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Gergely Zábrádi

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Gergely Zábrádi

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Eötvös Loránd University
Mathematical Institute
Department of Algebra and Number Theory
Pázmány Péter sétány 1/C
1117 Budapest
Hungary

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Gergely Zábrádi

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Abstract

We prove the exactness of the reduction map from étale (φ, Γ)-modules over completed localized group rings of compact open subgroups of unipotent p-adic algebraic groups to usual étale (φ, Γ)-modules over Fontaine's ring. This reduction map is a component of a functor from smooth p-power torsion representations of p-adic reductive groups (or more generally of Borel subgroups of these) to (φ, Γ)-modules. Therefore this gives evidence for this functor—which is intended as some kind of p-adic Langlands correspondence for reductive groups—to be exact. We also show that the corresponding higher Tor-functors vanish. Moreover, we give the example of the Steinberg representation as an illustration and show that it is acyclic for this functor to (φ, Γ)-modules whenever our reductive group is $\mathrm{GL}_{d+1}(\mathbb{Q}_p)$ for some $d \geq 1$.

1 Introduction

1.1 Colmez' work

In recent years it has become increasingly clear that some kind of p-adic version of the local Langlands conjectures should exist. However, a precise formulation is still missing. It is all the more remarkable that Colmez has recently managed to establish such a correspondence between 2-dimensional p-adic Galois representations of $\mathbb{Q}_p$ and continuous irreducible unitary p-adic representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. In fact, Colmez [3, 4] constructed a functor from smooth torsion P-representations to étale (φ, Γ)-modules where P is the standard parabolic subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$. Whenever we are given a unitary $\mathrm{GL}_2(\mathbb{Q}_p)$-representation V, we may find a $\mathrm{GL}_2(\mathbb{Q}_p)$-invariant lattice L inside it. Hence we can take the restriction to P of the reduction $L/p^m L$ mod p^m for some positive integer m and pass to (φ, Γ)-modules using Colmez' functor. The (φ, Γ)-module corresponding to the initial representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ will be the projective limit of these (φ, Γ)-modules when m tends to infinity. The miracle is that whenever we started with an irreducible supercuspidal GL_2-representation in characteristic p the resulting (φ, Γ)-module will be 2-dimensional and hence correspond to a 2-dimensional modulo p Galois representation of the field $\mathbb{Q}_p$. The image of 1-dimensional and pricipal series representations is, however, 0 and 1 dimensional, respectively (see Thm. 10.7 in [13]).

1.2 The Schneider-Vigneras functors

Even more recently, Schneider and Vigneras [10] managed to generalize Colmez' functor to general p-adic reductive groups. Their context is the following. Let G be the group of

$\mathbb{Q}_p$-points of a $\mathbb{Q}_p$-split connected reductive group over $\mathbb{Q}_p$ whose centre is also assumed to be connected for technical simplicity. To review their construction we fix a Borel subgroup $P = TN$ with split torus T and unipotent radical N. We also fix an appropriate compact open subgroup N_0 which gives rise to the 'dominant' submonoid $T_+ := \{t \in T \mid tN_0t^{-1} \subseteq N_0\}$ in T. On the one side we consider the abelian category $\mathcal{M}_{o-tor}(P)$ of all smooth o-torsion representations of the group P where o is the ring of integers in a fixed finite extension $K/\mathbb{Q}_p$. On the other side a monoid ring $\Lambda(P_+)$ is introduced for the monoid $P_+ := N_0T_+$ and we denote the category of all (left unital) $\Lambda(P_+)$-modules by $\mathcal{M}(\Lambda(P_+))$. Such a module M is called étale if every $t \in T_+$ acts, informally speaking, with slope zero on M. The universal δ-functor $V \mapsto D^i(V)$ for $i \geq 0$ from $\mathcal{M}_{o-tor}(P)$ to the category $\mathcal{M}_{et}(\Lambda(P_+))$ of étale $\Lambda(P_+)$-modules is constructed the following way. D^i are the derived functors of a contravariant functor $D\colon \mathcal{M}_{o-tor}(P) \to \mathcal{M}_{et}(\Lambda(P_+))$ which is not exact in the middle, but takes surjective, resp. injective maps to injective, reps. surjective maps. (Hence $D \neq D^0$ in general.) The modules $D^i(V)$ are not expected to have good properties in general. This is why it is natural to pass to some topological localization $\Lambda_\ell(P_\star)$ of the group ring $\Lambda(P_\star)$ of a submonoid $P_\star$ of P_+ generated by P_0, φ, and Γ. The corresponding abelian category $\mathcal{M}_{et}(\Lambda_\ell(P_\star))$ of étale $\Lambda_\ell(P_\star)$-modules is a generalization of Fontaine's (φ, Γ)-modules. Indeed, whenever $G = \mathrm{GL}_2(\mathbb{Q}_p)$ (in this case we denote by $S_\star$ the standard monoid inside $\mathrm{GL}_2(\mathbb{Q}_p)$ and note that $N_0 \cong \mathbb{Z}_p$) then the objects that are finitely generated over the smaller localized ring $\Lambda_\ell(N_0) \cong \Lambda_F(\mathbb{Z}_p)$ are exactly Fontaine's (φ, Γ)-modules. This construction leads to the universal δ-functor $D^i_\ell(V)$. The fundamental open question in [10] is for which class of P-representations V are the modules $D^i_\ell(V)$ finitely generated over $\Lambda_\ell(N_0)$. Moreover, with the help of a Whittaker type functional ℓ one may pass to the category $\mathcal{M}_{et}(\Lambda_F(S_\star))$ for the standard monoid $S_\star$ in $\mathrm{GL}_2(\mathbb{Q}_p)$. This way one obtains a δ-functor $D^i_{\Lambda_F(S_\star)}$ from $\mathcal{M}_{o-tor}(P)$ to the category of not necessarily finitely generated (φ, Γ)-modules à la Fontaine. For the group $G = \mathrm{GL}_2(\mathbb{Q}_p)$ Colmez' original functor coincides with $D^0_{\Lambda_F(S_\star)}$ and the higher $D^i_{\Lambda_F(S_\star)}$ vanish.

1.3 Outline of the paper

The aim of this short note is to investigate the exactness properties of the functors constructed by Schneider and Vigneras [10]. Whenever $G \neq \mathrm{GL}_2(\mathbb{Q}_p)$ then the reduction map

$$\ell\colon \Lambda_\ell(N_0) \twoheadrightarrow \Lambda_F(\mathbb{Z}_p)$$

has a nontrivial kernel and hence is not flat. However, the extra étale φ-structure allows us to show that the reduction functor from étale φ-modules over $\Lambda_\ell(N_0)$ to étale φ-modules over $\Lambda_F(\mathbb{Z}_p)$ induced by ℓ is still exact if we restrict ourselves to *pseudocompact* $\Lambda_\ell(N_0)$-modules which includes those finitely generated. The proof relies on the non-existence of nonzero maps from pure φ-modules of slope 0 to pure φ-modules of positive slope over Fontaine's ring. In section 3.4 we use this to show that in fact the higher Tor-functors $\mathrm{Tor}^i_{\Lambda_\ell(N_0)}(\Lambda_F(\mathbb{Z}_p), M)$ vanish for $i \geq 1$ whenever M is a pseudocompact étale φ-module over $\Lambda_\ell(N_0)$.

