Quasi-Hopf Algebras and R-Matrix Structures in Line Bundles over Flag Manifolds

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Abstract: We define a Poisson bracket on the space of sections of holomorphic line bundles over a flag manifold. This bracket is associated to the Drinfeld-Jimbo R-matrix. We quantize this bracket using the theory of quasi-Hopf algebras.

0. Introduction: Among all solutions of the modified classical Yang-Baxter equation on a simple complex Lie algebra $\mathfrak{g}$, there is one distinguished R-matrix, namely the Drinfeld-Jimbo R-matrix. In the Carlan-Weyl basis $\{H_\alpha, X_\alpha, X_{-\alpha}\}$ this R-matrix has the following form

$$r = \frac{1}{2}\sum_\alpha \frac{X_\alpha \wedge X_{-\alpha}}{\langle X_\alpha, X_{-\alpha}\rangle}, \tag{0.1}$$

where α are the positive roots, and $\langle , \rangle$ is the Killing form.

Given a representation $\rho : \mathfrak{g} \to \text{Der}\,(A)$ of $\mathfrak{g}$ into the algebra $\text{Der}\,(A)$ of derivations of some algebra A, one can define the operator

$$A \otimes A \ni a \otimes b \to \{a, b\} = \mu\rho(r)(a \otimes b) \in A \tag{0.2}$$

where $\mu : A \otimes A \to A$ is the multiplication in A. If A is a commutative algebra, then under certain conditions on A, this operator is a Poisson bracket (we call this one an R-matrix bracket). Such, for example, is the case of the space of functions on some orbits in representation spaces (we considered this case in [1]).

In this paper we define an R-matrix Poisson bracket in the space of global sections of holomorphic line bundles over G/B, and then we quantize this bracket. Here, G is the complex Lie group corresponding to the algebra $\mathfrak{g}$, and B is a Borel subgroup of G (G/B can also be realized as the quotient U/T of a compact form, but we shall not need this realization).

The quantization procedure is obtained as follows. As is well known from Borel-Weyl theory, each space V_α of an irreducible finite-dimensional representation of a group G can be realized as the space of global sections of some holomorphic line bundle over G/B. Under this realization, the algebra of holomorphic sections with pointwise multiplication is isomorphic to the algebra

$$\mathcal{M} = \oplus_\alpha V_\alpha$$

with the multiplication given by $a \otimes b \to \mu(a \otimes b)$. Here, α runs over the set of dominant weights, and μ is defined on $V_\alpha \otimes V_\beta$ as the projection $P_{\alpha,\beta}$ on the highest component $V_{\alpha+\beta}$.

We will show in Section 3 that the operator $\{,\}$ defined on this algebra by (0.2) is a Poisson bracket, and then we shall realize a deformation quantization of this bracket. This means that we construct a family of associative algebras with multiplication $\mu_\hbar$ depending on a parameter $\hbar$ and satisfying the correspondence principle

$$\mu_\hbar = \mu \bmod \hbar, \; \mu_\hbar - \mu_\hbar\sigma = \hbar\{,\} \bmod \hbar^2,$$

where μ is the initial commutative multiplication, and σ is the usual transposition: $\sigma(a \otimes b) = b \otimes a$.

The main idea of this quantization is based on the use of the theory of quasi-Hopf algebras developed by Drinfeld ([2], [3]; we recall the necessary aspects of this theory in Section 1), and on our observation that the associativity morphism Φ, induced from Drinfeld's construction on $\mathcal{M}$, is in some sense degenerate. We now describe this in more detail.

Since the initial R-matrix is a modified R-matrix, i.e. the element

$$[|r,r|] = \left[r^{12}, r^{13}\right] + \left[r^{12}, r^{23}\right] + \left[r^{13}, r^{23}\right],$$

where $r^{12} = r \otimes 1$, and so on, is $\mathfrak{g}$-invariant, the result of its quantization can be described in terms of quasi-Hopf algebras. The element $\Phi \in U(\mathfrak{g})^{\otimes 3}$ appears in their definition; this element measures the "defect of associativity" in the category of representations of the corresponding quasi-Hopf algebra.

