SPECTRAL PARAMETERIZATION FOR THE POWER SUMS OF QUANTUM SUPERMATRIX

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Abstract. A parameterization for the power sums of $GL(m|n)$ type quantum (super)matrix is obtained in terms of it's spectral values.

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1. Introduction

This paper is a complement to our previous works [GPS1, GPS2, GPS3] devoted to the quantum matrix algebras (QMA) of $GL(m|n)$ type. Here we continue investigation of the commutative characteristic subalgebra of the QMA. To be more precise, for the set of 'quantum' traces of 'powers' of the quantum (super)matrix we find out a parameterization in terms of spectral values which are the quantum analogs of the set of (super)matrix eigenvalues. Note, that the abovementioned set of quantum traces generates the characteristic subalgebra. To illustrate the statements let us briefly recall the corresponding facts from the classical matrix algebra.

As is well known, any $N \times N$ complex matrix $M \in \mathrm{Mat}_N(\mathbb{C})$ satisfies a polynomial Cayley-Hamilton (or, characteristic) identity, which can be presented in a factorized form

$$\prod_{i=1}^{N}(M - \mu_i I) = 0\,,$$

where I is a unit matrix, and $\mu_i,\ i = 1, \ldots, N,$ are the eigenvalues of M. Opening the brackets one can rewrite the identity in a form

$$\sum_{k=0}^{N}(-1)^k e_k(\mu) M^{N-k} = 0\,,$$

where $e_k(\mu),\ k = 0, 1, \ldots, N,$ are elementary symmetric polynomials[1] in variables $\{\mu_i\}_{1\le i\le N}$. The elementary symmetric polynomials generate the whole algebra of symmetric polynomials in the eigenvalues μ_i. Another well-known generating set for symmetric polynomials is given by the power sums

$$p_k(\mu) := \sum_{i=1}^{N}\mu_i^k \;\equiv\; \mathrm{Tr}(M^k)\,.$$

A relation between two sets of generators is provided by the Newton's recurrence

$$k\, e_k + \sum_{r=1}^{k}(-1)^r p_r e_{k-r} = 0 \quad \forall\, k \ge 1.$$

In papers [GPS1, GPS2] analogues of the above classical results were found for a family of Hecke type QMAs, which includes the q-generalizations of the $GL(m|n)$ type supermatrices for all integer $m \ge 0$ and $n \ge 0$ (see the next section for definitions). In particular, the Cayley-Hamilton identities for these algebras were derived, what clarified the way of introducing the spectral values for quantum matrices. It is remarkable that the coefficients of the Cayley-Hamilton polynomials commute among themselves. They generate a commutative characteristic subalgebra in the QMA, which serves as an analogue of the algebra of symmetric polynomials in the eigenvalues of matrix M. It is the Cayley-Hamilton identity which allows one to present the elements of the characteristic subalgebra as (super)symmetric polynomials in the spectral values of the quantum matrix.

As in the classical case, the quantum analogs of the power sums can be defined as some specific traces of 'powers' of the quantum matrix. By their construction the power sums belong to the characteristic subalgebra, but their explicit expression in terms of the spectral values was not known yet. The main goal of the present paper is to derive such an expression.

[1]For a review of basic results on symmetric functions see [Mac].

Our presentation is strongly based on the previous works cited above. In the next section we give a list of notations, definitions and main results which will be used below. For more detailed exposition, proofs and a short overview the reader is referred to [GPS1, GPS2].

2. Some basic results and definitions

Let V be a finite dimensional linear space over the field of complex numbers $\mathbb{C}$, $\dim V = N$. Let I denote the identity matrix (its dimension being clear from the context, if not explicitly specified), and $P \in \mathrm{Aut}(V^{\otimes 2})$ be the permutation automorphism: $P(u \otimes v) = v \otimes u$.

With any element $X \in \mathrm{End}(V^{\otimes p})$, $p = 1, 2, \ldots$, we associate a sequence of endomorphisms $X_i \in \mathrm{End}(V^{\otimes k})$, $k \geq p$, $i = 1, \ldots, k-p+1$, according to the rule

$$X_i = I^{\otimes(i-1)} \otimes X \otimes I^{\otimes(k-p-i+1)}, \quad 1 \leq i \leq k-p+1,$$

where I is the identical automorphism of V.

Consider a pair of invertible operators $R, F \in \mathrm{Aut}(V^{\otimes 2})$ subject to the following conditions:

(1) The operators R and F satisfy the *Yang-Baxter equations*

$$R_1R_2R_1 = R_2R_1R_2\,, \qquad F_1F_2F_1 = F_2F_1F_2\,. \tag{2.1}$$

Such operators are called R-*matrices*.

