Max-Planck-Institut für Mathematik
Bonn

The Eisenstein elements inside the space of modular symbols

by

Debargha Banerjee
Srilakshmi Krishnamoorthy

Max-Planck-Institut für Mathematik
Preprint Series 2013 (42)

The Eisenstein elements inside the space of modular symbols

Debargha Banerjee
Srilakshmi Krishnamoorthy

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

176, S. N. Roy Road
Kolkata, 700038
West Bengal
India

debargha.banerjee@gmail.com
lakshmi@imsc.res.in

MPIM 13-42

THE EISENSTEIN ELEMENTS INSIDE THE SPACE OF MODULAR SYMBOLS

DEBARGHA BANERJEE AND SRILAKSHMI KRISHNAMOORTHY

Abstract. We explicitly write down the *Eisenstein elements* inside the space of modular symbols for all Eisenstein series, giving a complete answer to a question of Merel. We also compute the winding element explicitly for level product of two distinct odd primes and power of odd primes, proving explicit versions of the *Manin-Drinfeld* theorem in these cases.

1. Introduction

Merel initiated the study of the Eisenstein elements inside the space of the modular symbols while proving *Uniform boundedness theorem* about torsion points of elliptic curves over number fields. His fundamental vision was to replace quotient of the Jacobian induced by Mazur's Eisenstein ideals [11] of the Hecke algebras with slightly bigger ideals, namely winding quotients induced by the winding elements of the space of modular symbols. Merel explicitly computed the winding element of the space of modular symbols using another explicit element of the space of modular symbols, which is the Eisenstein element corresponding to the unique Eisenstein series for the congruence subgroup $\Gamma_0(p)$ [15], [14]. Extending the fundamental work of Merel, we wrote down the *Eisenstein elements* and the *winding element* explicitly inside the space of modular symbols for the congruence subgroup $\Gamma_0(p^2)$ [2]. In the present paper [cf. Theorem 9], we write the Eisenstein elements completely inside the space of modular symbols for all Eisenstein series. By this explicit computation, we wish to understand the index of Eisenstein ideals and hence the possibility of congruence between cusp forms and Eisenstein series of general level.

The winding elements are the elements of the space of modular symbols whose annihilators define ideals of the Hecke algebras with the L-functions of the corresponding quotients of the Jacobian non-zero. In the present paper, we write down the winding elements explicitly for $N = pq$ with p and q distinct odd primes and also for a power of a distinct odd prime proving explicit versions of the *Manin-Drinfeld* theorem. We note that Manin-Drinfeld proved that the modular symbol $\{0, \infty\} \in \mathrm{H}_1(X_\Gamma, \mathbb{Q})$ using the theory of Hecke operator acting on the space of modular symbols. We follow the approach of Merel [cf. [15], Prop. 11]. Our explicit computation should be useful to compute certain quantities related to the statements and the reformulations of the Birch and Swinnerton-Dyer Conjecture. In particular, the explicit expression of winding element should be useful to understand the algebraic part of the *special values of the L-functions* [1].

In the last section, we attempt to generalize the Mazur invariant to the general level. If the level of the modular curves are not prime, then the orders of cuspidal subgroups and Shimura subgroups are

2010 *Mathematics Subject Classification.* Primary: 11F67, Secondary: 11F11, 11F20, 11F30.

Key words and phrases. Eisenstein series, Modular symbols, Special values of L-functions.

not same. In general, the order of the Shimura subgroups are not divisible by the order of the divisor $(0)-(\infty)$. As a consequence, we can write down the analogue of the Mazur invariant only if the order of the Shimura subgroup divides the order of the divisor $(0)-(\infty)$.

2. Acknowledgements

This paper owes it's existence to several e-mail communications, encouragements and several valuable suggestions of Professor Löic Merel. D. B. is grateful to Professor Joseph Oesterlé for sharing with him his preprints on the subject.

3. Cusps and Eisenstein series

Recall [4], the cusps of principal congruence subgroups $\Gamma(N)$ are $\Gamma(N)s$ with $s=\frac{a}{c}$ and $\pm[a,c]$ (mod N) pair of integers such that $gcd(a,c)=1$.

Lemma 1. *Let the integers a,c have images $\overline{a},\overline{c}$ in $\mathbb{Z}/N\mathbb{Z}$. Then the following are equivalent:*

- *$(\overline{a},\overline{c})$ has a lift $(a,c)\in\mathbb{Z}^2$ with $gcd(a,c)=1$,*
- *$gcd(a,c,N)=1$,*
- *$(\overline{a},\overline{c})$ has order N in the additive group $(\mathbb{Z}/N\mathbb{Z})^2$.*

For $v=(a_1,a_2)\in(\frac{1}{N}\mathbb{Z})^2$ and $v\notin\mathbb{Z}^2$, we define the Siegel functions by

$$g_v(\tau)=-q_\tau^{\frac{1}{2}B_2(a_1)}e^{2\pi i\frac{a_2(a_1-1)}{2}}(1-q_z)\prod_{n=1}^{\infty}(1-q_\tau^n q_z)(1-\frac{q_\tau^n}{q_z}).$$

Here, $B_2(X)=X^2-X+\frac{1}{6}$ is the second Bernoulli polynomial, $q_\tau=e^{2\pi i\tau}$ and $q_z=e^{2\pi iz}$ for $z=a_1\tau+a_2$.