For comparison, let us note that when quantizing a nonmodified R-matrix, i.e., if the classical Yang-Baxter equation $[|r,r|] = 0$ is satisfied, then the quantization can be realized with $\Phi = 1$. As a result, no problems arise with the associativity of the deformed multiplication $\mu_\hbar$. This approach was used in [4] to construct a "twisted" quantum mechanics.

In the situation considered in the present paper, $\Phi \neq 1$. Though, if the following relation

$$\mu^{123}\rho(\Phi)(a \otimes b \otimes c) = \mu^{123}(a \otimes b \otimes c), \text{ where } \mu^{123} = \mu(\mu \otimes \mathrm{id}) = \mu(\mathrm{id} \otimes \mu)$$

holds, then the deformed multiplication $\mu_\hbar$ remains associative. In Section 2 we establish the fact that in the algebra M the associativity morphism degenerates in this sense, and in Section 3 we use this fact to construct associative deformation structures in this algebra.

We thus realize a "quantum R-matrix structure" in the space of global sections of holomorphic line bundles over a flag manifold, i.e., a family of associative algebras which quantize the R-matrix Poisson bracket.

We think that the quantization method based on the application of quasi-Hopf algebras is remarkably clear and allows one to study the quantum deformations of various algebras form an unified point of view. This point of view reduces to the question: When is the morphism Φ degenerate?

We note that other approaches to the construction of quantum structures on homogeneous spaces were discussed in the papers, of [5] and [6], but there the flatness of the deformation (i.e. question on supply of elements of deformed structure) is not investigated. In frame of our approach we introduce a new multiplication on the same set of elements and the flatness arises automatically.

1. Quasi-Hopf algebras: necessary background

Recall that, according to [2], a quasitriangular quasi-Hopf algebra over a commutative ring k is a collection

$$\mathcal{A} = (A, \Delta, \varepsilon, \Phi, R),$$

where A is an associative algebra with unit, $\Delta : A \to A$ is, a comultiplication operator, $\varepsilon : A \to k$ is a counit operator, R is an invertible element in $A \otimes A$, Φ is an invertible element in $A \otimes A \otimes A$, and this collection satisfies a certain set of axioms.

These axioms are formulated in [2] and we shall not reproduce them here, but shall just clarify their meaning. The idea is that the representations of the algebra A form is quasi-tensor category Rep$\mathcal{A}$ in such a way that, given two representations $\rho_i : A \to \mathrm{End}_k(V_i)$, where $V_i, i = 1, 2$ are k-modules, then the representation of A on $V_1 \otimes V_2$ is given by the composition

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{\varphi_1 \otimes \varphi_2} \mathrm{End}_k(V_1 \otimes V_2).$$

The associativity morphism

$$(V_1 \otimes V_2) \otimes V_3 \to V_1 \otimes (V_2 \otimes V_3)$$

in this category is induced by the element Φ, and the commutativity morphism

$$V_1 \otimes V_2 \to V_2 \otimes V_1$$

is defined as the composition of the morphism induced by the element R with the usual transposition σ. In what follows, we shall use the symbols Φ and R both for the initial elements and for the morphisms that they induce. Moreover, as is customary in a deformation situation, we shall consider R (resp., Φ) as an element of the completion $A^{\widehat{\otimes 2}}$ (resp., $A^{\widehat{\otimes 3}}$) in the $\hbar$-adic topology, where $\hbar$ is the quantization parameter.

The element ε determines the identity representation of k and there are natural isomorphisms of representations

$$V \otimes k \xrightarrow{\sim} V \text{ and } k \otimes V \xrightarrow{\sim} V.$$

In the general situation, the commutativity morphism is not involutive; this is the difference between a tensor and a quasitensor category. If $\Phi = 1$, then this quasi-Hopf algebra is a Hopf algebra.

