INTEGRAL MOTIVES OF QUADRICS

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0. Introduction

In this paper we will construct the motivic decomposition of arbitrary quadric in the triangulated category of motives DM^{eff} (with $\mathbb{Z}$-coefficients) over a field k

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(see Definition 2.1.8), and then will give some applications of this decomposition to various questions from quadratic form theory.

We will work in the motivic category $DM^{eff}(k)$, constructed by V.Voevodsky in [17]. The usual category of *Chow motives* (see Definition 2.2.3) has natural full embedding into $DM^{eff}(k)$, and the image is closed under the taking of direct summands. In particular, for smooth projective variety Q, the corresponding motive $M(Q)$ has the same direct sum decomposition in $DM^{eff}(k)$, as in $Chow(k)$.

The motive of a quadric with rational coefficients is very simple: for odd-dimensional quadric it is the same as the motive of a projective space of the same dimension. With integral coefficients situation is much more subtle.

The investigation of the structure of a motive of a quadric (with integral coefficients) (in the category of *Chow motives*) was initiated by M.Rost in [14], who showed that the motive of a Pfister quadric (see Definition 2.4.3), corresponding to the *pure* symbol $\alpha = \{a_1, \ldots, a_n\} \in \mathrm{K}_n^M(k)/2$ (see Definition 2.4.4), can be decomposed in the category of *Chow motives* into a direct sum of 2^{n-1} simpler motives (so-called, Rost motives) $M_\alpha(i)[2i]$, each of which, shifted by appropriate Tate-motive $\mathbb{Z}(-i)[-2i]$ (see Definition 2.1.15), becomes, after extending k to $\bar{k}$, equal to the sum of just two Tate-motives: $\mathbb{Z}$ and $\mathbb{Z}(2^{n-1}-1)[2^n-2]$. As was shown by V.Voevodsky (see Theorem 4.5 from [18]), M_α itself, as an object of $DM^{eff}(k)$, can be constructed as an extension of two simpler things (each of which becomes a Tate-motive over $\bar{k}$); namely, $M_\alpha = \mathrm{Cone}[-1]\left(\mathcal{X}_Q \to \mathcal{X}_Q(2^{n-1}-1)[2^n-1]\right)$. Here $\mathcal{X}_Q$ is a motive of a standard simplicial scheme, corresponding to the morphism $Q \to \mathrm{Spec}(k)$ ($\mathcal{X}_Q^i = Q \times_k \cdots \times_k Q$ - $i+1$-times, with natural maps of faces and degenerations (see Definition 2.3.1)), in particular, we have a natural morphism from $\mathcal{X}_Q$ to $M(\mathrm{Spec}(k)) =: \mathbb{Z}$, which is an isomorphism iff Q has a rational k-point). Combining this result with the one of M.Rost about *excellent* quadrics (see [14], Proposition 4), we get that the motive $M(Q)$ is an *extension* (see Definition 2.5.6) of $2[\dim(Q)/2]+2$ "simple" parts for arbitrary *excellent* quadric (see [14], discussion after Proposition 4 for the definition).

The first attempts to try the non-excellent case were undertaken by D.Orlov in the case of 3-dimensional non-Pfister quadric, who shared his ideas with me.

In the Theorem 3.1 of Section 3 below we will show, how to generalize the motivic decomposition above (i.e.: the presentation of $M(Q)$ as an *extension* of $2[\dim(Q)/2]+2$ "simple" parts) to the case of arbitrary quadric. Then in Theorem 3.7 we will prove that this decomposition is completely canonical. Motive of Q appears in our construction equipped with the canonical Postnikov tower (*Postnikov tower* in the sense of triangulated category - just the system of distinguished triangles) with graded parts - forms of Tate-motives (i.e. motives, which become equal to $\mathbb{Z}(i)[2i]$ over $\bar{k}$). We can rearrange terms of this Postnikov tower (using octahedron axiom, see Definition 2.5.3 , axiom TR4) and get different presentations for $M(Q)$, which are sometimes useful. This is done in Proposition 3.4 and Proposition 3.6 . Theorem 3.7 on it's part give us more, than just canonicity, it shows, that any endomorphism of $M(Q)$ can be prolonged to our Postnikov tower in a unique way. This gives us that any nontrivial direct summand in $M(Q)$ should consist of the same "simple" parts as $M(Q)$, in particular, it should be non-zero over $\bar{k}$ (Lemma 3.23), and that idempotents of $\mathrm{End}(M(Q))$ are mapped surjectively under natural morphism $\alpha : \mathrm{End}(M(Q)) \to \mathrm{End}(M(Q|_{\bar{k}}))$ onto idempotents of $image(\alpha) \cap \mathrm{End}(M(Q|_{\bar{k}}))$ (Lemma 3.12). Using this, and the fact that the category of Chow motives $Chow^{eff}(k)$ is a full subcategory of $DM^{eff}(k)$ closed under direct summands, we give in Corollary 3.14 a criteria of decomposability of the motive of a quadric in the category of Chow motives $Chow(k)$ (see Definition 2.2.3 and Theorem 2.2.4). The same considerations also permit to bound the order of any element of $\mathrm{Aut}(M(Q))$, and bound "nonabelianness" of this group (Corollary 3.17 and Corollary 3.19).

In Section 4 I describe some operations F_α, $\alpha \in \mathrm{K}_n^M(k)/2$-*pure*, which act on the motives of quadrics (and, presumably, on all direct summands of them as well), producing *higher forms* of them (see Theorem 4.1 and discussion after it). This gives a generalization of Rost-motives, since $F_{\{a_2,\dots,a_n\}}(M(k(\sqrt{a_1}))) = M_{\{a_1,\dots,a_n\}}$.

In Section 5 I show that "elementary pieces" from Theorem 3.1 define motive of a quadric uniquely. Equivalently, if two quadrics have equivalent *universal splitting towers* (see Definition 2.4.19), then their motives are isomorphic. Taking into account the fact that we believe the motive of a quadric should define quadric itself up to isomorphism, this can provide us (after the proving of the last fact) with some method to check: are two quadratic forms proportional, or not.

In Section 6 I show some applications of our methods to the computation of the *Witt numbers* (see Definition 2.4.19), and formulate some open questions.

The main results of the paper are contained in sections 3,4,5 and 6.

Acknoledgements: This work uses the techniques, developed by V.Voevodsky in [18], [19], and by D.Orlov-V.Voevodsky in [13]. During my work I had constant discussions with Volodia Voevodsky, which cleared many things for me and helped me a lot. I'm very grateful to him, and to Dima Orlov (discussions with whom pushed me to consider the case of general quadric). Also, I would like to thank Lyonia Positselsky, Yuri Tschinkel and Vadim Vologodsky for very helpfull discussions.

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1. Some examples

Here we give few examples of to what in the classical world our methods can be applied. Everywhere we will assume that characteristic of our field k is 0.

1.1. **On the kernels in Milnor's K-theory under function field extensions.**

The first application settles some very particular case of the following question:

Question.

Let Q be a quadric, defined over the field k.
Is it true, that the ideal $\mathrm{Ker}_Q := \mathrm{Ker}\left(\mathrm{K}^M_*(k)/2 \to \mathrm{K}^M_*(k(Q))/2\right)$ *is generated by pure symbols?*

The positive answer to the above question would imply the one to the following two things:

1) Conjecture by B.Kahn, M.Rost and R.Sujatha, see the original version of [8], Conjecture on p.6 (the new version does not contain it). Here the authors basically conjectured the same for the Ker_Q in degrees $< \log_2(\dim(Q)+2)$ (among other things, which were settled in [13]).

2) Question: If any *conservative* quadric (see Definition 2.4.10) is *embedable*, i.e. can be represented as a subquadric in a *big Pfister quadric* (see Definition 2.4.3) ?

We have the positive answer to our question in the case of a *Pfister neighbor* (see Definition 2.4.6) in ([13], exact sequence (5) on p.19). Here we get it for the case of 2-dimensional non-Pfister quadric.

Statement 1.1.1 .

Let Q be a 2-dimensional quadric corresponding to the form $\langle 1, -a, -b, -c\rangle$, $-abc \notin (k^*)^2$. Then Ker_Q is generated by pure 3-symbols $\{a, b, x\}$, where $x \in \mathrm{Nrm}_{k\sqrt{-abc}/k}\left(\left(k\sqrt{-abc}\right)^*\right)$.

We will give a proof of Statement 1.1.1 in Section 2.6 , after we introduce some definitions.

1.2. **After the J-filtration conjecture.**

Another application is related to the J-filtration conjecture.

On the *Witt ring* of quadratic forms $W(k)$ (see Definition 2.4.2) we have two natural filtrations: I and J - filtration. I filtration corresponds to the powers of the ideal I of even-dimensional forms (see the discussion after Definition 2.4.2). And J - filtration can be constructed in the following way: let q be even-dimensional anisotropic quadratic form, and Q - corresponding projective quadric, then over $k(Q)$ Q will have a rational point, and so, q will represent 0, we have $q|_{k(Q)} = h_1 \cdot \mathbb{H} \perp q_1$, where $h_1 > 0$, $\mathbb{H}$ is the *elementary hyperbolic form* $\langle 1, -1\rangle$, and q_1 is some anisotropic form, defined over $k(Q)$ (such form is unique). Certainly, $\dim(q_1) < \dim(q)$. Over $k(Q)(Q_1)$ we have: $q_1|_{k(Q)(Q_1)} = h_2 \cdot \mathbb{H} \perp q_2$, where $h_2 > 0$, and q_2 is some anisotropic form defined over $k(Q)(Q_1)$, again, $\dim(q_2) < \dim(q_1)$. After few such steps we will have: $q_s = 0$. That means that q_{s-1} is hyperbolic over generic point of it's projective quadric. The only quadric which has such a property is so-called *Pfister quadric* (see Definition 2.4.3 , and [6], Theorem 5.8), i.e. quadric corresponding to the 2^n - dimensional quadratic form $q_{\{a_1,\dots,a_n\}} = \langle 1, -a_1\rangle \otimes \langle 1, -a_2\rangle \otimes \dots \otimes \langle 1, -a_n\rangle$, where $a_1, \dots, a_n$ is some set of nonzero field elements.

So, to each quadratic form q we can assign it's *degree* - the nonnegative integer - the number n, appearing in the *Pfister form* above. Now we can define $J_n(W(k))$ as the set of all quadratic forms of degree $\geqslant n$.

We have: $I^n \subset J_n$ ([6], Corollary 6.6), and it was conjectured (see [6], Question 6.7), that $I^n = J_n$.

This conjecture was settled in [13], Statement 2 of Section 3.3.
The numbers: $h_1, h_2, \dots, h_s$ above are called *Witt numbers* (see Definition 2.4.19). The J-filtration conjecture can be reformulated as: if $q \in I^n(W(k))\backslash I^{n+1}(W(k))$, then $h_s = 2^{n-1}$.

Now we can say something about h_{s-1}:

Statement 1.2.1 (see Statement 6.2).
Suppose our field k contains $\sqrt{-1}$, and $q \in I^n\backslash I^{n+1}$. Then $h_{s-1} \in \{2^r - 2^n, \text{ for } r > n; 2^m, \text{ for } 0 \leqslant m \leqslant n-1\}$.

We have some understanding of: what different values of h_{s-1} should mean - see Question 6.3 and Statement 6.2 ($j_i := h_s + h_{s-1} + \dots + h_{s-i+1}$, see the very end of Section 3.1). I would say that the Question 6.3 is an analog of the J-filtration conjecture for h_{s-1}.

Also we can give another bound on the possible behavior of the *Witt numbers.*

Statement 1.2.2 (see Statement 6.1).
Suppose our field k contains $\sqrt{-1}$. Let for some quadratic form q we have: $h_1 > h_i$, for all $i > 1$. Then $\dim(Q) - h_1 = 2^m - 2$ for some m.

For the proof of Statement 1.2.1 and Statement 1.2.2 - see Statement 6.2 and Statement 6.1 .

1.3. **Direct sum decomposition of the *Chow motive* of a quadric.** Using our methods we can give a criteria of decomposability of a *Chow motive* of a quadric.

Let P be a cycle of dimension $n = \dim(Q)$ on $Q \times Q$. Then it gives us a morphism $p : M(Q) \to M(Q)$ in the $Chow(k)$-category of *Chow motives* over k (see Definition 2.2.3). For all $0 \leqslant i < n/2$ we can define $p_i \in \mathbb{Z}$ as an intersection number of $P|_{\overline{k}}$ and $h^i \times l_i \subset (Q \times Q)_{\overline{k}}$, where h^i is a plane section of codimension i, and l_i is a projective subspace of dimension i on $Q|_{\overline{k}}$ (over $\overline{k}$ Q is hyperbolic); in the same way, we can define $p_i' \in \mathbb{Z}$ as an intersection number of $P|_{\overline{k}}$ and $l_i \times h^i$.

We get a map: $\alpha : CH^n(Q\times Q) \to \left(\times_{i=0}^{[\frac{n-1}{2}]}\mathbb{Z}\right)\times\left(\times_{j=0}^{[\frac{n-1}{2}]}\mathbb{Z}\right)$, and $\alpha/2 : CH^n(Q\times Q) \to \times_{l=1,\dots,2[n+1/2]}\mathbb{Z}/2$.

Statement 1.3.1 (Corollary 3.14).

$M(Q)$ is not decomposable into a direct sum in $Chow(k)$ if and only if: $image(\alpha/2) =$ diagonal $\mathbb{Z}/2$ (generated by the image of Δ_Q).

The same techniques gives us some generalization of the result of M.Rost. I remind, that in [14], Proposition 4 M.Rost proved that the *Chow motive* of 2^n-2-dimensional *Pfister quadric* $Q_{\{a_1,\dots,a_n\}}$ with quadratic form: $\langle\langle a_1,\dots,a_n\rangle\rangle := \langle 1,-a_1\rangle\times\cdots\times\langle 1,-a_n\rangle$ (see Definition 2.4.3) is a direct sum: $\oplus_{i=0}^{2^{n-1}-1} M_{\{a_1,\dots,a_n\}}(i)[2i]$, where $M_{\{a_1,\dots,a_n\}}(i)[2i]$ is a *Rost motive* $M_{\{a_1,\dots,a_n\}}$ (this is a definition of it) shifted by the *Tate motive* $\mathbb{Z}(i)[2i]$ (see Definition 2.1.15), and $M_{\{a_1,\dots,a_n\}}|_{\overline{k}} = \mathbb{Z}\oplus\mathbb{Z}(2^{n-1}-1)[2^n-2]$.

It appears that it is a particular case of the following:

Statement 1.3.2 (Theorem 4.1).

Let Q corresponds to the quadratic form q, and R corresponds to the quadratic form $q\times\langle\langle a_1,\dots,a_n\rangle\rangle$.

Then $M(R) = \oplus_{i=0}^{2^n-1}F_{\{a_1,\dots,a_n\}}(Q)(i)[2i]$, if $\dim(Q)$-even, and $= \oplus_{i=0}^{2^n-1}F_{\{a_1,\dots,a_n\}}(Q)(i)[2i] \oplus M(Q_{\{a_1,\dots,a_n\}})(\dim(R)/2-2^{n-1}+1)[\dim(R)-2^n+2]$, if $\dim(Q)$-odd.

And: $F_{\{a_1,\dots,a_n\}}(Q)|_{\overline{k}} = \left(\oplus_{0\leqslant k\leqslant[\dim(Q)/2]}\mathbb{Z}(k\cdot 2^n)[k\cdot 2^{n+1}]\right)\oplus \left(\oplus_{0\leqslant k\leqslant[\dim(Q)/2]}\mathbb{Z}((\dim(Q)+1-k)\cdot 2^n-1)[(\dim(Q)+1-k)\cdot 2^{n+1}-2]\right)$.

Notice, that the number of Tate motives in $F_{\{a_1,\dots,a_n\}}(Q)|_{\overline{k}}$ is the same as in $M(Q)|_{\overline{k}}$.

We get an action of the semigroup of *pure* symbols from $\mathrm{K}^M_*(k)/2$ (see Definition 2.4.4) on the *Chow motives* of quadrics. I would call $F_{\{a_1,\dots,a_n\}}(Q)$ - the *higher form* of Q.

In this language: the *Rost motive* $M_{\{a_1,\dots,a_n\}}$ is a *higher form* of a 0-dimensional quadric $k\sqrt{a_1}$: $M_{\{a_1,\dots,a_n\}} = F_{\{a_2,\dots,a_n\}}(k\sqrt{a_1})$.

1.4. ***Chow motive* of a quadric is determined by the *Witt indexes* of the corresponding form over all field extensions.** The following result gives a criteria for the *Chow motives* of two quadrics to be isomorphic.

Statement 1.4.1 (Proposition 5.1).

Let Q_1,Q_2 be projective quadrics. Then $M(Q_1) = M(Q_2)$ if and only if for every field extension E/k, and $i\geqslant 0$, the quadric $Q_1|_E$ contains a (rational) projective subspace of dimension i simultaneously with $Q_2|_E$ (equivalently, $q_1|_E = (i+1)\cdot\mathbb{H}\perp q_1'$ iff $q_2|_E = (i+1)\cdot\mathbb{H}\perp q_2'$).

In some interesting cases we can check that two quadrics have equal *Witt indexes* (see the text after Definition 2.4.2 for the definition), which gives an isomorphism of their *Chow motives* (see Corollary 5.2).

2. Basic definitions and facts

2.1. **Motivic category of V.Voevodsky.** In [17] V.Voevodsky introduced the triangulated category of motives over a field.

Let us briefly give the definition of this category (all the definitions are from [17], and [18]).

Definition 2.1.1 ([17], p.3).

Let X, Y are smooth schemes over k. Then denote as $c(X, Y)$ the free abelian group, generated by integral closed subschemes on $X \times Y$, which are finite over X and surjective over connected component of X.

As usually, as $Sm(k)$ we will denote the category of smooth schemes over k.

Then we can introduce the additive category $SmCor(k)$:

Definition 2.1.2 ([17], p.3).

The objects of $SmCor(k)$ are smooth schemes over k, and $\mathrm{Hom}_{SmCor(k)}(X, Y) := c(X, Y)$.

The composition of morphisms in $SmCor(k)$ is defined as follows: if $\varphi \in c(X, Y)$, and $\psi \in c(Y, Z)$, then $\psi \circ \varphi := p_{13*}(p_{12}^*(\varphi) \cap p_{23}^*(\psi))$, where p_{ij} - is a projection from $X \times Y \times Z$ onto corresponding double product.

The category $SmCor(k)$ has a natural tensor structure given by direct product of schemes on objects and by external multiplication of correspondences on morphisms.

We, certainly, have natural functor $Sm(k) \to SmCor(k)$.

Definition 2.1.3 ([17], Definition 3.1.1).

A presheaf with transfers is an additive contravariant functor from $SmCor(k)$ to the category of abelian groups.

A presheaf with transfers is called a Zariski sheaf with transfers, Nisnevich sheaf with transfers (see [4], Definition 3.2), Etale sheaf with transfers, if, restricted to $Sm(k)$ it is a sheaf in the corresponding topology.

Category of such sheaves will be denote as $Shv_{Zar}(SmCor(k))$, $Shv_{Nis}(SmCor(k))$, and $Shv_{et}(SmCor(k))$ respectively.

Theorem 2.1.4 ([17], Theorem 3.1.4).

The category of Nisnevich sheaves with transfers is abelian (see [5],II,§5,10).

Definition 2.1.5 .

Denote as $D^-(Shv_{Nis}(SmCor(k)))$ the derived category of complexes in $Shv_{Nis}(SmCor(k))$ bounded from above (see [5],III,§2,1).

Definition 2.1.6 ([17], Definition 3.1.9).

A presheaf with transfers F is called homotopy invariant if for any smooth scheme X over k the morphism $X \times \mathbb{A}^1 \to X$ induces isomorphism $F(X) \to F(X \times \mathbb{A}^1)$.

A Nisnevich sheaf with transfers is called homotopy invariant if it is homotopy invariant as a presheaf with transfers.

Theorem 2.1.7 ([17], Proposition 3.1.12).
For any perfect field k the full subcategory $HI(k)$ of the category $Shv_{Nis}(SmCor(k))$ which consists of homotopy invariant sheaves is abelian and the inclusion functor $HI(k) \to Shv_{Nis}(SmCor(k))$ is exact.

Definition 2.1.8 ([17], p.20).
Denote as $DM_-^{eff}(k)$ the full subcategory of $D^-(Shv_{Nis}(SmCor(k)))$ which consists of complexes with homotopy invariant cohomology sheaves.

Theorem 2.1.7 implies that $DM_-^{eff}(k)$ is a triangulated category (see Definition 2.5.3).

To see where the motives of schemes live in $DM_-^{eff}(k)$, we need to introduce another category, which will be a full subcategory of $DM_-^{eff}(k)$.

Definition 2.1.9 ([17], Definition 2.1.1).
Let $\mathcal{H}^b(SmCor(k))$ be the homotopy category of bounded complexes over $SmCor(k)$. Let T be the thick subcategory in it generated by complexes of the form $X \times \mathbb{A}^1 \to X$, and $U \cap V \to U \oplus V \to X$, for all smooth schemes X over k and all Zariski open coverings $X = U \cup V$.

The triangulated category of mixed motives $DM_{gm}^{eff}(k)$ is the pseudo-abelian envelope (see Definition 2.5.2) of the localization (see [5],III,§2,2) of $\mathcal{H}^b(SmCor(k))$ with respect to the morphisms with cones in T.

It is a tensor triangulated category.

The full embedding $DM_{gm}^{eff}(k) \to DM_-^{eff}(k)$ can be obtained in the following way:
Denote as $\Delta^{\cdot}$ the standard cosimplicial object in $Sm(k)$; i.e. $\Delta^n = \mathbb{A}^n$ given as a subscheme in $\mathbb{A}^{n+1}$: $\Delta^n = \operatorname{Spec}(k[x_1, \dots, x_n]/(\sum x_i - 1))$ with natural morphisms of faces and degenerations.

Definition 2.1.10 ([18], p.7).
For any presheaf on $Sm(k)$ denote by $\underline{C}_*(F)$ the bounded from above complex of presheaves on $Sm(k)$, s.t. for any smooth scheme U $\underline{C}_*(F)(U)$ is the normalization of the simplicial abelian group $F(U \times \Delta^{\cdot})$. $\underline{C}_*(F)$ will be a complex with homotopy invariant cohomology sheaves.

Definition 2.1.11 ([17], p.13).
For any variety X denote as $L(X)$ the presheaf with transfers, representable by X on $SmCor(k)$.

Theorem 2.1.12 ([17], Lemma 3.1.2).
For any smooth scheme X/k $L(X)$ is a sheaf in Nisnevich topology.

The rule: $X \mapsto \underline{C}_*(L(X))$ gives us an additive functor $i' : SmCor(k) \to DM_-^{eff}(k)$.

Theorem 2.1.13 ([17], Theorem 3.2.6).

i' defines the triangulated functor $i : DM^{eff}_{gm}(k) \to DM^{eff}_{-}(k)$, which is a full embedding.

Definition 2.1.14 .

Any smooth projective variety X has it's natural image $M(X)$ in $DM^{eff}_{gm}(k)$ (X is an object of the additive category $SmCor(k)$), and, via i, also in $DM^{eff}_{-}(k)$.

Now we can define *Tate-motives.*

Definition 2.1.15 .

Denote $M(\mathrm{Spec}(k)) := \mathbb{Z}$. This is a unit object with respect to the tensor structure on DM (any of them).

Denote $\mathbb{Z}(1)$ to be $\mathrm{Cone}(M(\mathbb{P}^1) \to M(k))[-3]$.

Denote by $\mathbb{Z}(m)[n]$ the motive $\mathbb{Z}(1)^{\otimes m}[n]$, where $[n]$ is a cohomological shift in triangulated category.

Also for arbitrary object X, denote $X(n)$ to be $X \otimes \mathbb{Z}(n)$.

Notice, that the map $M(\mathbb{P}^1) \to M(k)$ has a section given by any rational point on $\mathbb{P}^1$. Hence, $M(\mathbb{P}^1) = \mathbb{Z} \oplus \mathbb{Z}(1)[2]$.

Definition 2.1.16 .

Motivic cohomology of a motive Y are groups $\mathrm{H}^{j,i}_{\mathcal{M}}(Y) := \mathrm{Hom}_{DM^{eff}_{-}(k)}(Y, \mathbb{Z}(i)[j])$.

If Y is an object of $DM^{eff}_{gm}(k)$ then $\mathrm{H}^{j,i}_{\mathcal{M}}(Y) = \mathrm{Hom}_{DM^{eff}_{gm}(k)}(Y, \mathbb{Z}(i)[j])$, since $DM^{eff}_{gm}(k)$ is a full subcategory of $DM^{eff}_{-}(k)$.

For smooth projective varieties we have the following relation between *motivic cohomology* and *Higher Chow groups* of S.Bloch (see [2], p.1):

Theorem 2.1.17 ([17], Theorem 4.2.9).

Let X be smooth projective variety. Then $\mathrm{H}^{j,i}_{\mathcal{M}}(X) = CH^i(X, 2i-j)$.

In particular, $\mathrm{Hom}_{DM^{eff}_{gm}(k)}(X, \mathbb{Z}(i)[2i]) = CH^i(X)$.

From the previous theorem and the definition of *Higher Chow groups* (see [2], p.1) it follows that:

Theorem 2.1.18 .

For smooth projective variety X $\mathrm{Hom}_{DM^{eff}_{gm}(k)}(X, \mathbb{Z}(i)[j]) = 0$ in the following cases:

1) $i < 0$;

2) $j - i > \dim(X)$;

3) $j > 2i$;

Also, we have:

Theorem 2.1.19 ([15], Proposition 2.2).

For smooth connected scheme X

$$\mathrm{Hom}_{DM_{gm}^{eff}(k)}(X,\mathbb{Z}(0)[j])=\begin{cases}\mathbb{Z}, & \text{for } j=0;\\ 0, & \text{for } j\neq 0.\end{cases}$$

and

$$\mathrm{Hom}_{DM_{gm}^{eff}(k)}(X,\mathbb{Z}(1)[j])=\begin{cases}\mathcal{O}^{*}, & \text{for } j=1;\\ Pic(X), & \text{for } j=2;\\ 0, & \text{for } j\neq 1,2.\end{cases}$$

Some part of motivic cohomology of $\mathbb{Z}$ is given by the following:

Theorem 2.1.20 ([18], Proposition 2.7).

$\mathrm{Hom}_{DM}(\mathbb{Z},\mathbb{Z}(n)[n])=\mathrm{K}_n^M(k)$.

To introduce duality we need to enlarge the category $DM_{gm}^{eff}(k)$.

Definition 2.1.21 ([17], the end of p.5).

$DM_{gm}(k)$ is the following category: Objects of $DM_{gm}(k)$ are pairs (A,n), where A is an object of $DM_{gm}^{eff}(k)$, and n is integer number. The morphisms are defined by the following formula: $\mathrm{Hom}_{DM_{gm}(k)}((A,n),(B,m))=\lim_{k\geqslant -m,-n}\mathrm{Hom}_{DM_{gm}^{eff}(k)}(A(n+k),B(m+k))$

Theorem 2.1.22 ([17], p.6, Corollary 2.1.5, and Theorem 4.3.1).

Suppose k admits resolution of singularities. Then $DM_{gm}(k)$ is a tensor triangulated category and the functor $A\mapsto (A,0)$ is a full embedding.

Theorem 2.1.23 ([17], p.10, Theorem 4.3.7).

For any field k which admits resolution of singularities the category $DM_{gm}(k)$ is a "rigid tensor triangulated category". More precisely, one has:
1) For any pair of objects A,B in $DM_{gm}(k)$ there exists the internal Hom-object $\underline{\mathrm{Hom}}_{DM_{gm}(k)}(A,B)$. We set A^* to be $\underline{\mathrm{Hom}}_{DM_{gm}(k)}(A,\mathbb{Z})$.
2) For any object A in $DM_{gm}(k)$ the canonical morphism $A\to(A^*)^*$ is an isomorphism.
3) For any pair of objects A,B in $DM_{gm}(k)$ there are canonical isomorphisms: $\underline{\mathrm{Hom}}_{DM_{gm}(k)}(A,B)=A^*\otimes B$; $(A\otimes B)^*=A^*\otimes B^*$.
4) For smooth projective variety X/k of dimension n we have natural identification: $M(X)\to\underline{\mathrm{Hom}}_{DM_{gm}(k)}(M(X),\mathbb{Z}(n)[2n])$, given by the generic cycle of diagonal $X\to X\times X$, considered as an element of $\mathrm{Hom}(M(X)\otimes M(X),\mathbb{Z}(n)[2n])$ via Theorem 2.1.17 .

For each $n>0$ we have natural element $n\in\mathrm{Hom}(\mathbb{Z},\mathbb{Z})=\mathbb{Z}$.

Definition 2.1.24 .

$\mathbb{Z}/n:=\mathrm{Cone}(\mathbb{Z}\xrightarrow{n}\mathbb{Z})$.

$\mathrm{H}_{\mathcal{M}}^{j,i}(X,\mathbb{Z}/n):=\mathrm{Hom}_{DM_{gm}}(X,\mathbb{Z}/n)(i)[j])$.

We have action of motivic cohomological operations in motivic cohomology of motives of simplicial smooth schemes with finite coefficients.

From Theorem 2.1.20 , Theorem 2.1.19 it follows that $\mathrm{Hom}_{DM}(\mathbb{Z}/2,\mathbb{Z}/2(1)) = (\mathrm{Hom}_{DM}(\mathbb{Z},\mathbb{Z}(1)[1]))_2$, and the later is $\mathbb{Z}/2$ with the only nontrivial element represented by $\{-1\}$ via identification $\mathrm{Hom}_{DM}(\mathbb{Z},\mathbb{Z}(1)[1]) = \mathrm{K}_1^M(k) = k^*$

Definition 2.1.25 .

Denote as τ the element of the group $\mathrm{Hom}_{DM}(\mathbb{Z}/2,\mathbb{Z}/2(1))$, which correspond to $\{-1\}$ via identification above.

Denote $\rho := \beta(\tau) \in \mathrm{Hom}_{DM}(\mathbb{Z}/2,\mathbb{Z}/2(1)[1])$ (it will be also the image of $\{-1\}$ under natural map: $\mathrm{Hom}_{DM}(\mathbb{Z},\mathbb{Z}(1)[1]) \to \mathrm{Hom}_{DM}(\mathbb{Z}/2,\mathbb{Z}/2(1)[1])$.

We can consider cohomological operations - compositions with the above elements. We will denote them also as τ and ρ.

Theorem 2.1.26 ([20], Theorem 3.16 and Theorem 3.17).

Let k has characteristic 0.

There exist cohomological operations P^i, $i \geqslant 0$, where $P^i : \mathrm{Hom}(Y,\mathbb{Z}/2(a)[b]) \to \mathrm{Hom}(Y,\mathbb{Z}/2(a+i)[b+2i])$, s.t.:

1) $P^0 = id$;
2) *For any smooth simplicial scheme Y and any $u \in \mathrm{H}^{n,i}_{\mathcal{M}}(X,\mathbb{Z}/2)$, $P^i(u) = 0$ for $n < 2i$, and $P^i(u) = u^i$ for $n = 2i$.*
3) $\Delta(P^i) = \sum_{a+b=i} P^a \otimes P^b + \tau \sum_{a+b=i-2} \beta P^a \otimes \beta P^b$

Define the following operations Q_i of bidegree $(2^i-1)[2^{i+1}-1]$ inductively as follows: $Q_0 = \beta$-bokstein, and $Q_{i+1} = [Q_i, P^{2^i}]$. These operations have the following properties:

4) $Q_iQ_j = Q_jQ_i$, $Q_i^2 = 0$.
5) $\Delta(Q_i) = 1 \otimes Q_i + Q_i \otimes 1 + \sum \rho^{n_j}\varphi_j \otimes \psi_j$, *where $n_j > 0$, and φ_j, ψ_j are some operations of bidegree $(q)[p]$ with $p > 2q$.*
6) *for any $i > 0$ there exists operation q_i, s.t. $Q_i = [\beta, q_i]$.*

2.2. **Category of Chow motives.** Let us now define the category of *Chow motives.*

Definition 2.2.1 ([17], p.6).

Consider a category $\mathcal{C}$ whose objects are smooth projective varieties over k and morphisms are given by the formula $\mathrm{Hom}_{\mathcal{C}}(X,Y) = \oplus_{X_i} A_{\dim(X_i)}(X_i \times Y)$, where X_i are connected components of X, and $A_j(-)$ is the group of cycles of dimension j modulo rational equivalence.

The Pseudo Abelian envelope of $\mathcal{C}$ (see the Definition 2.5.2) is called the category of effective Chow motives over k: $Chow^{eff}(k)$.

We have natural functor $Chow : SmProj/k \to Chow^{eff}(k)$ from the category of smooth projective varieties to the category of *effective Chow motives.*

Theorem 2.2.2 ([17], Proposition 2.1.4).

There exists a functor $Chow^{eff}(k) \to DM_{gm}^{eff}(k)$, *s.t. the following diagram is commutative:*

$$\begin{array}{ccc} SmProj/k & \longrightarrow & Sm/k \\ {\scriptstyle Chow}\downarrow & & \downarrow \\ Chow^{eff}(k) & \longrightarrow & DM_{gm}^{eff}(k) \end{array}$$

As usually we will denote: *Chow motive* of $\mathrm{Spec}(k) := \mathbb{Z}$. In $Chow^{eff}(k)$ we have a decomposition: $Chow(\mathbb{P}^1) = \mathbb{Z} \oplus Y$, where direct summand $\mathbb{Z}$ is given by the idempotent $p \in \mathrm{Hom}_{Chow^{eff}(k)}(\mathbb{P}^1, \mathbb{P}^1) = CH^1(\mathbb{P}^1 \times \mathbb{P}^1)$ represented by the cycle $\mathrm{Spec}(k) \times \mathbb{P}^1 \subset \mathbb{P}^1 \times \mathbb{P}^1$. Define: $\mathbb{Z}(1)[2] := Y$. It is easy to see that corresponding idempotent is represented by the cycle $\mathbb{P}^1 \times \mathrm{Spec}(k) \subset \mathbb{P}^1 \times \mathbb{P}^1$.

In the same way as we obtained $DM_{gm}(k)$ from $DM_{gm}^{eff}(k)$, we can obtain category of *Chow motives* $Chow(k)$ from the category of *effective Chow motives* $Chow^{eff}(k)$:

Definition 2.2.3 .

Objects of $Chow(k)$ *are pairs* (X, n), *where* X *is an object of* $Chow^{eff}(k)$, *and* $n \in \mathbb{Z}$.

Morphisms are given by the formula:

$\mathrm{Hom}_{Chow(k)}((X, n), (Y, m)) = \lim_{l \geqslant -m,-n} \mathrm{Hom}_{Chow^{eff}(k)}(X(n + l), Y(m + l))$

Theorem 2.2.4 ([17], end of p.11, Corollary 4.2.6).

Let our field k *admits resolution of singularities.*

Then the natural functor $Chow^{eff}(k) \to Chow(k)$ *is a full embedding and the functor* $Chow^{eff}(k) \to DM_{gm}^{eff}(k)$ *from Theorem 2.2.2 can be prolonged to a functor* $Chow(k) \to DM_{gm}(k)$ *which is again a full embedding.*

The last functor performs an equivalence between $Chow(k)$ *and the full tensor additive subcategory of* $DM_{gm}(k)$ *which is closed under direct summands and is generated by the objects of the form* $M(X)(n)[2n]$, *where* X/k *is a smooth projective variety, and* $n \in \mathbb{Z}$.

Any object of the above category is called pure motive.

2.3. **Standard simplicial schemes.** Again, I remind that everywhere we assume that $char(k) = 0$.

Here we will describe some objects of $DM^{eff}(k)$ with which we will work.

Let $X^{\cdot}$ be a smooth simplicial scheme. Then it has it's image $M(X^{\cdot})$ in $DM^{eff}(k)$ in the following way: Let X^n be n-th simplicial part of $X^{\cdot}$.

$M(X^{\cdot})$ is equal to $\underline{C}_*(L(X^{\cdot}))$, where $L(X^{\cdot})$ is a complex of sheaves with r-th term $L(X^{-r})$, and morphisms given by the alternating sum of face maps.

Definition 2.3.1 .

Let P/k *be some smooth connected projective scheme. Then we will call standard simplicial scheme corresponding to the pair* $P \to \mathrm{Spec}(k)$ *to be the simplicial scheme*

$\mathcal{X}^{\cdot}{}_P$, *s.t.* $\mathcal{X}_P^n = P \times_k P \times_k \cdots \times_k P$ *- $n+1$-times, and the maps of faces and degenerations are given by partial projections, and partial diagonals.*

We will often denote the motive $M(\mathcal{X}^{\cdot}{}_P)$ by the same symbol $\mathcal{X}^{\cdot}{}_P$, or by $\mathcal{X}_P$.

We have natural projection $M(\mathcal{X}_P) \to \mathbb{Z}$ given by the natural morphism of simplicial schemes $\mathcal{X}^{\cdot}{}_P \to \mathrm{Spec}(k)$.

The following theorem describes how this map acts on the corresponding representable and corepresentable functors.

Theorem 2.3.2 ([18], Lemma 4.9).

Let P be some smooth projective scheme over k.

If A is an object of the thick subcategory of $DM^{eff}(k)$, generated by the objects of the form $P \otimes L$ for all L (i.e.: the minimal subcategory, containing specified set of objects and closed under taking direct summands), then $M(\mathcal{X}_P) \to \mathbb{Z}$ induces an isomorphism: $\mathrm{Hom}(\mathbb{Z}, A(i)[j]) \to \mathrm{Hom}(M(\mathcal{X}_P), A(i)[j])$.

If B is an object of localizing subcategory of $DM^{eff}(k)$, generated by the objects of the form $P \otimes L$ for all L (i.e.: the minimal subcategory, containing specified set of objects and closed under taking direct summands and arbitrary direct sums), and C some object of $DM^{eff}(k)$, then $\mathrm{Hom}(B(i)[j], M(\mathcal{X}_P) \otimes C) \to \mathrm{Hom}(B(i)[j], \mathbb{Z} \otimes C)$.

$M(\mathcal{X}_P)$ is an object of localizing subcategory of $DM^{eff}(k)$, generated by $M(P)$.

Remark In Lemma 4.9 of [18] everything is formulated in the context of $P = Q_\alpha$ - the *Pfister quadric* (and there is no C there), but the proof does not use any specific, and is general.

Theorem 2.3.3 .

Let P be smooth projective variety. Then:
1) $\mathrm{Hom}_{DM}(M(\mathcal{X}_P), \mathbb{Z}(i)[j]) = 0$ *for all $i < 0$, or $i = 0$ and $j < 0$ and*
$\mathrm{Hom}_{DM}(M(\mathcal{X}_P), \mathbb{Z}) = \mathrm{Hom}_{DM}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$ *(the first map is induced by the natural projection $M(\mathcal{X}_P) \to \mathbb{Z}$, and the second equality via Theorem 2.1.20).*
2) The image of $\mathrm{Hom}_{DM}(\mathbb{Z}, M(\mathcal{X}_P))$ *in* $\mathbb{Z} = \mathrm{Hom}_{DM}(\mathbb{Z}, \mathbb{Z})$ *(see Theorem 2.1.20) (via the map induced by the projection $\mathcal{X}^{\cdot}{}_P \to \mathrm{Spec}(k)$) is a subgroup which is generated by the greatest common divisor of the degrees of finite points on P, and it coincides with the image*$(\mathrm{Hom}_{DM}(\mathbb{Z}, M(P)) \to \mathrm{Hom}_{DM}(\mathbb{Z}, \mathbb{Z}))$.

Proof

1) The first part is just [20], Corollary 2.2, and the fact that $\mathrm{Hom}_{DM}(M(\mathcal{X}_P), \mathbb{Z}) = \mathbb{Z}$ follows from [15], Proposition 2.2.

2) Consider $Y := \mathrm{Cone}(M(P) \to M(\mathcal{X}^{\cdot}{}_P))$ (the map here is induced by natural isomorphism: $\mathcal{X}_P^0 = P$). Then by [17], Proposition 3.1.8, and since Y as a complex of Nisnevich sheaves with transfers is concentrated in negative degrees, we have that $\mathrm{Hom}(\mathbb{Z}, Y) = \mathrm{H}_{Nis}(\mathrm{Spec}(k), Y) = 0$ (the second group here is a hypercohomology of $\mathrm{Spec}(k)$ with coefficients in the specified complex of sheaves in Nisnevich topology) (see the proof of Proposition 3.1.8 from [17]).