Remark 2. By [[7], p. 31], g_v^{12N} are modular functions for the congruence subgroup $\Gamma(N)$. They have no zeros or poles on the upper half plane. The logarithmic derivative of the Siegel unit g_v is denoted by $2\pi i\phi_v$. The properties of the Eisenstein series ϕ_v are listed in [cf. [17], Section 2.4]. For any congruence subgroup Γ, let $E_2(\Gamma)$ be the space of Eisenstein series of weight two w. r. t. this congruence subgroup. The set of all ϕ_v's form a basis of $E_2(\Gamma(N))$.

Let $\epsilon_\infty(N)$ be the number of cusps for the congruence subgroup $\Gamma_0(N)$. Let $\alpha_k,\beta_l,\gamma_m$ and κ_s are the matrices $\begin{pmatrix}0&-1\\1&k\end{pmatrix}$, $\begin{pmatrix}-1&-l\\p&lp-1\end{pmatrix}$, $\begin{pmatrix}-1&-m\\q&mq-1\end{pmatrix}$, $\begin{pmatrix}1&0\\s&1\end{pmatrix}$ respectively. Let V be the set of all α_t's for t coprime to N.

We fix a point $z_0\in\mathbb{H}$. Let $c(\gamma)$ be the geodesic in Y_Γ joining z_0 and $\gamma(z_0)$. Let $\pi_E(\gamma)=\int_{c(\gamma)}E(z)dz$ be the period of the Eisenstein series E. The following proposition summarizes some well-known properties of the map π_E. The proofs are given in [[17], Prop. 2.3.3].

Proposition 3. *Let $\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ be an element of Γ.*

(1) *π_E is a homomorphism $\mathrm{H}_1(Y_\Gamma,K)\to K$.*

(2) $\pi_E(\gamma) = \frac{a+d}{c}a_0(E) - \frac{1}{2\pi i}L(E[\begin{pmatrix} 1 & -d \\ 0 & c \end{pmatrix}], 1)$ *if* $c \neq 0$ *and* $\pi_E(\gamma) = \frac{b}{d}a_0(E)$ *if* $c = 0$.

There is an isomorphism $\delta : E_2(\Gamma) \longrightarrow \mathrm{Div}^0(X_\Gamma, \mathrm{cusps}, K)$, where $\mathrm{Div}^0(X_\Gamma, \mathrm{cusps}, K)$ is the group of degree zero divisors supported on cusps. For all cusps x, let $e_\Gamma(x)$ denote the ramification index of x over $X(1)$. The Eisenstein series E corresponds to the divisor $\delta(E) = \sum_{x \in \Gamma\backslash\mathbb{P}^1(\mathbb{Q})} e_\Gamma(x)a_0(E[x])\{x\}$.

3.1. **Eisenstein series for $\Gamma_0(pq)$.** Consider the series $E_2'(\tau) = \frac{-1}{24} + \sum_n \sigma_1(n)q^n$, where $\sigma_1(n)$ denote the sum of the positive divisors of n. For $t \in \{p, q, pq\}$, let E_t be the Eisenstein series

$$E_t(\tau) = E_2'(\tau) - tE_2'(t\tau).$$

Lemma 4. *The set $\mathbb{E}_{pq} = \{E_p, E_q, E_{pq}\}$ represents a basis of $E_2(\Gamma_0(pq))$.*

Proof. Let $N = pq$ in [[4], Theorem 4.6.2]. □

Lemma 5. *The cusps of $\Gamma_0(pq)$ can be identified with the set $\{0, \infty, \frac{1}{p}, \frac{1}{q}\}$.*

Proof. If $\frac{a}{c}$ and $\frac{a'}{c'}$ are in $\mathbb{P}^1(\mathbb{Q})$, then $\Gamma_0(pq)\frac{a}{c} = \Gamma_0(pq)\frac{a'}{c'} \iff \begin{pmatrix} ay \\ c \end{pmatrix} \equiv \begin{pmatrix} a' + jc' \\ c'y \end{pmatrix} \pmod{pq}$, for some j and y such that $\gcd(y, pq) = 1$ [4]. A small check shows that the orbits $\Gamma_0(pq)0$, $\Gamma_0(pq)\infty$, $\Gamma_0(pq)\frac{1}{p}$, $\Gamma_0(pq)\frac{1}{q}$ are disjoint. □

The congruence subgroup $\Gamma_0(pq)$ has 4 cusps [Lemma 5]. By [[18], p. 538], we see that

$$e_{\Gamma_0(pq)}(x) = \begin{cases} q & \text{if } x = \frac{1}{p} \\ p & \text{if } x = \frac{1}{q} \\ 1 & \text{if } x = \infty \\ pq & \text{if } x = 0. \end{cases}$$

Noting $\sum_{x \in \mathrm{Cusps}\{\Gamma_0(pq)\}} e_{\Gamma_0(pq)}(x)a_0(E[x]) = 0$ $(\delta(E) \in \mathrm{Div}^0(X_0(pq), \mathrm{Cusps}, K))$,

$$\delta(E) = a_0(E)(\{\infty\} - \{0\}) + qa_0(E[\frac{1}{p}])(\{\frac{1}{p}\} - \{0\}) + pa_0(E[\frac{1}{q}])(\{\frac{1}{q}\} - \{0\}).$$

Lemma 6. *The constant Fourier coefficients of E_p, E_q and E_{pq} at cusps 0, $\frac{1}{p}$, $\frac{1}{q}$ and ∞ are as follows:*

	0	$\frac{1}{p}$	$\frac{1}{q}$	∞
E_p	$\frac{1-p}{24p}$	$\frac{p-1}{24}$	$\frac{1-p}{24p}$	$\frac{p-1}{24}$
E_q	$\frac{1-q}{24q}$	$\frac{1-q}{24q}$	$\frac{q-1}{24}$	$\frac{q-1}{24}$
E_{pq}	$\frac{1-pq}{24pq}$	0	0	$\frac{pq-1}{24}$