In addition, the category Rep$\mathcal{A}$ is usually assumed to be rigid. This means that to each object in this category there corresponds a dual object, also in this category. In terms of the algebra $\mathcal{A}$, this can be formulated as the condition of existence of an antipode. The antipode axioms are given in [2], but we shall not need them.

The category of quasi-Hopf algebras, in contrast to that of Hopf algebras, admits "gauge" transformations. Namely, let F be an inversible element in $A \otimes A$ such that

$$(\mathrm{id} \otimes \varepsilon)F = (\varepsilon \otimes \mathrm{id})F = 1.$$

Then one can define a new quasitriangular quasi-Hopf algebra

$$\mathcal{A}_F = (A, \Delta_F, \varepsilon, \Phi_F, R_F),$$

where

$$\Delta_F(a) = F\Delta(a)F^{-1}, \Phi_F = F^{23}\Delta^{23}F\Phi\left(\Delta^{12}F\right)^{-1}\left(F^{12}\right)^{-1}, R_F = F^{21}RF^{-1}.$$

Here, and below, we use the usual notations:

$$\Delta^{12} = \Delta \otimes \mathrm{id},\ F^{12} = F \otimes 1,\ F^{21} = F^2 \otimes F^1,\ \text{where } F = F^1 \otimes F^2,$$

etc.

In this case, it is customary to say that the algebra $\mathcal{A}_F$ is obtained from $\mathcal{A}$ by twisting via F. The meaning of these gauge transformations is that the quasitensor categories Rep$\mathcal{A}$ and Rep$\mathcal{A}_F$ are equivalent. Recall that the equivalence of two monoidal categories

$$(\mathcal{M}, \otimes) \to \left(\widetilde{\mathcal{M}}, \widetilde{\otimes}\right)$$

with tensor product functors $\otimes$ and $\widetilde{\otimes}$, respectively, is given by a pair (α, β), where $\alpha : \mathcal{M} \to \widetilde{\mathcal{M}}$ is a functor, and

$$\beta : \alpha(X \otimes Y) \to \alpha(X)\widetilde{\otimes}\alpha(Y)$$

is an isomorphism.

In our situation, the equivalence Rep$\mathcal{A} \to$ Rep$\mathcal{A}_F$ is given by the pair $(1, \beta)$, where 1 denotes the identity functor, and β is realized by the element F.

In this situation one can identify the set of objects of the equivalent categories and use the isomorphism F to transfer the operations from one category to the other. So, given an operation

$$\Gamma : V_1 \otimes V_2 \to V_3$$

in the category Rep$\mathcal{A}$, one can define an operation

$$\mu_F = \mu F^{-1}$$

in the category Rep$\mathcal{A}_F$.

Let $\mathfrak{g}$ be a simple Lie algebra over a field k of characteristic zero, let $A_0 = U(\mathfrak{g})$ be its universal enveloping algebra with the usual comultiplication $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g})^{\otimes 2}$, defined on the generators by

$$\Delta X = X \otimes 1 + 1 \otimes X$$

and the usual counit $\varepsilon : U(\mathfrak{g}) \to k$. If we also set $\Phi = 1$ and $R = 1$, then A_0 is equipped with the structure of a commutative triangular Hopf algebra, which we shall denote by

$$\mathcal{A}_0 = (A_0, \Delta, \varepsilon, 1, 1).$$

Let $t \in \mathfrak{g} \otimes \mathfrak{g}$ be the split Casimir element, i.e., The symmetric $\mathfrak{g}$-invariant tensor corresponding to the Casimir element $C \in U(\mathfrak{g})$. Consider the associative algebra

$$A_\hbar = U(\mathfrak{g})[[\hbar]] = U(\mathfrak{g})\hat{\otimes}k[[\hbar]],$$

equipped with the operators Δ and ε induced from the algebra A_0. Then, according to Drinfeld's results [2], [3], the following statements are true (here and below, $\widehat{\otimes}$ denotes the tensor product completed in the $\hbar$-adic topology).