So, we get a surjection: $\mathrm{Hom}_{DM}(\mathbb{Z}, M(P)) \to \mathrm{Hom}(\mathbb{Z}, M(\mathcal{X}^{\cdot}{}_P))$, and $image(\mathrm{Hom}(\mathbb{Z}, M(\mathcal{X}^{\cdot}{}_P)) \to \mathrm{Hom}_{DM}(\mathbb{Z}, \mathbb{Z})) = image(\mathrm{Hom}(\mathbb{Z}, M(P)) \to \mathrm{Hom}_{DM}(\mathbb{Z}, \mathbb{Z}))$.

Any element of $\mathrm{Hom}(\mathbb{Z}, M(P))$ is represented by a cycle of dimension 0 on P (see Theorem 2.1.17), and it's image in $\mathbb{Z} = \mathrm{Hom}_{DM}(\mathbb{Z}, \mathbb{Z})$ is equal to the degree of a cycle.

□

It appears that the motive $M(\mathcal{X}^{\cdot}{}_P)$ contains much less information, than P itself. Namely, we have the following:

Theorem 2.3.4 .

Let P have a zero-cycle of degree 1. Then $M(\mathcal{X}^{\cdot}{}_P) = \mathbb{Z}$.

Moreover, let P and Q are smooth (connected) projective varieties over k, s.t. Q has a zero cycle of degree 1 over $k(P)$, and P has a zero cycle of degree 1 over $k(Q)$.

Then there is unique up to change of sign isomorphism $M(\mathcal{X}^{\cdot}{}_P) \to M(\mathcal{X}^{\cdot}{}_Q)$.

Moreover, the condition above is equivalent to: $M(\mathcal{X}^{\cdot}{}_P) = M(\mathcal{X}^{\cdot}{}_Q)$.

Proof

Suppose P has a zero-cycle of degree 1, i.e. we have a section $j : \mathbb{Z} \to M(P)$ (see Theorem 2.1.17 , Theorem 2.1.23) of the standard projection, then j defines a direct sum decomposition $M(\mathcal{X}^{\cdot}{}_P) = \mathbb{Z} \oplus Y$, and a homotopy of id_Y to 0.

The second part follows from the following general result

Theorem 2.3.5 .

Suppose we have some morphism $\varphi : A \to B$ in $DM^{eff}_{-}(k)$, and smooth projective variety P/k. Then if $\varphi|_{k(P)}$ is an isomorphism (as a morphism in $DM^{eff}_{-}(k(P))$, then $\varphi \times id : A \otimes M(P) \to B \otimes M(P)$ is also an isomorphism.

Proof

$A \times M(P) \to B \times M(P)$ is an isomorphism $\Leftrightarrow$ $\mathrm{Cone}[-1](\varphi) \times M(P) \to 0$ is an isomorphism.

Now $\mathrm{Cone}[-1](\varphi)$ as an object of $DM^{eff}_{-}(k)$ is a bounded from above complex of Nisnevich sheaves with transfers with homotopy invariant cohomology sheaves. So, to prove that it is 0, it is equivalent to prove that all cohomology sheaves are.

We have a homotopy invariant Nisnevich sheaf with transfers, which is zero in the generic point. Since any Nisnevich covering contains some Zariski open subset (see [4], Definition 3.2), we have that our sheaf is zero on some Zariski open subscheme.

Then everything follows from [16], Corollary 4.18 (see also Definition 3.1 and Definition 3.6 there).

□

Now we have natural map: $M(\mathcal{X}^{\cdot}{}_P) \otimes M(Q) \to M(Q)$ (induced by the projection $M(\mathcal{X}^{\cdot}{}_P) \to \mathbb{Z}$). But if we change Q by $k(Q)$, then the corresponding map is an isomorphism (as we just saw), and so, our map is an isomorphism as well.

Since $M(\mathcal{X}^{\cdot}{}_Q)$ belongs to the *localizing category* generated by the object $M(Q)$, we have that the natural map $M(\mathcal{X}^{\cdot}{}_P) \otimes M(\mathcal{X}^{\cdot}{}_Q) \to M(\mathcal{X}^{\cdot}{}_Q)$ is an isomorphism as well. Similarly we can prove symmetric isomorphism.

So, $M(\mathcal{X}^{\cdot}{}_P) = M(\mathcal{X}^{\cdot}{}_Q)$. But $\mathrm{Hom}_{DM}(M(\mathcal{X}^{\cdot}{}_P), M(\mathcal{X}^{\cdot}{}_P)) = \mathrm{Hom}_{DM}(M(\mathcal{X}^{\cdot}{}_P), \mathbb{Z})$ by Theorem 2.3.2 , and the later group is identified with $\mathrm{Hom}_{DM}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$ (equality by Theorem 2.1.20) via the projection $M(\mathcal{X}^{\cdot}{}_P) \to \mathbb{Z}$ by Theorem 2.3.3 .

So, there are only two invertible elements there: 1 and -1.

Conversely, if $M(\mathcal{X}^{\cdot}{}_P) = M(\mathcal{X}^{\cdot}{}_Q)$, then $M(P) \times M(\mathcal{X}^{\cdot}{}_P) = M(P) \times M(\mathcal{X}^{\cdot}{}_Q)$, in particular, over $k(P)$: $M(\mathcal{X}_Q \times k(P)) = M(\mathcal{X}^{\cdot}{}_P \times k(P)) = \mathbb{Z}|_{k(P)}$ (as was proven above (P has a rational point over $k(P)$)). By Theorem 2.3.3 (2) it follows that over $k(P)$, Q has a zero-cycle of degree 1.

□

Remark From Theorem 2.3.4 , and the fact that over algebraically closed field any variety of finite type has a rational point, it follows that $M(\mathcal{X}^{\cdot}{}_P)$ is a *form* of Tate-motive, i.e. $M(\mathcal{X}^{\cdot}{}_P)|_{\overline{k}} = \mathbb{Z}$.

From the proof of Theorem 2.3.4 we by the way get the following fact:

Theorem 2.3.6 .

If P, Q are smooth projective varieties and P has a zero cycle of degree 1 over $k(Q)$, then $M(\mathcal{X}^{\cdot}{}_P) \otimes M(Q) = M(Q)$, and $M(\mathcal{X}^{\cdot}{}_P) \otimes M(\mathcal{X}^{\cdot}{}_Q) = M(\mathcal{X}^{\cdot}{}_Q)$.

The following result is very useful in computation of $ker(\mathrm{K}^M_*(k)/2 \to \mathrm{K}^M_*(k(Q))/2)$ for smooth projective variety Q/k.

Theorem 2.3.7 ([18], Lemma 6.4, [20], Theorem 4.1).

The map $\beta : \mathrm{Hom}_{DM}(M(\mathcal{X}_Q), \mathbb{Z}/2(n-1)[n]) \to \mathrm{Hom}_{DM}(M(\mathcal{X}_Q), \mathbb{Z}_{(2)}(n-1)[n+1])$ (see Definition 2.1.25) identifies the first group with the 2-torsion subgroup of the second.

$\tau : \mathrm{Hom}_{DM}(M(\mathcal{X}_Q), \mathbb{Z}/2(n-1)[n]) \to \mathrm{Hom}_{DM}(M(\mathcal{X}_Q), \mathbb{Z}/2(n)[n]) = \mathrm{K}^M_n(k)/2$ (see Definition 2.1.25 ,Theorem 2.1.18 and Theorem 2.1.20) identifies the first group with $ker(\mathrm{K}^M_n(k)/2 \to \mathrm{K}^M_n(k(Q))/2)$.

So, we get isomorphism: $\tau \circ \beta^{-1} : (\mathrm{Hom}_{DM}(M(\mathcal{X}_Q), \mathbb{Z}_{(2)}(n-1)[n+1]))_2 \to \mathrm{Ker}^Q_n := \mathrm{Ker}(\mathrm{K}^M_n(k)/2 \to \mathrm{K}^M_n(k(Q))/2)$.

Remark In Lemma 6.4 of [18] everything is formulated in the context of $Q = Q_\alpha$ - the *Pfister quadric*, but the proof does not use any specific, and is general.

Definition 2.3.8 .

Denote $M(\tilde{\mathcal{X}_P}) := \mathrm{Cone}[-1](M(\mathcal{X}_P) \to \mathbb{Z})$.

Since $M(\mathcal{X}_P)$ is a *form* of $\mathbb{Z}$ (i.e.: they are equal in étale topology), then, considering the just defined object, we get:

Theorem 2.3.9 ([20], Corollary 2.12 + the main result of the cited paper (Hilbert90 for $\mathbb{Z}/2$-coeff.).

For any smooth projective variety P/k, and any $n \in \mathbb{Z}$ we have:
$\mathrm{Hom}_{DM}(M(\mathcal{X}_P), \mathbb{Z}_{(2)}(n)[m]) = \mathrm{Hom}_{DM}(\mathbb{Z}, \mathbb{Z}_{(2)}(n)[m])$ *for* $m \leqslant n+1$.

In particular, by Theorem 2.1.18 we get:

Theorem 2.3.10 .
For any smooth projective variety P/k, and any $n \in \mathbb{Z}$ we have:
$\mathrm{Hom}_{DM}(M(\mathcal{X}_P), \mathbb{Z}_{(2)}(n)[n+1]) = 0$

Theorem 2.3.11 ([20], Theorem 3.25, Lemma 4.11).
Let P be a quadric of dimension $\geqslant 2^i - 1$. Then Q_i (see Theorem 2.1.26) (considered as a differential (it has square =0)) acts without cohomology on $\mathrm{H}^{*,*'}_{\mathcal{M}}(M(\tilde{\mathcal{X}}_P), \mathbb{Z}/2)$.

From the existence of *transfers* on $\mathrm{H}^{*,*'}_{\mathcal{M}}(M(\tilde{\mathcal{X}}_P), \mathbb{Z}_{(2)})$, and the fact that P has quadratic point, we have by Theorem 2.3.4 , that
$\mathrm{H}^{*,*'}_{\mathcal{M}}(M(\tilde{\mathcal{X}}_P), \mathbb{Z}_{(2)}) \subset \mathrm{H}^{*,*'}_{\mathcal{M}}(M(\tilde{\mathcal{X}}_P), \mathbb{Z}/2)$, and $\mathrm{H}^{*,*'}_{\mathcal{M}}(M(\tilde{\mathcal{X}}_P), \mathbb{Z}_{(2)})$ is a group of exponent 2 (in particular, we can change $\mathbb{Z}_{(2)}$ by $\mathbb{Z}$).

2.4. **Some facts about quadratic forms.** Let k be the field of characteristic different from 2. Quadratic form over k is pair (V, q), where V/k-finite dimensional vector space and $q : V \to k$ is a quadratic function (i.e., diagonal part of some symmetric bilinear function $V \times V \to k$). If $(e_1, \ldots, e_n)$ is a basis of V, then q is defined by it's matrix (q_{ij}). We, certainly, can always diagonalize the matrix of q. So, q is isomorphic to some form $\langle a_1, \ldots, a_n \rangle$, where the latest denotes quadratic form corresponding to the diagonal matrix with eigenvalues - a_i's . Of course, for two different sets $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ we can have: $\langle a_1, \ldots, a_n \rangle$ isomorphic to $\langle b_1, \ldots, b_n \rangle$ (for example, $\langle 1, -1 \rangle = \langle a, -a \rangle$); so, such presentation is not unique (not to say that each eigenvalue can be multiplied by a square). Due to Witt *Chain equivalence theorem*: we can transform one such presentation into arbitrary other by changing only two of our entries at a time.

The form $\langle 1, -1 \rangle := \mathbb{H}$ is called *elementary hyperbolic form.* Notice, that this form *represents* 0 (i.e., we can find nonzero vector $v \in V$, s.t. $q(v) = 0$; in our case $v = (1, 1)$). Direct sum of elementary hyperbolic forms is called *hyperbolic.* Notice: q is hyperbolic iff V_q contains *isotropic subspace* of dimension $\dim(V)/2$. The following observation is trivial: quadratic form q represents 0 iff it has $\mathbb{H}$ as a direct summand. It follows from the fact that every subform is a direct summand ($q = q' \perp q'^{\perp}$). The form is called *anisotropic*, if it does not represent 0.

Also we should take into account the *Witt cancellation theorem* :

Theorem 2.4.1 ([9], I, Theorem 4.2).
If for some quadratic forms q, s, s' we have $q \perp s = q \perp s'$, then $s = s'$.

Each form q can be represented as $q = d \cdot \mathbb{H} \perp q'$, where q' is the unique anisotropic form with such a property, and d is called the *Witt (isotropity) index.*

Definition 2.4.2 .

The Witt ring of quadratic forms over k is a Grothendieck group of quadratic forms over k modulo hyperbolic forms; addition is as in Groth. group, and multiplication via tensor product of quadratic spaces (in terms of diagonal presentations: $\langle a_1, \dots, a_n\rangle \cdot \langle b_1, \dots b_m\rangle =\perp_{i=1,\dots,n;j=1,\dots,m} \langle a_i \cdot b_j\rangle$).

Each element of $W(k)$ can be represented by a unique *anisotropic* quadratic form, i.e.: as a set, $W(k)$ is the same as the set of *anisotropic* forms.

$W(k)$ contains one natural ideal: the ideal of even-dimensional forms I (notice, that *hyperbolic* forms are even-dimensional). Powers of the ideal I provide us with the *multiplicative* decreasing filtration on $W(k)$: $W(k) \supset I \supset I^2 \supset \dots$. Denote as $gr^*(W(k))$ the associated graded ring. It is easy to see that $k^*/(k^*)^2 = I/I^2$ via map: $a \mapsto \langle 1, -a\rangle$.

Some quadratic forms are better than others. And the best possible forms are *Pfister forms*.

Definition 2.4.3 .

Let $a_1, a_2, \dots, a_n$ are elements of k^*. Then the (big) Pfister form, associated with this set is the form $\langle 1, -a_1\rangle \times \langle 1, -a_2\rangle \times \dots \times \langle 1, -a_n\rangle$. We will denote such form as $q_{\{a_1,\dots,a_n\}} = \langle\langle a_1, \dots, a_n\rangle\rangle$.

Definition 2.4.4 .

Milnor's K-theory of the field k is a homogeneous quadratic algebra (over $\mathbb{Z}$) with generators k^* and relations $a \otimes (1 - a)$, $a \in k^* \backslash 1$ $\quad (K_*^M(k) = T(k^*)/R$, where R is double-sided ideal generated by relations as above).

We will denote element of $K_1^M(k)$ as $\{a\}$. Element of $K_n^M(k)$ is called pure if it is multiplicative, i.e. the product of some elements of degree 1 : $\{a_1, \dots, a_n\}$.

It is not difficult to see that $\mathrm{K}_*^M(k)$ is a skew-symmetric ring (hence, $\mathrm{K}_*^M(k)/2$ is symmetric).

We have natural map $\rho : \mathrm{K}_*^M(k)/2 \to gr^*(W(k))$, given on the generators by the rule: $\{a\} \mapsto \langle 1, -a\rangle$. Since $gr^*(W(k))$ is generated by the first degree component, it will be a surjection. The well-known Milnor conjecture states that ρ is an isomorphism. Finally, it was proven by V.Voevodsky (see [13], p.14).

Notice, that $\rho(\{a_1, \dots, a_n\}) = \langle\langle a_1, \dots, a_n\rangle\rangle$. The following result is due to R.Elman and T.Y.Lam:

Theorem 2.4.5 ([3], Main Theorem 3.2).

The following conditions are equivalent:

1) $\{a_1, \dots, a_n\} = \{b_1, \dots, b_n\} \in \mathrm{K}_n^M(k)/2$
2) $\langle\langle a_1, \dots, a_n\rangle\rangle - \langle\langle b_1, \dots, b_n\rangle\rangle \in I^{n+1}(W(k))$
3) *The forms $\langle\langle a_1, \dots, a_n\rangle\rangle$ and $\langle\langle b_1, \dots, b_n\rangle\rangle$ are isomorphic.*

Moreover, if $\langle\langle a_1, \dots, a_n\rangle\rangle$ is isotropic, then it is hyperbolic.

Definition 2.4.6 .

A Pfister relative of the big Pfister form $q_{\{a_1,\dots,a_n\}} = \langle\langle a_1,\dots,a_n\rangle\rangle$ *is a subform of it which has dimension* $> 1/2\dim(\langle\langle a_1,\dots,a_n\rangle\rangle) = 2^{n-1}$.

A small Pfister form is a subform $q_{\{a_1,\dots,a_n\}}$ *of dimension* $2^{n-1}+1$ *(in particular, it is a relative).*

Two forms q_1 *and* q_2 *are called Pfister half-neighbors if* $\dim(q_1) = \dim(q_2)$*, and* $q_1 \perp q_2 = q_{\{a_1,\dots,a_n\}}$.

Let some element $h \in \mathrm{K}_n^M(k)/2$ be represented as a sum of two *pure symbols*: $h = \{a_1,\dots,a_n\} + \{b_1,\dots,b_n\}$.

Definition 2.4.7 .

An Albert form associated with h *is a form* $A_h = \langle\langle a_1,\dots,a_n\rangle\rangle \perp -\langle\langle b_1,\dots,b_n\rangle\rangle$.

(Notice that it is defined non-uniquely, since h *can have different presentations as a sum of two pure symbols.)*

Theorem 2.4.8 ([3], Theorem 4.5).

The only possible values for the $\dim((A_h)|_{anis.})$ *are:* $2^{n+1} - 2^{i+1}$*, where* $0 \leqslant i \leqslant n$.

Moreover, the following conditions are equivalent:

1) $\dim((A_h)|_{anis.}) = 2^{n+1} - 2^{i+1}$;
2) $\{a_1,\dots,a_n\}$ *and* $\{b_1,\dots,b_n\}$ *have common pure divisor* $\{c_1,\dots,c_i\}$ *of degree* i*, and do not have one of degree* $> i$.
3) h *has pure divisor* $\{c_1,\dots,c_i\}$ *of degree* i*, and does not have one of degree* $> i$.

Theorem 2.4.9 .

Let A_h *be an Albert form corresponding to the symbol* $h = \{a,b\} + \{c,d\}$ *in* $\mathrm{K}_2^M(k)/2$*. Then the following conditions are equivalent:*

1) h *is pure (see Definition 2.4.4);*
2) $(A_h)_{anis.}$ *has dimension* $\leqslant 4$;
3) A_h *is hyperbolic over some field* E*, where* $[E:k] = 2$;
4) A_h *is hyperbolic over some field* E*, where* $[E:k] = 2\cdot d$*, with* d*-odd.*

Proof

($1 \Leftrightarrow 2$) By Theorem 2.4.8 .

($1 \Rightarrow 3$) Again by Theorem 2.4.8 , we have that there exist $f_1, f_2, g \in k^*$, that $\{a,b\} = \{f_1, g\}$, $\{c,d\} = \{f_2, g\}$ (in $\mathrm{K}_2^M(k)/2$). Then $A_h = \langle 1, -f_1, -g, f_1\cdot g\rangle - \langle 1, -f_2, -g, f_2\cdot g\rangle$, and so it is hyperbolic over $k\sqrt{g}$.

($3 \Rightarrow 1$) Let A_h becomes hyperbolic over some quadratic extension $E = k\sqrt{g}$. This means that $\langle 1,-a,-b,a\cdot b\rangle|_E = \langle 1,-c,-d,c\cdot d\rangle|_E$, and by Theorem 2.4.5 $\{a,b\}|_E = \{c,d\}|_E$, i.e.: $h \in \mathrm{Ker}(\mathrm{K}_2^M(k)/2 \xrightarrow{j} \mathrm{K}_2^M(E)/2)$.

By the theorem of A.Merkurjev (see [11], Theorem and Proposition 2), we have the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{K}_1^M(k)/2 & \xrightarrow{\cdot\{g\}} & \mathrm{K}_2^M(k)/2 & \xrightarrow{j} & \mathrm{K}_2^M(k\sqrt{g})/2 \\ \| & & \| & & \| \\ \mathrm{H}^1_{et}(k,\mathbb{Z}/2) & \xrightarrow{\cdot\{g\}} & \mathrm{H}^2_{et}(k,\mathbb{Z}/2) & \xrightarrow{j} & \mathrm{H}^2_{et}(k,\pi_*(\mathbb{Z}/2)), \end{array}$$

where $\pi_*(\mathbb{Z}/2)$ is a $Gal(k)$-module, induced from the subgroup $Gal(k\sqrt{g}) \subset Gal(k)$. The lower row here is an exact sequence (coming from the short exact sequence $0 \to \mathbb{Z}/2 \to \pi_*(\mathbb{Z}/2) \to \mathbb{Z}/2 \to 0$). So, if $h \in \mathrm{Ker}(\mathrm{K}_2^M(k)/2 \to \mathrm{K}_2^M(E)/2)$, then h is divisible by $\{g\}$ and is *pure*.

$(3 \Rightarrow 4)$ Trivial.

$(4 \Rightarrow 2)$ $\dim((A_h)_{anis.}) \leqslant 4 \Leftrightarrow \langle -a, -b, ab, c, d, -cd\rangle$ is isotropic. Suppose A_h is hyperbolic over some field E of degree $2d$ with d-odd. We have a tower of fields $k \subset F \subset E$, where $[F : k] = d$ (F corresponds to some 2-Sylow subgroup of $Gal(E/k)$). Over F the form $\langle -a, -b, ab, c, d, -cd\rangle$ is isotropic (by $(3 \Rightarrow 2)$). But then by Springer's theorem (see Theorem 2.4.14) this form is isotropic over k.

$\square$

Definition 2.4.10 .

Projective quadric is called conservative, if $\mathrm{Ker}_*^Q := \mathrm{Ker}(\mathrm{K}_*^M(k)/2 \to \mathrm{K}_*^M(k(Q))/2)$ *is nontrivial.*

Theorem 2.4.11 ([13], Statement 1 of Section 3.2).

For any quadric Q of dimension $\geqslant 2^n - 1$ we have: $\mathrm{Ker}_n^Q = 0$.

Remark B.Kahn, M.Rost and R.Sujatha obtained the same result for $n \leqslant 4$ (see [8], Theorem 1).

We can introduce the notion of *Chern classes* of a quadratic form:

Definition 2.4.12 .

For n-dimensional quadratic form $q = \langle a_1, \dots, a_n\rangle$ the Chern character $Ch(q) \in \mathrm{K}_0^M(k)/2 \times \mathrm{K}_1^M(k)/2 \times \mathrm{K}_2^M(k)/2 \times \dots$ can be defined as $\prod_{1\leqslant j\leqslant n}(1+\{a_i\})\cdot(\prod_{1\leqslant i\leqslant [n/2]}(1+\{-1\}))^{-1}$. (In our definition Ch is multiplicative only on even-dimensional forms but it has the advantage of being equal to 1 on hyperbolic forms (even and odd-dimensional), in particular, $Ch(q) = Ch(q|_{anis.})$.)

$c_m(q)$ is a $\mathrm{K}_m^M(k)/2$-component of $Ch(q)$.

Definition 2.4.13 .

For an even-dimensional projective quadric Q define $\det(Q)$ as $c_1(q)$ for arbitrary form q representing Q.

For an odd-dimensional projective quadric Q define $c_2(Q)$ as $c_2(q)$, where q-unique form representing Q with $c_1(q) = 0$ (or, in other words: $\det(q) = (-1)^{\dim(q)-1/2}$).

The following theorem belongs to T.A.Springer:

Theorem 2.4.14 ([9], VII, Theorem 2.3).
Let quadric Q/k contains a point of odd degree. Then it contains rational point.

Let q be a quadratic form over k. We can associate with q the following varieties: Projective quadric Q, defined by this form; Q^1 - variety of lines on Q; Q^2 - variety of 2-dimensional planes on Q;....;Q^i - variety of i-dimensional planes on Q, for $0 \leqslant i \leqslant [n/2]$.

Also we can define the variety $\underline{Q^i}$ as a variety of flags: $(l_0 \subset l_1 \subset l_2 \subset \cdots \subset l_i)$, where l_j is a projective subspace of dimension j living on Q.

We have natural projection $\underline{Q^i} \to Q^i$, in particular, Q^i has a point over $k(\underline{Q^i})$. On the other hand, from the existence of rational i-dimensional plane on Q follows the existence of i-dimensional rational flag on it.

So, using Theorem 2.3.4 we get:

Theorem 2.4.15 .
$M(\mathcal{X}_{\underline{Q^i}}) = M(\mathcal{X}_{Q^i})$.

Since, evidently, from the existence of rational j-dimensional plane on Q follows the existence of rational i-dimensional plane on it, for $j \geqslant i$, we have by Theorem 2.3.6 :

Theorem 2.4.16 .
$M(\mathcal{X}^{\cdot}{}_{Q^j}) \otimes M(\mathcal{X}^{\cdot}{}_{Q^i}) = M(\mathcal{X}^{\cdot}{}_{Q^j})$ *for* $i \leqslant j$.

Since *big Pfister form* $q_{\{a_1,\dots,a_n\}}$ is hyperbolic as soon as it is isotropic (by Theorem 2.4.5), we have that $Q^{\dim(Q)/2}_{\{a_1,\dots,a_n\}}$ has a rational point over $k(Q_{\{a_1,\dots,a_n\}})$. So, by Theorem 2.3.4 we have that $M(\mathcal{X}^{\cdot}{}_{Q^{\dim(Q)/2}_{\{a_1,\dots,a_n\}}}) = M(\mathcal{X}^{\cdot}{}_{Q_{\{a_1,\dots,a_n\}}})$. Now, from Theorem 2.4.16 we have the following result.

Theorem 2.4.17 .
For big Pfister quadric $Q = Q_{\{a_1,\dots,a_n\}}$ *we have:* $M(\mathcal{X}^{\cdot}{}_Q) = M(\mathcal{X}^{\cdot}{}_{Q^1}) = \cdots = M(\mathcal{X}^{\cdot}{}_{Q^{\dim(Q)/2}})$. *(We remind that the dimension of the projective quadric is 2 less than of the corresponding form.)*

Remark Actually, *big Pfister quadric* and a hyperplane section in a *big Pfister quadric* are the only examples of quadrics having such a property. We will not use this fact.

Theorem 2.4.18 .
For projective quadric Q: the variety Q^i has a rational point if and only if it has a zero-cycle of degree 1.

In particular, $M(\mathcal{X}_{Q^i}) = M(\mathcal{X}_{R^j})$ for two quadrics Q and R, if and only if Q^i has a rational point over R^j and R^j has a rational point over Q^i.

Proof

If the variety $\underline{Q}^i$ has a zero-cycle of degree 1, then it has point of odd degree. So, projecting from $\underline{Q}^i$ to Q, we get a point of odd degree on Q. By Springer's theorem (see Theorem 2.4.14), q is isotropic. Let $q = \mathbb{H} \perp q'$. Then $M(\mathcal{X}_{Q'^{i-1}}) = M(\mathcal{X}_{Q^i})$ $= \mathbb{Z}$ (by Theorem 2.3.4). So, in the same way, $\underline{Q'}^{i-1}$ has a point of odd degree, and, again, that means that q' is isotropic by Springer theorem (Theorem 2.4.14). After i such steps we get: Q^i has a rational point. (See also Theorem 2.4.15 and discussion before it.)

The second statement follows from the first by Theorem 2.3.4 .

□

Definition 2.4.19 .

Let q be an anisotropic quadratic form, and Q-corresponding projective quadric. Then $q|_{k(Q)} = h_1 \cdot \mathbb{H} \perp q_1$, where q_1 is an anisotropic form defined over $k(Q)$ (such form is unique). Then we have: $q_1|_{k(Q)(Q_1)} = h_2 \cdot \mathbb{H} \perp q_2$, where q_2 is an anisotropic form defined over $k(Q)(Q_1)$, and we can continue this until we get $q_s = 0$.

The numbers $h_1, h_2, \ldots, h_s$ are called Witt numbers of a quadric Q.

The tower of fields: $k \subset k(Q) \subset k(Q)(Q_1) \subset \cdots \subset k(Q)\ldots(Q_{s-1})$ is called the Universal splitting tower of M.Knebusch.

We will say that for two quadrics Q, R such towers are equivalent, if their Witt numbers are equal (pairwise) (in particular, $\dim(Q) = \dim(R)$) and for each $i \geqslant 0$, $Q^{h_1+\cdots+h_i}$ has a rational point over $R^{h_1+\cdots+h_i}$ and for each j, $R^{h_1+\cdots+h_j}$ has a rational point over $Q^{h_1+\cdots+h_j}$ (see also discussion below).

From the definition of *Witt numbers* we see that: over $k(Q)$ all varieties: $Q, \ldots, Q^{h_1-1}$ have a rational point, i.e.: $M(\mathcal{X}_Q) = \cdots = M(\mathcal{X}_{Q^{h_1-1}})$ (by Theorem 2.4.16 and Theorem 2.3.6). Moreover, Q^{h_1} does not have a rational point over $k(Q)$ (again, by definition). Repeating this arguments with $k(Q^{h_1})$, $k(Q^{h_1+h_2})$, etc. ..., we get:

Theorem 2.4.20 .

$M(\mathcal{X}_Q) = \cdots = M(\mathcal{X}_{Q^{h_1-1}}) \neq M(\mathcal{X}_{Q^{h_1}}) = \cdots = M(\mathcal{X}_{Q^{h_1+h_2-1}}) \neq \ldots$
$\neq M(\mathcal{X}_{Q^{h_1+\cdots+h_{s-1}}}) = \cdots = M(\mathcal{X}_{Q^{[\dim(Q)/2]}})$

For two smooth projective varieties P and R we will say that $M(\mathcal{X}_P) \leqslant M(\mathcal{X}_R)$, if $M(\mathcal{X}_P) \times M(\mathcal{X}_R) = M(\mathcal{X}_R)$, and that $M(\mathcal{X}_P) < M(\mathcal{X}_R)$, if moreover, $M(\mathcal{X}_P) \times M(\mathcal{X}_R) \neq M(\mathcal{X}_P)$.

In this light we can put statement (using Theorem 2.4.16 and Theorem 2.4.18) in the following form: $M(\mathcal{X}_Q) = \ldots = M(\mathcal{X}_{Q^{h_1-1}}) < M(\mathcal{X}_{Q^{h_1}}) = \ldots = M(\mathcal{X}_{Q^{h_1+h_2-1}}) < \ldots < M(\mathcal{X}_{Q^{h_1+\cdots+h_{s-1}}}) = \ldots = M(\mathcal{X}_{Q^{[\dim(Q)/2]}})$.

Notice that if $P \subset Q$ is a subquadric of codimension m, then from the existence of rational point on Q^m follows the existence of rational point on P (just intersect corresponding rational m-dimensional projective subspace with P), hence, by Theorem 2.3.6 (since over $k(Q^m)$ P has a rational point) we get: $M(\mathcal{X}_P) \times M(\mathcal{X}_{Q^m}) = M(\mathcal{X}_{Q^m})$.

In the notations as above: $M(\mathcal{X}_P) \leqslant M(\mathcal{X}_{Q^m})$. On the other hand, since Q has a rational point over $k(P)$, we get (again by Theorem 2.3.6) that $M(\mathcal{X}_Q) \leqslant M(\mathcal{X}_P)$. So, $M(\mathcal{X}_Q) \leqslant M(\mathcal{X}_P) \leqslant M(\mathcal{X}_{Q^m})$.

Also, notice, that if for some smooth projective varieties P and R we have $M(\mathcal{X}_P) \leqslant M(\mathcal{X}_R)$ and $M(\mathcal{X}_R) \leqslant M(\mathcal{X}_P)$, then $M(\mathcal{X}_P) = M(\mathcal{X}_R)$ by the very definition.

Applying this result to the *big Pfister quadric* $Q_{\{a_1,\dots,a_n\}}$ (see Definition 2.4.3), and using the fact that $Q_{\{a_1,\dots,a_n\}}$ is hyperbolic as soon as it is isotropic (see Theorem 2.4.5), i.e. $Q^{\dim(Q)/2}_{\{a_1,\dots,a_n\}}$ has a rational point over $k(Q)$, and so, $M(\mathcal{X}_{Q_{\{a_1,\dots,a_n\}}}) = M(\mathcal{X}_{Q^{\dim(Q)/2}_{\{a_1,\dots,a_n\}}})$, we get:

Theorem 2.4.21 .

Let $Q_{\{a_1,\dots,a_n\}}$ be a big Pfister quadric, and $P \subset Q_{\{a_1,\dots,a_n\}}$ - it's relative (see Definition 2.4.6). Then $M(\mathcal{X}_{Q_{\{a_1,\dots,a_n\}}}) = M(\mathcal{X}_P)$.

Remark Actually, these two conditions are equivalent.

In conclusion, we will say few words about *Chow motives* of isotropic and hyperbolic quadrics. Using some abuse of notations we will call *odd* dimensional quadratic form $r \cdot \mathbb{H} \perp \langle 1 \rangle$ also *hyperbolic.*

Theorem 2.4.22 .

1) Let $q = r \cdot \mathbb{H}$ be a hyperbolic quadratic form of dimension $2r$, and Q -corresponding projective quadric.

Then

$$M(Q) = \sum_{\substack{i=0,\dots,r-2\\ r,\dots,2r-2}} \mathbb{Z}(i)[2i] \oplus (\mathbb{Z}(r-1)[2r-2] \oplus \mathbb{Z}(r-1)[2r-2]).$$

2) Let $q' = r \cdot \mathbb{H} \perp \langle 1 \rangle$ be a hyperbolic quadratic form of dimension $2r+1$, and Q' -corresponding projective quadric.

Then

$$M(Q') = \sum_{i=0,\dots,2r-1} \mathbb{Z}(i)[2i].$$

Proof

1) We have the following set of mutually orthogonal idempotents in $\mathrm{Hom}_{Chow}(Q, Q) = CH^{\dim(Q)}(Q \times Q)$: $h^i \times l_i$; $l_j \times h^j$ for $0 \leqslant i, j \leqslant r-2$, and $l'_{r-1} \times l''_{r-1}$, $l''_{r-1} \times l'_{r-1}$, or $l'_{r-1} \times l'_{r-1}$, $l''_{r-1} \times l''_{r-1}$ (depending on: is r - even, or odd), where h^i is a plane section of codimension i and l_j is a projective subspace of dimension j on Q (exists since Q is hyperbolic), l'_{r-1} and l''_{r-1} denote projective subspaces of middle dimension from different families.

These idempotents define our decomposition.

2) The same proof.

$\square$

The same result can be obtained from the iterated application of the following:

Suppose now that Q is just isotropic, i.e.: it contains a rational point, and corresponding quadratic form contains a *elementary hyperbolic* summand: $q = \langle 1, -1\rangle \perp q'$. Notice, that Q' will be precisely quadric of lines on Q passing some rational point p. Q is a union of three strata: p, $(T_p(Q)\cap Q)\backslash p$, and $Q\backslash(T_p(Q)\cap Q) = \mathbb{A}^{\dim(Q)}$. From this it is not difficult to deduce the following result (see Theorem 2.4.25 for the global result).

Theorem 2.4.23 ([14], Proposition 1).

$M(Q) = \mathbb{Z}\oplus M(Q')(1)[2]\oplus\mathbb{Z}(\dim(Q))[2\dim(Q)]$

Remark 1 By Theorem 2.2.4 $Chow(k)$ is a full subcategory of $DM_{gm}(k)$, closed under direct summands, so everything is valid in $DM_{gm}(k)$ as well.

Remark 2 Notice that the map $M(Q)\to\mathbb{Z}$ is given by the generic cycle (see Definition 2.2.1), and $\mathbb{Z}(\dim(Q))[2\dim(Q)]\to M(Q)$ is dual to it via duality: $\underline{\mathrm{Hom}}(M(Q),\mathbb{Z}(\dim(Q))[2\dim(Q)]) = M(Q)$ (see Theorem 2.1.23).

Definition 2.4.24 .

Let P - is a smooth projective variety.

We will say that n-dimensional quadric Q is globally $i+1$-times isotropic over P, if the natural projection $P\times Q^i\to P$ has a section, or, which is the same, there is a map $\varphi: P\to Q^i$.

We will say that n-dimensional quadric Q is globally $i+1$-times strongly isotropic over P, if the natural projection $P\times\underline{Q^i}\to P$ has a section.

Let P be a smooth projective variety, over which quadric Q is *globally $i+1$-times isotropic.*

Let R will be a variety of pairs: (p,π), where p is a point of P, and π is a $i+1$-dimensional projective plane on Q, containing i-dimensional plane $\varphi(p)$. R has natural projection to P with fibers - $n-2(i+1)$-dimensional projective quadrics. In particular, R is a smooth projective variety.

The same construction works in a *strongly isotropic* case.

Theorem 2.4.25 .

Let P be a smooth projective variety, over which quadric Q is globally $i+1$-times strongly isotropic.

Then $M(P\times Q) = (\oplus_{j=0,\dots,i}M(P)(j)[2j])\oplus M(R)(i+1)[2i+1]\oplus(\oplus_{j=0,\dots,i}M(P)(n-j)[2n-2j])$.

Moreover the maps: $M(P\times Q)\to M(P)(j)[2j]$ (resp. $M(P)(n-j)[2n-2j]\to M(P\times Q)$) are just $id_{M(P)}\times\alpha_j$ (resp. $id_{M(P)}\times\beta_j$) (see the definition before Theorem 3.1).

Proof

We have a map $\psi: P\to\underline{Q^i}$, and then a map $\varphi: P\to Q^i$ (by the natural projection $\underline{Q^i}\to Q^i$ (see the discussion before Theorem 2.4.15).

We have the following subvariety F of $P \times Q$: F consists of pairs (p, q), where q belongs to i-dimensional projective subspace $\varphi(p)$. So, F is a i-dimensional projective bundle over P (in Zariski topology).

Consider the variety G, which consists of pairs: (p, q), where p - point on P, and the $i+1$-dimensional projective space generated by q and $\varphi(p)$ lives on Q (in particular, $q \notin \varphi(p)$).

We have natural projection $G \to R$ with fibers - $\mathbb{P}^{i+1} \backslash \mathbb{P}^i = \mathbb{A}^{i+1}$.

Consider the complement to $G \cup F$ in $P \times Q$ - W. It will be a fibration over P and the fiber is the complement to the $T_{l_i,Q}$, where $T_{l_i,Q}$ is a set of points q, s.t. the subspace generated by l_i and q lives on Q; such $T_{l_i,Q}$ has dimension $n-i-1$.

Since over our P, Q is *strongly* $i+1$*-times isotropic*, we have flag of projective bundles $F_0 \subset F_1 \subset \cdots \subset F_i = F$ inside the bundle F. Complements to their $T_{-,Q}$'s will provide us with the filtration $W_0 \subset W_1 \subset \cdots \subset W_i = W$ on W. The complements: $W_{j+1} \backslash W_j$ will be a fibration over P with fibers $\mathbb{A}^{n-j}$.

By *Gysin distinguished triangle* (see [17], Proposition 3.5.4, and also: the Definition 2.1.1 there (for the definition of a motive of nonprojective smooth variety)) we have that $M(P \times Q)$ is an *extension* (see Definition 2.5.6) of: $M(F)(n-i)[2n-2i]$, $M(G)(i+1)[2i+2]$ and $M(W)$.

By [17], Proposition 3.5.1 we have: $M(F) = \oplus_{j=0,\ldots,i} M(P)(j)[2j]$. By homotopy invariance of a motive: $M(G) = M(R)$. And by homotopy invariance of a motive and by *Gysin distinguished triangle* ([17], Proposition 3.5.4) we have: $M(W) = \oplus_{j=0,\ldots,i} M(P)(j)[2j]$.

Since all the motives are *pure* (i.e.: the direct summands of the motives of smooth projective varieties), we have by duality (see Theorem 2.1.23) and Theorem 2.1.18 that this extension is trivial (for two pure motives A and B we have $\mathrm{Hom}_{DM}(A, B(i)[2i+j]) = 0$ for all $j > 0$ (can assume that A, B are motives of smooth projective varieties, let B has dimension n, then $\underline{\mathrm{Hom}}(B, \mathbb{Z}(n)[2n]) = B$, and $\mathrm{Hom}(A, B(i)[2i+j]) = \mathrm{Hom}(A \times B, \mathbb{Z}(n+i)[2n+2i+j]) = 0$ (by Theorem 2.1.18)), and an *elementary extension* (see Definition 2.5.6) is defined by some element $u \in \mathrm{Hom}(A, B[1])$).

