap Z(\alpha, P^{-v})$ is an $\mathfrak{o}_F$-order contained in $\mathfrak{A}$. Here the notation is:

$Z(\alpha, P^{-v}) := \{x \in A;\ x\alpha - \alpha x \in P^{-v}\}$,

$\mathfrak{J} := \sum_{v \geq 1} P^v \cap Z(\alpha, P^{-v})$ is the Jacobson radical of $\mathcal{O}$, and

$J^1 := U^1 \cap J = 1 + \mathfrak{J}$ are the principal units of $\mathcal{O}$.

Let $\mathfrak{K}_E, \mathfrak{A}_E$ be the intersections of $\mathfrak{K}, \mathfrak{A}$ resp. with the centralizer A_E of $E = F(\alpha)$ and let $L|E$ be a maximal extension such that $L^* \subset \mathfrak{K}_E$. As a matter of fact, we have:

$$\begin{aligned} &J = \mathfrak{K}_E J^1, \quad \text{hence} \quad J^0 = \mathfrak{A}_E^* \cdot J^1, \quad \text{and} \\ &J = L^* \cdot J^0 \end{aligned} \tag{3}$$

The third equation follows from the second because of $\mathfrak{K}_E = L^* \mathfrak{A}_E^*$.

2.2 (J^0, λ) is an admissible pair with respect to the sequence $\mathfrak{A}^* \supset U^1 \supset U^2 \supset \cdots$ of normal subgroups in $\mathfrak{K}$ (see [Zi$_{88}$], Section 2). Especially this means that $\lambda \in (J^0)^\frown$ is an irreducible representation with the additional property that for all integers $i \geq 1$ the restriction of λ to

$$J^i := U^i \cap J = 1 + \sum_{v \geq i} P^v \cap Z\left(\alpha, (P^{-v})\right)$$

is the multiple of an irreducible representation π_i of J^i and that all inductions $\mathrm{Ind}_{J^i \uparrow U^i}(\pi_i)$ for $i \geq 0$ ($\pi_0 = \lambda$, $U^0 = \mathfrak{A}^*$) are irreducible. As a consequence we get that π_1 is a Heisenberg representation because J^1 is nilpotent (see [Zi$_{88}$], Sections 4, 5). More precisely, let $\mathfrak{J}^\perp$ be the orthogonal complement of $\mathfrak{J}$ in P with respect to the alternating character

$$X_\alpha(x,y) = \psi_F \circ \mathrm{Tr}_{A/F}\left((xy - yx)\alpha\right)$$

(where $\psi_F \colon F^+ \to \mathbb{C}^*$ is an additive character of conductor $\mathfrak{p}_F$). Then $\mathfrak{J}^\perp \subseteq \mathfrak{J}$ is a ring too, namely:

$$\mathfrak{J}^\perp = \sum_{v \geq 1} P^v \cap Z\left(\alpha, P^{-v+1}\right)$$

and $1 + \mathfrak{J}^\perp \subseteq J^1$ is a subgroup which under π_1 is represented by scalar operators. The basic relation is:

$$\pi_1\left([1+x,\ 1+y]\right) = X_\alpha(x,y) \cdot \mathbf{1} \quad \text{for} \quad x, y \in \mathfrak{J},$$

where $[1+x,\ 1+y]$ denotes the commutator in J^1 and where $\mathbf{1}$ is the unit operator. With respect to $P = \mathrm{Jac}(\mathfrak{A})$, let $\nu_P(\alpha) = -j$ be the exponent of the simple stratum $\alpha + \mathfrak{A}$. Then $U^{[j/2]+1} \subseteq 1 + \mathfrak{J}^\perp \subseteq J^1$, and λ (or π_1) restricted to $U^{[j/2]+1}$ is a multiple of the character $\theta(1+x) = \psi_F \circ \mathrm{Tr}_{A/F}(\alpha x)$.

2.3 The intertwining of the pair (J^1, π_1) in G is

$$I_G(J^1, \pi_1) = J^1 \cdot GL_E \cdot J^1 \tag{4}$$

where GL_E is the centralizer of E^* in G, hence:

$$I_G(J^1, \pi_1) \cap \mathfrak{K} = J^1 \cdot \mathfrak{K}_E \cdot J^1 =_{(3)} J, \tag{5}$$

and because J normalizes J^1, we conclude that $I_G(J^1, \pi_1) \cap \mathfrak{K}$ is the $\mathfrak{K}$-normalizer of (J^1, π_1).

2.4 π_1 extends to J (especially to J^0), and there exists an extension $\pi \in J^\frown$ such that

$$I_G(J^0, \mathrm{Res}(\pi)) = I_G(J^1, \pi_1).$$

π is unique up to character twist with $\chi \circ \det_{A_E/E}$ of $\mathfrak{K}_E/U^1 \xrightarrow{\sim} J/J^1$ where χ is a tame character of E^*.

2.5 $\lambda = \mathrm{Res}(\pi) \otimes \sigma$, where σ is a "cuspidal" representation of $J^0/J^1 = \mathfrak{A}_E^*/U^1$ (see 4.1).

2.6 The intertwining of the simple type (J^0, λ) is:

$$I_G(J^0, \lambda) = J^0 \cdot GL_K \cdot J^0,$$

where $K|E$ is maximal unramified with the additional property $K^* \subset \mathfrak{K}$.

Unfortunately, the map $\alpha + \mathfrak{A} \mapsto (J^0, \lambda)$ depends on choices (to be made if one constructs λ), and $\alpha + \mathfrak{A}$ cannot be recovered from (J^0, λ). But it is possible to recover $\mathfrak{A}$ from J^0. Namely, $J^0 = \mathfrak{O}^*$ is the unit group of an $\mathfrak{o}_F$-order $\mathfrak{O}$ in A. Hence the $\mathfrak{o}_F$-module

$\mathfrak{L} \subseteq \mathfrak{O}$ which is generated by $\mathfrak{O}^*$ is an $\mathfrak{o}_F$-order too with the same Jacobson radical $\mathfrak{J}$ (see [BuF], (1.1.1)). And the normalizers $N_G(J^0) = N_G(\mathfrak{L})$ agree. Because $A = F \cdot \mathfrak{L}$, the centralizer of $\mathfrak{L}$ is F^*, hence

$$N_G(\mathfrak{L})/F^* \subseteq \mathrm{Aut}_{\mathfrak{o}_F}(\mathfrak{L})$$

is compact, and we see that $N_G(J^0)$ is a compact-modulo-center subgroup in G. Hence there is a maximal compact-modulo-center subgroup $\mathfrak{K}'$ such that $N_G(J^0) \subseteq \mathfrak{K}'$. On the other hand, we can take a maximal extension $L|E$ such that $L^* \subset \mathfrak{K}_E \subset J$ (see (3)). Then L^* normalizes J^0, hence $L^* \subset N_G(J^0) \subset \mathfrak{K}'$. But we have $L^* \subset \mathfrak{K}$, and from [Fr$_{87}$], Theorem 1 we know that there is precisely one maximal compact-modulo-center subgroup containing L^* because $[L{:}F] = N$, i.e. L is a maximal field in A. Hence $\mathfrak{K} = \mathfrak{K}'$, and we have

2.7 Proposition: *If (J^0, λ) is a simple type, then the normalizer $N_G(J^0)$ is contained in a uniquely determined maximal compact-modulo-center subgroup $\mathfrak{K}$ and $\mathfrak{K}$ is in 1-1 correspondence to a principal order $\mathfrak{A}$ (such that $\mathfrak{K} = N_G(\mathfrak{A})$). The normalizer of the pair (J^0, λ) is:*

$$N_G(J^0, \lambda) = J = I_G(J^0, \lambda) \cap \mathfrak{K}.$$

We are left to verify the equations. From 2.6 we obtain

$$I_G(J^0, \lambda) \cap \mathfrak{K} = J^0 \cdot \mathfrak{K}_K \cdot J^0 \subseteq_{(3)} J$$

because of $\mathfrak{K}_K \subseteq \mathfrak{K}_E$. Moreover J normalizes (J^0, λ), because J normalizes J^0 and J intertwines λ. Hence

$$J \subseteq N_G(J^0, \lambda) \subseteq I_G(J^0, \lambda) \cap \mathfrak{K} \subseteq J. \qquad \square$$

Now we are able to give a reasonable definition of extended simple types.

2.8 Definition: An extended simple type (EST) is a pair (J, Λ) where J is the G-normalizer of a simple type (J^0, λ) and where Λ is an extension of λ onto J.

Remark: Using (3) and 2.7, we see that there is a uniquely determined maximal compact-modulo-center subgroup $\mathfrak{K}$ containing J. The argument is the same as for $N_G(J^0)$. Because of $J/J^0 \subseteq \mathfrak{K}/\mathfrak{A}^*$, the factor group J/J^0 is cyclic such that an extension Λ always exists.

Now we come to the Main result of this section:

2.9 Theorem: *Let (J, Λ) be an extended simple type.*

(i) *There is a uniquely determined maximal compact modulo center subgroup $\mathfrak{K} \supset J$, and (J, Λ) is an admissible pair (in the sense of [Zi$_{88}$], 2.1) with respect to the descending sequence*

$$\mathfrak{K} \supset \mathfrak{A}^* \supset U^1 \supset U^2 \supset \cdots \tag{6}$$

of normal subgroups in $\mathfrak{K}$. Hence (J, Λ) corresponds to a representation filter of (6) and especially: $\mathrm{Ind}_{\mathfrak{K}}(\Lambda)$ is an irreducible representation of $\mathfrak{K}$.

(ii) *There exists precisely one discrete series representation* Π *of* G *such that*

$$\mathrm{Hom}_J(\Lambda, \Pi) \cong \mathrm{Hom}_G(\mathrm{ind}_G(\Lambda), \Pi) \neq 0,$$

and the multiplicity is one (i.e. $\dim_{\mathbb{C}} \mathrm{Hom} = 1$*).*

(iii) *The map* $(J, \Lambda) \to \Pi \in \widehat{G}_2$ *which is given by* (ii) *induces a bijection between* G*-conjugacy classes of extended simple types and the set of discrete series representations of* G*. The discrete series representation* Π *which is assigned to* (J, Λ) *is cuspical iff* $\Pi = \mathrm{ind}_G(\Lambda)$.

Remarks: 1. $\mathrm{ind}_G(\Lambda)$ denotes the induction with compact support.

2. (ii) may be reformulated by using $(\mathfrak{K}, \mathrm{Ind}_{\mathfrak{K}}(\Lambda))$ instead of (J, Λ). This point of view is adopted by Howe and Moy in the tame case.

3. A special instance of (ii) is $\Pi = \mathrm{ind}_G(\Lambda)$ which occurs iff Π is supercuspidal. Otherwise there is a uniquely determined maximal subspace in $\mathrm{ind}_G(\Lambda)$ such that the factor is a discrete series representation.

We are going to prove parts (ii), (iii) of 2.9. As to (i), the proof is essentially the same as that of [Zi$_{90}$], 4.6 but we will not go into that. According to (iii), it is enough to parametrize the conjugacy classes of EST's in order to get an overview over the discrete series representations. We will do that in the fourth section. We will approach 2.9 via a property of the simple types which is not mentioned explicitly in [BK] but is an easy consequence of §7. Namely:

2.10 Proposition: *Let* (J^0, λ) *be a simple type of* G*. All discrete series representations* $\Pi \in \widehat{G}_2$ *which contain* (J^0, λ) *form an equivalence class with respect to twisting by unramified characters.*

Proof: We consider $J^0 \subset J = N_G(J^0, \lambda) \subset \mathfrak{K} = \mathfrak{K}(\mathfrak{A})$ where the maximal compact-modulo-center group $\mathfrak{K}$ and the principal order $\mathfrak{A}$ resp. are uniquely determined. Then we have $J^0 = J \cap \mathfrak{A}^*$. If $\chi: F^*/U_F \to \mathbb{C}^*$ is an unramified character, then $\tilde{\chi} = \chi \circ \det_{A/F}$ vanishes on $\mathfrak{A}^*$ because $\det(\mathfrak{A}^*) \subseteq U_F$. Hence $\Pi \supset (J^0, \lambda)$ implies $\Pi \otimes \tilde{\chi} \supset (J^0, \lambda)$ for all unramified characters χ.

Now we need the following fundamental properties of simple types which we have not yet mentioned:

2.11 (see [BK]§§5,7) (i) Let (J^0, λ) be a simple type which is associated to a simple stratum $\alpha + \mathfrak{A}$ and consider $K \supseteq E \supseteq F$ as in (3) and in 2.6. Then there is a canonical family of Hecke algebra isomorphisms

$$\mathcal{H}(GL_K, \mathfrak{A}_K^*, \mathbf{1}) \xrightarrow{\sim} \mathcal{H}(G, J^0, \lambda)$$

where $\mathbf{1}$ denotes the unit representation of the Iwahori subgroup $\mathfrak{A}_K^*$ of GL_K.

(ii) Let $\mathfrak{A}d(G, J^0, \lambda)$ be the category of admissible G-representations of finite length such that each composition factor contains (J^0, λ). Then an equivalence

$$\mathfrak{A}d(GL_K, \mathfrak{A}_K^*, \mathbf{1}) \xrightarrow{\sim} \mathfrak{A}d(G, J^0, \lambda)$$

can be obtained which has the following properties:

— if $\Pi_K \mapsto \Pi$, then $|\det_{A_K/K}|_K^s \otimes \Pi_K \mapsto |\det_{A/F}|_F^s \otimes \Pi$ for all $s \in \mathbb{C}$ ([BK] 7.5.12).

— $\Pi_K \in GL_K{}^\frown$ is a discrete series representation iff $\Pi \in \widehat{G}$ is discrete series ([BK] 7.7.1).

We use 2.11 to reduce our Proposition to the case $(J^0, \lambda) = (\mathfrak{A}^*, \mathbf{1})$ where $\mathfrak{A}^*$ is an Iwahori subgroup of G, i.e. the period of the principal order $\mathfrak{A}$ is $e(\mathfrak{A}/\mathfrak{o}_F) = N$.

Lemma 2.12: *Let $\mathfrak{A}^*$ be an Iwahori subgroup of G. Then the irreducible discrete series representations from $\mathfrak{A}d(G, \mathfrak{A}^*, \mathbf{1})$ are precisely the unramified twists of the Steinberg representation St.*

Proof: Let $\Pi \in \mathfrak{A}d(G, \mathfrak{A}^*, \mathbf{1})$ be irreducible. If we apply [BK] (7.3.8) to the simple type $(\mathfrak{A}^*, \mathbf{1})$ with $e = e(\mathfrak{A}/\mathfrak{o}_F) = N$, then we see that the supercuspidal support of Π consists of N unramified characters of F^*. Moreover, if Π is a discrete series representation the supercuspidal support has to be a "segment" $\chi, \chi(1), \ldots, \chi(N-1)$, where $\chi(i) := \chi \cdot \|\|_F^i$. From Bernstein, Zelevinsky we know that the map

$$(\mathbf{Z}) \qquad \Pi \in (G)^\frown \longmapsto \text{supercuspidal support of } \Pi$$

is injective if we restrict to discrete series representations, and it is compatible with twisting by characters of F^*. We consider the special case where B is the Borel group of upper triangular matrices in G. For $\tau: F^* \to \mathbb{C}^*$ and $\tilde{\tau} = \tau \circ \det_{A/F}$ we obtain

$$\tilde{\tau} \otimes \mathrm{Ind}_{B\uparrow G}(\chi \otimes \chi(1) \otimes \cdots \otimes \chi(N-1)) = \mathrm{Ind}_{B\uparrow G}(\tau\chi \otimes (\tau\chi)(1) \otimes \cdots \otimes (\tau\chi)(N-1))$$

which proves the twist property of $(\mathbf{Z})$ in that special case. Because all unramified segments are obtained by twisting a fixed unramified segment with the unramified characters of F^*, our assertion follows from $(\mathbf{Z})$. □ (2.12), (2.10)

Now we come to the poof of 2.9 (ii). From (J, Λ) we go back to the simple type (J^0, λ) where $J^0 = J \cap \mathfrak{A}^*$ and $\lambda = \mathrm{Res}(\Lambda)$. Let Δ be the unramified twist class of all discrete series representations $\Pi \in \widehat{G}_2$ which contain (J^0, λ) (see 2.10). On the other hand, let Γ be the set of EST's (J, Λ') which extend (J^0, λ).

Step 1: We define a map

$$(7) \qquad \Pi \in \Delta \longmapsto (J, \Lambda_\Pi) \in \Gamma$$

Namely, if $\Pi \supset (J^0, \lambda)$ let V^λ be the λ-component of the representation space V_Π considered as a J^0-space. (V^λ is finite dimensional because Π is admissible and J^0 is an open subgroup of G). Using that J^0 is a normal subgroup in J and that λ is J-invariant, we conclude that V^λ is even a J-space. Moreover, we have

2.13 Lemma: *(J^0, λ) occurs in V_Π with multiplicity one.*

Proof: In view of the Hecke isomorphism 2.11 (i), the proof is the same as that of [HM] Cor. 5.5 (a). Namely, the Lemma is reduced to the case $(J^0, \lambda) = (\mathfrak{A}^*, \mathbf{1})$, where $\mathfrak{A}^*$ is an Iwahori subgroup of G. In that case Π is an unramified twist of the Steinberg representation (see 2.12), and we are left to show that $(\mathfrak{A}^*, 1)$ occurs in St with multiplicity one which is well known. □

From the Lemma we conclude that V^λ is irreducible as a J^0-space hence as a J-space too. And the map (7) is given by assigning the J-space V^λ to a $\Pi \in \Delta$.

Step 2: The map (7) is surjective.
Let $\chi: F^*/U_F \to \mathbb{C}^*$ be an unramified character. With $\tilde{\chi} = \chi \circ \det_{A/F}$, an easy property of (7) is

$$(8) \qquad \Lambda_{\tilde{\chi}\otimes\Pi} = \tilde{\chi} \otimes \Lambda_\Pi \in J^\frown$$

Using $(3)_3$, we have $J = L^* \cdot J^0$, and $[L: F] = N$ implies $\tilde{\chi}|_{L^*} = \chi \circ N_{L|F}$. But $\chi = \|_F^s$ for some $s \in \mathbb{C}$ and $|N_{L|F}|_F = \|_L$. Hence $\tilde{\chi} \otimes \Lambda_\Pi = \|_L^s \otimes \Lambda_\Pi$, where $\|_L^s$ acts on $L^*J^0/J^0 \cong L^*/U_L$. Therefore we see that the characters $\tilde{\chi} \otimes \Lambda_\Pi$ cover all extensions from λ onto J if χ runs over the unramified characters of F^*.

Step 3: The map (7) is injective.
Let $\chi: F^* \to \mathbb{C}^*$ be an unramified character such that $\Lambda_\Pi = \Lambda_{\tilde{\chi}\otimes\Pi}$. Then we have to show $\Pi = \tilde{\chi} \otimes \Pi$. Because of (8) and the identification $J/J^0 = L^*/U_L$ (see Step 2), our assumption implies:

$$(9) \qquad \Lambda_\Pi = (\chi \circ N_{L|F}) \otimes \Lambda_\Pi$$

Putting a prime element π_L into the argument and using $\chi \circ N_{L|F}(\pi_L) = \chi(\pi_F^f)$, we see that (9) is equivalent to $\chi^f \equiv 1$ where $f = f_{L|F}$ denotes the inertial degree. So we are left to show that $\chi^f \equiv 1$ implies $\tilde{\chi} \otimes \Pi = \Pi$. Here we can use (2.11) (ii). Namely, $\chi = \|_F^s$ for some $s \in \mathbb{C}$. Then $\chi^f \equiv 1$ iff $|\pi_F|_F^{s \cdot f} = |\pi_K|_K^s = 1$ (note that $f = f_{K/F}$ because $L|K$ is fully ramified) iff $\|_K^s \equiv 1$. Now if Π corresponds to $\Pi_K \in \mathfrak{A}d(GL_K, \mathfrak{A}_K^*, 1)$ then by 2.11 (ii) $\tilde{\chi} \otimes \Pi = |\det|_F^s \otimes \Pi$ corresponds to $|\det_{A_K|K}|_K^s \otimes \Pi_K = \Pi_K$ (because of $\|_K^s \equiv 1$), hence $\tilde{\chi} \otimes \Pi = \Pi$.

Altogether we see that (7) is a natural bijection, and that the extended simple type (J, Λ_Π) has multiplicity one in Π. Further from [BZ] or [Cas] 2.3.1, 2.3.2 we obtain the Frobenius reciprocity

$$\mathrm{Hom}_G(\mathrm{ind}_G(\Lambda), \Pi) \cong \mathrm{Hom}_J(\Lambda, (\tilde{\Pi}_J)^\sim)$$

where "$\sim$" denotes the contragredient admissible representation. But Π is from the discrete series. hence $\tilde{\Pi} = \chi \otimes \bar{\Pi}$, where $\bar{\Pi}$ is the complex conjugate representation and χ is a certain real character. Therefore $(\tilde{\Pi}_J)^\sim = \Pi_J$.