(2) The pair of R-matrices $\{R, F\}$ is *compatible*, that is

$$R_1F_2F_1 = F_2F_1R_2\,, \qquad F_1F_2R_1 = R_2F_1F_2\,. \tag{2.2}$$

(3) The matrices of both operators R and F are *strictly skew invertible*. Taking the operator R as an example, this requirement means the following:

a) R is skew invertible if there exists an operator $\Psi^R \in \mathrm{End}(V^{\otimes 2})$ such that

$$\mathrm{Tr}_{(2)} R_{12}\Psi^R_{23} = P_{13}\,,$$

where the subscript in the notation of the trace shows the number of the space V, where the trace is evaluated (the enumeration of the component spaces in the tensor product is taken as follows $V^{\otimes k} := V_1 \otimes V_2 \otimes \cdots \otimes V_k$).

b) The strictness condition implies additionally that the operator $D^R_1 := \mathrm{Tr}_{(2)}\Psi^R_{12}$ is invertible.

With the matrix D^R one defines the *R-trace* operation[2] $\mathrm{Tr}_R : \mathrm{Mat}_N(W) \to W$

$$\mathrm{Tr}_R\,(X) := \sum_{i,j=1}^{N} D^{R\,j}_{\ \,i} X^i_j, \quad X \in \mathrm{Mat}_N(W),$$

where W is any linear space.

Given a compatible pair $\{R, F\}$ of strictly skew invertible R-matrices the *quantum matrix algebra* $\mathcal{M}(R, F)$ is a unital associative algebra generated by N^2 components of the matrix $\|M^i_j\|^N_{i=1}$ subject to the relations

$$R_1M_{\overline{1}}M_{\overline{2}} = M_{\overline{1}}M_{\overline{2}}R_1\,. \tag{2.3}$$

Here we have introduced a notation

$$M_{\overline{1}} := M_1, \quad M_{\overline{k+1}} := F_kM_{\overline{k}}F_k^{-1}$$

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[2]In a literature on quantum groups the R-trace is usually named the quantum trace or, shortly, the q-trace. Giving the different name to this operation we hope to avoid misleading associations with a parameter q of the Hecke algebra (see below).

for the copies $M_{\overline{k}}$ of the matrix M. The defining relations (2.3) then imply the same type relations for consecutive pairs of the copies of M (see [IOP2])

$$R_k\, M_{\overline{k}} M_{\overline{k+1}} = M_{\overline{k}} M_{\overline{k+1}}\, R_k.$$

Specific subfamilies in a variety of QMAs are extracted by imposing additional conditions on the R-matrix R in the definition (2.3). These conditions we are now going to describe.

Suppose that R has a quadratic minimal polynomial which can be suitably normalized as[3]

$$(R - q\, I)(R + q^{-1}\, I) = 0\,, \quad q \in \mathbb{C} \setminus 0\,. \tag{2.4}$$

This relation in the present context is called the *Hecke condition* and the R-matrices satisfying it are called *the Hecke* R-matrices. We further assume that the parameter q in (2.4) is *generic*, that means, it does not coincide with the roots of equations

$$k_q := \frac{q^k - q^{-k}}{q - q^{-1}} = 0 \qquad \forall\, k = 2, 3, \dots\,. \tag{2.5}$$

Given any Hecke R-matrix R, one can construct a series of R-*matrix representations* ρ_R of the A type Hecke algebras $\mathcal{H}_k(q) \xrightarrow{\rho_R} \mathrm{End}(V^{\otimes k})$, $k = 2, 3, \dots$. The characteristic properties of these representations are used for a classification of the Hecke R-matrices[4]. Not going into details of the construction we only mention that under conditions (2.5) the Hecke algebra $\mathcal{H}_k(q)$ is isomorphic to the group algebra of the symmetric group $\mathbb{C}[S_k]$ and its irreducible representations are labelled by a set of partitions $\lambda \vdash k$, the corresponding central idempotents in $\mathcal{H}_k(q)$ are further denoted as e^λ. We fix some decomposition of e^λ into the sum of primitive idempotents $e^\lambda_a \in \mathcal{H}_k(q)$: $e^\lambda = \sum_{a=1}^{d_\lambda} e^\lambda_a$, where d_λ is the dimension of the representation with label λ. It is also suitable to introduce the following notations:

- Given two arbitrary integers $m \geq 0$ and $n \geq 0$, an infinite set of partitions $\lambda = (\lambda_1, \lambda_2, \dots)$, satisfying restriction $\lambda_{m+1} \leq n$ is denoted as $\mathsf{H}(m, n)$.
- The partition $((n + 1)^{m+1}) \vdash (m + 1)(n + 1)$ is shortly denoted as $\lambda_{m,n}$. The corresponding Young diagram is a rectangle with $m + 1$ rows of the length $n + 1$. Note that $\lambda_{m,n}$ is a minimal partition not belonging to the set $\mathsf{H}(m, n)$.