So, $M(P \times Q) = (\oplus_{j=0,\ldots,i} M(P)(j)[2j]) \oplus M(R)(i+1)[2i+1] \oplus (\oplus_{j=0,\ldots,i} M(P)(n-j)[2n-2j])$.

□

2.5. **Triangulated categories.**

Definition 2.5.1 ([5],II,§5,10).

An additive category is a category $\mathcal{C}$, s.t. the following properties are satisfied:

A1. For every $X, Y \in Ob(\mathcal{C})$, the set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ has a structure of an abelian group, and the composition map: $\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ is bilinear.

A2. There exist "zero object", i.e.: such object 0, that $\mathrm{Hom}_{\mathcal{C}}(0, 0) = 0$

A3. *For every pair of objects $X_1, X_2 \in \mathcal{C}$ there exists object Y (a direct sum), and morphisms $p_{1,2}$, $i_{1,2}$:*

$$X_1 \underset{i_1}{\overset{p_1}{\leftrightarrows}} Y \underset{i_2}{\overset{p_2}{\rightleftarrows}} X_2$$

that $p_1 i_1 = \mathrm{id}_{X_1}$, $p_2 i_2 = \mathrm{id}_{X_2}$, $i_1 p_1 + i_2 p_2 = \mathrm{id}_Y$, $p_2 i_1 = p_1 i_2 = 0$.

An additive functor is a functor $F : \mathcal{C}_1 \to \mathcal{C}_2$, which preserves group structure on Hom's, "zero object", and direct sums.

Definition 2.5.2 . *Let $\mathcal{C}$ be an additive category. The pseudo-abelian envelope of $\mathcal{C}$ is the category $PA(\mathcal{C})$, whose objects are pairs (X, p_X), where X is an object of $\mathcal{C}$, and $p_X \in \mathrm{End}(X)$ is a projector: $p^2 = p$. The morphism from (X, p_X) to (Y, p_Y) is a morphism $\varphi : X \to Y$, s.t. $p_Y \circ \varphi = \varphi$, and $\varphi = \varphi \circ p_X$. We have a natural functor $F : \mathcal{C} \to PA(\mathcal{C})$ $(F(X) = (X, id))$. $PA(\mathcal{C})$ is naturally an additive category, and F is a full embedding (by definition). For any idempotent $p \in \mathrm{End}(X)$ we have a direct sum decomposition in $PA(\mathcal{C})$: $F(X) = (X, p) \oplus (X, (1 - p))$.*

Definition 2.5.3 ([5],IV,§1,1).

The triangulated category is an additive category $\mathcal{D}$ together with:

a) *An additive automorphism $T : \mathcal{D} \to \mathcal{D}$, called: the shift functor. (We will denote $T^k(X)$ as $X[k]$.)*

b) *Class of distinguished (exact) triangles, (triangle is a diagram of the type:*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1];$$

the morphism of two triangles is a commutative diagram:

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle f[1]}\downarrow \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1] \end{array}).$$

This data should satisfy the following axioms:

TR1. a. *The triangle: $X \xrightarrow{id} X \to 0 \to X[1]$ is distinguished.*

b. *If triangle is distinguished, then any triangle isomorphic to it is distinguished as well.*

c. *Any morphism $X \xrightarrow{u} Y$ can be completed to a distinguished triangle: $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$.*

TR2. *The triangle: $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ is distinguished if and only if $Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$ is.*

$TR3$. *Let we have two distinguished triangles:* $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$, *and* $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} X'[1]$, *and a commutative diagram:* $\begin{array}{ccc} X & \xrightarrow{u} & Y \\ f\downarrow & & \downarrow g \\ X' & \xrightarrow{u'} & Y' \end{array}$. *Then there is morphism* h *(not necessarily unique), which together with* f *and* g *gives us the morphism of triangles:*

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ f\downarrow & & g\downarrow & & h\downarrow & & f[1]\downarrow \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1] \end{array}.$$

$TR4$. *(so-called: octahedron axiom) Let us denote as* $U \overset{[1]}{\to} V$ *the map from* U *to* $V[1]$. *Then the triangle can be denoted as:* $\begin{array}{ccc} & Z & \\ \nearrow & \downarrow [1] & \\ Y & \longleftarrow X & \end{array}$. *We will denote distinguished triangle as:* $\begin{array}{ccc} & Z & \\ \nearrow & \star\downarrow [1] & \\ Y & \longleftarrow X & \end{array}$.

Let we have a diagram of the type: $\begin{array}{ccccc} & & X' & & \\ & [1]\swarrow & [1]\downarrow\star & \nwarrow & \\ Z' & \longleftarrow & Y & \longrightarrow & Z \\ & [1]\searrow\star & \uparrow & \nearrow & \\ & & X & & \end{array}$, *where the "triangles" without* $\star$*'s are commutative. Then this diagram can be completed to a whole "octahedron", i.e.: there exists a diagram of the type:* $\begin{array}{ccccc} & & X' & & \\ & [1]\swarrow & \star\uparrow & \nwarrow & \\ Z' & \longrightarrow & Y' & \longleftarrow & Z \\ & [1]\searrow & [1]\downarrow\star & \nearrow & \\ & & X & & \end{array}$, *(with the same morphisms:* $X' \overset{[1]}{\to} Z'$, $Z' \overset{[1]}{\to} X$, $X \to Z$, $Z \to X'$*).*

From the axioms $TR1 - TR4$ it is easy to deduce the following:

Theorem 2.5.4 ([5], IV,§1, Proposition 3).

If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ *is a distinguished triangle, then for any object* U *we have exact sequences:*

$$\ldots \to \mathrm{Hom}(U, X[i]) \overset{f_*[i]}{\to} \mathrm{Hom}(U, Y[i]) \overset{g_*[i]}{\to} \mathrm{Hom}(U, Z[i]) \overset{h_*[i]}{\to} \mathrm{Hom}(U, X[i+1]) \to \ldots;$$
$$\ldots \to \mathrm{Hom}(X[i+1], U) \overset{h^*[i]}{\to} \mathrm{Hom}(Z[i], U) \overset{g^*[i]}{\to} \mathrm{Hom}(Y[i], U) \overset{f^*[i]}{\to} \mathrm{Hom}(X[i], U) \to \ldots.$$

From the above result, TR1 c) and TR3 follows that the following definition is correct.

Definition 2.5.5 .

Let we have some morphism $f : X \to Y$ in $\mathcal{D}$. Then: $\mathrm{Cone}(f)$ *is a unique up to noncanonical isomorphism object Z, s.t. there exists a distinguished triangle of the type: $X \overset{f}{\to} Y \to Z \to X[1]$.*

If $f = 0$, then $\mathrm{Cone}(f) = Y \oplus X[1]$; and f is an isomorphism iff $\mathrm{Cone}(f) = 0$.

Definition 2.5.6 .

Let $A, B, C \in Ob(\mathcal{C})$. We say that C is an elementary extension of A and B, if there exist either an exact triangle of the type: $A \to C \to B \to A[1]$, or of the type: $B \to C \to A \to B[1]$.

Let $A_1, \ldots, A_n, C \in Ob(\mathcal{C})$. We say that C is an extension of $A_1, \ldots, A_n$, if there exist $1 \leqslant j \leqslant n$, and $B \in Ob(\mathcal{C})$, s.t. C is an elementary extension of A_j and B, and B is an extension of $(A_i)_{1 \leqslant i \leqslant n; i \neq j}$.

From Theorem 2.5.4 it follows that if $\mathrm{Hom}(U, A_i) = 0$ for all $1 \leqslant i \leqslant n$, and some $U \in Ob(\mathcal{C})$, and C is an *extension* of $A_1, \ldots, A_n$, then $\mathrm{Hom}(U, C) = 0$ (the same about $\mathrm{Hom}(C, U)$).

From Theorem 2.5.4 we also have the following:

Theorem 2.5.7 .

Let we have pair of endomorphisms φ, ψ of some distinguished triangle $A \to C \to B \to A[1]$. If $\varphi_B = 0$, $\psi_A = 0$, then $(\psi \circ \varphi)_C = 0$.

From TR4 it is easy to get:

Theorem 2.5.8 .

Let we have a diagram:

$$\begin{array}{ccccccccc}
 & & A_1 & & & & A_2 & & \\
 & \nearrow \star & \downarrow [1] & \searrow [1] & & \nearrow \star & \downarrow [1] & \searrow [1] & \\
X & \longleftarrow & C_1 & \longrightarrow & D & \longleftarrow & C_2 & \longrightarrow & Y \\
 & \nwarrow & \uparrow \star & \swarrow [1] & & \nwarrow & \uparrow \star & \swarrow [1] & \\
 & & B_1 & & & & B_2 & &
\end{array} \quad ,$$

where all triangles without $\star$ are commutative. Then there exists a diagram of the type:

$$\begin{tikzcd}
 & A \arrow[d, "{[1]}"] \arrow[dr, "{[1]}"] \arrow[d, phantom, shift right=3ex, "\star"] & \\
X \arrow[ur] & D' \arrow[l] \arrow[r] & Y \arrow[dl, "{[1]}"] \\
 & B \arrow[ul] \arrow[u] \arrow[u, phantom, shift left=3ex, "\star"] &
\end{tikzcd} ,$$

where $A = \mathrm{Cone}[-1](A_1 \to A_2[1])$, *and* $B = \mathrm{Cone}[-1](B_2 \to B_1[1])$, *and* $A_1 \to A_2[1]$, $B_2 \to B_1[1]$ *are maps from the diagram.*

Moreover, if we had maps: $\alpha_1 : U \to C_1$, $\alpha_2 : U \to C_2$, *s.t. the diagram:*

$$\begin{tikzcd}
U \arrow[r, "\alpha_1"] \arrow[d, "\alpha_2"'] & C_1 \arrow[d] \\
C_2 \arrow[r] & D
\end{tikzcd}$$

is commutative, then there exists $\beta : U \to D'$, *s.t. the diagram:*

$$\begin{tikzcd}
U \arrow[r, "\beta"] \arrow[d, "\alpha_2"'] & D' \arrow[d] \\
C_2 \arrow[r] & Y
\end{tikzcd}$$

is commutative.

Applying few times the observation that the Cone of a zero map is a *direct sum*, we get the following:

Theorem 2.5.9 .

Suppose we have a diagram:

$$\begin{tikzcd}
 & A_1 \arrow[d, "{[1]}"] \arrow[d, phantom, shift right=3ex, "\star"] & A_2 \arrow[d, "{[1]}"] \arrow[d, phantom, shift right=3ex, "\star"] & \dots & & A_m \arrow[d, "{[1]}"] \arrow[d, phantom, shift right=3ex, "\star"] \\
A \arrow[ur] & Y_1 \arrow[l] \arrow[ur] & Y_2 \arrow[l] & & Y_{m-1} \arrow[ur] & Y_m \arrow[l]
\end{tikzcd} ,$$

s.t. for each $1 \leqslant i \leqslant m-1$ *the map* $Y_i \to A_{i+1}$ *can be lifted to a map from* A *to* A_{i+1}, *and* $Y_m = 0$.

Then $A = \oplus_{i=1,\dots m} A_i$.

In the same way, if in the diagram:

$$\begin{tikzcd}
 & B_1 \arrow[dl] \arrow[d, phantom, shift right=3ex, "\star"] & B_2 \arrow[dl] \arrow[d, phantom, shift right=3ex, "\star"] & \dots & & B_m \arrow[dl] \arrow[d, phantom, shift right=3ex, "\star"] \\
B \arrow[r] & Z_1 \arrow[u, "{[1]}"'] \arrow[r] & Z_2 \arrow[u, "{[1]}"'] & & Y_{m-1} \arrow[r] & Z_m \arrow[u, "{[1]}"']
\end{tikzcd} ,$$

with $Z_m = 0$, *all the maps:* $B_{i+1} \to Z_i$ *can be lifted to maps from* B_{i+1} *to* B, *then we have:* $B = \oplus_{i=1,\dots,m} B_i$.

From the previous two theorems we get:

Theorem 2.5.10 .

Let we have a diagram:

$$\begin{array}{ccccccccc}
 & A_1 & & A_2 & & A_3 & & A_m & \\
 & \nearrow \star \downarrow[1] \searrow[1] & & \nearrow \star \downarrow[1] \searrow[1] & & \nearrow \star \downarrow[1] \searrow[1] & & \nearrow \star \downarrow[1] \searrow[1] & \\
X & \longleftarrow C_1 \longrightarrow & X_1 & \longleftarrow C_2 \longrightarrow & X_2 & \longleftarrow C_3 \longrightarrow & X_3 \ldots X_{m-1} & \longleftarrow C_m \longrightarrow & X_m \\
 & \nwarrow \; \uparrow \star \swarrow[1] & & \nwarrow \; \uparrow \star \swarrow[1] & & \nwarrow \; \uparrow \star \swarrow[1] & & \nwarrow \; \uparrow \star \swarrow[1] & \\
 & B_1 & & B_2 & & B_3 & & B_m &
\end{array},$$

then there exists a diagram of the type:

$$\begin{array}{ccc}
 & A & \\
 & \nearrow \star \downarrow[1] \searrow[1] & \\
X & \longleftarrow Y \longrightarrow & X_m \\
 & \nwarrow \; \uparrow \star \swarrow[1] & \\
 & B &
\end{array},$$

where A and B have "decompositions":
$$\begin{array}{ccccccc}
 & A_1 & & A_2 & \ldots & & A_m \\
 & \nearrow \star \downarrow[1] & & \nearrow \star \downarrow[1] & & & \nearrow \star \downarrow[1] \\
A & \longleftarrow & Y_1 & \longleftarrow & Y_2 \qquad Y_{m-1} & \longleftarrow & Y_m
\end{array},$$

and
$$\begin{array}{ccccccc}
 & B_1 & & B_2 & \ldots & & B_m \\
 & \swarrow \star \uparrow[1] & & \swarrow \star \uparrow[1] & & & \swarrow \star \uparrow[1] \\
B & \longrightarrow & Z_1 & \longrightarrow & Z_2 \qquad Y_{m-1} & \longrightarrow & Z_m
\end{array}$$
, where $Z_m = Y_m = 0$.

Moreover, suppose in the original diagram for all $i = 2, \ldots, m$ the maps $B_i \to C_i$ "can be lifted to" the maps $B_i \to C_1$, i.e. for all $1 \leqslant j < i \leqslant m$ there exist maps $\gamma_{i,j} : B_i \to C_j$, that all the diagrams:
$$\begin{array}{ccc}
B_i & \xrightarrow{\gamma_{i,j}} & C_j \\
\Big\downarrow{\scriptstyle \gamma_{i,j+1}} & & \Big\downarrow \\
C_{j+1} & \longrightarrow & X_j
\end{array}$$
are commutative.

Then all the maps: $B_i \to Z_{i-1}$ can be lifted to the maps: $B_i \to B$, and, consequently, $B = \oplus_{i=0,\ldots,m} B_i$.

2.6. **Proof of the Statement 1.1.1.** Now we can give a proof of Statement 1.1.1 .

Proof

First of all we will construct some decomposition of the motive $M(Q)$ in $DM_{-}^{eff}(k)$ (see Definition 2.1.8).

Below we will omit symbols $M(-)$, and will denote the motives of varieties and simplicial schemes in the same way as the objects themselves. Also we will write just $\mathrm{Hom}(-,-)$ instead of $\mathrm{Hom}_{DM}(-,-)$.

So, let Q be a projective quadric over k given by the quadratic form $\langle 1, -a, -b, -c\rangle$.

Denote $\alpha_0 : Q \to \mathbb{Z}$ the element corresponding to the generic cycle in $CH^0(Q)$ via identification: $CH^0(Q) = \mathrm{Hom}(Q, \mathbb{Z})$ (see Theorem 2.1.17) (this map is also induced by projection $Q \to \mathrm{Spec}(k)$).

Denote $\beta_0 : \mathbb{Z}(2)[4] \to Q$ the dual map to α_0 via duality: $\mathrm{Hom}(\mathbb{Z}(2)[4], Q) = \mathrm{Hom}(\mathbb{Z}(2)[4], \underline{\mathrm{Hom}}(Q, \mathbb{Z}(2)[4])) = \mathrm{Hom}(Q(2)[4], \mathbb{Z}(2)[4]) = \mathrm{Hom}(Q, \mathbb{Z})$ (see Theorem 2.1.23).

By Theorem 2.3.2 : $\mathrm{Hom}(Q, \mathbb{Z}) = \mathrm{Hom}(Q, \mathcal{X}_Q)$, where $\mathcal{X}_Q$ - is the motive of the *standard simplicial scheme* corresponding to the pair $Q \to \mathrm{Spec}(k)$ (see Definition 2.3.1), and the map is induced by the natural projection $\mathcal{X}_Q \to \mathbb{Z}$. In the same way (by Theorem 2.3.2) we have: $\mathrm{Hom}(\mathbb{Z}(2)[4], Q) = \mathrm{Hom}(\mathcal{X}_Q(2)[4], Q)$, and the map is induced again by the projection $\mathcal{X}_Q \to \mathbb{Z}$. We can consider α_0 and β_0 as maps $Q \to \mathcal{X}_Q$ and $\mathcal{X}_Q(2)[4] \to Q$ respectively.

Denote $R := \mathrm{Cone}(\beta_0 : \mathcal{X}_Q(2)[4] \to Q)$. Consider the composition $\mathcal{X}_Q(2)[4] \to Q \to \mathcal{X}_Q$, it lives in the group $\mathrm{Hom}(\mathcal{X}_Q(2)[4], \mathcal{X}_Q) = \mathrm{Hom}(\mathcal{X}_Q(2)[4], \mathbb{Z})$ (by Theorem 2.3.2) $= 0$ (by Theorem 2.3.3). In the same way, $\mathrm{Hom}(\mathcal{X}_Q(2)[4], \mathcal{X}_Q[1]) = \mathrm{Hom}(\mathcal{X}_Q(2)[4], \mathbb{Z}[1]) = 0$. So, $\alpha_0 : Q \to \mathcal{X}_Q$ can be uniquely lifted to a map $\alpha_0' : R \to \mathcal{X}_Q$.

Denote $Q\langle 1\rangle(1)[2] := \mathrm{Cone}[-1](\alpha_0' : R \to \mathcal{X}_Q)$. We have the following diagram:

$$\begin{tikzcd}
 & \mathcal{X}_Q \arrow[dr, "{[1]}"] & \\
Q \arrow[ur] \arrow[r] & R \arrow[u, "\star"] \arrow[d, "\star{[1]}"'] & Q\langle 1\rangle(1)[2], \arrow[l] \arrow[dl, "{[1]}"] \\
 & \mathcal{X}_Q\,(2)\,[4] \arrow[ul] &
\end{tikzcd}$$

Let us multiply this diagram by Q. We have (since $\mathcal{X}_Q \times Q = Q$, by Theorem 2.3.6) the following diagram:

$$\begin{tikzcd}
 & Q \arrow[dr, "{[1]}"] & \\
Q \times Q \arrow[ur] \arrow[r] & R \arrow[u, "\star"] \arrow[d, "\star{[1]}"'] & Q \times Q\langle 1\rangle(1)[2], \arrow[l] \arrow[dl, "{[1]}"] \\
 & Q\,(2)\,[4] \arrow[ul] &
\end{tikzcd}$$

But, now the map $Q \times Q \overset{id\times\alpha_0}{\to} Q$ (which is induced by the projection on the first factor) has section - morphism $Q \to Q \times Q$ induced by the diagonal map. By duality (see Theorem 2.1.23) the map $Q(2)[4] \overset{id\times\beta_0}{\to} Q \times Q$ also has a splitting.

Hence, $Q \times Q = Q \oplus Q \times Q\langle 1\rangle(1)[2] \oplus Q(2)[4]$.

Consider now the variety $\underline{Q}^1$ of flags :(l, p), where l is a line on Q, and p is a point on this line. We have natural projection $\underline{Q}^1 \to Q$, and the fiber is constant - zero-dimensional quadric $k\sqrt{\det(Q)}$.

By Theorem 2.4.25 , where we take $P = Q$ with $\varphi : P \to Q$ - identical map, we have: $Q \times Q = Q \oplus R(1)[2] \oplus Q(2)[4]$. Notice that R here (see Theorem 2.4.25 for the definition) is nothing else but a variety $\underline{Q}^1$, and the maps $Q \times Q \to Q$ and $Q(2)[4] \to Q \times Q$ are just $id \times \alpha_0$ and $id \times \beta_0$.

So, we get that $Q \times Q\langle 1\rangle = \underline{Q}^1$.

Over $E = k\sqrt{\det(Q)}$ we have $\underline{Q}^1|_E = Q|_E \cup Q|_E$ (I mean varieties - not motives). Consider cycle $D \subset Q\times_k Q\times_k E\cup Q\times_k Q\times_E = \underline{Q}^1\times_k E\times_k Q$ - the diagonal embedding (over E) of $Q\times_k E$ into the first product. This gives us an isomorphism $Q\times E \to \underline{Q}^1$.

Finally, we get: $Q\times Q\langle 1\rangle = Q\times k\sqrt{\det(Q)}$.

Let us show that $Q\langle 1\rangle = \mathcal{X}_Q \times k\sqrt{\det(Q)}$ (see Definition 2.3.1).

First, we will construct a map $Q\langle 1\rangle \to k\sqrt{\det(Q)}$.

Notice, that by Theorem 2.3.6 $Q\times\mathcal{X}_Q = Q$, and $\mathcal{X}_Q\times\mathcal{X}_Q = \mathcal{X}_Q$ (since $\mathcal{X}_Q$ belongs to the *localizing* subcategory of $DM_-^{eff}(k)$ generated by Q, see Theorem 2.3.2). Hence, the natural map $Q\langle 1\rangle \times \mathcal{X}_Q \to Q\langle 1\rangle$ (induced by the projection $\mathcal{X}_Q \to \mathbb{Z}$) is an isomorphism as well ($Q\langle 1\rangle$ is a consecutive Cone of Q, $\mathcal{X}_Q$ and $\mathcal{X}_Q(2)[4]$).

Really, denote $Y := \mathrm{Cone}(Q \to \mathcal{X}_Q)$ ($Q = \mathcal{X}_Q^0$). Y is an extension of $Q\times Q[1]$, $Q\times Q\times Q[2]$, etc. ... (this has sense, since as a complexes of sheaves (see the definition of DM_-^{eff}: Definition 2.1.8) for each $k<0$ only finitely many of the above objects have the k-th cohomology), and we have natural map $Q\times Q[1] \to Y$, s.t. the composition $Q\times Q[1] \to Y \to Q[1]$ is the map induced by the difference of two projections (on different factors).

Consider the exact triangle: $Q\langle 1\rangle \times (Q \to \mathcal{X}_Q \to Y \to Q[1])$, which is the same as $Q\times Q\langle 1\rangle \to Q\langle 1\rangle \to Y\times Q\langle 1\rangle \to Q\times Q\langle 1\rangle[1]$. Consider Hom's from this triangle to E. Since $Q\times Q\times\cdots\times Q\times Q\langle 1\rangle = Q\times Q\times\cdots\times Q\times E$ (as was proven above), and $\mathrm{Hom}(S[j],E) = \mathrm{Hom}(S\times E,\mathbb{Z}[-j]) = 0$, for $j>0$ for arbitrary smooth projective variety S by duality (Theorem 2.1.23) and Theorem 2.1.19 , we have that the map $\mathrm{Hom}(Y[-1],E) \to \mathrm{Hom}(Q\times Q,E)$ is an embedding. Then the following sequence is exact: $\mathrm{Hom}(Q\langle 1\rangle,E) \to \mathrm{Hom}(Q\times Q\langle 1\rangle,E) \to \mathrm{Hom}(Q\times Q\times Q\langle 1\rangle,E)$, where the last map is induced by the difference of two projections.

Since $Q\times Q\langle 1\rangle = Q\times E$ (as was proven above), we have the map $\mu : Q\times Q\langle 1\rangle \to E$ (coming from the projection on E-factor). To show that it can be lifted to $Q\langle 1\rangle$, we need to prove that the restriction of μ to $Q\times Q\times Q\langle 1\rangle$ is trivial. But $Q\times Q\times Q\langle 1\rangle$ is a smooth projective variety (since $Q\times Q\langle 1\rangle = \underline{Q}^1$ is), and for any smooth projective variety S, $\mathrm{Hom}(S,\mathbb{Z}) = \oplus_{conn.comp.(S)}\mathbb{Z} \hookrightarrow \oplus_{conn.comp.(S|_{\overline{k}})}\mathbb{Z} = \mathrm{Hom}(S|_{\overline{k}},\mathbb{Z})$.

So, it is enough to check it over $\overline{k}$, where it is clear, since $E|_{\overline{k}} = Q\langle 1\rangle|_{\overline{k}} = \mathbb{Z}\oplus\mathbb{Z}$.

So, we get a map $\pi : Q\langle 1\rangle \to E$. Let us prove that $id_{\mathcal{X}_Q}\times\pi$ induces an isomorphism $Q\langle 1\rangle \to \mathcal{X}_Q\times Q\langle 1\rangle \to \mathcal{X}_Q\times E$.

Really, since $\mathcal{X}_Q$ belongs to the *localizing* subcategory of $DM^{eff}(k)$ generated by Q (see Theorem 2.3.2), it is enough to check that $Q\times Q\langle 1\rangle \to Q\times E$ is an isomorphism, but this follows from the way our morphism π was constructed (it was constructed as a lifting of the natural projection $Q\times E \to E$).

So, we proved that $Q\langle 1\rangle = \mathcal{X}_Q\times k\sqrt{\det(Q)}$. And we have the following decomposition for the motive of Q:

$$
\begin{array}{ccccc}
 & & \mathcal{X}_Q & & \\
 & \nearrow & \uparrow\star & \searrow[1] & \\
Q & \longrightarrow & R & \longleftarrow & k\sqrt{det(Q)}\times\mathcal{X}_Q(1)[2], \\
 & \nwarrow & \star\downarrow[1] & \swarrow[1] & \\
 & & \mathcal{X}_Q\,(2)\,[4] & &
\end{array}
$$

where morphisms $u \in \mathrm{Hom}(\mathcal{X}_Q, k\sqrt{det(Q)} \times \mathcal{X}_Q(1)[3])$, and $v \in \mathrm{Hom}(k\sqrt{det(Q)} \times \mathcal{X}_Q, \mathcal{X}_Q(1)[3])$ can be described in the following way: $\mathrm{Hom}(\mathcal{X}_Q, k\sqrt{det(Q)}\times\mathcal{X}_Q(1)[3]) = \mathrm{Hom}(k\sqrt{det(Q)} \times \mathcal{X}_Q, \mathcal{X}_Q(1)[3])$ by duality, and the later group is equal to $\mathrm{Hom}(k\sqrt{det(Q)} \times \mathcal{X}_Q, \mathbb{Z}(1)[3])$ (by Theorem 2.3.2). But over $E = k\sqrt{det(Q)} = k\sqrt{-abc}$ Q becomes 2-*fold Pfister* quadric, corresponding to the symbol $\{a,b\}|_E$ ($\{a,b\}|_E = \{b,c\}|_E = \{a,c\}|_E$) (see Definition 2.4.3).

$\mathrm{Hom}(\mathcal{X}_Q \times E, \mathbb{Z}(1)[3])$ is the same as $\mathrm{Hom}(\mathcal{X}_Q|_E, \mathbb{Z}(1)[3])$ in the category of motives over E. This group is $\mathbb{Z}/2$ with generator $\beta \circ \tau^{-1}(\{a,b\}|_E)$,
where $\beta : \mathrm{Hom}_E(\mathcal{X}_Q|_E, \mathbb{Z}/2(1)[2]) \to \mathrm{Hom}_E(\mathcal{X}_Q|_E, \mathbb{Z}(1)[3])$ is a *bokstein*, and $\tau : \mathrm{Hom}_E(\mathcal{X}_Q|_E, \mathbb{Z}/2(1)[2]) \to \mathrm{Hom}_E(\mathcal{X}_Q|_E, \mathbb{Z}/2(2)[2]) = \mathrm{Hom}_E(\mathbb{Z}_E, \mathbb{Z}/2(2)[2]) = \mathrm{K}_2^M(E)/2$ is a composition with the unique nonzero element $\tau \in \mathrm{Hom}(\mathbb{Z}/2, \mathbb{Z}/2(1))$ which corresponds to the choice of -1 as a square root of 1 (see Definition 2.1.25)(by Theorem 2.3.7). Really, $\{a,b\}$ is the only nonzero element from the $\mathrm{Ker}(\mathrm{K}_2^M(E)/2 \to \mathrm{K}_2^M(E(Q))/2)$ (see, for example, [13], exact sequence (1) on p.4).

By Theorem 2.3.7 we have an isomorphisms: τ (see Definition 2.1.25) from $\mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}/2(n-1)[n])$ to the subgroup of $\mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}/2(n)[n]) = \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/2(n)[n])$ (by Theorem 2.3.9) $= \mathrm{K}_n^M(k)/2$ (by Theorem 2.1.20 and Theorem 2.1.18) equal to the $\mathrm{Ker}(\mathrm{K}_n^M(k)/2 \to \mathrm{K}_n^M(k(Q))/2)$.

And the *bokstein* β performs an isomorphism: $\mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}/2(n-1)[n]) \to \mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}(n-1)[n+1])$ (by Theorem 2.3.7 , and since the last group is 2-torsion (by transfer arguments, since Q has a point of degree 2, see discussion after Theorem 2.3.11)).

So, to find a kernel $\mathrm{Ker}(\mathrm{K}_n^M(k)/2 \to \mathrm{K}_n^M(k(Q))/2)$ it is enough to compute the group $\mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}(n-1)[n+1])$

From the exact triangle $\mathcal{X}_Q(2)[4] \to Q \to R^1 \to \mathcal{X}_Q(2)[5]$ we have an exact sequence: $\mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}(n-2)[n-1]) \to \mathrm{Hom}(R^1, \mathbb{Z}(n)[n+4]) \to \mathrm{Hom}(Q, \mathbb{Z}/2(n)[n+4])$.

The first of these groups is zero by Theorem 2.3.10 , and by discussion after Theorem 2.3.11 . The last one is zero since $\dim(Q) = 2 < 4$ (see Theorem 2.1.18). So, $\mathrm{Hom}(R^1, \mathbb{Z}(n)[n+4]) = 0$. Now, from the exact triangle $R^1 \to \mathcal{X}_Q \to \mathcal{X}_Q \times k\sqrt{det(Q)}(1)[3] \to R^1[1]$, we get: $\mathrm{Hom}(\mathcal{X}_Q \times k\sqrt{det(Q)}, \mathbb{Z}(n-1)[n+1]) \to \mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}(n)[n+4]) \to 0 = \mathrm{Hom}(R^1, \mathbb{Z}(n)[n+4])$. The surjection $\mathrm{Hom}(\mathcal{X}_Q \times$

$k\sqrt{det(Q)}, \mathbb{Z}(n-1)[n+1]) \twoheadrightarrow \mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}(n)[n+4])$ here is given by multiplication by u, and, so, it is a composition: $\mathrm{Hom}(\mathcal{X}_Q|_E, \mathbb{Z}(n-1)[n+1]) \stackrel{\beta\circ\tau^{-1}(\{a,b\})}{\to} \mathrm{Hom}(\mathcal{X}_Q|_E, \mathbb{Z}(n)[n+4]) \longrightarrow \mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}(n)[n+4])$.

Again, since Q has a point of degree 2, by transfer arguments (and by Theorem 2.3.4) we have that $\mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}(a)[b])$, for all $b > a$ has exponent 2, and the map $\mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}(a)[b]) \to \mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}/2(a)[b])$ is injective.

We have an action of motivic operation Q_1 in $\mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}/2(a)[b])$ (see Theorem 2.1.26). Let us denote it as d^1 (we have too many Q's without it).

As usually, let $\widetilde{\mathcal{X}}_Q := \mathrm{Cone}[-1](\mathcal{X}_Q \to \mathbb{Z})$ (see Definition 2.3.8). We have: d^1 acts without cohomology on $\mathrm{Hom}(\widetilde{\mathcal{X}}_Q, \mathbb{Z}/2(a)[b])$ (see Theorem 2.3.11), and $\mathrm{Hom}(\widetilde{\mathcal{X}}_Q, \mathbb{Z}/2(a)[b]) = \mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}/2(a)[b])$ for $b > a$ (by Theorem 2.1.18). Since for $b \leqslant a$ $\mathrm{Hom}(\widetilde{\mathcal{X}}_Q, \mathbb{Z}(a)[b]) = 0$ (by Theorem 2.3.9), we get that d^1 performs an isomorphism from $\mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}/2(n-1)[n+1]) = \mathrm{Hom}(\widetilde{\mathcal{X}}_Q, \mathbb{Z}/2(n-1)[n+1])$ to the group $\mathrm{Ker}(d^1 : \mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}/2(n)[n+4]) \to \mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}/2(n+1)[n+7]))$.

By Theorem 2.1.26 (6), any element of the above subgroup of $\mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}/2(n)[n+4])$ comes from the cohomology with integral coefficients: $\mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}(n)[n+4])$.

Since our map $\mathrm{Hom}(\mathcal{X}_Q \times E, \mathbb{Z}(n-1)[n+1]) \to \mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}(n)[n+4])$ (induced by the multiplication by u) is surjective, the transfer map: $\mathrm{Hom}(\mathcal{X}_Q \times E, \mathbb{Z}(n)[n+4]) \to \mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}(n)[n+4])$ is surjective as well.

So, any element of the Ker : $(d^1 : \mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}/2(n)[n+4]) \to \mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}/2(n+1)[n+7]))$ can be lifted to the element of the group $\mathrm{Hom}(\mathcal{X}_Q \times E, \mathbb{Z}(n)[n+4])$, and, consequently, to some element of $\mathrm{Hom}(\mathcal{X}_Q \times E, \mathbb{Z}/2(n)[n+4])$.

Since $Q|_E$ is a 2-dimensional *Pfister quadric* (corresponding to the *pure symbol* $\{a, b\}$ (see Definition 2.4.3)), all the elements of the later group are killed by d^1 (see [13], Theorem 1'). Hence (since d^1 commutes with transfers in motivic cohomology (V.Voevodsky, unpublished)), $\mathrm{Ker}(d^1 : \mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}/2(n)[n+4]) \to \mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}/2(n+1)[n+7]))$ coincides with the image of $\mathrm{Tr}_{\mathrm{E/k}} : \mathrm{Hom}(\mathcal{X}_Q \times E, \mathbb{Z}(n)[n+4]) \to \mathrm{Hom}(\mathcal{X}_Q \times E, \mathbb{Z}/2(n)[n+4]) \to \mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}/2(n)[n+4])$.

On $\mathrm{Hom}(\mathcal{X}_Q \times E, \mathbb{Z}/2(n-1)[n+1])$ the map d^1 coincides with the multiplication with $\beta \circ \tau^{-1}(\{a, b\})$ (see [13], Theorem 1').

Now from the commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}(\mathcal{X}_Q \times E, \mathbb{Z}/2(n-1)[n+1]) & \longrightarrow & \mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}/2(n-1)[n+1]) \\ \Big\downarrow{\scriptstyle \cdot\beta\circ\tau^{-1}(\{a,b\})} & & \Big\downarrow{\scriptstyle d^1} \\ \mathrm{Hom}(\mathcal{X}_Q \times E, \mathbb{Z}/2(n)[n+4]) & \longrightarrow & \mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}/2(n)[n+4]) \end{array}$$

, and the fact that $d^1 : \mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}/2(n-1)[n+1]) \to \mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}/2(n)[n+4])$ is a monomorphism (see above), we get that the map $\mathrm{Hom}(\mathcal{X}_Q \times E, \mathbb{Z}(n-1)[n+1]) \xrightarrow{\mathrm{Tr}_{\mathrm{E/k}}} \mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}(n-1)[n+1]) \to \mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}/2(n-1)[n+1])$ is surjective.

Since transfers commute with τ and β, we get the surjection (see above):
$\mathrm{Ker}(\mathrm{K}_n^M(E)/2 \to \mathrm{K}_n^M(E(Q))/2) \twoheadrightarrow \mathrm{Ker}(\mathrm{K}_n^M(k)/2 \to \mathrm{K}_n^M(k(Q))/2)$.

But, by [13], exact sequence (1), p.4, we know that for *Pfister quadric* $Q|_E$ (see Definition 2.4.3), this kernel is generated by *pure* 2-symbol (see Definition 2.4.4) $\{a,b\}$:

$\mathrm{Ker}(\mathrm{K}_*^M(E)/2 \to \mathrm{K}_*^M(E(Q))/2) = \{a,b\} \cdot \mathrm{K}_{*-2}^M(E)/2$.

By the lemma of H.Bass and J.Tate (see [1], Corollary 5.3) we have that as a module over $\mathrm{K}_*^M(k)$ the Milnor's K-theory of E, $\mathrm{K}_*^M(E)$ is generated by the components of degree 0 and 1.

The transfer $\mathrm{Tr}_{\mathrm{E/k}} : \mathrm{K}_*^M(E) \to \mathrm{K}_*^M(k)$ is a homomorphism of $\mathrm{K}_*^M(k)$ - modules.

So, since the zero-degree component of $\mathrm{K}_*^M(E)/2$ is mapped to 0 via $\mathrm{Tr}_{\mathrm{E/k}}$, we get that the ideal $\mathrm{Ker}(\mathrm{K}_*^M(k)/2 \to \mathrm{K}_*^M(k(Q))/2)$ is generated by the elements of the form: $\{a,b\} \cdot \mathrm{Tr}_{\mathrm{E/k}}(E^*)$.

Statement 1.1.1 is proven.

□

3. General theorems

3.1. **Structure of the motive of a quadric.** In this section we will describe motivic decomposition for quadrics in the motivic category DM^{eff} (see Definition 2.1.8). Everywhere below we will assume that our field k has characteristic 0.

Below we will usually omit $M(-)$, and denote motives of schemes in the same way as schemes themselves.

Let Q be some quadric of dimension n . Denote as $Q^1,\ Q^2, \ldots, Q^{[n/2]}$ - varieties of lines, planes, etc. ... on a quadric Q . As usually, for a variety P we will denote as $\mathcal{X}_P$ the motive of a *standard simplicial scheme*, corresponding to the pair $P \to Spec\,(k)$ (see Definition 2.3.1). $\mathcal{X}_P$ is a *form* of Tate-motive (see remark after the proof of Theorem 2.3.4).

For 2-dimensional quadric Q we had decomposition, which can be drawn in the following way:

$$\begin{array}{ccccc} & & * & & \\ & \nearrow & \uparrow & \star\searrow & \\ * & \longrightarrow & * & \longleftarrow & * \\ & \nwarrow\star & \downarrow & \swarrow & \\ & & * & & \end{array}$$

or, considering the other side of the same octahedron:

$$\begin{array}{ccccc} & & * & & \\ & \nearrow\star & \downarrow & \searrow & \\ *\longleftarrow & & * & & \longrightarrow * \\ & \nwarrow & \uparrow & \star\swarrow & \\ & & * & & \end{array}$$

It appears that in the case of the quadric of arbitrary dimension we have similar decomposition:

$$\begin{array}{ccccccccccccccc} & & * & & & & * & & & & & & * & & \\ & \nearrow\star & \downarrow & \searrow & & \nearrow\star & \downarrow & \searrow & & & & \nearrow\star & \downarrow & \searrow & \\ *\longleftarrow & & * & & \longrightarrow *\longleftarrow & & * & & \longrightarrow * & \ldots & *\longleftarrow & & * & & \longrightarrow * \\ & \nwarrow & \uparrow & \star\swarrow & & \nwarrow & \uparrow & \star\swarrow & & & & \nwarrow & \uparrow & \star\swarrow & \\ & & * & & & & * & & & & & & * & & \end{array}$$

consisting of n exact triangles for $\dim(Q) = n$ - even, and of the form:

$$\begin{array}{ccccccccccccccccc} & & * & & & & * & & & & & & * & & & & * \\ & \nearrow\star & \downarrow & \searrow & & \nearrow\star & \downarrow & \searrow & & & & \nearrow\star & \downarrow & \searrow & & \nearrow & | \\ *\longleftarrow & & * & & \longrightarrow *\longleftarrow & & * & & \longrightarrow * & \ldots & *\longleftarrow & & * & & \longrightarrow * & \star & | \\ & \nwarrow & \uparrow & \star\swarrow & & \nwarrow & \uparrow & \star\swarrow & & & & \nwarrow & \uparrow & \star\swarrow & & \nwarrow & \downarrow \\ & & * & & & & * & & & & & & * & & & & * \end{array}$$

with n exact triangles for n-odd.

