HOMOLOGY OF IWAHORI-HECKE ALGEBRAS

PAUL BAUM AND VICTOR NISTOR

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Abstract. We determine the periodic cyclic homology of the Iwahori-Hecke algebras H_q, for $q \in \mathbb{C}^*$ not a "proper root of unity." (In this paper, by a *proper root of unity* we shall mean a root of unity other than 1.) Our method is based on a general result on periodic cyclic homology, which states that a "weakly spectrum preserving" morphism of finite type algebras induces an isomorphism in periodic cyclic homology. The concept of a weakly spectrum preserving morphism is defined in this paper, and most of our work is devoted to understanding this class of morphisms. Results of Kazhdan–Lusztig and Lusztig show that, for the indicated values of q, there exists a weakly spectrum preserving morphism $\phi_q : H_q \to J$, to a fixed finite type algebra J. This proves that ϕ_q induces an isomorphism in periodic cyclic homology and, in particular, that all algebras H_q have the same periodic cyclic homology, for the indicated values of q. The periodic cyclic homology groups of the algebra H_1 can then be determined directly, using results of Karoubi and Burghelea, because it is the group algebra of an extended affine Weyl group.

Contents

Introduction

Let G be the set of rational points of a reductive group defined over a p-adic field $\mathbb{F}$. Fix an Iwahori subgroup $I \subset G$. The space of complex, compactly supported functions on G that are right and left invariant with respect to I, with the convolution product, is called the *Iwahori-Hecke algebra of* G. Because the Iwahori subgroups of G are all conjugate, the isomorphism class of the Iwahori-Hecke algebra of G is independent of the choice of the Iwahori subgroup I.

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The Iwahori-Hecke algebras H_q were introduced in [25] by Iwahori and Matsumoto, who also provided in that paper a description of these algebras using generators and relations. Let $\widehat{W}$ be the extended affine Weyl group of G. As a vector space $H_q \cong \mathbb{C}[\widehat{W}]$, which provides us with a distinguished system of generators of H_q. We shall denote the elements of this system of generators by T_w, $w \in \widehat{W}$. The relations among the generators T_w of the Iwahori-Hecke algebra H_q, depend on the parameter q. By allowing q to be an arbitrary non-zero complex number, one obtains in this way a family of algebras H_q, which specialize to the Iwahori-Hecke algebra of G when q is the number of elements of the residue field of $\mathbb{F}$. (See Section 5, where we review these definitions and constructions in detail.)

Iwahori-Hecke algebras play a central role in the representation theory of p-adic group. For example, the classification of the irreducible, smooth representations of G with non-zero vectors fixed by I reduces to the classification of the irreducible representations of the Iwahori-Hecke algebra associated to G (see [8]). Other representations of G can also be classified in terms of Iwahori-Hecke algebras. Indeed, if $G = GL_n(\mathbb{F})$, the general linear group over the p-adic field $\mathbb{F}$, all irreducible, smooth representations of G are classified by representations of Iwahori-Hecke algebras [14].

The representations of Iwahori-Hecke algebras (for suitable values of q and suitable groups G) were classified by Kazhdan and Lusztig in a ground breaking paper, [28], using the K-homology of various varieties of Borel subgroups of the Langlands dual of G. The group G does not appear in the Kazhdan-Lusztig classification, which is based on the algebraic description of the Iwahori-Hecke algebras in terms of generators and relations. However, the study of the representation theory of G was certainly their main motivation.

In this paper, by a *proper root of unity* we shall mean a root of unity other than 1. Our work gives a complete determination of the periodic cyclic homology of the Iwahori-Hecke algebras H_q, provided that $q \in \mathbb{C}^*$ is not a proper root of unity. The first step of our computation is to show that the periodic cyclic homology groups $\mathrm{HP}_*(H_q)$ are naturally isomorphic for all these values of q. Because for $q = 1$ the corresponding Iwahori-Hecke algebra is the group algebra of an extended affine Weyl group, we can use the results of Karoubi [26] and Burghelea [13] to determine the groups $\mathrm{HP}_*(H_1)$ directly, which leads to a complete and natural determination of the groups $\mathrm{HP}_*(H_q)$, for $q \in \mathbb{C}^*$ not a root of unity. (The approach in [3] is in fact more convenient for the calculation, and we plan to discuss this in [4].) An important concept that facilitates our calculations is that of finite type algebra, see Definition 1. Actually, a crucial step in our calculations is the extension of a technical result on finite type algebras from [29] (see Theorem 6).

The idea of the proof is to study a certain class of morphisms of finite type algebras that we introduce in this paper, namely the class of "weakly spectrum preserving morphisms," Definition 4. The relevance of this class is two-fold. First, a spectrum preserving morphism defines an isomorphism in periodic cyclic homology (this is the content of Theorem 7), which is one of the main results of this paper. Second, Lusztig has constructed an algebra J, independent of q, and morphisms

$$\phi_q : H_q \to J, \tag{1}$$

which he used to classify the irreducible representations of H_q in terms of the irreducible representations of J. One of the main steps in this classification is a result, which, in our language, amounts to saying that ϕ_q is weakly spectrum preserving for all values of $q \in \mathbb{C}^*$ that are not proper roots of unity. See [28, 33, 34, 35, 36], and especially [37]. Thus, by Theorem 7,

$$(\phi_q)_* : \mathrm{HP}_*(H_q) \to \mathrm{HP}_*(J) \tag{2}$$

is an isomorphism, provided that $q \in \mathbb{C}^*$ is not a proper root of unity. Consequently, all groups $\mathrm{HP}_*(H_q)$ are isomorphic for our values of q, as mentioned also above. Moreover, these isomorphisms are natural.

The group G will not appear in the statements or the proofs of our results. Nevertheless, as in the Kazhdan-Lusztig theory, our main motivation is to study the representation theory of G. Let us briefly explain how our results could be related to the representation theory of G. Let $\mathcal{H}(G)$ be the (full) Hecke algebra of G, *i.e.,* the algebra of compactly supported, locally constant functions on G with the convolution product being defined using a fixed Haar measure. Then the groups $\mathrm{HP}_*(\mathcal{H}(G))$ have two different descriptions: one using the geometric (and algebraic) structure of G and another one using the representation theory of G.

The first, algebraic, description of the groups $\mathrm{HP}_*(\mathcal{H}(G))$ was obtained independently in [23, 46] using the action of G on a building, and thus not relying on the representation theory of G. An even more precise, geometric description of $\mathrm{HP}_*(\mathcal{H}(G))$ in terms of certain data that carry arithmetic information was obtained in [40]. This description is in terms of conjugacy classes of elements of G and the continuous cohomology of the stabilizers of the semi-simple conjugacy classes.

Let $\widehat{G}$ be the admissible dual of G, that is, the space of smooth, irreducible representations of G with the Jacobson topology. In [29], the groups $\mathrm{HP}_*(\mathcal{H}(G))$ were related to the de Rham cohomology of the strata of $\widehat{G}$ via a spectral sequence. This provides a representation theoretic, or spectral, description of $\mathrm{HP}_*(\mathcal{H}(G))$. Relating the geometric and spectral descriptions of the periodic cyclic cohomology groups of $\mathcal{H}(G)$ will thus translate into concrete relations between the geometric and spectral properties of G.

Let D be a connected (or Bernstein) component of $\widehat{G}$. As a topological space, each Bernstein component D is homeomorphic to the primitive ideal spectrum of a finite type algebra A_D, canonically associated to D, and hence we have the following disjoint union decomposition of $\widehat{G}$:

$$\widehat{G} = \cup_D \operatorname{Prim}(A_D). \tag{3}$$

The corresponding relation in periodic cyclic homology is that

$$\mathrm{HP}_*(\mathcal{H}(G)) \cong \oplus_D \mathrm{HP}_*(A_D), \tag{4}$$

which requires us however to use also the fact that the Hochschild homology of the full Hecke algebra of G vanishes above the split rank of G (a property not enjoyed by arbitrary finite type algebras).

One can formulate then a strategy of relating $\mathrm{HP}_*(\mathcal{H}(G))$ to the topology of $\widehat{G}$ as follows. First, using (4), we reduce the computation of $\mathrm{HP}_*(\mathcal{H}(G))$ to the computation of $\mathrm{HP}_*(A_D)$. Next, the periodic cyclic homology groups of the finite type algebras A_D are related to the topology of their primitive ideal spectrum by the results of [29] and of this paper. Then, finally, we determine the topology of $\widehat{G}$ from that of the individual Bernstein components using (3). For this strategy to be effective, we need a determination of the groups $\mathrm{HP}_*(A_D)$ that is as explicit as possible. Our results provide complete results for the Bernstein components whose commuting algebras are Iwahori-Hecke algebras (or tensor products of such algebras).

Another motivation to study the periodic cyclic homology of Iwahori-Hecke algebras is that, as explained by Baum, Higson, and Plymen in [6], the periodic cyclic homology of Iwahori-Hecke can be used to prove the Baum-Connes conjecture for $GL_n(\mathbb{F})$. (See [5] for a statement of the Baum-Connes conjecture for p-adic groups). See also [7]. It would be interesting to relate our approach to the computation of the periodic cyclic homology of Hecke algebras with the description of Iwahori-Hecke algebras in [21]. The implicit deformation idea used in this paper fits also with the philosophy in Manin's book [38].

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1. Finite type algebras

In this section, we recall some facts about finite type algebras and a prove a few needed properties of finite type algebras. We also use this opportunity to fix notation.

In what follows, $Z(\mathfrak{A})$ will denote the center of an algebra $\mathfrak{A}$. An algebra with unit will be called a *unital* algebra. By $\mathbf{k}$ we shall denote a commutative noetherian ring (with unit), which in most cases will be a finitely generated commutative algebra. By a $\mathbf{k}$–algebra (without unit in general), we shall mean an algebra A which is a $\mathbf{k}$–module such that its algebraic operations are $\mathbf{k}$-linear. If A is unital, this is the same as saying that there exists a unital morphism $\mathbf{k} \to Z(A)$.

For any $\mathbf{k}$-algebra A, we shall denote by $A^+ := A \oplus \mathbf{k}$, the algebra obtained from A by adjoining a (possibly new) unit. The multiplication is given by the straight-forward formula:

$$(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu).$$

A $\mathbf{k}$-linear morphism $A \to B$ of two $\mathbf{k}$-algebras extends in an obvious way to a unital $\mathbf{k}$-linear morphism $A^+ \to B^+$ of the algebras with adjoined unit.

Definition 1. *Let $\mathbf{k} \cong \mathbb{C}[X_1, \dots, X_r]/I$ be a finitely generated complex ring. A* finite type $\mathbf{k}$–algebra *is a (not necessarily unital) $\mathbf{k}$–algebra A that is a finitely generated $\mathbf{k}$–module. A* finite type algebra *is an algebra that is a finite type $\mathbf{k}$–algebra for some $\mathbf{k}$.*

Thus while in this paper $\mathbf{k}$ may occasionally denote more general commutative rings, when we are talking of a finite type $\mathbf{k}$-algebra, we always require $\mathbf{k}$ to be a finitely generated complex ring.