Now we are ready to formulate the classification of the Hecke R-matrices.

Proposition 1. ([H, GPS3]) *For a generic value of q the set of the Hecke* R-*matrices is separated into subsets labelled by an ordered pair of non-negative integers $\{m, n\}$.* R-*matrices belonging to the subset with label $\{m, n\}$ are called $GL(m|n)$* type *ones (alternatively, they are assigned a* bi-rank *$(m|n)$).* R-*matrix representations ρ_R generated by a $GL(m|n)$ type* R-*matrix R fulfill the following criterion: for all integer $k \geq 2$ and for any partition $\nu \vdash k$ the images of the idempotents $e^\nu \in \mathcal{H}_k(q)$ satisfy the relations*

$$\rho_R(e^\nu) = 0 \quad \textit{iff} \quad \nu \notin \mathsf{H}(m, n),$$

or, equivalently, iff $\lambda_{m,n} \subset \nu$, where the inclusion $\mu = (\mu_1, \mu_2, \dots) \subset \nu = (\nu_1, \nu_2, \dots)$ means that $\mu_i \leq \nu_i\ \forall\, i$.

The algebra $\mathcal{M}(R, F)$ defined by a Hecke ($GL(m|n)$ type) R-matrix R is further referred to as the *Hecke ($GL(m|n)$ type)* quantum matrix algebra.

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[3]Note that all the conditions (2.1), (2.2) do not depend on normalization of R.

[4] A brief description of the Hecke algebras and their R-matrix representations can be found in [GPS1]. For a more detailed exposition of the subject the reader is referred to [R, OP1] and to the references therein.

For the Hecke type QMA $\mathcal{M}(R,F)$ we consider a set of its elements $s_\lambda(M)$ called *the Schur functions*

$$s_0(M) := 1, \qquad s_\lambda(M) := \mathrm{Tr}_{R(1\dots p)}(M_{\overline{1}}\dots M_{\overline{p}}\,\rho_R(e_a^\lambda))\,, \quad \lambda\vdash p, \quad p=1,2,\dots\,,$$

where the latter formula does not depend on a particular choice of the primitive idempotent e_a^λ (actually, one can substitute it by $d_\lambda^{-1}e^\lambda$). As was shown in [IOP1], a linear span of the Schur functions $s_\lambda(M)\ \forall\,\lambda$, is an abelian subalgebra in $\mathcal{M}(R,F)$. We further call it the *characteristic subalgebra* of $\mathcal{M}(R,F)$. It follows that the characteristic subalgebra of the $GL(m|n)$ type QMA is spanned by the Schur functions $s_\lambda(M)$, $\lambda\in\mathsf{H}(m,n)$. The multiplication table for the elements $s_\lambda(M)\in\mathcal{M}(R,F)$ coincides with that for the basis of Schur functions in the ring of symmetric functions (see [Mac]) thus justifying the notation. One has [GPS2]

$$s_\lambda(M)s_\mu(M) = \sum_\nu C^\nu_{\lambda\mu}s_\nu(M), \tag{2.6}$$

where $C^\nu_{\lambda\mu}$ are the Littlewood-Richardson coefficients. Later on we shall need the information about generating sets of the characteristic subalgebra.

Proposition 2. ([IOP1, IOP2]) *For generic values of q the characteristic subalgebra of the Hecke QMA $\mathcal{M}(R,F)$ is generated by any one of the following three sets*

(1) *the single column Schur functions: $a_k(M) := s_{(1^k)}(M),\ k=0,1,2,\dots;$*
(2) *the single row Schur functions: $s_k(M) := s_{(k)}(M),\ k=0,1,2,\dots;$*
(3) *the set of* power sums*:*

$$p_0(M) := (\mathrm{Tr}_R I)\,1, \quad p_k(M) := \mathrm{Tr}_{R(1\dots k)}(M_{\overline{1}}\dots M_{\overline{k}}\,R_{k-1}\dots R_1)\,, \quad k\ge 1\,. \tag{2.7}$$

These sets are connected by a series of recursive Newton and Wronski relations

$$(-1)^k k_q\, a_k(M) + \textstyle\sum_{r=0}^{k-1}(-q)^r a_r(M)\,p_{k-r}(M) \;=\; 0\,, \tag{2.8}$$

$$k_q\, s_k(M) - \textstyle\sum_{r=0}^{k-1} q^{-r}s_r(M)\,p_{k-r}(M) \;=\; 0\,, \tag{2.9}$$

$$\textstyle\sum_{r=0}^{k}(-1)^r a_r(M)\,s_{k-r}(M) \;=\; 0 \qquad \forall\,k\ge 1\,. \tag{2.10}$$