So, the motive of our quadric Q is an extension (see Definition 2.5.6) of $n+1$ "elementary pieces", living in the upper and lower row of the diagram (also + the last element from the middle row, if n-even).

These "elementary pieces" appears to be *standard simplicial schemes* (see Definition 2.3.1), associated with the pairs $Q^i \to \mathrm{Spec}(k)$, for all $i = 0, \ldots, [n/2]$, where Q^i is a variety of i-dimensional planes on Q (see above).

More precisely, we have:

Theorem 3.1 .
$M(Q)$ is an extension (see Definition 2.5.6) of:
$\mathcal{X}_Q, \mathcal{X}_{Q^1}(1)[2], \ldots, \mathcal{X}_{Q^{(n-1)/2}}(n-1/2)[n-1], \mathcal{X}_{Q^{(n-1)/2}}(n+1/2)[n+1], \ldots,$
$\mathcal{X}_{Q^1}(n-1)[2n-2], \mathcal{X}_Q(n)[2n]$, *if n is odd,*
and an extension of $\mathcal{X}_Q, \mathcal{X}_{Q^1}(1)[2], \ldots, \mathcal{X}_{Q^{n/2-1}}(n/2-1)[n-2]$,
$\mathcal{X}_{Q^{n/2}}(n/2)[n] \times k\left(\sqrt{det(Q)}\right), \mathcal{X}_{Q^{n/2-1}}(n/2+1)[n+2], \ldots, \mathcal{X}_{Q^1}(n-1)[2n-2], \mathcal{X}_Q(n)[2n]$,
if n is even.

More precisely, there exist motives $Q\langle 1\rangle, Q\langle 2\rangle, \ldots, Q\langle [n/2]\rangle$, which fit in the following Postnikov tower (under this I mean nothing more than just a system of exact triangles in a triangulated category) for Q :

$$\begin{array}{ccccccccc}
 & \mathcal{X}_Q & & & & & \mathcal{X}_{Q^{[n/2]-1}}\left([n/2]-1\right)[2[n/2]-2] & & \\
\nearrow\star & \downarrow[1] & \searrow[1] & & & \nearrow\ \star & \downarrow[1] & \searrow[1] & \\
Q \longleftarrow & R^1 & \longrightarrow Q'\langle 1\rangle & ,\ldots, & Q'\langle [n/2]-1\rangle \longleftarrow & & R^{[n/2]} & \longrightarrow & Q'\langle [n/2]\rangle \\
\nwarrow & \uparrow\ \star & \swarrow[1] & & & \nwarrow & \uparrow\ \star & \swarrow[1] & \\
 & \mathcal{X}_Q(n)[2n] & & & & & \mathcal{X}_{Q^{[n/2]-1}}\left([n/2]+1\right)[2[n/2]+2] & &
\end{array}$$

where $Q'\langle i\rangle = Q\langle i\rangle(i)[2i]$.
and in the case n is even, $Q\langle n/2\rangle = \mathcal{X}_{Q^{n/2}} \times k\left(\sqrt{det\,(Q)}\right)$;
in the case n is odd there is an exact triangle:
$\mathcal{X}_{Q^{n-1/2}}(n+1/2)[n+1] \to Q'\langle (n-1)/2\rangle \to \mathcal{X}_{Q^{n-1/2}}(n-1/2)[n-1]$.

Proof of Theorem 3.1

The proof is similar to the one of Theorem 4.4 from [20].

Denote as α_i the standard morphism $Q \to \mathbb{Z}(i)[2i]$, corresponding to the plane section of codimension i (see Theorem 2.1.17), and as β_i the dual map $\mathbb{Z}(n-i)[2n-2i] \to Q$ (see (4) from Theorem 2.1.23).

Denote as R^1 the $Cone[-1](Q \to \mathcal{X}_Q)$ (map $Q \to \mathcal{X}_Q$ comes from the fact that $\mathcal{X}_Q^0 = Q$, or from α_0 via identification $\mathrm{Hom}(Q,\mathcal{X}_Q) = \mathrm{Hom}(Q,\mathbb{Z})$ (see Theorem 2.3.2)).

Since $\mathrm{Hom}(\mathcal{X}_Q, \mathcal{X}_Q(a)[b]) = \mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}(a)[b])$ (by Theorem 2.3.2), by Theorem 2.3.3 (1) we see that the composition: $\mathcal{X}_Q(n)[2n] \xrightarrow{\beta_0} Q \xrightarrow{\alpha_0} \mathcal{X}_Q$ is zero,

and we get a map $\mathcal{X}_Q(n)[2n] \to R^1$ (which is unique precisely by the same reason).

Denote as $Q\langle 1\rangle$ the $Cone\,(\mathcal{X}_Q(n)[2n] \to R^1)(-1)[-2]$.

We claim, that the natural map $Q\langle 1\rangle \times \mathcal{X}_{Q^1} \to Q\langle 1\rangle$ (coming from projection $\mathcal{X}_Q \to \mathbb{Z}$) is an isomorphism. Really, consider the following diagram :

$$\begin{array}{ccc}
Q\langle 1\rangle \times \mathcal{X}_{Q^1} \times \mathcal{X}_Q & \longrightarrow & Q\langle 1\rangle \times \mathcal{X}_Q \\
\downarrow & & \downarrow \\
Q\langle 1\rangle \times \mathcal{X}_{Q^1} & \longrightarrow & Q\langle 1\rangle
\end{array}$$

Both vertical maps are isomorphisms , since Q has point over $k(Q^1)$ and $Q\langle 1\rangle$ is constructed from Q (use Theorem 2.3.6) . So, it is enough to prove that upper horizontal map is an isomorphism, and for this it is enough to prove that $Q\langle 1\rangle \times \mathcal{X}_{Q^1} \times Q \to Q\langle 1\rangle \times Q$ is an isomorphism (since $\mathcal{X}_Q$ belongs to the *localizing* subcategory, generated by Q (see Theorem 2.3.2)) and the later is equivalent to the fact that $Q\langle 1\rangle \times \mathcal{X}_{Q^1} \times k(Q) \to Q\langle 1\rangle \times k(Q)$ is an isomorphism (by Theorem 2.3.5). But over $k(Q)$ Q has a point, and, so, $Q\langle 1\rangle \times k(Q)$ become just the quadric Q' of lines passing through this point (see Theorem 2.4.23 and Remark 2 after it), and $\mathcal{X}_{Q^1} \times k(Q) = \mathcal{X}_{Q'/k(Q)}$. Hence, the isomorphism in this case is evident.

Now, composition $\mathcal{X}_Q(n)[2n] \xrightarrow{\beta_0} R^1 \xrightarrow{\alpha_1|_{R^1}} \mathbb{Z}(1)[2]$ is zero by Theorem 2.3.2 and Theorem 2.3.3 (1), since $n > 1$. So, we get morphism $\alpha_1'' : Q\langle 1\rangle(1)[2] \to \mathbb{Z}(1)[2]$ and since $Q\langle 1\rangle \times \mathcal{X}_{Q^1} = Q\langle 1\rangle$ also a morphism $\alpha_1' : Q\langle 1\rangle(1)[2] \to \mathcal{X}_{Q^1}(1)[2]$ (by Theorem 2.3.2).

Since Q has a point over Q^1 ,we see that $\mathrm{Hom}(\mathcal{X}_{Q^1}, \mathcal{X}_Q(a)[b]) = \mathrm{Hom}(\mathcal{X}_{Q^1}, \mathbb{Z}(a)[b])$ by Theorem 2.3.2 and Theorem 2.3.6 , and β_1 can be lifted to $\beta_1' : \mathcal{X}_{Q^1}(n-1)[2n-2] \to Q\langle 1\rangle(1)[2]$. If $n > 2$, then the composition :

$\mathcal{X}_{Q^1}(n-1)[2n-2] \xrightarrow{\beta_1'} Q\langle 1\rangle(1)[2] \xrightarrow{\alpha_1'} \mathcal{X}_{Q^1}(1)[2]$ is zero, by Theorem 2.3.3 (1) and since

$\mathrm{Hom}(\mathcal{X}_{Q^1}, \mathcal{X}_{Q^1}(a)[b]) = \mathrm{Hom}(\mathcal{X}_{Q^1}, \mathbb{Z}(a)[b])$ (by Theorem 2.3.2), and if we denote

$R^2 = \mathrm{Cone}[-1](Q\langle 1\rangle(1)[2] \to \mathcal{X}_{Q^1}(1)[2])$, then β_1' can be lifted to a map from $\mathcal{X}_{Q^1}(n-1)[2n-2]$ to R^2

Denote as $Q\langle 2\rangle$ the $\mathrm{Cone}(\mathcal{X}_{Q^1}(n-1)[n-2] \to R^2)(-2)[-4]$. We claim, that the natural map : $Q\langle 2\rangle \times \mathcal{X}_{Q^2} \to Q\langle 2\rangle$ is an isomorphism. Again, consider a diagram

$$\begin{array}{ccc} Q\langle 2\rangle \times \mathcal{X}_{Q^2} \times \mathcal{X}_{Q^1} & \longrightarrow & Q\langle 2\rangle \times \mathcal{X}_{Q^1} \\ \downarrow & & \downarrow \\ Q\langle 2\rangle \times \mathcal{X}_{Q^2} & \longrightarrow & Q\langle 2\rangle \end{array}$$

Both vertical maps are isomorphisms, since Q^1 has point over $k(Q^2)$ and $Q\langle 2\rangle$ is made of $Q\langle 1\rangle$ (for which we have corresponding property) and Q^1 (we use Theorem 2.3.6 here). So, it is again enough to prove, that the upper horizontal map is an isomorphism, and in the same way as before for this it is enough to prove (using Theorem 2.3.5) that $Q\langle 2\rangle \times \mathcal{X}_{Q^2} \times k(Q^1) \to Q\langle 2\rangle \times k(Q^1)$ is an isomorphism. But over $k(Q^1)$ Q has a line and $Q\langle 2\rangle$ is isomorphic to the quadric Q'' of planes on Q ,containing this line (apply Theorem 2.4.23 twice). Hence, $\mathcal{X}_{Q^2} \times k(Q^1) = \mathcal{X}_{Q''/k(Q^1)}$, and we get what we need.

Continuing the same way we get $Q\langle i\rangle$,where $i < n/2$. Since $\mathrm{Hom}\left(\mathcal{X}_{Q^i}, \mathcal{X}_{Q^j}(a)[b]\right) = \mathrm{Hom}\left(\mathcal{X}_{Q^i}, \mathbb{Z}(a)[b]\right)$ for all $j < i$ by Theorem 2.3.2 and Theorem 2.3.6 (Q^j has a point over $k(Q^i)$), and $Q\langle i\rangle \times \mathcal{X}_{Q^i} = Q\langle i\rangle$ we can lift α_i and β_i to a maps : $\mathcal{X}_{Q^i}(n-i)[2n-2i] \xrightarrow{\alpha_i'} Q\langle i\rangle(i)[2i] \xrightarrow{\beta_i'} \mathcal{X}_{Q^i}(i)[2i]$ Since $n/2 > i$ the composition will be zero (by Theorem 2.3.2 and Theorem 2.3.3 (1)) and we can construct $Q\langle i+1\rangle$.

If $i + 1 < n/2$,then using precisely the same arguments as before we get that $Q\langle i+1\rangle \times \mathcal{X}_{Q^{i+1}} = Q\langle i+1\rangle$. Otherwise we have two possibilities :

1) n is odd and $i = (n-1)/2$

We claim that $Q\langle i+1\rangle = 0$. Really, Consider a diagram :

$$\begin{array}{ccc} Q\langle i+1\rangle \times \mathcal{X}_{Q^i} & \longrightarrow & 0 \\ \big\downarrow & & \big\downarrow \\ Q\langle i+1\rangle & \longrightarrow & 0 \end{array}$$

Vertical maps here are isomorphisms (since $Q\langle i+1\rangle$ is constructed from $Q\langle i\rangle$, for which we have this property, and $\mathcal{X}_{Q^i}$'s,), and to check that $Q\langle i+1\rangle \times \mathcal{X}_{Q^i} = 0$, it is enough to prove that $Q\langle i+1\rangle \times Q^i = 0$ or that $Q\langle i+1\rangle \times k\,(Q^i) = 0$ (by Theorem 2.3.5), which follows from Theorem 2.4.22 , since over $k(Q^{n-1/2})$ Q is hyperbolic.

2) n is even and $i = n/2 - 1$

Since $Q\langle n/2\rangle \times \mathcal{X}_{Q^{n/2}} = Q\langle n/2\rangle$, by Theorem 2.3.2 we have :

$\mathrm{Hom}\Big(Q'\langle n/2\rangle, \mathcal{X}_{Q^{n/2}}(n/2)[n] \times k\left(\sqrt{det\,(Q)}\right)\Big) = \mathrm{Hom}\Big(Q'\langle n/2\rangle, k\Big(\sqrt{det\,(Q)}\Big)(n/2)[n]\Big)$

Now we can take element of the $\mathrm{Hom}(Q'\langle n/2\rangle, \mathbb{Z}(n/2)[n])$, corresponding to the plane section of Q of dimension $n/2$ (see Theorem 2.1.17). This map is actually the composition: $Q'\langle n/2\rangle \to k\Big(\sqrt{det\,(Q)}\Big)\,(n/2)[n] \to \mathbb{Z}(n/2)[n]$. To see this we need the following:

Together with the varieties Q^i we can consider varieties $\underline{Q^i}$ - the varieties of flags *point* $\subset$ *line* $\subset \cdots \subset$ i-dim.*plane* on Q [2]. Evidently, $\underline{Q^i}$ is a grassmanian bundle over Q^i, and the existence of rational point on both varieties is equivalent. So, $\mathcal{X}_{Q^i} = \mathcal{X}_{\underline{Q^i}}$ (by Theorem 2.3.4 , see also Theorem 2.4.15).

Motive of the variety $\underline{Q^i}$ can be described as follows:

Claim 3.2 .

$\underline{Q^i} = Q \times Q\langle 1\rangle \times \cdots \times Q\langle i\rangle$.

Proof of Claim 3.2

Induction on i. Case $i = 0$ is trivial.

Since $\underline{Q^i} \times \mathcal{X}_{Q^i} = \underline{Q^i}$ (by Theorem 2.3.6 , for example), and $\mathcal{X}_{Q^r} \times \mathcal{X}_{Q^s} = \mathcal{X}_{Q^{\max(r,s)}}$ (by Theorem 2.3.6), we have: $\underline{Q^i} \times Q$ is an extension of $\underline{Q^i}(j)[2j]$, $\underline{Q^i}(\dim(Q) - j)[2\dim(Q) - 2j]$, $0 \leqslant j \leqslant i$, and $\underline{Q^i} \times Q\langle i+1\rangle(i+1)[2i+2]$ (by the already proven part of Theorem 3.1 (since Q is an extension of $\mathcal{X}_{Q^j}(j)[2j]$, $0 \leqslant j \leqslant i$, $Q\langle i+1\rangle(i+1)[2i+2]$, and $\mathcal{X}_{Q^j}(\dim(Q) - j)[2\dim(Q) - 2j]$, $0 \leqslant j \leqslant i$)). Certainly, $\underline{Q^{i+1}}$ is a fibration over $\underline{Q^i}$ with fibers - quadrics $Q(i+1, x)$ (quadric of $i+1$ dimensional projective subspaces on Q, containing fixed i-dimensional space, determined by $x \in \underline{Q^i}$).

Over $\underline{Q^i}$, Q is (globally) *i-times strongly isotropic* (see Definition 2.4.24), and, so the motive of $\underline{Q^i} \times Q$ is a direct sum of $\underline{Q^i}(j)[2j]$, $\underline{Q^i}(\dim(Q) - j)[2\dim(Q) - 2j]$, $0 \leqslant j \leqslant i$, and $\underline{Q^{i+1}}(i+1)[2i+2]$ (by Theorem 2.4.25).

Moreover, the maps $\underline{Q^i} \times Q \to \underline{Q^i}(i)[2i]$ which define this decomposition are just $id \times \alpha_i$ (see beginning of the proof of Theorem 3.1 for the definition of α_i). So, $\underline{Q^i} \times Q$

[2] I would like to thank V.Voevodsky for suggesting me the use of later varieties

will be also a direct sum of $\underline{Q^i}(j)[2j]$, $\underline{Q^i}(\dim(Q)-j)[2\dim(Q)-2j]$, $0\leqslant j\leqslant i$, and $\underline{Q^i}\times Q\langle i+1\rangle(i+1)[2i+2]$ (since the maps $\underline{Q^i}\times Q\to\underline{Q^i}(i)[2i]$ there are the same). And we get an isomorphism $\underline{Q^{i+1}}=\underline{Q^i}\times Q\langle i+1\rangle$.

Claim 3.2 is proven.

$\square$

Now, since $\underline{Q^{n/2-1}}\times\mathcal{X}_{Q^{n/2-1}}=\underline{Q^{n/2-1}}$, we have an exact triangle: $Y\times Q\langle n/2\rangle\to\underline{Q^{n/2-1}}\times Q\langle n/2\rangle\to Q\langle n/2\rangle\to Y[1]\times Q\langle n/2\rangle$, where $Y=\mathrm{Cone}[-1](\underline{Q^{n/2-1}}\to\mathcal{X}_{Q^{n/2-1}})$ (it is an extension of: $\underline{Q^{n/2-1}}\times\underline{Q^{n/2-1}}\times Q\langle n/2\rangle$, $\underline{Q^{n/2-1}}\times\underline{Q^{n/2-1}}\times\underline{Q^{n/2-1}}\times Q\langle n/2\rangle[1]$, etc. ...). But from Claim 3.2 we know that $\underline{Q^{n/2-1}}\times Q\langle n/2\rangle=\underline{Q^{n/2}}$, and also (since the last fibration $\underline{Q^{n/2}}\to\underline{Q^{n/2-1}}$ has fiber - 0-dimensional quadric $k\left(\sqrt{det\,(Q)}\right)$, which is defined over k) we have: $\underline{Q^{n/2-1}}\times k\left(\sqrt{det\,(Q)}\right)=\underline{Q^{n/2}}$. We have the commutative diagram :

$$\begin{array}{ccccc}
\underline{Q^{n/2-1}}\times\underline{Q^{n/2-1}}\times Q\langle n/2\rangle & \longrightarrow & \underline{Q^{n/2-1}}\times Q\langle n/2\rangle & \longrightarrow & Q\langle n/2\rangle \\
 & & \| & & \\
 & & \underline{Q^{n/2-1}}\times k\left(\sqrt{det\,(Q)}\right) & \longrightarrow & k\left(\sqrt{det\,(Q)}\right)
\end{array}$$

The upper row is not an exact triangle, but the sequence $\mathrm{Hom}(Q\langle n/2\rangle,k\left(\sqrt{det\,(Q)}\right))\to\mathrm{Hom}(\underline{Q^{n/2-1}}\times Q\langle n/2\rangle,k\left(\sqrt{det\,(Q)}\right))\to\mathrm{Hom}(\underline{Q^{n/2-1}}\times\underline{Q^{n/2-1}}\times Q\langle n/2\rangle,k\left(\sqrt{det\,(Q)}\right))$ is exact, since $Y\times Q\langle n/2\rangle\to\underline{Q^{n/2-1}}\times Q\langle n/2\rangle\to Q\langle n/2\rangle$ is an exact triangle and $\mathrm{Hom}(\underline{Q^{n/2-1}}\times\cdots\times\underline{Q^{n/2-1}}\times Q\langle n/2\rangle[j],k\left(\sqrt{det\,(Q)}\right))=\mathrm{Hom}(\underline{Q^{n/2-1}}\times\cdots\times\underline{Q^{n/2-1}}\times k\left(\sqrt{det\,(Q)}\right)[j],k\left(\sqrt{det\,(Q)}\right))=0$ for $j>0$ by Theorem 2.1.19 and Theorem 2.1.23.

Since $\underline{Q^{n/2-1}}\times\underline{Q^{n/2-1}}\times Q\langle n/2\rangle=\underline{Q^{n/2-1}}\times\underline{Q^{n/2-1}}\times k\left(\sqrt{det\,(Q)}\right)$, and the later is a smooth projective variety, it is enough to check that the composition map (not the projection): $\underline{Q^{n/2-1}}\times\underline{Q^{n/2-1}}\times Q\langle n/2\rangle\to k\left(\sqrt{det\,(Q)}\right)$ is zero over algebraic closure (for a smooth projective variety A $\mathrm{Hom}(A,\mathbb{Z})=CH^0(A)=\oplus_{con.comp.(A)}\mathbb{Z}\hookrightarrow\oplus_{con.comp.(A|_{\overline{k}})}\mathbb{Z}=CH^0(A|_{\overline{k}})=\mathrm{Hom}(A|_{\overline{k}},\mathbb{Z}))$,where it is evident since $Q\langle n/2\rangle|_{\overline{k}}=k\left(\sqrt{det\,(Q)}\right)|_{\overline{k}}$, and we get a map $Q\langle n/2\rangle\to k\left(\sqrt{det\,(Q)}\right)$, and, hence, a morphism from $Q\langle n/2\rangle$ to the $\mathcal{X}_{Q^{n/2}}\times k\left(\sqrt{det\,(Q)}\right)$ which will be an isomorphism by the same arguments as before (multiply everything by $\mathcal{X}_{Q^{n/2-1}}$, ...).

Theorem 3.1 is proven.

$\square$

Remark The motives $Q\langle i\rangle$ have the following geometric sense: as soon as our quadric Q is i-times isotropic (i.e.: Q^{i-1} has a rational point, and our quadric has $i-1$-dimensional projective subspace l_{i-1}) then $Q\langle i\rangle$ becomes just the $\dim(Q)-2i$ - dimensional quadric of i-dimensional projective subspaces on Q, containing l_{i-1} (it follows from Theorem 2.4.23 , and Theorem 3.1 , since if Q is i-times isotropic, then $\mathcal{X}_{Q^j}=\mathbb{Z}$ for all $0\leqslant j<i$ (see Theorem 2.3.4)). In the language of quadratic forms: we have $q=i\cdot\mathbb{H}\perp q'$, and $Q\langle i\rangle$ is just the quadric Q'.

Claim 3.3 .

Suppose n is odd. Then $Q\langle[n-1/2]\rangle=\mathrm{Cone}[-1]\left(\mathcal{X}_{Q^{[n-1/2]}}\xrightarrow{\gamma}\mathcal{X}_{Q^{[n-1/2]}}(1)[3]\right)$, where $\gamma\in\mathrm{Hom}\left(\mathcal{X}_{Q^{[n-1/2]}},\mathbb{Z}(1)[3]\right)$ (identification by Theorem 2.3.2) is equal to $\beta\circ\tau^{-1}(c_2(Q))$
(β is a bokstein, definition of τ is given in Definition 2.1.25 , and $c_2(Q)$ is defined in Definition 2.4.13 , see also Theorem 2.3.7).

Proof of the Claim 3.3 From Theorem 2.3.7 we know, that 2 - torsion part in $\mathrm{Hom}\left(\mathcal{X}_{Q^{[n-1/2]}},\mathbb{Z}(1)[3]\right)$ is equal to the Ker : $\mathrm{K}_2^M(k)/2\to\mathrm{K}_2^M\left(k\left(Q^{[n-1/2]}\right)\right)/2$ (via $\beta\circ\tau^{-1}$). Since $\mathcal{X}_{\underline{Q^{[n-1/2]}}}=\mathcal{X}_{Q^{[n-1/2]}}$, and $\underline{Q^{[n-1/2]}}$ is a consecutive fibration over Q with fibers - quadrics of dimensions $n-2$, $n-4$, $\ldots$, 1, we see that this kernel is $\mathbb{Z}/2$, or 0 (since by Theorem 2.4.11 for quadric P of dimension >2 there is no kernel at all, and for a conic C/E it consists of one pure 2-symbol $\{a,b\}$ (corresponding to the conic: C has form $\langle 1,-a,-b\rangle$)(see [13], exact sequence (1) on p.4).
We have two cases: 1) $c_2(Q)\neq 0$, 2) $c_2(Q)=0$.

1) Since, certainly, $c_2(Q)\in$ Ker : $\mathrm{K}_2^M(k)/2\to\mathrm{K}_2^M\left(k\left(Q^{[n-1/2]}\right)\right)/2$ (Q becomes hyperbolic over $k\left(Q^{[n-1/2]}\right)$; see Definition 2.4.12), and Ker : $\mathrm{K}_2^M(k)/2\to\mathrm{K}_2^M\left(k\left(Q^{[n-3/2]}\right)\right)/2=0$ (since the later variety is a consecutive fibration with fibers - quadrics of dimension $\geqslant 3$ (see Theorem 2.4.11)), we get that $Q\langle[n-1/2]\rangle\times k\left(Q^{[n-3/2]}\right)$ is a conic without a point (since conic *with a point*, i.e.: $\mathbb{P}^1$ do not have any kernel, but our conic does). So, $\gamma\times\mathrm{Id}_{Q^{[n-3/2]}}\neq 0$, and, hence, $\gamma\neq 0$. But the only nontrivial element in $\mathrm{Hom}\left(\mathcal{X}_{Q^{[n-1/2]}},\mathbb{Z}(1)[3]\right)$ is $\beta\circ\tau^{-1}(c_2(Q))$.

2) In the same manner, get: $Q\langle[n-1/2]\rangle\times k\left(Q^{[n-3]}\right)$ is a conic with trivial c_2 (since c_2 will be the same - see Definition 2.4.12), i.e. conic with a point. So, $\gamma\times\mathrm{Id}_{Q^{[n-3/2]}}=0$, but from the commutative diagram:

$$\begin{array}{ccc}
\mathrm{Hom}\left(\mathcal{X}_{Q^{[n-1/2]}},\mathbb{Z}(1)[3]\right) & \xrightarrow{\tau\circ\beta^{-1}} & \mathrm{K}_2^M(k)/2 \\
\downarrow & & \downarrow j \\
\mathrm{Hom}\left(\mathcal{X}_{Q^{[n-1/2]}}\times k\left(Q^{[n-3/2]}\right),\mathbb{Z}(1)[3]\right) & \xrightarrow{\tau\circ\beta^{-1}} & \mathrm{K}_2^M\left(k\left(Q^{[n-3/2]}\right)\right)/2
\end{array}$$

and injectivity of j (see Theorem 2.4.11) and $\tau\circ\beta^{-1}$ (see Theorem 2.3.7), we have: $\gamma=0$ ($\tau\circ\beta^{-1}$ is an injection).

Claim 3.3 is proven. $\square$

From the proof of Theorem 3.1 we can deduce more, namely, since the morphisms $\beta'_i:\mathcal{X}_{Q^i}(n-i)[2n-2i]\to Q'\langle i\rangle$ are actually liftings of the compositions $\mathcal{X}_{Q^i}(n-i)[2n-2i]\longrightarrow\mathbb{Z}(n-i)[2n-2i]\xrightarrow{\beta_i}Q$, by Theorem 2.5.10 , we get the following

Proposition 3.4 .

There is the following exact triangle:

$$Q\to P'\to\oplus_{i=0,\dots,[n-1/2]}\mathcal{X}_{Q^i}(n-i)[2n-2i+1]\to Q[1],$$

where in case n is odd, $P'=P$, and in case n- even we have triangle

$$P[-1]\to\left(\mathcal{X}_{Q^{n/2}}\times k\sqrt{det\,(Q)}\right)(n/2)[n]\to P'\to P,$$

where P fits into the following Postnikov tower:

$$\begin{array}{ccccccc} & \mathcal{X}_Q & & \mathcal{X}_{Q^1}(1)[2] & & & \mathcal{X}_{Q^{[n-1/2]}}([n-1/2])[2[n-1/2]] \\ \nearrow\star & \downarrow[1] & \nearrow\star & \downarrow[1] & \dots & \nearrow\ \star & \downarrow[1] \\ P & \longleftarrow P^1 & \longleftarrow & P^2 & & P^{[n-1/2]}\longleftarrow & 0. \end{array}$$

Remark

Unfortunately, in general, P is not equal to a direct sum $\oplus_{j=0,\dots,[n-1/2]}\mathcal{X}_{Q^j}(j)[2j]$, as already an example of a 3 - dimensional quadric shows. More precisely, we can state the following (for more general result - see Corollary 4.4):

Statement 3.5 .

A) *Suppose, Q is a 3 - dimensional quadric. Then the following is equivalent:*
  1) *Q is a small Pfister quadric, i.e.: Q corresponds to a quadratic form $\langle 1,-a,-b,ab,-c\rangle$ for some $a,b,c\in k^*$.*
  2) *$c_2(Q)$ is a pure symbol (see Definition 2.4.12 ,Definition 2.4.4).*
  3) *$P=\mathcal{X}_Q\oplus\mathcal{X}_{Q^1}(1)[2]$.*
  4) *$Q=\mathrm{Cone}[-1](\mathcal{X}_Q\to\mathcal{X}_Q(3)[7])\oplus\mathrm{Cone}[-1](\mathcal{X}_{Q^1}\to\mathcal{X}_{Q^1}(1)[3])(1)[2]$.*

B) *Q is hyperbolic iff $c_2(Q)=0$.*

Proof of the statement

($1\Leftrightarrow 2$) Let Q corresponds to the quadratic form q, then consider an Albert quadric Q' with quadratic form $q'=q+\langle -det\,(q)\rangle$. Everything follows from: a) $c_2(Q)=c_2(q')$, which is sum of two *pure* symbols, b) isotropness of q' is equivalent to the fact that this sum is a pure symbol (see Theorem 2.4.9), c) isotropness of q' is equivalent to the fact that q represents it's own determinant, i.e. there is a subform of dimension 4 and determinant 1 in it, i.e. q is a small Pfister form.

$(2 \Rightarrow 4)$ Consider decomposition of Q from Theorem 3.1 .

$$\begin{array}{ccccccc}
 & & \mathcal{X}_Q & & & & \mathcal{X}_{Q^1}(1)[2] \\
 & \nearrow & \star \downarrow[1] & \searrow[1] & & \nearrow & | \\
Q & \longleftarrow & R^1 & \longrightarrow & Q'\langle 1\rangle & \star & [1] \\
 & \nwarrow & \uparrow \;\star & \swarrow[1] & & \nwarrow & \downarrow \\
 & & \mathcal{X}_Q(3)[6] & & & & \mathcal{X}_{Q^1}(2)[4]
\end{array}$$

It is well known, that in the case of a 3 - dimensional quadric, Q^1 is a Severi-Brauer variety, corresponding to central simple algebra A of rank 4, which represents $c_2(Q) \in \mathrm{K}_2^M(k)/2 = \mathrm{H}^2_{et}(k, \mathbb{Z}/2) \subset \mathrm{H}^2_{et}(k, \mathcal{O}^*) = Br(k)$.

If $c_2(Q)$ is a pure symbol $\{a, b\}$, then this central simple algebra will be just $M_2(\mathrm{Quat}(\{a, b\}))$ and, hence, $\mathcal{X}_{Q^1} = \mathcal{X}_C$, where C is a Severi-Brauer conic, corresponding to the c.s.a. $\mathrm{Quat}(\{a, b\})$ (since existence of a rational point on these two varieties is equivalent (see Theorem 2.3.4).

By the way, notice, that $\mathrm{Hom}(\mathcal{X}_{Q^1}, \mathcal{X}_{Q^1}(1)[3]) = \mathrm{Hom}(\mathcal{X}_{Q^1}, \mathbb{Z}(1)[3]) = \mathbb{Z}/2$, with the only nontrivial element γ (see Theorem 2.3.2 and Theorem 2.3.7), and $C = \mathrm{Cone}[-1]\big(\mathcal{X}_{Q^1} \xrightarrow{\ \gamma\ } \mathcal{X}_{Q^1}(1)[3]\big)$ (see Claim 3.3).

From the Claim 3.3 we know that $Q\langle 1\rangle = C$.

$\mathrm{Hom}(\mathcal{X}_Q, C(1)[3]) = \mathrm{Hom}(C, \mathbb{Z}(2)[5]) = 0$, by duality (see Theorem 2.1.23), since $\mathcal{X}_Q \times Q\langle i\rangle = Q\langle i\rangle$ and $\dim(C) = 1 < 3$ (see Theorem 2.1.18). In the same way, since $\mathcal{X}_C = \mathcal{X}_{Q^1}$, we have (by duality, and Theorem 2.4.16): $\mathrm{Hom}(C, \mathcal{X}_Q(2)[5]) = \mathrm{Hom}(\mathbb{Z}, C(1)[3]) = 0$. So, $Q = C(1)[2] \oplus \mathrm{Cone}[-1](\mathcal{X}_Q, \mathcal{X}_Q(3)[7])$ (since Q is an extension of $\mathcal{X}_Q$, $C(1)[2]$, and $\mathcal{X}_Q(3)[6]$ (by Theorem 3.1), and as we see, the middle part can't be hooked to any other).

$(4 \Rightarrow 3)$ Evident.

$(3 \Rightarrow 2)$ Suppose $P = \mathcal{X}_Q \oplus \mathcal{X}_{Q^1}(1)[2]$,

$Q = \mathrm{Cone}[-1](P \to \mathcal{X}_{Q^1}(2)[5] \oplus \mathcal{X}_Q(3)[7])$ (see Proposition 3.4). Consider the map $\delta : \mathbb{Z}(1)[2] \to Q$ given by some subconic $C \subset Q$ (see Theorem 2.1.17 and Theorem 2.1.23). Then $\alpha_1 \circ \delta : \mathbb{Z}(1)[2] \to \mathbb{Z}(1)[2]$ (here α_1 as in Theorem 3.1 is given by some subquadric R of dimension 2 (see Theorem 2.1.17)) is just multiplication by 2=intersection number of R and C (everything can be considered in the category of *Chow motives* by Theorem 2.2.4).

Consider the composition $\mathbb{Z}(1)[2] \xrightarrow{\delta} Q \to P$. Since $P = \mathcal{X}_Q \oplus \mathcal{X}_{Q^1}(1)[2]$, we get a morphism $\delta' : \mathbb{Z}(1)[2] \to \mathcal{X}_{Q^1}(1)[2]$, s.t. it's composition with the natural projection $\mathcal{X}_{Q^1}(1)[2] \to \mathbb{Z}(1)[2]$ is multiplication by 2 (enough to check over $\overline{k}$).

By Theorem 2.3.3 (2), we have a zero-cycle of degree 2 on Q^1, i.e. there exists field extension E/k of degree $4k+2$ s.t. Q^1 has a point over E.

Consider again the quadric Q', corresponding to the form $q' = q \perp -\det(q)$. This is an Albert form corresponding to $c_2(q') = c_2(Q)$ (see Definition 2.4.7). $(Q'|_{k(Q')})_{anis.}$ is a 2-fold *Pfister form*. Hence (since *Pfister form* is hyperbolic over it's generic point (by Theorem 2.4.5)), we have $h_2(Q') = 2$ (see Definition 2.4.19), i.e. $\mathcal{X}_{Q'^1} = \mathcal{X}_{Q'^2}$ (see Theorem 2.4.20).

Since Q is a hyperplane section in Q', we have: $\mathcal{X}_{Q'^1} \leqslant \mathcal{X}_{Q^1} \leqslant \mathcal{X}_{Q'^2}$, and, consequently, $\mathcal{X}_{Q'^2} = \mathcal{X}_{Q^1}$ (see the discussion after Theorem 2.4.20).

So, if Q^1 has a point E of degree $4k+2$, then Q'^2 has a point of degree $4k+2$ (see Theorem 2.4.18) (since $\mathcal{X}_{Q'^2}|_E = \mathbb{Z}$). Then by Theorem 2.4.9 we get: $c_2(Q)$ is pure.

Statement 3.5 is proven. $\square$

In the same way as in Proposition 3.4 we see that $\alpha_i' \times \mathrm{Id}_{\mathcal{X}_{Q^j}}$ is a lifting of

$$Q \times \mathcal{X}_{Q^j} \xrightarrow{\alpha_i \times \mathrm{Id}} \mathcal{X}_{Q^j}\,(i)\,[2i] = \mathcal{X}_{Q^i}\,(i)\,[2i] \times \mathcal{X}_{Q^j}$$

for any $j \geqslant i$ (since $\mathcal{X}_{Q^i} \times \mathcal{X}_{Q^j} = \mathbb{Z} \times \mathcal{X}_{Q^j}$ (see Theorem 2.4.16), and morphisms $Q\langle i+1\rangle \to \mathcal{X}_{Q^i}(i)[2i]$ were obtained from the morphisms $Q\langle i+1\rangle \to \mathbb{Z}(i)[2i]$ (see the proof of Theorem 3.1), which, on their part, can be lifted to $\alpha_i : Q \to \mathbb{Z}(i)[2i]$ (see the beginning of the proof of Theorem 3.1 for the definition)).

On the other hand, $\mathcal{X}_{Q^i} \times \widetilde{\mathcal{X}}_{Q^j} = 0$ for any $j \leqslant i$ (by Theorem 2.4.16)(see Definition 2.3.8).

Let n be odd. Denote $\mathcal{X}_{Q^i} \times \widetilde{\mathcal{X}}_{Q^j}$ as $\mathcal{X}_{i,j}$, $\mathcal{X}_{Q^i}$ as $\mathcal{X}_i$, and put $\mathcal{X}_{[n+1/2]} = 0$. Then we have the following Postnikov system (the maps in the lower row come from the maps $\mathcal{X}_{Q^{i+1}} \to \mathbb{Z}$ (multiplied by $id_{\mathcal{X}_{Q^i}}$) via identification $\mathcal{X}_{Q^{i+1}} \times \mathcal{X}_{Q^i} = \mathcal{X}_{Q^{i+1}}$ (see Theorem 2.4.16)):

$$\begin{array}{ccccccc} & \mathcal{X}_{0,1} & & \mathcal{X}_{1,2} & & \mathcal{X}_{[n-1/2],[n+1/2]} & \\ [1]\nearrow\star & \downarrow & [1]\nearrow\star & \downarrow & \dots & [1]\nearrow \ \star & \downarrow \\ \mathcal{X}_0 & \longleftarrow & \mathcal{X}_1 & \longleftarrow & \mathcal{X}_2 \quad & \mathcal{X}_{[n-1/2]} & \leftarrow \ 0 \end{array}$$

Multiplying tower from Proposition 3.4 with this Postnikov system, and using arguments above and Theorem 2.5.10 , we get:

Proposition 3.6 .

Suppose n is odd. We have the following decomposition for Q:

$$\begin{array}{ccccccc} & Q \times \mathcal{X}_{0,1} & & Q \times \mathcal{X}_{1,2} & & Q \times \mathcal{X}_{[n-1/2],[n+1/2]} & \\ [1]\nearrow\star & \downarrow & [1]\nearrow\star & \downarrow & \dots & [1]\nearrow \ \star & \downarrow \\ Q = Q \times \mathcal{X}_0 & \longleftarrow & Q \times \mathcal{X}_1 & \longleftarrow & Q \times \mathcal{X}_2 \quad & Q \times \mathcal{X}_{[n-1/2]} & \leftarrow \ 0, \end{array}$$

where each $Q \times \mathcal{X}_{i,i+1}$ on it's part has a decomposition:

$$Q \times \mathcal{X}_{i,i+1} \to \bigoplus_{j=0,\dots,i} \mathcal{X}_{i,i+1}\,(j)\,[2j] \to \bigoplus_{j=0,\dots,i} \mathcal{X}_{i,i+1}\,(n-j)\,[2n-2j+1] \to Q \times \mathcal{X}_{i,i+1}[1].$$

Similar result can be formulated in the case n- even.

Remark Proposition 3.6 shows that if $\mathcal{X}_0 = \mathcal{X}_{[n-1/2]}$, that is: existence of a rational point on Q is equivalent to the existence of an $[n-1/2]$-dimensional plane on it, then there is the following exact triangle:

$$Q \to \bigoplus_{j=0,\ldots,[n-1/2]} \mathcal{X}_Q\,(j)\,[2j] \to \bigoplus_{j=0,\ldots,[n-1/2]} \mathcal{X}_Q\,(n-j)\,[2n-2j+1] \to Q[1].$$

From this it is quite easy to see that each $\mathcal{X}_Q\,(j)\,[2j]$ is actually hooked to only one other (i.e. Q is a direct sum of *pure* motives (see Theorem 2.2.4), each of which is an extension of just two $\mathcal{X}$'s), namely $\mathcal{X}_Q\,(j+[n-1/2])\,[2j+2[n-1/2]+1]$. This way we get Rost decomposition for Q - hyperplane section of a Pfister quadric.