Theorem 1.1.

a) *$A_\hbar$ admits a quasitriangular quasi-Hopf algebra structure*

$$\mathcal{A}_\hbar = (A_\hbar, \Delta, \varepsilon, \Phi_\hbar, R_\hbar)$$

where $R_\hbar \in A_0^{\widehat{\otimes}2}$ and $\Phi_\hbar \in A_0^{\widehat{\otimes}3}$ are elements of the following form:

$$R_\hbar = e^{\hbar t/2}, \Phi_\hbar = 1 + \frac{\hbar^2}{4}\varphi + 0(\hbar^2),$$

where $\varphi = [t^{12}, t^{23}] \in \wedge^3\mathfrak{g}$ is a $\mathfrak{g}$-invariant skew-symmetric tensor;

b) *moreover, Φ can be chosen of the form*

$$\exp P(\hbar t^{12}, \hbar t^{23}),$$

where P is a Lie formal series with rational coefficients, i.e., P is an element of the completed free Lie algebra over Q on two generators;

c) *there is an element $F = F_\hbar \in A_0^{\widehat{\otimes}2}$ such that $F = 1 \bmod \hbar$, and such that the quasitriangular algebra $\mathcal{A}_F$ obtained from $\mathcal{A}_\hbar$ by twisting via F is a quasitriangular Hopf algebra, i.e., it has the form*

$$\mathcal{A}_F = (A_\hbar, \Delta_F, \varepsilon, 1, R_F).$$

The latter in particular, means that R_F is a quantum R-matrix, i.e., satisfies the quantum Yang-Baxter equation

$$R_F^{12}R_F^{13}R_F^{23} = R_F^{23}R_F^{13}R_F^{12}.$$

This R-matrix is the result of quantizing the classical R-matrix

$$t_0/2 + t_{+-}.$$

Here, $t = t_0 + t_{+-} + t_{-+}$ is the decomposition of the split Casimir element corresponding to the triangular decomposition of the algebra

$$\mathfrak{g} : \mathfrak{g} = \mathfrak{h} + n_+ + n_-.$$

It follows from part (c) of this theorem, taking the equality

$$R_F = F^{21} e^{\hbar t/2} F^{-1}$$

into account, that if $\overline{r}$ is the linear term in the decomposition of $F_\hbar^{-1}\big(F_\hbar^{-1} = 1 + \hbar\overline{r} + 0(\hbar)\big)$, then the following equality

$$\overline{r} - \overline{r}^{21} = (t_{+-} - t_{-+})/2 \tag{1.1}$$

holds. The right-hand side of this equality is simply another form of writing the R-matrix (0.1).

We can assume that $\mathcal{A}_\hbar$ is obtained from $\mathcal{A}_0$ by a deformation of the associativity and commutativity structures while $\mathcal{A}_F$ is obtained by a deformation of the comultiplication and commutativity structure but the categories Rep$\mathcal{A}_\hbar$ and Rep$\mathcal{A}_F$ are equivalent.

The category Rep$\mathcal{A}_0$ is the usual tensor category of representations of the algebra $\mathfrak{g}$. Taking some liberty, we shall assume that the categories Rep$\mathcal{A}_\hbar$ and Rep$\mathcal{A}_F$ consists of the same objects as the category Rep$\mathcal{A}_0$ (in fact, the objects of the categories Rep$\mathcal{A}_\hbar$ and Rep$\mathcal{A}_F$ are the $k[[\hbar]]$-modules $V\hat{\otimes}_k k[(\hbar)]$, where V is a representation of $\mathfrak{g}$). In what follows, we shall write $\mathrm{Rep}_\hbar\mathfrak{g}$ and $\mathrm{Rep}_F\mathfrak{g}$, instead of Rep$\mathcal{A}_\hbar$ and Rep$\mathcal{A}_F$, respectively.