Recall that a primitive ideal of A is a two-sided ideal $\mathfrak{P} \subset A$ that is the kernel of a *non-zero* irreducible complex representation of A (so $\mathfrak{P} \neq A$). Denote by $\mathrm{Prim}(A)$ the primitive ideal spectrum of A endowed with the Jacobson topology. Recall that the closed sets of the Jacobson topology are the sets of the form $V(S_0)$,

$$V(S_0) := \{\mathfrak{P} \in \mathrm{Prim}(A),\ S_0 \subset \mathfrak{P}\},$$

$S_0 \subset A$ arbitrary. Clearly, $V(I) = \mathrm{Prim}(A/I)$, for any two-sided ideal I of A. Also, recall that the *Jacobson radical* of A, $\mathrm{Jac}(A)$, is the intersection of all primitive ideals of A. If $\mathrm{Jac}(A) = 0$, we say that A is *semi-primitive.*

We shall use several times the following well known result (see [45] or [29], Lemma 1).

Lemma 1. *Let A be a unital finite type algebra with center $Z = Z(A)$, and let $\mathfrak{P}$ be a primitive ideal of A, then $\Theta(\mathfrak{P}) := \mathfrak{P} \cap Z$ is a maximal ideal of Z and $A/\mathfrak{P} \cong M_k(\mathbb{C})$ for some k. If A is generated by q elements as a Z-module, then $k^2 \leq q$.*

Proof. Because Z has an at most countable dimension as a complex vector space, it follows that A and any simple left A–module V also have an at most countable $\mathbb{C}$-dimension. The center Z_0 of $\mathrm{End}_A(V)$ is a field extension of $\mathbb{C}$, because V is irreducible. Since the only field extension of finite or countable dimension of $\mathbb{C}$ is $\mathbb{C}$ itself, we obtain that $Z \to Z_0$ is surjective, and hence $\mathfrak{P} \cap Z$ is a maximal ideal of Z. Moreover, $Z/Z \cap \mathfrak{P} \cong \mathbb{C}$, (Hilbert's Nullstellensatz).

We obtain then that, $A/\mathfrak{P} \cong M_n(\mathbb{C})$, because all centrally simple complex algebras are matrix algebras. If A is generated as a Z–module by q elements, then $A/\mathfrak{P}$ is generated by q elements as a $Z/Z \cap \mathfrak{P}$-module, so $n \leq q^2$. ☐

See also [35] and [44].

Let A be a finite type $\mathbf{k}$-algebra. If $\mathfrak{P} \subset A$ is the kernel of an irreducible representation (π, V_π), then the isomorphism class of π and the dimension of the space V_π on which it acts are uniquely determined. This explains why studying the set $\mathrm{Prim}(A)$ of primitive ideals of A is the same thing as studying the irreducible representations of A.

We shall denote by $d_{\mathfrak{P}} := \dim V_\pi$ the dimension of the space V_π on which π acts and by $\mathrm{Prim}_n(A)$ the subset of $\mathrm{Prim}(A)$ consisting of those primitive ideals $\mathfrak{P}$ for which $d_{\mathfrak{P}} = n$. We know that $d_{\mathfrak{P}}$ is not greater than the number of generators of A as a $\mathbf{k}$–module, for all $\mathfrak{P} \in \mathrm{Prim}(A)$, and hence $\mathrm{Prim}_n(A)$ is empty if n is greater than a fixed number N that depends on A.

We shall denote by $\mathrm{Max}(\mathbf{k})$ the maximal ideal spectrum of a commutative ring $\mathbf{k}$ ($\mathrm{Max}(\mathbf{k}) = \mathrm{Prim}(\mathbf{k})$ if $\mathbf{k}$ is finitely generated). Let A be a finite type $\mathbf{k}$-algebra. Then Lemma 1 gives rise to a map

$$\Theta : \mathrm{Prim}(A) \ni \mathfrak{P} \longrightarrow \mathfrak{p} = Z(A) \cap \mathfrak{P} \in \mathrm{Max}(Z(A)) = \mathrm{Prim}(Z(A)), \tag{5}$$

called the *central character*. We can then define in a similar way a map

$$\Theta_A : \mathrm{Prim}(A) \ni \mathfrak{P} \longrightarrow \mathfrak{p} := \sqrt{\mathbf{k} \cap \mathfrak{P}} = \mathbf{k} \cap \mathfrak{P} \in \mathrm{Max}(\mathbf{k}), \tag{6}$$

also referred to as central character. The central character Θ_A map satisfies

$$\Theta_A^{-1}(V(I)) = V(IA),$$

for any ideal $I \subset \mathbf{k}$, and hence it is continuous and surjective (see [29], Lemma 1).

Every irreducible representation π of a finite type $\mathbf{k}$-algebra A extends uniquely to an irreducible representation π^+ of A^+. If $\mathfrak{P}$ is the kernel of π, then $a \in \mathbf{k}$ acts on the space of π via the scalar $\pi^+(a)$. In particular, this provides us with a way of defining the central character for non-unital algebras.

We now discuss the important and well known result on the nilpotency of the Jacobson radical of finite type algebras. We shall use this result several times in what follows, so we include also a proof, which is simpler in our setting of finite type algebras then in the most general case. See [11, 30] and [43]. The ideas of our proof will be useful again later on. A different proof of this result for finite type algebras was also given in [29] (note however that in that proof one has to first replace A with $\mathrm{Jac}(A)^+$, which is always possible).

If $\mathfrak{p} \subset \mathbf{k}$ is a prime ideal and M is a $\mathbf{k}$-module, then we shall denote by $M_\mathfrak{p}$, as it is customary, the *localization of* M *at* $\mathfrak{p}$, that is,

$$M_\mathfrak{p} = S^{-1}M,$$

where $S = \mathbf{k} \setminus \mathfrak{p}$. Recall then that the support of a $\mathbf{k}$-module M (a subset of $\mathrm{Max}(\mathbf{k})$) is defined as the set of maximal ideals $\mathfrak{p} \subset \mathbf{k}$ such that the localization $M_\mathfrak{p} \neq 0$. Also, recall that the dimension of $\mathbf{k}$ is the maximum length of chains of prime ideals in $\mathbf{k}$, see [2], Theorem 11.14, and the definitions before that theorem. (A chain consisting of $d+1$ distinct ideals has length d. Non-proper ideals, that is (0) and $\mathbf{k}$ are also allowed in such a chain.)

Lemma 2. *Let A be a finite type $\mathbf{k}$ algebra. Then $\mathrm{Jac}(A)$ is nilpotent. In particular, if $J_1 \subset J_2$ are two–sided ideals of A such that $V(J_1) = V(J_2)$, then J_2/J_1 is nilpotent.*

Proof. The result is well known if A is commutative (it is Hilbert's Nullstellensatz, see [2, 9]). We begin with two remarks.

Let $a \in A$ be an arbitrary element. Assume that a is not nilpotent, then there exists a maximal ideal $\mathfrak{m} \subset \mathbf{k}[a]$, $a \notin \mathfrak{m}$. We shall denote by $\chi : \mathbf{k}[a] \to \mathbb{C}$ the unique morphism with kernel $\mathfrak{m}$. Thus $\chi(a) \neq 0$. Then the left A-module

$$V := A \otimes_{\mathbf{k}[a]} (\mathbf{k}[a]/\mathfrak{m})$$

is a finite dimensional complex vector space and hence the vector $\xi = 1 \otimes 1 \in V$ is contained in an irreducible left A-submodule $W \subset V$. By construction, we have that

$$a(1 \otimes 1) = a \otimes 1 = 1 \otimes a = \chi(a) 1 \otimes 1,$$

so the element a will not act on W by a nilpotent endomorphism. Therefore a is not in the Jacobson radical of A. This shows that $\mathrm{Jac}(A)$ consists entirely of nilpotent elements.

Let $\mathbb{K}$ be the ring of fractions of $\mathbf{k}$ (the ring of fractions associated to the multiplicative subset of elements that are not zero-divisors). Denote by $Tor(A) \subset A$ the set of torsion elements of A, that is, the set of elements $a \in A$ such that there exists $f \in \mathbf{k}$, f not a divisor of zero, but $fa = 0$. By definition, $Tor(A)$ is the kernel of the map $A \to A \otimes_\mathbf{k} \mathbb{K}$, and hence $Tor(A)$ is a two-sided ideal of A and a $\mathbf{k}$-submodule. Because $A \otimes_\mathbf{k} \mathbb{K}$ is a finite dimensional algebra over a field, $\mathrm{Jac}(A \otimes_\mathbf{k} \mathbb{K})$ is nilpotent (Engel's theorem, [20]). Because $\mathrm{Jac}(A) \otimes_\mathbf{k} \mathbb{K}$ consists of nilpotent elements, we also have that $\mathrm{Jac}(A) \otimes_\mathbf{k} \mathbb{K} \subset \mathrm{Jac}(A \otimes_\mathbf{k} \mathbb{K})$, and hence $\mathrm{Jac}(A)^N \subset Tor(A)$, for some large N.

We shall prove our result by induction on the dimension of the support of A as a $\mathbf{k}$-module. By replacing $\mathbf{k}$ with a quotient, we can assume that the support of A is the whole of $\mathrm{Max}(\mathbf{k})$. When the dimension of $\mathrm{Max}(\mathbf{k})$ is zero, $\mathbf{k}$ is actually finitely dimensional, and the result again follows from Engel's theorem.

Because $\mathbf{k}$ is noetherian and A is finitely generated, the submodule $Tor(A)$ is also finitely generated. Hence we can find $f \in \mathbf{k}$, f not a zero divisor, such that $fTor(A) = 0$. Consequently, by a standard result (see [2], Corollary 11.9), the support of $Tor(A)$ has smaller dimension than the support of A. But

$$\mathrm{Jac}(A) \cap Tor(A) \subset \mathrm{Jac}(Tor(A)),$$

and hence $\mathrm{Jac}(A)^N \subset \mathrm{Jac}(Tor(A))$. This shows that our induction hypothesis can be applied to $Tor(A)$.

The last part follows because J_2/J_1 is contained in the Jacobson radical of the finite type algebra A/J_1. $\square$

We shall need a few simple facts from the theory of PI-algebras. Recall [45] that a *polynomial identity algebra* (or, simply, a *PI–algebra*) is an algebra A for which there exists a non-zero polynomial $P(X_1, X_2, \dots, X_m)$ in the non-commuting variables $X_1, X_2, \dots, X_m$, without constant term, such that

$$P(a_1, a_2, \dots, a_m) = 0,$$

for any $a_1, a_2, \dots, a_m \in A$. The polynomial P is called an *identity* of A. A finite type $\mathbf{k}$–algebra A satisfies an alternating multilinear identity of degree $q+1$, where q is the number of elements in a system of generators of A as a $\mathbf{k}$–module, so finite type algebras are PI-algebras.

The *PI–class* of a finite type algebra A is, by definition, the smallest integer n such that any identity of $M_n(\mathbb{C})$ is also an identity of A. The PI–class of A has representation theoretic significance because the largest integer n such that $\mathrm{Prim}_n(A)$ is not empty coincides with the PI–class of $A/\mathrm{Jac}(A)$.

We shall need a few basic results involving Azumaya algebras, whose definition we recall below.