Let us introduce the following set:
$J(Q) = \{\,\text{set of such}\, j,\ \text{that}\, \mathcal{X}_{Q[dim(Q)/2]-j+1} \neq \mathcal{X}_{Q[dim(Q)/2]-j}\,\} = \{j_1, j_2, \ldots, j_s\}$. Certainly, $j_s = [dim(Q)/2]$, and $j_{s-k+1} - j_{s-k} = h_k$ - k-th *Witt number* (see Definition 2.4.19 and Theorem 2.4.20). We have $\mathcal{X}_{Q[dim(Q)/2]-j_r} = \cdots = \mathcal{X}_{Q[dim(Q)/2]-j_{r+1}+1}$ (see Theorem 2.4.20), and we can denote it as $\mathcal{X}\{j_r\}$.

The same methods as in Proposition 3.6 , and Theorem 2.5.10 (with $X = Q\langle h_1 + \cdots + h_t\rangle(h_1 + \cdots + h_t)[2(h_1 + \cdots + h_t)]$ and $X_m = Q\langle h_1 + \cdots + h_{t+1}\rangle(h_1 + \cdots + h_{t+1})[2(h_1 + \cdots + h_{t+1})]$) give us the Postnikov tower for Q with graded parts
$\left(\oplus_{i=j_r,\ldots,j_{r+1}-1}\mathcal{X}\{j_r\}(*)[2*]\right)_{j_r \in J(Q)}$.

3.2. **Ring of endomorphisms of $M(Q)$.**

Theorem 3.7 .

Postnikov tower from Theorem 3.1 is compatible with the endomorphisms of Q. More precisely, for any $\varphi : Q \to Q$ there exists unique set of morphisms:
$\varphi_i : \mathcal{X}_{Q^i}(i)[2i] \to \mathcal{X}_{Q^i}(i)[2i]$, $\varphi'_i : \mathcal{X}_{Q^i}(n-i)[2n-2i] \to \mathcal{X}_{Q^i}(n-i)[2n-2i]$,
$\varphi\langle i\rangle : Q'\langle i\rangle \to Q'\langle i\rangle$, *and* $\varphi_{R^i} : R^i \to R^i$, *which commute with the maps from the tower.*

Proof of Theorem 3.7

Lemma 3.8 .

$\mathrm{Hom}(\mathcal{X}_{Q^i}(n-i)[2n-2i], Q'\langle i+1\rangle) = \mathrm{Hom}(\mathcal{X}_{Q^i}(n-i)[2n-2i], Q'\langle i+1\rangle[-1]) = 0.$

Proof of Lemma 3.8

In the same way as in the Proposition 3.4 (using Theorem 2.5.10) we can obtain the following decomposition for Q:

$$\begin{array}{ccc} & P_L & \\ \nearrow & \uparrow\star & \searrow[1] \\ Q \longrightarrow & \widetilde{R} & \longleftarrow Q'\langle i+1\rangle \\ \nwarrow\star & \downarrow[1] & \swarrow[1] \\ & P_U & \end{array} \quad \text{and} \quad \begin{array}{ccc} & P_L & \\ \nearrow\star & \downarrow[1] & \searrow[1] \\ Q \longleftarrow & \widetilde{R}' & \longrightarrow Q'\langle i+1\rangle \\ \nwarrow & \uparrow\star & \swarrow[1] \\ & P_U & \end{array}$$

(which are two halves of the same octahedron), where P_L is an extension of: $\mathcal{X}_Q, \mathcal{X}_{Q^1}(1)[2], \dots, \mathcal{X}_{Q^i}(i)[2i]$, and $P_U = \mathcal{X}_Q(n)[2n] \oplus \mathcal{X}_{Q^1}(n-1)[2n-2] \oplus \dots \oplus \mathcal{X}_{Q^i}(n-i)[2n-2i]$.

It is evident, that the pair $\mathcal{X}_{Q^i} \times (P_U \to Q)$ is isomorphic to the pair $\mathcal{X}_{Q^i} \times (\oplus_{j=0,\dots,i}\mathbb{Z}(n-j)[2n-2j] \xrightarrow{\oplus\beta_j} Q)$, and from the construction (see Theorem 3.1) we have, that the pair $\mathcal{X}_{Q^i} \times (Q \to P_L)$ is isomorphic to the pair $\mathcal{X}_{Q^i} \times (Q \xrightarrow{\oplus\alpha_j} \oplus_{j=0,\dots,i}\mathbb{Z}(j)[2j])$. Denote $Z_U := \oplus_{j=0,\dots,i}\mathbb{Z}(n-j)[2n-2j]$ and $Z_L := \oplus_{j=0,\dots,i}\mathbb{Z}(j)[2j]$.

Since the pairs $\oplus_{j=0,\dots,i}\mathbb{Z}(n-j)[2n-2j] \xrightarrow{\oplus\beta_j} Q$ and $Q \xrightarrow{\oplus\alpha_j} \oplus_{j=0,\dots,i}\mathbb{Z}(j)[2j]$ are dual to each other via duality $\underline{\mathrm{Hom}}(-, \mathbb{Z}(n)[2n])$, we have the commutative diagram:

$$\begin{array}{ccc}
\mathrm{Hom}(\mathcal{X}_{Q^i}(n-i)[2n-2i], P_U[b]) & \longrightarrow & \mathrm{Hom}(\mathcal{X}_{Q^i}(n-i)[2n-2i], Q[b]) \\
\| & & \| \\
\mathrm{Hom}(\mathcal{X}_{Q^i}(n-i)[2n-2i], P_U[b] \times \mathcal{X}_{Q^i}) & \longrightarrow & \mathrm{Hom}(\mathcal{X}_{Q^i}(n-i)[2n-2i], Q[b] \times \mathcal{X}_{Q^i}) \\
\| & & \| \\
\mathrm{Hom}(\mathcal{X}_{Q^i}(n-i)[2n-2i], Z_U[b] \times \mathcal{X}_{Q^i}) & \longrightarrow & \mathrm{Hom}(\mathcal{X}_{Q^i}(n-i)[2n-2i], Q[b] \times \mathcal{X}_{Q^i}) \\
\| & & \| \\
\mathrm{Hom}(\mathcal{X}_{Q^i}(n-i)[2n-2i], Z_U[b]) & \longrightarrow & \mathrm{Hom}(\mathcal{X}_{Q^i}(n-i)[2n-2i], Q[b]) \\
\| & & \| \\
\mathrm{Hom}(Z_L \times \mathcal{X}_{Q^i}, \mathbb{Z}(i)[2i+b]) & \longrightarrow & \mathrm{Hom}(Q \times \mathcal{X}_{Q^i}, \mathbb{Z}(i)[2i+b]) \\
\| & & \| \\
\mathrm{Hom}(P_L \times \mathcal{X}_{Q^i}, \mathbb{Z}(i)[2i+b]) & \longrightarrow & \mathrm{Hom}(Q \times \mathcal{X}_{Q^i}, \mathbb{Z}(i)[2i+b])
\end{array}$$

(first and third identifications - by Theorem 2.3.2)

Consider two exact sequences, corresponding to Hom's from $P_L \times \mathcal{X}_{Q^i}(n-i)[2n-2i] \to Q \times \mathcal{X}_{Q^i}(n-i)[2n-2i] \to \widetilde{R}' \times \mathcal{X}_{Q^i}(n-i)[2n-2i]$ to $\mathbb{Z}(n)[2n+b]$, and from $\mathcal{X}_{Q^i}(n-i)[2n-2i]$ to $P_U[b] \to Q[b] \to \widetilde{R}[b]$.

Considering the first and the last row of the diagram above (together with the isomorphism between them), we see, that $\mathrm{Hom}(\mathcal{X}_{Q^i}(n-i)[2n-2i], \widetilde{R}[b])$ has as many elements as $\mathrm{Hom}(\widetilde{R}' \times \mathcal{X}_{Q^i}, \mathbb{Z}(i)[2i+b])$.

But: $\mathrm{Hom}(\widetilde{R}' \times \mathcal{X}_{Q^i}, \mathbb{Z}(i)[2i+b]) = 0$. Really, $\widetilde{R}' \times \mathcal{X}_{Q^i}$ is an extension of: $\mathcal{X}_{Q^j}(j)[2j]$, $\mathcal{X}_{Q^j}(n-j)[2n-2j]$ for $i < j \leqslant [n/2]$, and $\mathcal{X}_{Q^i}(n-r)[2n-2r]$ for $0 \leqslant r \leqslant i$ (by Theorem 3.1), and all "round numbers" (-) are $> i$, by Theorem 2.3.3 (1), we get what we need.

So, we have: $\mathrm{Hom}(\mathcal{X}_{Q^i}(n-i)[2n-2i], \widetilde{R}[b]) = 0$.

But since $\mathrm{Hom}(\mathcal{X}_{Q^i}(n-i)[2n-2i], P_L[b]) = \mathrm{Hom}(\mathcal{X}_{Q^i}(n-i)[2n-2i], P_L[b] \times \mathcal{X}_{Q^i}) = \mathrm{Hom}(\mathcal{X}_{Q^i}(n-i)[2n-2i], (\oplus_{0 \leqslant j \leqslant i}\mathbb{Z}(j)[2j])[b] \times \mathcal{X}_{Q^i}))$ (by Theorem 2.3.2) $= \mathrm{Hom}(\mathcal{X}_{Q^i}(n-i)[2n-2i], (\oplus_{0 \leqslant j \leqslant i}\mathbb{Z}(j)[2j])[b]) = 0$, (since $n-i > j$, by the same arguments as above), we have: $\mathrm{Hom}(\mathcal{X}_{Q^i}(n-i)[2n-2i], Q'\langle i+1\rangle[b]) = 0$ for all b.

Lemma 3.8 is proven.

$\square$

Suppose we already found φ_j, φ'_j, $\varphi_{R^{j+1}}$ and $\varphi\langle j+1\rangle$ for all $j < i$.

Consider the diagram:

$$\begin{array}{ccccc} R^{i+1} & \longrightarrow & Q'\langle i\rangle & \longrightarrow & \mathcal{X}_{Q^i}(i)[2i] \\ & & \big\downarrow{\scriptstyle \varphi\langle i\rangle} & & \\ R^{i+1} & \longrightarrow & Q'\langle i\rangle & \longrightarrow & \mathcal{X}_{Q^i}(i)[2i] \end{array}$$

Since $\mathrm{Hom}(R^{i+1}, \mathcal{X}_{Q^i}(i)[2i]) = \mathrm{Hom}(R^{i+1}, \mathcal{X}_{Q^i}(i)[2i-1]) = 0$ (since R^{i+1} is an extension of $\mathcal{X}_{Q^m}(m)[2m]$ and $\mathcal{X}_{Q^m}(n-m)[2n-2m]$, $m > i$ and $\mathcal{X}_{Q^i}(n-i)[2n-2i]$), and $\mathrm{Hom}(\mathcal{X}_{Q^m}(*)[*'], \mathcal{X}_{Q^i}(i)[2i+b]) = \mathrm{Hom}(\mathcal{X}_{Q^m}(*)[*'], \mathbb{Z}(i)[2i+b])$ (by Theorem 2.3.2) $= 0$ (since $* > i$, and for any smooth projective variety S, $\mathrm{Hom}(\mathcal{X}_S, \mathbb{Z}(-t)[*]) = 0$ for all $t > 0$ (see Theorem 2.3.3)), we get unique morphisms φ_i and $\varphi_{R^{i+1}}$.

Now, consider the diagram:

$$\begin{array}{ccccc} \mathcal{X}_{Q^i}(n-i)[2n-2i] & \longrightarrow & R^{i+1} & \longrightarrow & Q'\langle i+1\rangle \\ & & \big\downarrow{\scriptstyle \varphi_{R^{i+1}}} & & \\ \mathcal{X}_{Q^i}(n-i)[2n-2i] & \longrightarrow & R^{i+1} & \longrightarrow & Q'\langle i+1\rangle \end{array}$$

By Lemma 3.8 we get unique morphisms $\varphi\langle i+1\rangle$ and φ'_i (which make it commutative).

Theorem 3.7 is proven.

$\square$

From the uniqueness it follows that the maps : $\varphi \mapsto \varphi_i, \varphi'_i, \varphi_{R^{i+1}}, \varphi\langle i\rangle$ (see Theorem 3.7 for the definition) are homomorphisms of rings.

The same considerations as in the proof of Theorem 3.7 give the following:

Corollary 3.9 . *The Postnikov tower from Theorem 3.1 is defined up to unique isomorphism by the quadric Q. In particular, the motives $Q\langle i\rangle$, R^i are defined uniquely by the quadric Q.*

Since $\mathrm{Hom}(\mathcal{X}_{Q^i}(i)[2i], \mathcal{X}_{Q^i}(i)[2i]) = \mathrm{Hom}(\mathcal{X}_{Q^i}, \mathbb{Z}) = \mathbb{Z}$ (see Theorem 2.3.2 and the end of the proof of Theorem 2.3.4), we see that φ_i and φ'_i, for $i \leqslant [(n-1)/2]$ are, actually, just integers. For even n, $\varphi\langle n/2\rangle$ can be considered as an element of $M_2(\mathbb{Z})$ (We fix some algebraic closure $k \subset \overline{k}$, and a basis in $Q'\langle n/2\rangle|_{\overline{k}} = \mathbb{Z}(n/2)[n] \oplus \mathbb{Z}(n/2)[n]$ consisting of planes of middle dimension (via identification: $CH^{n/2}(Q|_{\overline{k}}) = \mathrm{Hom}(\mathbb{Z}(n/2)[n], Q|_{\overline{k}}) = \mathrm{Hom}(\mathbb{Z}(n/2)[n], Q'\langle n/2\rangle|_{\overline{k}}) = \mathbb{Z} \oplus \mathbb{Z})$ (it is unique up to permutation); $Gal(\overline{k}/k)$ acts on $M_2(\mathbb{Z})$ through $Gal(k\sqrt{\det(Q)}/k)$, and if $\det(Q) \neq 1$, the generator of the $Gal(k\sqrt{\det(Q)}/k)$ acts on $M_2(\mathbb{Z})$ via conjugation with $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$).

We have a natural map

$\alpha : \mathrm{End}(Q) \to \times_{i=0,\dots,[n/2]} \left(\mathrm{End}(\mathcal{X}_{Q^i}(i)[2i]) \times \mathrm{End}(\mathcal{X}_{Q^i}(n-i)[2n-2i])\right) = \mathrm{End}(Q|_{\bar{k}})$

in the case n-odd (equality - since both groups are isomorphic to $\prod_{l=1,\dots,2[n/2]+2}\mathbb{Z}$ - by Theorem 2.4.22 , and the fact that $\mathrm{Hom}(\mathbb{Z},\mathbb{Z}(i)[2i])=0$ for any $i\neq 0$, and $\mathrm{Hom}(\mathbb{Z},\mathbb{Z})=\mathbb{Z}$ (see Theorem 2.1.18 , Theorem 2.1.20)), and
$\alpha:\mathrm{End}(Q)\to\times_{i=0,\dots,n/2-1}\left(\mathrm{End}(\mathcal{X}_{Q^i}(i)[2i])\times\mathrm{End}(\mathcal{X}_{Q^i}(n-i)[2n-2i])\right)\times$
$\times\,\mathrm{End}(\mathcal{X}_{Q^{n/2}}(n/2)[n]\times k\left(\sqrt{\det(Q)}\right))\subset\mathrm{End}(Q|_{\bar{k}})$ in the case n-even (the same references).

Lemma 3.10 .
$(\mathrm{End}(Q))_{tors}=\mathrm{Ker}(\mathrm{End}(Q)\to\mathrm{End}(Q|_{\bar{k}}))$
Moreover, $2^{[n/2]+1}\cdot(\mathrm{End}(Q))_{tors}=0$.
1) If $q\notin I^2(W(k))$, *then* $(\mathrm{End}(Q))_{nilp}=\mathrm{Ker}(\mathrm{End}(Q)\to\mathrm{End}(Q|_{\bar{k}}))$
Moreover, $((\mathrm{End}(Q))_{nilp})^{2[n/2]+1}=0$.
2) If $q\in I^2(W(k))$, *then* $(\mathrm{Ker}(\mathrm{End}(Q)\to M_2(\mathbb{Z})))_{nilp}=\mathrm{Ker}(\alpha)$, *and, moreover,* $\mathrm{Ker}(\alpha)^{2[n/2]+1}=0$.

Proof of Lemma 3.10
We will consider only the case n-odd, other cases are similar.
Since $\mathrm{End}(Q')$ is torsion free for any hyperbolic Q', and on $Q^{[n/2]}$ there exists point of degree $2^{[n/2]+1}$, from the existence of transfers we have that $(\mathrm{End}(Q))_{2^{[n/2]+1}}=\mathrm{Ker}(\mathrm{End}(Q)\to\mathrm{End}(Q|_{\bar{k}}))$.
Let $\varphi\in\mathrm{Ker}(\alpha)$. Since φ acts trivially on each "elementary" piece $\mathcal{X}_{Q^i}(*)[2*]$, and Q is an *extension* of $2[n/2]+2$ such "elementary" pieces (see Definition 2.5.6), we have (by Theorem 2.5.7): $\varphi^{2[n/2]+1}=0$ (we just need to mention that on $Q\langle[n/2]\rangle$ φ will also act as 0 (to improve number by 1)).
Lemma 3.10 is proven.

□

Corollary 3.11 .
For any nontrivial idempotent $u\in\mathrm{End}(Q)$, $\alpha(u)$ *is again a nontrivial idempotent of* $\mathrm{End}(Q|_{\bar{k}})$.
Proof Evident. □
Remark Using transfer arguments, one can easily show, that $2^k\cdot(\mathrm{Ker}(\alpha))^l=0$, if $k+\frac{1}{2}l>[n/2]$; then also: $2^k\cdot(\mathrm{Ker}(\alpha))^{2^l}=0$, if $k+l>[n/2]$.

Lemma 3.12 .
α *maps idempotents of* $\mathrm{End}(Q)$ *surjectively onto idempotents of* $\mathrm{End}(Q|_{\bar{k}})\cap(\mathrm{Image}(\alpha))$.
Proof of Lemma 3.12
Suppose, $\overline{u}$ is some idempotent in the image of α, then there exists $u\in\mathrm{End}(Q)$, such that $u^2-u=x\in\mathrm{Ker}(\alpha)$. Since x commutes with u, we have that $u^{2^{[n/2]+2}}=(u+x)^{2^{[n/2]+1}}=u^{2^{[n/2]+1}}$ by remark above. So, $u^{2^{[n/2]+1}}$ will be an idempotent, and $\alpha(u^{2^{[n/2]+1}})=\overline{u}$. Lemma 3.12 is proven. □

Lemma 3.13 .

For any $0 \leqslant i \leqslant [(n-1)/2]$ *there exist* $\varphi(i), \varphi'(i) \in \mathrm{End}(Q)$, *s.t.* $\varphi(i)_j = 2\delta_{ij}$, $\varphi(i)'_j = 0$, $\varphi'(i)'_j = 2\delta_{ij}$, $\varphi'(i)_j = 0$.

For n*-even there exists* $\varphi, \psi \in \mathrm{End}(Q)$, *s.t.* $\varphi\langle n/2\rangle = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $\psi\langle n/2\rangle = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, *and* $\varphi_i = \varphi'_i = \psi_i = \psi'_i = 0$.

Proof of Lemma 3.13

Evidently, $\varphi(i) = h^{dim(Q)-i} \times h^i$, $\varphi'(i) = h^i \times h^{dim(Q)-i}$, $\psi = h^{dim(Q)/2} \times h^{dim(Q)/2}$ and $\varphi = 2\Delta - \sum_{i<n/2}(\varphi(i) + \varphi'(i))$, where h^i is a plane section of Q of codimension i, will work (we use identification: $\mathrm{End}(Q) = CH^{\dim(Q)}(Q \times Q)$ - see Theorem 2.1.23 , Theorem 2.1.17). $\square$

Since the category $Chow(k)$ is closed under taking kernels and cokernels of projectors (see Definition 2.2.1), and $Chow(k) \to DM_{gm}(k)$ is a full embedding (see Theorem 2.2.4), we see that Q is decomposable (i.e. can be represented as a direct sum of two nonzero motives in $DM_{gm}(k)$ (which will be automatically in $Chow(k)$ by Theorem 2.2.4)), if and only if $\mathrm{End}(Q)$ contains a nontrivial idempotent.

Corollary 3.14 .

For n*-odd: the motive of* Q *is decomposable into a direct sum iff the image of* $\alpha/2 : \mathrm{End}(Q) \to \mathbb{Z}/2 \times \cdots \times \mathbb{Z}/2$, *is bigger than the diagonal* $\mathbb{Z}/2$ *(generated by id).*

Proof

From Lemma 3.13 it follows that the image of α contains $2 \cdot \mathrm{End}(Q|_{\overline{k}})$, so, each element of $image(\alpha/2) \subset \mathrm{End}(Q|_{\overline{k}})/2 \cdot \mathrm{End}(Q|_{\overline{k}})$ corresponds to an idempotent of $\mathrm{End}(Q|_{\overline{k}}) \cap image(\alpha)$, and, hence, to an idempotent of $\mathrm{End}(Q)$.

$\square$

Remark The same is true for $\dim(Q) > 0$ - even by Lemma 3.23 , Proposition 4.2 and the fact that if $Q\langle n/2\rangle \times \mathcal{X}_{Q^{n/2}}(n/2)[n]$ is a direct summand in Q, then Q is hyperbolic over the quadratic extension $k\sqrt{\det(Q)}$.

Corollary 3.15 .

Consider two fields: K_1 *and* K_2 *over* k. *Call* $K_1 \sim K_2$, *if each of them can be represented as an extension of odd degree of the other.*
Let $\dim(Q_1) = \dim(Q_2)$*-odd, and* $M(Q_1)$ *is undecomposable.*
Then the following conditions are equivalent:

1) $M(Q_1) = M(Q_2)$

2) $k(Q_1) \sim k(Q_2)$

3) $\mathcal{X}_{Q_1} = \mathcal{X}_{Q_2}$

Proof (2 $\Rightarrow$ 1) If $k(Q_1) \sim k(Q_2)$, then we have morphism $f : Q_1 \to Q_2$ which is of odd degree on the *generic cycle*. Consider the closure of it's graph in $Q_1 \times Q_2$. We get some morphism $\varphi : M(Q_1) \to M(Q_2)$ (via identification: $\mathrm{Hom}(Q_1, Q_2) =$

$CH^{\dim(Q_i)}(Q_1 \times Q_2)$, see Theorem 2.1.23 , Theorem 2.1.17); analogously, we have $\psi : M(Q_2) \to M(Q_1)$. Now, we have $(\psi \circ \varphi)_0$ and $(\psi \circ \varphi)'_0$ are odd (for definition of φ_j, φ'_j see Theorem 3.7). If for some i we would have $(\psi \circ \varphi)_i$, or $(\psi \circ \varphi)'_i$ - even, then we would have a nontrivial idempotent in the $M(Q_1)$ (by Corollary 3.14). So, all of them are odd. Now, using Lemma 3.13 , we can correct φ, to make all $\varphi_i = \varphi'_i = 1$, and φ itself - an isomorphism (by Lemma 3.10 (consider $1 - \varphi$)).

($1 \Rightarrow 2$) If $M(Q_1) = M(Q_2)$, then we have some cycles $P_{1,2}$ on $Q_1 \times Q_2$ of dimension $= \dim(Q_i)$, which have an odd degree over corresponding factors. Now we need to use the Springer's theorem (see Theorem 2.4.14), to get rational morphism $Q_1 \to Q_2$ and back. This morphisms should give us desired extension by the same arguments as above.

($3 \Rightarrow 2$) If $\mathcal{X}_{Q_1} = \mathcal{X}_{Q_2}$, then we have rational morphism from Q_1 to Q_2 and back (by Springer's theorem), which should cover the generic point "oddly" (or we will have an idempotent).

($2 \Rightarrow 3$) By Theorem 2.3.4 and Springer's theorem.

□

Remark 0 The same is true for $\dim(Q_i)$-even - by Lemma 3.25 .

Remark 1 In particular, we see that if $k(Q_1) \sim k(Q_2)$, then $M(Q_1)$ is decomposable iff $M(Q_2)$ is.

Remark 2 For decomposable quadrics it is completely wrong. For example, for two *small Pfister* subquadrics of the same *big Pfister* quadric we have: $k(Q_1) \sim k(Q_2)$, though, in general, $M(Q_1) \neq M(Q_2)$; more trivially, for two isotropic quadrics we have $k(Q_1) = k(Q_2)$-purely transcendental$/k$, but motives can be different (see Theorem 2.4.23). For general statement (independent of decomposability) see Proposition 5.1.

Corollary 3.16 .

Let $h_1(Q) > 1$ (i.e. $\mathcal{X}_Q = \mathcal{X}_{Q^1}$). Then $M(Q)$ is decomposable.

Proof Consider the odd-dimensional case first.

Over $k(Q)$ Q has a line, so we have rational map $Q \to Q^1$, which gives us some cycle of dimension $\dim(Q) + 1$ on $Q \times Q$ ("graph"), and, hence a morphism $\pi : M(Q) \to M(Q)(-1)[-2]$ (see Theorem 2.1.23 , Theorem 2.1.17). Then the composition: $h \circ \pi$ (where h is a hyperplane section embedded diagonally) is a morphism $\varphi : Q \to Q$, s.t. $\varphi'_0 = 0$ and $\varphi_0 = 1$ (see Theorem 3.7 for the definition of φ_i) (to compute this numbers you should change k by $\overline{k}$, and then compute the intersection numbers of cycle corresponding to φ with $Q \times \mathrm{Spec}(\overline{k})$ and $h \times l_1$ (h here is a hyperplane section, and l_1 is some line on $Q|_{\overline{k}}$)). By Corollary 3.14 $M(Q)$ is decomposable.

In the case of even-dimensional quadric we have to add that in the basis of $CH^{n/2}(Q|_{\overline{k}})$ consisting of $n/2$ - dimensional planes T, T' the matrix of φ (which is just $\varphi\langle n/2\rangle$, see discussion after Corollary 3.9) has the form: $\begin{pmatrix} a & c \\ a & c \end{pmatrix}$ for some a, c (since $CH^{n/2-1}(Q|_{\overline{k}}) =$

$\mathbb{Z}$ is generated by the class of subquadric S of dimension $n/2+1$, and $h(S)=T+T'$) So, we can use Lemma 3.13 , to correct φ (by subtracting $[(a+c)/2]\cdot\psi$ (notations from Lemma 3.13)), and make either $\varphi\langle n/2\rangle$ - an idempotent in $M_2(\mathbb{Z})$, or $(\varphi\langle n/2\rangle)^2=0$ (in which case we can substitute φ by φ^2). Now, again using Lemma 3.13 , we can correct φ, to make all φ_i, φ'_j equal to 1, or 0, which means that $\varphi|_{\overline{k}}$ - is an idempotent. By Lemma 3.12 we have a nontrivial idempotent in $\mathrm{End}(Q)$ (it will be equal to $\varphi^{2^{[n/2]+1}}$ (see the proof of Lemma 3.12)).

$\square$

Corollary 3.17 . *For any automorphism* $\varphi\in\mathrm{Ker}(\mathrm{Aut}(M(Q))\to GL_2(\mathbb{Z}))$ *:* $\varphi^{2^{[n/2]+2}}=1$.

Proof Since $\varphi\in\mathrm{Aut}(Q)$, $\varphi_i,\varphi'_i=\pm1$, for $0\leqslant i\leqslant[(n-1)/2]$ (see Theorem 3.7 for the definition of φ_i and φ'_i). So, $(\varphi^2)_i=(\varphi'^2)_i=1$. Hence, (since $\varphi\in\mathrm{Ker}$) $\varphi^2=1+y$, where $y\in\mathrm{Ker}(\alpha)$. Now, the same arguments as in the proof of Lemma 3.12 above show, that $(1+y)^{2^{[n/2]+1}}=1$, and $\varphi^{2^{[n/2]+2}}=1$. $\square$

We should mention that usual "algebro-geometric" automorphisms of a quadric almost don't act on it's motive.

Observation 3.18 .
The map $\mathrm{SO}(q)\to\mathrm{Aut}(M(Q))$ *is trivial.*

Proof
By the theorem of Cartan-Deudonne (see [9], p.27) we know that $\mathrm{O}(q)$ is generated by reflections $\tau_y: x\mapsto x-\frac{B(x,y)}{q(y)}\cdot y\ \ (q(y)\neq0)$. In particular, $\mathrm{SO}(q)$ is generated by pairs of reflections. It is easy to see that graphs of τ_y and τ_z are rationally equivalent: just consider the graph of $\tau_{\lambda_0\cdot y+\lambda_1\cdot z}$. So $\mathrm{SO}(q)$ does not act on $M(Q)$ (since $\tau_y^2=1$).
$\square$

Remark The group $\mathbb{Z}/2=\mathrm{O}(q)/\mathrm{SO}(q)$ acts nontrivially on the motive of the even-dimensional quadric Q, even when Q is hyperbolic (for the generator τ we have $\tau\langle\dim(Q)/2\rangle=\begin{pmatrix}0&1\\1&0\end{pmatrix}$). On the other hand, evidently, this $\mathbb{Z}/2$ does not act on the motive of odd-dimensional hyperbolic quadric.

If we consider natural Lie algebra structure on $End(Q)$ and denote as $D^k(End(Q))$ the k-th derivative of $End(Q)$, then we have:

Corollary 3.19 .
1) In the case $q\notin I^2(W(k))$*:* $D^{2+[\log_2(n)]}(End(Q))=0$, *and* $D^{[n/2]}(Aut(Q))=1$.
2) In the case $q\in I^2(W(k))$*:* $D^{2+[\log_2(n)]}(\mathrm{Ker}(End(Q)\to M_2(\mathbb{Z})))=0$, *and* $D^{[n/2]}(\mathrm{Ker}(Aut(Q)\to GL_2(\mathbb{Z})))=1$.

More generally, $D^{[n/2]-i}(\mathrm{Ker}(End(Q\langle i\rangle)\to M_2(\mathbb{Z})))=0$,
$D^{2+[\log_2(n-2i)]}(\mathrm{Ker}(Aut(Q\langle i\rangle)\to GL_2(\mathbb{Z})))=1$.

Proof Consider the case $q\notin I^2(W(k))$.
$D(End(Q))\subset \mathrm{Ker}(\alpha)$, now, the fact that $D^{2+[\log_2(n)]}(\mathrm{Ker}(\alpha))=0$ is evident (since for any elements $x_1,\dots,x_{2[n/2]+1}$ from $\mathrm{Ker}(\alpha)$ we have $\prod x_i=0$ (see the proof of Lemma 3.10)). Analogously in the case of $\mathrm{Aut}(Q)$. □

3.3. **Some unpleasant computations.** In this section we will prove that, roughly speaking, it is enough to have reasonable (motivic) morphism between two quadrics, in order to show that they contain isomorphic direct summands. We have so many computations here because of the even-dimensional case.

Let Q_1 and Q_2 be two quadrics of the same dimension n. We can fix an algebraic closure $k\subset\overline{k}$ and bases in $CH^i(Q_1|_{\overline{k}})$, $CH^i(Q_2|_{\overline{k}})$, consisting of: projective subspace $l_{n-i}(1,2)$ of dimension $n-i$ for $n\geqslant i>n/2$, plane section $h^i(1,2)$ of codimension i for $0\leqslant i<n/2$, and of two projective subspaces $l_{n/2}(1,2)_1$, $l_{n/2}(1,2)_2$ of dimension $n/2$ from different families (i.e. whose intersection is 1) for $i=n/2$ (if n is even).

For an element $\varphi\in\mathrm{Hom}(Q_1,Q_2)$ we can define integers φ_i, φ'_i, and $\varphi\langle n/2\rangle\in M_2(\mathbb{Z})$ in the same way as in Theorem 3.7 . I.e.: if φ is represented by a cycle $F\subset Q\times Q$, then $\varphi_i=(F,h^i(1)\times l_i(2))$, $\varphi'_i=(F,l_i(1)\times h^i(2))$ for all $0\leqslant i<n/2$, and $\varphi\langle n/2\rangle_{a,b}=(F,l_{n/2}(1)_a\times l_{n/2}(2)_b)$, where (A,B) for two cycles A,B of complementary dimension is their intersection number.

The generator of $Gal(k\sqrt{\det(Q_1)}/k)$ acts on $M_2(\mathbb{Z})$ via multiplication by $\begin{pmatrix}0&1\\1&0\end{pmatrix}$ on the right (if $\det(Q_1)\neq 1$), and generator of $Gal(k\sqrt{\det(Q_2)}/k)$ acts on $M_2(\mathbb{Z})$ via multiplication by the same matrix on the left (since such a generator $\sigma_{1,2}$ intercanges the $l_{n/2}(1,2)_1$ and $l_{n/2}(1,2)_2$). $Gal(\overline{k}/k)$ does not act on those $\mathbb{Z}$'s, where φ_i's and φ'_i's live (since it acts trivially on corresponding basis elements).

Up to duality (see Theorem 2.1.23), we have 4 cases:
1) $\det(Q_1)=\det(Q_2)=1$;
2) $\det(Q_1)=1$, $\det(Q_2)\neq 1$;
3) $\det(Q_1)=\det(Q_2)\neq 1$;
4) $1\neq\det(Q_1)\neq\det(Q_2)\neq 1$.

Let U_2 be the subgroup of $M_2(\mathbb{Z})$, consisting of matrices $\begin{pmatrix}a&b\\a&b\end{pmatrix}$. Analogously, let U_1 be a subgroup of $M_2(\mathbb{Z})$, consisting of matrices $\begin{pmatrix}a&a\\c&c\end{pmatrix}$. Let V be a subgroup of $M_2(\mathbb{Z})$, consisting of matrices $\begin{pmatrix}a&b\\b&a\end{pmatrix}$.

Let $\overline{\langle n/2\rangle}$ be the natural map $\mathrm{Hom}(Q_1,Q_2)\to M_2(\mathbb{Z})$.

We have the following analog of Lemma 3.13 .

Lemma 3.20 .

A) For any $0 \leqslant i < n/2$ *there exist elements* u_i, u'_i *in* $\mathrm{Hom}(Q_1, Q_2)$, *s.t.* $(u_i)_j = 2\delta_{ij}$, $(u_i)'_j = 0$, $(u'_i)_j = 0$, $(u'_i)'_j = 2\delta_{ij}$, *and* $u_i\langle n/2\rangle = u'_i\langle n/2\rangle = 0$.

There exists $w_{n/2} \in \mathrm{Hom}(Q_1, Q_2)$, *s.t.* $(w_{n/2})_i = (w_{n/2})'_i = 0$, *and* $w_{n/2}\langle n/2\rangle = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

B) In the four cases specified above we have the following:

1) $2^r \cdot M_2(\mathbb{Z}) \subset image(\overline{\langle n/2\rangle})$, *for some large* r.
2) $2^r \cdot U_2 \subset image(\overline{\langle n/2\rangle}) \subset U_2$, *for some large* r.
3) $2^r \cdot V \subset image(\overline{\langle n/2\rangle}) \subset V$, *for some large* r.
4) $image(\overline{\langle n/2\rangle}) = \mathbb{Z} \cdot w_{n/2}$.

Proof

A) u_i corresponds to the cycle $h^{n-i} \times h^i \subset Q_1 \times Q_2$, u'_i - to $h^i \times h^{n-i} \subset Q_1 \times Q_2$, and $w_{n/2}$ - to $h^{n/2} \times h^{n/2} \subset Q_1 \times Q_2$.

B) Since φ is defined over k, $\varphi\langle n/2\rangle$ will be stable under $Gal(\overline{k}/k)$, so we get inclusions: in the second case - into U_2, in the third - into V, and in the fourth - into $\mathbb{Z} \cdot w_{n/2}$.

Since Q_1, Q_2 are hyperbolic over some field extension E of degree 2^r over k, and if $Q_1|_E$, $Q_2|_E$ are hyperbolic, then $image(\overline{\langle n/2\rangle}|_E) = M_2(\mathbb{Z})$, we get the first inclusions in all 3 cases: consider the composition $\widetilde{\psi} : Q_1 \to Q_1 \times E \stackrel{\psi}{\to} Q_2 \times E \to Q_2$, then $\widetilde{\psi}\langle n/2\rangle = \sum_{\sigma \in Gal(E/k)} \sigma(\psi\langle n/2\rangle)$. In the fourth case we have an equality, since $w_{n/2} \in image(\overline{\langle n/2\rangle})$ by A).

□

Let Q_1, Q_2 be two quadrics of the same dimension n.
Let J^d, J^u be subsets of $[0, 1, \ldots, [n-1/2]$. For a subset I of $[0, 1, \ldots, [n-1/2]]$ put $\delta_I(j) = 1$, if $j \in I$, and 0, if $j \notin I$, for all $0 \leqslant j < n/2$. Let $J = (J^d, J^u)$. Denote $\mathrm{Hom}(Q_1, Q_2)_J\langle n/2\rangle$ the subset of $M_2(\mathbb{Z})$, consisting of the images of maps $\varphi \in \mathrm{Hom}(Q_1, Q_2)$, s.t. $\varphi_i = \delta_{J^d}(i)$, $\varphi'_j = \delta_{J^u}(j)$. Denote $\mathrm{Hom}(Q_1, Q_2)_0\langle n/2\rangle$ the subgroup of $M_2(\mathbb{Z})$, generated by the images of maps $\varphi \in \mathrm{Hom}(Q_1, Q_2)$, s.t. $\varphi_i = 0$, $\varphi'_j = 0$ for all $0 \leqslant i, j < n/2$.

Lemma 3.21 .

Let N, N' *be undecomposable direct summands in* Q, *s.t. for corresponding idempotents* $p_N, p_{N'}$ *we have* $(p_N)_i = (p_{N'})_i = 1$ *for some* i *(see above, or Theorem 3.7 for the definition). Then there are morphisms* $u : N \to N'$ *and* $v : N' \to N$ *defining the isomorphism* $N = N'$.

In particular, $(p_N)_j = (p_{N'})_j$ *and* $(p_N)'_j = (p_{N'})'_j$ *for all* j.

Proof

Let $j_N : N \to Q$, $\pi_N : Q \to N$, and $j_{N'} : N' \to Q$, $\pi_{N'} : Q \to N'$ be maps defining the decompositions (in particular, $j_N \circ \pi_N = p_N$ and $j_{N'} \circ \pi_{N'} = p_{N'}$).

Let $\dim(Q) = n$ is odd. Then $\mathrm{End}(N)|_{\overline{k}}$ is $(\prod_{l:(p_N)_l=1} \mathbb{Z}) \times (\prod_{j:(p_N)'_j=1} \mathbb{Z})$. Consider $u = \pi_{N'} \circ j_N$ and $v = \pi_N \circ j_{N'}$. Then $\varphi = \pi_N \circ p_{N'} \circ j_N \in \mathrm{End}(N)$ has the property that: φ_l and φ'_j is either 0, or 1 (since p_N, $p_{N'}$ have such property). Hence, $\varphi|_{\overline{k}}$ is an idempotent. By the proof of Lemma 3.12 , φ^r will be an idempotent in $\mathrm{End}(N) \subset \mathrm{End}(Q)$. Since N is undecomposable, it should be p_N (since $\varphi_i = 1$ by condition). Hence φ^r is an isomorphism, and φ is an isomorphism. In the same way, considering $\psi = \pi_{N'} \circ p_N \circ j_{N'} \in \mathrm{End}(N')$, we get that $\pi_{N'} \circ j_N$ and $\pi_N \circ j_{N'}$ perform an isomorphism $N = N'$.

Let $\dim(Q) = n$ is even. We have 3 cases:

1) $rank(p_N\langle n/2\rangle) = 0$, or $rank(p_{N'}\langle n/2\rangle) = 0$;

2) $rank(p_N\langle n/2\rangle) = 2$, or $rank(p_{N'}\langle n/2\rangle) = 2$;

3) $rank(p_N\langle n/2\rangle) = rank(p_{N'}\langle n/2\rangle) = 1$.

1) Consider $u = \pi_{N'} \circ j_N$ and $v = \pi_N \circ j_{N'}$. Can assume that $rank(p_N\langle n/2\rangle) = 0$. We have: $p_N\langle n/2\rangle = 0$. Then
$\mathrm{End}(N)|_{\overline{k}} = (\prod_{l:(p_N)_l=1;\, 0\leqslant l<n/2} \mathbb{Z}) \times (\prod_{j:(p_N)'_j=1;\, 0\leqslant j<n/2} \mathbb{Z})$. Let $\varphi = v \circ u : N \to N$, and $\psi = u \circ v : N' \to N'$. Since φ_l, φ'_j are 0, or 1 (since p_N, $p_{N'}$ have such a property), we have that $\varphi|_{\overline{k}}$ will be an idempotent in $\mathrm{End}(N)|_{\overline{k}} \subset \mathrm{End}(Q)|_{\overline{k}}$. In the same way as before, we get that $\varphi : N \to N$ is an isomorphism.