In section 4 we investigate the example of the Steinberg representation V_{St}. We show that in this case we have $D^0(V_{St}) = D(V_{St})$ and, in particular $D^0(V_{St})$ is finitely generated over $\Lambda_\ell(N_0)$. Moreover, we prove that all the higher $D^i(V_{St})$ vanish for $i \geq 1$. This is the first known example of a smooth o-torsion P-representation with finitely generated D^0_ℓ and with known D^i_ℓ for all $i \geq 0$. Hence V_{St} is acyclic for the functor D_ℓ and also for the functor $D_{\Lambda_F(S_\star)}$ by the first part of the paper. It also follows that the functor in the other direction from $\mathcal{M}_{et}(\Lambda(P_+))$ to $\mathcal{M}_{o-tor}(P)$ sends $D^0(V_{St})$ back to V_{St}. We expect that the method of computing $D^i(V_{St})$ for $i \geq 0$ generalizes to a wider class of smooth o-torsion P-representations. For technical reasons we restrict ourselves in this section to the general linear group $\mathrm{GL}_{d+1}(\mathbb{Q}_p)$ with $d \geq 1$. The case $\mathrm{GL}_2(\mathbb{Q}_p)$ is also formally included—however, the functor D is known [3, 4] to be exact in this case.

2 Preliminaries and notations

2.1 Basic notations

We are going to use the notations of [10], but for the convenience of the reader we recall them here, as well. Let G be the group of $\mathbb{Q}_p$-rational points of a $\mathbb{Q}_p$-split connected reductive group over $\mathbb{Q}_p$. Assume further that the centre of this reductive group is also connected. We fix a Borel subgroup $P = TN$ in G with maximal split torus T and unipotent radical N. Let Φ^+ denote, as usual, the set of roots of T positive with respect to P and let $\Delta \subseteq \Phi^+$ be the subset of simple roots. For any $\alpha \in \Phi^+$ we have the root subgroup $N_\alpha \subseteq N$. We recall that $N = \prod_{\alpha \in \Phi^+} N_\alpha$ for any total ordering of Φ^+. Let $T_0 \subseteq T$ be the maximal compact subgroup. We fix a compact open subgroup $N_0 \subseteq N$ which is totally decomposed, ie. $N_0 = \prod_\alpha N_0 \cap N_\alpha$ for any total ordering of Φ^+. Then $P_0 := T_0 N_0$ is a group. We introduce the submonoid $T_+ \subseteq T$ of all $t \in T$ such that $tN_0t^{-1} \subseteq N_0$, or equivalently, such that $\alpha(t)$ is integral for any $\alpha \in \Delta$. Then $P_+ := N_0 T_+ = P_0 T_+ P_0$ is obviously a submonoid of P.

We also fix a finite extension $K/\mathbb{Q}_p$ with ring of integers o, prime element π, and residue class field k. For any profinite group H let $\Lambda(H) := o[[H]]$, resp. $\Omega(H) := k[[H]] = \Lambda(H)/\pi\Lambda(H)$ be the Iwasawa algebra of H with coefficients in o, resp. k.

2.2 The functors D and D^i

By a representation we will always mean a linear action of the group (or monoid) in question in a torsion o-module V. It is called smooth if the stabilizer of each element in V is open in the group. We put $V^* := \mathrm{Hom}_o(V, K/o)$ the Pontryagin dual of V which is a compact linear-topological o-module. Following [10] we define

$$D(V) := \varinjlim_M M^*$$

where M runs through all the generating P_+-subrepresentations of V. Whenever V is compactly induced it is equipped with an action of the ring $\Lambda(P_+)$ which is by definition the image of the natural map

$$\Lambda(P_0) \otimes_{o[P_0]} o[P_+] \to \varprojlim_Q o[Q \setminus P_+]$$

where Q runs through all open normal subgroups $Q \subseteq P_0$ which satisfy $bQb^{-1} \subseteq Q$ for any $b \in P_+$ (cf. Proposition 3.4 in [10]). The $\Lambda(P_+)$-modules $D^i(V)$ for a general smooth P-representation V and $i \geq 0$ are obtained as the cohomology groups $D^i(V) := h^i(D(\mathcal{I}_\bullet(V)))$ for some resolution

$$\mathcal{I}_\bullet(V)\colon \cdots \to \mathrm{ind}_{P_0}^P(V_n) \to \cdots \to \mathrm{ind}_{P_0}^P(V_0) \to V \to 0$$

of V by compactly induced representations. This is independent of the choice of the resolution by Corollary 4.4 in [10]. Since D is not exact in the middle, we do not have $D(V) = D^0(V)$ in general.

2.3 The ring $\Lambda_\ell(N_0)$

As in [10] we fix once and for all isomorphisms of algebraic groups

$$\iota_\alpha \colon N_\alpha \overset{\cong}{\to} \mathbb{Q}_p$$

for $\alpha \in \Delta$, such that

$$\iota_\alpha(tnt^{-1}) = \alpha(t)\iota_\alpha(n)$$

for any $n \in N_\alpha$ and $t \in T$. Since $\prod_{\alpha\in\Delta} N_\alpha$ is naturally a quotient of $N/[N,N]$ we now introduce the group homomorphism

$$\ell := \sum_{\alpha\in\Delta} \iota_\alpha \colon N \to \mathbb{Q}_p.$$

Moreover, for the sake of convenience we normalize the ι_α such that

$$\iota_\alpha(N_0 \cap N_\alpha) = \mathbb{Z}_p$$

for any α in Δ. In particular, we then have $\ell(N_0) = \mathbb{Z}_p$. We put $N_1 := \operatorname{Ker}(\ell_{|N_0})$. The group homomorphism ℓ also induces a map

$$\Lambda(N_0) \twoheadrightarrow \Lambda(\mathbb{Z}_p)$$

which we still denote by ℓ. By [2] the multiplicatively closed subset $S := \Lambda(N_0) \setminus (\pi, \operatorname{Ker}(\ell))$ is a left and right Ore set in $\Lambda(N_0)$ and we may define the localization $\Lambda(N_0)_S$ of $\Lambda(N_0)$ at S. We define the ring $\Lambda_\ell(N_0) := \Lambda_{N_1}(N_0)$ as the completion of $\Lambda(N_0)_S$ with respect to the ideal $(\pi, \operatorname{Ker}(\ell))\Lambda(N_0)_S$. This is a strict-local ring with maximal ideal $(\pi, \operatorname{Ker}(\ell))\Lambda_\ell(N_0)$. Moreover, it is pseudocompact (c.f. Thm 4.7 in [9]).

2.4 Generalized (φ, Γ)-modules

Now since we assume that the centre of G is connected, the quotient $X^*(T)/\bigoplus_{\alpha\in\Delta}\mathbb{Z}\alpha$ is free. Hence we find a cocharacter ξ in $X_*(T)$ such that $\alpha \circ \xi = \mathrm{id}_{\mathrm{G}_m}$ for any α in Δ. It is injective and uniquely determined up to a central cocharacter. We fix once and for all such a ξ. It satisfies

$$\xi(\mathbb{Z}_p \setminus \{0\}) \subseteq T_+$$

and

$$\ell(\xi(a)n\xi(a^{-1})) = a\ell(n)$$

for any a in $\mathbb{Q}_p^\times$ and n in N. Put $\Gamma := \xi(1 + p^{\epsilon(p)}\mathbb{Z}_p)$ and $\varphi := \xi(p)$.

The group Γ and the semigroup generated by φ naturally act on the ring $\Lambda_\ell(N_0)$. Hence we may define (φ, Γ)-modules (resp. φ-modules) over $\Lambda_\ell(N_0)$ as $\Lambda_\ell(N_0)$-modules together with a commuting and compatible action of φ and Γ (resp. just a compatible action of φ). The notion of $(\Lambda_\ell(N_0),\Gamma,\varphi)$-module refers to (φ,Γ)-modules that are *finitely generated* over $\Lambda_\ell(N_0)$. We call a φ-module M étale if the map

$$\begin{aligned}\Lambda_\ell(N_0)\otimes_\varphi M &\to M\\ \nu\otimes m &\mapsto \nu\varphi_M(m)\end{aligned}$$

is bijective.