Proof. We first prove that the constant coefficient for the Fourier expansion of E_{pq} at the cusp $\frac{1}{p}$ is 0. As usual, the constant term of the Fourier expansion of E_{pq} at the cusp $\frac{1}{p}$ is the constant term at ∞ of $E_{pq}[\beta_0]$. Similarly, the constant term of the Fourier expansion of E_{pq} at the cusp $\frac{1}{q}$ is the constant term at ∞ of $E_{pq}[\gamma_0]$. Let Δ be the Ramanujan's cusp form of weight 12. We write

$\frac{d}{dz}\log\Delta(\beta(z)) = 12\frac{d}{dz}\log(pz+1) + \frac{d}{dz}\log\Delta(z)$ for $\beta = \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}$. A simple calculation shows that $\Delta(\frac{pqz}{pz+1}) = \Delta(\begin{pmatrix} q & 0 \\ 1 & 1 \end{pmatrix} pz) = (\frac{pz+1}{q})^{12}\Delta(\frac{pz+1}{q})$. By taking logarithmic derivative, we deduce that

$$\frac{d}{dz}log\Delta\begin{pmatrix} q & -1 \\ 1 & 0 \end{pmatrix}(\frac{pz+1}{q}) = 12\frac{d}{dz}\log(pz+1) + \frac{d}{dz}\log\Delta(\frac{pz+1}{q}).$$

Since $E_{pq}(z) = \frac{1}{2\pi i}\frac{d}{dz}\log\frac{\Delta(pqz)}{\Delta(z)}$, the above calculation shows that the constant term of E_{pq} at the cusp $\frac{1}{p}$ is 0. Similarly, the constant term of E_{pq} at the cusp $\frac{1}{q}$ is 0. The constant term of E_{pq} is $\frac{pq-1}{24}$ at the cusp ∞ and $\frac{1-pq}{24pq}$ at 0. For the Eisenstein series $E_p \in E_2(\Gamma_0(p))$, $\frac{1}{p}$ represents the cusp ∞ and $\frac{1}{q}$ represents the cusp 0. We deduce that $a_0(E_p[\beta_0]) = \frac{p-1}{24}$ and $a_0(E_p[\gamma_0]) = \frac{1-p}{24p}$.

For the other Eisenstein series $E_q \in E_2(\Gamma_0(q))$, $\frac{1}{q}$ represents the cusp ∞ and $\frac{1}{p}$ represents the cusp 0. We deduce that $a_0(E_1[\gamma_0]) = \frac{q-1}{24}$ and $a_0(E_1[\beta_0]) = \frac{1-q}{24q}$. □

4. Modular Symbols

Let T, S and $R = ST$ be the matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$respectively. Let $\zeta : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{H}_1(X_\Gamma, \mathrm{cusps}, K)$ be the map that takes the matrix $g \in \mathrm{SL}_2(\mathbb{Z})$ to the class in $\mathrm{H}_1(X_\Gamma, \mathrm{cusps}, K)$ of the image in X_Γ of the geodesic in $\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$ joining $g.0$ and $g.\infty$.

Theorem 7.
- *The map ζ is surjective.*
- *For all $g \in \Gamma\backslash\mathrm{SL}_2(\mathbb{Z})$, $\zeta(g) + \zeta(gS) = 0$ and $\zeta(g) + \zeta(gR) + \zeta(gR^2) = 0$.*

Proof. Manin [10]. □

Let $\rho = \frac{1+\sqrt{-3}}{2}$ and $\rho^* = -\bar{\rho}$ be the points on the boundary of the fundamental domain. For all $g \in \mathrm{SL}_2(\mathbb{Z})$, let $\{g\rho, g\rho^*\}$ be the image in $X_\Gamma(\mathbb{C})$ of the geodesic in $\mathbb{H}$ joining the points $g.\rho$ and $g.\rho^*$.

Lemma 8. *Let $K \subset \mathbb{Z}^{\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})}$ be the set consisting of all formal sums $\sum_g \lambda_g g$ with $\lambda_g + \lambda_{gS} = 0$ and $\lambda_g + \lambda_{gR} + \lambda_{gR^2} = 0$. The map $K \to \mathrm{H}_1(Y_\Gamma, \mathbb{Z})$ given by $\sum_g \lambda_g g \to \sum_g \lambda_g g\{i, \rho\}$ is an isomorphism. Every closed path in $\mathrm{H}_1(Y_\Gamma, \mathbb{Z})$ is generated by $\{g\rho, g\rho^*\}$.*

Proof. By [[13], [12]], every element of $\mathrm{H}_1(Y_\Gamma, \mathbb{Z})$ can be written as $\sum_g \lambda_g\{gi, g\rho\}$ with $\lambda_g + \lambda_{gS} = 0$. Since $\{gi, g\rho\} + \{g\rho^*, gi\} = \{g\rho^*, g\rho\}$, hence the lemma follows. □