We are left with the proof of 2.9 (iii). By [BK] (8.4.3) any discrete series representation contains a simple type, hence an extended simple type too, i.e. our map is surjective. On the other hand, the simple type (J^0, λ) in Π is unique up to conjugation, and from 2.13 we see that (J^0, λ) has a unique extensioin (J, Λ) in Π. Hence the EST (J, Λ) in Π is unique up to conjugation too. If $\Pi = \mathrm{ind}_G(\Lambda)$ then by a well known argument Π is cuspidal. On the other hand from [BK] we know that the simple type $(J^0, \lambda) \subset \Pi$ is maximal if Π is supercuspidal. This means $L = K$ in the notation of $(3)_3$ and 2.6, hence $I_G(J^0, \lambda) = K^*J^0 = J$, hence $\mathrm{ind}_G(\Lambda) = \Pi$ is irreducible. □ (2.9)

2.14 Corollary: *For $\Pi \in \widehat{G}_2$ let f_Π be the cardinal number of unramified characters $\chi: F^* \to \mathbb{C}^*$ such that $\tilde{\chi} \otimes \Pi = \Pi$, and let e be the period of the principal order $\mathfrak{A}$ such that Π contains an extended simple type (J, Λ) which is in $\mathfrak{K} = \mathfrak{K}(\mathfrak{A})$. Then:*

$$e \cdot f_\Pi = N.$$

Proof: We fix a simple type $(J^0, \lambda) \subset \Pi$ and consider the bijection (7). Because it is compatible with unramified character twist, we conclude:

$$f_\Pi = \{\chi; \tilde{\chi} \otimes \Lambda = \Lambda\} = f_{L|F}$$

as we have seen in Step 3 of the proof of 2.9 (ii). But $[L: F] = N$ and $L^* \subset \mathfrak{K}$ implies $N/f_{L|F} = e_{L|F} = e$. □

For an extended simple type (J, Λ) define:

$$j(\Lambda) := \begin{cases} 0 \text{ if } \Lambda|_{J^1} \equiv 1 \\ \max\{i; \Lambda|_{J^i} \not\equiv 1\} \text{ otherwise.} \end{cases}$$

Note that $j(\Lambda) = 0$ implies $\pi_1 \equiv 1$ hence $J^1 = U^1$ because $\mathrm{Ind}_{U^1}(\pi_1)$ is irreducible. But if $(J^1, \pi_1) = (U^1, \mathbf{1})$, then from (4), (5) follows $E = F$ and $J = \mathfrak{K}$, such that Λ is a "cuspidal" representation of $\mathfrak{K}/U^1$. On the other hand if $j(\Lambda) = j \neq 0$ then $J^{[j/2]+1} = U^{[j/2]+1}$ as we have stated at the very end of 2.2. Moreover $j \neq 0$ corresponds to $\alpha + \mathfrak{A} \neq \mathfrak{A}$, i.e. $\alpha \notin \mathfrak{A}$ for an associated simple stratum, namely $\nu_P(\alpha) = -j$ because

$$\Lambda(1 + x) = \psi_F \circ \mathrm{Tr}_{A|F}(\alpha x) \cdot \mathbf{1}$$

for $x \in P^{[j/2]+1}$. Especially $\Lambda|_{U^j}$ corresponds to the stratum $\alpha + P^{-j+1}$. (Note that ψ_F has conductor 1, i.e. $\psi_F(\mathfrak{p}_F) \equiv 1$, $\psi_F(\mathfrak{o}_F) \not\equiv 1$). This stratum is fundamental because α is $\mathfrak{A}$-pure, hence $\alpha + P^{-j+1} = \alpha \cdot U^1 \subseteq \mathfrak{K}$ such that $\alpha + P^{-j+1}$ contains no nilpotent elements (see [KM] 1.3). In terms of [BFr] (2.2.8) this also means that $\alpha + P^{-j+1}$ is non-degenerate which we will use in the next section. The level of $\alpha + P^{-j+1}$ is by definition:

$$\ell(\alpha + P^{-j+1}) = -\nu_P(\alpha)/e = j/e,$$

where e is the period of $\mathfrak{A}$. Identifying $\alpha + P^{-j+1}$ with $\Lambda|_{U^j}$, we may say that

$$\Pi \supseteq \Lambda \supseteq \alpha + P^{-j+1} \text{ if } \Pi \in \widehat{G}_2 \text{ corresponds to } (J, \Lambda).$$

Now it is known that all fundamental strata in a representation $\Pi \in \widehat{G}$ have the same level, and this is by definition the level $\ell(\Pi)$. If $j(\Lambda) = 0$, i.e. if Π contains a "cuspidal" representation of some $\mathfrak{K}/U^1$, then the level $\ell(\Pi)$ is defined to be zero (see [KM], 1.3, 1.4 for more details and hints). Thus we have:

2.15 Corollary: *If $\Pi \in \widehat{G}_2$ corresponds to the extended simple type (J, Λ), then it has the level:*

$$\ell(\Pi) = \begin{cases} 0 \text{ if } \Lambda_{J^1} \equiv 1, \text{ hence } J^1 = U^1,\ J = \mathfrak{K} \\ j(\Lambda)/e \text{ otherwise} \end{cases}$$

(where e is the period of the uniquely determined principal order $\mathfrak{A}$ which is normalized by J).

3. Interdependence of the exponential Artin conductor and the level of a discrete series representation

In this section we want to compare the level of $\Pi \in \widehat{G}_2$ with its exponential Artin conductor $a(\Pi)$ which yields another interpretation of $a(\Pi)$ than that of [J, P, Sh] – at least for the discrete series representations. (In loc. cit. $a(\Pi)$ has been interpreted for all generic representations.) Let us recall the definition of $a(\Pi)$:

Let $B|F$ be a central simple algebra of index N and let $\Pi \in B^{*\wedge}$ be an irreducible admissible representation. The exponential Artin conductor $a(\Pi)$ is given by the equation:

(1) $\varepsilon(s, \Pi, \psi_F) = \varepsilon(0, \Pi, \psi_F) \cdot q_F^{-a(\Pi)\cdot s}$ for all $s \in \mathbb{C}$, where the ε-factor is defined as in [GJ], Theorem 3.3 (4), where $\psi_F \colon F^+ \to \mathbb{C}^*$ is an additive character of conductor 0 and where $q_F = |k_F|$ is the order of the residue field.

We remark that $a(\Pi \otimes |Nrd|_F^t) = a(\Pi)$ for all $t \in \mathbb{C}$ because $\varepsilon(s, \Pi \otimes |Nrd|_F^t, \psi_F) = \varepsilon(s+t, \Pi, \psi_F)$, i.e. $a(\Pi)$ is invariant with respect to unramified character twist. And in the case $B = F$, i.e. Π is a character of F^*, the usual exponential Artin conductor is recovered.

We want to compare $a(\Pi)$ and $\ell(\Pi)$ in the case where $B = A = M_N(F)$ and $B = D$ a division algebra of index N resp. Before stating our main result we have still to define $\ell(\Pi)$ for division algebra representations. Namely:

(2) For $\Pi \in D^{*\wedge}$ define $j(\Pi) = 0$ if $U_D^1 \subset \operatorname{Ker}\Pi$ and $j(\Pi) = \max\{i; \Pi|_{U^i} \not\equiv i\}$ otherwise and define the level to be
$$\ell(\Pi) := j(\Pi)/N.$$

The main result of this section is:

3.1 Theorem: *(i) Let $\Pi \in \widehat{G}_2$ be a discrete series representation. If Π is not an unramified twist of the Steinberg representation St_N, then: $a(\Pi) = N(\ell(\Pi)+1)$. On the other hand, we have*
$$\varepsilon(s, St_N, \psi_F) = (-1)^{N-1} \cdot q_F^{(\frac{1}{2}-s)(N-1)}$$
hence $a(\Pi) = N-1$ if Π is an unramified twist of St_N.

(ii) Assume that $\Pi \in D^{*\wedge}$ is not an unramified character. Then:
$$a(\Pi) = N(\ell(\Pi)+1).$$

On the other hand, if $\mathbf{1}$ is the unit representation of D^* then:
$$\varepsilon(s, \mathbf{1}, \psi_F) = (-1)^{N-1} \cdot q_F^{(\frac{1}{2}-s)(N-1)},$$

hence $a(\tilde{\chi}) = N-1$ if $\tilde{\chi} = \chi \circ Nrd$ is an unramified character of D^* (i.e. $\chi \colon F^*/U_F \to \mathbb{C}^*$).

Proof: To begin with we prove (i) in the case where $\Pi \in \widehat{G}_2$ is not an unramified twist of St_N. Consider an extended simple type (J, Λ) which corresponds to Π by 2.9 (iii).

Then we will apply [BFr] (3.3.8) to $\rho = \mathrm{Ind}_{\mathfrak{K}}(\Lambda)$, where $\mathfrak{K} \supset J$ is the uniquely determined maximal compact modulo center subgroup. Namely as we have seen before 2.15, Λ (hence ρ) contains a fundamental stratum $\alpha + P^{-j+1} \subseteq \mathfrak{K}$ which means that ρ is nondegenerate. Note that the case $j(\Lambda) = 0$ is included into the definition of "non-degenerate" too ([BFr] 2.2.8 ff). Therefore [BFr] (3.3.8) gives us the formula:

$$\varepsilon(\Pi, s) = N(\mathfrak{D}_{\mathfrak{A}} \cdot \mathfrak{f}(\rho))^{(\frac{1}{2}-s)/N} \cdot W(\rho), \tag{3}$$

where $\mathfrak{D}_{\mathfrak{A}}$ is the differente of $\mathfrak{A}/\mathbb{Z}_p$, ($\mathfrak{A}$ being the principal order that corresponds to $\mathfrak{K}$) where $N(\mathfrak{d}) = (\mathfrak{A}\colon \mathfrak{d})$ for an ideal $\mathfrak{d} \subset \mathfrak{A}$, where $\mathfrak{f}(\rho) = P^{\nu_P(\mathfrak{f}(\rho))}$ is the conductor of ρ and where $W(\rho)$ is the root number of ρ which does not depend on s (see [BFr] (2.8.9)).

3.2 Remark: We have used [BFr] 3.4.(C). According to that remark, (3) is valid not only for supercuspidal Π but for all Π such that $L(\Pi, s) = L(\Pi^\vee, s) = 1$. Therefore from [J] (3.1) we see that (3) applies to discrete series representations too if we exclude unramified twists of the Steinberg representation.

Because of the definition of $\mathfrak{f}(\rho)$ in [BFr] (2.2.3) we see that

$$\nu_P(\mathfrak{f}(\rho)) = e \cdot \ell(\Pi) + 1 \tag{4}$$

where $\ell(\Pi)$ is the level and e is the period of $\mathfrak{A}$. Note that in our case $\nu_p(\mathfrak{f}(\rho)) \neq 0$ because Π is cuspidal. Further, if $\mathfrak{d}$ is a power of the Jacobson radical P of $\mathfrak{A}$, then an easy comptution gives us:

$$N(\mathfrak{d})^{1/N} = q_F^{\nu_P(\mathfrak{d})\cdot N/e}. \tag{5}$$

Because $\mathfrak{D}_{\mathfrak{A}} = P^{e-1} \cdot \Delta$, where Δ is the differente of $\mathfrak{o}_F/\mathbb{Z}_p$, and because $\nu_P(\Delta) = e \cdot \nu_F(\Delta)$, we obtain from (4), (5):

$$N(\mathfrak{D}_{\mathfrak{A}} \cdot \mathfrak{f}(\rho))^{1/N} = q_F^{N[\nu_F(\Delta)+\ell(\Pi)+1]}. \tag{6}$$

Now we remark that the ε-factor of [BFr] and that of [GJ] Theorem 3.3 (4) are related by the formula:

$$\varepsilon(\Pi, s) = \varepsilon(s, \Pi, \psi') \tag{7}$$

where $\psi' = \psi_{\mathbf{Q}_p} \cdot Tr_{F|\mathbf{Q}_p}$ is an additive character of F, which is lifted from the additive character $\psi_{\mathbf{Q}_p}$ having conductor 0 (i.e. $\psi_{\mathbf{Q}_p}$ vanishes on $\mathbb{Z}_p$ but not on $p^{-1}\mathbb{Z}_p$). According to (1) we need $\varepsilon(s, \Pi, \psi_F)$ where ψ_F has conductor 0 with respect to F. Hence:

$$\psi'(x) = \psi_F(x\delta) \tag{8}$$

where $\delta \in F$ is a generator of the absolute differente Δ. Using [GJ] (3.3.5), we conclude from (7), (8):

$$\varepsilon(s, \Pi, \psi_F) = \omega(\delta^{-1}) \cdot |\delta^{-1}|_F^{N(s-1/2)} \cdot \varepsilon(\Pi, s) \tag{9}$$

where ω is the central character of Π. But $|\delta^{-1}|_F = q_F^{\nu_F(\Delta)}$, such that from (3), (6), (9) we obtain:

$$\varepsilon(s, \Pi, \psi_F) = \omega(\delta^{-1}) \cdot q_F^{N(\ell(\Pi)+1)(1/2-s)} \cdot W(\rho) = q_F^{-sN(\ell(\Pi)+1)} \cdot \varepsilon(0, \Pi, \psi_F),$$

hence $a(\Pi) = N(\ell(\Pi) + 1)$.

We come to the proof of 3.1 (i) in the case where $\Pi \in \widehat{G}_2$ is an unramified twist of the Steinberg representation. Using a Theorem of Zelevinsky, we know that every discrete series representation Π is the unique irreducible quotient of $\mathrm{Ind}_{P\uparrow G}(\sigma_1 \otimes \cdots \otimes \sigma_{N/n})$, where n is a divisor of N, $P \subseteq G$ is a parabolic subgroup with the Levi component $GL_n(F) \times \cdots \times GL_n(F) \subseteq G$ (where the number of factors is N/n), and

$$\sigma_i = \sigma_1 |\det|_F^{i-1} \quad \text{for} \quad i = 1, \ldots, N/n$$

are cuspidal representations of $GL_n(F)$.

The Steinberg representation $\Pi = St_N$ is obtained for $n = 1$ and $\sigma_1 = |\det|_F^{-1/2(N-1)}$. This ensures that the supercuspidal support of St_N has the property $\{\sigma_1, \ldots, \sigma_N\} = \{\tilde{\sigma}_1, \ldots, \tilde{\sigma}_N\}$ because $\tilde{\sigma}_i = \sigma_i^{-1}$. Hence St_N is selfdual.

The γ-factor of $\Pi \in \widehat{G}$ is by definition:

$$\gamma(s, \Pi, \psi_F) = \varepsilon(s, \Pi, \psi_F) \cdot L(1 - s, \tilde{\Pi}) / L(s, \Pi).$$

From [J] (2.7.3), (2.3) we know that in the situation of Zelevinsky's Theorem we have

$$\gamma(s, \Pi, \psi_F) = \prod_{i=1}^{N/n} \gamma(s, \sigma_i, \psi_F). \tag{10}$$

In the case $\Pi = St_N$ one can use (10) with $n = 1$ and $\sigma_i = |\det|_F^{-1/2(N+1)+i}$. Then the computation of [GJ], p. 97 (where the notation is $\varepsilon'(s, \sigma, \psi)$ instead of $\gamma(s, \sigma, \psi)$) gives us the formula for $\varepsilon(s, St_N, \psi_F)$.

Next we apply (3) to prove 3.1 (ii). In the case of a *division algebra* we simply have $\rho = \Pi$ and $\mathfrak{A} = \mathfrak{O}$ (= integers of D) because D^* is compact modulo center. Therefore (3) turns into:

$$\varepsilon(\Pi, s) = N(\mathfrak{D}_{\mathfrak{O}} \cdot \mathfrak{f}(\Pi))^{(\frac{1}{2}-s)/N} \cdot W(\Pi), \tag{3$_D$}$$

where $\mathfrak{D}_{\mathfrak{O}}$ is the differente of $\mathfrak{O}/\mathbb{Z}_p$, and $\mathfrak{f}(\Pi)$ is the conductor of Π. If Π is not an unramified character, then from [BFr] (2.2.3) we see that

$$\nu_P(\mathfrak{f}(\Pi)) = N \cdot \ell(\Pi) + 1,$$

where P is the prime ideal of $\mathfrak{O}$ and $\ell(\Pi)$ is the level. If $\mathfrak{d}$ is a power of P, then:

$$N(\mathfrak{d}) := (\mathfrak{O} : \mathfrak{d}) = q_F^{N \cdot \nu_P(\mathfrak{d})}.$$

Using $\mathfrak{D}_{\mathcal{O}} = P^{N-1} \cdot \Delta$, where Δ is as before, we conclude that:

$$(11) \qquad N(\mathfrak{D}_{\mathcal{O}} \cdot \mathfrak{f}(\Pi))^{1/N} = q_F^{N[\nu_F(\Delta)+\ell(\Pi)+1]}.$$

Again we have to use the formula (9) such that $(3)_{\mathrm{D}}$, (11) imply:

$$\varepsilon(s, \Pi, \psi_F) = \omega(\delta^{-1}) \cdot q_F^{N(\ell(\Pi)+1)(\frac{1}{2}-s)} W(\Pi),$$

where ω is the central character of Π, hence $a(\Pi) = N(\ell(\Pi)+1)$.

We are left with the case where Π is an unramified character of D^*, and here we have to compute $\varepsilon(s, \mathbf{1}, \psi_F)$. Computing $\varepsilon(\mathbf{1}, s)$ by means of $(3)_{\mathrm{D}}$, we can use [BFr] (2.8.13) (ii), which gives us:

$$W(\mathbf{1}_{D^*}) \cdot y(\mathbf{1}_{D^*}) = [W(\mathbf{1}_{F^*}) \cdot y(\mathbf{1}_{F^*})]^N,$$

where for the moment we have written $\mathbf{1}_{D^*}$ instead of $\mathbf{1}$. But apparently $y(\mathbf{1}_{D^*}) = y(\mathbf{1}_{F^*}) = -1$ and $W(\mathbf{1}_{F^*}) = 1$, hence $W(\mathbf{1}_{D^*}) = (-1)^{N-1}$, and because of $\mathfrak{f}(\mathbf{1}_{D^*}) = \mathcal{O}$ the formula $(3)_{\mathrm{D}}$ yields:

$$\varepsilon(\mathbf{1}, s) = (-1)^{N-1} \cdot N(\mathfrak{D}_{\mathcal{O}})^{(1/2-s)/N}.$$

Again we use $\mathfrak{D}_{\mathcal{O}} = P^{N-1} \cdot \Delta$ and proceed as before to obtain

$$\varepsilon(s, \mathbf{1}, \psi_F) = (-1)^{N-1} \cdot q_F^{(\frac{1}{2}-s)(N-1)}. \qquad \square$$

Using the congruence Gauss sums of [BFr] we want to make the formulas for $\varepsilon(s, \Pi, \psi_F)$ a little more precise by reducing them to a congruence Gauss sum for the extended simple type (J, Λ) corresponding to Π. In the division algebra case, an extended simple type is nothing else than an admissible pair with respect to the principal unit filtration $D^* \supset U_D^1 \supset U_D^2 \supset \cdots$ which gives rise to Π.

We begin by defining the congruence Gauss sums for an arbitrary additive character $\psi: F^+ \to \mathbb{C}^*$ with conductor $\mathfrak{f}_F(\psi) = \mathfrak{p}_F^v$.

3.3 Definition: Let $\mathfrak{K} = \mathfrak{K}(\mathfrak{A})$, $P = \mathrm{Jac}(\mathfrak{A})$ be a maximal compact mod center subgroup in the central simple algebra B/F, and let $\rho \in \mathfrak{K}^\wedge$ be an irreducible representation. The congruence Gauss sum $\tau(\rho, \psi)$ is the value of the scalar operator

$$T(\rho, \psi) = \sum_{u \in (\mathfrak{A}/\mathfrak{f}(\rho))^*} \rho(c^{-1}u) \cdot \psi_B(c^{-1}u)$$

where $\psi_B = \psi \circ Trd_{B/F}$, $\mathfrak{f}(\rho) = P^{j+1}$ is the conductor of ρ (we will always assume $\rho|_{\mathfrak{A}^*} \not\equiv 1$, i.e. $j \geq 0$), and where c is any generator of the ideal $\mathfrak{f}(\psi_B)^{-1}\mathfrak{f}(\rho) = P^{e-1} \cdot \mathfrak{f}_F(\psi)^{-1} \cdot \mathfrak{f}(\rho)$ (which is a power of P). $e := e(\mathfrak{A}/\mathfrak{o}_F) = e(\mathfrak{A}/\mathcal{O}_D) \cdot e(\mathcal{O}_D/\mathfrak{o}_F)$, if B is a matrix algebra over a division algebra D.

For $a \in F^*$ define ψ_a by: $\psi_a(x) = \psi(ax)$. Then $\mathfrak{f}_F(\psi_a)^{-1} = a \cdot \mathfrak{f}_F(\psi)^{-1}$, hence a small computation yields:

3.4 Lemma: *$T(\rho,\psi_a) = \omega_\rho(a)^{-1}\cdot T(\rho,\psi)$, where $\omega_\rho\colon F^* \to \mathbb{C}^*$ is the central character defined by ρ.*

Now the formula of [BFr] (3.3.8) reads:

$$\epsilon(s,\Pi,\psi) = N(\mathfrak{f}(\rho)\cdot\mathfrak{f}(\psi_B)^{-1})^{(\frac{1}{2}-s)/N}\cdot W(\rho,\psi) \tag{12}$$

where:

$$W(\rho,\psi) = (-1)^{N-m}\cdot N\mathfrak{f}(\rho)^{-1/2}\cdot\tau(\rho^\vee,\psi) \tag{13}$$

if $\rho\in\mathfrak{K}^\wedge$ is a nondegenerate representation occurring in Π and if $L(\Pi,s) = L(\Pi^\wedge,s) = 1$. N is the index of B/F and m is determined by $B\cong M_m(D)$.

Note that as a consequence of 3.4 we obtain:

$$W(\rho,\psi_a) = \omega_\rho(a)\cdot W(\rho,\psi). \tag{14}$$

We consider the case that $\rho\in\mathfrak{K}^\wedge$ is induced by $\Lambda\in J^\wedge$, where $F^*\subset J\subset\mathfrak{K}$ is a subgroup of finite index. Modulo twist with $\tilde{\chi} = \chi\circ Nrd_{B/F}$, where $\chi\colon F^*/U_F\to\mathbb{C}^*$ is an unramified character, we can assume that ρ and Λ are finite representations. Hence we have the usual formula connecting the characters χ_ρ and χ_Λ. Now we apply this formula to

$$\tau(\rho,\psi)\cdot\dim\rho = \sum_{u\in(\mathfrak{A}/\mathfrak{f}(\rho))^*}\chi_\rho(c^{-1}u)\psi_B(c^{-1}u)$$

i.e. we substitute $\chi_\rho(c^{-1}u) = \sum_{r\in R}\chi^0_\Lambda(r^{-1}c^{-1}ur)$, where R is a residue system of $\mathfrak{K}/J$. Then we interchange the sums and replace $r^{-1}c^{-1}ur = (r^{-1}c^{-1}r)(r^{-1}ur)$ by $r^{-1}c^{-1}ru$ because the summation is over u. We find:

$$\tau(\rho,\psi)\dim\rho = \sum_r\left\{\sum_u\chi^0_\Lambda(r^{-1}c^{-1}ru)\cdot\psi_B(r^{-1}c^{-1}ru)\right\}.$$

Now we assume $c\in J$, then:

$$r^{-1}c^{-1}ru\in J \qquad \text{iff} \qquad r^{-1}c^{-1}ru\in c^{-1}J^0,$$

where $J^0 = J\cap\mathfrak{A}^*$. Hence

$$\sum_u\chi^0_\Lambda(r^{-1}c^{-1}ru)\psi_B(r^{-1}c^{-1}ru) = \sum_{u\in J^0/1+\mathfrak{f}(\rho)}\chi_\Lambda(c^{-1}u)\cdot\psi_B(c^{-1}u)$$

is independent from $r\in R$, and we conclude:

$$\tau(\rho,\psi)\cdot\dim\rho = (\mathfrak{K}\colon J)\tau(\Lambda,\psi)\cdot\dim\Lambda.$$

3.5 Proposition: *If $\rho = Ind_{J\uparrow\mathfrak{K}}(\Lambda)$ and if $\mathfrak{f}(\psi_B)^{-1} \cdot \mathfrak{f}(\rho)$ has a generator in J (i.e. $c\mathfrak{A}^* \subset J\mathfrak{A}^*$ for any c as in 3.3), then:*

$$\tau(\rho, \psi) = \tau(\Lambda, \psi) \tag{15}$$

where $\tau(\Lambda, \psi)$ is the value of the scalar operator $\sum_u \Lambda(c^{-1}u)\psi_B(c^{-1}u)$ where the sum is over $u \in J^0/1 + \mathfrak{f}(\rho)$ and c is now a generator of $\mathfrak{f}(\psi_B)^{-1} \cdot \mathfrak{f}(\rho)$ in J. □

Note that (15) holds for all additive characters because $F^* \subset J$ and that $\mathfrak{f}(\rho) = \mathfrak{f}(\Lambda)$.