2. Degeneration of the associativity morphism

In this section we shall formulate conditions on a morphism

$$\nu : V_1 \otimes V_2 \otimes V_3 \to W$$

in the category $\mathrm{Rep}_\hbar\mathfrak{g}$ sufficient for the equality

$$\nu\Phi = \nu.$$

This equality means that, modulo the morphism ν, the "associativity defect" Φ degenerates into the identity morphism.

Definition 2.1 *Let $\nu : V \to W$ be a map of vector spaces, $A : V \to V$ be an operator on V. We say that ν is primitive with respect to A if $\nu A = q\nu$, where q is a constant, called the primitivity constant.*

Lemma 2.2. *Let $\nu : V \to W$ be a primitive map with respect to the operators $A_1, \dots, A_n$ with primitivity constants $q_1, \dots, q_n$, respectively. Let $L = L(A_1, \dots, A_n)$ be a polynomial in the operators A_i, $i = 1, \dots, n$. Then $\nu L = q_L\nu$, where $q_L = L(q_1, \dots, q_n)$. Thus, ν is primitive with respect to L with primitivity constant q_L.*

The proof is obvious.

Proposition 2.3. *Let V_1, V_2, V_3, W be $\mathfrak{g}$-modules*

$$\nu : V_1 \otimes V_2 \otimes V_3 \to W$$

be a map of $\mathfrak{g}$-modules such that ν is primitive with respect to t^{12} and t^{23}. Then $\nu\varphi = 0$ and $\nu\Phi = \nu$, where $\Phi = \Phi_{[\hbar]} \in \widehat{U(\mathfrak{g})^{\otimes 3}}$ is of the form described in Theorem 1.1 (b).

Proof. By theorem 1.1, Φ is of the form

$$1+\sum_{i\geq/2}\hbar^i L_i = 1+\sum_{i\geq/2}\hbar^i A_i\varphi B_i,$$

where A_i and B_i are polynomials in t^{12} and t^{23}. By Lemma 2.2, $\nu L_i = 0$ and, in particular, $\nu\varphi = 0$ (note that $L_2 = \varphi/4$). This finishes the proof.

A $\mathfrak{g}$-module will be calls isotopic if it is the direct sum of isomorphic irreducible $\mathfrak{g}$-modules. In what follows, we shall denote by $\widetilde{V}$ the highest weight of the isotopic $\mathfrak{g}$-module V.

Proposition 2.4. *Let V_1, V_2, W be isotopic $\mathfrak{g}$-modules, $\mu : V_1\otimes V_2 \to W$ a morphism of $\mathfrak{g}$-modules. Then μ is primitive with respect to t with primitivity constant*

$$q\left(\widetilde{V_1},\widetilde{V_2},\widetilde{W}\right) = \left(\left\langle\widetilde{W},\widetilde{W}\right\rangle - \left\langle\widetilde{V_1},\widetilde{V_1}\right\rangle - \left\langle\widetilde{V_2},\widetilde{V_2}\right\rangle + \left\langle\widetilde{W}-\widetilde{V_1}-\widetilde{V_2},2\rho\right\rangle\right)/2$$

where $\langle,\rangle$ is the form induced by the Killing form on the set of roots of the algebra $\mathfrak{g}$, and ρ is half the sum to the positive roots of the algebra $\mathfrak{g}$.

Proof. We decompose $V_1\otimes V_2$ into a sum of isotopic components: $V_1\otimes V_2 = \sum W_i$. The operator t applied to each component reduces to multiplication by a constant. Indeed,

$$t = (\Delta(C) - C\otimes 1 - 1\otimes C)/2,$$

where C is the Casimir operator. The operator $C\otimes 1$ reduces to multiplication of each element of $V_1\otimes V_2$ by $\left\langle\widetilde{V},\widetilde{V_1}+2\rho\right\rangle$, and $1\otimes C$ multiplies each element by $\left\langle\widetilde{V_2},\widetilde{V_2}+2\rho\right\rangle$. If $\widetilde{W_i}\neq\widetilde{W}$, then $\mu(W_i) = 0$. This implies the proposition.