Definition 2. *An* Azumaya algebra [of rank r] *over the commutative algebra Z is a unital Z–algebra A which is a finitely generated projective module* [of rank r] *over Z such that*

$$A \otimes_Z A^{\mathrm{op}} \cong \mathrm{End}_Z(A).$$

It follows that Z coincides with the center of A. In general, the rank of A can be different on different connected components of the maximal ideal spectrum of its center. Assume that A has rank r. The dimension $[A_{\mathfrak{p}} : Z_{\mathfrak{p}}]$ of the localization $A_{\mathfrak{p}} = (Z \setminus \mathfrak{p})^{-1}A$ is equal to r, for all maximal ideals $\mathfrak{p} \subset A$, and $r = n^2$, where n is the PI–class of A. Moreover, the central character defines a homeomorphism $\Theta : \mathrm{Prim}(A) \to \mathrm{Prim}(Z)$, see [41]. Azumaya algebras will play an important rôle in what follows, because of the following basic result.

Theorem 1 (Artin-Procesi). *Suppose that $\mathfrak{A}$ is a unital PI–algebra of PI–class n and that* $\mathrm{Prim}(\mathfrak{A}) = \mathrm{Prim}_n(\mathfrak{A})$*, then $\mathfrak{A}$ is an Azumaya algebra of rank n^2.*

See [45, 1.8.48] or the original papers [1, 42] for a proof.

We shall now prove two lemmas on finite type algebras. We shall assume in the two lemmas below (and in a few other lemmas later on) that $\mathbf{k}$ is an *integral domain*, that is, that $\mathbf{k}$ has no zero-divisors. We shall do so even when the results hold in greater generality, because this level of generality is enough for us and the proofs are shorter. Moreover, the proof of our final results can be reduced, from the general case of a finitely generated ring $\mathbf{k}$, to the case when $\mathbf{k}$ is an integral domain.

Recall that if $\mathfrak{p} \subset \mathbf{k}$ is a prime ideal and M is a $\mathbf{k}$-module, then $M_{\mathfrak{p}}$ denotes the localization of M at $\mathfrak{p}$.

Lemma 3. *Let A be a semi-primitive, finite type* $\mathbf{k}$*-algebra, and assume that* $\mathbf{k}$ *is an integral domain. Then the set*

$$S_1 := \{\mathfrak{p} \in \mathrm{Max}(\mathbf{k}),\ A_{\mathfrak{p}} \text{ is unital}\}$$

is open and dense in $\mathrm{Max}(\mathbf{k})$.

We agree that the algebra $\{0\}$ *does have* a unit.

Proof. To prove that S_1 is dense, it is enough to show that it is a non-empty open set, because $\mathbf{k}$ is an integral domain.

Denote by $\mathbb{K}$ the ring of fractions of $\mathbf{k}$. Then $A \otimes_{\mathbf{k}} \mathbb{K}$ is a semi-primitive finite dimensional $\mathbb{K}$-algebra, and hence it has unit. Let $a \in A$ and $f \in \mathbf{k}^*$ be such that $f^{-1}a$ is the unit of $A \otimes_{\mathbf{k}} \mathbb{K}$. Let $b_1, \ldots, b_l$ be generators of A as a $\mathbf{k}$-module. Then there exists $g_1, \ldots, g_l \in \mathbf{k}^*$ such that $g_j(ab_j - fb_j) = 0$ in A, for any $j = 1, \ldots, l$. If we let then S to be the principal open set associated to $h = fg_1 \ldots g_l \neq 0$ (that is, the set of maximal ideals of $\mathbf{k}$ not containing h), we get $S \subset S_1$. This shows that S_1 is not empty.

To prove that S_1 is open, we proceed in an analogous manner. Let $\mathfrak{p}$ be a maximal ideal such that $A_{\mathfrak{p}}$ is unital. Let $f^{-1}a$ (where $a \in A$ and $f \in \mathbf{k}^*$, $f \notin \mathfrak{p}$) be that unit. Define $g_1, \ldots, g_l \in \mathbf{k}^*$ and $h = fg_1 \ldots g_l \neq 0$ as above. Then the principal open set associated to h contains $\mathfrak{p}$ and is in turn contained in S_1. This completes the proof. $\square$

The following lemma is a well known consequence of the Artin-Procesi theorem.

Lemma 4. *Let A be a unital, semi-primitive, finite type $\mathbf{k}$-algebra, and assume that $\mathbf{k}$ is an integral domain. Then the set*

$$S_2 := \{\mathfrak{p} \in \mathrm{Max}(\mathbf{k}),\ A_{\mathfrak{p}} \ \text{ is an Azumaya algebra}\}$$

is open and dense in $\mathrm{Max}(\mathbf{k})$.

Proof. Let $Z = Z(A)$ be the center of A. By standard commutative algebra, the set of maximal ideals $\mathfrak{p} \in \mathrm{Max}(\mathbf{k})$ such that $A_{\mathfrak{p}}$ is a projective $Z_{\mathfrak{p}}$-module is an open subset of $\mathrm{Max}(\mathbf{k})$. Because A is a finitely generated Z-module, $\left[\mathrm{End}_Z(A)\right]_{\mathfrak{p}} = \mathrm{End}_{Z_{\mathfrak{p}}}(A_{\mathfrak{p}})$. Then, the set of maximal ideals $\mathfrak{p} \in \mathrm{Max}(\mathbf{k})$ such that

$$A_{\mathfrak{p}} \otimes_{Z_{\mathfrak{p}}} A_{\mathfrak{p}} = (A \otimes_Z A)_{\mathfrak{p}} \cong \left[\mathrm{End}_Z(A)\right]_{\mathfrak{p}} = \mathrm{End}_{Z_{\mathfrak{p}}}(A_{\mathfrak{p}})$$

is an open subset of $\mathrm{Max}(\mathbf{k})$. This shows that S_2 is open. To prove that S_2 is dense, it is enough then to prove that it is also not empty.

Let n be the PI-type of A. Then $\mathrm{Prim}_n(A) \neq 0$, because A is semi-primitive. Let $\mathfrak{I} \subset A$ be the intersection of the kernels of all irreducible representations of A of dimension $< n$. Then Z, the center of A, intersects $\mathfrak{I}$ non-trivially. (This is a deep fact that is proved using central polynomials as follows: if g_n is a central polynomial, see [19], then $g_n(\mathfrak{I}) \subset \mathfrak{I} \cap Z$ and $g_n(\mathfrak{I}) \neq \{0\}$.) Let $I := Z \cap \mathfrak{I}$, then an irreducible complex representation of A is in $\mathrm{Prim}_n(A)$ if, and only if, it does not vanish on I. Let $\mathfrak{q}$ be a maximal ideal of $\mathbf{k}$ *not* containing $\mathbf{k} \cap \mathfrak{I}$. Then $A_{\mathfrak{q}}$ is a PI-algebra of PI-type n and all irreducible representations of $A_{\mathfrak{q}}$ are of dimension n. The Artin-Procesi theorem gives then that $A_{\mathfrak{q}}$ is an Azumaya algebra. This verifies that S_2 is not empty and hence it completes the proof. $\square$

2. Spectrum preserving morphisms

We shall now study a class of morphisms implicitly appearing in Lusztig's work on the representation of Iwahori-Hecke algebras, see [35]. More precisely, we shall study the class of "spectrum preserving" morphisms of finite type algebras, our main reason for studying this class of morphisms being that they induce an isomorphism in periodic cyclic homology (Theorem 7).

Denote by $\Theta_L : \mathrm{Prim}(L) \to \mathrm{Max}(\mathbf{k})$ and by $\Theta_J : \mathrm{Prim}(J) \to \mathrm{Max}(\mathbf{k})$ the central character maps of the finite type $\mathbf{k}$-algebras L and J. If $\phi : L \to J$ is a $\mathbf{k}$-linear morphism, we define

$$\mathcal{R}_\phi := \{(\mathfrak{P}', \mathfrak{P}) \subset \mathrm{Prim}(J) \times \mathrm{Prim}(L),\ \phi^{-1}(\mathfrak{P}') \subset \mathfrak{P}\}. \tag{7}$$

Lemma 5. *If $(\mathfrak{P}', \mathfrak{P}) \in \mathcal{R}_\phi$, then $\Theta_L(\mathfrak{P}) = \Theta_J(\mathfrak{P}')$.*

Proof. If $\mathfrak{P}$ and $\mathfrak{P}'$ are as in the statement of the lemma, then an irreducible representation with kernel $\mathfrak{P}$ identifies with a subspace of an irreducible representation with kernel $\mathfrak{P}'$. This shows that any $a \in \mathbf{k}$ will act on both spaces via the same scalar $\chi(a)$. Consequently, $\Theta_L(\mathfrak{P}) := \ker(\chi) =: \Theta_J(\mathfrak{P}')$. □

We shall need the following lemma.

Lemma 6. *Suppose $\phi : L \to J$ and $\psi : J \to A$ are morphisms of finite type $\mathbf{k}$-algebras. Then $\mathcal{R}_{\psi\circ\phi} = \mathcal{R}_\psi \circ \mathcal{R}_\phi$.*

Proof. The inclusion $\mathcal{R}_{\psi\circ\phi} \supset \mathcal{R}_\psi \circ \mathcal{R}_\phi$ follows directly from the definition.

To prove the other inclusion, suppose (π, V_π) is an irreducible representation of L (on the vector space V_π) with kernel $\mathfrak{P}$. Let $\mathfrak{P}'' \subset A$ be the kernel of an irreducible representation $(\pi'', V_{\pi''})$ of A. If $(\psi \circ \phi)^{-1}(\mathfrak{P}'') \subset \mathfrak{P}$, then V_π identifies with a subrepresentation of $\pi'' \circ \psi \circ \phi$. Restrict π'' to J, and let W be an irreducible J-module containing V_π. Then the annihilator $\mathfrak{P}'$ of W satisfies $\phi^{-1}(\mathfrak{P}') \subset \mathfrak{P}$ and $\psi^{-1}(\mathfrak{P}'') \subset \mathfrak{P}'$. This proves that $\mathcal{R}_{\psi\circ\phi} \subset \mathcal{R}_\psi \circ \mathcal{R}_\phi$. □

We now introduce the class of morphisms we are interested in.

Definition 3. *Let $\phi : L \to J$ be a $\mathbf{k}$-linear morphism of unital, finite type $\mathbf{k}$-algebras. We say that ϕ is a* spectrum preserving morphism *if, and only if, the set $\mathcal{R}_\phi$ defined in Equation* (7) *is the graph of a bijective function*

$$\phi^* : \mathrm{Prim}(J) \to \mathrm{Prim}(L).$$

More concretely, we see that $\phi : L \to J$ is spectrum preserving if, and only if, the following two conditions are satisfied:

1. *For any primitive ideal $\mathfrak{P}$ of J, the ideal $\phi^{-1}(\mathfrak{P})$ is contained in a unique primitive ideal of L, namely $\phi^*(\mathfrak{P})$*, and
2. *The resulting map $\phi^* : \mathrm{Prim}(J) \to \mathrm{Prim}(L)$ is a bijection.*

It is instructive to look at the case when both L and J are finite dimensional, semi-simple. Then any morphism $\phi : L \to J$ gives rise to a morphism

$$\phi_* : K_0(L) \to K_0(J), \tag{8}$$

and the conjugacy class of ϕ is determined by ϕ_*. By identifying the K_0-groups with the free abelian subgroups generated by the simple factors, we see that ϕ_* is determined by a matrix with integer entries, called the *Bratteli diagram* of ϕ [10] (see also [18]). A morphism $\phi : L \to J$ of finite dimensional semi-simple algebras will then be spectrum preserving if, and only if, it decomposes up to conjugacy as a direct sum of (not necessarily unital) injective morphisms $M_r(\mathbb{C}) \to M_q(\mathbb{C})$. This condition is equivalent to saying that the Bratteli diagram of ϕ is a diagonal matrix with non-zero diagonal entries (after reordering the simple factors of J, if necessary). The definition of the Bratteli diagram extends also to other classes of rings L and J in the obvious way. We shall be interested below (Lemma 9) in the case when L and J are direct sums of matrix algebras over some completion of $\mathbf{k}$.