The map ℓ induces a φ- and Γ-equivariant ring homomorphism

$$\Lambda_\ell(N_0)\twoheadrightarrow\Lambda_F(\mathbb{Z}_p)$$

onto Fontaine's ring $\Lambda_F(\mathbb{Z}_p)$ which is the p-adic completion of the Laurent series ring $o[[T]][T^{-1}]$. Hence it gives rise to a functor from (étale) (φ,Γ)-modules over $\Lambda_\ell(N_0)$ to not necessarily finitely generated (étale) (φ,Γ)-modules over $\Lambda_F(\mathbb{Z}_p)$. We may restrict this functor to pseudocompact (or less generally to finitely generated) étale modules. The main result of this short note is that this restriction is exact.

3 Exactness of reduction on pseudocompact modules

3.1 A p-valuation on N_1

We fix a simple root α_0 in Δ. Since N_0 is totally decomposed we can fix topological generators n_α of $N_0\cap N_\alpha$ for any α in Δ such that $\ell(n_\alpha)=1$. Further, we fix topological generators n_β of $N_0\cap N_\beta$ for each $\beta\in\Phi^+\setminus\Delta$. Hence the set

$$A:=\{n_{\alpha_0}\}\cup\{n_{\alpha_0}^{-1}n_\alpha\}_{\alpha\in\Delta\setminus\{\alpha_0\}}\cup\{n_\beta\}_{\beta\in\Phi^+\setminus\Delta}$$

is a minimal set of topological generators of the group N_0. Moreover, $A\setminus\{n_{\alpha_0}\}$ is a minimal set of generators of the group $\mathrm{Ker}(\ell)\cap N_0$. Further, we put

$$b_\alpha:=\begin{cases}n_\alpha-1 & \text{if } \alpha\in(\Phi^+\setminus\Delta)\cup\{\alpha_0\}\\ n_{\alpha_0}^{-1}n_\alpha-1 & \text{if } \alpha\in\Delta\setminus\{\alpha_0\}.\end{cases}$$

Now we define a p-valuation ω on N_1 as follows. Any β in Φ^+ can be written as a positive integer combination $\beta=\sum_{\alpha\in\Delta}m_{\alpha\beta}\alpha$ of simple roots α. We denote by $m_\beta:=\sum_\alpha m_{\alpha\beta}$ the degree of $\beta\circ\xi$ which is a positive integer and is equal to 1 if and only if β lies in Δ. Further, we fix a total ordering $<$ of Φ^+ such that the minimal element of Φ^+ is α_0 and whenever $m_\alpha<m_\beta$ for roots α,β in Φ^+ then also $\alpha<\beta$. As N_0 is totally decomposed we may write any element g in N_1 as a product

$$g=\prod_{\alpha\in\Delta\setminus\{\alpha_0\}}(n_{\alpha_0}^{-1}n_\alpha)^{g_\alpha}\prod_{\beta\in\Phi^+\setminus\Delta}n_\beta^{g_\beta}$$

where g_α and g_β are in $\mathbb{Z}_p$ and the product is taken in the ordering $<$ of Φ^+ defined above. We put

$$\omega(g):=\min_{\beta\in\Phi^+\setminus\{\alpha_0\}}m_\beta(v_p(g_\beta)+1)$$

for any $1\neq g$ in N_1. Here v_p denotes the additive p-adic valuation on $\mathbb{Z}_p$.

Lemma 1. *The function ω is a p-valuation on $N_1 \setminus \{1\}$. In other words, we have*

(i) $\omega(gh^{-1}) \geq \min(\omega(g), \omega(h))$.

(ii) $\omega(g^{-1}h^{-1}gh) \geq \omega(g) + \omega(h)$.

(iii) $\omega(g^p) = \omega(g) + 1$.

Proof. For the proof of (i) we are going to use triple induction. At first by induction on the number of non-zero coordinates among $(h_\beta)_{\beta\in\Phi\setminus\{\alpha_0\}}$ we are reduced to the case when h is of the form $(n_{\alpha_0}^{-1}n_\alpha)^{h_\alpha}$ or $n_\beta^{h_\beta}$. For simplifying notation we put

$$n'_\alpha := \begin{cases} n_{\alpha_0}^{-1}n_\alpha & \text{if } \alpha \in \Delta \setminus \{\alpha_0\} \\ n_\alpha & \text{if } \alpha \in \Phi^+ \setminus \Delta. \end{cases}$$

So we have $h = n_{\alpha(h)}^{h_{\alpha(h)}}$ for some $\alpha(h)$ in $\Phi^+ \setminus \Delta$. Now we use (descending) induction on $m_{\alpha(h)}$ and suppose that the statement (i) is true for any $\alpha(h)$ with $m_{\alpha(h)} > m_0$ and we are given an h with $m_{\alpha(h)} = m_0$. For this we remark that once N_0 is fixed the set $\{m_\beta\}_{\beta\in\Phi^+}$ is finite. Note that for any β in $\Phi^+ \setminus \{\alpha_0\}$ the commutator $[{n'_\beta}^{g_\beta}, {n'_{\alpha(h)}}^{-h_{\alpha(h)}}]$ is a product of elements $\prod_\alpha {n'_\alpha}^{i_\alpha}$ with $m_\alpha > m_{\alpha(h)}$ by the commutator formula in Proposition 8.2.3 in [11]. Hence by further induction on the number of non-zero coordinates of g for a fixed m_0 we are finally reduced to the case when $g = {n'_{\alpha(g)}}^{g_{\alpha(g)}}$ (and $h = {n'_{\alpha(h)}}^{h_{\alpha(h)}}$). The statement follows applying the commutator formula in Proposition 8.2.3 in [11] once again.

Since we know (i) it suffices to check (ii) in the case $g = {n'_{\alpha(g)}}^{g_{\alpha(g)}}$ and $h = {n'_{\alpha(h)}}^{h_{\alpha(h)}}$. For these this is another application of the commutator formula cited above.

The assertion (iii) is clear from the definition using (i) and (ii). □

Remark. The p-valuation ω extends to N_0 by putting $\omega(n_{\alpha_0}) := m_{\alpha_0} = 1$.

3.2 The ideals J_n

In view of Lemma 1 we define for each positive integer n the normal subgroup $N_{1,n}$ in N_1 as the set of elements g in N_1 with $\omega(g) \geq n$ together with 1. In particular, $N_{1,1} = N_1$. We define $J_n(\Lambda(N_1))$ to be the kernel of the natural surjection $\Lambda(N_1) \twoheadrightarrow \Lambda(N_1/N_{1,n})$. Moreover, we denote by J_n the ideal generated by $J_n(\Lambda(N_1))$ in $\Lambda_\ell(N_0)$. We further have the following

Lemma 2. *$N_{1,n}$ is a normal subgroup in P_0 for any $n \geq 1$. In particular, J_n is the kernel of the natural surjection from $\Lambda_\ell(N_0) = \Lambda_{N_1}(N_0)$ onto $\Lambda_{N_1/N_{1,n}}(N_0/N_{1,n})$. Further, we have $\varphi N_{1,n}\varphi^{-1} \subseteq N_{1,n+1}$. Therefore there is an induced φ-action on each $\Lambda_\ell(N_0)/J_n$ such that the module J_n/J_{n+1} is killed by φ for any $n \geq 1$.*

Proof. The proof of the fact that $N_{1,n}$ is normalized by n_{α_0} is similar to the proof of Lemma 1. If t is in T_0 then we have $tn_\alpha t^{-1} = n_\alpha^{t_\alpha}$ with t_α in $\mathbb{Z}_p^\times$. Hence the first part of the statement. For the second part we note that $\varphi n_\alpha \varphi^{-1} = n_\alpha^{p^{m_\alpha}}$. □

Note that the Jacobson radical $Jac(\Lambda_\ell(N_0))$ is equal to the ideal (π, J_1) by definition of J_1. Moreover, for any element g in $N_{1,n\max_{\beta\in\Phi^+} m_\beta}$ is a product of p^nth powers of elements in N_1, hence $J_{n\max_{\beta\in\Phi^+} m_\beta} \subseteq Jac(\Lambda_\ell(N_0))^n$. In particular, $\bigcap_n J_n = 0$.