We have 3 choices for the $rank(p_{N'}\langle n/2\rangle)$: a) 0; b) 1; c) 2.

a) In the same way as before we get that $\psi = u \circ v$ is an isomorphism $N' \to N'$. So, u, v are isomorphisms.

b) We have: $\mathrm{End}(N')|_{\overline{k}} = (\prod_{l:(p_{N'})_l=1;\, 0\leqslant l<n/2} \mathbb{Z}) \times (\prod_{j:(p_{N'})'_j=1;\, 0\leqslant j<n/2} \mathbb{Z}) \times \mathbb{Z}$, where the last term corresponds to the endomorphisms of $\mathbb{Z}(n/2)[n]$ as a direct summand of $N'|_{\overline{k}}$ (and the generator is represented by $p_{N'}\langle n/2\rangle \in M_2(\mathbb{Z}) = \mathrm{End}(Q\langle n/2\rangle|_{\overline{k}})$). ψ_l, ψ'_j are 0, or 1 as before, and $\psi\langle n/2\rangle = 0$ (since $p_N\langle n/2\rangle = 0$). So, $\psi|_{\overline{k}}$ is an idempotent in $\mathrm{End}(N'|_{\overline{k}}) \subset \mathrm{End}(Q|_{\overline{k}})$, and it can be lifted to a nontrivial idempotent in $\mathrm{End}(N')$ (nontrivial, since $\psi_i = 1$ and $\psi\langle n/2\rangle = 0$, while $p_{N'}\langle n/2\rangle \neq 0$). Contradiction ($N'$ is undecomposable).

c) We have $p_{N'}\langle n/2\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Hence, $\psi|_{\overline{k}}$ is an idempotent in $\mathrm{End}(N|_{\overline{k}})$. An it should be nontrivial, since $\psi\langle n/2\rangle = 0$ and $\psi_i = 1$, but $p_{N'}\langle n/2\rangle = Id_2$. Contradiction.

2) Consider, again, $u = \pi_{N'} \circ j_N$ and $v = \pi_N \circ j_{N'}$.

We can assume that $rank(p_N\langle n/2\rangle) = 2$, i.e.: $p_N\langle n/2\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

We have 3 choices for the $rank(p_{N'}\langle n/2\rangle)$: a) 0; b) 1; c) 2.

a) The same as 1)c).

b) We have $\mathrm{End}(N)|_{\overline{k}} = (\prod_{l:(p_N)_l=1;\,0\leqslant l<n/2}\mathbb{Z}) \times (\prod_{j:(p_N)'_j=1;\,0\leqslant j<n/2}\mathbb{Z}) \times M_2(\mathbb{Z})$, where the last term corresponds to the endomorphisms of $\mathbb{Z}(n/2)[n] \oplus \mathbb{Z}(n/2)[n]$ as a direct summand of $N|_{\overline{k}}$. φ_l, φ'_j are 0, or 1 as before, and $\varphi\langle n/2\rangle = p_{N'}$. Using Lemma 3.12 we can lift it to the idempotent on N (equal to φ^r, see the proof of Lemma 3.12), which should be nontrivial, since $\varphi_i = 1$, $rank(\varphi\langle n/2\rangle) = 1$, but $rank(p_N\langle n/2\rangle) = 2$.

c) The same considerations as above shows that φ, ψ will be the isomorphisms, and so will be u and v.

3) We have: $\mathrm{End}(N)|_{\overline{k}} = (\prod_{l:(p_N)_l=1;\,0\leqslant l<n/2}\mathbb{Z}) \times (\prod_{j:(p_N)'_j=1;\,0\leqslant j<n/2}\mathbb{Z}) \times \mathbb{Z}$, where the last term corresponds to the endomorphisms of $\mathbb{Z}(n/2)[n]$ as a direct summand of $N|_{\overline{k}}$ (and the generator is represented by $p_N\langle n/2\rangle \in M_2(\mathbb{Z}) = \mathrm{End}(Q\langle n/2\rangle|_{\overline{k}})$), and $\mathrm{End}(N')|_{\overline{k}} = (\prod_{l:(p_{N'})_l=1;\,0\leqslant l<n/2}\mathbb{Z}) \times (\prod_{j:(p_{N'})'_j=1;\,0\leqslant j<n/2}\mathbb{Z}) \times \mathbb{Z}$, where the last term corresponds to the endomorphisms of $\mathbb{Z}(n/2)[n]$ as a direct summand of $N'|_{\overline{k}}$ (and the generator is represented by $p_{N'}\langle n/2\rangle \in M_2(\mathbb{Z}) = \mathrm{End}(Q\langle n/2\rangle|_{\overline{k}})$).

Take $\varphi = p_N \circ p_{N'} \circ p_N$, and $\varphi' = p_{N'} \circ p_N \circ p_{N'}$. We have: $p_N\langle n/2\rangle \circ p_{N'}\langle n.2\rangle \circ p_N\langle n/2\rangle = \lambda \cdot p_N\langle n/2\rangle$, and $p_{N'}\langle n/2\rangle \circ p_N\langle n/2\rangle \circ p_{N'}\langle n/2\rangle = \lambda \cdot p_{N'}\langle n/2\rangle$. By Lemma 3.13 we can correct φ into $\widetilde{\varphi}$ (without changing φ_l and φ'_j), and make $\lambda = 1$, or 0 (we can't get 0 by 1)b), so, λ is odd).

We have: $\widetilde{\varphi}|_{\overline{k}}$ an idempotent. Using Lemma 3.12 we can lift it to the idempotent on N (equal to $\widetilde{\varphi}^r$, see the proof of Lemma 3.12), which should be an identity, since $\varphi_i = 1$ and N is undecomposable. Then $\widetilde{\varphi}$ will be an isomorphism $N \to N$. The same can be done with N' instead of N.

In particular, we see that $(p_{N'})_l = (p_N)_l$ and $(p_{N'})'_j = (p_N)'_j$ for all $0 \leqslant l, j < n/2$. Denote J^d - the set of such i, that $(p_N)_i = 1$, and J^u - the set of such j, that $(p_N)'_j = 1$, and $J = (J^d, J^u)$.

Let $(p_N)\langle n/2\rangle = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since $rank(p_N\langle n/2\rangle) = 1$ (and p_N is an idempotent in $M_2(\mathbb{Z})$), we have: $a + d = 1$.

By Lemma 3.13 we have element $\rho \in \mathrm{End}(Q)$, s.t. $\rho_i = \rho'_i = 1$, for all $0 \leqslant i < n/2$, and $\rho\langle n/2\rangle = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

Since $a - d$ is odd, we have: either $a - c + b - d$, or $a - b + c - d$ is not divisible by 4. Suppose that the first one is not divisible (the other case is completely symmetric).

We have: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{End}(Q)_J\langle n/2\rangle$. Then (multiplying by ρ) we have: $\begin{pmatrix} -b & -a \\ -d & -c \end{pmatrix} \in$ $\mathrm{End}(Q)_J\langle n/2\rangle$. Subtracting, we get: $\begin{pmatrix} a+b & a+b \\ c+d & c+d \end{pmatrix} \in \mathrm{End}(Q)_0\langle n/2\rangle$. Using Lemma 3.13 , we get: $\begin{pmatrix} a+b-c-d & a+b-c-d \\ 0 & 0 \end{pmatrix} \in \mathrm{End}(Q)_0\langle n/2\rangle$.

Since $rank(p_N) = 1$, we have that $\det(Q) = 1$ (really, if $\det(Q) \neq 1$, then $\mathrm{End}(Q)_J\langle n/2\rangle$ is contained in the subgroup of $M_2(\mathbb{Z})$, generated by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (by Lemma 3.20)). Since $\det(Q) = 1$, by Lemma 3.20 , we have that $\begin{pmatrix} 2^r & 2^r \\ 0 & 0 \end{pmatrix} \in \mathrm{End}(Q)_0\langle n/2\rangle$, for large r.

Since $a+b-c-d$ is not divisible by 4, we have: $\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} \in \mathrm{End}(Q)_0\langle n/2\rangle$. Then $\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix} \in \mathrm{End}(Q)_0\langle n/2\rangle$ (by duality), and so, $\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{End}(Q)_0\langle n/2\rangle$.

We can assume that a is odd and d is even. If $b-c$ is even, we can use Lemma 3.13 , and our matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{End}(Q)_J\langle n/2\rangle$, and $\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} \in \mathrm{End}(Q)_0\langle n/2\rangle$, to find some matrix of the type $\begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}$ in $\mathrm{End}(Q)_J\langle n/2\rangle$. Multiplying by ρ, and considering square, we get that $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{End}(Q)_J\langle n/2\rangle$. Since, by Lemma 3.12 , we can lift the corresponding element to an idempotent, we get a contradiction by 1)b).

So, $b-c$ is odd, and without loss of generality we can assume that a, b are odd, and c, d are even.

Using matrices: $\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$, p_N, and Lemma 3.13 (or Lemma 3.20), we get that $\mathrm{End}(Q)_J\langle n/2\rangle$ contains one of the following matrices: $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$, or $\begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}$. Considering squares, we reduce the choice to the first two cases. Consider the second case (the first one is similar). Let $\psi \in \mathrm{End}(Q)$ be the corresponding representative.

Consider $\psi \circ \psi^{\vee} \in \mathrm{End}(Q)$. We have $\psi \circ \psi^{\vee}\langle n/2\rangle = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$. I.e.: $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{End}(Q)_I\langle n/2\rangle$, where $I = (J^d \cap J^u, J^d \cap J^u)$. Multiplying by ρ, we get: $\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \in \mathrm{End}(Q)_I\langle n/2\rangle$, and considering square: $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{End}(Q)_I\langle n/2\rangle$. Considering powers of the corresponding representative, we get an idempotent $\gamma \in \mathrm{End}(Q)$, s.t. $\gamma_i = \gamma'_i = \delta_{J^d \cap J^u}(i)$. Let M be corresponding direct summand.

If $J^d \cap J^u$ is nonempty, and $j \in J^d \cap J^u$, then considering N and undecomposable direct summand M' in M "containing j", we get (by what was proven above $((p_N)_l = (p_{M'})_l$ and $(p_N)'_s = (p_{M'})'_s)$) that $J^d \cap J^u = J^d = J^u$. This is impossible by 1)b), since $rank(\gamma) = 0$.

So, $J^d \cap J^u$ is empty, which means that $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{End}(Q)_0\langle n/2\rangle$. Then $2 \cdot M_2(\mathbb{Z}) \subset \mathrm{End}(Q)_0\langle n/2\rangle$. Then $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in \mathrm{End}(Q)_J\langle n/2\rangle$. Let β be corresponding representative, which we can assume to be an idempotent (by Lemma 3.12); let N_1 be the corresponding direct summand.

Since $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an idempotent of $rank = 1$ in $M_2(\mathbb{Z})$, we have that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm \begin{pmatrix} x & y \\ z & t \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} t & -y \\ -z & x \end{pmatrix}$, for some matrix $\begin{pmatrix} x & y \\ z & t \end{pmatrix}$ with $xt - yz = \pm 1$. I.e.: $a = \pm x(t-z)$, $b = \pm x(x-y)$, $c = \pm z(t-z)$, $d = \pm z(x-y)$. Since a, b are odd, and c, d are even, we get: x, t are odd, and y, z are even.

Since $2 \cdot M_2(\mathbb{Z}) \subset \mathrm{End}(Q)_0\langle n/2\rangle$, and we have an identity element in $\mathrm{End}(Q)$, we get an automorphism $\varepsilon : Q \to Q$, s.t. $\varepsilon\langle n/2\rangle = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$ (use Lemma 3.12).

ε and ε^{-1} gives us an isomorphism between N and N_1.

In the same way, we can construct an isomorphism between N' and some direct summand N'_1, given by an idempotent β', where $\beta'\langle n/2\rangle$ is $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, or $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ (depending on: is $a' - b'$ even or odd). The last case can't happen, since in this case $\beta + \beta'$ would provide us (using Lemma 3.13) with an idempotent $\alpha \in \mathrm{End}(Q)$, s.t. $\alpha_i = \alpha'_i = 0$ for all i, and $\alpha\langle n/2\rangle = Id$. Then $p_N \circ (1-\alpha)$ would give us an idempotent $p_{N''}$ in Q, s.t. $(p_{N''})_i = 1$, but $rank(p_{N''}\langle n/2\rangle) = 0$. This contradicts 1)b).

So, N_1 and N'_1 are isomorphic, and hence, N and N' are isomorphic.

□

Lemma above explains the following definition.

Definition 3.22 .

Let $0 \leqslant i, j < n/2$. We say that $\mathcal{X}_{Q^i}(i)[2i]$ is hanged to $\mathcal{X}_{Q^j}(j)[2j]$ (respectively, $\mathcal{X}_{Q^j}(n-j)[2n-2j]$), if for some undecomposable direct summand N of Q we have for corresponding idempotent p_N: $(p_N)_i = 1$ and $(p_N)_j = 1$ (resp. $(p_N)'_j = 1$).

Let we have some idempotent $p_N \in \mathrm{End}(Q)$. Then by Theorem 3.7 it gives us the set of idempotents: $(p_N)_i$, $(p_N)'_i$, $(p_N)_{R^j}$, $(p_N)\langle j\rangle$ for all $0 \leqslant i < n/2$, $1 \leqslant j \leqslant n/2$. And our whole *Postnikov tower* is a direct sum of two towers, corresponding to the decomposition: $Q = N \oplus N^{\perp}$.

This gives us that N is an extension of "elementary pieces": $(p_N)_i \cdot \mathcal{X}_{Q^i}(i)[2i]$, $(p_N)'_i \cdot \mathcal{X}_{Q^i}(n-i)[2n-2i]$, and $(p_N)\langle n/2\rangle \cdot Q'\langle n/2\rangle$ (if n is even) in the same sense as Q is an extension of $\mathcal{X}_{Q^i}(i)[2i]$, $\mathcal{X}_{Q^i}(n-i)[2n-2i]$, and $Q'\langle n/2\rangle$, see Theorem 3.1 .

Let $J^d(N)$ be the set of such $0 \leqslant i < n/2$, that $(p_N)_i = 1$, and $J^u(N)$ be the set of such $0 \leqslant i < n/2$, that $(p_N)'_i = 1$.

Since all idempotents in $\mathcal{X}_{Q^i}(i)[2i]$ and $\mathcal{X}_{Q^i}(n-i)[2n-2i]$ are either 0, or identity maps, we have that N is an extension of: $\mathcal{X}_{Q^i}(i)[2i]$, for $i \in J^d(N)$, $\mathcal{X}_{Q^i}(n-i)[2n-2i]$, for $i \in J^u(N)$, and $p_N\langle n/2\rangle \cdot Q'\langle n/2\rangle$.

We have three cases:

1) $rank(p_N\langle n/2\rangle) = 0$; 2) $rank(p_N\langle n/2\rangle) = 1$; 3) $rank(p_N\langle n/2\rangle) = 2$.

The considerations above give us the following

Lemma 3.23 .

Let Q be projective quadric of dimension n, and N be a direct summand in Q, then N is an extension (in the same sense as in Theorem 3.1) of $\mathcal{X}_{Q^i}(i)[2i]$, for $i \in J^d(N)$, $\mathcal{X}_{Q^j}(n-j)[2n-2j]$, for $j \in J^u(N)$ in the first case specified above; the same objects $+\ \mathcal{X}_{Q^{n/2}}(n/2)[n]$ in the second case ($\det(Q)$ is necessarily 1 in such case); and the same objects $+\ \mathcal{X}_{Q^{n/2}}(n/2)[n] \times k\sqrt{\det(Q)}$ in the third case.

Remark We could just say that N is an extension of it's "elementary pieces" in the sense of Definition 2.5.6 . The picture from Theorem 3.1 just clarify: in which order elementary pieces appear.

Let J be a subset of $[0, 1, \ldots, n/2-1]$. We will denote the pair (J, J) in the same way: J.

Lemma 3.24 .

Let Q_1, Q_2 be projective quadrics, s.t.: $\dim(Q_1) = \dim(Q_2) = n$.

Let $\varphi \in \mathrm{Hom}(Q_1, Q_2)$ be such element, that $\varphi_i = \varphi'_i = \delta_J(i)$ (see the definition before Lemma 3.21). Then there exist direct summands: N_1 in Q_1, and N_2 in Q_2, s.t. for corresponding idempotents p_{N_1}, p_{N_2} we have: $(p_{N_1})_i = (p_{N_1})'_i = (p_{N_2})_i = (p_{N_2})'_i = \delta_J(i)$. And there exist isomorphisms $u : N_1 \to N_2$, $v : N_2 \to N_1$.

Proof

The case n - odd is completely trivial: by Lemma 3.12 , $(\varphi\circ\varphi^{\vee})^{2^s}$ and $(\varphi^{\vee}\circ\varphi)^{2^s}$ will be the idempotents for large s, and φ, $\varphi^{\vee}$ will be isomorphisms of corresponding direct summands.

Let now n is even.

By Lemma 3.13 , in $\mathrm{End}(Q_{1,2})$ we have element $\rho_{1,2}$, s.t. $(\rho_{1,2})_i=(\rho_{1,2})'_i=1$, for all $0\leqslant i<n/2$, and $(\rho_{1,2})\langle n/2\rangle=\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$.

Up to duality we have four cases: A) $1\neq\det(Q_1)\neq\det(Q_2)\neq 1$; B) $\det(Q_1)=\det(Q_2)\neq 1$; C) $\det(Q_1)=1$, $\det(Q_2)\neq 1$; D) $\det(Q_1)=\det(Q_2)=1$.

A) By Lemma 3.20 we can correct φ to make $\varphi\langle n/2\rangle=0$, and then $p_{N_2}:=(\varphi\circ\varphi^{\vee})^{2^s}$ and $p_{N_1}:=(\varphi^{\vee}\circ\varphi)^{2^s}$ will be the idempotents for large s (by Lemma 3.12), with $(p_{N_j})_i=(p_{N_j})'_i=\delta_J(i)$, $(p_{N_j})\langle n/2\rangle=0$, and corresponding direct summands will be isomorphic via φ and $\varphi^{\vee}$.

B) Using Lemma 3.20 , we can assume that $\varphi\langle n/2\rangle=\begin{pmatrix}a&0\\0&a\end{pmatrix}$. $(\varphi\circ\varphi^{\vee})\langle n/2\rangle=\begin{pmatrix}a^2&0\\0&a^2\end{pmatrix}$. If a is even then, by Lemma 3.13 , we have $0\in\mathrm{End}(Q_2)_J\langle n/2\rangle$ (the same with Q_1). So, by Lemma 3.12 , we have idempotents $p_{N_1}\in\mathrm{End}(Q_1)$, $p_{N_2}\in\mathrm{End}(Q_2)$, s.t. $(p_{N_j})_i=(p_{N_j})'_i=\delta_J(i)$, $(p_{N_j})\langle n/2\rangle=0$. And $p_{N_2}\circ\varphi\circ p_{N_1}$, $p_{N_1}\circ\varphi^{\vee}\circ p_{N_2}$ will give an isomorphism between N_1 and N_2.

If a is odd, then since (by Lemma 3.13) $2\cdot\begin{pmatrix}a&0\\0&a\end{pmatrix}\in\mathrm{Hom}(Q_1,Q_2)_0\langle n/2\rangle$, by Lemma 3.20 we have: $\begin{pmatrix}2&0\\0&2\end{pmatrix}\in\mathrm{Hom}(Q_1,Q_2)_0\langle n/2\rangle$. So, $\begin{pmatrix}1&0\\0&1\end{pmatrix}\in\mathrm{Hom}(Q_1,Q_2)_J\langle n/2\rangle$.

If ψ is corresponding representative, then by Lemma 3.12 $p_{N_2}:=(\psi\circ\psi^{\vee})^{2^s}$ and $p_{N_1}:=(\psi^{\vee}\circ\psi)^{2^s}$, for large s, will be idempotents, and corresponding direct summands will be isomorphic.

C) By Lemma 3.20 we can assume that $\varphi\langle n/2\rangle=\begin{pmatrix}a&0\\a&0\end{pmatrix}$.

Then $(\varphi^{\vee}\circ\varphi)\langle n/2\rangle=\begin{pmatrix}2a^2&0\\0&0\end{pmatrix}\in\mathrm{End}(Q_1)_J\langle n/2\rangle$. Then, using ρ and considering square, we have $0\in\mathrm{End}(Q_1)_J\langle n/2\rangle$. Then $0\in\mathrm{Hom}(Q_1,Q_2)_J\langle n/2\rangle$, and $0\in\mathrm{End}(Q_2)_J\langle n/2\rangle$. In the same way as before this gives us isomorphic direct summands $N_{1,2}$ in $Q_{1,2}$.

D) Let $\varphi\langle n/2\rangle=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{Hom}(Q_1,Q_2)_J\langle n/2\rangle$.

Since $\det(Q_1)=\det(Q_2)=1$, by Lemma 3.20 we have: $2^r\cdot M_2(\mathbb{Z})\subset\mathrm{Hom}(Q_1,Q_2)_0\langle n/2\rangle$.

Considering $\varphi\langle n/2\rangle\,(\,mod\,2\,) \in M_2(\mathbb{Z}/2)$, and using $\rho_{1,2}$, and the fact above, we get that either $\varphi\langle n/2\rangle\,(\,mod\,2\,) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, or $= \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, or we have: $\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} \in \mathrm{Hom}(Q_1, Q_2)_J\langle n/2\rangle \subset M_2(\mathbb{Z})$.

In the last case $p_{N_2} := (\varphi \circ \varphi^\vee)^{2^s}$ and $p_{N_1} := (\varphi^\vee \circ \varphi)^{2^s}$ will be idempotents in $\mathrm{End}(Q_2)$ and $\mathrm{End}(Q_1)$ for large s (see the proof of Lemma 3.12), and φ, $\varphi^\vee$ will give us an isomorphism of corresponding direct summands, and $(p_{N_1})_i = (p_{N_1})'_i = (p_{N_2})_i = (p_{N_2})'_i = \delta_J(i)$.

Because of the $\rho_{1,2}$, it is enough to consider the case $\varphi\langle n/2\rangle\,(\,mod\,2\,) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$.

In particular, $a-c$ is odd.

We have two cases: 1) $a+b-c-d$ is not divisible by 4, 2) $a-b-c+d$ is not divisible by 4.

1) Considering $\varphi - \varphi\circ\rho_1$, we have: $\begin{pmatrix}a+b & a+b\\ c+d & c+d\end{pmatrix} \in \mathrm{Hom}(Q_1, Q_2)_0\langle n/2\rangle$. Then, by Lemma 3.20 , $\begin{pmatrix}a+b-c-d & a+b-c-d\\ 0 & 0\end{pmatrix} \in \mathrm{Hom}(Q_1, Q_2)_0\langle n/2\rangle$.

Since $\det(Q_{1,2}) = 1$, by Lemma 3.20 we have that $\begin{pmatrix}2^r & 2^r\\ 0 & 0\end{pmatrix} \in \mathrm{Hom}(Q_1, Q_2)_0\langle n/2\rangle$. Hence, $\begin{pmatrix}2 & 2\\ 0 & 0\end{pmatrix} \in \mathrm{Hom}(Q_1, Q_2)_0\langle n/2\rangle$.

Using matrices: $\begin{pmatrix}2 & 2\\ 0 & 0\end{pmatrix}$, $\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \in \mathrm{Hom}(Q_1, Q_2)_0\langle n/2\rangle$, and the fact that $a-c$ and $b-d$ are odd, we can correct φ, to make $\varphi\langle n/2\rangle = \begin{pmatrix}g & 0\\ 0 & f\end{pmatrix} \in \mathrm{Hom}(Q_1, Q_2)_J\langle n/2\rangle$ for some odd g and f. Then also, $\varphi^\vee\langle n/2\rangle = \begin{pmatrix}g & 0\\ 0 & f\end{pmatrix} \in \mathrm{Hom}(Q_2, Q_1)_J\langle n/2\rangle$.

We have $\begin{pmatrix}2g & 2g\\ 0 & 0\end{pmatrix} \in \mathrm{Hom}(Q_1, Q_1)_0\langle n/2\rangle$. Again, by Lemma 3.20 , $\begin{pmatrix}2 & 2\\ 0 & 0\end{pmatrix} \in \mathrm{Hom}(Q_1, Q_1)_0\langle n/2\rangle$. By duality, $\begin{pmatrix}2 & 0\\ 2 & 0\end{pmatrix} \in \mathrm{Hom}(Q_1, Q_1)_0\langle n/2\rangle$. And then: $\begin{pmatrix}4 & 0\\ 0 & 0\end{pmatrix} \in \mathrm{Hom}(Q_1, Q_1)_0\langle n/2\rangle$. Then $\begin{pmatrix}4g & 0\\ 0 & 0\end{pmatrix} \in \mathrm{Hom}(Q_1, Q_2)_0\langle n/2\rangle$. And by Lemma 3.20 , $\begin{pmatrix}4 & 0\\ 0 & 0\end{pmatrix} \in \mathrm{Hom}(Q_1, Q_2)_0\langle n/2\rangle$.

Then we can reduce everything to: $g = \pm 1$, $f = \pm 1$.

Denoting corresponding element as u, and dual as v, we get: $(u\circ v)_i = (u\circ v)'_i = (v\circ u)_i = (v\circ u)'_i = \delta_J(i)$, and $(u\circ v)\langle n/2\rangle = (v\circ u)\langle n/2\rangle = Id$. So (by the proof of

Lemma 3.12), $u \circ v$ and $v \circ u$ define direct summands in Q_2 and Q_1, and u, v give an isomorphism between them.

2) Since $(\varphi + \rho_2 \circ \varphi \circ \rho_1)\langle n/2\rangle = \begin{pmatrix} a+d & b+c \\ b+c & a+d \end{pmatrix}$, by Lemma 3.20 we have that $\begin{pmatrix} a+d-b-c & 0 \\ 0 & a+d-b-c \end{pmatrix} \in \mathrm{Hom}(Q_1, Q_2)_0\langle n/2\rangle$.

Since $2^r \cdot M_2(\mathbb{Z}) \subset \mathrm{Hom}(Q_1, Q_2)_0\langle n/2\rangle$, we have: $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \in \mathrm{Hom}(Q_1, Q_2)_0\langle n/2\rangle$.

Using the later matrix, $\rho_{1,2}$, and the fact that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $(\,mod\, 2\,)$, we can assume that $\begin{pmatrix} 1+x & y \\ 0 & 1 \end{pmatrix} \in \mathrm{Hom}(Q_1, Q_2)_J\langle n/2\rangle$, where x and y are even.

Then (using $\rho_{1,2}$ and duality) $\begin{pmatrix} 1 & y \\ 0 & 1+x \end{pmatrix} \in \mathrm{Hom}(Q_2, Q_1)_J\langle n/2\rangle$. Then (multiplying) $\begin{pmatrix} 1+x & 2y(1+x) \\ 0 & 1+x \end{pmatrix} \in \mathrm{End}(Q_2)_J\langle n/2\rangle$, and (since $2^r \cdot M_2(\mathbb{Z}) \subset \mathrm{End}(Q_2)_0\langle n/2\rangle$) $\begin{pmatrix} 1 & 2y \\ 0 & 1 \end{pmatrix} \in \mathrm{End}(Q_2)_J\langle n/2\rangle$.

Then: $\begin{pmatrix} 1+x & 3y \\ 0 & 1 \end{pmatrix} \in \mathrm{Hom}(Q_1, Q_2)_J\langle n/2\rangle$, and $\begin{pmatrix} 0 & 2y \\ 0 & 0 \end{pmatrix} \in \mathrm{Hom}(Q_1, Q_2)_0\langle n/2\rangle$.

In the same way, since (using ρ and Lemma 3.20) $\begin{pmatrix} 1+y & x \\ 0 & 1 \end{pmatrix} \in \mathrm{Hom}(Q_1, Q_2)_J\langle n/2\rangle$, we can get: $\begin{pmatrix} 0 & 2x \\ 0 & 0 \end{pmatrix} \in \mathrm{Hom}(Q_1, Q_2)_0\langle n/2\rangle$. Since $2^r \cdot M_2(\mathbb{Z}) \subset \mathrm{Hom}(Q_1, Q_2)_0\langle n/2\rangle$, we get that $2^{k+1} \cdot M_2(\mathbb{Z}) \subset \mathrm{Hom}(Q_1, Q_2)_0\langle n/2\rangle$, where 2^k is the maximal power of 2, which divides both x and y.

So, everything can be reduced to the following 3 cases: 1) $x = 2^k$, $y = 0$; 2) $x = 0$, $y = 2^k$; 3) $x = 2^k$, $y = 2^k$.

First two cases are equivalent (using $\rho_{1,2}$ and matrices: $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$).

Consider the second case. If $\varphi\langle n/2\rangle = \begin{pmatrix} 1 & 2^k \\ 0 & 1 \end{pmatrix}$, then (using duality, $\rho_{1,2}$, and matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$) we get some $\psi \in Q_2 \to Q_1$: $\psi\langle n/2\rangle = \begin{pmatrix} 1 & -2^k \\ 0 & 1 \end{pmatrix} \in \mathrm{Hom}(Q_2, Q_1)_J\langle n/2\rangle$.

Then $(\varphi \circ \psi)\langle n/2\rangle = Id$, and $(\psi \circ \varphi)\langle n/2\rangle = Id$. So, $p_{N_1} := (\psi \circ \varphi)^{2^s}$ and $p_{N_2} := (\varphi \circ \psi)^{2^s}$, for large s, will be the idempotents in Q_1 and Q_2 with desired properties. φ, ψ will give an isomorphism of corresponding direct summands.

In the third case (by Lemma 3.20), we get $\begin{pmatrix} 1+2^{k-1} & 2^{k-1} \\ -2^{k-1} & 1-2^{k-1} \end{pmatrix} \in \mathrm{Hom}(Q_1, Q_2)_J\langle n/2\rangle$.
On the other hand (using $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$), we have $\begin{pmatrix} 1 & 2^k \\ 0 & 1-2^k \end{pmatrix} \in \mathrm{Hom}(Q_1, Q_2)_J\langle n/2\rangle$.
Since $\begin{pmatrix} 0 & 0 \\ 0 & 2^{k+1} \end{pmatrix} \in \mathrm{Hom}(Q_1, Q_2)_0\langle n/2\rangle$, we get $\begin{pmatrix} 1 & 2^k \\ 0 & 1+2^k \end{pmatrix} \in \mathrm{Hom}(Q_1, Q_2)_J\langle n/2\rangle$.
Then, again using Lemma 3.20 , we have $\begin{pmatrix} 1-2^{k-1} & 2^{k-1} \\ -2^{k-1} & 1+2^{k-1} \end{pmatrix} \in \mathrm{Hom}(Q_1, Q_2)_J\langle n/2\rangle$,
which means that there is some $\psi \in \mathrm{Hom}(Q_2, Q_1)$, s.t. $\psi\langle n/2\rangle = \begin{pmatrix} 1-2^{k-1} & -2^{k-1} \\ 2^{k-1} & 1+2^{k-1} \end{pmatrix}$
$\in \mathrm{Hom}(Q_2, Q_1)_J\langle n/2\rangle$.

Since $\psi\langle n/2\rangle = \varphi\langle n/2\rangle^{-1}$, we get that $p_{N_1} := (\psi\circ\varphi)^{2^s}$ and $p_{N_2} := (\varphi\circ\psi)^{2^s}$, for large s, will be the idempotents in Q_1 and Q_2 with $(p_{N_1})_i = (p_{N_1})'_i = (p_{N_2})_i = (p_{N_2})'_i = \delta_J(i)$, and φ, ψ perform isomorphism of corresponding direct summands.

□

Lemma 3.25 .

Let Q_1, Q_2 be projective quadric, s.t. $\dim(Q_1) = \dim(Q_2)$. *Suppose we have* $\varphi \in \mathrm{Hom}(Q_1, Q_2)$, $\psi = \mathrm{Hom}(Q_2, Q_1)$, *s.t.* $\varphi_i = \psi_i = 1$ *for some* $0 \leqslant i < n/2$ *(see the beginning of this subsection). Then there exist direct summands N_1 in Q_1, and N_2 in Q_2, s.t. N_t contains $\mathcal{X}_{Q_t^i}(i)[2i]$ (in the sense of Lemma 3.23), and there exist an isomorphism: $N_1 = N_2$.*

Proof

By Lemma 3.20 we can assume that φ_i and φ'_i are either 0, or 1. The same about ψ. As usually, we will denote: $J^d(\varphi)$ - the set of such $0 \leqslant l < n/2$, that $\varphi_l = 1$; $J^u(\varphi)$ - the set of such $0 \leqslant j < n/2$, that $\varphi'_j = 1$.

We, certainly, have $i \in J^d(\varphi) \cap J^d(\psi)$

We have two cases: 1) $i \in J^d(\varphi) \cap J^d(\psi) \cap J^u(\varphi) \cap J^u(\psi)$, 2) $i \notin J^d(\varphi) \cap J^d(\psi) \cap J^u(\varphi) \cap J^u(\psi)$

1) Take $\varepsilon := \varphi\circ\psi\circ\varphi\circ\varphi^\vee\circ\psi^\vee$, then $i \in J^d(\varepsilon) = J^u(\varepsilon)$, and so, ε satisfies the conditions of Lemma 3.24 .

2) Take $\widetilde{\varepsilon} := \varphi\circ\psi\circ\varphi + \psi^\vee\circ\varphi^\vee\circ\psi^\vee$. By Lemma 3.20 we can correct $\widetilde{\varepsilon}$ into ε, to make ε_l and ε'_j be 0 or 1 (instead of 0, 1 and 2). Then $i \in J^d(\varepsilon) = J^u(\varepsilon)$, and again, ε satisfies the conditions of Lemma 3.24 .

□

We will also need some generalization of the above result.

Lemma 3.26 .

Let Q_1, Q_2 be two projective quadrics. Suppose we have two morphisms: $\varphi : Q_1 \to Q_2(l)[2l]$, and $\psi : Q_2(l)[2l] \to Q_1$, s.t. $(\psi\circ\varphi)_i = 1$ for some $i \neq n_{1,2}/2$, where

$n_1 = \dim(Q_1)$, $n_2 = \dim(Q_2)$. *Then there are direct summands* N_1 *in* Q_1, N_2 *in* Q_2, *s.t.* $(p_{N_1})_i = 1$, *and* N_1 *is isomorphic to* $N_2(l)[2l]$.

Proof

Using Theorem 2.4.23 , we can change $q_{1,2}$ by $d_{1,2} \cdot \mathbb{H} \perp q_{1,2}$, and assume that $l = 0$. We have two cases: 1) $\dim(Q_1) = \dim(Q_2)$; 2) $\dim(Q_1) \neq \dim(Q_2)$.

1) In this case everything follows from Lemma 3.25 .

2) Let us choose, as usually, some algebraic closure $k \subset \overline{k}$, and bases in $CH^{n_t-p}(Q_t|_{\overline{k}})$ (consisting of plane section h^{n_t-p} of dimension p, if $n_t \geqslant p > n_t/2$; of projective subspace l_p of dimension p, if $0 \leqslant p < n_t/2$; and of two projective subspaces $l'_{n_t/2}$, $l''_{n_t/2}$ of dimension $n_t/2$ from different families, if $p = n_t/2$).

Let $f : Q_1 \to Q_2$ will be some morphism. Denote as f_p the matrix of f, corresponding to the bases above in $CH^{n_1-p}(Q_1|_{\overline{k}})$ and $CH^{n_2-p}(Q_2|_{\overline{k}})$. The matrix f_p will be an integer, if $0 \leqslant p \leqslant min(n_1, n_2)$, and $p \neq n_{1,2}$; and it will be the matrix 2 by 1 (resp. 1 by 2), if $p = n_1/2$ (resp. $n_2/2$). Since $n_1 \neq n_2$ we will not have other cases.

Let $n_1 > n_2$.

In the same way, as in Lemma 3.20 , we have elements u_l in $\mathrm{Hom}(Q_1, Q_2)$, for all $0 \leqslant l \leqslant n_2$, $l \neq n_{1,2}$ s.t. u_l acts trivially on all $CH^{n_1-p}(Q_1|_{\overline{k}})$ with $p \neq l$, and the matrix of the action of u_i on $CH^{n_1-l}(Q_1|_{\overline{k}})$ is 4 if $n_2/2 < l < n_1/2$, and 2 in all other cases. Also, we have elements $w_{n_1/2}$, $w_{n_2/2}$ (if n_1 and n_2 are even), s.t. $w_{n_t/2}$ acts trivially on $CH^{n_1-l}(Q_1|_{\overline{k}})$ for $0 \leqslant l \leqslant n_2$, $l \neq n_t$, and the matrices of the action of $w_{n_t/2}$ on $CH^{n_1-n_t/2}(Q_1|_{\overline{k}})$ are $\begin{pmatrix} 2 & 2 \end{pmatrix}$ (if $n_2 \geqslant n_1/2$, and 0 if not), for $t = 1$, and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, for $t = 2$. Really, u_l and w_l are represented by the cycles $h^l \times h^{n_2-l} \subset Q_1 \times Q_2$.

In the same way, we have elements v_l in $\mathrm{Hom}(Q_2, Q_1)$, for all $0 \leqslant l \leqslant n_2$, $l \neq n_{1,2}$ s.t. v_l acts trivially on all $CH^{n_2-p}(Q_2|_{\overline{k}})$ with $p \neq l$, and the matrix of the action of v_l on $CH^{n_2-l}(Q_2|_{\overline{k}})$ is 1 if $n_2/2 < l < n_1/2$, and 2 in all other cases; and we have $y_{n_1/2}$, $y_{n_2/2}$, s.t. $y_{n_t/2}$ acts trivially on $CH^{n_2-l}(Q_2|_{\overline{k}})$ for $0 \leqslant l \leqslant n_2$, $l \neq n_t$, and the matrices of the action of $y_{n_t/2}$ on $CH^{n_2-n_t/2}(Q_2|_{\overline{k}})$ are $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (if $n_2 \geqslant n_1/2$, and 0 if not), for $t = 1$, and $\begin{pmatrix} 1 & 1 \end{pmatrix}$, for $t = 2$.

Using Lemma 3.13 we can find $\pi \in \mathrm{End}(Q_1)$, s.t. $\pi_j = 1$, for all $j \neq n_{1,2}$, s.t. $(\varphi \circ \psi)_j$ is odd, and $\pi_j = 0$, for all $j \neq n_{1,2}$, s.t. $(\varphi \circ \psi)_j$ is even. Changing φ by $\varphi \circ \pi$, we can assume, that φ_j is either odd, or 0, for all $j \neq n_{1,2}$.

Using the elements u_l and v_l above, we can correct φ and ψ, to make φ_j be ± 1, or 0, and ψ_j be 1, or 0 (depending on the evenness of ψ_j) for all $0 \leqslant j \leqslant n_2$, $j \neq n_{1,2}$. Changing φ by $\varphi \circ \psi \circ \varphi$, and ψ by $\psi \circ \varphi \circ \psi \circ \varphi \circ \psi$, we can assume that $\varphi_j = \psi_j$ and $= 0$, or 1, for all $0 \leqslant j \leqslant n_2$, $j \neq n_{1,2}$.

Let $\varphi_{n_2/2} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, and $\psi_{n_2/2} = \begin{pmatrix} \gamma & \delta \end{pmatrix}$. Using $w_{n_2/2}$ and $y_{n_2/2}$, we can correct φ and ψ, to make $\alpha = 0$, and $\delta = 0$. Then $(\psi \circ \varphi)_{n_2/2} = 0$, and changing φ by $\varphi \circ \psi \circ \varphi$ and ψ by $\psi \circ \varphi \circ \psi$, we can assume that $\varphi_{n_2/2} = 0$, and $\psi_{n_2/2} = 0$.

Let $\varphi_{n_1/2} = \begin{pmatrix} a & b \end{pmatrix}$, and $\psi_{n_1/2} = \begin{pmatrix} c \\ d \end{pmatrix}$.

Using $y_{n_1/2}$, we can assume that $d = 0$, and because of $w_{n_1/2}$, we can assume that a is 0, or 1.