On introducing the generating functions for these sets of generators

$$A(t) := \sum_{k\ge 0} a_k(M)\,t^k, \quad S(t) := \sum_{k\ge 0} s_k(M)\,t^k, \quad P(t) := 1 + (q-q^{-1})\sum_{k\ge 1}p_k(M)\,t^k\,, \tag{2.11}$$

one can rewrite relations (2.8)–(2.10) in a compact form [Mac, I][5]

$$P(-t)A(qt) = A(q^{-1}t)\,, \qquad P(t)S(q^{-1}t) = S(qt)\,, \qquad A(t)S(-t) = 1\,. \tag{2.12}$$

For the $GL(m|n)$ type QMA the zeroth power sum equals [GPS3]

$$p_0(M) = q^{n-m}(m-n)_q 1\,. \tag{2.13}$$

One of the remarkable properties of the Hecke QMA $\mathcal{M}(R,F)$ is the existence of the characteristic identity for the matrix M of it's generators. To formulate the result we introduce a notion of the *matrix $\star$-product* of *quantum matrices* (for a detailed exposition see [OP2], section 4.4). Namely, starting with the quantum matrix of generators M and the scalar quantum matrices $s_\lambda(M)I\ \ \forall\,\lambda\vdash k,\ k\ge 0,$ we construct the whole set of quantum matrices

[5]The first of these relations in the limiting case $\mathcal{M}(R,R)\xrightarrow{q\to 1}U(\mathfrak{gl}_n)$ was also derived in [U].

by the following recursive procedure: given any quantum matrix N, its $\star$-multiplication by $s_\lambda(M)I$ and left $\star$-multiplication by M are also quantum matrices defined as

$$M\star(s_\lambda(M)I)=(s_\lambda(M)I)\star M:=M\cdot s_\lambda(M),$$
$$M\star N:=M\cdot\phi(N),\quad \text{where}\quad \phi(N)_1:=\mathrm{Tr}_{R(2)}N_{\overline{2}}R_{12},$$

Here the dot-product means the usual multiplication of a matrix by a scalar(matrix). The $\star$-product of the quantum matrices is demanded to be associative and, by construction, it is commutative. The $\star$-powers of the quantum matrix M read

$$M^{\overline{0}}:=I,\quad M^{\overline{1}}:=M,\quad M^{\overline{k}}:=\underbrace{M\star\cdots\star M}_{k\text{ times}}=\mathrm{Tr}_{R(2\dots k)}(M_{\overline{1}}\dots M_{\overline{k}}R_{k-1}\dots R_1)\ \forall\, k>1.$$

We note that for the family of the so-called *reflection equation algebras* — these are the QMAs of the form $\mathcal{M}(R,R)$ — the $\star$-product is identical to the usual matrix product.

The characteristic identity depends essentially on a type of the quantum matrix algebra. For the $GL(m|n)$ type QMA it is an $(m+n)$-th order polynomial identity in $\star$-powers of the matrix M with coefficients in the characteristic subalgebra. One has the following q-analogue of the classical Cayley-Hamilton theorem.

Theorem 3. ([GPS1, GPS2]) *The characteristic identity for the matrix of generators of the $GL(m|n)$ type QMA $\mathcal{M}(R,F)$ reads*

$$\Big(\sum_{k=0}^{m}(-q)^k\,M^{\overline{m-k}}s_{[m|n]^k}(M)\Big)\star\Big(\sum_{r=0}^{n}q^{-r}\,M^{\overline{n-r}}s_{[m|n]_r}(M)\Big)\equiv 0\,, \tag{2.14}$$

where we used a shorthand notation for the partitions

$$[m|n]^k:=\big((n+1)^k,n^{m-k}\big),\qquad [m|n]_r:=\big(n^m,r\big).$$

Remarkably enough, for generic type QMA (i.e., if $mn>0$) the characteristic identity (2.14) factorizes in two parts. Therefore, when setting factorization problem for the characteristic polynomial one is forced to separate all the $m+n$ roots into two parts of sizes m and n.

Let $\mathbb{C}[\mu,\nu]$ be an algebra of polynomials in two sets of mutually commuting and algebraically independent variables $\mu:=\{\mu_i\}_{1\le i\le m}$ and $\nu:=\{\nu_j\}_{1\le j\le n}$. Consider a map of the coefficients of the characteristic polynomial into $\mathbb{C}[\mu,\nu]$

$$s_{[m|n]^k}(M)\ \mapsto\ s_{[m|n]^k}(\mu,\nu):=s_{[m|n]}(\mu,\nu)\,e_k(q^{-1}\mu)\,,\quad 1\le k\le m\,, \tag{2.15}$$
$$s_{[m|n]_r}(M)\ \mapsto\ s_{[m|n]_r}(\mu,\nu):=s_{[m|n]}(\mu,\nu)\,e_r(-q\nu)\,,\quad 1\le r\le n\,, \tag{2.16}$$