Let now $\mathfrak{g}$ be a complex semisimple Lie algebra. Denote by

$$P_+ = P_+(\mathfrak{g})$$

the semigroup of dominant weights of the algebra $\mathfrak{g}$. For each weight $\alpha\in P_+$, fix an irreducible representation V_α of the given highest weight and highest vector $v_\alpha\in V_\alpha$. For $\alpha,\beta\in P_+$, define a morphism of $\mathfrak{g}$-modules

$$\mu : V_\alpha\otimes V_\beta \to V_{\alpha+\beta}$$

as follows. The module $V_\alpha\otimes V_\beta$ contains a unique submodule with highest weight $\alpha+\beta$. The element $v_\alpha\otimes v_\beta$ is the highest vector of this submodule. Define μ as the composition $\lambda\circ\pi$, where

$$\pi : V_\alpha\otimes V_\beta \to W$$

is the natural projection to the module with highest vector $v_\alpha\otimes v_\beta$, and $\lambda : W\to V_{\alpha+\beta}$ is the morphism of $\mathfrak{g}$-modules defined by

$$\lambda\left(v_\alpha\otimes v_\beta\right) = v_{\alpha+\beta}.$$

Thus, if $a \in V_\alpha$, $b \in V_\beta$, then the product

$$ab = \mu(a \otimes b) \in V_{\alpha+\beta}$$

is defined. This product is obviously commutative and associative.

Define the graded algebra $\mathcal{M} = \mathcal{M}(g)$ as the direct sum

$$\mathcal{M} = \bigoplus_{\alpha \in P_+} V_\alpha$$

with multiplication induced by the multiplication μ on the homogeneous components.

It follows from Proposition 2.4 that $\mu : V_\alpha \otimes V_\beta \to V_{\alpha+\beta}$ is a primitive morphism with respect to t and that

$$\mu t(a \otimes b) = \left\langle \tilde{a}, \tilde{b} \right\rangle \mu(a \otimes b) = \left\langle \tilde{a}, \tilde{b} \right\rangle ab,$$

where we use the notation $\tilde{a} = \alpha$ if $a \in V_\alpha$.

Definition 2.5. Let $\nu : V_1 \otimes V_2 \otimes V_3 \to W$ be a morphism of $\mathfrak{g}$-modules. We say that ν is φ-degenerate if $\nu\varphi = 0$; we say that ν is Φ-degenerate if $\nu\Phi = \nu$.

Set

$$\mu^{123} = \mu(\mu \otimes id) = \mu(id \otimes \mu).$$

Proposition 2.6. *The map*

$$\mu^{123} : \mathcal{M} \ni (a \otimes b \otimes c) \to abc \in \mathcal{M}$$

is φ-degenerate and Φ-degenerate.

Proof. It suffices to check the statement for the homogeneous components in $\mathcal{M}$. Let $V_\alpha, V_\beta, V_\gamma$ be homogeneous components. We shall prove that the map

$$\mu^{123} : V_\alpha \otimes V_\beta \otimes V_\gamma \to V_{\alpha+\beta+\gamma}$$

is primitive with respect to t^{12}. Indeed, as the map μ^{123} factors through $V_\alpha \otimes V_\beta \otimes V_\gamma \to V_{\alpha+\beta} \otimes V_\gamma$, i.e., the diagram

$$\begin{array}{ccc} V_\alpha \otimes V_\beta \otimes V_\gamma & \overset{\mu^{123}}{\longrightarrow} & V_{\alpha+\beta+\gamma} \\ \mu \otimes id \searrow & & \nearrow \mu \\ & V_{\alpha+\beta} \otimes V_\gamma & \end{array}$$

is commutative, the following relation

$$\mu^{123} t^{12} = \langle \alpha, \beta \rangle \mu^{123}$$

holds. The proof that μ is primitive with respect to t^{23} is analogous. Now the proposition follows from Proposition 2.3.