5. Eisenstein elements inside the space of modular symbols

Following [15] and [12], we briefly recall the concept of Eisenstein elements of the space of Modular symbols. Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. This subgroup acts on the upper half plane $\mathbb{H}$ in the usual way. The quotient space $\Gamma\backslash\mathbb{H}$ is denoted by Y_Γ. Let X_Γ be the compactification of Y_Γ obtained by adjoining the set of cusps $\Gamma\backslash\mathbb{P}^1(\mathbb{Q})$. Let $E_2(\Gamma)$ be the space of weight 2 Eisenstein series for this congruence subgroup. Suppose the Fourier coefficients at ∞ of an Eisenstein series E are in a fixed number field K. Let $\pi_E : \mathrm{H}_1(Y_\Gamma, K) \to K$ be the "period" homomorphism of E. The intersection pairing $\circ$ [[12]] induces a perfect, bilinear pairing of K-vector spaces

$$\mathrm{H}_1(X_\Gamma, \mathrm{cusps}, K) \times \mathrm{H}_1(Y_\Gamma, K) \to K.$$

Since $\circ$ is a non-degenerate bilinear pairing, there is a unique element $\mathcal{E}$ such that $\mathcal{E} \circ c = \pi_E(c)$. The modular symbol $\mathcal{E}$ is the *Eisenstein element* corresponding to the Eisenstein series E.

We write down Eisenstein elements for all Eisenstein series w. r. t. the principal congruence subgroup $\Gamma(N)$. Recall, we have an isomorphism $\Gamma(N)\backslash \mathrm{SL}_2(\mathbb{Z}) \cong \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. Define a function $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) \to K$ by

$$G_v(g) = \begin{cases} \frac{1}{12} & \text{if } \overline{vg} = (0, \frac{s}{N}) \\ -\frac{1}{12} & \text{if } \overline{vg} = (\frac{r}{N}, 0) \\ \frac{1}{2}B_2(\frac{r}{N}) + E(\frac{r}{N}, \frac{s}{N}) & \text{Otherwise.} \end{cases}$$

Theorem 9. *Let $E \in E_2(\Gamma(N))$ be an Eisenstein series such that the Fourier coefficients at ∞ of E are in a fixed number field K. If ϕ_v is a well-known basis element of $E_2(\Gamma(N))$(Remark 2), then the Eisenstein element is completely determined by the function G_v. The error terms $E(\frac{r}{N}, \frac{s}{N})$ are always bounded by* $\log(N^2)$.

Proof. Let $\mathcal{E} = \sum_{g \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})} F_E(g)\zeta(g)$ be an element inside the space of modular symbols corresponding to the Eisenstein series $\phi_v \in E_2(\Gamma(N))$. Since $\zeta(g) = -\zeta(gS)$, the functions $G_E(g) = F_E(g) - F_E(gS)$ determine uniquely an element of the space of modular symbol. Let $c \in \mathrm{H}_1(Y(N), \mathbb{Z})$ be an arbitrary closed cycle. By Lemma 8, we write $c = \sum_g \lambda_g g\{i, \rho\}$ with $\lambda_g + \lambda_{gS} = 0$ and hence $\mathcal{E} \circ c = \sum_g F_E(g)\lambda_g$[[13], Cor. 3]. From the consideration of the Fundamental domain [16], we conclude that

$$g\{0, \infty\} \circ h\{\rho, \rho^*\} = \begin{cases} 1 & \text{if } \Gamma g = \Gamma h \\ -1 & \text{if } \Gamma g = \Gamma h s \\ 0 & \text{otherwise.} \end{cases}$$

If $\mathcal{E}$ is the Eisenstein element inside the space of modular symbol corresponding to the Eisenstein series E, then $\mathcal{E} \circ c = \int_c E_2^v(z)dz$. On the other hand, we conclude that $\int_c E_2^v(z) = \sum_g \frac{\lambda_g}{2\pi i}[\log g_v(gi) - \log g_v(g\rho)]$.

Since $\mathrm{H}_1(Y(N), \mathbb{Z})$ is generated by the images of the path of the form $g\{\rho, \rho^*\}$, the function $G_v(g) := \frac{1}{2\pi i}[\log g_{vg}(\rho) - \log g_{vg}(\rho^*)]$ uniquely determines the Eisenstein element of the modular symbol for all Eisenstein series ϕ_v.

We now compute the values of the functions G_v explicitly for all Eisenstein series ϕ_v with $v \in (\mathbb{Z}/N\mathbb{Z})^2 - \{0\}$. For all $h \in \Gamma(N)\backslash \mathrm{SL}_2(\mathbb{Z})$, we compute

$$G_v(h) = F_E(h) - F_E(hS) = \int_{h\rho^*}^{h\rho} \phi_v(z)dz = \int_{\rho^*}^{\rho} \phi_v[h](z)dz = \frac{1}{2\pi i}[\log g_{vh}(\rho) - \log g_{vh}(\rho^*)].$$

The Siegel unit has a decomposition as a product of the Klein forms t_{vh} and the 12 th root of Ramanujan's Δ function. By taking the logarithm, we get an expression $\log g_{vh}(\tau) = \log t_{vh}(\tau) + \frac{1}{12}\log \Delta(\tau)$.

Since $\log\Delta(\rho)-\log\Delta(\rho^*)=2\pi i$, we conclude that $\log g_{vh}(\rho)-\log g_{vh}(\rho^*)=\log t_{vh}(\rho)-\log t_{vh}(\rho^*)+\frac{\pi i}{6}$. By the well-known decomposition of the Klein form,

$$\log t_{(\frac{r}{N},\frac{s}{N})}(\tau,1)=-(\frac{r\eta_1(\tau)+s\eta_2(\tau)}{2N})(\frac{r\tau+s}{N})+\log\sigma(\frac{r\tau+s}{N},\tau,1).$$

Here, η_1 and η_2 are the quasi-periods attached to the lattice $[\eta,1]$. The above calculation shows,

$$[\log g_{vg}(\rho)-\log g_{vg}(\rho^*)]=-\pi irt+E(\frac{r}{N},\frac{s}{N})+\frac{\pi i}{6}$$

for $E(\frac{r}{N},\frac{s}{N})=\log\sigma(\frac{r\rho+s}{N},[\rho,1])-\log\sigma(\frac{r\rho^*+s}{N},[\rho,1])$ and some number t depending on $\overline{vg}$.