Recall that $\Lambda_\ell(N_0)$ is a pseudocompact ring (c.f. [9] Thm. 4.7).

Lemma 3. *If M is any pseudocompact module over $\Lambda_\ell(N_0)$ then J_nM and M/J_nM are also pseudocompact in the subspace, resp. quotient topologies.*

Proof. It suffices to show that J_nM is closed in M. By Lemma 1.6 in [1] and by the fact that the pseudocompact modules form an abelian category ([6] IV.3. Thm. 3) we are reduced to the case when $M = \prod_{i\in I}\Lambda_\ell(N_0)$ with the product topology. However, in this case we have $J_n\prod_{i\in I}\Lambda_\ell(N_0) = \prod_{i\in I} J_n$ as J_n is finitely generated ($\Lambda_\ell(N_0)$ is noetherian), and this is closed in the product topology as J_n is closed in $\Lambda_\ell(N_0)$ using once again that it is finitely generated and hence pseudocompact in the subspace topology of $\Lambda_\ell(N_0)$. □

Lemma 4. *If M is any pseudocompact module over the ring $\Lambda_\ell(N_0)$ then the natural map induces an isomorphism*

$$M \cong \varprojlim_n M/J_nM.$$

Proof. By Lemma 3 the submodules J_nM are closed, and since $J_{n\max_{\beta\in\Phi^+} m_\beta} \subseteq Jac(\Lambda_\ell(N_0))^n$ we have $\bigcap_n J_nM = 0$. The statement follows from IV.3. Proposition 10 in [6]. □

3.3 Main result

Proposition 5. *Let M and N be pseudocompact étale φ-modules over $\Lambda_\ell(N_0)$. Then injective continuous maps (in the pseudocompact topology) $M \hookrightarrow N$ reduce to injective maps $M/J_1M \hookrightarrow N/J_1N$ between the φ-modules over $\Lambda_F(\mathbb{Z}_p)$.*

Proof. Let K_n be the kernel of the induced map from M/J_nM to N/J_nN. We assume indirectly that $K_1 \neq 0$. We show that the natural map from K_n to K_1 is surjective for any n. For this we are going to use the following commutative diagram with some X_n and Y_n.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_n & \longrightarrow & K_n & \longrightarrow & K_1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J_1M/J_nM & \longrightarrow & M/J_nM & \longrightarrow & M/J_1M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J_1N/J_nN & \longrightarrow & N/J_nN & \longrightarrow & N/J_1N & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & Y_n & & & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array} \tag{1}$$

We remark immediately that by Lemma 3 all the modules in the diagram (1) are pseudocompact modules over $\Lambda_\ell(N_0)$, and all the maps are continuous in the pseudocompact topologies. Indeed, the pseudocompact modules form an abelian category ([6] IV.3. Thm. 3).

By the snake lemma we obtain the exact sequence

$$0 \to X_n \to K_n \to K_1 \xrightarrow{\delta_n} Y_n.$$

We claim that there does not exist any nonzero map from K_1 to Y_n. This would show that K_n surjects onto K_1 for any n. As φ is flat over $\Lambda_F(S_\star)$, étale modules form an abelian category over $\Lambda_F(S_\star)$. In particular, K_1 is étale as it is the kernel of a homomorphism between the étale modules M/J_1M and N/J_1N. Therefore if there is a surjective φ-equivariant $\Lambda_F(S_\star)$-homomorphism from K_1 to some module A then we also have that $\varphi(A)$ generates A as a $\Lambda_F(S_\star)$-module. On the other hand, J_1N/J_nN admits the filtration $Fil^k(J_1N/J_nN) := J_kN/J_nN$ for $1 \leq k \leq n$. This induces a filtration $Fil^k(Y_n)$ on Y_n via the above surjection in (1). Let us assume now that δ_n is nonzero. Then there is an integer $k < n$ such that $\delta_n(K_1) \subseteq Fil^k(Y_n)$ but $\delta_n(K_1) \not\subseteq Fil^{k+1}(Y_n)$. Hence we get a nonzero map from K_1 to $Fil^k(Y_n)/Fil^{k+1}(Y_n)$ which we denote by δ'_n. However, we claim that φ acts as zero on the latter which will contradict to the fact that $\varphi(\delta'_n(K_1))$ generates $\delta'_n(K_1)$. Indeed, we have a surjective composite map

$$(J_k/J_{k+1}) \otimes_{\Lambda_\ell(N_0)} N \twoheadrightarrow J_kN/J_{k+1}N \twoheadrightarrow Fil^k(Y_n)/Fil^{k+1}(Y_n),$$

hence $\varphi(Fil^k(Y_n)/Fil^{k+1}(Y_n)) = 0$ as we have $\varphi(J_k) \subseteq J_{k+1}$ by Lemma 2.

Now we have a map from the projective system $(K_n)_n$ to the projective system $(K_1)_n$ which is surjective on each layer, hence its projective limit is also surjective by the exactness of $\varprojlim$ on pseudocompact modules ([6] IV.3. Thm. 3). The statement follows from the completeness of M (Lemma 4). $\square$

Whenever M and N are finitely generated over $\Lambda_\ell(N_0)$ then they admit a unique pseudocompact topology (since $\Lambda_\ell(N_0)$ is pseudocompact and noetherian [9] Thm. 4.7 and Lemma 4.2(ii)) and any homomorphism between them is continuous. So we obtain the main result of this paper as corollary of Proposition 5.

Proposition 6. *The functor from the category of étale $(\Lambda_\ell(N_0), \Gamma, \varphi)$-modules to the category of étale $(\Lambda_F(S_0), \Gamma, \varphi)$-modules induced by the natural surjection*

$$\ell \colon \Lambda_\ell(N_0) \twoheadrightarrow \Lambda_F(\mathbb{Z}_p)$$

is exact.

3.4 Vanishing of higher Tor-functors

Let M be a pseudocompact étale φ-module over $\Lambda_\ell(N_0)$. Then $M/(\pi, J_1)M$ is also a pseudocompact étale φ-module over the field $\Lambda_\ell(N_0)/(\pi, J_1) \cong k((t))$. Hence there is an index set I such that we have an isomorphism of pseudocompact modules

$$M/(\pi, J_1)M \cong \prod_{i \in I} \Lambda_\ell(N_0)/(\pi, J_1)$$

by Lefschetz's Structure Theorem for linearly compact vector spaces ([8], p. 83 Thm. (32.1), see also [5]). Moreover, we have $(\pi, J_1) = Jac(\Lambda_\ell(N_0))$, therefore we obtain a minimal projective cover of M

$$f \colon \prod_{i \in I} \Lambda_\ell(N_0) \twoheadrightarrow M$$

which is an isomorphism modulo (π, J_1).

In this section we need to assume that φ acts continuously on the pseudocompact module M. Note that this is automatic if M is finitely generated over $\Lambda_\ell(N_0)$.