3. R-matrix structures in the algebra $\mathcal{M}$

In this section we shall define an R-matrix Poisson bracket in the algebra $\mathcal{M}$ and then quantize it. We shall then realize the algebra $\mathcal{M}$ itself in terms of bundles over G/B.

In the previous section we defined the commutative algebra $\mathcal{M}$. It is not difficult to see that the action of an arbitrary element $X \in \mathcal{G}$ on $\mathcal{M}$ agrees with the multiplication on $\mathcal{M}$ via the Leibniz identity. Consider the R-matrix (0.1) and define a bilinear operation

$$\mathcal{M} \otimes \mathcal{M} \ni a \otimes b \to \{a, b\} \in \mathcal{M} \tag{3.1}$$

according to formula (0.2).

Clearly, this operator is skew-symmetric and satisfies the Leibniz identity $\{ab, c\} = \{a, c\}b + a\{b, c\}$.

Proposition 3.1. *The operator (3.1) satisfies the Jacobi identity and therefore is a Poisson bracket.*

Proof. A direct computation shows that the Jacobi identity is equivalent to the equation $\mu^{123}\varphi = 0$, which was proved in Proposition 2.6.

We have thus defined a Poisson structure on $\mathcal{M}$ via the R-matrix bracket (3.1).

We now quantize this bracket in the sense of deformation quantization. In order to do this, we introduce a new multiplication $\mu_F = \mu F^{-1}$, where F is as in Theorem 1.1, and we prove that this multiplication is associative.

Indeed,

$$\begin{aligned} \mu_F \mu_F^{12} &= \mu F^{-1} \mu^{12} (F^{12})^{-1} = \mu \mu^{12} (\Delta^{12} F)^{-1} (F^{12})^{-1} = \\ &= \mu \mu^{12} \Phi (\Delta^{12} F)^{-1} (F^{12})^{-1} = \mu \mu^{23} (\Delta^{23} F)^{-1} (F^{23})^{-1} = \\ &= \mu F^{-1} \mu^{23} (F^{23})^{-1} = \mu_F \mu_F^{23}. \end{aligned}$$

Here, we have used the Φ-degeneracy of the multiplication μ^{123}.

We shall write $\mathcal{M}_F$ for $\mathcal{M}$ equipped with the multiplication μ_F. It is not difficult to see that the trivial representation is the unit in this algebra.

Since $F = 1 \bmod \hbar$, equation (1.1) implies that μ_F indeed quantizes the R-matrix Poisson structure in the sense of deformation quantization, i.e., the correspondence principle mentioned in the Introduction holds.

The new algebra is no longer commutative. In order to describe its properties, we shall introduce the operator $\overline{S} = F\sigma F^{-1}$.

Clearly, this operator is involutive, i.e. $\overline{S}^2 = id$, but, unlike $S = \sigma R = F\sigma e^{\hbar t/2} F^{-1}$, it does not satisfy the quantum Yang-Baxter equation (we note here that the quantum Yang-Baxter equation for the operator S has the form

$$S^{12} S^{23} S^{12} = S^{23} S^{12} S^{23}).$$

Proposition 3.2.

(a) *The multiplication* μ_F *is* $\overline{S}$*-commutative, i.e.,*

$$\mu_F \overline{S} = \mu_F;$$

(b) $\overline{S}$ *satisfies the Yang-Baxter equation in a weakened form:*

$$\mu^{123}\overline{S}^{12}\overline{S}^{23}\overline{S}^{12} = \mu_F^{123}\overline{S}^{23}\overline{S}^{12}\overline{S}^{23};$$

(c) *the multiplication* μ_F *is a morphism in the category* Rep ${}_F\mathcal{G}$ *considered as the category of representations of the quasitriangular Hopf algebra (quantum group)* $\mathcal{A}_F$ *(cf. Theorem 1.1).*

All the assertions of Proposition 3.2 are obvious (cf. also [1]).