Now we consider $\Pi \in \widehat{G}_2$, $D^{*\wedge}$ resp., and we take (J, Λ) to be a corresponding EST. Moreover let ψ be an additive character of conductor 0. Then from (8) we see $\psi = \psi'_{\delta^{-1}}$, and the computations in proving Theorem 3.1 give us:

$$\varepsilon(s, \Pi, \psi) = q_F^{N(\ell(\Pi)+1)\cdot(\frac{1}{2}-s)} \cdot W(\rho, \psi), \tag{16}$$

where $\rho = \mathrm{Ind}_{J\uparrow\mathfrak{K}}(\Lambda)$. We check that Proposition 3.5 is applicable, i.e. that $\mathfrak{f}(\psi_B)^{-1}\mathfrak{f}(\rho)$ has a generator in J. For that we may assume that ψ has conductor $\mathfrak{p}_F$. Then $\mathfrak{f}(\psi_B)^{-1} \cdot \mathfrak{f}(\rho) = P^{-1} \cdot \mathfrak{f}(\rho) = P^{j(\rho)}$, where $j(\rho) = j(\Lambda)$ is the index of ρ, Λ resp. Now if (J, Λ) is related to the simple stratum $\alpha + \mathfrak{A}$ then on one hand $j(\Lambda) = -\nu_P(\alpha)$ and on the other hand $\mathfrak{K}_E \subset J$ (see 2.(3)) where $E = F(\alpha)$. (In the division algebra case we have $D_E^* \subset J$). Hence we can take $c^{-1} = \alpha \in J$. Therefore:

3.6 Theorem: *If $\Pi \in \widehat{G}_2$, $D^{*\wedge}$ resp. corresponds to an extended simple type (J, Λ) and if $\psi_F: F^+ \to \mathbb{C}^*$ is an additive character of conductor $\mathfrak{p}_F$, then:*

$$\varepsilon(s, \Pi, \psi_{F,\pi_F}) = q_F^{N(\ell(\Pi)+1)(\frac{1}{2}-s)} \cdot \omega_\Pi(\pi_F) \cdot W(\Lambda, \psi_F)$$

where $W(\Lambda, \psi_F) = (-1)^{N-m} \cdot N\mathfrak{f}(\Lambda)^{-1/2} \cdot \tau(\Lambda^\vee, \psi_F)$, with $m = 1, N$ if B/F is the division algebra and matrix algebra resp.

Remark: Note that Λ and $\rho = \mathrm{Ind}_{J\uparrow\mathfrak{K}}(\Lambda)$ have the same conductor $\mathfrak{f}(\Lambda) = \mathfrak{f}(\rho)$ and that in case $\mathfrak{f}(\rho) = P$ we have $\rho = \Lambda$.

Proof: ψ_{F,π_F} has conductor $\mathfrak{o}_F$. Hence we can apply (16) and (15). The result follows from $W(\Lambda, \psi_{F,\pi_F}) = \omega_\Pi(\pi_F) \cdot W(\Lambda, \psi_F)$. □

4. Parametrizing extended simple types

In view of 2.9 we have to find a parameter system for the G-conjugacy classes of extended simple types. We consider $\mathcal{T}^-$ from 1.(12) and

$$\mathcal{T}_N^- := \{t \in \mathcal{T}^-;\ \deg(t) \mid N\}$$

where the degree of an R-polynomial has been defined in 1.2. We are going now to construct an injection of $\mathcal{T}_N^-$ into the set of G-conjugacy classes of extended simple types (which in fact is a bijection). Of course we could replace $\mathcal{T}_N^-$ by $\mathcal{S}_N^-$, $\mathcal{P}_N^-$ resp. (which are defined in the same way as $\mathcal{T}_N^-$). Taking $\mathcal{P}_N^-$ is especially useful for counting procedures. We will not go into that but will work only with $\mathcal{T}_N^-$. Before going ahead two remarks might be in order:

Remarks: 1. $\mathcal{T}_N^-$ will turn out to be not a canonical system of parameters because the construction of the conjugacy class of EST's (J, Λ) which is assigned to $t \in \mathcal{T}_N^-$ will depend on certain choices.

2. According to a conjecture of Langlands it should be possible to parametrize the irreducible admissible representations of the absolute Weil group $W_F \subseteq \mathrm{Gal}(\bar{F}/F)$ which are of dimension dividing N in terms of $\mathcal{T}_N^-$ too. But here it seems to be more advantageous to use $\mathcal{S}_N^-$, $\mathcal{P}_N^-$ resp. (see §6 of [Zi$_{89}$] for some more precise conjectures. The case $p \nmid N$ can be settled in an easy way as it is explained there after 6.6).

For all $e|N$ we *fix* a principal order $\mathfrak{A}_e$ in A and a map

$$f(T) \in F[T]^-_{e,N/e} \longmapsto \alpha_f \in \Delta_e^- \longmapsto (J_f, \pi_f) \tag{1}$$

as in Part I, Prop. 2.4, where (J_f, π_f) is an admissible pair (in the sense of [Zi$_{88}$], section 2) with respect to the filtration

$$\mathfrak{K} \supset \mathfrak{A}_e^* \supset U^1 \supset U^2 \supset \cdots,$$

with $\mathfrak{K}$ being the G normalizer of $\mathfrak{A}_e$. Δ_e^- denotes a fundamental domain for $\mathrm{Ad}(\mathfrak{K})\backslash A(\mathfrak{A}_e)$ $/\mathfrak{A}_e$, hence $\alpha_f \in \Delta_e^-$ is a uniquely determined root of $f(T)$ in A which is $\mathfrak{A}_e$-pure, and $\alpha_f + \mathfrak{A}_e$ is a simple stratum. Mostly we are considering a fixed order $\mathfrak{A} = \mathfrak{A}_e$ and therefore omit the subscript e (= period of $\mathfrak{A}$).

(J_f, π_f) is a pair associated to the simple stratum $\alpha_f + \mathfrak{A}$ which has the properties 2.1, 2.2 (if we take the sequence $\mathfrak{K} \supset \mathfrak{A}^* \supset U^1 \supset U^2 \supset \cdots$), 2.3 and 2.4. Note that π_f is just one of the representations π occurring in 2.4, i.e. π_f restricted to J_f^1 is irreducible and is a Heisenberg representation π_1 of J_f^1. Now we have $E = F(\alpha_f) \subset A$, and we fix a maximal extension $L|E$ with the property $L^* \subset \mathfrak{K}_E = \mathfrak{K} \cap A_E$. Then we write $\mathfrak{K}_{L|E}$, $\mathfrak{A}_{L|E}$ instead of $\mathfrak{K}_E$, $\mathfrak{A}_E$ resp. and thereby we think of the construction in the proof of 1.1 in Part I. Hence

$$J_f^0/J_f^1 \cong \mathfrak{A}_{L|E}^*/U^1 \cong [GL_{k_E}(k_L)]^{e_{L|E}} \tag{2}$$

is the product of $e_{L|E}$ exemplars of $GL_{k_E}(k_L)$, and

$$(3) \qquad J_f/J_f^1 \cong \mathfrak{K}_{L|E}/U^1 = L^*\mathfrak{A}^*_{L|E}/U^1 \cong \langle \pi_L \rangle \ltimes [GL_{k_E}(k_L)]^{e_{L|E}}.$$

According to 2.5 (which is [BK] (5.5.10)) a simple type is obtained ty tensoring $\mathrm{Res}(\pi_f) \in (J_f^0)^\wedge$ with the inflation (see (2)) of a representation $(\otimes\sigma_0)^{e_{L|E}}$, which is to denote the external tensor power of an irreducible cuspidal representation σ_0 of $GL_{k_E}(k_L)$. Hence an extended simple type (J_f, Λ) is obtained by tensoring π_f with a representation $R_{\mathfrak{K}}$ of (3) that extends some $(\otimes\sigma_0)^{e_{L|E}}$. Here our main result in this section is:

4.1 Main Lemma: *Let $L|F$ be a finite extension and let $\mathfrak{K}_{L|F} \subseteq GL_F(L)$ be the maximal compact-modulo-center subgroup associated to $\mathfrak{A}_{L/F}$. Let $\Gamma_F \subseteq Gal(\tilde{F}|F)$ be the tame Weil group. Then there is a well defined injection*

$$(4) \qquad \Gamma_F{}^\wedge(\dim = f_{L|F}) \longrightarrow (\mathfrak{K}_{L|F}/U^1)^\wedge, \quad R \longmapsto R_{\mathfrak{K}}$$

of irreducible admissible representations $R \in \Gamma_F{}^\wedge$ of dimension $f_{L|F}$, which has the following properties:

(i) *The image consists of all irreducible representations $R_{\mathfrak{K}}$, the restriction of which to $\mathfrak{A}^*_{L|F}/U^1 \cong [GL_{k_F}(k_L)]^{e_{L|F}}$ is an external tensor power of a cuspidal representation of $GL_{k_F}(k_L)$.*

(ii) *The map (4) lifts the bijection (due to J. Green)*

$$(5) \qquad G(k_L|k_F)\backslash\{k_F - \text{regular characters of } k_L^*\} \longleftrightarrow GL_{k_F}(k_L)^\wedge \text{ cuspidal}$$

between Galois orbits of regular characters and irreducible cuspidal representations of $GL_{k_F}(k_L)$. Namely, let K be the inertial field in $L|F$, and via

$$\phi: K^*/U_K^1 \to \mathbb{C}^* \longmapsto R = Ind_F(\phi)$$

(where K^*/U_K^1 is identified with Γ_K^{ab} and Ind_F denotes the induction from Γ_K to Γ_F) identify the range of (4) with the set of $G_{K/F}$-orbits of regular characters ϕ. Then if $R_{\mathfrak{K}}|_{\mathfrak{A}^*_{L|F}} = (\otimes\sigma_0)^{e_{L|F}}$, the Galois orbit of $\phi_0 = \phi|_{U_K}$ (which is a regular character of $k_K^* = k_L^*$) corresponds to σ_0 under (5).

(iii) *If $\chi: F^*/U_F^1 \to \mathbb{C}^*$ is a tame character, then*

$$(\chi \otimes R)_{\mathfrak{K}} = \tilde{\chi} \otimes R_{\mathfrak{K}},$$

where $\tilde{\chi} = \chi \circ \det$ is the corresponding character of $GL_F(L)$.

Before proving the Main Lemma, we will explain how it is used. Looking at (3), from the Main Lemma we conclude that $R_{\mathfrak{K}}$ should be related to an irreducible representation R of the tame Weil group $\Gamma_E \subset \mathrm{Gal}(\tilde{E}|E)$ which is of dimension $f_{L|E}$. On the other hand, we know that $f_{L|F} = N/e$, where e is the period of $\mathfrak{A}$ (see at the very end of Section 2). Hence:

$$(6) \qquad \dim R = N/ef_{E|F}, \qquad e = N/\dim R \cdot f_{E|F}.$$

Now we consider an R-polynomial $t = [R, \beta] \in \mathfrak{T}_N^-$. To avoid confusion let $f(T)$ be the polynomial and let f_t be the inertial degree which are associated to t (see 1.(3) and 1.2). Because $f_t = \dim(R)\ f_{F(\beta)/F}$ and $F(\beta) \cong F(\alpha_f) = E$ (see (1)), we see from $(6)_2$ that the extended simple type (J, Λ) corresponding to t should be related to the principal order $\mathfrak{A} = \mathfrak{A}_e$ where $e = N/f_t$. Namely, for that e we have $f(T) \in F[T]^-_{e,N/e}$ because $f_{F(\beta)/F}$ divides $f_t = N/e$ and $e_{F(\beta)/F} = \deg(t)/f_t$ divides $N/f_t = e$. Moreover, if we choose $L|F$ as in (2), then $e_{L|F} = e$ implies $f_{L|F} = N/e = f_t$ and $f_{L|E} = f_t/f_{E|F} = \dim R$. So we have:

$$(7)_1 \qquad t = [R, \beta] \in \mathfrak{T}_N^- \longmapsto f(T) \in F[T]^-_{e,N/e} \overset{(1)}{\longmapsto} (J_f, \pi_f)$$

where $e = N/f_t$, and we use the isomorphism $F(\beta) \simeq E = F(\alpha_f)$ which takes β to α_f to transfer $R \in \Gamma_{F(\beta)}^{\wedge}$ into a representation $R \in \Gamma_E{}^{\wedge}(\dim = f_{L|E})$. If we start from $(\sigma R \sigma^{-1}, \sigma(\beta)) \in t$ we get the same $R \in \Gamma_E{}^{\wedge}$ because then we have to use the isomorphism $F(\sigma(\beta)) \cong E = F(\alpha_f)$ which takes $\sigma(\beta)$ to α_f. Now we can apply the Main Lemma:

$$(7)_2 \qquad R \in \Gamma_E{}^{\wedge}(\dim = f_{L|E}) \longmapsto R_{\mathfrak{K}} \in (\mathfrak{K}_{L|E}/U^1)^{\wedge}$$

and by means of (3) we arrive at the following

4.2 Definition: The extended simple type (J, Λ) which is associated to $t = [R, \beta] \in \mathfrak{T}_N^-$ is contained in $\mathfrak{K}_e = \mathfrak{K}(\mathfrak{A}_e)$ where $e = N/f_t$ and is given by $J = J_f$, $\Lambda = \pi_f \otimes R_{\mathfrak{K}}$, where $f(T)$ and $R_{\mathfrak{K}}$ are as in $(7)_1$, $(7)_2$.

Before stating some properties of the map $t \mapsto (J, \Lambda)$ we return to the

Proof of the Main Lemma: We begin with *the case where $L|F$ is unramified.* Then $R \in \Gamma_F{}^{\wedge}(\dim = f_{L|F})$ has the form $R = \mathrm{Ind}_F(\phi)$ where $\phi: L^*/U_L^1 \to \mathbb{C}^*$ is an F-regular character, i.e. all conjugates are different. $\mathfrak{K}_{L|F} \subset GL_F(L)$ has the period $e = 1$, $\mathfrak{K}_{L|F} = L^* \cdot \mathfrak{A}^*_{L|F} = F^* \cdot \mathfrak{A}^*_{L|F}$,

$$(8) \qquad \mathfrak{K}_{L|F}/U^1 = F^* \cdot \mathfrak{A}^*_{L|F}/U^1 = \langle \pi_F \rangle \cdot GL_{k_F}(k_L)$$

and there exists a uniquely determined irreducible representation $R_{\mathfrak{K}}$ of (8) such that:

(4.1.1) (i) On $GL_{k_F}(k_L)$, $R_{\mathfrak{K}}$ is the irreducible cuspidal representation which is assigned to the Galois orbit of the regular character $\phi_0 \in (k_L^*)^{\wedge}$.

(ii) On the center $(F^*/U_F^1) \subseteq \mathfrak{K}_{L|F}/U^1$, $R_{\mathfrak{K}}$ gives rise to the character $\phi_F = \phi|_{F^*}$.

Remark: Note that the intersection $(F^*/U_F^1) \cap GL_{k_F}(k_L)$ of subgroups in $\mathfrak{K}_{L|F}/U^1$ consists of the scalar matrices $k_F^* \subseteq GL_{k_F}(k_L)$. Hence, according to the results of J. Green, the definition (4.1.1) is consistent (see [Rei], 2.7 for some further details).

From the definition the property 4.1 (ii) is obvious. We verify 4.1 (iii): Note that $\chi \otimes R = \mathrm{Ind}_F(\chi \cdot N_{L|F} \otimes \phi)$ and $(\chi \circ N_{L|F} \otimes \phi)_0 = (\chi_0 \circ N_{k_L|k_F}) \otimes \phi_0$, where $\chi_0 = \chi|_{k_F^*}$. As we know from J. Green, the cuspical representation of $GL_{k_F}(k_L)$ assigned to $(\chi_0 \circ N_{k_L|k_F}) \otimes \phi_0$ is $\tilde{\chi}_0 \otimes R_{\mathfrak{K}}$ where $\tilde{\chi}_0 = \chi_0 \circ \det$ is a character of $GL_{k_F}(k_L)$. Hence 4.1 (iii) is correct on $GL_{k_F}(k_L) \subseteq \mathfrak{K}_{L|F}/U^1$. On the other hand:

$$(\chi \circ N_{L/F}) \cdot \phi|_{F^*} = \chi^{[L:F]} \cdot \phi_F = (\tilde{\chi} \otimes R_{\mathfrak{K}})|_{F^*}$$

because $\tilde{\chi}|_{F^*} = \chi^N$. Therefore, 3.1 (iii) is correct on F^*/U^1 too.

Now we come to the *general case*. Then we have

$$\mathfrak{K}_{L/F}/U^1 = L^*/U^1 \,.\, [GL_{k_F}(k_L)]^e \qquad \text{(see (3))} \tag{9}$$

where $e = e_{L/F}$ and the intersection of the two factors is k_L^* diagonally embedded into $[GL_{k_F}(k_L)]^e$. Hence:

$$\mathfrak{K}_{L/F}/U^1 = \langle \pi_L \rangle \ltimes [GL_{k_F}(k_L)]^e \tag{9'}$$

is a semidirect product. We are going to compute the π_L-conjugation on $[GL_{k_F}(k_L)]^e$. Write $\mathfrak{A}, P$ instead of $\mathfrak{A}_{L/F}, P_{L/F}$ resp. Then we have a natural isomorphism

$$\mathfrak{A}/P \cong \mathrm{End}_{k_F}(\mathfrak{o}_L/\mathfrak{p}_L) \oplus \cdots \oplus \mathrm{End}_{k_F}(\mathfrak{p}_L^{e-1}/\mathfrak{p}_L^e).$$

Let $x = \varphi_0 + \cdots + \varphi_{e-1}$ be the corresponding decomposition of $x \in \mathfrak{A}/P$. φ_i can be given as: $\varphi_i = \{\pi_L^i \zeta \mapsto \pi_L^i \cdot x_i(\zeta)\}$ where $x_i \in \mathrm{End}_{k_F}(\mathfrak{o}_L/\mathfrak{p}_L)$, which gives rise to:

$$\mathfrak{A}/P \cong [\mathrm{End}_{k_F}(\mathfrak{o}_L/\mathfrak{p}_L)]^e, \quad x \longmapsto (x_0, \ldots, x_{e-1}). \tag{10}$$

We want to compute $\pi_L x \pi_L^{-1}$. Obviously:

$$\pi_L \varphi_i \pi_L^{-1} = \{\pi_L^{i+1} \cdot \zeta \longmapsto \pi_L^{i+1} \cdot x_i(\zeta)\} \quad \text{for} \quad 0 \le i < e-1.$$

Now we consider

$$\pi_L \varphi_{e-1} \pi_L^{-1} = \{\pi_L^e \zeta \longmapsto \pi_L^e \cdot x_{e-1}(\zeta)\}. \tag{10'}$$

We have to identify (10)′ with an element from $\mathrm{End}_{k_F}(\mathfrak{o}_L/\mathfrak{p}_L)$. This is done by using

$$\pi_L^e \equiv \pi_F \cdot \lambda \bmod U_L^1, \quad \text{where} \quad \pi_F \in F,\ \lambda \in k_L^*. \tag{11}$$

Then (10)′ turns into

$$\pi_L \varphi_{e-1} \pi_L^{-1} \hat{=} \{\lambda\zeta \longmapsto \lambda x_{e-1}(\zeta)\} \in \mathrm{End}_{k_F}(\mathfrak{o}_L/\mathfrak{p}_L),$$

and by substituting $u = \lambda\zeta \in \mathfrak{o}_L/\mathfrak{p}_L$ we obtain

$$\pi_L \varphi_{e-1} \pi_L^{-1} \hat{=} \{u \longmapsto \lambda x_{e-1}(\lambda^{-1} u)\} \in \mathrm{End}_{k_F}(\mathfrak{o}_L/\mathfrak{p}_L).$$

So we may state:

4.3 Lemma: *Consider* $x \mapsto (x_0, \dots, x_{e-1})$ *as in (10). Then*

$$\pi_L x \pi_L^{-1} \longmapsto (\lambda x_{e-1} \lambda^{-1}, x_0, \dots, x_{e-2})$$

if $\pi_L^e \equiv \pi_F \cdot \lambda \bmod U_L^1$, *where* $\lambda \in k_L^* \subset GL_{k_F}(k_L)$. □

Remark: π_L being fixed, the result in 4.3 does not depend on the choice of the prime element π_F. Another choice of π_F changes λ by a factor from k_F^* but x_{e-1} is a k_F-endomorphism.

Now we are going to define $\Gamma_F\hat{}(\dim = f_{L/F}) \to (\mathfrak{K}_{L/F}/U^1)\hat{}$ in the general case. Let K be the inertial field in L/F. According to (4.1.1) we get

(12) $$R \in \Gamma_F\hat{}(\dim = f_{L/F}) \longmapsto R_0 \in (\mathfrak{K}_{K/F}/U^1)\hat{}$$

To avoid any confusion we write here R_0 instead of $R_{\mathfrak{K}}$. We note $\mathfrak{K}_{K/F}/U^1 = F^*/U_F^1 \cdot GL_{k_F}(k_L)$. Let V be the representation space of R_0. From (12) we deduce a representation of $F^* \cdot \mathfrak{A}_{L/F}^*/U^1 = (F^*/U_F^1) \cdot [GL_{k_F}(k_L)]^e$ in the space $(\otimes V)^e$, namely:

(13) $$\begin{aligned} R_{\mathfrak{K}}(\alpha \cdot (x_0, \dots, x_{e-1})) :&= R_0(\alpha x_0) \otimes \cdots \otimes R_0(\alpha x_{e-1}) \\ &= \phi_F^e(\alpha) \cdot (R_0(x_0) \otimes \cdots \otimes R_0(x_{e-1})) \end{aligned}$$

where we have used (4.1.1) (ii). $F^* \cdot \mathfrak{A}_{L/F}^* \subset \mathfrak{K}_{L/F} = L^* \cdot \mathfrak{A}_{L/F}^*$ is a normal subgroup with a cyclic factor which is $L^*/F^*U_L \cong \mathbb{Z}/e\mathbb{Z}$ and is generated by π_L. We extend $R_{\mathfrak{K}}$ onto $\mathfrak{K}_{L/F}$ as follows:

(4.1.2) Definition: Let $R_{\mathfrak{K}}(\pi_L)$ be the operator on $(\otimes V)^e$ which is given by $R_{\mathfrak{K}}(\pi_L) := [R_0(\pi_F\lambda') \otimes \mathbf{1} \cdots \otimes \mathbf{1}] \cdot S$, where $\lambda' \in k_L^* \subset GL_{k_F}(k_L)$ is determined by:

$$\pi_L^e \equiv \pi_F \cdot \lambda' \cdot \delta \bmod U_L^1$$
$$\delta = \begin{cases} +1 & \text{if } e' \text{ is odd} \\ -1 & \text{if } e' \text{ is even} \end{cases}$$

(e' denotes the prime-to-p factor of e), where $\mathbf{1}$ is the unit operator on V and where S is the operator on $(\otimes V)^e$ which is given by $S(v_0 \otimes \cdots \otimes v_{e-1}) = v_{e-1} \otimes v_0 \cdots \otimes v_{e-2}$.