If $a = 0$, or c is even, using $(\varphi \circ \psi)$ and Lemma 3.13 , we can find $\eta \in \mathrm{End}(Q_2)$, s.t. $\eta_j = (\varphi \circ \psi)_j = 1$, or 0, for all $0 \leqslant j \leqslant n_2$, $j \neq n_1/2$, and $\eta_{n_1/2} = 0$. Then changing φ by $\eta \circ \varphi$, and ψ by $\psi \circ \eta$, we can assume that $\varphi_{n_1/2} = 0$, and $\psi_{n_1/2} = 0$.

If $a = 1$, and c is odd, we have that $(\psi \circ \varphi \circ \psi - \psi)_{n_1/2} = \begin{pmatrix} c(c-1) \\ 0 \end{pmatrix}$, and $(\psi \circ \varphi \circ \psi - \psi)_l = 0$, for all $l \neq n_1/2$. Since c is odd (and $d = 0$), we have: $\det(Q_1) = 0$ (since if $\det(Q_1) \neq 1$, then $\begin{pmatrix} c \\ d \end{pmatrix}$ should be stable under $Gal(k\sqrt{\det(Q)}/k)$, which acts via: $\begin{pmatrix} c \\ d \end{pmatrix} \mapsto \begin{pmatrix} d \\ c \end{pmatrix}$). Then by the same arguments as in Lemma 3.20 , we have $\mu \in \mathrm{Hom}(Q_2, Q_1)$, s.t. $\mu_{n_1/2} = \begin{pmatrix} (c-1)\cdot 2^r \\ 0 \end{pmatrix}$, for large r, and $\mu_l = 0$ for all $l \neq n_1/2$. Then we have $\nu \in \mathrm{Hom}(Q_2, Q_1)$, s.t. $\nu_{n_1/2} = \begin{pmatrix} c-1 \\ 0 \end{pmatrix}$, and $\nu_l = 0$ for all $l \neq n_1/2$. Then we can correct ψ, to make $\psi_{n_1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Since $a = 1$, we will have: $(\varphi \circ \psi)_{n_1/2} = 1$, and $(\psi \circ \varphi)_{n_1/2} = \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix}$.

So, we get $(\varphi \circ \psi)_l$ and $(\psi \circ \varphi)_l$ are idempotents for all l. So, by Lemma 3.12 we have direct summands $N_{1,2}$ in Q_1 and Q_2, which will be isomorphic via φ and ψ. And $(p_{N_1})_i = (\psi \circ \varphi)_i = 1$ (since we did not change the evenness of this number).

Lemma is proven.

□

4. Higher forms - generalization of Rost motives

It is well known that if quadratic form r is divisible by the Pfister form $\langle\langle a_1, \dots, a_n\rangle\rangle$, then the Witt numbers of r are all divisible by 2^n (it follows from the fact that if q is divisible by $\langle\langle a_1, \dots, a_n\rangle\rangle$, then $(q|_{k(Q)})|_{anis.}$ is also divisible by $\langle\langle a_1, \dots, a_n\rangle\rangle$ (by the theorem of Pfister, q is divisible by $\langle\langle a_1, \dots, a_n\rangle\rangle$ if and only if $q|_{k(Q_{\{a_1,\dots,a_n\}})}$ is hyperbolic (follows from [9], IX, Corollary 2.9, VII, Theorem 3.2, since the Pfister form is hyperbolic over it's generic point (see Theorem 2.4.5))). So, $\mathcal{X}_Q = \ldots = \mathcal{X}_{Q^{2^n-1}}$, $\mathcal{X}_{Q^{2^n}} = \ldots = \mathcal{X}_{Q^{2\cdot 2^n-1}}$, etc ... (see Theorem 2.4.20).

In this section we will show that this fact reflects the motivic structure of such a quadrics. Namely, the motive of a quadric R with quadratic form $r = \langle\langle a_1, \ldots, a_n\rangle\rangle \cdot q$ can be decomposed (in $DM_{gm}(k)$ as well as in $Chow(k)$ (see Theorem 2.2.4)) into a direct sum of 2^n motives if $\dim(q)$-even, and 2^n+1 motives if $\dim(q)$ is odd, s.t. for each $0 \leqslant d \leqslant [\dim(q)/2]-1$ all $\mathcal{X}$'s from the set $\mathcal{X}_{Q^{d\cdot 2^n}}, \ldots, \mathcal{X}_{Q^{(d+1)\cdot 2^n-1}}$ belong to 2^n different isomorphic (up to shift) summands (for $\dim(q)$-odd "in the middle" we will have also the motive of a Pfister quadric $Q_{\{a_1,\ldots,a_n\}}$).

More precisely, the following is true:

Theorem 4.1 .

Let quadric R corresponds to the quadratic form $r = q \times \langle\langle a_1, \ldots, a_n\rangle\rangle$ for some form q. Then the motive of R can be decomposed into a direct sum:

$$R = \oplus_{i=1,\ldots,2^n} F_\alpha(Q)(i)[2i] \qquad \textit{in the case } \dim(q) \textit{ is even, and}$$

$$R = (\oplus_{i=1,\ldots,2^n} F_\alpha(Q)(i)[2i]) \oplus Q_\alpha(\dim(R)/2 - 2^{n-1}+1)[\dim(R) - 2^n + 2],$$

if $\dim(q)$ is odd (here Q_α is the Pfister quadric, corresponding to the symbol $\alpha = \{a_1, \ldots, a_n\} \in \mathrm{K}_n^M(k)/2$), for some pure motive (see Theorem 2.2.4) $F_\alpha(Q)$. Moreover, $F_\alpha(Q)$ is an extension of: $\mathcal{X}_{R^i}(i)[2i]$, $\quad \mathcal{X}_{R^l}(dim(R)-l)[2dim(R)-2l]$, where $l = i + 2^n - 1$, and i runs through all numbers of the form $k \cdot 2^n$, $0 \leqslant k \leqslant [dim(Q)/2]$.

So, we get some operations F_α which act on quadrics, producing *higher forms* of them. Notice, that *higher form* $F_\alpha(Q)$ consists of the same number of "elementary" pieces ($\mathcal{X}_{Q^i}(*)[2*]$'s) as Q itself (but they are 2^n-times as apart as they were in Q (I mean difference between round (-), or square [-] numbers)).

This generalizes the *Rost motives*, since $M_{\{a_1,\ldots,a_n\}} = F_{\{a_2,\ldots,a_n\}}\left(k\left(\sqrt{a_1}\right)\right)$.

We expect that F_α's are, actually, functors on the pairs $Q \to \mathbb{P}^{\dim(Q)}$, and, in particular, they preserve direct sum decomposition, and so, act on all *pure* motives coming from quadrics.

We will begin with the case of 1-symbol $\{a\} \in \mathrm{K}_1^M(k)/2$.

Proposition 4.2 .

Suppose $R \subset \mathbb{P}^{m+1}$ - is a quadric of dimension m. Then the following conditions are equivalent:

(1) There exists some quadratic extension $k(\sqrt{a})$, s.t. R is hyperbolic over it.

(2) $r = q \times \langle 1, -a\rangle$ for some form q.

(3) There exists $\varphi \in \mathrm{PGL}_{m+1}$ - automorphism of order 2, which acts on $\mathbb{P}^{m+1}$ without rational fixed points, and an operator $r'(x,y) := r(x, \varphi(y))$ is skew-symmetric.

Any of the above conditions implies the following one :

(4) *The motive of* R *can be decomposed into a direct sum:* $R = R_1 \oplus R_1(1)[2]$ *in the case* $m \equiv 2\,(mod\,4)$*, and* $R = R_1 \oplus k\left(\sqrt{det(R)}\right)(dim(R)/2)[dim(R)] \oplus R_1(1)[2]$*, if* $m \equiv 0\,(mod\,4)$*, where* R_1 *is an extension of:* $\mathcal{X}_{R^i}(i)[2i], \mathcal{X}_{R^l}(dim(R)-l)[2dim(R)-2l]$*, where* i *(* l *) run through all even (respectively, odd) numbers* $\leqslant 2[dim(R)/4]$.

Proof of Proposition 4.2

($1 \Leftrightarrow 2$) This is classical (see, for example [9], VII, Theorem 3.2).

($2 \Rightarrow 3$) If $r = q \times \langle 1, -a\rangle$ for some quadratic form q, take $\varphi(v_1, v_2) = (a \cdot v_2, v_1)$.

($3 \Rightarrow 2$) Since $\varphi^2 = a\cdot \mathrm{Id}$, we have that $m+2$ is even and φ has eigenvalues: $\sqrt{a}$ and $-\sqrt{a}$ - $(m+2)/2$ of each. Let A_1 and A_2 are corresponding eigenspaces (defined over $k(\sqrt{a})$). Since r' is skew-symmetric, we can always find an r'-isotropic subspace V_1 of dimension equal to $(m+1)/2$, such that $V_1 \cap A_j = 0$. Then $V_1 \cap \varphi(V_1) = 0$. And we have $V = V_1 \oplus \varphi(V_1)$. Evidently, $r(V_1, \varphi(V_1)) = 0$, and $r(\varphi(v_1), \varphi(v_1')) = -a\cdot r(v_1, v_1')$. Now it is enough to take $q = r|_{V_1}$.

($3 \Rightarrow 4$)

Consider the following cycle $P \subset R \times R$: $(x, y) \in P$ iff x and y are contained in some line, stable under φ. $\dim(P) = \dim(R) + 1$, so it defines the morphism $p : R \to R(-1)[-2]$ (by Theorem 2.1.23 , Theorem 2.1.17).

Since the condition on x and y above is symmetric, we have $p^\vee(-1)[-2] = p$, where $p^\vee \in \mathrm{Hom}(R(1)[2], R)$ is the element dual to p via duality:
$R = \underline{\mathrm{Hom}}(R, \mathbb{Z}(\dim(R))[2\dim(R)])$.

Let H be a cycle in $R\times R$, represented by a hyperplane in R, embedded diagonally, and $h : R \to R(1)[2]$ - the corresponding morphism (see Theorem 2.1.17).

Consider the following endomorphism of R: $\pi = p(1)[2] \circ h$. Let us compute numbers: π_i, π_i' (see Theorem 3.7). All the computations can be, evidently, performed over $\overline{k}$. (to compute these numbers we have to find the intersection of our cycle with the cycles of the type : $h^i \times l_i$, $l^j \times h^j$, and $l'_{\dim(R)/2} \times l''_{\dim(R)/2}$ over $\overline{k}$, see the proof of Theorem 2.4.22 and the discussion before Lemma 3.10).

Notice, that $p = p^\vee(-1)[-2]$, and $h^\vee = h(-1)[-2]$ (if morphism $\psi : R \to R(m)[2m]$ is represented by a cycle Ψ, then the dual morphism $\psi^\vee : R(-m)[-2m] \to R$ will be represented by the cycle $\Psi^\vee$ obtained from Ψ by the reflection in the diagonal).

Then we have: $\pi^\vee = h(-1)[-2] \circ p$, and, hence, $\pi_i = (\pi^\vee)_i' = \pi_{i+1}'$, for $i < \dim(R)/2-1$, and for $i = \dim(R)/2-1$: $2\cdot\pi_{\dim(R)/2-1} = s(\pi\langle n/2\rangle)$ (here $s\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = a+b+c+d$) (shift by 1 in numbers, since the h is situated from the different sides of p in π and $\pi^\vee$).

So, it is enough to compute π_i' (and $\pi\langle\dim(R)/2\rangle$).

Consider the locus of fixed points of φ: $A_1 \cup A_2$, where A_1 and A_2 are projective spaces in $\mathbb{P}^{m+1}$ of dimension $\dim(R)/2$ (I will denote them in the same way as corresponding linear spaces). Evidently, $A_j \cap R$ is A_j. Since $A_1 \cap A_2 = \emptyset$, we can find space

T of dimension $\dim(R)/2$ on Q if $\dim(R) \equiv 2\,(mod\,4)$, and of dimension $\dim(R)/2-1$ if $\dim(R) \equiv 0\,(mod\,4)$ on R, such that $T \cap A_j = \emptyset$ and (consequently) $\varphi(T) \cap A_j = \emptyset$. Then $T \cap \varphi(T) = \emptyset$, i.e. corresponding affine spaces are linearly independent. We have a complete flag of subspaces $T_0 \subset T_1 \subset \cdots \subset T$ in T.

Since T_{i-1} does not intersect A_j, π'_i is equal to the degree of the variety S_{i-1}, consisting of all lines $l(t, \varphi(t))$, $t \in T_{i-1}$ (this degree is the same as the intersection number of the cycle representing π with the cycle $h^i \times l_i$).

Consider arbitrary point $t_{i-1} \in T_{i-1} \backslash T_{i-2}$. Then projection of S_{i-1} from the point t_{i-1} will be equal to the cone over S_{i-2}, from which it follows, that the degree of S_{i-1} minus the degree of S_{i-2} is equal to 1. Since, evidently, $\pi'_0 = 0$, $\pi'_1 = 1$, we have that $\pi'_i = i$, for all $i < dim(R)/2$, and in the basis of $CH^{\dim(R)/2}(R|_{\overline{k}}) = \mathbb{Z} \oplus \mathbb{Z}$, consisting of planes, $\pi\langle\dim(R)/2\rangle$ looks as $\begin{pmatrix} a & a \\ c & c \end{pmatrix}$, where $a+c = \dim(R)/2$; moreover, in the case $\dim(R) \equiv 2\,(mod\,4)$, the same considerations give us that for some half-dimensional plane T in R (the one which does not intersect A_j), the cycle $p(T)$ (i.e: the union of all lines $l(t, \varphi(t))$ for $t \in T$) has the degree $\dim(R)/2 + 1$.

Using Lemma 3.13 and our π, we can find $\psi \in \mathrm{End}(R)$, s.t.: $\psi'_i = (1+(-1)^{i+1})/2$, $\psi_i = (1+(-1)^i)/2$, for all $i < \dim(R)/2$, and $\psi\langle\dim(R)/2\rangle$ looks as: $\begin{pmatrix} b & b \\ 1-b & 1-b \end{pmatrix}$, if $\dim(R) \equiv 2\,(mod\ 4)$, for some b, and as: $\begin{pmatrix} b & b \\ -b & -b \end{pmatrix}$ if $\dim(R) \equiv 0\,(mod\ 4)$ (b here is equal to $(a-c+1)/2$). Since in the last case $image(\overline{\langle\dim(R)/2\rangle})$ (see the definition of $\overline{\langle\dim(R)/2\rangle}$ before Lemma 3.20) is contained in the subring of $M_2(\mathbb{Z})$, generated by id and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ($\det(R) \neq 1$, and we can go first to the field where anisotropic part of q is equal to $\lambda \cdot \langle 1, \det(R)\rangle$, before passing to the algebraic closure, but for 0-dimensional anisotropic quadric P/E, $\mathrm{End}(P)$ is $\mathbb{Z} \oplus \mathbb{Z}$, additively generated by the matrices specified above (in $\mathrm{End}(P|_{\overline{E}})$)), the corresponding matrix must be 0.

Now, using Lemma 3.13 and Lemma 3.12 , we can correct ψ to make it an idempotent ($\psi\langle n/2\rangle$ is already an idempotent, and we need only to make ψ_i, ψ'_i be equal 0, or 1, then use Lemma 3.12). We get a decomposition: $R = R_1 \oplus R_2(1)[2]$, or $R = R_1 \oplus k(\sqrt{a})(dim(R)/2)[dim(R)] \oplus R_2(1)[2]$ (in the second case we have: $\psi\langle n/2\rangle = 0$ (see above), and $1 - \psi - \psi^\vee$ will give us the idempotent for the middle summand (in both cases: $\psi_i + \psi_i^\vee = 1$ for all $0 \leqslant i < n/2$)). Let us prove, that $R_1 = R_2$.

Let T', T-basis of $CH^{\dim(R)/2}(R|_{\overline{k}})$ consisting of planes, and $T_{\dim(R)/2-1}$ and $h^{\dim(R)/2-1}$ are bases of $CH^{\dim(R)/2-1}(R|_{\overline{k}})$ and $CH^{\dim(R)/2+1}(R|_{\overline{k}})$, consisting of a plane and a subquadric, correspondingly. Let us find the matrices of $p : CH^{\dim(R)/2}(R|_{\overline{k}}) \to CH^{\dim(R)/2+1}(R|_{\overline{k}})$, and $p : CH^{\dim(R)/2-1}(R|_{\overline{k}}) \to CH^{\dim(R)/2}(R|_{\overline{k}})$ in those bases.

Since cycle p is symmetric in $R \times R$, both matrices will be the same. We know that one of them will be $\begin{pmatrix} a \\ c \end{pmatrix}$, where $a + c = \dim(R)/2$. But in the case $\dim(R) \equiv 2\,(mod\ 4)$, we also have $2 \cdot c = \dim(R)/2 + 1$ (since for some plane T we have degree $p(T) = \dim(R)/2 + 1$, see above), i.e. $c - a = 1$, and $b = 0$. So, $b = 0$ always.

We will consider the case: $\dim(R) \equiv 2\,(mod\,4)$. Consider the following map $\alpha : R \to R(-1)[-2]$. $\alpha = p - \sum_{i=0,\dots,dim(R)/2-1}(v(i) + v'(i+1)) \cdot ([i + 1/2]))$, where $v(i)$, $v'(i)$ are defined by the cycles $h^{\dim(R)-i} \times h^{i+1}$ and $h^i \times h^{\dim(R)-i+1}$, respectively.

We can construct two maps: $\beta : R_1 \to R_1$ and $\gamma : R_2(1)[2] \to R_2(1)[2]$ in the following way.

$\beta = \psi \circ \alpha(1)[2] \circ (1-\psi)(1)[2] \circ h \circ \psi$, $\gamma = (1-\psi) \circ h(-1)[-2] \circ \psi(-1)[-2] \circ \alpha \circ (1-\psi)$. $(\beta \oplus \gamma)$ defines a map $R = R_1 \oplus R_2(1)[2] \to R_1 \oplus R_2(1)[2]$, s.t. $(\beta \oplus \gamma)_i = (\beta \oplus \gamma)'_i = 1$ for all i, and it is not difficult to check that $(\beta \oplus \gamma)\langle \dim(R)/2 \rangle = id$, and, hence, it is an isomorphism (since $1 - (\beta \oplus \gamma)$ is a nilpotent by Lemma 3.10). Then $(1-\psi)(1)[2] \circ h \circ \psi : R_1 \to R_2$ is also an isomorphism.

Proposition is proven.

□

Higher forms have some kind of *relatives*. Namely, we have the following:

Lemma 4.3 .

Let $r' = q' \cdot \langle 1, -a \rangle + \langle x \rangle$, and $q = q' + \langle x \rangle$. Then we have a decomposition: $R' = R'_1 \oplus R'_2$, where $R'_1 = F_a(Q)$ and $R'_2 = F_a(Q')(1)[2]$. Here R'_1 is an extension of $\mathcal{X}_{R'^i}(i)[2i]$, $\mathcal{X}_{R'^i}(\dim(R') - i)[2\dim(R') - 2i]$ for all even i, and R'_2 - for all odd i.

Proof

We will consider the case $\dim(r') \equiv 3\,(mod\,4)$, other one is similar. Let $r = q \cdot \langle 1, -a \rangle$ (as usually). Then the idempotent $\psi : R \to R$, which gives the decomposition $R = R_1 \oplus R_1(1)[2]$, is actually a composition $R \xrightarrow{h} R'(1)[2] \to R(1)[2] \xrightarrow{\alpha} R$, where α was defined in the end of the proof of Proposition 4.2 , and the middle map is induced by the embedding of the subquadric R' into R. (since the map $R \xrightarrow{h} R(1)[2]$ has such a decomposition, because the map h is given by a cycle $\Delta_{R'} \subset R \times R$, which lives in $R \times R'$ (we can take arbitrary hyperplane section here)).

This gives us a map $g : R' \to R'$ (equal to a composition $R' \to R \xrightarrow{\alpha(-1)[-2]} R(-1)[-2] \xrightarrow{h(-1)[-2]} R'$), and it is easy to see that $g_i = g'_i = 1$ if i-even, and 0-otherwise (g_i, g'_i here are numbers associated with the morphism g (see Theorem 3.7), they are equal to the intersection numbers of the cycle representing our morphism (see Theorem 2.1.23 (4) and Theorem 2.1.17) with the cycles $h^i \times l_i$ and $l_i \times h^i$ (see the proof of Theorem 2.4.22)) (since $(h \circ \alpha)_i = (h \circ \alpha)'_{i+1} = 1$ for odd i, and $= 0$ for even i).

So, we get a decomposition: $R' = R'_1 \oplus R'_2$.

From the above it is clear that $R'_1 = R_1 = F_a(Q)$ (via the maps $t: R'_1 \to R' \to R \to F_a(Q)$, and $s: F_a(Q) \to R \overset{\alpha(-1)[-2]}{\to} R(-1)[-2] \overset{h(-1)[-2]}{\to} R' \to R'_1$). Really, the two composition maps $u := s \circ t: R'_1 \to R'_1$, and $v := t \circ s: F_a(Q) \to F_a(Q)$ are automorphisms of R'_1 and $F_a(Q)$ respectively (consider the morphisms $u \oplus id_{R'_2}: R' \to R'$, and $v \oplus id_{F_a(Q)(1)[2]}: R \to R$, they are isomorphisms by Lemma 3.10 (they are equal to the identity maps modulo nilpotents)).

Consider $r'' = q' \cdot \langle 1, -a \rangle$. Then the composition $R' \overset{h}{\to} R''(1)[2] \overset{\alpha''(1)[2]}{\to} R'' \to R'$, where α'' is morphism constructed in the same way as α but for R'', is the same as $1 - g$ modulo nilpotents (we need just to compute numbers). Hence, in the same way as above: $R'_2 = F_a(Q')(1)[2]$.

$\square$

Results above permit us to clarify the picture in the Proposition 3.4 :

Corollary 4.4 .

Suppose n-dimensional quadric R is such that (in the notations of Proposition 3.4) $P = \oplus_{i=0,\dots,[\dim(R)-1/2]} \mathcal{X}_{R^i}(i)[2i]$. Then:

1) If $r \in I^2(W(k))$, then $R = R_1 \oplus R_1(1)[2]$, where R_1 is an extension of the same $\mathcal{X}$'s as in Proposition 4.2 .

2) If $r \notin I^1(W(k))$, then R has a decomposition as in Lemma 4.3 .

Proof

We remind that $P' := \mathrm{Cone}((\oplus_{i=0,\dots,[n-1/2]} \mathcal{X}_{R^i}(n-i)[2n-2i]) \to R)$, where the maps $\mathcal{X}_{R^i}(n-i)[2n-2i] \to R$ are the compositions: $\mathcal{X}_{R^i}(n-i)[2n-2i] \to \mathbb{Z}(n-i)[2n-2i] \to R$, where the first map is the natural projection $\mathcal{X}_{R^i} \to \mathbb{Z}$ shifted by $(n-i)[2n-2i]$, and the second map correspond to the plane section of codimension i via identification: $\mathrm{Hom}(\mathbb{Z}(n-i)[2n-2i], R) = CH^i(R)$ (see Theorem 2.1.17 , Theorem 2.1.23 (4)), and: $P = P'$ for n-odd, and $P = \mathrm{Cone}(\mathcal{X}_{R^{n/2}} \times k\sqrt{\det(R)}(n/2)[n] \overset{\eta}{\to} P')$, for some morphism η when n is even.

1) Consider hyperplane section of dimension $\dim(R)/2 - 1$.
It defines some morphism $\rho: \mathbb{Z}(\dim(R)/2-1)[\dim(R)-2] \to R \to P$.

Since $P = \oplus_{i=0,\dots,[\dim(R)-1/2]} \mathcal{X}_{R^i}(i)[2i]$, we get some morphism $\varepsilon: \mathbb{Z}(\dim(R)/2 - 1)[\dim(R)-2] \to \mathcal{X}_{R^{\dim(R)/2-1}}(\dim(R)/2-1)[\dim(R)-2]$. This morphism, composed with the standard map $\mathcal{X}_{R^{\dim(R)/2-1}}(\dim(R)/2-1)[\dim(R)-2] \to \mathbb{Z}(\dim(R)/2 - 1)[\dim(R)-2]$, gives $2: \mathbb{Z} \to \mathbb{Z}$ (shifted).

Really, this can be checked over algebraic closure, where $\mathcal{X}_{R^i}|_{\overline{k}}(i)[2i]) = \mathbb{Z}(i)[2i]$, and we don't have any maps $\mathbb{Z}(\dim(R)/2-1)[\dim(R)-2] \to \mathbb{Z}(i)[2i]$ for $i < \dim(R)/2 - 1$ by Theorem 2.1.18 , so $\varepsilon|_{\overline{k}}$ is the only nontrivial component of ρ. But the composition $R|_{\overline{k}} \to P|_{\overline{k}} \to \mathcal{X}_{R^{\dim(R)/2-1}}|_{\overline{k}}(\dim(R)/2-1)[\dim(R)-2] \to \mathbb{Z}|_{\overline{k}}(\dim(R)/2-1)[\dim(R)-2]$ is just the map, corresponding to the plane section of codimension $\dim(R)/2 - 1$ (via the identification: $\mathrm{Hom}(R, \mathbb{Z}(i)[2i]) = CH^i(R)$, see

Theorem 2.1.17)(see the definition of P above, and the proof of Theorem 2.4.22). So, the composition:
$\mathbb{Z}|_{\overline{k}}(\dim(R)/2-1)[\dim(R)-2] \to R|_{\overline{k}} \to P|_{\overline{k}} \to \mathcal{X}_{R^{\dim(R)/2-1}}|_{\overline{k}}(\dim(R)/2-1)[\dim(R)-2] = \mathbb{Z}|_{\overline{k}}(\dim(R)/2-1)[\dim(R)-2]$ is the multiplication by the intersection number of subquadrics of dimensions $\dim(R)/2-1$ and $\dim(R)/2+1$, which is 2.

That means that $R^{\dim(R)/2-1}$ contains a point E of degree $2 \cdot d$ with d-odd (by Theorem 2.3.3 (2)); i.e. over E R becomes hyperbolic (since it becomes $\dim(R)/2-1$-times isotropic, and since $\det(R) = 1$).

We have tower of fields: $k \subset F \subset E$, where $[F : k] = d$, and $[E : F] = 2$. So, over F we have a decomposition (by Proposition 4.2): $R|_F = \widetilde{R_1} \oplus \widetilde{R_1}(1)[2]$ (since $\det(R) = 1$, we must have $\dim(R) \equiv 2\,(mod\,4)$ (since our form over F appears to be divisible by binary one (see equivalence ($1 \Leftrightarrow 2$) in Proposition 4.2))).

I.e.: we have $\psi \in \mathrm{End}(R|_F)$ that $\psi_i = \psi'_{i-1} = 1$, for odd i, and 0 for even i, and $\psi\langle\dim(R)/2\rangle = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ (see the proof of Proposition 4.2 (notice, $b = 0$ there (notations from there))).

But, now, the existence of transfers from F to k gives us that in the $\mathrm{End}(R)$ there is element φ, s.t. $\varphi_i = \varphi'_{i-1} = d$, for odd i, and $= 0$ for even i; and $\varphi\langle\dim(R)/2\rangle$: $\varphi\langle\dim(R)/2\rangle = \begin{pmatrix} d & d \\ 0 & 0 \end{pmatrix}$. Using Lemma 3.13 and Lemma 3.12 we can correct φ to make it an idempotent. We get: $R = R_1 \oplus R_1(1)[2]$.

2) The same considerations give us that over some E of degree $2 \cdot d$ with d-odd R becomes hyperbolic. Again, using transfers, we get that $R = R_1 \oplus R_2$.

□

Remark In particular, in the case 1) above all *Witt numbers* are divisible by 2.

Consider now the case of a *pure* symbol $\{a_1, \ldots, a_n\}$ (see Definition 2.4.4) of higher dimension.

Proof of Theorem 4.1

Let us first prove the following:

Lemma 4.5 .

Suppose for some quadric R we have undecomposable direct summand $N(i_0)[2i_0]$ which is contained in $R'\langle i_0\rangle$ (i.e.: for the corresponding idempotent g we have $g_j = g'_j = 0$ for all $j < i_0$ (see Theorem 3.7 for the definition of g_j, g'_j)), "the smallest elementary piece" of $N(i_0)[2i_0]$ is $\mathcal{X}_{R^{i_0}}(i_0)[2i_0]$ (i.e.: $g_j = 0$ for $j < i_0$), and $\mathcal{X}_{R^{i_0+t}} = \mathcal{X}_{R^{i_0}}$ (see in this connection: Definition 2.4.19 , Theorem 2.4.20 , and discussion after it).

Then in R we have also direct summands isomorphic to $N(i_0 + t)[2i_0 + 2t]$.

Proof

Consider the projection: $\pi : \underline{R^{i_0}} \to \underline{R^{i_0-1}}$. It's fibers are quadrics $R(i_0,x)$ of i_0-dimensional planes on R, containing fixed i_0-1-dimensional plane on R (defined by the point x from $\underline{R^{i_0-1}}$). Let F be the preimage of $\Delta_{\underline{R^{i_0-1}}} \subset \underline{R^{i_0-1}} \times \underline{R^{i_0-1}}$ of the diagonal, under the projection: $\pi\times\pi : \underline{R^{i_0}} \times \underline{R^{i_0}} \to \underline{R^{i_0-1}} \times \underline{R^{i_0-1}}$. $F = \underline{R^{i_0}} \times_{\underline{R^{i_0-1}}} \underline{R^{i_0}}$ has natural projection $\psi : F \to \underline{R^{i_0-1}}$ with fibers $R(i_0,x)\times R(i_0,x)$.

Since $\mathcal{X}_{R^{i_0+t}} = \mathcal{X}_{R^{i_0}}$, we have that each $R(i_0,x)$ has a rational t-dimensional plane as soon as it has a rational point (see remark before Claim 3.3 , and also: Definition 2.4.19 , Theorem 2.4.20 , and discussion after it), hence, there is a rational map $R(i_0,x) \to R(i_0,x)^t$, so we have cycle P_t of dimension $\dim(\underline{R^{i_0}})+t$ on F, and, hence on $\underline{R^{i_0}} \times \underline{R^{i_0}}$, which over generic point of $\underline{R^{i_0-1}}$ corresponds to the rational map $\varphi : R(i_0, k(\underline{R^{i_0-1}})) \to R(i_0, k(\underline{R^{i_0-1}}))^t$ (this cycle is a closure of the cycle over $k(\underline{R^{i_0-1}})$, consisting of pairs $(x,y) \in R(i_0,k(\underline{R^{i_0-1}})) \times R(i_0,k(\underline{R^{i_0-1}}))$, s.t. $y \in \varphi(x)$, where $\varphi(x)$ is a t-dimensional plane on $R(i_0,k(\underline{R^{i_0-1}}))$, obtained via map φ).

This gives us the map $p_t : \underline{R^{i_0}} \to \underline{R^{i_0}}(-t)[-2t]$ (by Theorem 2.1.26 (4), Theorem 2.1.17).

By the Claim 3.2 we know that $\underline{R^{i_0}} = \underline{R^{i_0-1}} \times R\langle i_0\rangle$, and $\underline{R^{i_0-1}} \times R\langle i_0\rangle$ is a direct summand in $\underline{R^{i_0-1}} \times R(-i_0)[-2i_0]$, in particular, we have map $\varphi : \underline{R^{i_0-1}} \times R\langle i_0\rangle \to \underline{R^{i_0-1}} \times R(-i_0)[-2i_0]$, given by the cycle: $\Phi \subset \underline{R^{i_0}} \times \underline{R^{i_0-1}} \times R$, consisting of triples: $(F_{i_0}^{\cdot}, F_{i_0}^{\leqslant i_0-1}, y)$, where $F_{i_0}^{\cdot}$ is a point of $\underline{R^{i_0}}$, and $y \in F_{i_0}^{i_0}$. Over $\overline{k}$, φ maps $\mathbb{Z}\times\mathbb{Z}(t)[2t]$ to $\mathbb{Z}\times\mathbb{Z}(t)[2t]$ identically.

Let $\psi : \underline{R^{i_0-1}} \to \mathbb{Z}$ denote the natural projection; $h^t : R \to R(t)[2t]$ denotes the morphism, corresponding (by Theorem 2.1.17 and Theorem 2.1.23) to the plane section of codimension t embedded diagonally into $R\times R$; and $\pi : R(-i_0)[-2i_0] \to N$ comes from $N(i_0)[2i_0]$ being a direct summand in R.

Let $d := \dim(\underline{R^{i_0-1}})$, and, as usually, $n := \dim(R)$.

Sublemma 4.6.

Let N be undecomposable direct summand of R, contained in $R'\langle i_0\rangle$. Then the natural maps: $\underline{R^{i_0-1}} \times N \to N$ and $N(d)[2d] \to \underline{R^{i_0-1}} \times N$ have splittings: ρ and ν.

Really, first of all, since $N(i_0+t)[2i_0+2t]$ is a direct summand in R, it will be also a direct summand in
$R\langle 1\rangle(1)[2], ..., R\langle i_0\rangle(i_0)[2i_0]$ (by Theorem 3.7 the corresponding idempotent of $\mathrm{End}(R)$ will give us idempotents in $\mathrm{End}(R\langle 1\rangle), \dots, \mathrm{End}(R\langle i_0\rangle)$).

Consider $R\langle j\rangle \times N$ for $0 \leqslant j \leqslant i_0$. It is an extension on $\mathcal{X}_{R^j}\times N$, $R\langle j+1\rangle(1)[2]\times N$ and $\mathcal{X}_{R^j}(n-2j)[2n-4j]\times N$ (in the sense specified in Theorem 3.1).

But N itself is an extension of $\mathcal{X}_{R^m}(*)[2*]$'s with $m \geqslant i_0$. Hence, $\mathcal{X}_{R^j}\times N = N$ (by Theorem 2.4.16).

Motive N is *pure* (as a direct summand of R). So, $\mathrm{Hom}(N, \mathcal{X}_P(k)[2k+1]) = 0$ for any smooth projective P.

Really, it is enough to prove that $\mathrm{Hom}(R, \mathcal{X}_P(k)[2k+1]) = 0$. We can consider exact triangle $SK^r(\mathcal{X}_P) \to \mathcal{X}_P \to Z_r$, where $SK^r(\mathcal{X}_P)$ is the motive of the r-th skeleton of $\mathcal{X}_P$. For large r, Z_r as a complex of sheaves with transfers in Nisnevich topology (see Definition 2.1.8 , and Definition 2.3.1 (and text before it)), does not have cohomology sheaves with numbers $> -T$, for any given T. From [17], Proposition 3.1.8 it follows that for large r $\mathrm{Hom}(R, Z_r(k)[2k+1]) = 0$. So, we can change $\mathcal{X}_P$ by it's finite piece $SK^r(\mathcal{X}_P)$, which is an *extension* (see Definition 2.5.6) of: P, $P \times P[1], \ldots, P \times \cdots \times P[r-1]$. But $\mathrm{Hom}(R, P \times \cdots \times P[s](k)[2k+1]) = 0$ for $s \geqslant 0$ (for $k < 0$ - by Theorem 2.1.18 (1) and duality (Theorem 2.1.23), for $k \geqslant 0$ - by Theorem 2.1.18 (3) and duality). So, $\mathrm{Hom}(R, SK^r(\mathcal{X}_P)(k)[2k+1]) = 0$, and $\mathrm{Hom}(N, \mathcal{X}_P(k)[2k+1]) = 0$.

Since $R\langle j+1\rangle(1)[2] \times N$ and $\mathcal{X}_R(n-2j)[2n-4j] \times N$ are extensions of $\mathcal{X}_P(*)[2*]$'s (use Theorem 2.4.16), and there are no 1-st ext's between N and $\mathcal{X}_P(k)[2k]$ for any smooth projective P and any k (see above), we have that the map $R\langle j\rangle \times N \to N$ has a splitting, and N is a direct summand in $R\langle j\rangle \times N$.

Now, using the fact, that $\underline{R^{i_0-1}} = R \times R\langle 1\rangle \times \cdots \times R\langle i_0 - 1\rangle$, we get that the natural map $\underline{R^{i_0-1}} \times N \to N$ has a splitting.

By duality: $\underline{\mathrm{Hom}}(\underline{R^{i_0-1}}, \mathbb{Z}(d)[2d]) = \underline{R^{i_0-1}}$, we get a splitting for the natural map: $N(d)[2d] \to \underline{R^{i_0-1}} \times N$.

□

Let us consider the following diagram:

$$\begin{array}{ccc}
\underline{R^{i_0-1}} \times R(-i_0-t)[-2i_0-2t] & \xrightarrow{\psi\times id} & R(-i_0-t)[-2i_0-2t] \\
\uparrow{\scriptstyle \varphi(-t)[-2t]} & & \downarrow{\scriptstyle h^t(-i_0-t)[-2i_0-2t]} \\
\underline{R^{i_0-1}} \times R\langle i_0\rangle(-t)[-2t] & & R(-i_0)[-2i_0] \\
\uparrow{\scriptstyle p_t} & & \downarrow{\scriptstyle \pi} \\
\underline{R^{i_0-1}} \times R\langle i_0\rangle & \xleftarrow{\rho} & N
\end{array}$$

It is not difficult to see, that over $\overline{k}$ the maps: $\alpha := (\psi\times id)\circ\varphi(-t)[-2t]\circ p_t\circ\rho : N \to R(-i_0-t)[-2i_0-2t]$, and $\beta := \pi \circ h^t(-i_0-t)[-2i_0-2t] : R(-i_0-t)[-2i_0-2t] \to N$ map $\mathbb{Z}$ to $\mathbb{Z}$ identically.

By Lemma 3.26 we have a direct summand in $R(-t-i_0)[-2t-2i_0]$, isomorphic to N.

Lemma is proven.

□

Now we can prove Theorem 4.1 .

Let quadric R corresponds to the quadratic form $r = q \times \langle\langle a_1, \ldots, a_n \rangle\rangle$. By induction we can assume that

$$R = \oplus_{i=1,\ldots,2^{n-1}} F_{\alpha_{n-1}}(P)(i)[2i],$$

where $\alpha_{n-1} = \{a_1, \ldots, a_{n-1}\}$ and quadric P corresponds to quadratic form $p = \langle 1, -a_n \rangle \times q$ (dimension of P is even).

Since all Witt numbers of r are divisible by 2^n (see the discussion before Theorem 4.1) we have that for even number of k's : $\mathcal{X}_R = \mathcal{X}_{R^{k\cdot 2^{n-1}}}$. Let N_1 be an undecomposable direct summand in $F_{\alpha_{n-1}}(P)$, containing $\mathcal{X}_R$. By Lemma 4.5 we have the even number of undecomposable direct summands in $F_{\alpha_{n-1}}(P)$, each isomorphic to $N_1(k \cdot 2^{n-1})[k \cdot 2^n]$, and containing $\mathcal{X}_{R^{k\cdot 2^{n-1}}}(k \cdot 2^{n-1})[k \cdot 2^n]$.

By duality, we have also the even number of undecomposable direct summands in $F_{\alpha_{n-1}}(P)$, isomorphic to $N_1^{up}(-k \cdot 2^{n-1})[-k \cdot 2^n]$, where N_1^{up} is an undecomposable direct summand in $F_{\alpha_{n-1}}(P)$, containing the "upper elementary piece" $\mathcal{X}_R(\dim(R) - 2^{n-1} + 1)[2\dim(R) - 2^n + 2]$.

It could be that either each of $N_1(*)[2*]$'s coincide with some of $N^{up}(*)[2*]$'s, or they are all different (actually, only the first case is possible), but in any case there are even number of such identifications, since Witt numbers are divisible by 2^n.

Let k_1 be the smallest number such that $\mathcal{X}_{R^{k_1\cdot 2^{n-1}}}(k_1 \cdot 2^{n-1})[k_1 \cdot 2^n]$ is not contained in any of $N_1(*)[2*]$'s or $N^{up}(*)[2*]$'s. By considerations above, it should be even.

Then we can consider the undecomposable direct summand N_{k_1} in $F_{\alpha_{n-1}}(P)$, containing $\mathcal{X}_{R^{k_1\cdot 2^{n-1}}}(k_1 \cdot 2^{n-1})[k_1 \cdot 2^n]$. It will be evidently contained in $R\langle k_1 \cdot 2^{n-1}\rangle$.

We can again apply Lemma 4.5 , etc ... .