where $e_k(\cdot)$ are the elementary symmetric polynomials in their arguments (e.g., $e_k(\mu)\equiv e_k(\mu_1,\dots,\mu_m)=\sum_{1\le i_1<\dots<i_k\le m}\mu_{i_1}\dots\mu_{i_k}$). For the moment we don't specify an explicit expression for the polynomial $s_{[m|n]}(\mu,\nu)$. We now define a central extension of the $\star$-product algebra of the quantum matrices by the scalar matrices of the form $p(\mu,\nu)I,\ p(\mu,\nu)\in\mathbb{C}[\mu,\nu]$, such that $s_\lambda(M)I\equiv s_\lambda(\mu,\nu)I$. In the extended algebra the characteristic identity (2.14) takes a completely factorized form

$$\prod_{i=1}^{m}(M-\mu_iI)\star\prod_{j=1}^{n}(M-\nu_jI)\cdot(s_{[m|n]}(\mu,\nu))^2\equiv 0\,.$$

Assuming that $s_{[m|n]}(\mu,\nu)\neq 0$ we can interpret the variables $\mu_i,\ i=1,\dots,m,$ and $\nu_j,\ j=1,\dots,n,$ as eigenvalues of the quantum matrix M. They are called, respectively, *"even"* and *"odd" spectral values* of M.

The map (2.15), (2.16) admits a unique extension to a homomorphic map of the characteristic subalgebra into the algebra $\mathbb{C}[\mu,\nu]$ of polynomials in spectral values μ_i and ν_j. Using the Littlewood-Richardson multiplication rule (2.6) we obtain (see [GPS2])

$$a_k(M)\equiv s_{[k|1]}(M) \quad\mapsto\quad a_k(\mu,\nu) := \sum_{r=0}^{k} e_r(q^{-1}\mu)\, h_{k-r}(-q\nu)\,, \tag{2.17}$$

$$s_k(M)\equiv s_{[1|k]}(M) \quad\mapsto\quad s_k(\mu,\nu) := \sum_{r=0}^{k} e_r(-q\nu)\, h_{k-r}(q^{-1}\mu)\,, \tag{2.18}$$

where $h_k(\dots)$ stands for the complete symmetric polynomial in its variables: $h_k(\mu)\equiv h_k(\mu_1,\dots,\mu_m)=\sum_{1\le i_1\le\dots\le i_k\le m}\mu_{i_1}\dots\mu_{i_k}$. Since each of the sets $\{a_k(M)\}_{k\ge0}$, $\{s_k(M)\}_{k\ge0}$ generates the characteristic subalgebra, the homomorphism is completely defined by (2.17) or by (2.18). In particular, formulas (2.17), (2.18) prescribe an explicit expression for the unspecified polynomial $s_{[m|n]}(\mu,\nu)$ in (2.15), (2.16), which is the image of $s_{[m|n]}(M)$:

$$s_{[m|n]}(M) \;\mapsto\; s_{[m|n]}(\mu,\nu)=\prod_{i=1}^{m}\prod_{j=1}^{n}\left(q^{-1}\mu_i-q\nu_j\right).$$

This homomorphic map induced by (2.17), or (2.18) is called the *spectral parameterization* of the characteristic subalgebra. In the next section we derive the spectral parameterization for the third generating set of the characteristic subalgebra, that is the set of power sums $\{p_k(M)\}_{k\ge0}$.

3. Spectral parameterization of power sums

In this section we are working with the $GL(m|n)$ type QMA $\mathcal{M}(R,F)$ defined by relations (2.3) with an R-matrix R satisfying the criterion of the proposition 1. During the considerations we assume that the parameter q is generic (see (2.5)), although afterwards this restriction can be waived out: unlike the cases of $a_k(M)$ and $s_k(M)$ the power sums $p_k(M)$ are consistently defined for all $q\in\mathbb{C}\setminus 0$.

For the particular case of the $GL(m)\equiv GL(m|0)$ type reflection equation algebra $\mathcal{M}(R,R)$ the spectral parameterization of the power sums was found in [GS]. Taking into account that in the $GL(m)$ case the quantum matrix has only "even" eigenvalues $\{\mu_i\}_{1\le i\le m}$, the result reads

$$p_k(M)\mapsto\sum_{i=1}^{m} d_i\,\mu_i^k\,,\qquad \text{where}\qquad d_i:=q^{-1}\prod_{j\neq i}^{m}\frac{\mu_i-q^{-2}\mu_j}{\mu_i-\mu_j}\,.$$

Our goal is to extend the this formula for the general case. To this end we introduce an auxiliary set of polynomials $\{\pi_k(\mu,\nu)\}_{k\ge1}\subset\mathbb{C}[\mu,\nu]$ defined by relations

$$\pi_k(\mu,\nu):=\sum_{i=1}^{m}(q^{-1}\mu_i)^k-\sum_{j=1}^{m}(q\nu_j)^k\,,\qquad k\ge1\,. \tag{3.1}$$

In the subsequent considerations we shall use the following property of these polynomials.