Remark: Note that for $p = 2$ we have always $\delta = 1$. The reason for modifying (11) is the property 4.5 below.

We are going to verify

4.4 Lemma: (i) $R_{\mathfrak{K}}(\pi_L) \cdot R_{\mathfrak{K}}(x) \cdot R_{\mathfrak{K}}(\pi_L)^{-1} = R_{\mathfrak{K}}(\pi_L x \pi_L^{-1})$ *for* $x = \alpha(x_0, \dots, x_{e-1}) \in F^*/U^1 \cdot [GL_{k_F}(k_L)]^e$.

(ii) $R_{\mathfrak{K}}(\pi_L)^e = R_{\mathfrak{K}}(\pi_F(\lambda, \dots, \lambda))$ *with the notations from 4.3.*

Proof: (i) Obviously we have $S \cdot R_{\mathfrak{K}}(x) \cdot S^{-1} = R_{\mathfrak{K}}(\alpha(x_{e-1}, x_0, \dots, x_{e-2}))$, hence:

$$\begin{aligned} R_{\mathfrak{K}}(\pi_L) \cdot R_{\mathfrak{K}}(x) \cdot R_{\mathfrak{K}}(\pi_L)^{-1} &= [R_0(\pi_F\lambda') \otimes \mathbf{1} \cdots \otimes \mathbf{1}] \cdot R_{\mathfrak{K}}(\alpha(x_{e-1}, x_0, \dots, x_{e-2})) \cdot \\ &\quad \cdot [R_0(\pi_F\lambda') \otimes \mathbf{1} \cdots \otimes \mathbf{1}]^{-1} \\ &= R_{\mathfrak{K}}(\alpha(\lambda' x_{e-1} \lambda'^{-1}, x_0, \dots, x_{e-2})). \end{aligned}$$

Because of $\lambda' = \pm\lambda$ we obtain $\lambda' x_{e-1} \lambda'^{-1} = \lambda x_{e-1} \lambda^{-1}$ such that (i) follows from 4.3.

(ii) Define $T := R_0(\pi_F \lambda') \otimes \mathbf{1} \cdots \otimes \mathbf{1}$. Hence:

$$R_{\mathfrak{R}}(\pi_L)^e = (TS)^e = T(STS^{-1}) \cdots (S^{e-1} T S^{e-1}) \cdot S^e$$

Now we use $S^e = \mathbf{1}$, $S^i T S^{-i} = \mathbf{1} \otimes \cdots R(\pi_F \lambda') \otimes \cdots \otimes \mathbf{1}$ where $R_0(\pi_F \lambda')$ is shifted from the first to the $(i+1)$ position. Hence

$$(TS)^e = R_{\mathfrak{R}}(\pi_F(\lambda', \dots, \lambda')).$$

If e' is odd we are done because $\lambda' = \lambda$. If e' is even we get $\lambda' = -\lambda$ and e is even too. Hence:

$$\begin{aligned} R_{\mathfrak{R}}(\pi_F(\lambda', \dots, \lambda')) = (\otimes R_0(\pi_F \lambda'))^e = \phi_F(-1)^e \cdot R_{\mathfrak{R}}(\pi_F(\lambda, \dots, \lambda)) = \\ = R_{\mathfrak{R}}(\pi_F(\lambda, \dots, \lambda)) \end{aligned}$$

where we have used (4.1.1) (ii). □

From the lemma we see that, by (4.1.1), (4.1.2), we have a well defined representation $R_{\mathfrak{R}}$ such that the map (4) has been established.

Our next aim is the proof of 4.1 (iii) in the general case. We need the following lemma:

4.5 Lemma: *Let K be the inertial field in L/F, $e = e_{L/F}$ and e' the prime-to-p component of e. Then:*

$$\pi_L^e \equiv \delta \cdot N_{L/K}(\Pi_L) \bmod U_L^1$$

where δ is defined as in (4.1.2).

Proof: We can assume $K = F$, i.e. L/F is fully unramified. Let $\hat{L}/F$ be the normal closure of L/F in $\bar{F}/F$. Then

$$\pi_L^e / N_{L/F}(\pi_L) = \prod_{\sigma} \pi_L^{1-\sigma}$$

where σ runs over a system of representatives of $G_{\hat{L}/F}/G_{\hat{L}/L}$. We put $G_{\hat{L}/F} = G$, $G_{\hat{L}/L} = H$. then $H_1 = G_1 \cap H$ where H_1, G_1 are the subgroups of wild ramification, hence $G_1/H_1 \hookrightarrow G/H$. And if $\sigma_1 \in G_1$ then $\pi_L^{1-\sigma_1} \in U_{\hat{L}}^1$. Hence if $L_1 \mid F$ is the maximal tame subextension of $L \mid F$, then:

$$\begin{aligned} \pi_L^{e/e'} &\equiv N_{L/L_1}(\pi_L) \bmod U_L^1 \\ \pi_L^e &\equiv N_{L/L_1}(\pi_L)^{e'} \bmod U_L^1. \end{aligned}$$

Now $\pi_1 = N_{L/L_1}(\pi_L)$ is a prime element of L_1, and we are left to show that:

$$\pi_1^{e'} \equiv \delta \cdot N_{L_1/F}(\pi_1) \bmod U_L^1.$$

Our computation is in $L_1^*/U_{L_1}^1 \subseteq L^*/U_L^1$. By an unramified base change we can shift L_1/F into a cyclic extension EL_1/E. Then we know that

$$\sigma \in G_{EL_1/E} \longmapsto \pi_1^{1-\sigma} \in k_E^* = U_{EL_1}/U_{EL_1}^1$$

is an injective homomorphism of groups. Therefore

$$\left\{\pi_1^{1-\sigma};\, \sigma \in G_{EL_1/E}\right\} = \left\{1, s, \ldots, s^{e'-1}\right\}$$

is a cyclic group of order e' in k_E^*. Hence

$$\pi_1^{e'}/N_{L_1/F}(\pi_1) = 1 \cdot s \cdot s^2 \cdots s^{e'-1} = s^{\frac{1}{2}e'(e'-1)} = \delta \in k_E^*. \quad \square$$

Now we come to the proof of 4.1 (iii). It is enough to show that

(i) $(\chi \otimes R)_{\mathfrak{K}} = \tilde{\chi} \otimes R_{\mathfrak{K}}$ on $F^*\mathfrak{A}_{L/F}^* \subset \mathfrak{K}_{L/F}$

(ii) $(\chi \otimes R)_{\mathfrak{K}} = \tilde{\chi} \otimes R_{\mathfrak{K}}$ if the argument is π_L.

As to (i), we use (13) and the property 4.1 (iii) of $R \mapsto R_0$:

$$\begin{aligned}(\chi \otimes R)_{\mathfrak{K}}(\alpha(x_0, \ldots, x_{e-1})) &= (\chi \otimes R)_0(\alpha x_0) \otimes \cdots \otimes (\chi \otimes R)_0(\alpha x_{e-1}) = \\ &= \tilde{\chi}(\alpha(x_0, \ldots, x_{e-1})) \cdot R_0(\alpha x_0) \otimes \cdots \otimes R_0(\alpha x_{e-1}) = \\ &= \tilde{\chi}(\alpha(x_0, \ldots, x_{e-1})) \cdot R_{\mathfrak{K}}(\alpha(x_0, \ldots, x_{e-1})).\end{aligned}$$

Note that $\tilde{\chi}(U^1) \equiv 1$ for $U^1 = U_{L/F}^1 \subset \mathfrak{K}_{L/F}$, because χ is a tame character. Therefore it is possible to consider $\tilde{\chi}$ as a character of $\mathfrak{K}_{L/F}/U^1$. As to (ii), we use (4.1.2) which gives us:

$$(14) \qquad (\chi \otimes R)_{\mathfrak{K}}(\pi_L) = [(\chi \otimes R)_0(\pi_F\lambda') \otimes \mathbf{1} \cdots \otimes \mathbf{1}] \cdot S.$$

Moreover $(\chi \otimes R)_0(\pi_F\lambda') = \chi(N_{K/F}(\pi_F\lambda')) \cdot R_0(\pi_F\lambda')$ where K is the inertial field. This is because $R \mapsto R_0$ has the property 4.1 (iii) and "det" restricted to $K^* \subseteq GL_F(K)$ agrees with $N_{K/F}$.

Now from $\pi_L^e \equiv \pi_F \cdot \lambda' \cdot \delta \bmod U_L^1$ and 4.5 we conclude:

$$\pi_F\lambda' \equiv N_{L/K}(\pi_L) \bmod U_L^1, \quad \text{hence}$$
$$N_{K/F}(\pi_F\lambda') \equiv N_{L/F}(\pi_L) \bmod U_L^1$$

On the other hand $\pi_L \in L^* \subseteq \mathfrak{K}_{L/F} \subseteq GL_F(L)$ yields $N_{L/F}(\pi_L) = \det(\pi_L)$ such that:

$$(\chi \otimes R_0)(\pi_F\lambda') = \tilde{\chi}(\pi_L) \cdot R_0(\pi_F\lambda').$$

Substituting this into (14) we obtain (ii).

To complete the proof of 4.1 we are left to show that (4) is an injection and to determine its image.

Consider $R_1 = \mathrm{Ind}_{K\uparrow F}(\phi_1)$; $R_2 = \mathrm{Ind}_{K\uparrow F}(\phi_2) \in \Gamma_F\hat{}(\dim = f_{L/F})$ and assume $R_{1\mathfrak{K}} = R_{2\mathfrak{K}}$.

Then the property 4.1 (ii) implies that $\phi_1|_{U_K}$ and $\phi_2|_{U_K}$ are conjugated with respect to $G_{K/F}$. Therefore by conjugating ϕ_2 we can assume $\phi_1|_{U_K} = \phi_2|_{U_K}$, and using that $N_{K/F}: K^*/U_K \to F^*/U_F$ is injective, hence $N^*_{K/F}: (F^*/U_F)\hat{} \to (K^*/U_K)\hat{}$ is surjective, we see:

$$R_2 = \chi \otimes R_1, \quad \text{where } \chi: F^*/U_F \to \mathbb{C}^* \text{ is unramified.} \tag{15}$$

Now we apply 4.1 (iii) and obtain

$$R_{1\mathfrak{K}} = R_{2\mathfrak{K}} = \tilde{\chi} \otimes R_{1\mathfrak{K}}. \tag{16}$$

Let $W = (\otimes V)^e$ be a representation space of $R_{1\mathfrak{K}}$ as in (4.1.2). The equivalence of representations means for the operators:

$$\tilde{\chi} \otimes R_{1\mathfrak{K}} = A \cdot R_{1\mathfrak{K}} \cdot A^{-1} \quad \text{where } A \in \mathrm{Aut}_{\mathbf{C}}(W).$$

We restrict to $\mathfrak{A}^*_{L/F}/U^1 \subset \mathfrak{K}_{L/F}/U^1$. Because $\tilde{\chi}$ vanishes on $\mathfrak{A}^*_{L/F}$ we obtain:

$$R_{1\mathfrak{K}} = A \cdot R_{1\mathfrak{K}} A^{-1} \quad \text{on} \quad \mathfrak{A}^*_{L/F}/U^1.$$

But on $\mathfrak{A}^*_{L/F}/U^1$ we know that $R_{1\mathfrak{K}} = (\otimes\sigma_0)^e$ is still irreducible. Hence A has to be a scalar operator, and we see that (16) is even an equation of operators. Now we check the argument π_L. Then we obtain $1 = \tilde{\chi}(\pi_L) = \chi(N_{L/F}(\pi_L)) = \chi(\pi_L^{f_{L/F}})$, hence $\chi^{f_{L/F}} \equiv 1$ because χ is unramified. But this implies $\chi \otimes R_1 = R_1$ because $f_{L/F} = [K : F]$; and from (15) we conclude $R_2 = R_1$.

Finally we come to the proof of 4.1 (i). Because of 4.1 (ii) we see that $R \mapsto R_{\mathfrak{K}}|_{\mathfrak{A}^*_{L/F}}$ covers all external tensor powers $(\otimes\sigma_0)^e$ of cuspidal representations $\sigma_0 \in GL_{k_F}(k_L)\hat{}$. Fix one $(\otimes\sigma_0)^e$ and consider

$$\Sigma := \left\{\sigma \in (\mathfrak{K}_{L/f}/U^1)\hat{};\ \sigma|_{\mathfrak{A}^*_{L/F}} = (\otimes\sigma_0)^e\right\}$$

We use $\mathfrak{K}_{L/F}/\mathfrak{A}^*_{L/F} \tilde{\to} L^*/U_L$, the surjectivity of $N^*_{L/F}: (F^*/U_F)\hat{} \to (L^*/U_L)\hat{}$ and the property $\det|_{L^*} = N_{L/F}$ with respect to $L^* \subset GL_F(L)$. Then we see that starting from any $\sigma \in \Sigma$ the whole set Σ is covered by the representations $\tilde{\chi} \otimes \sigma$, where χ runs over the unramified characters of F^*. But we have at least one $R_{\mathfrak{K}} \in \Sigma$. Therefore 4.1 (i) is a consequence of 4.1 (iii). This completes the proof of our Main Lemma. □

Now we proceed to prove some properties of 4.2.

4.6 Theorem: *The map*

$$t = [R, \beta] \in \mathfrak{T}_N^- \longmapsto (J_f, \pi_f \otimes R_{\mathfrak{K}}) \in \{EST\}_N$$

into the set of extended simple types in G which has been defined by 4.2 is injective and meets each conjugacy class of extended simple types at most once. Moreover it has the properties:

(i) *For a tame character $\chi: F^*/U^1 \to \mathbb{C}^*$ let be $\chi \otimes t := [(\chi \circ N_{F(\beta)/F}) \otimes R, \beta]$. Then $\chi \otimes t$ is mapped onto $(J_f, \tilde{\chi}(\pi_f \otimes R_{\mathfrak{K}}))$, where $\tilde{\chi} = \chi \cdot \det$ is the associated character of G.*

(ii) *$\Pi_t := Ind_{\mathfrak{K}}(\pi_f \otimes R_{\mathfrak{K}}) \in \mathfrak{K}^\wedge$ (where $\mathfrak{K} = \mathfrak{K}_e$, $e = N/f_t$) has the index $j(\Pi_t) := \max\{i \in \mathbb{Z}; \Pi_{t|U^i} \not\equiv 1\} = -e \cdot \nu_F(\beta)$ if $\beta \neq 0$, and Π_t vanishes on U^1 iff $\beta = 0$.*

Proof: (ii) From the constructions of Part I, 2.4 ff we know that π_f vanishes on $J_f^1 = J_f \cap U^1$ iff $f(T) = T$, i.e. $\beta = 0$. Moreover this is equivalent to $\Pi_{t|U^1} \equiv 1$ because $\Pi_{t|U^1}$ contains $\operatorname{Ind}_{J_f^1 \uparrow U^1}(\operatorname{Res} \pi_f)$ as an irreducible component. If $\pi_f \not\equiv 1$ on J_f^1, i.e. $\beta \neq 0$, then we get

$$j(\Pi_t) = j(\operatorname{Ind}_{\mathfrak{K}}(\pi_f)) = -e \cdot \nu_F(\beta)$$

where the last equation follows again from Part I, loc. cit.

(i) $\chi \otimes t$ has the same polynomial $f(T) \in F[T]^-$ as t. Hence J_f, π_f are unchanged. Consider

$$(R, \beta) \longmapsto R \in \Gamma_E^\wedge(\dim = f_{L/E}) \longmapsto R_{\mathfrak{K}} \in (\mathfrak{K}_{L/E}/U^1)^\wedge.$$

Then $(\chi \circ N_{F(\beta)/F}) \otimes R$ is mapped to $(\chi \circ N_{E/F}) \otimes R \in \Gamma_E^\wedge$, and by 4.1 (iii) we have:

$$((\chi \circ N_{E/F}) \otimes R)_{\mathfrak{K}} = (\chi \circ N_{E/F} \circ \det_{A_E/E}) \otimes R_{\mathfrak{K}} = \chi \circ \det_{A/F} \otimes R_{\mathfrak{K}}$$

because $R_{\mathfrak{K}} \in \widehat{\mathfrak{K}_{L/E}}$ and $\mathfrak{K}_{L/E} \subset GL_E$. Furthermore if "inf" denotes the inflation from $\mathfrak{K}_{L/E}$ to $J_f = \mathfrak{K}_{K/E} \cdot J_f^1$, then:

$$\inf(\chi \circ N_{E/F} \circ \det_{A_E/E} \otimes R_{\mathfrak{K}}) = (\chi \circ \det_{A/F}) \otimes \inf(R_{\mathfrak{K}})$$

because χ is a tame character.

Now we consider $t_1 \neq t_2 \in \mathfrak{T}_N^-$ and we will show that the corresponding simple types are not G-conjugated. If the inertial degrees f_{t_1}, f_{t_2} are different then $e_1 \neq e_2$ for $e_i = N/f_{t_i}$. Hence the groups J_1, J_2 which correspond to t_1, t_2 resp. normalize orders of different periods namely $\mathfrak{A}_{e_1}$, $\mathfrak{A}_{e_2}$ resp., and from the remark following 2.8 we see that they cannot be conjugated.

Finally we assume $f_{t_1} = f_{t_2}$ hence $e_1 = e_2 = e$, and we will show that $t_1 = t_2$ if the corresponding extended simple types are conjugated. Namely then we have J_1, $J_2 \subseteq \mathfrak{K}_e$, and $xJ_1x^{-1} = J_2$ for some $x \in G$. Therefore J_2 is in $\mathfrak{K}_e$ and in $x\mathfrak{K}_ex^{-1}$ too. Because of the remark following 2.8 we conclude $x\mathfrak{K}_ex^{-1} = \mathfrak{K}_e$, hence $x\mathfrak{A}_ex^{-1} = \mathfrak{A}_e$ and $x \in \mathfrak{K}_e$. Thus we obtain:

$$x(\pi_1 \otimes R_{1\mathfrak{K}})x^{-1} = \pi_2 \otimes R_{2\mathfrak{K}} \quad \text{with} \quad x \in \mathfrak{K}_e \tag{17}$$

Restricting to $J_2^1 = J_2 \cap U^1$, we see that $x\pi_1x^{-1} = \pi_2$ on J_2^1 because $R_{i\mathfrak{K}}$ vanishes on J_i^1. Hence for $\Pi_i := \operatorname{Ind}_{J_i \uparrow \mathfrak{K}}(\pi_i)$ we will have

$$\operatorname{Hom}_{U^1}(\Pi_1, \Pi_2) \neq 0. \tag{18}$$

In Part I, 2.(12) we have established a distance relation for the map

$$f(T) \in F[T]^-_{e,N/e} \longmapsto \alpha_f \in \Delta_e^- \longmapsto (J_f, \pi_f) \longmapsto \Pi_f = \mathrm{Ind}_{\mathfrak{K}_e}(\pi_f).$$

According to that relation (18) is equivalent to $f_1(T) = f_2(T)$. But then we have $(J_1, \pi_1) = (J_2, \pi_2)$ too and we denote this pair (J, π). Now (17) turns into

$$x(\pi \otimes R_{1\mathfrak{K}})x^{-1} = \pi \otimes R_{2\mathfrak{K}} \quad \text{with} \quad x \in \mathfrak{K}_e, \tag{19}$$

and restricting to J^1 we see that $x \in \mathfrak{K}$ normalizes the pair $(J^1, \mathrm{Res}(\pi))$. But then $x \in J$ as we see from 2.(5). Hence $x\pi x^{-1} = \pi$, $R_{2\mathfrak{K}} = xR_{1\mathfrak{K}}x^{-1} = R_{1\mathfrak{K}}$ and from the injectivity of (4) (Main Lemma) we conclude that $t_1 = [R_1, \beta_1]$ and $t_2 = [R_2, \beta_2]$ coincide. □

Combining Theorems 2.9 and 4.6, we obtain

Theorem 4.7: *The map*

$$t = [R, \beta] \in \mathfrak{T}_N^- \longmapsto (J, \Lambda)_t = (J_f, \pi_f \otimes R_{\mathfrak{K}}) \in \{EST\}_N \longmapsto \Pi_t^{GL} \in \widehat{G}_2$$

is an injection of $\mathfrak{T}_N^-$ *into the set of discrete series representations of* G. *It has the following properties:*

(i) $\Pi^{GL}_{\chi\otimes t} = \tilde{\chi} \otimes \Pi_t^{GL}$ *for tame characters* $\chi: F^*/U_F^1 \to \mathbb{C}^*$. *Especially we have* $f_t = f_\Pi$ *for* $\Pi = \Pi_t^{GL}$.

(ii) Π_t^{GL} *is cuspidal iff* $\deg(t) = N$.

(iii) *The choices can be made such that* Π_t^{GL} *is the Steinberg representation if* $t = [\mathbf{1}, 0] \in \mathfrak{T}_N^-$.

(iv) Π_t^{GL} *is a character twist of the Steinberg representation iff* $\deg(t) = 1$.

(v) *For* $t = [R, \beta]$ *the level of* Π_t^{GL} *is* $\ell(\Pi_t^{GL}) = \begin{cases} 0 & \text{if } \beta = 0 \\ -\nu_F(\beta) & \text{otherwise} \end{cases}$

(vi) *The exponential Artin conductor is*

$$a(\Pi_t^{GL}) = \begin{cases} N - 1 & \text{if } t = [\chi, 0] \text{ where } \chi \text{ is an unramified character} \\ N & \text{if } t = [R, 0] \text{ and } R \neq \chi \\ N(1 - \nu_F(\beta)) & \text{if } t = [R, \beta] \text{ and } \beta \neq 0. \end{cases}$$

Remark: In 6.9 below we will see that the map of 4.7 is in fact bijective.