If $\overline{vg}=(0,\frac{s}{N})$, then $\log g_{(0,\frac{s}{N})}(\rho)-\log g_{(0,\frac{s}{N})}(\rho^*)=\frac{\pi i}{6}$. Since $[\rho,1]$ and $[\rho^*,1]$ generate the same lattice, there is no contribution from the sigma function. Note that $\overline{vg}=(\frac{r}{N},0)$ if and only if $\overline{vg}S=(0,-\frac{r}{N})$ and hence by [[6], Prop. 2.1], $\log g_{(\frac{r}{N},0)}(S\rho^*)-\log g_{(\frac{r}{N},0)}(S\rho)=\log g_{(0,-\frac{r}{N})}(\rho^*)-\log g_{(0,-\frac{r}{N})}(\rho)=-\frac{\pi i}{6}$. In the remaining cases, we use [[3], Prop. 2.3] to compute the bounds of the error terms as in the statement. □

Unfortunately, $\log(N^2)$ is not a very good bound and we loose lot of information by keeping the error terms. In the next corollary, we improve the error terms.

Corollary 10. *If $\overline{vg}\neq(0,\frac{r}{N})$ or $(\frac{r}{N},0)$, we simplify the function as $G_E(g)=-\frac{s^2}{2N^2}+\frac{1}{12}+T(\frac{r}{N},\frac{s}{N})$. The error term $T(\frac{r}{N},\frac{s}{N})$ depends on the theta functions.*

Proof. By [[4], Chap. 4.8], we have $\eta_2(\Lambda_\tau)=G_2(\tau)$ and $\eta_1(\Lambda_\tau)=\tau G_2(\tau)-2\pi i$. From $T\rho^*=\rho$ and the well-known transformation formula for G_2[[4] Ch. 1], we deduce that $G_2(\rho)=G_2(\rho^*)$. The matrix ST fixes the root of unity ρ^*. By the same transformation formula, we conclude that $G_2(\rho^*)=2\pi ix$. From Theorem 9, $[\log g_{vg}(\rho)-\log g_{vg}(\rho^*)]=-\frac{\pi ir}{N^2}[(x\sqrt{-3}-1)r+2sx]+E(\frac{r}{N},\frac{s}{N})+\frac{\pi i}{6}$. We calculate the values of the $E(\frac{r}{N},\frac{s}{N})$. Following [[19], p. 473], we write the sigma function in terms of theta functions as $\sigma(z,[\omega_1,\omega_2])=\frac{2\omega_1}{\pi\nu_1'}\exp(\frac{\eta_1z^2}{2\omega_1})\nu_1(\frac{\nu z}{2\omega_1},[\frac{\omega_2}{\omega_1},1])$. By putting $u=\rho\rho^*=-1$, we deduce that

$$G_E(g)=\frac{1}{12}-\frac{1}{2N^2}[(xr^2\sqrt{-3}-r^2)+2srx-r^2(\sqrt{-3}x-1)-2srx-\frac{s^2}{u}]+T(\frac{r}{N},\frac{s}{N}).$$

By a further obvious simplification, we get the function as in the statement. □

Corollary 11. *For any congruence subgroup Γ and any Eisenstein series $E\in E_2(\Gamma)$, we can explicitly determine the Eisenstein elements inside the space of modular symbols.*

Proof. Since the Eisenstein series of all levels of weight two can be written as a linear combination of logarithmic derivatives of g_v and the intersection pairing is a non-degenerate bilinear pairing, the Eisenstein element corresponding to any Eisenstein series is completely determined by the above theorem. For any congruence subgroup Γ and $E\in E_2(\Gamma)$, we define a function $H_E:\mathrm{SL}_2(\mathbb{Z})\to\mathbb{C}$ by $H_E(g)=\int_\rho^{\rho^*}E[g]dz$. By the proof of the above theorem, the Eisenstein elements inside the space of modular symbol are completely determined by the function H_E. □

5.1. **Calculation of the error terms.** The mathematical software GP/PARI may be used to calculate the error terms of the Corollary 10. We compute the error term using the command $\log(theta(x,s))$.

5.2. **Remarks about the other expressions of the Eisenstein elements.** If $\Gamma = \Gamma_0(p)$ or $\Gamma = \Gamma_0(p^2)$, then Eisenstein elements are calculated in [15] and [2]. We note that the expressions of the Eisenstein elements in these cases are more explicit. We list the expression of the Eisenstein elements in these cases for the sake of completeness. Let $F_p : \mathbb{P}^1(\mathbb{Z}/p\mathbb{Z}) \to \mathbb{Z}$ be the function defined as follows:

$$F_p(x) = \begin{cases} \sum_{h=0}^{p-1} B_1(\frac{hr}{2p}) & \text{if } x = (r-1, r+1), \\ 0 & \text{if } x = (\pm 1, 1). \end{cases}$$

Theorem 12 ([15]). *If $\Gamma = \Gamma_0(p)$ and E is a well-known basis element of $E_2(\Gamma_0(p))$, then*

$$\mathcal{E} = \sum_{g \in \mathbb{P}^1(\mathbb{Z}/p^2\mathbb{Z})} F_p(g)\zeta(g).$$