¿From Lemma 6 we get the following proposition, which implies, among other things, that the composition of two spectrum preserving morphisms is again spectrum preserving.

Proposition 1. *Suppose that $\phi : L \to J$ and $\psi : J \to A$ are morphisms of finite type $\mathbf{k}$-algebras. If two of the morphisms ϕ, ψ, or $\psi \circ \phi$ are spectrum preserving, then the third morphism is spectrum preserving as well and $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.*

Proof. The relation $\mathcal{R}_{\psi\circ\phi} = \mathcal{R}_\psi \circ \mathcal{R}_\phi$ of the above lemma tells us right away that if two of the relations $\mathcal{R}_\phi$, $\mathcal{R}_\psi$, or $\mathcal{R}_{\psi\circ\phi}$ are the graphs of bijective functions, then the third one is also the graph of a bijective function. Moreover, the relation $(\psi \circ \phi)^* = \phi^* \circ \psi^*$ also follows. □

Assuming that the morphism $\phi : L \to J$ is spectrum preserving, we obtain the following description of ϕ^*. If $\mathfrak{P}$ is a primitive ideal of J, denote by $p : L \to L/\phi^{-1}(\mathfrak{P})$ the natural projection. Then

$$\phi^*(\mathfrak{P}) = p^{-1}(\mathrm{Jac}(L/\phi^{-1}(L))).$$

An explicit description of $(\phi^*)^{-1}$ will be obtained later on.

It is easy to check using the definition that $\phi : L \to J$ is a spectrum preserving morphism if, and only if, $\phi^+ : L^+ \to J^+$ is a spectrum preserving morphism, and hence we can take the latter to be the definition of a spectrum preserving morphism in the non-unital case. It also follows directly from the definition that $L \to J$ is spectrum preserving if, and only if, $L \to J/\mathrm{Jac}(J)$ is spectrum preserving.

Lemma 7. *The kernel of a spectrum preserving morphism $\phi : L \to J$ is contained in the Jacobson radical of L. Consequently, a morphism $\phi : L \to J$ is spectrum preserving if, and only if, $\ker(\phi) \subset \mathrm{Jac}(L)$ and the induced map $L/\ker(\phi) \to J$ is spectrum preserving.*

Proof. Let $\mathfrak{Q} \subset L$ be an arbitrary primitive ideal. We need to show that $\ker(\phi) \subset \mathfrak{Q}$. By the surjectivity of the map $\phi^* : \mathrm{Prim}(J) \to \mathrm{Prim}(L)$, there exists a primitive ideal $\mathfrak{P} \subset J$ such that $\phi^{-1}(\mathfrak{P}) \subset \mathfrak{Q}$. Then $\ker(\phi) := \phi^{-1}(0) \subset \mathfrak{Q}$. □

¿From the above discussion and the above lemma, we get the following.

Corollary 1. *A morphism $\phi : L \to J$ of finite type algebras is spectrum preserving if, and only if, $\phi^{-1}(\mathrm{Jac}(J)) \subset \mathrm{Jac}(L)$ and the induced morphism*

$$L/\phi^{-1}(\mathrm{Jac}(J)) \to J/\mathrm{Jac}(J)$$

is spectrum preserving.

Spectral preserving morphisms behave well when we restrict them to ideals and when we take quotients.

Proposition 2. *Let $\phi : L \to J$ be a $\mathbf{k}$-linear morphism of (not necessarily unital) finite type algebras, and let $J' \subset J$ be a two-sided ideal. If ϕ is spectrum preserving, then the induced morphisms $\phi^{-1}(J') \to J'$ and $L/\phi^{-1}(J') \to J/J'$ are spectrum preserving.*

Proof. By considering algebras with adjoined unit, we can assume that both L and J are unital. The Jacobson spectrum $\mathrm{Prim}(J)$ decomposes as a disjoint union of the set of primitive ideals containing J' and the set of primitive ideals *not* containing J'. These two sets identify with the Jacobson spectrums of J/J' and J', respectively.

By Lemma 7, we can assume that ϕ is an inclusion (so L identifies with a subalgebra of J). We can restrict the morphism ϕ^* to each of $\mathrm{Prim}(J/J')$ and $\mathrm{Prim}(J')$, the two sets considered above.

It is clear from definitions that ϕ^* maps a primitive ideal of J containing J' to a primitive ideal of L containing $L \cap J'$. This means that ϕ^* maps $\mathrm{Prim}(J/J')$ to $\mathrm{Prim}(L/(L \cap J'))$. It is then enough to show that the induced map

$$\phi^* : \mathrm{Prim}(J/J') \to \mathrm{Prim}(L/(L \cap J'))$$

is a bijection.

Lemma 5 gives that ϕ^* induces a bijection of the sets $\Theta_J^{-1}(\mathfrak{p})$ and $\Theta_L^{-1}(\mathfrak{p})$. We also know that ϕ^* maps

$$S := \Theta_J^{-1}(\mathfrak{p}) \cap \mathrm{Prim}(J/J') \quad \text{to} \quad S' := \Theta_L^{-1}(\mathfrak{p}) \cap \mathrm{Prim}(L/(L \cap J')).$$

To prove that ϕ^* maps $\mathrm{Prim}(J/J')$ to $\mathrm{Prim}(L/(L \cap J'))$ bijectively, it is enough to prove that $\phi^* : S \to S'$ is bijective.

Now we have that $S = \mathrm{Prim}(J/\mathfrak{p}J)$ and $S' = \mathrm{Prim}(L/\mathfrak{p}L)$. Moreover, it is clear from Lemma 5 that the induced map $\phi_0 : L/\mathfrak{p}L \to J/\mathfrak{p}J$ is spectrum preserving and that $\phi_0^* = \phi^*$ as maps from S to S'. Because both $J/\mathfrak{p}J$ and $L/\mathfrak{p}L$ are finite dimensional, this reduces our discussion to the finite dimensional case.

So assume now that L and J are finite dimensional, that J is semi-primitive, and that $L \subset J$. Then J is semi-simple. Because the exact sequence

$$0 \to \mathrm{Jac}(L) \to L \to L/\,\mathrm{Jac}(L) \to 0 \tag{9}$$

splits, we can also assume that L is semi-primitive. If we write $J = \oplus J_\alpha$ as a direct sum of simple factors, the assumption that $L \to J$ is spectrum preserving means that we can write $L = \oplus\,(L \cap J_\alpha)$, from which the result follows. ∎

It is interesting to mention the following consequence of the (proof of the) above proposition.

Corollary 2. *If $\phi : L \to J$ is spectrum preserving and $J' \subset J$ is an ideal, then $\phi^*(V(J')) = V(\phi^{-1}(J'))$.*

Similarly, we have the following property.

Lemma 8. *Assume that $\phi : L \to J$ is spectrum preserving and that all irreducible representations of J have dimension $\leq n$, then $(\phi^*)^{-1}(V(L')) = V(J\phi(L')^n J)$, for any two-sided ideal $L' \subset L$.*

Proof. Let $\mathfrak{P} \subset J$ be an arbitrary primitive ideal. Because $L/\phi^{-1}(\mathfrak{P})$ identifies with a subalgebra of the algebra of $m \times m$ matrices for some $m \leq n$, the Jacobson radical of $L/\phi^{-1}(\mathfrak{P})$ is nilpotent of order at most m, by Engel's theorem. Hence, if $L_1 := \mathrm{Jac}(L/\phi^{-1}(\mathfrak{P}))$, then $L_1^n = 0$. This shows that $\phi^*(\mathfrak{P})$, which is the inverse image of L_1 in L, satisfies $\phi^*(\mathfrak{P})^n \subset \phi^{-1}(\mathfrak{P})$.

We have $\mathfrak{P} \in (\phi^*)^{-1}(V(L'))$ if, and only if, $L' \subset \phi^*(\mathfrak{P})$. The above discussion shows that $L' \subset \phi^*(\mathfrak{P}) \Leftrightarrow L'^n \subset \phi^{-1}(\mathfrak{P})$. Since

$$L'^n \subset \phi^{-1}(\mathfrak{P}) \Leftrightarrow \phi(L')^n \subset \mathfrak{P} \Leftrightarrow J\phi(L')^n J \subset \mathfrak{P},$$

the result follows. ∎

By applying the above lemma to a primitive ideal $\mathfrak{P}'$ of L, we obtain the following description of $(\phi^*)^{-1}$. Denote by $p : J \to J/J\phi(\mathfrak{P}')J$ the natural projection, and let n be as in the above lemma. Then

$$(\phi^*)^{-1}(\mathfrak{P}') = p^{-1}(\mathrm{Jac}(J/J\phi(\mathfrak{P}')^n J)).$$

Together, the above results give the following theorem.

Theorem 2. *Let L and J be finite type algebras and $\phi : L \to J$ be a spectrum preserving morphism. Then the induced map $\phi^* : \mathrm{Prim}(J) \to \mathrm{Prim}(L)$ is a homeomorphism.*

The following result, which is, in a way, a converse of Corollary 2, will be needed later on.

Corollary 3. *Suppose $\phi : L \to J$ is a spectrum preserving morphism and let $L_1 \subset L$ and $J_1 \subset J$ be two-sided ideals such that $\phi(L_1) \subset J_1$. Then the induced map $L_1 \to J_1$ is spectrum preserving if, and only if, $\phi^*(V(J_1)) = V(L_1)$. Similarly, the $L/L_1 \to J/J_1$ is spectrum preserving if, and only if, $\phi^*(V(J_1)) = V(L_1)$.*

Proof. We know that $L_2 := \phi^{-1}(J_1) \to J_1$ is spectrum preserving (by Proposition 2) and that $V(L_2) = \phi^*(V(J_1))$. Because $L_1 \subset L_2$ is a two-sided ideal of L_2, the inclusion $L_1 \to L_2$ is spectrum preserving if, and only if, $V(L_1) = V(L_2)$. The fact that the map $L_1 \to J_1$ is spectrum preserving then follows from Proposition 1.

Let us prove that $L/L_1 \to J/J_1$ is spectrum preserving. Indeed, Corollary 2 and the assumption that $\phi^*(V(J_1)) = V(L_1)$ show that $L/L_1 \to L/\phi^{-1}(J_1)$ is spectrum preserving. The result then follows from Proposition 1 and Proposition 2. $\square$

3. Periodic cyclic homology

In this section, we recall a few basic results on cyclic and Hochschild homology necessary for the proof, in the next section, of the fact that a spectrum preserving morphism induces an isomorphism in periodic cyclic cohomology. The account of Hochschild and cyclic homology in [29] is more complete, and we refer the reader without prior exposure to cyclic homology to that paper for more homological results relevant to our setting. See also [15, 16, 26, 32], and [48].