Lemma 4. *The sets of polynomials* $\{a_k(\mu,\nu)\}_{k\geq 0}$, $\{s_k(\mu,\nu)\}_{k\geq 0}$ *(see (2.17), (2.18)) and* $\{\pi_k(\mu,\nu)\}_{k\geq 1}$ *satisfy the Newton's recurrent relations*

$$(-1)^k k\, a_k(\mu,\nu) + \sum_{r=0}^{k-1} (-1)^r a_r(\mu,\nu)\, \pi_{k-r}(\mu,\nu) \;=\; 0\,, \tag{3.2}$$

$$k\, s_k(\mu,\nu) - \sum_{r=0}^{k-1} s_r(\mu,\nu)\, \pi_{k-r}(\mu,\nu) \;=\; 0 \quad \forall\, k\geq 1\,. \tag{3.3}$$

Proof. First, we recall relations among the generating functions for the power sums and the elementary symmetric and complete symmetric polynomials in a finite set of variables $x := \{x_i\}_{1\leq i\leq p}$ (see [Mac], section I.2)

$$\begin{aligned} E(x|t) &:= \sum_{k=0}^{p} e_k(x)t^k = \prod_{i=1}^{p}(1+x_i t)\,, \quad H(x|t) := \sum_{k\geq 0} h_k(x)t^k = \prod_{i=1}^{p}(1-x_i t)^{-1}\,, \\ P(x|t) &:= \sum_{k\geq 1} p_k(x)t^{k-1} = -\frac{d}{dt}\log E(x|-t) = \frac{d}{dt}\log H(x|t)\,. \end{aligned} \tag{3.4}$$

Consider three functions depending on two sets of variables $x := \{x_i\}_{1\leq i\leq m}$ and $y := \{y_i\}_{1\leq i\leq n}$:

$$\begin{aligned} A(x,y|t) &:= E(x|t)H(-y|t), \qquad S(x,y|t) := H(x|t)E(-y|t), \\ \Pi(x,y|t) &:= P(x|t) - P(y|t). \end{aligned} \tag{3.5}$$

These functions serve as super-matrix analogues of, respectively, generating functions of the elementary and complete symmetric polynomials and the power sums (see [Mac], section I.5, exercise 27 and bibliographic references for it). Indeed, using (3.4) it is easy to check that these functions satisfy relations similar to (3.4)

$$\Pi(x,y|t) = -\frac{d}{dt}\log A(x,y|-t) = \frac{d}{dt}\log S(x,y|t)\,. \tag{3.6}$$

Now the assertion of the lemma follows from an observation that $A(q^{-1}\mu, q\nu|t)$, $S(q^{-1}\mu, q\nu|t)$ and $\Pi(q^{-1}\mu, q\nu|t)$ are, respectively, generating functions for the sets of polynomials $\{a_k(\mu,\nu)\}_{k\geq 0}$, $\{s_k(\mu,\nu)\}_{k\geq 0}$ and $\{\pi_k(\mu,\nu)\}_{k\geq 1}$. The relations (3.2) and (3.3) are just expansions of (3.6) in powers of t. ∎

Now we can formulate the main result on the power sums.

Proposition 5. *The spectral parameterization (2.17) (or (2.18)) for the power sums* $p_k(M)$ *(2.7) in the* $GL(m|n)$ *type quantum matrix algebra is given by the formulas*

$$p_k(M) \mapsto p_k(\mu,\nu) = \sum_{i=1}^{m} d_i \mu_i^k + \sum_{j=1}^{n} \tilde{d}_j \nu_j^k \quad \forall\, k\geq 0\,, \tag{3.7}$$

where the "weight" coefficients d_i and $\tilde{d}_j$ explicitly read

$$d_i \;:=\; q^{-1}\prod_{\substack{p=1\\ p\neq i}}^{m}\frac{\mu_i-q^{-2}\mu_p}{\mu_i-\mu_p}\,\prod_{j=1}^{n}\frac{\mu_i-q^{2}\nu_j}{\mu_i-\nu_j}\,, \tag{3.8}$$

$$\tilde{d}_j \;:=\; -\,q\prod_{i=1}^{n}\frac{\nu_j-q^{-2}\mu_i}{\nu_j-\mu_i}\,\prod_{\substack{p=1\\ p\neq j}}^{n}\frac{\nu_j-q^{2}\nu_p}{\nu_j-\nu_p}\,. \tag{3.9}$$

Recall, that the spectral values $\{\mu_i\}$ and $\{\nu_j\}$ are supposed to be algebraically independent, and therefore all the coefficients d_i and $\tilde{d}_j$ are nonzero and well defined.