We have thus defined a classical R-matrix structure on $\mathcal{M}$ via deformation quantization.

These structures can be naturally interpreted in terms of line bundles over the flag manifold G/B. According to Borel-Weyl theory, to each weight α of g there corresponds a holomorphic bundle $L_{\alpha'}$ and the space $\Gamma(L_\alpha, G/B)$ of global sections of this bundle is nontrivial if and only if α is in a dominant weight; moreover, there is an isomorphism $\Gamma(L_\alpha, G/B) \widetilde{\rightarrow} V_\alpha$. We leave it to the reader to check the fact that pointwise multiplication of the sections in the algebra of all global sections corresponds to the multiplication in the algebra $\mathcal{M}$. Thus, all the R-matrix structures constructed above for the algebra $\mathcal{M}$ carry over the algebra $\mathcal{M}(G/B)$ generated by all the global sections. I.e., we have defined in $\mathcal{M}(G/B)$ a (pointwise) multiplication

$$\mu : \mathcal{M}(G/B)^{\otimes 2} \to \mathcal{M}(G/B),$$

an R-matrix Poisson bracket $\{,\}$ of the form (0.2), and an associative multiplication $\mathcal{M}_F$ which is the result of a deformation of the multiplication μ "in the direction" of the bracket $\{,\}$.

Note that the algebra $\mathcal{M}$ can be also regarded as algebra of functions on G/N where $N \subset G$ is a maximal nilpotent subgroup.

Let us now formulate the resulting theorem.

Theorem 3.3. *The R-matrix bracket introduced above (cf. Proposition 3.1) can be quantized in sense of deformation quantization which preserves the natural graduation (by the highest weights) of the algebra* $\mathcal{M}$.

In conclusion, we make some remarks.

Remark 1. Let V_α be the space of an irreducible representation of g with highest weight $\alpha \in P_+(g)$. Consider the orbit $\mathcal{O}_\alpha$ of the highest vector $v_\alpha \in V_\alpha$ of this representation. The algebra $\mathcal{M}_\alpha$ of holomorphic algebraic functions on $\mathcal{O}_\alpha$ naturally embeds in $\mathcal{M}(G/B)$ as a subalgebra. Namely, it is the subalgebra of $\mathcal{M}(G/B)$ generated by the sections $\Gamma(L_{n\alpha'}, G/B)$ of the bundles $L_{n\alpha'}$, where n is a natural number, and α' is the weight of the representation that is dual to V_α.

We note that the R-matrix bracket and its quantizing algebraic structure can be extended naturally to the fields of quotients of the algebras $\mathcal{M}$ and $\mathcal{M}_\alpha$ (it is not difficult to see that they do not contain zero-devizors).

Remark 2. On all orbits as above, the stabilizer algebra of an arbitrary point contains some maximal nilpotent subalgebra of $\mathfrak{g}$. There is another natural class of orbits on which an expression of the form (0.2) is a Poisson bracket. These are the orbits with stabilizers H such that G/H is a symmetric space (here $\mathfrak{g}$ can be compact form of a simple Lie algebra). The quantization scheme proposed above no longer extends to such spaces. We propose to consider the questions connected with the quantization of R-matrix brackets on such spaces in subsequent publications.

Remark 3. The sphere S^2 is an orbit in g^*, where $g = su_2$ and is a symmetric space $S^2 = SU_2/T$ of the type described above. An R-matrix bracket is thus defined on S^2 as on a symmetric space. The 2-sphere is also a flag manifold SL_2/B, so an R-matrix bracket is defined as above on the line bundles over S^2. We stress the fact that these are different brackets and that they are defined on different objects.

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