Lemma 7. *Let $F = \prod_{i\in I}\Lambda_\ell(N_0)$ be a φ-module over $\Lambda_\ell(N_0)$. Then F is étale if and only if so is $F/Jac(\Lambda_\ell(N_0))F$ over $k((t))$.*

Proof. If F is étale then by definition so is $F/Jac(\Lambda_\ell(N_0))F$. Now assume that $F/Jac(\Lambda_\ell(N_0))F$ is étale. In other words the map

$$1\otimes\varphi\colon \Lambda_\ell(N_0)\otimes_{\varphi,\Lambda_\ell(N_0)} F \to F \tag{2}$$

is isomorphism modulo $Jac(\Lambda_\ell(N_0))$. Therefore (2) is for instance surjective as its cokernel is pseudocompact and killed by $Jac(\Lambda_\ell(N_0))$. On the other hand, since F is topologically free, we have a continuous section of the map (2). Since (2) is an isomorphism modulo $Jac(\Lambda_\ell(N_0))$, so is this section. However, by the same argument as above this section also has to be surjective and therefore is an inverse to the map (2). □

Proposition 8. *Let M be an étale pseudocompact φ-module over $\Lambda_\ell(N_0)$ with continuous φ-action. Then the action of φ on M can be lifted to $F := \prod_{i\in I}\Lambda_\ell(N_0)$ via the surjection f in (3.4). Any such lift makes F an étale φ-module.*

Proof. Let us define another continuous $\Lambda_\ell(N_0)$-homomorphism

$$\begin{array}{rcl} g\colon \prod_{i\in I}\Lambda_\ell(N_0) & \to & M \\ e_i & \mapsto & \varphi(f(e_i)). \end{array}$$

We need to check that $\lim_{i\in I}\varphi(f(e_i)) = 0$ in the pseudocompact topology of M so that g really defines a continuous homomorphism. This is, however, clear by the continuity of φ and f. By the projectivity of F (Lemma 1.6 in [1]) we obtain a lift φ_{lin}

$$\begin{array}{ccc} & & F \\ & \overset{\varphi_{\mathrm{lin}}}{\nearrow} & \downarrow f \\ F & \xrightarrow{g} & M \end{array}$$

which we define as the linearization of φ on F. Hence we define

$$\varphi(e_i) := \varphi_{\mathrm{lin}}(e_i)$$

and extend it σ_φ-linearly and continuously to the whole F. By construction this is a lift of $\varphi_{|M}$. The étaleness follows from Lemma 7 noting that by construction of (3.4) we have $F/Jac(\Lambda_\ell(N_0))F = M/Jac(\Lambda_\ell(N_0))M$ and the latter is étale as so is M. □

Corollary 9. *For any pseudocompact étale φ-module M over $\Lambda_\ell(N_0)$ with continuous φ and any $i \geq 1$ we have*

$$\mathrm{Tor}^i_{\Lambda_\ell(N_0)}(\Lambda_\ell(N_0)/J_1, M) = 0.$$

Proof. By Proposition 8 there is a projective resolution $(F_i)_{i\in\mathbb{N}}$ of M in the category of pseudocompact $\Lambda_\ell(N_0)$-modules, such that the F_i are étale φ-modules and the resolution is φ-equivariant. By Proposition 5, the functor $\Lambda_\ell(N_0)/J_1 \otimes_{\Lambda_\ell(N_0)} \cdot$ is exact on this resolution. The result follows noting that the modules $\prod_{i\in I}\Lambda_\ell(N_0)$ are flat over the noetherian ring $\Lambda_\ell(N_0)$ as in this case an arbitrary direct product of flat modules is flat again. □

Corollary 10. *Let M be a pseudocompact étale module over $\Lambda_\ell(N_0)$ with continuous φ such that $\pi M = 0$. Then there exists an index set I such that $M \cong \prod_{i\in I}\Lambda_\ell(N_0)/\pi$. In particular, M is a projective object in the category of pseudocompact modules over $\Lambda_\ell(N_0)/\pi$.*

Proof. By Proposition 8 we obtain a minimal projective cover F of M with F admitting an étale lift of the φ-action on M. Since $\pi M = 0$ this factors through $F/\pi F$ which is also étale in the induced φ-action. Now we denote by K the kernel of the map from $F/\pi F$ onto M. Then K is also étale as these form an abelian category. Hence by Proposition 5 we obtain an exact sequence

$$0 \to K/J_1K \to F/(\pi, J_1)F \to M/J_1M \to 0.$$

However, the map $F/(\pi, J_1)F \to M/J_1M$ is an isomorphism by the construction of F (3.4) showing that $K/J_1K = 0$ whence $K = 0$ as K is pseudocompact. □

4 An example

In this section we are going to investigate the so called Steinberg representation. For the sake of simplicity (of the Bruhat-Tits building) we let G be $\mathrm{GL}_{d+1}(\mathbb{Q}_p)$ in this section for some $d \geq 1$ and P be its standard Borel subgroup of lower triangular matrices. Recall that the group $P = NT$ acts on N by $(nt)(n') = ntn't^{-1}$. This induces an action of P on the vector space $V_{St} := C_c^\infty(N)$ of k-valued locally constant functions with compact support on N. It is straightforward to see (cf. Example on p. 8 in [10] and [12] Lemme 4) that the subspace $M := C^\infty(N_0)$ of locally constant functions on N_0 is generating and P_+-invariant. Moreover, it is shown in [10] Lemma 2.6 that we have $D(V_{St}) = \Lambda(N_0)/\pi\Lambda(N_0)$. We have the following refinement of this.

Proposition 11. *Let V_{St} be the smooth modulo p Steinberg representation of the group P. Then we have $D^0(V_{St}) = D(V_{St}) = \Lambda(N_0)/\pi\Lambda(N_0)$, and $D^i(V_{St}) = 0$ for any $i \geq 1$.*

For the proof of Proposition 11 we are going to construct an explicit resolution

$$\mathcal{I}_\bullet\colon 0 \to \mathrm{ind}_{P_0Z}^P(V_d) \to \cdots \to \mathrm{ind}_{P_0Z}^P(V_1) \to \mathrm{ind}_{P_0Z}^P(V_0) \to V_{St} \to 0$$

of V_{St} using the Bruhat-Tits building of G. Here Z denotes the centre of G that will act trivially on each V_i $(0 \leq i \leq d)$. Since $Z \cong \mathbb{Q}_p^\times$, Lemma 11.8 in [10] generalizes to this case with the same proof, so we have $D^0(\mathrm{ind}_{P_0Z}^P(V_i)) = D(\mathrm{ind}_{P_0Z}^P(V_i))$ and $D^i(\mathrm{ind}_{P_0Z}^P(V_i)) = 0$ for all $0 \leq i \leq d$. In particular, we may compute $D^i(V_{St}) = h^i(D(\mathcal{I}_\bullet))$.