Proof: (v), (vi): As we have seen in 2.15, the level is

$$\ell(\Pi_t^{GL}) = j(\Lambda_t)/e = -\nu_P(\alpha_f)/e,$$

because Λ_t is associated to the simple stratum $\alpha_f + \mathfrak{A}$. But $\alpha_f \in A$ and $\beta \in \bar{F}$ are roots of the same irreducible polynomial $f(T)$. Therefore $\nu_P(\alpha_f)/e = \nu_F(\beta)$. And using 3.1 we see that (vi) is a consequence of (v) and of (iii).

(i) The twist property follows because $\Lambda_{\chi\otimes t} = \tilde{\chi} \otimes \Lambda_t \subset \tilde{\chi} \otimes \Pi_t^{GL}$. From the Remark 2.1.1 in Part I we conclude $f_t = f_\Pi$.

(ii) Let $(\mathfrak{K}_e, J, \Lambda)$ be the extended simple type which is associated to t. The corresponding simple type is $(J^0, \lambda) = (J \cap \mathfrak{A}_e^*, \mathrm{Res}(\Lambda))$. From [BK] (5.5.17) and the remark after that, we obtain the G-intertwining of (J^0, λ) to be: $J^0 GL_K J^0$ $(\supseteq J)$ where $E = F(\alpha_f) \cong F(\beta)$ is the subfield in A associated to $t = [R, \beta]$ and K/E is maximally unramified with the property $K^* \subset \mathfrak{K}_e$. But this means in the notation of $(7)_2$:

$$[K{:}E] = f_{L|E},\ [K{:}F] = \dim(R) \cdot [E{:}F] = \deg(t).$$

By 2.5 Π_t^{GL} is cuspidal iff $\Pi_t^{GL} = \mathrm{ind}_G(\Lambda)$ is induced from the corresponding EST. And this happens iff the intertwining of the simple type is $J^0 GL_K J^0 = J$, i.e. $[K{:}F] = N$ (see [BK] section (6.2).)

(iii) From [Bo] we know that the extended simple type of the Steinberg representation St is $(J, \Lambda) = (\mathfrak{K}_N, 1)$ (see 2.12 and 2.9 (ii)). For the corresponding parameter $t = [R, \beta]$ this implies $\beta = 0$ (see Part I, proof of 2.2 and what is said there on the construction of the systems of characters X_D, Y_D and X_e, Y_e resp.) and $f_t = \dim R = 1$. $\Lambda_t \in \mathfrak{K}_N\hat{}$ is given as $\Lambda_t = \pi_f \otimes R_{\mathfrak{K}}$. And π_f has to be an extension of the 1-representation of U^1 onto $\mathfrak{K}_N$ which has the intertwining G. Hence

$$\begin{aligned} &\pi_f = \chi \circ \det, \quad \text{where } \chi \cdot F^*/U_F^1 \to \mathbb{C}^* \text{ is tamely ramified,} \\ &R = \chi^{-1}. \end{aligned}$$

Establishing a map $t \in \mathcal{T}_N^- \mapsto (J, \Lambda)_t \in \{EST\}_N$, we have to fix one extension π_f and we do this by taking $\chi \equiv 1$.

(iv) Let Π_t^{GL} be a character twist of St. Then we have $\mathfrak{K}_e = \mathfrak{K}_N = J$ because a character twist does not change the pair $\mathfrak{K}_e \supset J$. From $e = N$ we conclude $f_t = \dim R \cdot f_{F(\beta)/F} = 1$. Hence $t = [\chi, \beta]$ where $F(\beta)/F$ is fully ramified and $\chi \in \Gamma_{\widehat{F(\beta)}}$ (dim $= 1$). We consider the minimal polynomial $f(T)$ of β. The definition of J_f in Part I, 2.4 ff implies $J_f = \mathfrak{K}$ iff $\deg f(T) = 1$. Hence $\beta \in F$, $\deg(t) = 1$.

Conversely, assume $\deg(t) = 1$. With the notations of Part I, 2.4 ff this implies $J_f = \mathfrak{K}_N$ and $J_f^1 = H_f^1 = U^1$. Now we look at the character $\theta_f{:}\, U^1 \to \mathbb{C}^*$ as it is given in Part I, 2.4.2. The polynomial $f = f(T)$ associated to t has degree 1, hence $\alpha_f = \alpha_0 \in F$, and the relation of Part 1, 2.4.2 for $v = 0$ gives us:

$$\theta_f(1 + x) = \lambda_f \circ \det_{A/F}(1 + x) \quad \text{for } 1 + x \in U^1,$$

because $A_0 = A \supset U^1$ and $\lambda_0 = \lambda_f$. And because π_f has to be an extension of θ_f onto $\mathfrak{K}_N$ which has the intertwining G, we find $\pi_f = \hat{\lambda}_f \circ \det$, where $\hat{\lambda}_f{:}\, F^* \to \mathbb{C}^*$ is an extension of λ_f. Therefore:

$$(J, \Lambda)_t = (\mathfrak{K}_N, (\hat{\lambda}_f \circ \det) \otimes R_{\mathfrak{K}}),$$

where $R = \chi$ is a tame character of F^*. Using 4.1 (iii) we see $R_{\mathfrak{K}} = \tilde{\chi} \otimes \mathbf{1}_{\mathfrak{K}} = \tilde{\chi}$, because the unit representation $\mathbf{1} \in \Gamma_F\hat{}$ gives rise to $\mathbf{1}_{\mathfrak{K}} = \mathbf{1} \in (\mathfrak{K}_N/U^1)\hat{}$. Now $\Lambda_t = (\hat{\lambda}_f \chi)^\sim$ implies $\Pi_t^{GL} = (\hat{\lambda}_f \chi)^\sim \otimes St$. □

5. Computation of formal degrees of the discrete series representations in terms of their parameters

Consider the injective map

$$(1) \qquad t \in \mathfrak{T}_N^- \longmapsto \Pi_t^{GL} \in \widehat{G}_2$$

which has been obtained by combining Theorems 2.9 and 4.6. While the map of 2.9 has been completely canonical, 4.6 has depended on several choices, the first one of which has been to fix an approximation procedure on $F[T]$, just as to say what $\mathfrak{T}_N^-$ is (see 1.(12)). The other choices we have logged in Part I after 2.4. In section 1 we have reminded that an approximation procedure on $F[T]$ is given by fixing an appropriate bijection 1.(7)$_*$. Such a bijection we had constructed in [Zi89] by pursuing the methods of H. Koch. We stress that the approximation procedure on $F[T]$ refers to an unconventional exponential distance w_F on $F[T]$ which has been defined in [Zi89] too. The key role of w_F has become visible in Part I, 1.8.

5.1 Theorem: *Assume the Haar measure on $G/\langle \pi_F \rangle$ to be normalized in such a way that the Steinberg representation $St = \Pi_{t_0}^{GL}$ corresponding to $t_0 = [\mathbf{1}, 0] \in \mathfrak{T}_N^-$ has formal degree 1. Then the discrete series representation Π_t^{GL} has the formal degree:*

$$\deg \Pi_t^{GL} = f_t \cdot \frac{q^N - 1}{q^{N/e_t} - 1} \cdot q^{(1/2)\cdot m_{f(T)}}, \quad \textit{where:}$$

- *e_t, f_t are defined in 1.2*
- *$q = |k_F|$*
- *$f(T) \in F[T]^-$ is the polynomial associated to $t = [R, \beta] \in \mathfrak{T}_N^-$ (see 1.3)*
- *$m_{f(T)} = N \left[\sum_{v \geq 0, v \in 1/n\mathbf{Z}} (1 - 1/\deg f^{-v}(T)) - (1 - 1/e_t) \right]$ with $f^{-v}(T) :=$ approximation polynomial of $f(T)$.*

The formula for $m_{f(T)}$ is in fact independent from the approxiamtion procedure because:

$$\deg f^{-v}(T) = gcd\{deg\, g(T);\ w_F(g(T), f(T)) \geq -v\}.$$

Remarks: 1. For the invariant description of deg $f^{-v}(T)$ see Remark 5 following 1.9 of Part I.

2. In the next section we see that Divisionalgebra representations have the same dimension formula.

3. $m_{f(T)}$ is a number depending not only on $f(T)$ but on N, too. It is easy to see that $m_{f(T)} = -N(1 - 1/e_t) + N^2 \cdot d_{f(T)}$, where

$$d_{f(T)} = \frac{1}{e_t} \sum_{v \geq 0, v \in \frac{1}{e_t} Z} (1 - 1/\deg f^{-v}(T))$$

only depends on $f(T)$. Then we obtain

$$\deg \Pi_t^{GL} = f_t \cdot \frac{q^{N/2} - q^{-N/2}}{q^{N/2e_t} - q^{-N/2e_t}} \cdot q^{\frac{1}{2}N^2 d_{f(T)}}$$

Before we can prove 5.1 we need two propositions.

5.2 Proposition: *Let (J, Λ) be an extended simple type for G. Then it is possible to find a field extension $L|F$ of degree N in A and an intermediate field $L \supseteq E \supseteq F$, such that*

(i) $L^* \subset J \subset \mathfrak{K}_e$, *where* $e = e_{L/F}$

(ii) $J = \mathfrak{K}_{L/E} J^1$, *where* $J^1 = J \cap U^1(\mathfrak{K}_e)$.

The maximal compact modulo center subgroup $\mathfrak{K}_e$ and the inertial degrees and ramification exponents in the tower $L \supseteq E \supseteq F$ are uniquely determined.

Proof: The existence of E, L follows from the results of [BK], §5 on simple types which we have recorded in section 2. The uniqueness of $\mathfrak{K}_e$, hence of $e = e_{L/F}$, has been remarked after 2.8. The uniqueness of $f_{L/E}$, $e_{L/E}$ follows from

$$J/J^1 \cong \langle \pi_L \rangle \ltimes [GL_{k_E}(k_L)]^{e_{L/E}} \quad \text{(see 4.(3))},$$

and finally

$$\begin{aligned} f_{E/F} &= f_{L/F}/f_{L/E} = N/e \cdot f_{L/E} \\ e_{E/F} &= e/e_{L/E}. \qquad \square \end{aligned}$$

5.3 Proposition: *Let (J, Λ) be an extended simple type for G and let $L \supseteq E \supseteq F$, $e = e_{L/F}$ be as in 5.2. Consider the discrete series representation Π which is assigned to (J, Λ) via 2.9 (ii). The formal degree of Π is:*

$$\text{(2)} \qquad \deg(\Pi, G/\langle \pi_F \rangle) \cdot vol(\mathfrak{A}^*_{L/F}) =$$
$$= \dim \{Ind_{\mathfrak{K}_e}(\Lambda)\} \cdot \frac{1}{e} \cdot \prod_{k=1}^{e_{L/E}-1} (q_L^k - 1) \cdot vol(\mathfrak{A}^*_{L/K})/vol(GL_{\mathfrak{o}_K}(\mathfrak{o}_L)),$$

where $K|E$ is maximal unramified in $L|E$ and $q_L = |k_L|$.

Remark: As we know from [BK], §6, Π is supercuspidal iff $L|E$ is unramified, i.e. $L = K$, $e_{L/E} = 1$. In that case the formula reduces to:

$$\deg(\Pi, G/\langle \pi_F \rangle) \cdot \text{vol}(\mathfrak{A}^*_{L/F}) = \frac{1}{e} \cdot \dim \{\text{Ind}_{\mathfrak{K}_e}(\Lambda)\}.$$

Proof: We start out from [BK] (7.7.9): Let $(J^0, \lambda) = (J \cap \mathfrak{A}^*_e, \text{Res}\,\Lambda)$ be the corresponding simple type. Then:

$$\text{(3)} \qquad \deg(\Pi, G/F^*) \cdot \text{vol}(J^0 F^*/F^*) =$$
$$= (1/e_{K/F}) \cdot \dim(\lambda) \cdot \deg(\sigma, GL_K/K^*) \cdot \text{vol}(\mathfrak{A}^*_{L/K} K^*/K^*),$$

where σ is the discrete series representation of GL_K containing the simple type $(\mathfrak{A}^*_{L/K}, \mathbf{1})$ that under a Hecke algebra isomorphism 2.11 (i) corresponds to Π. Hence σ is (up to twist by an unramified character of K^*) the Steinberg representation.

Therefore from [CMS] (2.2.2) we deduce that

$$\deg(\sigma, GL_K/K^*) \cdot \mathrm{vol}(K^* \cdot GL_{\mathfrak{o}_K}(\mathfrak{o}_L)/K^*) = \frac{1}{n} \cdot \prod_{k=1}^{n-1}(q_K^k - 1) \tag{4}$$

where $n = [L : K]$ and $q_K = |k_K|$. But K is the inertial field in $L|E$. Hence $n = e_{L|E}$, $n \cdot e_{K/F} = e$ and $q_K = q_L$. Therefore substituting (4) into (3), we obtain

(5)
$$\deg(\Pi, G/F^*) \cdot \mathrm{vol}(J^0F^*/F^*) =$$
$$= (1/e) \cdot \dim(\lambda) \cdot \prod_{k=1}^{e_{L/E}-1} (q_L^k - 1) \cdot \mathrm{vol}(\mathfrak{A}^*_{L/K} \cdot K^*/K^*)/\mathrm{vol}(GL_{\mathfrak{o}_K}(\mathfrak{o}_L)K^*/K^*).$$

The last factor on the right hand side can be replaced by $\mathrm{vol}(\mathfrak{A}^*_{L/K})/\mathrm{vol}(GL_{\mathfrak{o}_K}(\mathfrak{o}_L))$. Moreover:

$$\dim\left(\mathrm{Ind}_{\mathfrak{A}^*_e}\lambda\right) = (\mathfrak{A}^*_e \colon J^0) \cdot \dim(\lambda).$$

Therefore, multiplying (5) with $(\mathfrak{A}^*_e \cdot F^* \colon J^0F^*) = (\mathfrak{A}^*_e \colon J^0)$, we obtain

$$\deg(\Pi, G/F^*) \cdot \mathrm{vol}(\mathfrak{A}^*_e \cdot F^*/F^*) = \frac{1}{e} \dim(\mathrm{Ind}_{\mathfrak{A}^*_e}\lambda) \cdot \prod_{k=1}^{e_{L/E}-1} (q_L^k - 1) \cdot \frac{\mathrm{vol}(\mathfrak{A}^*_{L/K})}{\mathrm{vol}(GL_{\mathfrak{o}_K}(\mathfrak{o}_L))}. \tag{6}$$

Finally, we note that:

$$\deg(\Pi, G/F^*) \cdot \mathrm{vol}(\mathfrak{A}^*_e \cdot F^*/F^*) = \deg\left(\Pi, G/\langle\pi_F\rangle\right) \cdot \mathrm{vol}\left(\mathfrak{A}^*_e \cdot F^*/\langle\pi_F\rangle\right)$$

and $\dim(\mathrm{Ind}_{\mathfrak{A}^*_e}(\lambda)) = \dim(\mathrm{Ind}_{\mathfrak{K}_e}(\Lambda))$. □

Proof of 5.1: We consider $t = [R, \beta] \in \mathfrak{T}_N^- \mapsto (J, \Lambda)_t \in \{EST\}_N$ and we will apply (2) to $(J, \Lambda)_t$. Then we have $\Lambda_t = \pi_f \otimes R_{\mathfrak{K}}$ and we are going to compute $\dim(\mathrm{Ind}_{\mathfrak{K}_e}(\Lambda_t))$ in three steps.

Step 1: The restriction of π_f to J^1 is a Heisenberg representation π_1, and we compute:

$$d_1 := \dim(\mathrm{Ind}_{U^1}(\pi_1)) = \dim(\mathrm{Ind}_{\mathfrak{K}_{L/E}U^1}(\pi_f))$$

because 5.2 (ii) implies $J \cdot U^1 = \mathfrak{K}_{L/E}U^1$.

Step 2: We compute

$$d_2 := \dim(\mathrm{Ind}_{J\uparrow\mathfrak{K}_{L/E}U^1}(R_{\mathfrak{K}} \otimes \pi_f)) = d_1 \cdot \dim(R_{\mathfrak{K}}).$$

The equation follows because $R_{\mathfrak{K}}$ lifts to a representation of $\mathfrak{K}_{L/E}U^1/U^1$.

Step 3: Finally, we get:

$$d_3 := \dim(\mathrm{Ind}_{\mathfrak{K}_e}(\Lambda)) = d_2 \cdot (\mathfrak{K}_e : \mathfrak{K}_{L/E} U^1).$$

To compute d_1 we use 2.2. It implies that the Heisenberg representation π_1 has

$$\dim(\pi_1) = \sqrt{(1+\mathfrak{d}_f : 1+\mathfrak{d}_f^{\perp})} = \sqrt{\mathfrak{d}_f : \mathfrak{d}_f^{\perp}},$$

hence

$$d_1 = (1+P : 1+\mathfrak{d}_f)\sqrt{\mathfrak{d}_f : \mathfrak{d}_f^{\perp}} = \sqrt{P : P^{\perp}}, \qquad \text{where}$$

$P^{\perp} = \{x \in P, X_f(x,P) \equiv 1\}$ is the radical of the alternating character $X_f(x,y) = \psi_F \circ Tr((xy-yx)\alpha_f)$ on P. Explicitly it is given in the same way as in the division algebra case (see [Zi90], 2.3, the formula for $P_{\beta}^{\perp}$), namely:

$$P^{\perp} = \sum_{v>0,\, v\in\frac{1}{e}\mathbf{Z}} (P^{ve} \cap A_{-v}), \tag{7}$$

where A_{-v} is the centralizer of α_{-v} (= zero of the approximation polynomial $f^{-v}(T)$ in Δ_e). Note that $\alpha_0 = \alpha_f$ and that $\alpha_{-v} \in \alpha_f + P^{-ve}$ is a simple representative for all $v \geq 0$. Hence:

$$d_1^2 = (P : P^{\perp}) = \prod_{v>0,\, v\in\frac{1}{e}\mathbf{Z}} \frac{(P^{ve} : P^{ve+1})}{(P^{ve} \cap A_{-v} : P^{ve-1} \cap A_{-v})}. \tag{8}$$

$\mathfrak{A}_e$ being a principal order, we have:

$$\mathfrak{A}_e/P \cong \bigoplus_{i=1}^{e-1} M_{N/e}(k_f)$$

$$(P^i : P^{i+1}) = (\mathfrak{A}_e : P) = q^{N^2/e} \quad \text{for all integers } i.$$

To obtain $(P^{ve} \cap A_{-v} : P^{ve+1} \cap A_{-v})$ we switch from the principal order $\mathfrak{A}_e = \mathfrak{A}_{L/F}$ to the principal order $\mathfrak{A}_{L/F(\alpha_{-v})}$ of A_{-v}. Then we have to replace:

$$N \ \text{ by } \ N/[F(\alpha_{-v}) : F], \ q \ \text{ by } \ q^{f_{F(\alpha_{-v})/F}},$$
$$e = e_{L/F} \ \text{ by } \ e_{L/F(\alpha_{-v})}.$$

This gives us:

$$(P^i \cap A_{-v} : P^{i+1} \cap A_{-v}) = q^{n_v}, \quad \text{where:}$$
$$n_v = f_{F(\alpha_{-v})/F} \cdot (N/[F(\alpha_{-v}) : F])^2 \cdot e_{L/F(\alpha_{-v})}^{-1} = N^2/[F(\alpha_{-v}) : F] \cdot e.$$

Therefore:

$$\frac{(P^{ve}:P^{ve+1})}{(P^{ve}\cap A_{-v}:P^{ve+1}\cap A_{-v})}=q^{m_v},\qquad \text{where}$$

$m_v=\frac{N^2}{e}\left(1-\frac{1}{[F(\alpha_{-v}):F]}\right)$. Substituting this in (8), we get:

$$d_1=q^{(1/2)\cdot\mu},\ \text{where } \mu=\mu_{f(T)}=\frac{N^2}{e}\sum_{v>0,\,v\in\frac{1}{e}\mathbf{Z}}(1-1/\deg f^{-v}(T)). \tag{9}$$

In step 2 we have to compute $\dim(R_{\mathfrak{K}})$, where $R_{\mathfrak{K}}$ is the image of

$$R\in\Gamma_E\hat{}(\dim=f_{L/E})\longmapsto R_{\mathfrak{K}}\in(\mathfrak{K}_{L/E}/U^1)\hat{}$$

(see the Main Lemma 4.1). From 4.(13) we see that

$$\dim R_{\mathfrak{K}}=(\dim R_0)^{e_{L/E}}.$$

Using now the dimension formula for cuspidal representations of $GL_{k_E}(k_L)$ we obtain: $\dim R_0=\prod_{i=1}^{f_{L/E}-1}(q_E^i-1)$

$$\dim R_{\mathfrak{K}}=\prod_{i=1}^{f_{L/E}-1}(q_E^i-1)^{e_{L/E}},\quad \text{where}\quad q_E=|k_E|=q^{f_{E/F}}. \tag{10}$$

In step 3 we compute:

$$\begin{aligned}(\mathfrak{K}_e:\mathfrak{K}_{L/E}\cdot U^1)&=(\mathfrak{K}_e:U^1)/(\mathfrak{K}_{L/E}U^1:U^1)\\&=(\mathfrak{A}^*_{L/F}:U^1)/(\mathfrak{A}^*_{L/E}\cdot U^1:U^1)\\&=|GL_{k_F}(k_L)|^e/|GL_{k_E}(k_L)|^{e_{L/E}}\\&=|GL_{f_t}(q)|^e/|GL_{f_{L/E}}(q_E)|^{e_{L/E}}.\end{aligned}$$

Now we use the formula: $\mathrm{ord}(N,q):=|GL_N(q)|=\prod_{i=1}^N(q^i-1)\cdot q^{\frac{1}{2}N(N-1)}$. Then we obtain:

$$\begin{aligned}\dim(\mathrm{Ind}_{\mathfrak{K}_e}(\Lambda))&=d_1\cdot\dim R_{\mathfrak{K}}\cdot\mathrm{ord}(f_t,q)^e/\mathrm{ord}(f_{L/E},q_E)^{e_{L/E}}\\&=q^{(\frac{1}{2})\mu}\cdot\mathrm{ord}(f_t,q)^e/(q_L-1)^{e_{L/E}}\cdot q_L^{(1/2)(f_{L/E}-1)e_{L/E}}\end{aligned} \tag{11}$$

where we have used (9), (10) and the fact that $\dim R_{\mathfrak{K}}\mid\mathrm{ord}(f_{L/E},q_E)^{e_{L/E}}$, and $q_E^{f_{L/E}}=q_L$.