Proof. For the proof we need yet another recursive set of formulas for the power sums $\{p_k(\mu,\nu)\}_{k\ge 1}$ (3.7) and the polynomials $\{\pi_k(\mu,\nu)\}_{k\ge 1}$ (3.1).

Lemma 6. *The following relations hold true*

$$k\,p_k(\mu,\nu)=k_q\,\pi_k(\mu,\nu)+(q-q^{-1})\sum_{r=1}^{k-1}r_q\,\pi_r(\mu,\nu)p_{k-r}(\mu,\nu)\quad\forall\,k\ge 1\,. \tag{3.10}$$

In terms of the generating functions

$$P(\mu,\nu|t):=1+(q-q^{-1})\sum_{k\ge 1}p_k(\mu,\nu)t^k\qquad \Pi(q^{-1}\mu,q\nu|t):=\sum_{k\ge 1}\pi_k(\mu,\nu)t^{k-1}$$

($P(\mu,\nu|t)$ is the spectral parameterization of $P(t)$, see (2.11); $\Pi(\mu,\nu|t)$ was first defined in (3.5)) the relations (3.10) shortly read

$$P(\mu,\nu|t)\Big(q\,\Pi(q^{-1}\mu,q\nu|qt)\;-\;q^{-1}\,\Pi(q^{-1}\mu,q\nu|q^{-1}t)\Big)=\frac{d}{dt}P(\mu,\nu|t)\,. \tag{3.11}$$

Proof of the lemma. Introduce a meromorphic function $f:\mathbb{C}[\mu,\nu]\to\mathbb{C}(\mu,\nu)$ by the formula

$$f(z):=\prod_{i=1}^{m}\frac{(z-q^{-2}\mu_i)}{(z-\mu_i)}\prod_{j=1}^{n}\frac{(z-q^{2}\nu_j)}{(z-\nu_j)}\,.$$

Since the spectral values are algebraically independent the above function has a first order pole at each point μ_i and ν_j. Besides, as can be easily seen

$$f(0)=q^{2(n-m)}\qquad\text{and}\qquad \lim_{z\to\infty}f(z)=1\,.$$

Taking into account the above limit at infinity, we expand the function $f(z)$ into the sum of simple fractions

$$f(z)=1+\sum_{i=1}^{m}\frac{1}{z-\mu_i}\,\mathrm{Res}f(z)\big|_{z=\mu_i}+\sum_{j=1}^{n}\frac{1}{z-\nu_j}\,\mathrm{Res}f(z)\big|_{z=\nu_j}\,, \tag{3.12}$$

where the residues at the poles read

$$\mathrm{Res}f(z)\big|_{z=\mu_i}=(q-q^{-1})\,\mu_i\,d_i\,,\qquad \mathrm{Res}f(z)\big|_{z=\nu_j}=(q-q^{-1})\,\nu_j\,\tilde{d}_j\,.$$

Evaluating the right hand side of (3.12) at $z=0$ we get

$$f(0)=q^{2(n-m)}=1-(q-q^{-1})\Big(\sum_{i=1}^{m}d_i+\sum_{j=1}^{n}\tilde{d}_j\Big):=1-(q-q^{-1})\,p_0(\mu,\nu)$$

and therefore,

$$p_0(\mu,\nu) = q^{n-m}(m-n)_q 1\,.$$

So, we have verified the consistency of (3.7) with our previous result (2.13) for $p_0(M)$.

In order to prove relations (3.10) we expand functions $z^k f(z)$, $k \ge 1$, into the sum of simple fractions. Besides the simple poles in μ_i and ν_j, the function $z^k f(z)$ possesses the k-th order pole at $z=\infty$, or, introducing a new variable $y = z^{-1}$, at the point $y=0$. Taking into account that

$$\mathrm{Res}\, z^k f(z)\big|_{z=\mu_i} = (\mu_i)^k\, \mathrm{Res} f(z)\big|_{z=\mu_i}\,, \qquad \mathrm{Res}\, z^k f(z)\big|_{z=\nu_j} = (\nu_j)^k\, \mathrm{Res} f(z)\big|_{z=\nu_j}\,,$$

we come to the corresponding expansion

$$z^k f(z) = \sum_{r=0}^{k} \frac{z^{k-r}}{r!}\, f_r + (q-q^{-1})\Bigl(\sum_{i=1}^{m} \frac{d_i \mu_i^{k+1}}{z-\mu_i} + \sum_{j=1}^{n} \frac{\tilde{d}_i \nu_j^{k+1}}{z-\nu_i}\Bigr)\,, \tag{3.13}$$

where the coefficients f_r are the following derivatives

$$f_r := \frac{d^r f(y)}{dy^r}\Big|_{y=0}\,.$$

Evaluating (3.13) at the point $z=0$ and taking into account (3.7) we find the relation

$$p_k(\mu,\nu) = \frac{f_k}{(q-q^{-1})\,k!} \quad \forall\, k \ge 1\,. \tag{3.14}$$