Let $F_{p^2} : \mathbb{P}^1(\mathbb{Z}/p^2\mathbb{Z}) \to K$ be the function defined as follows:

$$F_{p^2}(x) = \begin{cases} \frac{1}{2\pi i}(2L(E[\alpha_r], 1) - L(E[\beta_r], 1)) & \text{if } x = (r-1, r+1), \\ \int_0^\infty 2(2E[\nabla_k] - E[\kappa_k])dz & \text{if } x = (1+kp, 1), \\ -F_E((kp+1, 1)) & \text{if } x = (kp-1, 1), \\ 0 & \text{if } x = (\pm 1, 1). \end{cases}$$

Theorem 13 ([2]). *If $\Gamma = \Gamma_0(p^2)$ and E is a well-known basis element of $E_2(\Gamma_0(p^2))$([2]), then*

$$\mathcal{E} = \sum_{g \in \mathbb{P}^1(\mathbb{Z}/p^2\mathbb{Z})} F_{p^2}(g)\zeta(g).$$

In these papers, the strategy was to calculate the Eisenstein elements by intersecting with $\Gamma(2)$ and then prove that these are actually the Eisenstein elements by showing they have the same boundary. Unfortunately, it may be difficult to use the same method for other congruence subgroups. In the present paper, we calculate the Eisenstein elements without using boundary calculation.

6. Explicit computation of winding elements

We now recall the definition of the *winding element*. Let $\{0, \infty\}$ denote the projection of the path from 0 to ∞ in $\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$ to $X_0(N)(\mathbb{C})$. We have an isomorphism $\mathrm{H}_1(X_0(N), \mathbb{Z}) \otimes \mathbb{R} = \mathrm{Hom}_{\mathbb{C}}(\mathrm{H}^0(X_0(N), \Omega^1), \mathbb{C})$. Let $e_N \in \mathrm{H}_1(X_0(N), \mathbb{Z}) \otimes \mathbb{R}$ corresponds to the homomorphism $\omega \to \int_0^\infty \omega$. The element e_N is called the *winding element*. The explicit expression of e_p is the key tool in the proof of the Uniform boundedness theorem [14]. In [2], we explicitly computed e_{p^2}. We may compute the winding elements explicitly for all congruence subgroups using Theorem 9. In the present paper, we write down the winding elements explicitly for two important cases, namely for $N = pq$ and $N = p^r$.

6.1. **Level product of two distinct odd primes.** In this section, we compute the winding element for the level $N = pq$ product of 2 distinct odd primes. There is a canonical bijection $\Gamma_0(pq)\backslash \mathrm{SL}_2(\mathbb{Z}) \cong \mathbb{P}^1(\mathbb{Z}/pq\mathbb{Z})$ given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to (c, d)$.

Lemma 14. *The set $T = \{I, \alpha_k, \beta_l, \gamma_m | 0 \le k \le pq-1, 0 \le l \le (q-1), 0 \le m \le (p-1)\}$ forms a complete set of coset representatives of $\mathbb{P}^1(\mathbb{Z}/pq\mathbb{Z})$.*

Proof. The orbits $\Gamma_0(pq)\alpha_k$, $\Gamma_0(pq)\beta_l$ and $\Gamma_0(pq)\gamma_m$ are disjoint since ab^{-1} do not belong to $\Gamma_0(pq)$ for two distinct matrices a, b from the set T. There are $1+pq+p+q = |\mathbb{P}^1(\mathbb{Z}/pq\mathbb{Z})|$ coset representatives. $\square$

The following lemma will be useful in the computation of the *winding elements*.

Lemma 15. *We list rational numbers coming from coset representatives as equivalence classes of cusps as follows:*

0	$\frac{1}{p}$	$\frac{1}{q}$
$\frac{-l}{lp-1}$, $(lp-1,q)=1$	$\frac{-1}{k}$, $(k,p)>1$	$\frac{-1}{k}$, $(k,q)>1$
$\frac{-m}{mq-1}$, $(mq-1,p)=1$	$\frac{-m}{mq-1}$, $(mq-1,p)>1$	$\frac{-l}{lp-1}$, $(lp-1,q)>1$

.

Proof. Follows from a careful analysis using Lemma 5. $\square$

We have a short exact sequence,

$$0 \to \mathrm{H}_1(X_0(N),\mathbb{Z}) \to \mathrm{H}_1(X_0(N), cusps, \mathbb{Z}) \to \widetilde{\mathrm{H}_0}(cusps) \to 0.$$

The first map is a canonical injection. The second map (boundary map) δ takes a geodesic, joining the cusps r and s to the formal symbol $[r]-[s]$ and the third map is the sum of the coefficients.

The following proposition presents a sharp difference between the level $N = p^2$ and $N = pq$. If the level is $N = pq$, then to find the winding element we need to use Eisenstein elements corresponding to all three Eisenstein series of the well-known basis. If the level is p^2, then the winding element is determined by exactly one Eisenstein series. This proposition is an explicit version of the *Manin-Drinfeld* theorem for the congruence subgroup $\Gamma_0(pq)$ with p and q distinct odd primes.