According to (2) we go ahead to compute $\mathrm{vol}(\mathfrak{A}^*_{L/F})$, referring to a Haar measure of G and $\mathrm{vol}(\mathfrak{A}^*_{L/K})/\mathrm{vol}(GL_{\mathfrak{o}_K}(\mathfrak{o}_L))$ referring to a Haar measure of GL_K.

If the Haar measure of $G/\langle\pi_F\rangle$ is normalized in such a way that the Steinberg representation gets $\deg(St)=1$, then:

$$\mathrm{vol}(GL_N(\mathfrak{o}_F))=\frac{1}{N}\cdot\prod_{k=1}^{N-1}(q^k-1) \tag{12}$$

as we see from [CMS] (2.2.2). Using (12), we have to compute $\mathrm{vol}(\mathfrak{A}^*_{L/F})$. To this end we determine:

$$\begin{aligned}\mathrm{vol}(GL_N(\mathfrak{o}_F))/\mathrm{vol}(\mathfrak{A}^*_{L/F}) &= (GL_N(\mathfrak{o}_F): 1 + M_N(\mathfrak{p}_F))/(\mathfrak{A}^*_e: 1 + P_e)(1 + P_e: 1 + M_N(\mathfrak{p}_F)) \\ &= |GL_N(q)|/|GL_{N/e}(q)|^e \cdot [P_e: M_N(\mathfrak{p}_F)].\end{aligned}$$

We describe $P_e/M_N(\mathfrak{p}_F) \subseteq M_N(k_F)$. It consists of matrices which are divided into blocks of size $f_t \times f_t$ (note $f_t = N/e$). All blocks on the diagonal and below have zero entries. All blocks above the diagonal have arbitrary entries from k_F. Hence the k_F-dimension of $P_e/M_N(\mathfrak{p}_F)$ is:

$$\dim_{k_F} = f_t^2(1 + 2 + \cdots + (e-1)) = \frac{1}{2}N^2\left(1 - \frac{1}{e}\right), \quad \text{because} \quad N = f_t \cdot e$$

Substituting this into the formula above, we obtain:

$$\text{(13)} \qquad \begin{aligned}\mathrm{vol}(GL_N(\mathfrak{o}_F))/\mathrm{vol}(\mathfrak{A}^*_{L/F}) &= \mathrm{ord}(N,q)/\mathrm{ord}(f_t,q)^e \cdot q^{(1/2)N^2(1-1/e)} = \\ &= \prod_{i=1}^{N}(q^i - 1)\Big/\prod_{i=1}^{f_t}(q^i-1)^e.\end{aligned}$$

And from (12) we deduce:

$$\text{(14)} \qquad \mathrm{vol}(\mathfrak{A}^*_{L/F}) = \frac{1}{N} \cdot \prod_{i=1}^{f_t}(q^i - 1)^e/(q^N - 1).$$

To compute $\mathrm{vol}(\mathfrak{A}^*_{L/F})/\mathrm{vol}(GL_{\mathfrak{o}_K}(\mathfrak{o}_L))$ we may use (13) with K instead of F, i.e. we have to replace:

$$\begin{array}{rll} N \text{ and } e & \text{by} & [L:K] = e_{L/K} = e_{L/E} \\ q & \text{by} & q_K = q_L \\ f_t & \text{by} & 1. \end{array}$$

Then we get:

$$\text{(15)} \qquad \mathrm{vol}(\mathfrak{A}^*_{L|K})/\mathrm{vol}(GL_{\mathfrak{o}_K}(\mathfrak{o}_L)) = (q_L - 1)^{e_{L/E}}\Big/\prod_{i=1}^{e_{L/E}}(q_L^i - 1).$$

Now, substituting our results into (2), we obtain

$$\text{(16)} \qquad \deg\left(\Pi_t^{GL}\right) = \frac{(15)}{(14)} \cdot \frac{1}{e} \cdot \prod_{k=1}^{e_{L/E}-1}(q_L^k - 1) \cdot (11) = \frac{(15)'}{(14)} \cdot \frac{1}{e} \cdot (11),$$

where $(15)' = (q_L - 1)^{e_{L/E}}/(q_L^{e_{L/E}} - 1)$. The numerator of (14) is in the numerator of (11), and the numerator of (15)′ is in the denominator of (11). Hence, (16) implies:

$$\text{(17)} \qquad \deg\left(\Pi_t^{GL}\right) = \frac{N}{e} \cdot \frac{(q^N - 1)}{(q_L^{e_{L/E}-1)}} \cdot q^{(1/2)d}, \qquad \text{where}$$

$$d = ef_t(f_t - 1) - f_t(f_{L/E} - 1)e_{L/E} + \mu.$$

Now we use $f_t = N/e$, $f_t \cdot [L\colon E] = \frac{N}{e} \cdot \frac{N}{\deg f(T)}$, $f_t \cdot e_{L/E} = N/e_t$ (because $e_t \cdot e_{L/E} = e$). Then we get:

$$\begin{aligned} d &= N^2/e - N - N^2/e \cdot \deg f(T) + N/e_t + (N^2/e) \cdot \sum_{v>0,\, v\in\frac{1}{e}\mathbf{Z}} (1 - 1/\deg f^{-v}(T)) \\ &= -N(1-1/e_t) + (N^2/e) \sum_{v\geq 0,\, v\in\frac{1}{e}\mathbf{Z}} (1 - 1/\deg f^{-v}(T)), \end{aligned}$$

where the sum now starts from $v = 0$. Because of $N/e = \left(\frac{1}{N}\mathbb{Z} \colon \frac{1}{e}\mathbb{Z}\right)$ and because $\deg f^{-v}(T)$ does not change for $v \in \frac{1}{N}\mathbb{Z} \setminus \frac{1}{e}\mathbb{Z}$, the sum can be replaced by $N \cdot \sum_{v\geq 0,\, v\in\frac{1}{N}\mathbf{Z}} (1 - 1/\deg f^{-v}(T))$, and we see that

$$d = m_{f(T)} = N \left\{ \sum_{v\geq 0,\, v\in\frac{1}{N}\mathbf{Z}} (1 - 1/\deg f^{-v}(T)) - (1 - 1/e_t) \right\}$$

which proves our Theorem. □

6. Comparison of division algebra representations and discrete series representations of G

We start from an analogue of 4.7 in the division algebra case. Let D/F be a central division algebra of index N and let $D^{*\wedge}$ be the set of equivalence classes of irreducible admissible representations of D^*.

6.1 Theorem: *The map*

$$t = [R, \beta] \in \mathfrak{T}_N^- \longmapsto \Pi_t^D = Ind_{J_f \uparrow D^*}(\pi_f \otimes R_D) \in D^{*\wedge} \tag{1}$$

which has been defined in Part 1, is a bijection with the following properties:

(i) $\pi^D_{\chi\otimes t} = \tilde{\chi} \otimes \Pi_t^D$ *for tame characters* $\chi: F^*/U_F^1 \to \mathbb{C}^*$, *where* $\tilde{\chi} = \chi \circ Nrd$. *Hence* $f_t = f_\Pi$ *for* $\Pi = \Pi_t^D$.

(ii) The unit representation $\Pi_t^D \equiv 1$ *is assigned to* $t = [1, 0]$ *and* $\dim(\Pi_t^D) = 1$ *iff* $\deg(t) = 1$.

(iii) The level of Π_t^D *is* $\ell(\Pi_t^D) = -\nu_F(\beta)$ *and the exponential Artin conductor is* $a(\Pi_t^D) = N(1 - \nu_F(\beta))$ *if* $\beta \neq 0$. *If* $\beta = 0$ *then* $\ell(\Pi_t^D) = 0$ *and for* $t = [R, 0]$ *we have either* $\ell(\Pi_t^D) = N - 1$ *or* $= N$ *depending on* R *being (via class field theory) an unramified character of* F^* *or not.*

(iv) $\dim(\Pi_t^D) = f_t \frac{q^N - 1}{q^{N/e_t} - 1} \cdot q^{(1/2)m_{f(T)}}$
with the same notation as in 4.1.

Proof: Everything but the dimension formula is clear, either from Part 1 or from Section 3. f_Π is defined in the same way as in 2.14 for G. The index $j(\Pi_t^D)$ (compare 3.(2)) has been seen in 2.2 (i) of Part 1 to be $-N \cdot \nu_F(\beta)$ if $t = [R, \beta]$ with $\beta \neq 0$ and $j(\Pi_t^D) = 0$ if $t = [R, 0]$. Hence (iii) follows from Section 3. And (ii) can be derived from (iv) which we are going to see is an easy consequence of [Zi$_{90}$] 6.6 and 2.4. Namely $\dim(\Pi_t^D)$ is computed in precisely the same way as dim $\mathrm{Ind}_{\mathfrak{K}_e}(\Lambda)$ in the proof of 5.1.

At first we compute

$$d_1 := \dim(\mathrm{Ind}_{U^1}(\pi_1)) = \dim(\mathrm{Ind}_{J_1 \uparrow D_0^* U^1}(\pi_f)),$$

where U^1 are the principal units in D, J_f and J_f^1 are as in 2.(3) ff of Part 1, the Heisenberg representation $\pi_1 \in J_f^{1\wedge}$ is the restriction of $\pi_f \in J_f^\wedge$ and D_0 is the centralizer of $\alpha_f \in \Delta_D^-$. In [Zi$_{90}$], 6.6 and 2.4 we have seen that

$$\begin{aligned} &d_1 = q^{(1/2)m_{f(T)}}, \qquad \text{where} \\ &m_{f(T)} = (N/e_0)(1 - 1/f_0) + N \sum_{v>0,\, v\in\frac{1}{N}\cdot\mathbf{Z}} (1 - 1/\deg f^{-v}(T)) \end{aligned} \tag{2}$$

with $e_0 = e_F(f(T)) = e_{F(\beta)/F}$, $f_0 = f_F(f(T)) = f_{F(\beta)/F}$. Note that here $f(T) = f^0(T)$ and the notation β of loc. cit. 2.4 has to be replaced by $\alpha_f \in \Delta_D^-$, hence $\alpha_f \in \alpha_f + \mathcal{O}_D$ is a minimal representative, and deg $f^{-v}(T)$ is the degree of a minimal representative from $\alpha_f + P_D^{-vN}$. Moreover:

$$(N/e_0)(1-1/f_0) = N(1-1/e_0f_0) - N(1-1/e_0) = N(1-1/\deg f(T)) - N(1-1/e_t),$$

which transforms (2) into the exponent of 4.1.

In the second step we compute:

$$d_2 := \dim(\mathrm{Ind}_{J_f\uparrow D_0^*U^1}(\pi_f \otimes R_D)) = d_1 \cdot \dim R_D,$$

because R_D is a representation of $J_f/J_f^1 = D_0^*U^1/U^1$. But from the last step in the proof of 2.2. (i) in Part 1 we see that $\dim R_D = \dim R$.

Finally we get $\dim(\Pi_t^D) = d_2 \cdot (D^*: D_0^*U^1)$, and

$$(D^*: D_0^*U^1) = e_{D/D_0} \cdot (k_D^*: k_{D_0}^*) = f_0 \cdot \frac{q^N-1}{q^x-1}$$

with $x = [k_{D_0}: k_F] = f_{D_0/F} = N/f_{D/D_0} = N/e_0$. This gives us the result because $e_0 = e_t$ and $\dim R \cdot f_0 = f_t$. □

For further use we want to express the invariants f_t, e_t, deg $f^{-v}(T)$ of $t \in \mathfrak{T}_N^-$ completely in terms of $\Pi = \Pi_t^D$. Here we have:

(3) $$f_t = f_\Pi \qquad \text{(see 5.1 (i))}$$

(4) $$e_t = N/\log_q\left\{\frac{q^N-1}{[\dim(\Pi)/f_\Pi]_{p'}} + 1\right\}$$

where $[\dim(\Pi)/f_\Pi]_{p'}$ denotes the prime-to-p-factor (see 5.1 (iv)). Moreover let $\Pi \supset \Pi_0 \supset \Pi_1 \supset \Pi_2 \supset \cdots$ be any representation filter with respect to

(5) $$D^* \supset \mathcal{O}_D^* \supset U^1 \supset U^2 \supset \cdots,$$

which has the leading term Π. Then as we have seen in [Zi$_{90}$] 6.2, (2):

(6) $$N_{D^*}(\Pi_0) \cap N_{D^*}(\Pi_1) = D_K^* \cdot U^1$$

where K/F is a field extension with $f_{K/F} = f_\Pi$ and $e_{K/F} = e_t$, $[K: F] = \deg(t)$ if $\Pi = \Pi_t^D$. Namely in that case $(J_f, \pi_f \otimes R_D)$ is an admissible pair with respect to $D^* \supset U^1 \supset U^2 \supset \cdots$. We look at $J_f = D_0^* \cdot J_f^1$ where D_0 is the centralizer of $E = F(\alpha_f)$ and we take K/E to be unramified of degree $\dim(R_D) = \dim(R)$. Then D_E/D_K is fully ramified, $D_K^* \cdot J_f^1 \subset J_f$ is a subgroup of index $[K: E] = \dim R$, and $(D_K^* \cdot J_f^1, \mathrm{Res}(\pi_f) \otimes \chi)$ – where χ is a character contained in R_D restricted to $D_K^* J_f^1$ – is an admissible pair with respect to (5). Therefore we obtain (6) in the case where $\Pi = \Pi_t^D$ and $K/F(\alpha_f)$ is an unramified extension of degree $\dim(R)$. Hence:

6.2 Proposition: *Consider* $\Pi \in (D^*)^\frown$ *and let* $\Pi \supset \Pi_0 \supset \Pi_1$ *be a representation filter with respect to* $D^* \supset \mathfrak{O}_D^* \supset U^1$. *Then there exists an extension* K/F *in* D *such that:*

$$N_{D^*}(\Pi_0) \cap N_{D^*}(\Pi_1) = D_K^* \cdot U^1,$$

and for any such K/F *we have:*

$$\begin{aligned} f_{K/F} &= f_\Pi \\ e_{K/F} &= N/\log_q \left\{ \frac{q^N - 1}{[\dim(\Pi)/f_\Pi]_{p'}} + 1 \right\} \end{aligned}$$

Proof: Because of 6.1 there exists $t \in \mathfrak{T}_N^-$ such that $\Pi = \Pi_t^D$. In view of (3), (4) and what we have said before we are left to show that $D_K^* \cdot U^1 = D_{K'}^* \cdot U^1$ implies $e_{K/F} = e_{K'/F}$, $f_{K/F} = f_{K'/F}$ which is an easy exercise. □

6.3 Definition: For $\Pi \in D^{*\frown}$ we define

$$e_\Pi := e_{K/F}, \qquad \dim_{\mathrm{Gal}}(\Pi) := [K:F]$$

to be the ramificatioin exponent and the Galois dimension of Π, where K/F is any field as in 6.2.

Concerning the other invariants, we see from [Zi$_{90}$], 6.2.(3), that for all integers $i \geq 0$ there exist field extensions K_{-i}/F in D such that

$$N_{D^*}(\Pi_{i+1}) \cdot U^1 = D_{-i} \cdot U^1, \tag{7}$$

where $\Pi_{i+1} \in U^{i+1\frown}$ is from a representation filter with leading term Π and where D_{-i} is the centralizer of K_{-i}.

If $\Pi = \Pi_t^D$ and $f(T)$ is the polynomial of $t = [R, \beta] \in \mathfrak{T}_N^-$, then it is $\alpha_f \in \Delta_D^-$ which plays the role of the element β from loc. cit. 6.2. And if $\alpha_{-i/N} \in \Delta_D^-$ is a zero of the approximation polynomial $f^{-i/N}(T)$, then $\alpha_{-i/N} \in \alpha_f + P^{-i}$ is a representative of minimal degree, and we conclude:

6.4 Proposition: *If* $\Pi = \Pi_t^D$ *and if* K_{-i}/F *is any field extension in* D *whose centralizer* D_{-i} *fulfills (7), then* K_{-i}/F *has the same ramification exponent and inertial degree as* $F(\alpha_{-i/N})/F$, *hence* $[K_{-i}:K] = \deg f^{-i/N}(T)$. □

6.5 Corollary: *If* $\Pi \in D^{*\frown}$ *and if* $\Pi \supset \Pi_0 \supset \Pi_1 \supset \cdots$ *is any representation filter with respect to (5) and if* K/F, K_{-i}/F *are field extensions such that (6), (7) resp. are fulfilled: then:*

$$\dim(\Pi) = f_{K/F} \cdot \frac{q^N - 1}{q^{N/e} - 1} \cdot q^{(1/2)m},$$

where $e = e_{K/F}$, $m = N\Big[\sum_{i\geq 0}(1 - 1/[K_{-i}:F]) - (1 - 1/e)\Big]$. □

Now we want to play the same game in the GL_N-case. If $\Pi = \Pi_t^{GL} \in \widehat{G}_2$, then Π corresponds to the EST $(J, \Lambda)_t = (J_f, \pi_f \otimes R_{\mathfrak{K}})$ which represents a representation filter

$$(\Pi^{\mathfrak{K}}, \Pi_0, \Pi_1, \ldots) \quad \text{of} \quad \mathfrak{K} \supset \mathfrak{A}^* \supset U^1 \supset \cdots, \tag{8}$$

where $\mathfrak{K} = \mathfrak{K}_e$ for $e = N/f_\Pi = N/f_t$. Namely we have: $\Pi^{\mathfrak{K}} = \mathrm{Ind}_{\mathfrak{K}}(\Lambda)$, $\Pi_0 = \Pi^{\mathfrak{K}}|_{\mathfrak{A}^*} = \mathrm{Ind}_{\mathfrak{A}^*}(\lambda)$, $\Pi_i = \mathrm{Ind}_{U^i}(\pi_i)$, where $\pi_i \in J^{i\wedge}$ is the support of the isotypic representation $\pi_f|_{J^i}$ (for all $i \geq 1$).

The construction of (J_f, π_f) in 2.4 ff of Part 1 implies that $\Pi_{i+1} \in U^{i+1\wedge}$ corresponds to the double class $\mathrm{Ad}\, U^{i+1} \circ (\alpha_{-i/e} + P^{-i}) \in \mathrm{Ad}\, U^{i+1} \backslash A(\mathfrak{A})/P^{-i}$. Hence from [BK] (3.2.2) we see that the G-intertwining $I_G(\Pi_{i+1})$ coincides with the intertwining $I_G(\mathrm{Ad}\, U^{i+1} \circ (\alpha_{-i/e} + P^{-i})) = U^{i+1} I_G(\alpha_{-i/e} + P^{-i}) U^{i+1}$, where $I_G(\beta + P^v) := \{x \in G,\, x(\beta + P^v)x^{-1} \cap (\beta + P^v) \neq \phi\}$.

The reason is that $\alpha_{-i/e} + P^{-i}$, hence $\alpha_{-i/e} + \mathfrak{A}$ are simple strata, and by 2.4.1 of Part 1 we have $\theta_f = \theta_g$ on $H_f^1 \cap U^{i+1}$ if $f = f(T)$ and if $g = f^{-i/e}(T)$ is the approximation polynomial. Therefore if we are interested only in θ_f on $H_f^1 \cap U^{i+1}$ we can use 2.4.2 of Part 1 with $f^{-i/e}(T)$ instead of $f(T)$. Therefore [BK] (3.3.2) applies with i instead of m and $\alpha_{-i/e} + \mathfrak{A}$ instead of $\beta + \mathfrak{A}$. Moreover $\pi_{i+1} \in J^{i+1\wedge}$ is the Heisenberg representation which is given by $(H_f^1 \cap U^{i+1}, \mathrm{Res}\,\theta_f)$ (see section 4 of [Z88]), and $\Pi_{i+1} = \mathrm{Ind}_{U^{i+1}}(\pi_{i+1})$. Thus we have:

6.6 Proposition: *If $\Pi = \Pi_t^{GL} \in \widehat{G}_2$ and if (8) is the representation filter which is determined by the EST $(J, \Lambda)_t$, then:*

$$U^1 \cdot I_G(\Pi_{i+1}) U^1 = U^1 \cdot GL(\alpha_{-i/e}) \cdot U^1$$

for all integers $i \geq 0$, where $\alpha_{-i/e} \in \Delta_e^-$ is a zero of the approximation polynomial $f^{-i/e}(T)$, and $GL(\alpha_{-i/e})$ denotes the centralizer of $\alpha_{-i/e}$ in G. Moreover

$$I_G(\Pi_0) = \mathfrak{A}^* GL_K \mathfrak{A}^*,$$

where $K/F(\alpha_0)$ is a maximal unramified extension such that $K^* \subset \mathfrak{K}$.

Proof: We know that $I_G(\Pi_{i+1}) = U^{i+1} I_G(\alpha_{-i/e} + P^{-i}) \cdot U^{i+1}$ and $GL(\alpha_{-i/e}) \subset I_G(\alpha_{-i/e} + P^{-i}) \subset U^1 \cdot GL(\alpha_{-i/e}) \cdot U^1$ because of [BK] (1.5.8). $I_G(\Pi_0)$ follows from the intertwining $I_G(\lambda)$ of the simple type (J^0, λ) which has been computed in [BK] §5. □

6.7 Corollary: *Let $\Pi \in \widehat{G}_2$, let (J, Λ) be a corresponding EST and consider the representation filter (8) of $\mathfrak{K}_e$ ($e = N/f_\Pi$) which is assigned to (J, Λ). Further let K_{-i}/F be any field extension such that:*

$$(9) \qquad K_{-i}^* \subset \mathfrak{K}_e \quad \textit{and} \quad U^1 \cdot I_G(\Pi_{i+1}) \cdot U^1 = U^1 \cdot GL_{K_{-i}} \cdot U^1.$$

Then for a normalized Haar measure we have:

$$\deg(\Pi) = f_\Pi \cdot \frac{q^N - 1}{q^{N/e'} - 1} \cdot q^{(1/2)m}$$

where $e' = e_{K_0/F}$ and $m = N\Big[\sum_{i \geq 0} f_\Pi (1 - 1/[K_{-i}: F]) - (1 - 1/e')\Big]$.