Let A be a complex unital algebra. Define the operators s, t, B, b', b, acting on $A^{\otimes n+1} := A \otimes A \otimes \ldots \otimes A$ ($n+1$ times), as follows.

$$s(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = 1 \otimes a_0 \otimes a_1 \otimes \ldots \otimes a_n,$$
$$t(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes \ldots \otimes a_{n-1},$$
$$B(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = (1-t)s \sum_{k=0}^{n} t^k (a_0 \otimes a_1 \otimes \ldots \otimes a_n),$$
$$b'(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n, \quad \text{and}$$
$$b(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = b'(a_0 \otimes a_1 \otimes \ldots \otimes a_n) + (-1)^n a_n a_0 \otimes \ldots \otimes a_{n-1}.$$

Here we have used the notation of [16]. Then the *Hochschild homology groups of A*, denoted $\mathrm{HH}_q(A)$, can be computed as the homology groups of the complex $(A^{\otimes n+1}, b)$. If A is a $\mathbf{k}$-algebra, then $\mathrm{HH}_q(A)$ is a $\mathbf{k}$-module. The *cyclic homology* groups of A, denoted $\mathrm{HC}_n(A)$, are the homology groups of the cyclic complex $(\mathcal{C}(A), b+B)$, where

$$\mathcal{C}_n(A) = \bigoplus_{k\geq 0} A \otimes (A/\mathbb{C}1)^{\otimes n-2k}.$$

The complex $\mathcal{C}_n$ has an endomorphism $S : \mathcal{C}_n \to \mathcal{C}_{n-2}$. The inverse limit of this complex is called the *periodic cyclic complex.* Its homology is called the *periodic cyclic cohomology* of A and is denoted by $\mathrm{HP}_*(A)$.

Let J be a two–sided ideal of A. By replacing in the definitions above $A^{\otimes n+1}$ with

$$A^{\widehat{\otimes} n+1} := \lim_{\leftarrow} (A/J^k)^{\otimes n+1}, \quad k \to \infty,$$

we obtain the definitions of the groups $\mathrm{HH}_*^{\mathrm{top}}(\widehat{A})$ and $\mathrm{HP}_*^{\mathrm{top}}(\widehat{A})$. We shall use these constructions for A a $\mathbf{k}$-algebra, where $\mathbf{k}$ is a Noetherian ring, and $J = IA$, for some ideal I of $\mathbf{k}$. Also, denote, as usual by

$$\widehat{M} := \lim_{\leftarrow} M/I^nM, \quad n \to \infty,$$

the I-adic completion of a $\mathbf{k}$–module M. If M is a finitely generated $\mathbf{k}$-module, then $\widehat{M} \cong M \otimes_{\mathbf{k}} \widehat{\mathbf{k}}$ [2, 9].

The following theorem from [29] describes the effect of I–adic completion on Hochschild homology.

Theorem 3. *Assume that $\mathbf{k}_0$ is a noetherian ring and that A_0 is a unital $\mathbf{k}_0$–algebra, which is a finitely generated $\mathbf{k}_0$-module. Let $I \subset \mathbf{k}_0$ be an ideal, and let $\widehat{A}_0 := \lim_{\leftarrow} A_0/I^kA_0$ be the completion of A_0 in the I-adic topology. Then the natural map $\mathrm{HH}_*(A_0) \to \mathrm{HH}_*^{\mathrm{top}}(\widehat{A}_0)$ and the $\mathbf{k}_0$–module structure on $\mathrm{HH}_*(A_0)$ define an isomorphism*

$$\mathrm{HH}_*(A_0) \otimes_{\mathbf{k}_0} \widehat{\mathbf{k}}_0 \cong \mathrm{HH}_*^{\mathrm{top}}(\widehat{A}_0)$$

of $\widehat{\mathbf{k}}_0$–modules.

In [29], the above theorem was proved when $\mathbf{k}_0$ a finitely generated algebra, however, the proof works without any change in the case of a noetherian ring $\mathbf{k}_0$. We shall need this extension when $\mathbf{k}_0 = \mathbf{k}_{\mathfrak{p}}$ is the localization of a finitely generated commutative ring at a maximal ideal $\mathfrak{p}$ and $I = \mathfrak{p}\mathbf{k}_{\mathfrak{p}}$. In that case the following simple argument reduces the proof of Theorem 3 to a simple application of the original theorem in [29], so we do not have to go over the whole proof of that theorem again. Indeed,

$$\mathrm{HH}_*^{\mathrm{top}}(\widehat{A}_0) = \mathrm{HH}_*^{\mathrm{top}}(\widehat{A}) \cong \mathrm{HH}_*(A) \otimes_{\mathbf{k}} \widehat{\mathbf{k}} \cong \mathrm{HH}_*(A) \otimes_{\mathbf{k}} \widehat{\mathbf{k}_0}.$$

On the other hand, we can argue using a result of Brylinski [12] (see Proposition 3 below), that

$$\mathrm{HH}_*(A_0) \otimes_{\mathbf{k}_0} \widehat{\mathbf{k}}_0 \cong (\mathrm{HH}_*(A) \otimes_{\mathbf{k}} \mathbf{k}_0) \otimes_{\mathbf{k}_0} \widehat{\mathbf{k}}_0 \cong \mathrm{HH}_*(A) \otimes_{\mathbf{k}} \widehat{\mathbf{k}}_0.$$

Thus $\mathrm{HH}_*(A_0) \otimes_{\mathbf{k}_0} \widehat{\mathbf{k}}_0 \cong \mathrm{HH}_*^{\mathrm{top}}(\widehat{A}_0)$ for this particular choice, for $\mathbf{k}_0 = \mathbf{k}_{\mathfrak{p}}$, $A_0 = A_{\mathfrak{p}}$, and $I = \mathfrak{p}A_{\mathfrak{p}}$. This particular case is enough for our purposes.

Recall now the proposition from [12] used above.

Proposition 3. *Let S be a multiplicative subset of the center Z of the algebra A. Then $\mathrm{HH}_*(S^{-1}A) \cong S^{-1}\,\mathrm{HH}_*(A)$.*

The following result is well known (a proof of it can be found in [29], for example). A proof can be obtained from the above proposition too.

Proposition 4. *Let Z be a commutative ring and A be an Azumaya algebra over Z. Then the inclusion $j : Z \to A$ defines an isomorphism $j_* : \mathrm{HH}_*(Z) \to \mathrm{HH}_*(A)$.*

We now two basic results in cyclic homology. We begin with the following result of Goodwillie [22].

Theorem 4. *If $I \subset A$ is a nilpotent ideal, then the quotient morphism $A \to A/I$ induces an isomorphism $\mathrm{HP}_*(A) \to \mathrm{HP}_*(A/I)$.*

A deep result of Cuntz and Quillen, which we shall often use below, is the Excision property in periodic cyclic homology [17].

Theorem 5 (Excision)**.** *Any two-sided ideal J of an algebra A over a characteristic zero field gives rise to a natural, six-term periodic exact sequence*

$$\begin{tikzcd} \mathrm{HP}_0(J) \arrow[r] & \mathrm{HP}_0(A) \arrow[r] & \mathrm{HP}_0(A/J) \arrow[d, "\partial"] \\ \mathrm{HP}_1(A/J) \arrow[u, "\partial"] & \mathrm{HP}_1(A) \arrow[l] & \mathrm{HP}_1(J). \arrow[l] \end{tikzcd} \tag{10}$$

An immediate consequence of the above two results, also due to Cuntz and Quillen, states that if I is a nilpotent algebra, then $\mathrm{HP}_*(I) = 0$.

4. A comparison theorem

We shall prove now our main result on cyclic homology, Theorem 7. Theorem 7 formulates a general principle, which, in rough terms, states that a morphism of finite type algebras that preserves the spectrum induces an isomorphism of periodic cyclic homology groups. Thus the periodic cyclic homology of a finite type algebra is a "spectral invariant."

We begin with a sequence of lemmas. Recall that we denote by $M_{\mathfrak{p}} = S^{-1}M$ the localization of a $\mathbf{k}$-module M with respect to a maximal ideal $\mathfrak{p} \subset \mathbf{k}$. Denote by $\widehat{\mathbf{k}}$ the completion of $\mathbf{k}$ with respect to (the powers of) $\mathfrak{p}$.

Lemma 9. *Assume that $\widehat{L}$ and $\widehat{J}$ are Azumaya algebras and that their centers are isomorphic to finite direct sums of copies of $\widehat{\mathbf{k}}$. Let $\widehat{\phi} : \widehat{L} \to \widehat{J}$ be a $\widehat{\mathbf{k}}$-linear algebra morphism. If the induced morphism $\phi_1 : \widehat{L}/\mathfrak{p}\widehat{L} \to \widehat{J}/\mathfrak{p}\widehat{J}$ is spectrum preserving, then $\widehat{\phi}_* : \mathrm{HH}_q^{\mathrm{top}}(\widehat{L}) \to \mathrm{HH}_q^{\mathrm{top}}(\widehat{J})$ is an isomorphism for all q.*

Proof. Assume that the center of $\widehat{L}$ has m direct summands factors isomorphic to $\widehat{\mathbf{k}}$. Then $\mathrm{HH}_*^{\mathrm{top}}(\widehat{L}) \cong \mathrm{HH}_*^{\mathrm{top}}(\widehat{\mathbf{k}})^m \cong \widehat{\mathbf{k}}^m$, each direct summand corresponding to a direct summand of the center of $\widehat{L}$. The morphism $\widehat{\phi}_* : \mathrm{HH}_q^{\mathrm{top}}(\widehat{L}) \to \mathrm{HH}_q^{\mathrm{top}}(\widehat{J})$ identifies then with a matrix, which is immediately seen to be isomorphic to the Bratteli diagram of ϕ_1. ☐

We shall use this lemma to prove the following result.

Proof. The assumptions imply that $\widehat{L}$ is isomorphic to a direct sum of matrix algebras over $\widehat{\mathbf{k}}$. Moreover, any $\widehat{\mathbf{k}}$-linear morphism $\widehat{\phi} : \widehat{L} \to \widehat{J}$ is uniquely determined by its Bratteli diagram. ☐

Lemma 10. *Let $\phi : L \to J$ be a unital, injective, spectrum preserving morphism of semi-primitive, finite type* $\mathbf{k}$*-algebras. Assume that* $\mathbf{k}$ *is an integral domain. Then the set*

$$S_3 := \{\mathfrak{p} \in \mathrm{Max}(\mathbf{k}),\ \phi_* : \mathrm{HH}_*(L_\mathfrak{p}) \to \mathrm{HH}_*(J_\mathfrak{p}) \ \textit{ is an isomorphism } \}$$

is open and dense in $\mathrm{Max}(\mathbf{k})$.