Proposition 16. *If $N = pq$, the winding element is determined by the Eisenstein elements corresponding to all 3 Eisenstein series of the basis $\mathbb{E}_{pq}$.*

Proof. Since the map $\zeta : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{H}_1(X_0(pq), \mathrm{cusps}, \mathbb{Z})$ is surjective, we write the Eisenstein element corresponding to $E \in E_2(\Gamma_0(pq))$ as $\mathcal{E} = \sum_{g\in\Gamma_0(pq)\backslash \mathrm{SL}_2(\mathbb{Z})} F_E(g)\zeta(g)$ for some $F_E : \mathbb{P}^1(\mathbb{Z}/pq\mathbb{Z}) \to \mathbb{Q}$.

$$\mathcal{E} = \sum_{g\in\mathbb{P}^1(\mathbb{Z}/pq\mathbb{Z})} F_E(g)\{g.0, g.\infty\} = F_E(0,1)\{0\ ,\infty\} + F_E(1,0)\{\infty, 0\} + \sum_{x_a\in V} F_E(x_a)\{0, \frac{1}{a}\}$$
$$+\sum_{k=1}^{p-1} F_E(1,kq)\{\frac{-1}{kq}, 0\} + \sum_{k=1}^{q-1} F_E(1,kp)\{\frac{-1}{kp}, 0\}$$
$$+\sum_{l=0}^{q-1} F_E(p,lp-1)\{\frac{-l}{lp-1}, \frac{-1}{p}\} + \sum_{m=0}^{p-1} F_E(q,mq-1)\{\frac{-m}{mq-1}, \frac{-1}{q}\}.$$

By Lemma 15, there exists exactly one r such that $1 \le r \le (q-1)$, $rp-1$ is a multiple of q, and there exists exactly one j such that $1 \le j \le (p-1)$, $jq-1$ is a multiple of p. By Corollary 11, the Eisenstein element is determined by numbers $F_E(h) - F_E(hs) = \int_{h\rho}^{h\rho^*} E(z)dz$. Hence $[F_E(0,1) - F_E(1,0)] = -\int_{\rho^*}^{T\rho^*} E(z)dz = -a_0(E)$. Using the lemma 15,

$$\mathcal{E} = -a_0(E)\{0,\infty\} + \sum_{x_a\in V} F_E(x_a)\{0, \frac{1}{a}\} + \sum_{k=1}^{p-1} F_E(1,kq)\{\frac{1}{q}, 0\} + \sum_{k=1}^{q-1} F_E(1,kp)\{\frac{1}{p}, 0\} +$$

$$+\sum_{l=0,l\neq r}^{q-1} F_E(p,lp-1)\{0,\frac{1}{p}\}+\sum_{m=0,m\neq j}^{p-1} F_E(q,mq-1)\{0,\frac{1}{q}\}+F_E(p,rp-1)\{\frac{1}{q},\frac{1}{p}\}+F_E(q,jq-1)\{\frac{1}{p},\frac{1}{q}\}.$$

Simplifying further, $\mathcal{E}=-a_0(E)\{0,\infty\}+A(E)\{0,\frac{1}{p}\}+B(E)\{0,\frac{1}{q}\}+\sum_{x_a\in V}F_E(x_a)\{0,\frac{1}{a}\}$, where

$$A(E)=\sum_{l=0}^{q-1}F_E(p,lp-1)-\sum_{k=1}^{q-1}F_E(1,kp)-F_E(q,jq-1),$$

$$B(E)=\sum_{m=0}^{p-1}F_E(q,lq-1)-\sum_{k=1}^{q-1}F_E(1,kq)-F_E(p,rp-1).$$

Note that for $0\le l\le(q-1)$ and $l\neq r$, there are integers k and a such that $k(lp-1)+aq=1$ with $\Gamma_0(pq)\beta_l S=\alpha_{kp}$. A simple calculation shows that $\Gamma_0(pq)\gamma_j S=\beta_r$. Similarly for $0\le m\le(p-1)$ with $m\neq j$, we have $\Gamma_0(pq)\gamma_m=\alpha_{tq}$. Here, t and b are some integers satisfying $t(mq-1)+bp=1$. We also see that $\Gamma_0(pq)\gamma_j S=\beta_r$. Thus, $A(E)=\sum_{l=0}^{q-1}[F_E(\beta_l)-F_E(\beta_l.s)], B(E)=\sum_{m=0}^{p-1}[F_E(\gamma_m)-F_E(\gamma_m.s)]$.

For $t\in\{p,q,pq\}$, let $\Delta_t(z)=\frac{\Delta(tz)}{\Delta(z)}$ be the modular units on $Y_0(pq)$. We deduce that,

$$A(E)=\sum_{l=0}^{q-1}\int_{\beta_l\rho}^{\beta_l\rho^*}E(z)dz=\sum_{l=0}^{q-1}\int_{\beta_lT\rho^*}^{\beta_l\rho^*}\frac{1}{2\pi i}\frac{d}{dz}\log\Delta_t(z)dz=\sum_{l=0}^{q-1}\frac{1}{2\pi i}[\log\Delta_t(\beta_l\rho^*)-\log\Delta_t(\beta_lT\rho^*)].$$

Note that for all $0\le l<(q-1)$, $\beta_lT=\beta_{l+1}$ and $\beta_{q-1}T=\gamma\beta_0$ for $\gamma=\begin{pmatrix}1+pq & q\\ -qp^2 & 1-qp\end{pmatrix}$. Hence, we prove that $A(E)=\pi_E(\gamma)$. Similarly, we prove $B(E)=\pi_E(\gamma_0)$ for $\gamma_0=\begin{pmatrix}1+pq & p\\ -q^2p & 1-qp\end{pmatrix}$. The Eisenstein element is given by $\mathcal{E}=-a_0(E)\{0,\infty\}+A(E)\{0,\frac{1}{p}\}+B(E)\{0,\frac{1}{q}\}+\sum_{x_a\in V}F_E(x_a)\{0,\frac{1}{a}\}$. Since $\int_{\mathcal{E}}\omega=0$ for all holomorphic differentials ω, the winding element is determined by the above three equations in three unknowns corresponding to 3 Eisenstein series in $\mathbb{E}_{pq}$. If the determinant of the matrix

$$\begin{pmatrix}-a_0(E_p) & \pi_{E_p}(\gamma) & \pi_{E_p}(\gamma_0)\\ -a_0(E_q) & \pi_{E_q}(\gamma) & \pi_{E_q}(\gamma_0)\\ -a_0(E_{pq}) & \pi_{E_{pq}}(\gamma) & \pi_{E_{pq}}(\gamma_0)\end{pmatrix}$$

is non-zero, then we explicitly write down the winding element in $\mathrm{H}_1(X_0(pq),\mathbb{R})$. □