Recall that the Bruhat-Tits building $\mathcal{BT}$ of G is the simplicial complex whose vertices are the similarity classes $[L]$ of $\mathbb{Z}_p$-lattices in the vector space $\mathbb{Q}_p^{d+1}$ and whose q-simplices are given by families $\{[L_0], \ldots, [L_q]\}$ of similarity classes such that

$$pL_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_q \subsetneq L_0.$$

Let $\mathcal{BT}_q$ denote the set of all q-simplices of $\mathcal{BT}$. We also fix an orientation of $\mathcal{BT}$ with the corresponding incidence numbers $[\eta : \eta']$. We choose a basis $e_0, \dots, e_d$ of $\mathbb{Q}_p^{d+1}$ in which P is the Borel subgroup of lower triangular matrices and denote the origin of $\mathcal{BT}$ by $x_0 := [\sum_{i=0}^d \mathbb{Z}_p e_i]$. Further, for all $1 \leq i \leq d$ let φ_i be the dominant diagonal matrix $\mathrm{diag}(1, \dots, 1, p, \dots, p)$ with i entries equal to 1 and $d+1-i$ entries equal to p and put $x_i := \varphi_i x_0$. Then T_+/T_0Z is clearly generated by the elements $\{\varphi_i T_0 Z\}_{i=1}^d$ as a monoid. Moreover, for each subset $J = \{j_1 < \dots < j_q\} \subseteq \{1, \dots, d\}$ we define the (oriented) q-simplex

$$\eta_J := \{x_0, x_{j_1}, \dots, x_{j_q}\}.$$

Now we define the coefficient system

$$V_{nt\eta_J} := C_c^\infty\left(nt\left(\bigcap_{j\in J}\varphi_j N_0 \varphi_j^{-1}\right)t^{-1}\right)$$

for any n in N, t in T, and $J \subseteq \{1, \dots, d\}$; and $V_x := 0$ if $\eta \neq b\eta_J$ for any b in P and $J \subseteq \{1, \dots, d\}$. The restriction maps are the natural inclusion maps. Indeed, for any two simplices $\eta_1 \subseteq \eta_2$ such that $V_{\eta_2} \neq 0$ we have a $b = nt$ in P such that $\eta_i = b\eta_{J_i}$ for $J_1 \subseteq J_2 \subseteq \{1, \dots, d\}$ and $i = 1, 2$ therefore $V_{\eta_2} = ntV_{\eta_{J_2}}$ is naturally contained in $V_{\eta_1} = ntV_{\eta_{J_1}}$ by extending the functions f in $V_{nt\eta_J}$ to the whole N by putting $f_{|N\setminus nt\left(\bigcap_{j\in J}\varphi_j N_0\varphi_j^{-1}\right)t^{-1}} = 0$. Later on we will often view elements of $V_{nt\eta_J}$ as functions on N with support in $\mathrm{supp}(V_{nt\eta_J}) = nt\left(\bigcap_{j\in J}\varphi_j N_0 \varphi_j^{-1}\right)t^{-1}$.

Note that V_η is either zero or equal to $\bigcap_{x\in\eta\cap\mathcal{BT}^0} V_x$. It might, however, happen that this intersection is nonzero but $V_\eta = 0$ as η is not in the P-orbit of η_J for any $J \subseteq \{1, \dots, d\}$. We also see immediately that P acts naturally on the coefficient system (V_η) and this action is compatible with the boundary maps. Moreover, we claim

Lemma 12. *We have*

$$\bigoplus_{\eta\in\mathcal{BT}_q} V_\eta \cong \mathrm{ind}_{P_0Z}^P(V_q) \tag{3}$$

with

$$V_q := \sum_{\substack{b_0\in P_0,\ |J|=q\\ J\subseteq\{1,\dots,d\}}} V_{b_0\eta_J} = \bigoplus_{|J|=q, J\subseteq\{1,\dots,d\}} \ \bigoplus_{n_0\in N_0/\bigcap_{j\in J}\varphi_j N_0 \varphi_j^{-1}} V_{n_0\eta_J}.$$

Proof. By construction V_q is a P_0-subrepresentation of $\bigoplus_{\eta\in\mathcal{BT}_q} V_\eta$ so we clearly have a P-equivariant map from the right hand side of (3) to the left hand side. Since V_q contains V_{η_J} for any q-element subset J of $\{1, \dots, d\}$ this map is surjective.

For the injectivity let b be in P with $b\eta_{J_1} = \eta_{J_2}$ for two (not necessarily distinct) subsets J_1 and J_2 of $\{1, \dots, d\}$. Assume that b does not lie in P_0Z. Then we have $bx_0 = \varphi_i x_0$ and $b\varphi_j x_0 = x_0$ for some $1 \leq i, j \leq d$. Hence $b = \varphi_i b_0$ for some b_0 in $\mathrm{Stab}_P(x_0) = P_0Z$ with $\varphi_i b_0 \varphi_j$ lying also in P_0Z. This is a contradiction as $\varphi_i b_0 \varphi_i^{-1}$ is in P_0Z, but $\varphi_i\varphi_j$ is not. It follows that η_{J_1} and η_{J_2} are in different P-orbits of $\mathcal{BT}$ if $J_1 \neq J_2$ (since $\dim_{\mathbb{F}_p} L_{\varphi_i x_0}/pL_0 = p^i$ for all $1 \leq i \leq d$) and $\mathrm{Stab}_P(\eta_J) \subseteq P_0Z$. The statement follows. □

Lemma 13. *The coefficient system $(V_\eta)_\eta$ defines an acyclic resolution of the representation V_{St}, ie. $H_0((V_\eta)_\eta) = V_{St}$ and $H_i((V_\eta)_\eta) = 0$ for all $i \geq 1$.*

Proof. By Lemma 12 we note immediately that the natural map

$$\bigoplus_{\eta\in\mathcal{BT}_0} V_\eta \cong \mathrm{ind}_{P_0Z}^P(V_0) = \mathrm{ind}_{P_0Z}^P(M) \to V_{St} \tag{4}$$

is surjective since M generates V_{St}. On the other hand, if an element f in $\mathrm{ind}_{P_0Z}^P(M)$ lies in the kernel of the above map (4) then for some t in T_+ the support of tf lies in P_+. Hence for proving that f lies in the image of $\bigoplus_{\eta\in\mathcal{BT}_1} V_\eta$ we may assume that f has support in P_+. However, we claim that for any b in P_+ and any element v in V_{bx_0} there is an element v_0 in V_{x_0} such that $v - v_0$ lies in the image of $\bigoplus_{\eta\in\mathcal{BT}_1} V_\eta$. Indeed, if $b = n_0t$ for some n_0 in N_0 and t in T_+ (since $P_+ = N_0T_+$) then v has support in $n_0tN_0t^{-1} \subseteq n_0t\varphi_j^{-1}N_0\varphi_jt^{-1}$ for any j with $t\varphi_j^{-1} \in T_+$. Hence v lies in $V_{\{n_0t\varphi^{-1}x_0, n_0tx_0\}}$ and the claim follows by induction on $K = \sum_{i=1}^d k_i$ with $tT_0Z = \prod_{i=1}^d \varphi_i^{k_i}T_0Z$. This shows that $H_0((V_\eta)_\eta) = V_{St}$.