Proof. For any morphism $f : M \to M'$ of finitely generated $\mathbf{k}$-modules, it is known that the set

$$\{\mathfrak{p} \in \mathrm{Max}(\mathbf{k}),\ f_\mathfrak{p} : M_\mathfrak{p} \cong M'_\mathfrak{p}\ \}$$

is open in $\mathrm{Max}(\mathbf{k})$, the maximal ideal spectrum of $\mathbf{k}$. Because both $\mathrm{HH}_*(L)$ and $\mathrm{HH}_*(J)$ are finitely generated $\mathbf{k}$-modules (see [29] for a proof of this simple fact), this observation and Proposition 4 give that S_3 is open.

To prove that S_3 is dense, it is enough to prove that it is not empty, because $\mathrm{Max}(\mathbf{k})$ is irreducible. Since L and J are unital and semi-primitive, using Lemma 5 we get that we can also assume $\mathbf{k}$ to map injectively to each of L and J. By Lemma 4, we know that we can find an open dense set $S_2 \subset \mathrm{Max}(\mathbf{k})$ such that both $L_\mathfrak{p}$ and $J_\mathfrak{p}$ are Azumaya algebras.

Let $Z(L)$ be the center of L and $\pi_L : \mathrm{Max}(Z(L)) \to \mathrm{Max}(\mathbf{k})$ be the central character map defined by the inclusion $\mathbf{k} \to Z(L)$, which is defined because $Z(L)$ is a finite type $\mathbf{k}$-algebra. Define $\pi_J : \mathrm{Max}(Z(J)) \to \mathrm{Max}(\mathbf{k})$ similarly.

Let $\widetilde{\mathbf{k}}$ be the field of fractions of the integral domain $\mathbf{k}$, and let n_L be the dimension of $Z(L) \otimes_\mathbf{k} \widetilde{\mathbf{k}}$ as a $\widetilde{\mathbf{k}}$-vector space. Define n_J similarly. We can choose a maximal ideal $\mathfrak{p} \in S_2$ such that the following three conditions are satisfied: $Z(L)/\mathfrak{p}Z(L)$ has dimension n_L, $Z(J)/\mathfrak{p}Z(J)$ has dimension n_J, and all points of $\pi_L^{-1}(\mathfrak{p})$ and $\pi_J^{-1}(\mathfrak{p})$ are regular. (Recall that a maximal ideal $\mathfrak{m} \subset \mathbf{k}$ is called *regular* if $\dim(\mathfrak{m}/\mathfrak{m}^2) = \dim \mathbf{k}$, see [2], Theorem 11.22. Geometrically, this means that the associated point in the spectrum of $\mathbf{k}$ is non-singular.) We shall prove then that $\mathfrak{p} \in S_3$.

Let $\widehat{\mathbf{k}}$ be the completion of $\mathbf{k}$ with respect to $\mathfrak{p}$. Because $\widehat{\mathbf{k}}$ is faithfully flat over $\mathbf{k}_\mathfrak{p}$ ([2], Theorem 10.17) it is enough to prove then that the map ϕ induces an isomorphism

$$\mathrm{HH}_*(L_\mathfrak{p}) \otimes_\mathbf{k} \widehat{\mathbf{k}} \cong \mathrm{HH}_*(J_\mathfrak{p}) \otimes_\mathbf{k} \widehat{\mathbf{k}}.$$

Let $\widehat{L}$ and $\widehat{J}$ be the completions of L and J with respect to $\mathfrak{p}$, too. Because $L_\mathfrak{p} \otimes_\mathbf{k} \widehat{\mathbf{k}} \cong \widehat{L}$ and $J_\mathfrak{p} \otimes_\mathbf{k} \widehat{\mathbf{k}} \cong \widehat{J}$, Theorem 3 gives that it is enough to prove that

$$\mathrm{HH}_*^{\mathrm{top}}(\widehat{L}) \cong \mathrm{HH}_*^{\mathrm{top}}(\widehat{J}).$$

By assumptions, $\widehat{L}$ is an Azumaya algebra over $\widehat{Z(L)} := Z(L) \otimes_\mathbf{k} \widehat{\mathbf{k}}$. Because $\mathfrak{p}$ is regular, the ring $\widehat{\mathbf{k}}$ is isomorphic to the ring of formal power series. Similarly, because the preimage of $\mathfrak{p}$ in $\mathrm{Max}(Z(L))$ consists of regular points, the ring $\widehat{Z(L)}$ is a direct sum of rings of power series. More precisely, we have that $\widehat{Z(L)} \cong \widehat{\mathbf{k}}^m$, where m is the degree of the field of fractions of $Z(L)$ over the field of fractions of $\mathbf{k}$. The result then follows from Lemma 9 and Lemma 5. □

We shall need also the following easy lemma from [29]

Lemma 11. *Let L be a $\mathbf{k}$–algebra and $\mathfrak{p} \subset \mathbf{k}$ be a maximal ideal such that $L_\mathfrak{p}$ is unital. If $B = \mathbf{k} + L$ is the $\mathbf{k}$–algebra with adjoined unit, then the inclusions $L \to B$ and $\mathbf{k} \to B$ induce isomorphisms $\mathrm{HH}_*(B)_\mathfrak{p} \cong \mathrm{HH}_*(\mathbf{k}_\mathfrak{p}) \oplus \mathrm{HH}_*(L_\mathfrak{p})$.*

In particular, $\mathrm{HH}_*(L_\mathfrak{p}) = \ker[\mathrm{HH}_*(B)_\mathfrak{p} \to \mathrm{HH}_*(\mathbf{k})_\mathfrak{p}]$. The following proposition puts together the above results.

Proposition 5. *Let $\phi : L \to J$ be an injective spectrum preserving morphism of semi-primitive, finite type $\mathbf{k}$-algebras, with $\mathbf{k}$ an integral domain. Then the set*

$$S_4 := \{\mathfrak{p} \in \mathrm{Max}(\mathbf{k}),\ L_\mathfrak{p} \text{ and } J_\mathfrak{p} \text{ are unital and } \mathrm{HH}_*(L_\mathfrak{p}) \cong \mathrm{HH}_*(J_\mathfrak{p})\,\}$$

is open and dense in $\mathrm{Max}(\mathbf{k})$.

Proof. We know that the set

$$S_3 := \{\mathfrak{p} \in \mathrm{Max}(\mathbf{k}),\ \phi_* : \mathrm{HH}_*(L_\mathfrak{p}^+) \cong \mathrm{HH}_*(J_\mathfrak{p}^+)\,\}$$

is open and dense in $\mathrm{Max}(\mathbf{k})$ by Lemma 10. By Lemma 3, we know that the set

$$S_1 := \{\mathfrak{p} \in \mathrm{Max}(\mathbf{k}),\ L_\mathfrak{p} \text{ and } J_\mathfrak{p} \text{ are unital}\,\}$$

is also open and dense in $\mathrm{Max}(\mathbf{k})$. Then Lemma 11 gives that $S_4 = S_1 \cap S_3$, from which the result follows. ☐

We shall need also one of the main results from [29], Theorem 8 (or rather Corollary 5 of that paper) which gives a criterion for two algebras to have the same periodic cyclic homology.

Theorem 6 (Kazhdan-Nistor-Schneider)**.** *Let A be a unital finite type $\mathbf{k}$–algebra and $L \subset J \subset IA$ be inclusions of finite type $\mathbf{k}$–algebras, where $I \subset \mathbf{k}$ is an ideal. Suppose that, for all maximal ideals $\mathfrak{p}$, $I \not\subset \mathfrak{p}$, the localizations $L_\mathfrak{p}$ and $J_\mathfrak{p}$ are unital and $\mathrm{HH}_q(L_\mathfrak{p}) \to \mathrm{HH}_q(J_\mathfrak{p})$ is an isomorphism. Then the inclusion $L \to J$ gives an isomorphism*

$$\mathrm{HP}_q(L) \cong \mathrm{HP}_q(J).$$

We are ready now to prove the main result on cyclic homology of this paper, a result that reflects the *spectral invariance of periodic cyclic cohomology* in the class of finite type algebras.

Theorem 7. *Let $\phi : L \to J$ be a spectrum preserving morphism of finite type $\mathbf{k}$-algebras. Then the induced map $\phi_* : \mathrm{HP}_*(L) \to \mathrm{HP}_*(J)$ is an isomorphism.*

Proof. By Lemma 5, the $\mathbf{k}$-modules L and J have the same support

$$Y := \mathrm{supp}\, L = \mathrm{supp}\, J,$$

a closed subset of $\mathrm{Max}(\mathbf{k})$ in the Zariski topology. we can replace $\mathbf{k}$ with a quotient and assume that $Y = \mathrm{Max}(\mathbf{k})$.

We shall prove the theorem by induction on $n = \dim \mathrm{Max}(\mathbf{k})$. If $\dim \mathrm{Max}(\mathbf{k}) = 0$, then $\mathbf{k}$ is finite dimensional and hence J and L are also finite dimensional algebras. We can then use the argument at the end of the proof of Proposition 2 to replace L and J by their semi-simple quotients, in which case the result is straightforward.

We now proceed with the inductive step. Assume that

> (A) *if $\phi_0 : L_0 \to J_0$ is a spectrum preserving morphism of finite type $\mathbf{k}_0$-algebras and $\dim \mathbf{k}_0 < n$, then $(\phi_0)_* : \mathrm{HP}_*(L_0) \to \mathrm{HP}_*(J_0)$ is an isomorphism.*

We want to prove then that $\phi_* : \mathrm{HP}_*(L) \to \mathrm{HP}_*(J)$ is an isomorphism for all spectral preserving morphisms between finite type $\mathbf{k}$-algebras with $\dim \mathbf{k} = n$.

We shall use the induction assumption and excision in periodic cyclic homology to successively reduce the proof of the isomorphism $\phi_* : \mathrm{HP}_*(L) \to \mathrm{HP}_*(J)$ to the proof of an isomorphism of the same kind, but with different algebras L and J, which can then be handled directly. We now begin our sequence of (six) reductions.

Step 1. First, let $\mathrm{nil}(\mathbf{k})$ be the nilradical of $\mathbf{k}$. Because $\mathrm{nil}(\mathbf{k})L$ and $\mathrm{nil}(\mathbf{k})J$ are nilpotent, they are contained in the Jacobson radicals of L and, respectively, J and have vanishing periodic cyclic homology groups. By Corollary 3, the induced morphism $\phi_1 : L/\mathrm{nil}(\mathbf{k})L \to J/\mathrm{nil}(\mathbf{k})J$ is also spectrum preserving. By excision and Lemma 2, ϕ_* is an isomorphism if, and only if, $(\phi_1)_*$ is an isomorphism. We can then replace L by $L/\mathrm{nil}(\mathbf{k})L$, J by $J/\mathrm{nil}(\mathbf{k})J$, and $\mathbf{k}$ by $\mathbf{k}/\mathrm{nil}(\mathbf{k})$. This shows that we can also assume that $\mathbf{k}$ is reduced.

Step 2. Then, using Lemma 2, the fact that a nilpotent ideal has vanishing periodic cyclic homology, and Lemma 7, we see that, after replacing L by $L/\ker(\phi)$, if necessary, we can also assume that the morphism ϕ is injective.