Proof: We use that $t \in \mathfrak{T}_N^- \mapsto \Pi_t^{GL} \in \widehat{G}_2$ is surjective (which has not yet been proved). Then $\Pi = \Pi_t^{GL}$, and by comparing $(9)_2$ with the first equation in 6.6 we see that K_{-i}/F

has the same ramification exponent and the same inertial degree as $F(\alpha_{-i/e})/F$. The argument is similar as in the proof of 6.2 if we use the equation

$$(GL(\alpha_{-i/e}) \cap \mathfrak{K}) \cdot U^1 = (GL_{K_{-i}} \cap \mathfrak{K}) \cdot U^1.$$

Especially we have $[K_{-i}: F] = \deg f^{-i/e}(T)$ for all integers $i \geq 0$, and $e_t = e_{K_0/F}$. Moreover the sequence $[f^v(T)\}_{v \in \frac{1}{N}\mathbb{Z}}$ of approximation polynomials has jumps at most for $v \in 1/e \cdot \mathbb{Z}$ because $f(T) \in F[T]^-_{e,N/e}$. Therefore $\deg f^v(T)$ is constant for $-i/e < v \leq -(i-1)/e$, hence in the sum $m = m_{f(T)}$ of 5.1 we can write

$$m = N \left[\sum_{v \geq 0, v \in \frac{1}{e}\mathbb{Z}} \frac{N}{e}(1 - 1/\deg f^{-v}(T)) - (1 - 1/e_t) \right]$$

and then by means of $e_t = e_{K_0/F}$ and $\frac{N}{e} = f_\Pi$ we get our result. □

Now we are ready to compare division algebra representations and discrete series representations of GL_N. Namely combining 4.7, 5.1, 6.1 we obtain:

6.8 Theorem: *There exist injections $D^{*\wedge} \to \widehat{G}_2$ $\Pi' \mapsto \Pi$, which have the following properties:*

(i) *If $\Pi' \mapsto \Pi$ and if $\chi: F^*/U^1 \to \mathbb{C}^*$ is a tame character, then: $(\chi \circ Nrd) \otimes \Pi' \mapsto (\chi \circ \det) \otimes \Pi$. Especially $f_{\Pi'} = f_\Pi$, i.e. Π corresponds to an EST (J, Λ) for a maximal compact-modulo-center subgroup $\mathfrak{K}_e$ with $e = N/f_{\Pi'}$.*

(ii) *If we consider a representation filter $\Pi' \supset \Pi'_0 \supset \Pi'_1 \supset \cdots$ with respect to (5) and a representation filter (8) of $\mathfrak{K}_e$ which corresponds to (J, Λ) then from $I_G(\Pi_{i+1})$ and $N_{D^*}(\Pi'_{j+1})$ via (9), (7) resp. we are led to the same numerical invariants if $i/e = j/N$. Especially the projective level of Π: $= \frac{1}{e} \cdot \min\{i, I_G(\Pi_{i+1}) = G\}$ and the projective level of Π': $= \frac{1}{N}\min\{j; N_{D^*}(\Pi'_{j+1}) = D^*\}$ coincide.*

(iii) *Π is cuspidal iff the Galois dimension of Π' (which has been defined in 5.3) is equal to N.*

(iv) *Π is a character twist of the Steinberg representation iff $\dim(\Pi') = 1$.*

(v) *If the measure on $GL_N(F)/\langle \pi_F \rangle$ is chosen appropriately, then $\deg(\Pi) = \dim(\Pi')$.*

(vi) *$\ell(\Pi) = \ell(\Pi')$ and $a(\Pi) = a(\Pi')$.*

Proof: The injection is $\Pi' = \Pi_t^D \mapsto \Pi = \Pi_t^{GL}$ for all $t \in \mathfrak{T}_N^-$. Π_t^{GL} cuspidal means $\deg(t) = N$ as we have seen in 4.7 (ii) and this is equivalent to $\dim_{\mathrm{Gal}}(\Pi') = N$. □

6.9 Complement: *Any injection $i: D^{*\wedge} \to \widehat{G}_2$ which is compatible with unramified character twists and which preserves the exponential Artin conductor is necessarily a bijection. Especially the maps of 6.8 and 4.7 are bijective.*

Proof: We consider the bijection $b: D^{*\wedge} \to \widehat{G}_2$ of [BDKV] (and of J. Rogawski). It is compatible with character twist and preserves the exponential Artin conductor. But from 6.1 we see that the set of unramified twist classes $\{\Pi'\}$, $\Pi' \in D^{*\wedge}$ where the exponential

Artin conductor (or the level, compare 3.1 (ii)) is fixed, has only finitely many elements. Namely by 6.1 (i), (iii) it corresponds to the set of unramified twist classes $\{t\}$, $t = [R, \beta] \in \mathcal{T}_N^-$ where $\nu_F(\beta)$ is fixed. Now we have the bijection

$$\mathcal{T}_N^- \longleftrightarrow \mathcal{P}_N^- = \{[\rho, x] \in \mathcal{P}^-;\ \dim(\rho) \cdot [K_x : F] | N\} \tag{10}$$

which is given via 1.4 and 1.(9). Remember that $K_x = F(x_v; v \in \mathbb{Q})$ is generated by all terms x_v of the C-expansion $x = \sum_{v \in \mathbf{Q}} x_v$ of $x \in \mathcal{K}$. Because $\nu_F(\beta) = \nu_F(x)$ if $t = [R, \beta]$ corresponds to $[\rho, x]$ and because $\mathcal{P}^- = \{[\rho, x] \in \mathcal{P};\ x \in \mathcal{K}^-\}$ (compare at the very end of section 1), we are left to show that:

$$\mathcal{P}_{N,j}^- = \{[\rho, x] \in \mathcal{P};\ x \in \mathcal{K}^-,\ \nu_F(x) = -j,\ \dim(\rho) \cdot [K_x : F] | N\}$$

consists of finitely many unramified twist classes. If we fix an $x \subset \mathcal{K}^-$ then ρ has to be an irreducible representation of the tame Weil-group $\Gamma_x \subset G(\tilde{F}/\tilde{F} \cap K_x)$, hence the condition $\dim(\rho) \cdot [K_x : F] | N$ implies that the set of admissible ρ consists of finitely many unramified twist classes. Thus we are left to show that $\{x \in \mathcal{K}^-;\ \nu_F(x) = -j,\ [K_x : F] | N\}$ is a finite set. We consider the C-expansion of x:

$$x = \sum_{-j \leq v < 0} x_v \quad \text{where} \quad x_v \in C \cup \{0\},\ x_{-j} \neq 0. \tag{11}$$

Note that the indices v are rational numbers and that $\nu_F(x_v) = v$ if $x_v \neq 0$. The decisive remark is that the ramification exponent of $K_x = F(x_v;\ v \in \mathbb{Q})$ is

$$e = \ell.c.m.\{\text{denominators of all } v \in \mathbb{Q} \text{ such that } x_v \neq 0\},$$

which can be easily proved by induction on the number of nonvanishing terms x_v in the C-expansion of x. Hence $[K_x : F] | N$ implies $v \in \frac{1}{N}\mathbb{Z}$ if $x_v \neq 0$, and in (11) we are restricted to the finite set $v \in \frac{1}{N}\mathbb{Z} \cap [-j, 0)$. Therefore it is enough to show that for a fixed $v \in \frac{1}{N}\mathbb{Z}$ the set

$$C_{v,N} = \{x \in C;\ \nu_F(x) = v \quad \text{and} \quad [F(x) : F] | N\}$$

is finite, which is obvious.

Now using the bijection b, we see that the set of unramified twist classes $\{\Pi\}, \Pi \in \widehat{G}_2$ with a fixed exponential Artin conductor is finite too and has the same cardinal number as in the division algebra case. Hence 6.9 follows. □

6.10 Corollary: *Let* $\Pi' \in D^{*\wedge} \mapsto \Pi \in \widehat{G}_2$ *be a bijection which is compatible with unramified character twist and such that* $\dim(\Pi') = \deg(\Pi)$ *(if we consider the normalized formal degree of the discrete series representations). Then* Π *is cuspidal iff* Π' *has the Galois dimension* N *(see 5.3).*

We need the following Proposition which is the analogy of 6.2 in the GL_N-case.

6.11 Proposition: *Consider* $\Pi \in \widehat{G}_2$ *with a corresponding extended simple type* $(\mathfrak{K}_e, J, \Lambda)$, $e = N/f_\Pi$, *and let* $\Pi_0 \in \mathfrak{A}^{*\wedge}$ *be from the representation filter which is given by* (J, Λ). *Further consider a field extension* K/F *in* A *such that (i)* $K^* \subset \mathfrak{K}_e$, *(ii)* $GL_K \cap \mathfrak{K}_e$ *is an extended Iwahori subgroup in* GL_K, *(iii)* $I_G(\Pi_0) = \mathfrak{A}^* GL_K \mathfrak{A}^*$ *and* $\mathfrak{A}^* \backslash GL_K / \mathfrak{A}^* \leftrightarrow \mathfrak{A}_K^* \backslash GL_K / \mathfrak{A}_K^*$ *is a natural bijection. Then:*

$$f_{K/F} = f_\Pi, \quad e_{K/F} = N / \log_q \left\{ \frac{q^N - 1}{[\deg(\Pi)/f_\Pi]_{p'}} + 1 \right\}. \tag{11}$$

Proof: Because of (ii) K/F has the maximal inertial degree which is possible for fields normalizing $\mathfrak{A} = \mathfrak{A}_e$. Hence $f_{K/F} = N/e = f_\Pi$. Further, because of 6.9 we may assume $\Pi = \Pi_t^{GL}$. Then from 5.1 we see that the right hand side of $(11)_2$ coincides with e_t. On the other hand, the field K appearing at the very end of 6.6 has the degree $[K{:}\,F] = \deg t$ and $e_{K/F} = e_t$. And if L, K are two field extensions of F in A normalizing $\mathfrak{A}$ and such that:

$$\begin{aligned} \mathfrak{A}_L^* \backslash GL_L / \mathfrak{A}_L^* &\longleftrightarrow \mathfrak{A}^* \backslash GL_L / \mathfrak{A}^* = \mathfrak{A}^* \backslash GL_K / \mathfrak{A}^* \\ &\longleftrightarrow \mathfrak{A}_K^* \backslash GL_K / \mathfrak{A}_K^* \end{aligned}$$

is a natural bijection, then we will have $[L{:}\,F] = [K{:}\,F]$. Hence the field K from our Proposition has the same degree as the field from 6.6 and it has the same inertial degree too, namely f_Π. Hence it has the same ramification exponent which is e_t. □

Proof of 6.10: Let K'/F be a field extension in D which is related to Π' as in 6.2 and let K/F be a field extension in A which is related to Π as in 6.11. Then we get $f_{K'/F} = f_{\Pi'} = f_\Pi = f_{K/F}$ because $\Pi' \mapsto \Pi$ is compatible with tame character twist. On the other hand $\dim(\Pi') = \deg(\Pi)$, the last equation of 6.2 and (11), imply $e_{K'/F} = e_{K/F}$. Especially we have $[K'{:}\,F] = [K{:}\,F]$. But from [BK] we know that Π is cuspidal iff the simple types contained in Π are maximal which means $[K{:}\,F] = N$. □

Remark: 6.10 especially applies to the bijection "b" of [BDKV]. The idea of proof was submitted to the author in a letter of C. Bushnell [B].

7. Some further computations concerning the ε-factor

In 4.7 and 6.1 the maps $t \in \mathcal{T}_N^- \mapsto \Pi_t \in \widehat{G}_2$, $D^{*\wedge}$ resp. have been given via the extended simple types $(J_f, \pi_f \otimes R_{\mathfrak{K}})$ in the matrix algebra case (see 4.6) and $(J_f, \pi_f \otimes R_D)$ in the division algebra case (see Part 1, proof of 2.2 (i)). Now we use our information on the EST's for some further remarks on the root nubmer $W(\Lambda, \psi_F)$ of (J, Λ), which by 3.6 the comptuation of the ε-factor has been reduced to.

Up to now the EST's have been given in terms of the polynomial $f(T)$ of the parameter $t \in \mathcal{T}_N^-$, namely

$$f(T) \in F[T]_{e,N/e}^- \longmapsto \alpha_f \in \Delta_e^-,\ \lambda_f \in X_e,\ \theta_f \in Y_e,\ \pi_f \in \mathcal{K}_e^{\wedge}, \tag{1}$$

$$f(T) \in F[T]_N^- \longmapsto \alpha_f \in \Delta_D^-,\ \lambda_f \in X_D,\ \theta_f \in Y_D,\ \pi_f \in D^{*\wedge}. \tag{2}$$

(For the chracter sets X_e, Y_e, X_D, Y_D see the proof of 2.2 in Part 1.) To simplify the notations we will avoid now the polynomials and we will write λ_α, θ_α, π_α instead of λ_f, θ_f, π_f because α is the unique root of $f(T)$ in Δ^- such that we can introduce α as the parameter. And we will speak of the approximation procedure on Δ_e^-, Δ_D^-, resp., which of course is obtained by transport from the approximation procedure for polynomials. We will use the integral numeration of approxiamtions, i.e. for $\alpha, \alpha' \in \Delta^-$:

$$\nu_P(\alpha - \alpha') \geq j \qquad \text{iff} \qquad \alpha_j = \alpha'_j, \tag{3}$$

where P is the Jacobson radical of $\mathfrak{A}_e$ and the prime ideal of $\mathcal{O}_D$ resp. The maps of 4.7 and 6.1 can be expressed then as:

$$\begin{aligned} t = [R, \beta] \in \mathcal{T}_N^- \longmapsto (J_\alpha, \pi_\alpha \otimes R_{\mathfrak{K}}) &\longmapsto \Pi_t \in \widehat{G}_2 \\ \longmapsto (J_\alpha, \pi_\alpha \otimes R_D) &\longmapsto \Pi_t \in D^{*\wedge} \end{aligned}$$

where $\alpha \in \Delta_e^-$, $e = N/f_t$ and $\alpha \in \Delta_D^-$ resp. is the uniquely determined element which has the same minimal polynomial over F as β. Our additive character ψ_F has conductor $\mathfrak{p}_F$, and we write $\psi_B = \psi_F \circ Trd$ for $B = A$, D resp. Now we are ready to state the main results of this section:

7.1 Theorem: *If $(J, \Lambda) = (J_\alpha, \pi_\alpha \otimes R_{\mathfrak{K}})$, $(J_\alpha, \pi_\alpha \otimes R_D)$ is an EST in the matrix and division algebra case resp., then:*

(o) $W(\Lambda, \psi_F) = (-1)^{N-1} q^{-N/2} \tau(\phi_0^{-1} \circ Norm, \psi_{k_N})$

if $\alpha = 0$, i.e. $t = [R, 0]$, and R is not an unramified character of F^*. Here k_N is the extension of degree N over the residue field k of F, the regular character ϕ_0 of k_K^* is related to $R \in \hat{\Gamma}_F$ as in 4.1(ii), $Norm$ denotes the norm with respect to the extension $k_N | k_K$ of finite fields, and

$$\tau(\phi_0^{-1} \circ Norm, \psi_{k_N}) = \sum_{x \in k_N^*} \phi_0^{-1} \circ Norm(x).\psi_F \circ Tr_{k_N|k}(x)$$

is a Gauss sum.

If $\alpha \neq 0$, i.e. $j(\Lambda) = -\nu_P(\alpha) \geq 1$, we have:

(i) $W(\Lambda, \psi_F) = (-1)^{N-m}\Lambda^\vee(\alpha)\psi_B(\alpha)$ *if* $2 \nmid j = -\nu_P(\alpha)$. *In this case* $\Lambda^\vee(\alpha)$ *is a scalar operator which in the equation is to be identified with its value.*

If $2|j = -\nu_P(\alpha)$ *then let* $a = \alpha_{-j+1} \in \Delta^-$ *be the approxiamtion of* α, *i.e.* $\nu_P(\alpha - a) \geq -j+1$. *Moreover let* B_a *be the centralizer of* a *in* A, D *resp., let* $B'_a := (P^{j/2} \cap B_a) + P^{j/2+1}$ *be the intermediate ring* $P^{j/2} \supset B'_a \supset P^{j/2+1}$ *and let* M', M *be the first and second index resp. Then:*

(ii) $W(\Lambda, \psi_F) = sgn \cdot (M')^{-1/2} \cdot \frac{\dim \Lambda}{\chi_\Lambda(\alpha)} \cdot \psi_B(\alpha) \cdot \delta(\lambda_a)^{N_a}$ *if* $F(a)/F$ *is tamely ramified (which includes the case* $a \in F$*), where* $\delta(\lambda_a) \in \mathbb{C}^*$ *depends on the character* $\lambda_a \in X_e$, X_D *resp. (*$\delta(\lambda_a)$ *will be defined in (16), (18) below), where* $N_a = N/[F(a){:}\,F]$ *and where* $sgn = 1$, $(-1)^{N-N_a}$ *in the cases* $B = A$, D *resp.*

(iii) $W(\Lambda, \psi_F) = (-1)^{N-m} \cdot \left(\frac{M}{M'}\right)^{1/2} \cdot \frac{\dim \Lambda}{\chi_\Lambda(\alpha)} \cdot \psi_B(\alpha)$, *if* $F(a)/F$ *is wildly ramified.*

Remarks: The sign in (i), (iii) is determined by $m = N, 1$ in the cases $B = A$, D resp. The theorem reduces the computation of $W(\Lambda, \psi_F)$ to that of the character value $\chi_\Lambda(\alpha)$ because the numbers $\delta(\lambda_a)$ can be considered to be known. If $a \in F$ then (ii) applies with $M' = 1$, $N_a = N$ and $\Lambda(\alpha)$ is a scalar operator, such that $W(\Lambda, \psi_F) = \Lambda^\vee(\alpha) \cdot \psi_\Lambda(\alpha) \cdot \delta(\lambda_a)^N$.

Proof: (o) We begin with the division algebra case. Here we have $J = D^*$ and $\Lambda = R_D = Ind(\tilde{\phi})$, where $\phi : K^*/U_K^1 \to C^*$ is related to R as in 4.1(ii),$\tilde{\phi} = \phi \circ Nrd_{D_K|K}$ is the corresponding character of the centralizer of K in D, and the induction is from $D_K^* U_D^1$ onto D^*. According to 3.(13) and 3.3 with $(\mathfrak{K}, \rho) = (D^*, R_D)$ we obtain

$$W(\Lambda, \psi_F) = (-1)^{N-1}(\mathcal{O}_D : P)^{-1/2}\tau(R_D^\vee, \psi_F)$$

where $\tau(R_D^\vee, \psi_F)$ is the value of the scalar operator
$T(R_D^\vee, \psi_F) = \sum_{u \in (\mathcal{O}_D/P)^*} R_D^\vee(u)\psi_D(u)$. Note that ψ_F is of conductor $\mathfrak{p}_F$, hence $\psi_D = \psi_F \circ Trd$ is of conductor P. Because $T(R_D, \psi_F)$ is a scalar operator and because the restriction of R_D onto the units $\mathcal{O}_D^*$ contains $\phi_0 \circ Norm$ (which is a consequence of $R_D = Ind(\tilde{\phi})$), we obtain $\tau(R_D, \psi_F) = \tau(\phi_0 \circ Norm, \psi_{k_N})$, hence the result.

In the matrix algebra case we have $J = \mathfrak{K}_e$, $e = N/dim(R)$, and $R_\mathfrak{K} \in \hat{\mathfrak{K}}_e$ is an extension of the external tensor power $(\otimes\sigma_0)^e \in (\mathfrak{A}_e/P)^{*\wedge}$(see 4.1(ii)). From 3.(13) with $m = N$ and 3.3 we obtain

$$W(\Lambda, \psi_F) = (\mathfrak{A}_e : P)^{-1/2}\tau(R_\mathfrak{K}^\vee, \psi_F) \qquad \text{(a)}$$

where $\tau(R_\mathfrak{K}^\vee, \psi_F)$ is the value of the scalar operator
$T(R_\mathfrak{K}^\vee, \psi_F) = \sum_{u \in (\mathfrak{A}_e/P)^*} (\otimes\sigma_0^\vee)^e(u)\psi_A(u)$, with $\psi_A = \psi_F \circ Tr$. Therefore:
$\tau(R_\mathfrak{K}, \psi_F)dim(R_\mathfrak{K}) = \sum tr_{(\otimes\sigma_0)^e}(u)\psi_A(u)$,
and because of $(\mathfrak{A}_e/P)^* \cong [GL_f(k)]^e$, where $f = dim(R) = [K : F]$ (see 4.1), we conclude $\tau(R_\mathfrak{K}, \psi_F)dim(R_\mathfrak{K}) = [\sum_{u \in GL_f(k)} tr_{\sigma_0}(u) \cdot \psi_F \circ Tr(u)]^e$, hence

$$\tau(R_\mathfrak{K}, \psi_F) = \tau(\sigma_0, \psi_F)^e \qquad \text{(b)}$$

is the e-th power of the value of the scalar operator

$T(\sigma_0, \psi_F) = \sum_{u \in GL_f(k)} \sigma_0(u)\psi_F \circ Tr(u)$. Because $\sigma_0 \in GL_f(k)\hat{}$ is the cuspidal representation which is associated to the regular character ϕ_0 of k_K^* we can apply Kondo's formula:

$\tau(\sigma_0, \psi_F) = (-1)^f.q^{(f^2-f)/2}.[-\tau(\phi_0, \psi_{k_K})]$,

where $\tau(\phi_0, \psi_{k_K}) = \sum_{x \in k_K^*} \phi_0(x) \cdot \psi_F \circ Tr_{k_K|k}(x)$. Now from (a),(b) we obtain

$W(\Lambda, \psi_F) = (\mathfrak{A}_e : P)^{-1/2}(-1)^N q^{e(f^2-f)/2}[-\tau(\phi_0^{-1}, \psi_{k_K})]^e$

$= (-1)^{(N-1)} \cdot q^{-N/2}\tau(\phi_0^{-1} \circ Norm, \psi_{k_N})$, where we have applied the Hasse-Davenport formula for Gauss sums.

(i)–(iii) The first informations we use are that the restrictions $\Lambda|_{U^i}$ to the principal unit groups $U^i = 1 + P^i$ are isotypic, that $j = -\nu_P(\alpha)$ is the index of Λ and that

$$\Lambda|_{U^i}(1 + x) = \psi_F(\alpha x)\mathbf{1} \qquad \text{for} \qquad i = [j/2] + 1$$

is a multiple of the character $1 + x \mapsto \psi_F(\alpha x)$. Therefore the same argument as in the proof of Proposition 1 of [T], §1 can be applied to obtain:

7.2 Proposition: *With the assumptions of 7.1 we have:*

$$N\mathfrak{f}(\Lambda)^{-1/2}\tau(\Lambda^\vee, \psi_F) \cdot \mathbf{1} = \begin{cases} \Lambda^\vee(\alpha) \cdot \psi_B(\alpha) & \text{if } 2 \nmid j(\Lambda) \\ \Lambda^\vee(\alpha)\psi_B(\alpha) \cdot NP^{-1/2}\Big[\sum_x \Lambda^\vee(1 + x)\psi_B(\alpha x)\Big] & \text{if } 2|j(\Lambda) \end{cases}$$

where the sum is over $x \in P^{j/2}/P^{j/2+1}$. In the first case $\Lambda^\vee(\alpha)$ has to be a scalar operator because the left hand side is known to be scalar. □

Remark: See section 3 for further notations. By 3.6 we have $W(\Lambda, \psi_F) = (-1)^{N-m}N\mathfrak{f}(\Lambda)^{-1/2} \cdot \tau(\Lambda^\vee, \psi_F)$.