For the acyclicity of the resolution $(V_\eta)_\eta$ we are going to use Grosse-Klönne's local criterion [7]. To recall his result we need to introduce some terminology. Let $\hat\eta$ be a pointed $(q-1)$-simplex with underlying $(q-1)$-simplex η. Let $N_{\hat\eta}$ be the set of vertices z of $\mathcal{BT}$ such that $(\hat\eta, z)$ is a pointed q-simplex. Each element z in $N_{\hat\eta}$ corresponds to a lattice L_z with $L_{q-1} \subsetneq L_z \subsetneq L_0$ where $(L_0, \dots, L_{q-1})$ represents η. We call a subset M_0 of $N_{\hat\eta}$ stable with respect to $\hat\eta$ if for any two z, z' in M_0 the lattice $L_z \cap L_{z'}$ represents an element in M_0, as well. (By Lemma 2.2 in [7] this is equivalent to the original definition of stability in the case of the Bruhat-Tits building.) By Theorem 1.7 in [7] we need to verify that for any $1 \leq q \leq d$, any pointed $(q-1)$-simplex $\hat\eta$, and any subset M_0 of $N_{\hat\eta}$ that is stable with respect to $\hat\eta$ the sequence

$$\bigoplus_{\substack{z,z'\in M_0\\ \{z,z'\}\in\mathcal{BT}_1}} V_{\{z,z'\}\cup\eta} \to \bigoplus_{z\in M_0} V_{\{z\}\cup\eta} \to V_\eta \tag{5}$$

is exact. Since our coefficient system is P-equivariant, we may assume without loss of generality that $\eta = \eta_J$ for some subset $J \subseteq \{1, \dots, d\}$ with $|J| = q-1$. Let $M_0 \subseteq N_{\hat{\eta_J}}$ be stable with respect to $\hat{\eta_J}$ (here $\hat{\eta_J}$ corresponds to any fixed vertex of η_J). Since the stabilizer of $\eta = \eta_J$ is contained in P_0Z, for any simplex $\nu \supset \eta$ we have $\nu = n_\nu\eta_{J'}$ for some $J' \supset J$ and n_ν in N_0 stabilizing η. In particular, $\mathrm{supp}(V_\nu) = n_\nu\left(\bigcap_{j\in J'}\varphi_jN_0\varphi_j^{-1}\right)$. Hence for any n_0 in N_0 the coset $n_0\bigcap_{j=1}^d \varphi_jN_0\varphi_j^{-1}$ is either contained in $\mathrm{supp}(V_\nu)$ or disjoint from $\mathrm{supp}(V_\nu)$. This means that we have

$$V_\nu = C_c^\infty\left(n_\nu\bigcap_{j\in J'}\varphi_jN_0\varphi_j^{-1}\right) = \bigoplus_{n_0\in n_\nu\bigcap_{i\in J'}\varphi_iN_0\varphi_i^{-1}/\bigcap_{j=1}^d\varphi_jN_0\varphi_j^{-1}} C_c^\infty\left(n_0\bigcap_{j=1}^d\varphi_jN_0\varphi_j^{-1}\right)$$

and it suffices to check the exactness of the restriction of (5) to each coset $n_0\bigcap_{j=1}^d\varphi_jN_0\varphi_j^{-1}$. For any fixed n_0 we multiply the restriction of (5) to the coset $n_0\bigcap_{j=1}^d\varphi_jN_0\varphi_j^{-1}$ by n_0^{-1} and obtain the sequence

$$\bigoplus_{z\neq z'\in n_0^{-1}M_0\cap\{x_0,\dots,x_d\}} C_c^\infty(\bigcap_{j=1}^d \varphi_j N_0\varphi_j^{-1}) \to \bigoplus_{z\in n_0^{-1}M_0\cap\{x_0,\dots,x_d\}} C_c^\infty(\bigcap_{j=1}^d \varphi_j N_0\varphi_j^{-1}) \to C_c^\infty(\bigcap_{j=1}^d \varphi_j N_0\varphi_j^{-1}) \tag{6}$$

since the condition on n_0 lying in $n_\nu \bigcap_{i\in J'} \varphi_i N_0\varphi_i^{-1}/\bigcap_{j=1}^d \varphi_j N_0\varphi_j^{-1}$ is equivalent to that $n_0^{-1}\nu$ is a subsimplex of $\{x_0,\dots,x_d\}$. However, (6) is clearly exact and the lemma follows. □

Proof of Proposition 11. At first we note that Lemma 11.8 in [10] generalizes to our case with the same proof, ie. $D(\mathrm{ind}_{P_0Z}^P(V)) = D^0(\mathrm{ind}_{P_0Z}^P(V))$ and $D^i(\mathrm{ind}_{P_0Z}^P(V)) = 0$ for $i\geq 1$ for any smooth P-representation V with central character since $Z\cong\mathbb{Q}_p^\times$ here, as well. So by Lemmas 12 and 13 (and noting that Z acts trivially on each V_q) we may compute

$$D^i(V_{St}) = h^i(D(\bigoplus_{\eta\in\mathcal{BT}_\bullet} V_\eta)).$$

By Lemma 2.5 in [10] it suffices to show that for any $0\leq q\leq d-1$ and any generating P_+-subrepresentation M_{q+1} of $\mathrm{ind}_{P_0Z}^P(V_{q+1})$ there exists a generating P_+-subrepresentation M_q of $\mathrm{ind}_{P_0Z}^P(V_q)$ such that $M_q\cap\partial_{q+1}(\mathrm{ind}_{P_0Z}^P(V_{q+1}))\subseteq\partial_{q+1}(M_{q+1})$. By (the analogue of) Lemma 3.2 in [10] (see the proof of Lemma 11.8 in [10]) we may assume that M_{q+1} is of the form $M_{q+1} = M_{q+1,\sigma}$ for some order reversing map σ from T_+/T_0Z to $\mathrm{Sub}(V_{q+1})$ satisfying

$$\bigcup_{t\in T_+/T_0Z} \sigma(t) = V_{q+1}.$$

Here $\mathrm{Sub}(V_{q+1})$ denotes the partially ordered set of P_0-subrepresentations of V_{q+1} and

$$M_{q+1,\sigma} = \bigoplus_{t\in T_+/T_0Z} \mathrm{ind}_{P_0Z}^{N_0tP_0Z}\sigma(t)$$

where $\mathrm{ind}_{P_0Z}^X(V)$ denotes the set of functions with support in X from P to V as a subset of $\mathrm{ind}_{P_0Z}^P(V)$ for any P_0Z-representation V and P_0Z-invariant subset X of P.

Moreover, since we have for any n_0 in N_0

$$V_{n_0\eta_J} = C_c^\infty\left(n_0\bigcap_{j\in J}\varphi_jN_0\varphi_j^{-1}\right) = \bigcup_{n=0}^\infty C_c^\infty\left(n_0\bigcap_{j\in J}\varphi_jN_0\varphi_j^{-1}/\bigcap_{j'=1}^d \varphi_{j'}N_0^{p^n}\varphi_{j'}^{-1}\right)$$

with finite sets

$$C_c^\infty\left(n_0\bigcap_{j\in J}\varphi_jN_0\varphi_j^{-1}/\bigcap_{j'=1}^d \varphi_{j'}N_0^{p^n}\varphi_{j'}^{-1}\right),$$

we may further assume (making M_{q+1} possibly even smaller) that σ is induced by an unbounded order reversing map $\sigma_0\colon T_+/T_0Z\to\mathbb{N}\cup\{-1\}$ with

$$\sigma(t) = \sum_{\substack{n_0\in N_0,\ |J|=q+1\\ J\subseteq\{1,\dots,d\}}} V_{n_0\eta_J}(\sigma_0(t))$$

where

$$V_{n_0\eta_J}(\sigma_0(t)) := C_c^\infty\left(n_0\bigcap_{j\in J}\varphi_j N_0\varphi_j^{-1}/\bigcap_{j'=1}^{d}\varphi_{j'}N_0^{p^{\sigma_0(t)}}\varphi_{j'}^{-1}\right) \tag{7}$$

for $\sigma_0(t)\geq 0$ and $V_{n_0\eta_J}(-1):=0$. Now we put

$$M_q := M_{q,\sigma_0} := \bigoplus_{t\in T_+/T_0Z}\mathrm{ind}_{P_0Z}^{N_0tP_0Z}\sum_{\substack{n_0\in N_0,\ |J|=q\\ J\subseteq\{1,\dots,d\}}} V_{n_0\eta_J}(\sigma_0(t))$$

with $V_{n_0\eta_J}(\sigma_0(t))$ defined as in (7). We claim that

$$M_q\cap\partial_{q+1}(\mathrm{ind}_{P_0Z}^P(V_{q+1}))=\partial_{q+1}(M_{q+1}). \tag{8}$$