Step 3. Using induction on the number of irreducible components of $\mathrm{Max}(\mathbf{k})$, we can further reduce to the case when $\mathbf{k}$ is an integral domain, as follows. Let $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_r$ be the minimal prime ideals of $\mathbf{k}$. Then $\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \dots \cap \mathfrak{p}_r = 0$, because we have assumed $\mathbf{k}$ to be reduced. Let

$$J_j := (\mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_j)J$$

and $L_j := \phi^{-1}(J_j)$. Then the induced maps $L_j/L_{j+1} \to J_j/J_{j+1}$ are injective and spectrum preserving, by Proposition 2. Moreover, the $\mathbf{k}$-module structure on J_j/J_{j+1} descends to a $\mathbf{k}/\mathfrak{p}_{j+1}$-module structure. This shows that, after replacing $\mathbf{k}$ by $\mathbf{k}/\mathfrak{p}_{j+1}$ and $L \to J$ by $L_j/L_{j+1} \to J_j/J_{j+1}$, we can also assume that $\mathbf{k}$ is an integral domain.

Step 4. Using Lemma 2, the fact that a nilpotent ideal has vanishing periodic cyclic homology, and Corollary 1, we can also assume that J is semi-primitive, eventually after replacing J by $J/\mathrm{Jac}(J)$ and L by $L/\phi^{-1}(\mathrm{Jac}(J))$.

Step 5. For any $\mathbf{k}$-module M, we shall denote by $Tor(M)$ the set of torsion elements of M, that is, the set of elements whose annihilator contains at least one non-zero element. We already know that $Tor(L)$ and $Tor(J)$ are two-sided ideal in L and, respectively, J (see the proof of Lemma 2). Because $\phi : L \to J$ is injective, $Tor(L) = \phi^{-1}(Tor(J))$. Proposition 2 implies then that both $Tor(L) \to Tor(J)$ and $L/Tor(L) \to J/Tor(J)$ are injective and spectrum preserving. Because the support of $Tor(L)$ (= the support of $Tor(J)$, by Lemma 5) has smaller Krull dimension, using the excision property of periodic cyclic cohomology, we see from the inductive hypothesis that it is enough to prove that $L/Tor(L) \to J/Tor(J)$ induces an isomorphism in periodic cyclic cohomology. In other words, we can also assume that both L and J are torsion free.

We need to check however that this reduction does not affect the previous reductions. More precisely, we need to check that the quotient $J/Tor(J)$ is still semi-primitive. Indeed, $\mathrm{Jac}(J/Tor(J))$ coincides with the nil-radical of $J/Tor(J)$, so it is enough to prove that if $I \subset J$ is a two-sided ideal such that $I^n \subset Tor(J)$, then $I \subset Tor(J)$. If $I^n \subset Tor(J)$, because $\mathbf{k}$ is noetherian, there exists $f \neq 0$ such that $fI^n = 0$. But then $(fI)^n = 0$, so $fI = 0$, because J is semi-primitive.

Step 6. Our next reduction is to show that we can assume that L is semi-primitive too. Let $\mathbb{K} = S^{-1}\mathbf{k}$ be the field of fractions of $\mathbf{k}$ (so $S = \mathbf{k}^*$ because we have reduced to the case when $\mathbf{k}$ is an integral domain). The algebra $B := L \otimes_{\mathbf{k}} \mathbb{K}$ is a finite dimensional algebra over the field $\mathbb{K}$ and it satisfies

$$\operatorname{Jac}(B) \cong \operatorname{Jac}(L) \otimes_{\mathbf{k}} \mathbb{K} \quad \text{and} \quad B/\operatorname{Jac}(B) \cong (L/\operatorname{Jac}(L)) \otimes_{\mathbf{k}} \mathbb{K}.$$

Moreover, B splits, in the sense that B contains a subalgebra which maps isomorphically to $B/Jac(B)$. Since L is torsion free, $L/\operatorname{Jac}(L)$ is also torsion free, and hence B will also contain an algebra L_1 which projects isomorphically onto $L/\operatorname{Jac}(L)$ via the natural map $L_1 \to B \to B/\operatorname{Jac}(B)$.

Because L_1 is a finitely generated $\mathbf{k}$-module, there exists then $f \in \mathbf{k}$, $f \neq 0$, such that $fL_1 \subset L$ (we used here the fact that L is torsion free, so it identifies with a subalgebra of B).

Let $J_1 := fJ$. Then $\phi : L \to J$ maps $L_2 := f^2 L_1 \subset fL$ to J_1. We have that $V(L_2) = \Theta_L^{-1}(V(f^2\mathbf{k}))$ and $V(J_1) = \Theta_J^{-1}(V(f\mathbf{k}))$. Consequently,

$$\phi^*(V(J_1)) = V(L_2),$$

and hence the induced map $L_2 \to J_1$ is spectrum preserving, by Corollary 3. By construction, both L_2 and J_1 are semi-primitive.

Moreover, the $\mathbf{k}$-module structures on L/L_2 and J/J_1 descend to $\mathbf{k}/(f^2)$-module structures, and the maximal ideal spectrum of $\mathbf{k}/(f^2)$ has dimension. Using the excision property of periodic cyclic cohomology and the inductive hypothesis, we see that it is enough to prove that $L_2 \to J_1$ induces an isomorphism in periodic cyclic cohomology. We obtain finally that in the induction step we can also assume both L and J to be semi-primitive. This completes our sequence of reductions.

We stress that each reduction is in addition to the previous reductions. In this way, we obtained that – in order to establish the inductive step – it is enough to prove the following:

> (R)*: if* $\phi : L \to J$ *is a spectrum preserving morphism of finite type* $\mathbf{k}$-*algebras, where* $\mathbf{k}$ *is an integral domain,* ϕ *is injective, and* L *and* J *are semi-primitive, then* $\phi_* : \mathrm{HP}_*(L) \to \mathrm{HP}_*(J)$ *is an isomorphism.*

(We nolonger need L or J to be torsion free as $\mathbf{k}$-modules, that assumption was necessary only to reduce to the case where we can assume L to be semi-primitive.)

We now prove (R), assuming (A) (the inductive assumption). The advantage of (R) is that we can now use Proposition 5 to conclude that the set

$$S_4 := \{\mathfrak{p} \in \operatorname{Max}(\mathbf{k}),\ L_{\mathfrak{p}} \ \text{ and } J_{\mathfrak{p}} \ \text{ are unital and } \phi_* : \mathrm{HH}_*(L_{\mathfrak{p}}) \cong \mathrm{HH}_*(J_{\mathfrak{p}})\,\}$$

is open and dense in $\operatorname{Max}(\mathbf{k})$. Let $I \subset \mathbf{k}$ be the ideal corresponding to this open set. Define $L_1 := L \cap IJ^+$ and $J_1 := J \cap IJ^+ = IJ$. Then $L_1 \subset J_1 \subset IJ^+$ satisfy the assumptions of Theorem 6 for the ideal I (we use here also that $I_{\mathfrak{p}} \cong \mathbf{k}_{\mathfrak{p}}$, if $I \not\subset \mathfrak{p}$), and hence the inclusion $L_1 \to J_1$ induces an isomorphism $\mathrm{HP}_*(L_1) \to \mathrm{HP}_*(J_1)$.

The induced morphism $\phi_1 : L/L_1 \to J/J_1$ is spectrum preserving, by Proposition 2. By construction, the $\mathbf{k}$-module structure on the quotients L/L_1 and J/J_1 descends to a $\mathbf{k}/I$-module structure, and the maximal ideal spectrum of $\mathbf{k}/I$ has smaller dimension than the dimension of $\operatorname{Max}(\mathbf{k})$. By the induction hypothesis, ϕ_1 induces an isomorphism in periodic cyclic cohomology. The excision property then gives that $L \to J$ induces also an isomorphism in periodic cyclic cohomology. The induction step is now established, so our theorem is proved. $\square$

5. Homology of Hecke algebras

Because the converse of Proposition 2 is not true, the assumption that a morphism be spectrum preserving is usually too restrictive in applications. However, the following weaker version of the spectrum preserving property is usually enough.

Definition 4. *A morphism $\phi : L \to J$ of finite type algebras is called* weakly spectrum preserving *if, and only if, there exist increasing filtrations*

$$(0) = L_0 \subset L_1 \subset \ldots \subset L_n = L \quad \text{and} \quad (0) = J_0 \subset J_1 \subset \ldots \subset J_n = J$$

of two-sided ideals such that $\phi(L_k) \subset J_k$ and the induced morphisms $L_k/L_{k-1} \to J_k/J_{k-1}$ are spectrum preserving.

Using the excision property of periodic cyclic homology, we see that Theorem 7 immediately implies the following only slightly stronger, but much more useful, form of that theorem:

Theorem 8. *Let $\phi : L \to J$ be a weakly spectrum preserving morphism of finite type $\mathbf{k}$-algebras. Then the induced map $\phi_* : \mathrm{HP}_*(L) \to \mathrm{HP}_*(J)$ is an isomorphism.*

It is interesting to note that if we denote by $R(A)$ the Grothendieck group of finite dimensional modules over A and if $\phi : L \to J$ is a weakly spectrum preserving morphism, then we obtain an induced map $\phi^* : R(J) \otimes \mathbb{Q} \to R(L) \otimes \mathbb{Q}$, which is an isomorphism. (Compare to [35] and note that in that case there is no need to tensor with $\mathbb{Q}$.) Also, we obtain bijections $\mathrm{Prim}(J_k/J_{k-1}) \to \mathrm{Prim}(L_k/L_{k-1})$, which in turn induce a bijection $\mathrm{Prim}(J) \to \mathrm{Prim}(L)$. This bijection is not continuous, in general, and it may also depend on the choice of the filtrations used to check the weakly spectrum preserving property.

We now explain how the above theorem and the results of Kazhdan-Lusztig [28] and Lusztig [33, 34, 35, 36] and [37] lead to a determination of the periodic cyclic homology of Iwahori-Hecke algebras H_q, for $q \in \mathbb{C}^*$ not a proper root of unity. (See also [47] for $q = 1$.) We begin by recalling the classical definition of Hecke algebras (we shall work here with affine Hecke algebras and extended affine Hecke algebras).

Fix a root system $\Phi \subset E$ in a Euclidean space and let W be the affine Weyl group generated by Φ, that is, the group of affine isometries of E generated by the reflections with respect to all hyperplanes $H_{\alpha,k} := \{\lambda \in E, \, (\lambda, \alpha) = k\}$, where $k \in \mathbb{Z}$ and $\alpha \in \Phi$. (See [24]). Let W_0 be the finite Weyl group associated to Φ (it is the group generated by the reflections relative to the planes $H_{\alpha,0}$. It is well known that W is a Coxeter group (see [24], Theorem 4.6). We fix a system of generators $S \subset W$ such that (W, S) is a Coxeter system. The resulting length function on W will be denoted by l, as in [37].

Then the *Iwahori-Hecke algebra H_q associated to W* is the algebra generated by T_x, $x \in W$, with relations

$$\begin{aligned} T_x T_y = T_{xy}, \ \ &\text{if } l(xy) = l(x) + l(y), \quad \text{and} \\ (T_s - q)(T_s + 1) = 0, \ \ &\text{if } s \in S. \end{aligned} \tag{11}$$

Let Ω be a group acting by automorphisms on (W, S). That is, Ω acts by automorphisms on W and maps S to itself. Let $\widehat{W} := W \rtimes \Omega$. We extend l to $\widehat{W}$ by $l(w\omega) = l(w)$, if $w \in W$ and $\omega \in \Omega$. Then the *Iwahori-Hecke algebra H_q associated* *to* $\widehat{W}$ is the algebra generated by T_x, $x \in \widehat{W}$, and with exactly the same relations as those of the Equation 11.