Let us turn to the calculation of f_k. Recalling that $y = z^{-1}$ we write the first order derivative $f'(y)$ in the form

$$\frac{df(y)}{dy} = (q-q^{-1})\, u(y) f(y)\,, \tag{3.15}$$

where the function $u(y)$ reads

$$u(y) := \sum_{i=1}^{m} \frac{q^{-1}\mu_i}{(1-\mu_i y)(1-q^{-2}\mu_i y)} - \sum_{j=1}^{n} \frac{q\nu_j}{(1-\nu_j y)(1-q^2\nu_j y)}\,.$$

By a simple induction in k one can check that

$$\begin{aligned} u_k(y) \;:=\; \frac{d^k u(y)}{dy^k} \;&=\; k!\Bigl\{\sum_{i=1}^{m}\sum_{a=0}^{k} \frac{q^{-1-2(k-a)}\mu_i^{k+1}}{(1-\mu_i y)^{1+a}(1-q^{-2}\mu_i y)^{1+k-a}} \\ &\qquad - \sum_{j=1}^{n}\sum_{a=0}^{k} \frac{q^{1+2(k-a)}\nu_j^{k+1}}{(1-\nu_j y)^{1+a}(1-q^{2}\nu_j y)^{1+k-a}}\Bigr\}. \end{aligned}$$

The above relation leads immediately to

$$u_k(0) = k!(k+1)_q\Bigl\{\sum_{i=1}^{m}(q^{-1}\mu_i)^{k+1} - \sum_{j=1}^{n}(q\nu_j)^{k+1}\Bigr\} := k!(k+1)_q \pi_{k+1}(\mu,\nu) \quad \forall\, k \ge 0. \tag{3.16}$$

Now, differentiating the relation (3.15) at $y=0$ gives rise to

$$f_k = (q-q^{-1})\sum_{r=0}^{k-1} \binom{r}{k-1} u_r(0) f_{k-r-1}\,, \quad k \ge 1\,,$$

where we use an obvious condition $f_0 = 1$. On taking into account relations (3.14) and (3.16), we can easily prove the assertion of the lemma. Indeed,

$$\begin{aligned} kp_k(\mu,\nu) &= \frac{f_k}{(q-q^{-1})(k-1)!} = \frac{1}{(k-1)!}\sum_{r=0}^{k-1}\binom{r}{k-1}u_r(0)f_{k-r-1} \\ &= \sum_{r=0}^{k-1}\frac{(r+1)_q}{(k-r-1)!}\,\pi_{r+1}(\mu,\nu)\,f_{k-r-1} = \sum_{r=1}^{k} r_q\,\pi_r(\mu,\nu)\frac{f_{k-r}}{(k-r)!} \\ &= k_q\pi_k(\mu,\nu) + (q-q^{-1})\sum_{r=1}^{k-1} r_q\,\pi_r(\mu,\nu)\,p_{k-r}(\mu,\nu)\,. \end{aligned}$$

At last, the equivalence of (3.10) and (3.11) is verified by a direct calculation. ■

Now we are ready to prove the proposition. We shall verify that relations (2.8) and (2.9) turn into identities if we substitute $a_r(M)$, $s_r(M)$ and $p_{k-r}(M)$ by their spectral parameterizations (2.17), (2.18) and (3.7)–(3.9) and take into account the equalities (3.2), (3.3) together with (3.10). For the definiteness we shall verify the relation (2.8).

To simplify the calculation we shall work with the generating functions: $A(q^{-1}\mu, q\nu|t)$ for $\{a_k(\mu,\nu)\}_{k\ge 0}$, $\Pi(q^{-1}\mu, q\nu|t)$ for $\{\pi_k(\mu,\nu\}_{k\ge 1}$, and $P(\mu,\nu|t)$ for $\{p_k(\mu,\nu)\}_{k\ge 1}$. Substituting the expression of $\Pi(\dots)$ through $A(\dots)$ (see (3.6)) into the equality (3.11) and gathering together similar terms we get

$$\frac{d}{dt}\log\Big(P(\mu,\nu|t)\,A(q^{-1}\mu, q\nu|-qt)\Big) \;=\; \frac{d}{dt}\log\Big(A(q^{-1}\mu, q\nu|-q^{-1}t)\Big)\,.$$

Integrating this equation leads to

$$P(\mu,\nu|t)\,A(q^{-1}\mu, q\nu|-qt) \;=\; C A(q^{-1}\mu, q\nu|-q^{-1}t)\,.$$

The obvious boundary condition $P(\mu,\nu|0) = A(q^{-1}\mu, q\nu|0) = 1$ fixes the integration constant C to unity and we come to a desired relation: the first one from (2.12). ■

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