We now distinguish two cases whether $q=0$ or bigger. In the case $q>0$ the proof of (8) is completely analogous to that of Lemma 13. We see by construction that $\partial_{q+1}(M_{q+1})\subseteq M_q$. Hence we have the following coefficient system on $\mathcal{BT}$ concentrated in degrees $q+1$, q, and $q-1$. In degrees $q+1$ and q we put M_{q+1} and M_q, respectively as subspaces of $\bigoplus_{\eta\in\mathcal{BT}_{q+1}}V_\eta=\mathrm{ind}_{P_0Z}^P(V_{q+1})$ and $\bigoplus_{\eta\in\mathcal{BT}_q}V_\eta=\mathrm{ind}_{P_0Z}^P(V_q)$, respectively. Indeed, we have by construction

$$\begin{aligned} M_{q+1} &= \bigoplus_{\eta\in\mathcal{BT}_{q+1}} M_{q+1}\cap V_\eta;\\ M_q &= \bigoplus_{\eta\in\mathcal{BT}_q} M_q\cap V_\eta. \end{aligned}$$

In degree $q-1$ we put the whole $\mathrm{ind}_{P_0Z}^P(V_{q-1})$. We use Grosse-Klönne's criterion in order to show that the sequence

$$M_{q+1}\to M_q\to\mathrm{ind}_{P_0Z}^P(V_{q-1})$$

is exact which implies (8) as the kernel of the map from M_q to $\mathrm{ind}_{P_0Z}^P(V_{q-1})$ is exactly the left hand side of (8) by Lemma 13. The proof proceeds the same way as in Lemma 13, but here all the functions are constant modulo the subgroup $\bigcap_{j=1}^d\varphi_jN_0^{p^{\sigma_0(t)}}\varphi_j^{-1}$ where t only depends on η (except for the case $\sigma_0(t)=-1$ whence all the functions are zero and the exactness is trivial). The sequence (6) remains exact if we replace $C_c^\infty(\bigcap_{j=1}^d\varphi_jN_0\varphi_j^{-1})$ by

$$C_c^\infty\left(\bigcap_{j=1}^d\varphi_jN_0\varphi_j^{-1}/\bigcap_{j=1}^d\varphi_jN_0^{p^{\sigma_0(t)}}\varphi_j^{-1}\right)$$

hence the statement.

For $q=0$ we have to be a bit more careful, since the inductional argument in the proof of Lemma 13 does not work here as it is not true that any v in $M_0\cap V_{n_0tx_0}$ is equivalent to some v_0 in $M_0\cap V_{x_0}$ modulo $\partial_1(M_1)$. (Note that $M_0\cap V_{x_0}=V_{x_0}(\sigma_0(1))$ but $M_0\cap V_{n_0tx_0}=V_{n_0tx_0}(\sigma_0(t))$ and $\sigma_0(t)$ could be much bigger than $\sigma_0(1)$.) However, we claim that for any v_t in $M_0\cap V_{n_0tx_0}$ with n_0 in N_0 and any $t'\leq t$ in T_+ there exists an element $v_{t'}$ in

$$M_0\cap\left(\bigoplus_{n_1\in N_0/t'N_0t'^{-1}}V_{n_1t'x_0}\right)$$

such that $v_t - v_{t'}$ lies in $\partial_1(M_1)$. The statement is derived from this the following way. Any element m in M_0 is supported on finitely many vertices $\{b_i t_i x_0\}_{i=1}^l$ of $\mathcal{BT}$ with t_i in T_+ and b_i in N_0. Moreover, there is a common t' in T_+ with $t' \leq t_i$ for any $1 \leq i \leq l$. Now if m lies in $M_0 \cap \partial_1(\mathrm{ind}_{P_0Z}^P(V_1))$ then by our claim there exists an m' in

$$M_0 \cap \left(\bigoplus_{n_1 \in N_0/t'N_0t'^{-1}} V_{n_1t'x_0} \right) \tag{9}$$

such that $m - m'$ lies in $\partial_1(M_1)$. However, the map from (9) to V_{St} is injective since the supports of functions in $V_{n_1t'x_0}$ and in $V_{n_1't'x_0}$ are disjoint for $n_1n_1'^{-1}$ not in $t'N_0t'^{-1}$. It follows that $m' = 0$ hence m is in $\partial_1(M_1)$.

For the proof of the claim let v_t be in $M_0 \cap V_{n_0tx_0}$ for some n_0 in N_0 and t in T_+. Then by definition of $V_{n_0tx_0}$ the function v_t is supported on

$$n_0tN_0t^{-1} = \dot{\bigcup_{n_1 \in tN_0t^{-1}/t'N_0t'^{-1}}} n_0n_1t'N_0t'^{-1} \tag{10}$$

since $t' \leq t$ implies $t'N_0t'^{-1} \subseteq tN_0t^{-1}$. Moreover, v_t is constant on the cosets of

$$t\left(\bigcap_{j=1}^d \varphi_j N_0^{p^{\sigma_0(t)}} \varphi_j^{-1}\right)t^{-1}$$

by the definition of M_0. We may assume by induction that $t' = t\varphi_i$ for some $1 \leq i \leq d$. Hence for any n_1 in $tN_0t^{-1}/t\varphi_iN_0\varphi_i^{-1}t^{-1}$ the pair $\{x_0, t^{-1}n_1t\varphi_ix_0\}$ represents an edge of $\mathcal{BT}$. Therefore we have

$$M_1 \cap V_{\{n_0tx_0, n_0n_1t\varphi_ix_0\}} = C_c^\infty(n_0n_1(t\varphi_iN_0\varphi_i^{-1}t^{-1}/t\bigcap_{j=1}^d \varphi_jN_0^{p^{\sigma_0(t)}}\varphi_j^{-1}t^{-1})) \tag{11}$$

and the map

$$\pi_{n_0tx_0} \circ \partial_1 \colon M_1 \cap \left(\bigoplus_{n_1 \in tN_0t^{-1}/t'N_0t'^{-1}} V_{\{n_0tx_0, n_0n_1t\varphi_ix_0\}} \right) \to M_0 \cap V_{n_0tx_0}$$

is surjective comparing (10) and (11). (Here $\pi_{n_0tx_0}$ denotes the projection of M_0 onto $M_0 \cap V_{n_0tx_0}$.) The claim follows noting that

$$\partial_1(M_1 \cap V_{\{n_0tx_0, n_0n_1t\varphi_ix_0\}}) \subseteq M_0 \cap (V_{n_0tx_0} \oplus V_{n_0n_1t\varphi_ix_0}).$$

□

The following is an immediate corollary of Remark 6.4 in [10] using Proposition 11.

Corollary 14. *The natural transformation a_V defined in section 6 of [10] gives an isomorphism*

$$a_{V_{St}} \colon V_{St}^* \to \psi^{-\infty}(D^0(V_{St})).$$

Remark. Proposition 11 (and also Lemmas 12 and 13) remain true in the following more general setting with basically the same proof. Let G still be $\mathrm{GL}_{d+1}(\mathbb{Q}_p)$ and V be a smooth o-torsion P-representation with a unique minimal generating P_+-subrepresentation M. Assume further that $nM \cap M = 0$ for any n in $N \setminus N_0$. Then we have $D^0(V) = D(V)$ and $D^i(V) = 0$ for all $i \geq 1$.

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Gergely Zábrádi
Eötvös Loránd University, Mathematical Institute
Department of Algebra and Number Theory
Budapest
Pázmány Péter sétány 1/C
H-1117
Hungary
zger@cs.elte.hu