In the case $2|j(\Lambda)$ we write the right hand side of 7.2 as the product of two operators, namely

$$D_1 = \Lambda^\vee(\alpha), \quad D_2 = \psi_B(\alpha) \cdot NP^{-1/2}\Big[\sum_x \Lambda^\vee(1 + x)\psi_B(\alpha x)\Big].$$

Because $D_1 D_2 = \mu\mathbf{1}$ is a scalar operator and because D_1 is invertible, we get

$$tr(D_2) = \mu \cdot tr(D_1^{-1}). \tag{4}$$

Moreover $tr(D_1^{-1}) = \chi_\Lambda(\alpha)$ is the character value of α with respect ot Λ. Now we use $\Lambda = \pi_\alpha \otimes R_{\mathfrak{K}}$, $\pi_\alpha \otimes R_D$ resp., where $\pi_\alpha \in J_\alpha\hat{}$ is an appropriate extension of the Heisenberg representation $\eta_\alpha = \pi_{\alpha 1} \in (J_\alpha^1)\hat{}$ corresponding to the Heisenberg pair $(H_\alpha^1, \theta_\alpha)$, where $\theta_\alpha \in Y_e$, Y_D resp. (See the proof of 2.2 in Part 1 and 2.3, 2.4 above.) According to [H], we have:

7.3 Proposition: *Let $H_\alpha \subseteq J_\alpha$ be the subgroup which is given by $H_\alpha = \mathfrak{K}_E H^1_\alpha$, $H_\alpha = D^*_E H^1_\alpha$ resp., where $E = F(\alpha)$. Then $\pi_\alpha \in J_\alpha^\frown$ has the property:*

$$\pi_\alpha \otimes \overset{\vee}{\pi}_\alpha = Ind_{H_\alpha \uparrow J_\alpha}(1). \tag{5}$$

Later we are going to make some remarks on the proof of 7.3 because our situation is a little different from that in [H]. At the moment we see from (5) that:

$\chi_\alpha(\alpha) \cdot \overset{\vee}{\chi}_\alpha(\alpha) \neq 0$, where χ_α denotes the character of π_α, because $\alpha \in E^* \subseteq H_\alpha$. Further $\Lambda = \pi_\alpha \otimes R$ with $R = R_{\mathfrak{K}}$, R_D resp., and $R(\alpha)$ is a scalar operator because R is irreducible on $J/J^1 \cong H/H^1$ and α is in the center of $\mathfrak{K}_E$, D^*_E resp. Therefore we see that $\chi_\Lambda(\alpha) = \chi_\alpha(\alpha)\chi_R(\alpha) \neq 0$, and we can divide (4) by $tr(D_1^{-1}) = \chi_\Lambda(\alpha)$. Hence:

7.4 Proposition: *If $2|j = j(\Lambda)$ we obtain*

$$N\mathfrak{f}(\Lambda)^{-1/2}\tau(\Lambda^\vee, \psi_F) = \frac{\psi_B(\alpha)}{\chi_\Lambda(\alpha)} \cdot NP^{-1/2}\Big[\sum_x \overline{\chi_\Lambda(1+x)}\psi_B(\alpha x)\Big]$$

where the sum is over $x \in P^{j/2}/P^{j/2+1}$.

Next we note that $\Lambda|_{U^{j/2}}$ is a multiple of the Heisenberg representation $\eta = (U^{j/2}, 1 + B'_a, \text{res}(\theta_\alpha))$. Namely because of $a = \alpha_{-j+1}$ we have

$$B'_a = (P^{j/2} \cap B_a) + P^{j/2+1} = P^{j/2} \cap \mathfrak{d}^\perp_\alpha, \quad 1 + B'_a = U^{j/2} \cap H^1_\alpha,$$

where we have used the notations H^1_α, $\mathfrak{d}_\alpha$ instead of H^1_f, $\mathfrak{d}_f$ in Part 1. Thus we have $\eta(z) = \theta_\alpha(z)1$ for $z \in 1 + B'_a$, and the alternating character

$$\theta_\alpha([1+x, 1+y]) = \psi_B((xy - yx)\alpha)$$

is nondegenerate on $U^{j/2}/1 + B'_a$. The character of η is:

$$\chi_\eta(u) = \begin{cases} 0 & \text{for } u \in U^{j/2} - (1 + B'_a) \\ \dim(\eta) \cdot \theta_\alpha(u) & \text{for } u \in 1 + B'_a. \end{cases}$$

$\Lambda|_{U^{j/2}}$ being a multiple of η, we conclude:

$$\chi_\Lambda(u) = \begin{cases} 0 & \text{for } u \in U^{j/2} - (1 + B'_a) \\ \dim \Lambda \cdot \theta_\alpha(u) & \text{for } u \in 1 + B'_a. \end{cases}$$

Putting this into 7.4, we see:

$$\sum_x \overline{\chi_\Lambda(1+x)}\,\psi_B(\alpha x) = \dim \Lambda \sum_{x \in B'_a/P^{j/2+1}} \overline{\theta_\alpha(1+x)}\,\psi_B(\alpha x). \tag{6}$$

Now we prove:

7.5 Lemma: *If the extension $F(a)|F$ is wildly ramified, then*

$$1+x \longmapsto \overline{\theta_\alpha(1+x)}\,\psi_B(\alpha x)$$

is a character on $1+B'_a/1+P^{j/2+1}$.

Proof: We use the abbreviation $A(1+x) = \overline{\theta_\alpha(1+x)}\,\psi_B(\alpha x)$. Then we find:

$$\frac{A(1+x+y+xy)}{A(1+x)\cdot A(1+y)} = \psi_B(\alpha xy) = \psi_B(axy)$$

because $a = \alpha_{-j+1}$, $xy \in P^j$ and ψ_B has the conductor P. We want to show that $\psi_B(axy) \equiv 1$ for $x,y \in B'_a$, and because of $\psi_B(aP^{j+1}) \equiv 1$ we may assume $x,y \in B_a \cap P^{j/2}$, hence:

$$\psi_B(axy) = \psi_F \circ Tr_{K/F} \circ Trd_{B_a|K}(axy), \tag{7}$$

where $K = F(a)$ is the center of B_a. We compute the conductor

$$\mathfrak{f}_1 := \mathfrak{f}(\psi_F \circ Tr_{K/F} \circ Trd_{B_a/K}) = \mathfrak{D}^{-1}_{\mathfrak{A}\cap B_a/\mathfrak{o}_F} \cdot \mathfrak{f}(\psi_F). \tag{8}$$

In the matrix algebra case $B = A$ we have $\mathfrak{A} = \mathfrak{A}_e$ and $e(\mathfrak{A}\cap B_a \mid \mathfrak{o}_K) = e/e_a$ with $e_a = e_{K/F}$. Therefore

$$(9)_1 \qquad \mathfrak{D}_{\mathfrak{A}\cap B_a/\mathfrak{o}_K} = (P\cap B_a)^{e'-1} \quad \text{with} \quad e' = e/e_a$$

$$(9)_2 \qquad \mathfrak{D}_{\mathfrak{o}_K/\mathfrak{o}_F} = \mathfrak{p}_K^{e_a-1+\delta}.$$

Using $\mathfrak{p}_K(P\cap B_a) = (P\cap B_a)^{e'}$, we see that the exponent of $\mathfrak{D}_{\mathfrak{A}\cap B_a/\mathfrak{o}_F} = (9)_1\cdot(9)_2$ with respect to $P\cap B_a$ is:

$$e' - 1 + e'(e_a - 1 + \delta) = -1 + e + e'\delta.$$

Because of $\mathfrak{f}(\psi_F) = \mathfrak{p}_F = (P\cap B_a)^e$ we see now from (8):

$$\mathfrak{f}_1 = (P\cap B_a)^{1-e'\delta}.$$

In the division algebra case $B = D$, hence $\mathfrak{A} = \mathfrak{O}_D$, a similar computation yields:

$$\mathfrak{f}_1 = (P\cap B_a)^{1-N'\delta}, \quad \text{where} \quad N' = N/[K:F].$$

Now from $(9)_2$ we conclude $\delta > 0, \in \mathbb{Z}$, because $K = F(a)$ is wildly ramified over F. Hence

$$\mathfrak{f}_1 \supset \mathfrak{A}\cap B_a,\ \mathfrak{O}_D \cap B_a \quad \text{resp.},$$

and from $\nu_P(axy) \geq 0$ for $x,y \in B_a \cap P^{j/2}$, we see that $\psi_B(axy) \equiv 1$. □

Because of 7.5, the sum on the right hand side of (6) is over the values of a character of the abelian group $1 + B'_a/1 + P^{j/2+1} \cong B'_a/P^{j/2+1}$. Hence either

$$(10) \qquad \overline{\theta_\alpha(1+x)} \cdot \psi_B(\alpha x) \equiv 1 \quad \text{for} \quad x \in B'_a$$

or the sum vanishes, which is impossible because $\tau(\Lambda^\vee, \psi_F) \neq 0$. (See [BFr] (2.3.8).) Therefore (10) applies, and from 7.4 and (6) we conclude 7.1 (iii).

We are left with the case $2 \mid j = j(\Lambda)$ and $F(a) \mid F$ tamely ramified which is 7.1 (ii). In that case we use Part 1, 2.3.1, 2.4.1. Namely $\nu_P(\alpha - a) \geq -j + 1$ implies:

$$(11) \qquad (\theta_\alpha \cdot \theta_a^{-1})(1+x) = \psi_B((\alpha - a)x) \quad \text{for} \quad 1 + x \in H^1_\alpha \cap U^{[j\frac{-1}{2}]+1}.$$

But because of $2 \mid j$ the intersection is nothing else than $1 + B'_a$. Therefore using (11), (6) turns into

$$(12) \qquad \sum_x \overline{\chi_\Lambda(1+x)}\, \psi_B(\alpha x) = \dim \Lambda \sum_{x \in B'_a/P^{j/2+1}} \overline{\theta_a(1+x)}\, \psi_B(ax).$$

For the right hand side we can take $x \in B_a \cap P^{j/2}/B_a \cap P^{j/2+1}$. And from Part 1, 2.3.2, 2.4.2 with a instead of α_f and with $v = 0$ we see:

$$\theta_a(1+x) = \lambda_a \circ Nrd_{B_a|K}(1+x) \quad \text{on} \quad U^1 \cap B_a,$$

where $K = F(a)$ and $\lambda_a \in X_e$, X_D resp. With the notation $\tilde{\lambda}_a = \lambda_a \circ Nrd_{B_a|K}$, (12) turns into:

$$(13) \qquad \sum_x \overline{\chi_\Lambda(1+x)}\, \psi_B(\alpha x) = \dim \Lambda \cdot \sum_x \overline{\tilde{\lambda}_a(1+x)}\, \psi_B(ax),$$

where the right hand sum S is over $x \in P^{j/2} \cap B_a/P^{j/2+1} \cap B_a$. Now we can apply [BFr] (2.8.13) (ii) for the central simple algebra $B_a|K$, where the group G is the normalizer of the principal order $\mathfrak{A} \cap B_a$. This gives us:

$$(14) \qquad W(\tilde{\lambda}_a, \psi_K) = W(\lambda_a, \psi_K)^{N_a}, \quad \text{where} \quad N_a = N/[K:F],\ \psi_K = \psi_F \circ Tr_{K|F},$$

for every extension λ_a from U^1_K onto K^*. Note that $\lambda_a(U^1_K) \not\equiv 1$, because $\nu_P(a) = \nu_P(\alpha) = -j < 0$.

Further, from 7.2 with $B_a|K$ instead of $B|F$ and $\tilde{\lambda}_a$ instead of Λ, we see:

$$(15) \qquad W(\tilde{\lambda}_a, \psi_K) = (-1)^{N_a - m_a} \tilde{\lambda}_a^{-1}(a) \psi_B(a) N(P \cap B_a)^{-1/2} \cdot S,$$

where S is precisely the sum from (13). Therefore, introducing the notation

$$(16) \qquad \delta(\lambda_a) := W(\lambda_a, \psi_K) \cdot \lambda_a(a) \cdot \psi_K(-a)$$

from (14), (15) we obtain:

$$(17) \qquad S = (-1)^{N_a - m_a} \delta(\lambda_a)^{N_a} \cdot M^{1/2}, \quad \text{where} \quad m_a = 1, N_a \quad \text{if} \quad B = D, A \quad \text{resp.},$$

and 3.6, 7.4, (13), (17) imply 7.1 (ii). □

7.6 Remarks: *1.* Note that $\delta(\lambda_a)$ is independent from the extension of λ_a onto K^*. More precisely, the argument of [T] loc. cit. yields:

$$(18)\quad \delta(\lambda_a) = \begin{cases} 1 & \text{if } P^{j/2} \cap K = P^{j/2+1} \cap K \\ N(\mathfrak{p}_K)^{-1/2} \sum_{x \in P^{j/2} \cap K / P^{j/2+1}} \overline{\lambda_a(1+x)}\, \psi_K(ax), & \text{otherwise.} \end{cases}$$

In the case $a \in F$ we have $B'_a = P^{j/2}$ and the Heisenberg representation η is simply a character of $U^{j/2}$. Hence $\chi_\Lambda|_{U^{j/2}}$ is a multiple of θ_α, and we see that D_2 is a scalar operator, hence D_1 too.

2. A flaw of 7.1 is that $j = j(\Lambda)$ is not invariant. If we start from $t = [R, \beta] \in \mathcal{T}_N^-$ with inertial degree f_t, then in the division algebra case we have $j(\Lambda_t) = N \cdot \nu_F(\beta)$, but in the matrix algebra case we have $j(\Lambda_t) = \frac{N}{f_t} \cdot \nu_F(\beta)$.

Remarks on the proof of 7.3: There is an easy reduction to the case where J is a finite group. Therefore we may assume:

(i) $J = J^1 \cdot H$ is a finite group and $J^1 \subset J$ is normal.

(ii) $H^1 = J^1 \cap H$ is normal in J and $J/H^1 = J^1/H^1 \rtimes H/H^1$, where $V = J^1/H^1$ is an $\mathbb{F}_p$-vectorspace.

(iii) $\pi \in J^\wedge$ has the property that $\pi_{J^1} = (H^1, \theta)$ is a Heisenberg representation of J^1, i.e. θ applied to the commutator of J^1 is a non-degenerate alternating character X on V which is invariant with respect to the action of $J/J^1 \cong H/H^1$ by conjugation.

We want to prove that (i)–(iii) imply $\pi \otimes \check{\pi} = \mathrm{Ind}_{H \uparrow J}(1)$. The proof consists of two steps:

Step 1: The character χ_π vanishes on conjugacy classes of J which do not intersect H.

Step 2: The assertion follows if χ_π has the property proved in step 1.

Step 2 is explained in [H] at the end of section I (after Proposition 2). The argument with minor changes applies to our case too. Therefore we restrict to Step 1.

We start from the following remark of [H]:

7.7 Lemma: *$a \cdot h \in J^1 \cdot H$ is in a conjugacy class of J which intersects H, iff $\bar{a} = v - \bar{h}(v)$ for some $v \in V$.*

We will write the operation on V additively and the conjugation we will interpret as an $\mathbb{F}_p$-endomorphism on V.

Because of 7.7 we can restrict our considerations to the case $J = J^1 \cdot \langle h \rangle$, i.e. $H/H^1 = \langle \bar{h} \rangle$ is a cyclic group. Then we have to prove:

$$(19)\qquad a \cdot h \in J^1 \cdot \langle h \rangle \quad \text{with} \quad \bar{a} \in V - (1-h) \circ V \quad \text{implies} \quad \chi_\pi(ah) = 0.$$

We consider

$$(20)\qquad \langle g = \bar{h} \rangle \longrightarrow Sp_X(V),$$

where X is as in (iii) and $\mathrm{Sp}_X(V)$ is the corresponding symplectic group. Let g^r generate the kernel of (20). Then we may replace J^1, H^1 by $J^1 \langle h^r \rangle$, $H^1 \cdot \langle h^r \rangle$ resp. and therefore come down to the case where (20) is injective. Thus we may consider g as an element of $\mathrm{Sp}_X(V)$.

Write $V = V_1 + V_2$ where V_1 is the 1-eigenspace of g (that is, $g-1$ is unipotent on V_1) and $g-1$ is non-singular on V_2. Then on V_1 we have $(g-1)^{p^b} = g^{p^b} - 1 = 0$ for some b. And $V = V_1 + V_2$ is an orthogonal direct sum with respect to X (see [H]). We may assume

$$\bar{a} = v_1 + v_2 \quad \text{with} \quad v_1 \in V - (1-g) \circ V_1, \tag{21}$$

because otherwise $\bar{a} \in (1-g) \circ V_1 + V_2 = (1-g) \circ V$ which contradicts to our assumption (19). Let $g = s \cdot u$ be the decomposition of g into its regular and unipotent component. Then:

$$s|_{V_1} = id_{V_1}, \quad s(V_i) = V_i, \quad u(V_i) = V_i \quad \text{for} \quad i = 1,2.$$

Therefore:

$$\bar{J} = V \rtimes \langle g \rangle = V_2 \cdot s \rtimes V_1 \cdot u,$$

where $V_2 \cdot s$ (and $V_1 \cdot u$ resp.) denotes the subgroup which is generated by V_2 and s (by V_1 and u resp.). Let $Y \subseteq Y_s \subseteq J$ be the preimage of $V_2 \subseteq V_2 \cdot s \subseteq \bar{J}$ resp., and let $\rho \in Y_s\widehat{\ }$ be an extension of the Heisenberg representation $\eta = (H^1, \theta)$ of Y. Note that θ applied to the commutator of Y gives the non-degenerate alternating character X on $V_2 = Y/H^1$. And η extends to Y_s because it is s-invariant. Moreover we have the p-group $V_1 \cdot u$ acting on the set of extensions ρ of η. But the number of such extensions is prime to p, hence we find $\rho \in Y_s\widehat{\ }$ which is $V_1 \cdot u$-invariant. Therefore:

$$\pi = \tilde{\rho} \otimes \mu \tag{22}$$

where $\tilde{\rho}$ is an extension of ρ to a projective representation of J and μ is a projective representation of $J/Y_s \cong V_1 \cdot u$. Now let $[V_1 \cdot u]$ be a central extension of $V_1 \cdot u$ such that μ lifts to an ordinary representation $\tilde{\mu}$ of $[V_1 \cdot u]$. Then, from (21) and $\bar{h} = g = s \cdot u$ we see that $a \cdot h \equiv v_1 u \bmod Y_s$, hence:

$$\chi_{\tilde{\mu}}(v_1 u) = 0 \quad \text{implies} \quad \chi_\pi(a \cdot h) = 0, \tag{23}$$

because from (22) we conclude $\chi_\pi \cdot \overline{\chi_\pi} = \chi_{\tilde{\rho} \otimes \tilde{\rho}^\vee} \cdot \chi_{\mu \otimes \check{\mu}}$, and $\chi_{\mu \otimes \check{\mu}} = \chi_{\tilde{\mu}} \cdot \overline{\chi_{\tilde{\mu}}}$. (Note that $\tilde{\rho} \otimes \tilde{\rho}^\vee$ and $\mu \otimes \check{\mu}$ are ordinary representations.)

Therefore we can reduce to the case $J = [V_1 \cdot u]$ and $\pi = \tilde{\mu}$, i.e. we assume that $V = V_1$ and that H/H^1 is a cyclic p-group with generator $g = u$. We consider the subspace V^g of g-invariant vectors. With respect to X we have $(V^g)^\perp = (g-1) \circ V$, and we let $W \subset V^g$ be any complement of $V^g \cap (g-1) \circ V$. Then:

$V = W + W^\perp$ is an orthogonal direct sum with respect to X, and

$\bar{a} = w + w' \in V - (1-g) \circ V$.

Case 1: Assume $w \neq 0$. This ensures $\bar{a} \notin (1-g) \circ V$ because $W \subset V^g$ implies $W^\perp \supset (1-g) \circ V$. Let $Y \subset J$ be the preimage of the normal subgroup $W^\perp \subset J$. (H^1, θ) determines a Heisenberg representation η of Y because X is non-degenerate on $W^\perp$. And

η is $W \cdot g$-invariant. Hence $\pi = \rho \otimes \mu$, where ρ is a projective representation of J extending η and μ is a projective representation of $J/Y \cong W \cdot g$. As before, let $\tilde{\mu}$ be an ordinary representation of an appropriate central extension $[W \cdot g]$ of $W \cdot g$, which lifts μ. Again, we conclude that $\chi_{\tilde{\mu}}(w \cdot g) = 0$ implies $\chi_\pi(ah) = 0$ because $ah \equiv w \cdot g \bmod Y$. But $\tilde{\mu}$ is a Heisenberg representation of $[W \cdot g]$ with center $[g]$. Hence $w \neq 0$ implies $\chi_{\tilde{\mu}}(w \cdot g) = 0$.

Case 2: $\bar{a} = w' \in W^\perp - (g-1) \circ V$. Let $Y \subset J$ be the preimage of the normal subgroup $W \subset \bar{J}$ and let $\eta \in Y^\wedge$ be the Heisenberg representation which corresponds to (H^1, θ). η is $W^\perp \cdot g$-invariant, hence

$$\pi = \rho \otimes \mu,$$

where ρ is a projective representation of J extending η and μ is a projective representation of $J/Y = W^\perp \cdot g$. And it is enough to show $\chi_{\tilde{\mu}}(w' \cdot g) = 0$ for the representation $\tilde{\mu}$ of $[W^\perp \cdot g]$. Therefore we can reduce to the case $J = [W^\perp \cdot g]$, $V = W^\perp$, $\pi = \tilde{\mu}$, hence $V^g \subset (g-1) \circ V$ and $\bar{a} = v \in V - (g-1) \circ V$. Then we have:

- $V^g = ((g-1) \circ V)^\perp$ is normal in $\bar{J}$ and isotropic in V.
- Let $Y \subset J$ be the preimage of V^g. Then $\pi|_Y$ is a multiple m of a character orbit of length $|V^g|$, hence $m = \dim \pi / |V^g|$.
- If $\chi: Y \to \mathbb{C}^*$ is from that character orbit, then its normalizer $N_J(\chi)$ has the property $N_J(\chi)/H^1 = (V^g)^\perp \cdot g = ((g-1) \circ V) \cdot \langle g \rangle$.

Hence $N_J(\chi) \subset J$ is a normal subgroup and π is induced from $N_J(\chi)$. But $ah \mapsto v \cdot g \in \bar{J}$ is not in $N_J(\chi)/H^1$, because $v \notin (g-1) \circ V$. Hence $\chi_\pi(ah) = 0$. $\square$

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