We now proceed to recall the definition of Lusztig's "asymptotic Iwahori-Hecke algebra" J. We need first to recall the definition of the generic Hecke algebra $\mathbb{H}_{\overline{\mathbf{q}}}$ and of the Kazhdan-Lusztig polynomials. Recall that $\mathbb{H}_{\overline{\mathbf{q}}}$ is the (complex) algebra with generators $\overline{\mathbf{q}}$, r, and T_x, $x \in W$, satisfying the relations

$$T_xT_y = T_{xy}, \text{ if } l(xy) = l(x) + l(y), \quad rT_x = T_xr,\ r^2 = \overline{\mathbf{q}} \quad \text{and} \\ (T_s - \overline{\mathbf{q}})(T_s + 1) = 0, \quad \text{if } s \in S. \tag{12}$$

The map $r \to r^{-1}$ and $T_w \to T_{w^{-1}}^{-1}$, $w \in W$, defines an involution $\kappa : \mathbb{H}_{\overline{\mathbf{q}}} \to \mathbb{H}_{\overline{\mathbf{q}}}$. Then [27], for any $w \in W$, there exists a unique element $C_w \in \mathbb{H}_{\overline{\mathbf{q}}}$ such that $\kappa(C_w) = C_w$ and

$$C_w = r^{-l(w)} \sum_{y \leq w} P_{y,w} T_y, \quad y \in W, \tag{13}$$

where $P_{w,w} = 1$ and $P_{y,w}$ is a polynomial in r of degree $\leq l(w) - l(y) - 1$, if $y < w$. The polynomials $P_{y,w}$ are called the *Kazhdan-Lusztig* polynomials.

Let $h_{w,u,v} \in \mathbb{C}[r, r^{-1}]$ be defined by

$$C_wC_u = \sum_{v \in W} h_{w,u,v} C_v. \tag{14}$$

For any $v \in W$, let $a(v)$ be the smallest non-negative integer i with the property that $r^i h_{w,u,v}$ has no strictly negative powers of r. Denote then by $\gamma_{w,u,v}$ the constant term of $(-r)^i h_{w,u,v}$.

The *asymptotic Hecke algebra* J is defined then as the algebra generated by t_w, $w \in W$, and relations

$$t_xt_y = \sum_{z \in W} \gamma_{x,y,z} t_z.$$

Let $\mathcal{D} := \{w \in W, \deg P_{e,w} = l(w) - a(w)\}$. Define the map

$$\phi : \mathbb{H}_{\overline{\mathbf{q}}} \to J \otimes \mathbb{C}[r, r^{-1}], \quad \phi(C_w) = \sum_{a(d)=a(u)} h_{w,d,u} t_u, \tag{15}$$

where in the above sum $w \in W$ and $d \in \mathcal{D}$ are arbitrary subject to the condition that $a(d) = a(u)$. Then ϕ is a morphism of $\mathbb{C}[r, r^{-1}]$ algebras, see [34]. (See also [49], page 23.) Denote by $\phi_q : H_q \to J$ the morphism obtained by specializing at $\overline{\mathbf{q}} = q$ (using a fixed choice of a square root of q, which thus induces a morphism $\mathbb{C}[r, r^{-1}] \to \mathbb{C}$.)

Fix now $q \in \mathbb{C}^*$. We know that if we denote by $\mathbf{k}$ the center of H_q, then H_q and J are finite type $\mathbf{k}$-algebras, see [35], Proposition 1.6; the $\mathbf{k}$-module structure on J is obtained via ϕ_q. The following theorem is due to Lusztig.

Theorem 9 (Lusztig). *If $q = 1$ or $q \in \mathbb{C}^*$, then the morphism $\phi_q : L = H_q \to J$ is weakly spectrum preserving.*

Proof. Let $J^{(i)} \subset J$ be the ideal generated by t_w, $a(w) = i$. Then $J^{(i)} \subset J$ is a two-sided ideal and $J \cong \oplus_i J^{(i)}$. Define $J_k = \oplus_{i \geq k} J^{(i)}$ and $L_k \subset L := H_q$ by $L_k = \phi_q(J_k)$. Then the results of [37], Theorem 8.1, prove the theorem. ☐

Remark. The filtration J_k used in the above proof should be compared with the filtrations from [35], Theorem 3.4 and Corollary 3.6, or [49], 5.9. The result in [37] does not involve taking the "dual" $E \to E^*$ of a J-module, so we get our chosen inequality sign, the other two references provide the opposite inequality sign, that is, we have to define then $J_k = \oplus_{i \leq k} J^{(i)}$.

A consequence of the above computation is the following result on the periodic cyclic homology of the algebras J and H_q, for $q \in \mathbb{C}^*$ not a proper root of unity.

Theorem 10. *Assume that $q \in \mathbb{C}^*$ is not a proper root of unity. Then Lusztig's morphisms $\phi_q : H_q \to J$, see Equation* (15)*, induce isomorphisms*

$$(\phi_q)_* : \mathrm{HP}_*(H_q) \to \mathrm{HP}_*(J).$$

For $q = 1$ we can find $\mathrm{HP}_*(H_q)$ explicitly, so this leads to a determination of the groups $\mathrm{HP}_*(H_q)$, provided that $q \in \mathbb{C}^*$ is not a proper root of unity. Indeed, let $\widehat{W}$ be the extended affine Weyl group used in the definition of the algebras H_q and J. Then $H_1 \cong \mathbb{C}[\widehat{W}]$.

Assume now that $\widehat{W}$ acts properly and isometrically on a Euclidean space E (this is the case for example if Ω is commutative). The space E contains the root system Φ, but is not necessarily generated by Φ. It follows then that any subgroup $G \subset \widehat{W}$ will have finite cohomological dimension over a characteristic zero field. Indeed, this is because

$$\mathrm{H}_*(G; \mathbb{C}) \cong \mathrm{H}_*(E/G; \mathbb{C}). \tag{16}$$

We continue to assume that $\widehat{W}$ acts properly and isometrically on a Euclidean space E. Denote by $o(g)$ the linear part of an isometry g, defined by writing the group of isometries as the direct product of the group of linear isometries (=isometries fixing the origin) with the group of translations. If $g \in G$, we shall denote by $\widehat{W}_g := \{g' \in \widehat{W},\, gg' = g'g\}$, that is the *centralizer of g in $\widehat{W}$*. If $g \in \widehat{W}$ is an element of infinite order, let E_0 be the subspace fixed by $o(g)$. If $g' \in G$ commutes with g, then g' maps E_0 to itself. By definition, g acts by translations on E_0. Let E' be the one-dimensional subspace generated by the direction of this translation and let $(g) \cong \mathbb{Z}$ be the subgroup generated by g. Then E_0/E' is acted upon *properly* by $\widehat{W}/(g)$. This shows that all groups of the form $\widehat{W}_g/(g)$ have finite cohomological dimension.

Let $\langle G\rangle'$ be the set of finite order conjugacy classes of a group G, let $\langle G\rangle'$ be the set of infinite order conjugacy classes of G and $\langle G\rangle := \langle G\rangle' \cup \langle G\rangle''$. If $g \in G$, we shall denote by $G_g := \{g' \in G,\, gg' = g'g\}$, that is the *centralizer of g in G*. Recall now a result of Burghelea [13] (see also [26] and [39]), which states that the *cyclic homology groups* of $\mathbb{C}[G]$ are given by

$$\mathrm{HC}_p(\mathbb{C}[G]) \cong \bigoplus_{\langle g\rangle \in \langle G\rangle'} (\oplus_{k \geq 0} \mathrm{H}_{p-2k}(G_g; \mathbb{C})) \bigoplus \bigoplus_{\langle g\rangle \in \langle G\rangle''} \mathrm{H}_p(G_g/(g); \mathbb{C}). \tag{17}$$

Moreover, the natural morphism $S : \mathrm{HC}_{p+2}(\mathbb{C}[G]) \to \mathrm{HC}_p(\mathbb{C}[G])$ preserves the above direct sum decomposition according to conjugacy classes and is the natural projection on each component corresponding to a finite order conjugacy class.

This gives the following description of the groups $\mathrm{HP}_p(H_q)$ and $\mathrm{HP}_*(\mathbb{C}[\widehat{W}])$.

Theorem 11. *Assume that $q \in \mathbb{C}^*$ is not a proper root of unity and let $\widehat{W}$ be an extended affine Weyl group acting properly on a Euclidean space E. Then the map*

$S : \mathrm{HC}_{p+2}(\mathbb{C}[\widehat{W}]) \to \mathrm{HC}_p(\mathbb{C}[\widehat{W}])$ *is surjective for* $p \geq \dim E$ *and*

$$\mathrm{HP}_p(H_q) \cong \mathrm{HP}_p(\mathbb{C}[\widehat{W}]) \cong \bigoplus_{\langle g\rangle \in \langle \widehat{W}\rangle'} \Big(\oplus_{k\in\mathbb{Z}} \mathrm{H}_{p-2k}(\widehat{W}_g;\mathbb{C}) \Big)$$
$$\cong \bigoplus_{\langle g\rangle \in \langle \widehat{W}\rangle'} \Big(\oplus_{k\in\mathbb{Z}} \mathrm{H}_{q-2k}(E/\hat{W}_g;\mathbb{C}) \Big) .$$

All these isomorphisms are natural.

We note that it is sometimes convenient to use $\mathrm{H}_*(E/\widehat{W}_g) \cong \mathrm{H}_*(E^g/\widehat{W}_g)$, if g is a finite order element of $\hat{W}$. We conclude this section with a few remarks.

Let Ω be a discrete group acting by automorphisms on the finite type $\mathbf{k}$–algebras L and J, and let $L \rtimes \Omega$ and $J \rtimes \Omega$ be the corresponding crossed-products. If $\phi : L \to J$ is Ω-equivariant, spectrum preserving morphism, then the induced map $\phi \rtimes \Omega : L \rtimes \Omega \to J \rtimes \Omega$ is also spectrum preserving. This is proved using an explicit description of the primitive ideal spectra of $L \rtimes \Omega$ and $J \rtimes \Omega$. If ϕ is only weakly spectrum preserving, but the defining filtrations of Definition 4 are G-invariant, then $\phi \rtimes \Omega : L \rtimes \Omega \to J \rtimes \Omega$ will be again weakly spectrum preserving. The example we have in mind is when L is the Iwahori-Hecke algebra associated to an affine Weyl group W and J is the corresponding asymptotic Hecke algebra. Then $L \rtimes \Omega$ and $J \rtimes \Omega$ are the Hecke, respectively, asymptotic Hecke algebras of $\widehat{W}$, for a suitable action of Ω.

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Pennsylvania State University, Math. Dept., University Park, PA 16802, USA
E-mail address: baum@math.psu.edu

Pennsylvania State University, Math. Dept., University Park, PA 16802, USA
E-mail address: nistor@math.psu.edu