If $\dim(q)$-even in this way we get that for each $0 \leqslant k \leqslant [\dim(q)/2]$, there are undecomposable direct summands N_k and $N_k(2^{n-1})[2^n]$, and M_k and $M_k(2^{n-1})[2^n]$ in R, s.t. N contains $\mathcal{X}_{R^{k\cdot 2^n}}(k \cdot 2^n)[2k \cdot 2^n]$ (in the sense of Lemma 3.23), and M_k contains $\mathcal{X}_{R^{(k+1)\cdot 2^n - 1}}(\dim(R) - (k+1) \cdot 2^n + 1)[2\dim(R) - (k+1) \cdot 2^{n+1} + 2]$. By Lemma 3.21 , two such direct summands either coincide, or do not "intersect" (in the sense of Lemma 3.23). Then $F_{\alpha_{n-1}}(P) = S \oplus S(2^{n-1})[2^n]$, and S is an extension of $\mathcal{X}_{R^{k\cdot 2^n}}(k\cdot 2^n)[2k\cdot 2^n]$ and $\mathcal{X}_{R^{(k+1)\cdot 2^n - 1}}(\dim(R) - (k+1)\cdot 2^n + 1)[2\dim(R) - (k+1)\cdot 2^{n+1} + 2]$ for $0 \leqslant k \leqslant [\dim(q)/2]$.

If $\dim(q)$-odd in the same way we get $F_{\alpha_{n-1}}(P) = S \oplus S(2^{n-1})[2^n] \oplus A$, where A is an extension of $\mathcal{X}_{R^{\dim(R)/2 - 2^{n-1} + 1}}(\dim(R)/2 - 2^{n-1} + 1)[\dim(R) - 2^n + 2]$ and $\mathcal{X}_{R^{\dim(R)/2}}(\dim(R)/2)[\dim(R)]$.

But we have $\mathcal{X}_{\dim(R)/2 - j} = \mathcal{X}_{Q_\alpha}$ for all $0 \leqslant j \leqslant 2^{n-1} - 1$. Really, $q \cdot \langle\langle a_1, \ldots, a_n \rangle\rangle - \langle\langle a_1, \ldots, a_n \rangle\rangle \in I^{n+1}(W(k))$ (see Definition 2.4.2). So, $q \cdot \langle\langle a_1, \ldots, a_n \rangle\rangle \in I^{n+1}(W(k)) \Leftrightarrow \langle\langle a_1, \ldots, a_n \rangle\rangle \in I^{n+1}(W(k)) \Leftrightarrow Q_\alpha$ has a rational point (by Theorem 2.4.5). So, if $q \cdot \langle\langle a_1, \ldots, a_n \rangle\rangle$ is hyperbolic, then Q_α has a rational point. And, conversely, if Q_α has a rational point, then it is hyperbolic (by Theorem 2.4.5), and so is $q \cdot \langle\langle a_1, \ldots, a_n \rangle\rangle$. So, by Theorem 2.4.18 , $\mathcal{X}_{\dim(R)/2 - j} = \mathcal{X}_{Q_\alpha}$.

A should be a nontrivial extension of the specified above objects, or $\mathcal{X}_{R^{\dim(R)/2}}(\dim(R)/2)[\dim(R)]$ will be a direct summand in the motive of R, but we know that $\mathrm{Hom}(\mathcal{X}_{Q_\alpha}, \mathbb{Z}(2^{n-1}-1)[2^n-1]) = \mathbb{Z}/2$ (easily follows from the fact that M_α is an extension of $\mathcal{X}_{Q_\alpha}$ and $\mathcal{X}_{Q_\alpha}(2^{n-1}-1)[2^n-2]$ (see [20], Theorem 4.4), see the cited place), and *pure motives* (i.e.: direct summands in the motives of smooth projective varieties) do not have such cohomology groups (by Theorem 2.1.18). And as I mentioned, there is only one nontrivial element in $\mathrm{Hom}(\mathcal{X}_{Q_\alpha}, \mathcal{X}_{Q_\alpha}(2^{n-1}-1)[2^n-1]) = \mathrm{Hom}(\mathcal{X}_{Q_\alpha}, \mathbb{Z}(2^{n-1}-1)[2^n-1])$ (by Theorem 2.3.2).

Hence, $A = M_\alpha(\dim(R)/2 - 2^{n-1}+1)[\dim(R) - 2^n + 2]$.

Denote $F_{\{a_1,\dots,a_n\}}(Q) := S$.

Theorem 4.1 is proven.

□

It seems to me that the following question could have the positive answer (at least, I don't have any counterexample at the moment).

Question 4.7 .

Is it true that any undecomposable direct summand in the motive of a quadric is of one of the two following types:

1) *Tate motive:* $\mathbb{Z}(i)[2i]$*;*
2) *Higher form of a quadric (shifted):* $F_\alpha(Q)(j)[2j]$*, where* $\alpha \in \mathrm{K}^M_n(k)/2$ *is pure and* Q *is some undecomposable quadric (certainly, these conditions do not guarantee that higher form will be undecomposable, but they are necessary).*

?

5. Back from $\mathcal{X}$'s to quadratic forms

Using the same methods as in Corollary 3.15 , we can generalize it to the case of arbitrary quadric. This shows that "elementary pieces" from Theorem 3.1 define the motive of a quadric uniquely. Since those "elementary pieces" contain just the same information as *Universal Splitting Tower* of Manfred Knebusch (see Definition 2.4.19), we get that the motive of a quadric is defined by the equivalence class of $UST(Q)$ (see Definition 2.4.19 for the definition of equivalence). In many interesting cases we can compare UST for two quadrics, which gives isomorphism for their motives.

Proposition 5.1 .

Let Q_1*,* Q_2 *be projective quadrics, s.t.* $\dim(Q_1) = \dim(Q_2)$*. Then the following conditions are equivalent:*

1) $M(Q_1) = M(Q_2)$.
2) $\mathcal{X}_{Q_1} = \mathcal{X}_{Q_2}$, $\mathcal{X}_{Q_1^1} = \mathcal{X}_{Q_2^1}, \dots, \mathcal{X}_{Q_1^{[\dim(Q_1)/2]}} = \mathcal{X}_{Q_2^{[\dim(Q_2)/2]}}$.
3) *Universal splitting towers for* Q_1 *and* Q_2 *are equivalent (see Definition 2.4.19).*

Proof

(1 $\Rightarrow$ 2) It follows from the proof of Theorem 3.7 .

(2 $\Leftrightarrow$ 3) Follows from Theorem 2.4.18 .

(2 $\Rightarrow$ 1)

Let $\dim(Q) = n$. Denote as I the set of such $0 \leqslant i < n/2$ that there exist no direct summands in $M(Q_1)$, which are isomorphic to some direct summand in $M(Q_2)$ and which contain $\mathcal{X}_{Q_1^i}(i)[2i]$ (as usually, we say that some direct summand N of Q contains $\mathcal{X}_{Q^i}(i)[2i]$ if for the corresponding idempotent p_N we have $(p_N)_i = 1$ (see Theorem 3.7 for the definition), see also Lemma 3.23). By duality, I will be also the set of such j that there exist no direct summands in $M(Q_1)$, which are isomorphic to some direct summand in $M(Q_2)$ and which contain $\mathcal{X}_{Q_1^j}(\dim(Q_1)-j)[2\dim(Q_1)-2j]$ (in the sense of Lemma 3.23). We want to show that I is empty. Let i_0 will be smallest element of I.

For any $j \notin I$ there exist idempotents $u_j\{1\}$, $u'_j\{1\} \in \mathrm{End}(Q_1)$, s.t. $(u_j\{1\})_j = 1$ and $(u'_j\{1\})'_j = 1$, and corresponding direct summands are isomorphic to some direct summands in $M(Q_2)$. Denote the set of such $u_j\{1\}$, $u'_j\{1\}$'s as S. So, we have $u_s\{1\}$, $s \in S$.

Consider $U\{1\} = \prod_{s\in S}(1 - u_s\{1\})$ (we choose some order on S (in which we multiply u_s)). Evidently, $U\{1\}_j = U\{1\}'_j = 0$, if $j \notin I$, and $U\{1\}_i = U\{1\}'_i = 1$ for all $i \in I$.

So, by Lemma 3.24 , we have a direct summand M_1 in Q_1 (I will write Q instead of $M(Q)$ from this point), which is an extension of all $\mathcal{X}_{Q_1^i}(i)[2i]$, $\mathcal{X}_{Q_1^i}(\dim(Q_1) - i)[2\dim(Q_1) - 2i]$, $i \in I$, and may be something from $Q_1\langle n/2\rangle$, see Lemma 3.23 .

The same can be done with Q_2.

Denote $N = M(-i_0)[-2i_0]$, and $\dim(N) := \dim(Q) - 2i_0$.

Since $\mathcal{X}_{\underline{Q}_1^{i_0}} = \mathcal{X}_{\underline{Q}_2^{i_0}}$, we have morphisms: $v : \underline{Q}_1^{i_0} \to \underline{Q}_2^{i_0}$ and $w : \underline{Q}_2^{i_0} \to \underline{Q}_1^{i_0}$ which are of degree 1 on zero-dimensional cycle $\mathbb{Z} \times \cdots \times \mathbb{Z}$ (by Theorem 2.4.18 and duality).

From Claim 3.2 we know that $\underline{Q}^{i_0} = Q \times Q\langle 1\rangle \times \cdots \times Q\langle i_0\rangle$.

By Sublemma 4.6we get morphisms $\rho : N \to Q \times Q\langle 1\rangle \times \cdots \times Q\langle i_0 - 1\rangle \times Q\langle i_0\rangle$, s.t. over $\overline{k}$ $\mathbb{Z}$ goes to $\mathbb{Z} \times \cdots \times \mathbb{Z}$ identically, and $\nu : Q \times Q\langle 1\rangle \times \cdots \times Q\langle i_0 - 1\rangle \times Q\langle i_0\rangle \to N$, which makes just the opposite.

Consider the following morphism from N_1 to N_2:

$\varphi_{1\to2} : N_1 \stackrel{\rho_1}{\to} Q_1 \times Q_1\langle 1\rangle \times \cdots \times Q_1\langle i_0\rangle \stackrel{v}{\to} Q_2 \times Q_2\langle 1\rangle \times \cdots \times Q_2\langle i_0\rangle \stackrel{\nu_2}{\to} N_2$.

Analogously, one can construct the opposite morphism $\varphi_{2\to1}$.

We, evidently, have: $(\varphi_{2\to1} \circ \varphi_{1\to2})_0 = (\varphi_{1\to2} \circ \varphi_{2\to1})_0 = 1$. This gives us that some direct summand of N_1 is isomorphic to some direct summand of N_2(by Lemma 3.25). We get a contradiction if I is not empty.

So, I is empty, and $S = [0, \ldots, [n-1/2]] \cup [[n-1/2]', \ldots, 0']$. Let (in the notations as above) $u_s\{1\} = u_s^{\leftarrow} \circ u_s^{\rightarrow}$; $\quad u_s\{2\} = u_s^{\rightarrow} \circ u_s^{\leftarrow}$.

For every subset $J \subset S$ choose some $s_J \in J$, and let $J' = J \backslash s_J$. Choose some order on each J'.

Put $U^{\rightarrow} = \sum_{J \subset S} (-1)^{\#(J)-1} u^{\rightarrow}_{s_J} \prod_{s \in J'} u_s\{1\}$, and
$U^{\leftarrow} = \sum_{J \subset S} (-1)^{\#(J)-1} u^{\leftarrow}_{s_J} \prod_{s \in J'} u_s\{2\}$ (product in the order chosen).

Evidently, $(U^{\rightarrow} \circ U^{\leftarrow})_i = (U^{\rightarrow} \circ U^{\leftarrow})'_i = (U^{\leftarrow} \circ U^{\rightarrow})_i = (U^{\leftarrow} \circ U^{\rightarrow})'_i = 1$, for all $0 \leqslant i < n/2$.

Hence, by Lemma 3.24 , we have that $U^{\rightarrow}$ and $U^{\leftarrow}$ give us an isomorphism of some direct summands L_1 in Q_1 and L_2 in Q_2, where L_1 contains $\mathcal{X}_{Q^i_1}(i)[2i]$ and $\mathcal{X}_{Q^i_1}(n-i)[2n-2i]$ for all $0 \leqslant i < n/2$ (in the sense of Lemma 3.23), and L_2 contains $\mathcal{X}_{Q^i_2}(i)[2i]$ and $\mathcal{X}_{Q^i_2}(n-i)[2n-2i]$ for all $0 \leqslant i < n/2$.

The complement summands: $L_1^{\perp}$ and $L_2^{\perp}$ could be nontrivial only if n is even, and then they could contain only $\mathcal{X}_{Q_1^{n/2}}(n/2)[n]$, or $\mathcal{X}_{Q_1^{n/2}}(n/2)[n] \times k\sqrt{\det(Q_1)}$, and $\mathcal{X}_{Q_2^{n/2}}(n/2)[n]$, or $\mathcal{X}_{Q_2^{n/2}}(n/2)[n] \times k\sqrt{\det(Q_2)}$, respectively. Since $\det(Q_1) = \det(Q_2)$ (because $k\sqrt{\det(Q_t)} = k(Q_t^{n/2}) \cap \overline{k}$), and $\mathcal{X}_{Q_1^{n/2}} = \mathcal{X}_{Q_2^{n/2}}$, those summands will be also isomorphic.

So, $M(Q_1) = M(Q_2)$.

Proposition 5.1 is proven.

□

Remark Using almost the same arguments as in the proof above one can get *Rost decomposition* for a motive of a *Pfister neighbor* (i.e. subquadric of dimension $\geqslant 1/2$ in a *big Pfister quadric*) (see [14], Proposition 4).

Corollary 5.2 .

a) Let q_1 and q_2 are Pfister half-neighbors (see Definition 2.4.6), i.e. such forms that $q_1 \perp q_2 = q_\alpha$ and $\dim(q_1) = \dim(q_2)$ ($\alpha \in \mathrm{K}^M_n(k)/2$-pure). Then $M(Q_1) = M(Q_2)$. (This is, in some sense, the extreme case of the Remark above.)

b) Let for some $h \in \mathrm{K}^M_n(k)/2$ of length 2 we have two representations: $h = \alpha_1 + \beta_1 = \alpha_2 + \beta_2$ with α_i, β_j-pure; and A_1, A_2 - corresponding Albert quadrics (i.e. quadrics corresponding to the forms $q_{\alpha_i} \perp -q_{\beta_i}$, see Definition 2.4.7). Then $M(A_1) = M(A_2)$.

Proof a) Over $k(Q_i)$ the *Pfister form* q_α get a rational point, and so, becomes hyperbolic, which means that $q_1|_{k(Q_i)} = -q_2|_{k(Q_i)}$, and *Universal splitting towers* (see Definition 2.4.19) for q_1 and q_2 are equivalent.

b) By result of R.Elman and T.Y.Lam (see Theorem 2.4.8): q_i is 2^s-times isotropic iff α_i and β_i have common *pure* divisor of degree s (and these are the only possible values of *isotropity index* (i.e.: the number of *elementary hyperbolic* summands in A_i)), and iff h itself is divisible by *pure* symbol of degree s (which does not depend on the choice of i, and is itself very interesting fact, expressing some outstanding properties of the ring $\mathrm{K}^M_*(k)/2$). Hence, A_1^j has a rational point iff A_2^j does.

U.s.towers are equivalent, and motives are isomorphic.

$\square$

Remark If we would be able to prove that the motive of a quadric defines quadric itself up to isomorphism, then we would have affirmative answer to the question of T.Y.Lam (see [10], (6.6) on p.28): Are two *Albert* forms, corresponding to the same symbol of *length* 2 proportional?

6. Few remarks about the Witt numbers

Let q/k be an anisotropic quadratic form. We have: $q|_{k(Q)} = h_1 \cdot H \perp q_1$, where q_1 is some anisotropic form defined over $k(Q)$; $q_1|_{k(Q)(Q_1)} = h_2 \cdot H \perp q_2$; etc. ... Continuing this way we get *Witt numbers*: $h_1, h_2, \ldots, h_s$.

This numbers contain the same information as our set $J(Q)$ (introduced in the very end of Section 1.1).

We can apply methods from previous sections to get some information on the possible behavior of the *Witt numbers*.

Statement 6.1 .

Suppose our field k contains $\sqrt{-1}$.

Suppose that $h_1 > h_i$ for every $i > 1$. Then $\mathcal{X}_Q$ is hooked only to $\mathcal{X}_Q(\dim(Q) - h_1 + 1)[2\dim(Q) - 2h_1 + 2]$ (i.e. there is a direct summand in the motive of Q, which is an extension of $\mathcal{X}_Q$ and $\mathcal{X}_Q(\dim(Q) - h_1 + 1)[2\dim(Q) - 2h_1 + 2]$), and $\dim(Q) - h_1 = 2^m - 2$ for some m .

Proof

From Lemma 4.5 it follows that for each $0 \leqslant j < h_1$ we have an undecomposable direct summand N_j in Q, which contains $\mathcal{X}_Q(j)[2j]$ (in the sense of Lemma 3.23) and does not contain any $\mathcal{X}_Q(j')[2j']$, for any $0 \leqslant j' < h_1$, $j' \neq j$. Moreover, $N_j = N_0(j)[2j]$. Since $h_1 > h_i$ for every $i > 1$, the only way it could be possible is: N does not contain any $\mathcal{X}_{Q^l}(l)[2l]$ and $\mathcal{X}_{Q^l}(\dim(Q)-l)[2\dim(Q)-2l]$ for $l \geqslant h_1$. So, $\mathcal{X}_Q$ is hooked (in the sense of Definition 3.22) only to something from *outer shell* (I mean to some $\mathcal{X}_{Q^j}(*)[2*]$ with $\mathcal{X}_{Q^j} = \mathcal{X}_Q$); and using again the fact that $N_j = N_0(j)[2j]$, we get that it is hooked only to $\mathcal{X}_Q(\dim(Q)-h_1+1)[2\dim(Q)-2h_1+2]$ (it should be hooked to something if Q is anisotropic (if the natural morphism $Q \to \mathcal{X}_Q$ has splitting, then the projection $Q \to \mathbb{Z}$ has splitting as well (by Theorem 2.3.2), which means that Q has a zero-cycle of degree 1, then Q has a rational point by Springer's theorem)). So, we get some *pure motive* $N = \mathrm{Cone}[-1](\mathcal{X}_Q \to \mathcal{X}_Q(\dim(Q)-h_1+1)[2\dim(Q)-2h_1+3])$. Denote $n := \dim(Q) - h_1 + 1$.

$N := Cone[-1](\mathcal{X}_Q \xrightarrow{\mu} \mathcal{X}_Q(n)[2n+1])$, where μ is the only nontrivial element from $\mathrm{Hom}(\mathcal{X}_Q, \mathcal{X}_Q(n)[2n+1])$ (consider Hom's from our exact triangle to $\mathcal{X}_Q(n)[2n+1]$, and take into account that N is a direct summand in $Q(-h_1+1)[-2h_1+2]$); we have that N is a direct summand in Q. In particular, $\mathrm{Hom}(N, \mathbb{Z}(a)[b]) = 0$, for

$b > 2a$, or $b - a > n$ (here we need inclusion into $Q(-h_1+1)[-2h_1+2]$). Considering an exact triangle $N \to \mathcal{X}_Q \to \mathcal{X}_Q(n)[2n+1]$, it is easy to see that multiplication by μ performs an isomorphism: $\mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}(c)[d]) \to \mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}(c+n)[d+2n+1])$ for any $c > 0, d - c > 0$ (since $\mathrm{Hom}(Q, \mathbb{Z}(i)[j]) = 0$ for $j - i > \dim(Q)$ (see Theorem 2.1.18), and N is a direct summand in $Q(-h_1+1)[-2h_1+2]$).

For finite field extension E/k we have action of *transfers* on motivic cohomology. Transfer is a map $\mathrm{H}^{b,a}_{\mathcal{M}}(X|_E, \mathbb{Z}) = \mathrm{H}^{b,a}_{\mathcal{M}}(X, E) \to \mathrm{H}^{b,a}_{\mathcal{M}}(X, \mathbb{Z})$, where the first equality is duality (see Theorem 2.1.26), and the second is induced by the projection $\pi : E \to \mathbb{Z}$ (E-finite field extension)(see also Definition 2.1.24); the main property of transfer is: $\mathrm{Tr} \circ j = \cdot[E : k]$, where $j : \mathrm{H}^{b,a}_{\mathcal{M}}(X, \mathbb{Z}) \to \mathrm{H}^{b,a}_{\mathcal{M}}(X, E) = \mathrm{H}^{b,a}_{\mathcal{M}}(X|_E, \mathbb{Z})$ is induced by the map $j : \mathbb{Z} \to E$ dual to π via duality (see Theorem 2.1.26).

Consider $\widetilde{\mathcal{X}}_Q := \mathrm{Cone}[-1](\mathcal{X}_Q \to \mathbb{Z})$ (see Definition 2.3.8).

Quadric has a point E of degree 2, and over E, $\mathcal{X}_Q$ becomes $\mathbb{Z}$ (see Theorem 2.3.4), we have by Theorem 2.1.18 (see also Definition 2.1.24) that $\mathrm{Hom}(\mathcal{X}_Q|_E, \mathbb{Z}(a)[b]) = 0$ for $b - a > 0$.

Also we have that $\widetilde{\mathcal{X}}_Q|_E = 0$, and so, $\mathrm{Hom}(\widetilde{\mathcal{X}}_Q|_E, \mathbb{Z}(a)[b]) = 0$ for all a, b.

Now, considering the composition $\mathrm{Tr} \circ j = \cdot[E : k]$ we get that $\mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}(a)[b])$ is a 2-torsion group for $b > a$, and $\mathrm{Hom}(\widetilde{\mathcal{X}}_Q, \mathbb{Z}(a)[b])$ is a 2-torsion group for all a and b.

In particular, we get that $\mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}(a)[b])$ embeds into $\mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}/2(a)[b])$ for $b > a$, and $\mathrm{Hom}(\widetilde{\mathcal{X}}_Q, \mathbb{Z}(a)[b])$ embeds into $\mathrm{Hom}(\widetilde{\mathcal{X}}_Q, \mathbb{Z}/2(a)[b])$ for all a and b.

Moreover, for $b > a$ $\mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}/2(a)[b])$ coincides with $\mathrm{Hom}(\widetilde{\mathcal{X}}_Q, \mathbb{Z}/2(a)[b])$ (by Theorem 2.1.18).

From Theorem 2.3.11 we know, that the differential Q_i acts without cohomology on $\mathrm{Hom}(\widetilde{\mathcal{X}}_Q, \mathbb{Z}/2(*)[*'])$ for any $i \leqslant [\log_2(n+1)]$ (see Theorem 2.1.26), Denote $\eta := \mu(mod 2)$, i.e. the image of μ in cohomology with $\mathbb{Z}/2$ coefficients.

$Q_i(\eta) = 0$, for all $i \leqslant [\log_2(n)]$. Really, $Q_i(\eta) \in \mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}/2(n+2^i-1)[2n+2^{i+1}])$, and the later group is an extension of 2-cotorsion in $\mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}(n+2^i-1)[2n+2^{i+1}])$, and 2-torsion in $\mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}(n+2^i-1)[2n+2^{i+1}+1])$.

But as was shown above, μ performs a surjections $\mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}(2^i-1)[2^{i+1}-1]) \to \mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}(n+2^i-1)[2n+2^{i+1}])$, and $\mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}(2^i-1)[2^{i+1}]) \to \mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}(n+2^i-1)[2n+2^{i+1}+1])$.

And the groups: $\mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}(2^i-1)[2^{i+1}-1])$, $\mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}(2^i-1)[2^{i+1}])$ are zero.

Really, from the exact triangle $N \to \mathcal{X}_Q \to \mathcal{X}_Q(n)[2n+1]$, we have exact sequence: $\mathrm{Hom}(N, \mathbb{Z}(2^i-1)[2^{i+1}-1]) \leftarrow \mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}(2^i-1)[2^{i+1}-1]) \leftarrow \mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}(2^i-1-n)[2^{i+1}-2n])$ The first group is zero since N is a *pure* motive (i.e.: a direct summand in the motive of some smooth projective variety), and (consequently) $\mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}(a)[b]) = 0$ for $b > 2a$ (see Theorem 2.1.18). The third group is zero by Theorem 2.3.3 (since $n > 2^i - 1$). So, the second is zero as well. Analogously, in the case of $\mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}(2^i-1)[2^{i+1}])$.

Denote $k = [\log_2(n)]$. We have $\eta = Q_i(x_i)$, for all $1 \leqslant i \leqslant k$. Let us prove that there exists such $\gamma \in \mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}/2(n-2^{k+1}+2+k)[2n-2^{k+2}+4+k])$, that $\eta = Q_k \circ Q_{k-1} \circ \cdots \circ Q_1 \circ Q_0(\gamma)$.

Multiplication by μ is an isomorphism on "positive" diagonals ("number of the diagonal" is the difference between "square" [-] and "round" (-) number)(see above); $\mu(mod2) = \eta = Q_i(x_i)$; and Q_j's does not act from under the first diagonal to above it (since by Theorem 2.3.9 , the natural projection $\mathcal{X}_Q \to \mathbb{Z}$ defines an isomorphism $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/2(a)[b]) \to \mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}/2(a)[b])$, and since Q_j is functorial with respect to the morphisms of simplicial schemes, we have that the images of such an action would come from $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/2(c)[d])$ with $d > c$; but such groups are trivial (by Theorem 2.1.18)). From all this: Q_j is a monomorphism on the diagonals from $n+2$ to $n+2^j+1$. Really, if $Q_j(\overline{y}) = 0$ for $\overline{y}$ from such a diagonal, then $\overline{y}$ is in the image of Q_j, but by Theorem 2.1.26 (6), in such case $\overline{y}$ comes from y - element of integral cohomology, then $y = \mu \cdot z$, where $z \in \mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}(a)[b])$ with $1 \leqslant b-a \leqslant 2^j$, and $Q_j(\overline{\mu \cdot z}) = \eta \cdot Q_j(\overline{z})$ (since $\eta = Q_j(x_j)$, and by Theorem 2.1.26 (5) (we should remind that ρ is a class of $\{-1\} \in \mathrm{K}_1^M(k)/2 = \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/2(1)[1])$, see Definition 2.1.25 , and it is 0 in our case)). But on the diagonals from 1-st to 2^j-th the Q_j is injective (since it does not act from under the first diagonal to above it, as was explained above). And multiplication by $\eta = \overline{\mu}$ is monomorphic on the diagonals beginning from the 1-st. Contradiction.

Let us prove by induction on m that $\eta = Q_m \circ Q_{m-1} \circ \cdots \circ Q_1 \circ Q_0 \eta_m$, for some η_m. The first step is trivial since $Q_0\eta = bokst(\eta) = 0$, since η comes from integral cohomology. But now, $Q_m(\eta_{m-1})$ lives on the $n+2$-nd diagonal, and, hence, is equal to 0 (since Q_m is monomorphic on such a diagonal and $(Q_m)^2 = 0$). So, $\eta_{m-1} = Q_m(\eta_m)$.

Finally, we get: $\eta = Q_k \circ \cdots \circ Q_1 \circ Q_0(\gamma)$, and $\gamma \in \mathrm{Hom}(\mathcal{X}_Q, \mathbb{Z}/2(n-2-2^{k+1}+2+k)[2n-2^{k+2}+4+k])$.

But, $(2n-2^{k+2}+4+k)-(n-2^{k+1}+2+k) = n-2^{k+1}+2$, and $k = [\log_2(n)]$, so: $2^k \leqslant n < 2^{k+1}$. Since we know, that Q_i's don't act from under the first diagonal to above it, the only possible choice for n is $n = 2^{k+1}-1$.

□

Remark 0 The condition $\sqrt{-1} \in k$ is added only because of the lack of the appropriate reference (in Theorem 2.1.26 it is not specified that φ_j, ψ_j are expressible in terms of Q_t's, though they are).

Remark 1

I think, one can prove that the condition: $h_1 > h_i$, $\forall i > 1$ in Statement 6.1 can be replaced by: h_i is not divisible by h_1 $\forall i > 1$.

Another observation is a some step on the road paved by the *J-filtration conjecture*. The *J-filtration conjecture* states basically, that for $q \in I^n \backslash I^{n+1}$, $j_1(Q) = 2^{n-1}$. This conjecture was settled in [13], Statement 2 of Section 3.3. Now I can move a bit further and make some attempt to describe: what $j_2(Q)$ stays for (see the definition of j_i in the very end of Section 3.1). Consider the natural map $\pi : I^n(W(k)) \to I^n(W(k))/I^{n+1}(W(k)) = \mathrm{K}_n^M(k)/2$ (by the result of V.Voevodsky (Milnor's conjecture on quadratic forms), see [13], p.14).

Statement 6.2 .
Suppose our field k contains $\sqrt{-1}$.
Let $q \in I^n \backslash I^{n+1}$ (i.e.: $j_1(Q) = 2^{n-1}$).
Then $j_2(Q) \in \{2^r - 2^{n-1}, r > n;\ 2^n;\ 2^{n-1} + 2^m, m < n-1\}$ (see the very end of Section 3.1 for the definition of j_i).
Moreover, for $j_2(Q) < 2^n$ we have: $\pi(q)$ is not pure (see Definition 2.4.4).

Proof
Since we are interested only in j_1 and j_2, we can assume that $\#J(Q) = 2$ (just change k by $k(Q^{\dim(Q)/2-j_2})$ and q by $(q|_{k(Q^{\dim(Q)/2-j_2})})_{anis.}$). Now, $j_2 - j_1 = h_1$. So, by Statement 6.1 we have that if $j_2 > 2^n$, then $\dim(Q) - j_2 + j_1 = 2^r - 2$. Since $\dim(Q) = 2j_2 - 2$ (because $\#J(Q) = 2$), we get: $j_2(Q) = 2^r - j_1 = 2^r - 2^{n-1}$ (since $j_1 = 2^{n-1}$ by the J-filtration conjecture (see discussion above)).
In the case $j_2 < 2^n$, using Lemma 4.5 , we get that if $h_2 = j_1$ is not divisible by $h_1 = j_2 - j_1$, then we will have a direct summand in the motive of Q, consisting of $\mathcal{X}$'s only from the *inner shell*, (i.e. of $\mathcal{X}_{Q^i}(*)[2*]$'s with $\mathcal{X}_{Q^i} = \mathcal{X}_{Q^{\dim(Q)/2}}$).
Evidently, if such summand contains $\mathcal{X}_{Q^i}(i)[2i]$, then it contains $\mathcal{X}_{Q^{\dim(Q)-(i+j_1-1)}}(i+j_1-1)[2i+2j_1-2]$ as well (since the same holds over $k(Q^{\dim(Q)-j_1})$, where anisotropic part of Q is just *Pfister quadric*).
Again, using Lemma 4.5 , we get that the entire *inner shell* will be a direct summand in Q, and, so, *outer shell* will be a direct summand (i.e. there is a direct summand, consisting (in the sense of Lemma 3.23) of $\mathcal{X}_{Q^i}(*)[2*]$'s with $\mathcal{X}_{Q^i} = \mathcal{X}_Q$) and a *pure motive* as well. The proof of Statement 6.1 shows that in this case $\dim(Q) - h_1 + 1 = 2^m - 2$, which is impossible if $j_2 < 2^n$.
So, h_2 is divisible by h_1, and we get desired choice for $j_2(Q)$.
For the last question, changing k by $k(Q^{\dim(Q)/2-j_2})$, we can again assume that $\#J(Q) = 2$ (if $h \in \mathrm{K}_n^M(k)/2$ and for some extension E/k: $h|_E$ is *non-pure*, then h is *non-pure*).
Suppose for $j_2(Q) < 2^n$ we would have $\pi(q) = \alpha$ is *pure symbol*, then, $\mathcal{X}_{Q^{\dim(Q)/2}} = \mathcal{X}_{Q_\alpha}$, where Q_α is a *Pfister quadric*, corresponding to α.
Really, it follows from the fact that $J(Q)$ does not contain any number of the form 2^s except $j_1 = 2^{n-1}$, since by J-filtration conjecture: $\pi(q)|_E = 0 \Leftrightarrow j_1(Q|_E) = 2^s$, for some $s > n-1$, or $q|_E$ is hyperbolic (one could say: $s = \infty$ in this case), and since $J(Q|_E) \subset J(Q)$, in our situation only the last case is possible, so we get: q is hyperbolic over some extension E/k iff $\pi(q)|_E = 0$. Existence of a rational point on

$Q^{\dim(Q)/2}$ is equivalent to the fact that Q is hyperbolic. So, by definition we have: $\mathcal{X}_{Q^{\dim(Q)/2}} = \mathcal{X}_{Q_\alpha}$.

Then we would have that the *inner shell* is a direct summand. Really, we know that $Q\langle h_1\rangle = \mathrm{Cone}[-1](\oplus_{i=0,\dots,2^{n-1}-1}\mathcal{X}_{\dim(Q)/2}(i)[2i] \to \oplus_{i=0,\dots,2^{n-1}-1}\mathcal{X}_{\dim(Q)/2}(2^n-2-i)[2^{n+1}-3-2i])$ (by Theorem 3.1 and Theorem 2.5.10 (with $X = Q\langle h_1\rangle(h_1)[2h_1]$, $X_m = 0$)) $= \mathrm{Cone}[-1](\oplus_{i=0,\dots,2^{n-1}-1}\mathcal{X}_{Q_\alpha}(i)[2i] \to \oplus_{i=0,\dots,2^{n-1}-1}\mathcal{X}_{Q_\alpha}(2^n-2-i)[2^{n+1}-3-2i])$ (by what was just proven).

Let $\alpha = \pi(q)$ - our *pure symbol.* We have a decomposition of the Rost motive M_α: $M_\alpha = \mathrm{Cone}[-1](\mathcal{X}_{Q_\alpha} \xrightarrow{\mu} \mathcal{X}_{Q_\alpha}(2^{n-1}-1)[2^n-1])$, for some morphism $\mu \in \mathrm{Hom}(\mathcal{X}_{Q_\alpha}, \mathcal{X}_{Q_\alpha}(2^{n-1}-1)[2^n-1]) = \mathrm{Hom}(\mathcal{X}_{Q_\alpha}, \mathbb{Z}(2^{n-1}-1)[2^n-1])$ (equality by Theorem 2.3.2) (see [14], Proposition 4 and [18] Theorem 4.5). And M_α is a direct summand in the motive of *small Pfister* quadric of dimension $2^{n-1}-1$ (see [14], Proposition 4; and Definition 2.4.6).

Considering the Hom's from the exact triangle $M_\alpha = \mathrm{Cone}[-1](\mathcal{X}_{Q_\alpha} \xrightarrow{\mu} \mathcal{X}_{Q_\alpha}(2^{n-1}-1)[2^n-1])$ to $\mathbb{Z}(k)[2k+1]$, and using the fact that $\mathrm{Hom}(M_\alpha, \mathbb{Z}(k)[2k+1]) = 0$ for all k (since M_α is *pure motive* (i.e.: a direct summand in the motive of a smooth projective variety)) (see Theorem 2.1.18), we get that the map: $\mathrm{Hom}(\mathcal{X}_{Q_\alpha}, \mathbb{Z}(k-2^{n-1}+1)[2k-2^n+2]) \to \mathrm{Hom}(\mathcal{X}_{Q_\alpha}, \mathbb{Z}(k)[2k+1])$ is surjective. In particular, $\mathrm{Hom}(\mathcal{X}_{Q_\alpha}, \mathbb{Z}(k)[2k+1]) = 0$ for $k < 2^{n-1}-1$ (by Theorem 2.3.3). That means that $\mathcal{X}_{Q_\alpha}(2^{n-1}-1)[2^n-2]$ - the most upper summand of the lower part of $Q\langle h_1\rangle$ can be hanged only to $\mathcal{X}_{Q_\alpha}(2^n-2)[2^{n+1}-4]$, and the extension is given by μ, which is the only nontrivial element of the group $\mathrm{Hom}(\mathcal{X}_{Q_\alpha}, \mathbb{Z}(2^{n-1}-1)[2^n-1])$ (see surjection above + Theorem 2.3.2 + Theorem 2.1.20 + the fact that our group has exponent 2 (by transfer arguments (see the proof of Statement 6.1 for the definition of transfer)). But then the Cone$[-1]$ of this extension will be just Rost motive (shifted by $(2^{n-1}-1)[2^n-2]$)(see above), so, it is *pure*, and consequently, a direct summand in $Q\langle h_1\rangle$ (since there are no 1-st Ext's from M_α to $\mathcal{X}_{Q_\alpha}(k)[2k]$, as well as from $\mathcal{X}_{Q_\alpha}$ to $M_\alpha(l)[2l]$; really, $\mathrm{Hom}(M_\alpha, \mathcal{X}_{Q_\alpha}(k)[2k+1]) = \mathrm{Hom}(M_\alpha, \mathbb{Z}(k)[2k+1])$ (by Theorem 2.3.2) $= 0$ since M_α is *pure* by Theorem 2.1.18 , and $\mathrm{Hom}(\mathcal{X}_{Q_\alpha}, M_\alpha(l)[2l+1]) \subset \mathrm{Hom}(\mathcal{X}_{Q_\alpha}, Q_{small}(l)[2l+1])$ (since M_α is a direct summand in Q_{small}) $= \mathrm{Hom}(\mathcal{X}_{Q_\alpha} \times Q_{small}, \mathbb{Z}(l+2^{n-1}-1)[2l+2^n-1])$ (by duality Theorem 2.1.23) $= \mathrm{Hom}(Q_{small}, \mathbb{Z}(l+2^{n-1}-1)[2l+2^n-1])$ (since $\mathcal{X}_{Q_{small}} = \mathcal{X}_{Q_\alpha}$ by Theorem 2.4.21) $= 0$ (by Theorem 2.1.18)).

So, by Lemma 4.5 $Q\langle h_1\rangle$ is a direct sum of $M_\alpha(i)[2i],\ i = 0,\dots,2^{n-1}-1$.

So, our *inner shell* $Q\langle h_1\rangle(h_1)[2h_1]$ is a *pure motive* and a direct summand in Q, and the complement summand will be *outer shell.*

Hence, we get that *outer shell* is a direct summand, which is impossible if $j_2(Q) < 2^n$ (in the proof of Statement 6.1 we use just the fact that *outer shell* is a direct summand).

Statement is proven.

$\square$

Remark Actually, quadrics with $j_2(Q) = 2^{n-1} + 2^m$ with $m < n-2$ should not exist, but it is not proven.

The j_2 analog of the *J-filtration conjecture* would be the following question:

Question 6.3 .

Let $q \in I^n \backslash I^{n+1}$, then:

a) *Are the following conditions equivalent ?*
1. *$\pi(q)$ is not pure.*
2. *$j_2(Q) = 3 \cdot 2^{n-2}$.*

b) *Are the following conditions equivalent ?*
1. *$\pi(q)$ is pure.*
2. *$j_2(Q) = 2^n$, or $j_2(Q) = 2^k - 2^{n-1}$, $k > n$.*

In connection with the above question we can state the following hypothetical description of quadrics with $\#J(Q) = 2$ (It seems that M.Knebusch understood it, and was able to prove it in some cases (see [7])):

Question 6.4 .

Is it true that any quadric Q with $\#J(Q) = 2$ is of one of the three following types:

1) $q = \langle\langle a_1, \ldots, a_{n-2}\rangle\rangle \cdot \langle b, c, -bc, -d, -e, de\rangle$, *where* $\pi(q) = \{a_1, \ldots, a_{n-2}\} \cdot (\{b, c\} + \{d, e\}) \in \mathrm{K}_n^M(k)/2$ *is not pure (length of* $\pi(q)$ *is* 2*).* $j_2(Q) = 3 \cdot 2^{n-2}$;
2) $q = \langle\langle a_1, \ldots, a_{n-1}\rangle\rangle \cdot \langle b, -c, -d, cd\rangle$, *where* $\pi(q) = \{a_1, \ldots, a_{n-1}, b\} \neq 0$, *and* $\{a_1, \ldots, a_{n-1}, c, d\}$ *is not divisible by* $\{a_1, \ldots, a_{n-1}, b\}$ *(*$\pi(q)$ *here is pure).* $j_2(Q) = 2^n$;
3) $q = \langle\langle a_1 \ldots, a_n\rangle\rangle \cdot (\langle\langle b_1, \ldots, b_k\rangle\rangle - \langle 1\rangle)$ *(here, again,* $\pi(q) = \{a_1, \ldots, a_n\}$ *is pure).* $j_2(Q) = 2^{n+k-1} - 2^{n-1}$.

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