We have an inclusion $\Gamma_0(pq)\subset\Gamma_0(p)$, which induces a map between Riemann surfaces $\pi:Y_0(pq)\to Y_0(p)$. We have induced maps $\pi_*:\mathrm{H}_1(X_0(pq),cusps,\mathbb{Z})\to\mathrm{H}_1(X_0(p),cusps,\mathbb{Z})$ and

$$\pi^*:\mathrm{H}_1(X_0(p),cusps,\mathbb{Z})\to\mathrm{H}_1(X_0(pq),cusps,\mathbb{Z})$$

induced by the isomorphism $\mathrm{H}_1(X_\Gamma,cusps,\mathbb{Z})\cong\mathrm{H}^1(Y_\Gamma,\mathbb{Z})$. Now, by the general property of the Riemann surface $\pi_*\circ\pi^*=deg(\pi)$. Let $\mathcal{E}^{(p)}$ be the Eisenstein element in $\mathrm{H}_1(X_0(p),cusps,\mathbb{Z})$ obtained from the differential form $E_p(z)dz$ on $Y_0(p)$ and $\mathcal{E}^{(pq)}$ be the Eisenstein element in $\mathrm{H}_1(X_0(pq),cusps,\mathbb{Z})$ obtained from the same differential form $E_p(z)dz$ on the Riemann surface $Y_0(pq)$.

Lemma 17. *If $E=E_p$, then $\mathcal{E}^{(pq)}=\frac{1}{deg\pi}\pi^*(\mathcal{E}^{(p)})$.*

Proof. Let $c \in \mathrm{H}_1(Y_0(pq), \mathbb{Z})$, then we have $\pi^*(\mathcal{E}^{(p)}) \circ c = \pi_*\pi^*(\mathcal{E}^{(p)} \circ \pi_*(c) = deg(\pi)\mathcal{E}^{(p)} \circ \pi_*(c) = deg(\pi)\int_{\pi_*(c)} E_p(z)dz$ The second equality follows from [5], p. 198. But then $deg(\pi)\int_{\pi_*(c)} E_p(z)dz = \int_c \pi^*(E_1^{(p)}dz) = \mathcal{E}^{(pq)} \circ c$. By the uniqueness of the Eisenstein elements, we have, $\mathcal{E}^{(pq)} = \frac{1}{deg(\pi)}\pi^*(\mathcal{E}^{(p)})$. □

The following proposition presents a sharp difference between the Eisenstein elements for level $N = p^2$ and level $N = pq$.

Proposition 18. *For $E \in \mathbb{E}_{pq}$, the boundary of the Eisenstein element inside the space of modular symbols is not a constant multiple of the boundary of E.*

Proof. For $E \in \mathbb{E}_2(\Gamma_0(pq))$, the boundary of E is

$$\delta(E) = a_0(E)(\{\infty\} - \{0\}) + qa_0(E[\frac{1}{p}])(\{\frac{1}{p}\} - \{0\}) + pa_0(E[\frac{1}{q}])(\{\frac{1}{q}\} - \{0\}).$$

From Lemma 6, we deduce that $\delta(E_p) = \frac{p-1}{24}[\{\infty\} - \{0\} + q\{\frac{1}{p}\} - \{\frac{1}{q}\}]$, $\delta(E_q) = \frac{q-1}{24}[\{\infty\} - \{0\} - \{\frac{1}{p}\} + p\{\frac{1}{q}\}]$, and $\delta(E_{pq}) = \frac{pq-1}{24}[\{\infty\} - \{0\}]$. We now calculate the boundary of the corresponding Eisenstein elements inside the space of modular symbols. From Proposition 16, the Eisenstein element corresponding to an Eisenstein series is always given by $\mathcal{E} = -a_0(E)\{0, \infty\} + A(E)\{0, \frac{1}{p}\} + B(E)\{0, \frac{1}{q}\} + \sum_{x_a \in V} F_E(x_a)\{0, \frac{1}{a}\}$. Clearly, if $E \in \mathbb{E}_{pq}$ then the boundary of the Eisenstein elements are not constant multiples of the boundary of the corresponding Eisenstein series. □

6.2. **Level power of an odd prime.**

Lemma 19. *The set $T = \{I, \alpha_t, \kappa_{kp^s} | 0 \leq t \leq p^r - 1, 1 \leq s \leq (r-1), 1 \leq k \leq (p-1)\}$ forms a complete set of coset representatives of $\mathbb{P}^1(\mathbb{Z}/p^r\mathbb{Z})$.*

Proof. The orbits $\Gamma_0(p^r)\alpha_t$ and $\Gamma_0(p^r)\kappa_{kp^s}$ are disjoint since the matrices $\alpha_t\alpha_{t'}^{-1}, \kappa_{kp^s}\kappa_{k'p^{s'}}^{-1}$ and $\alpha_t\kappa_{kp^s}^{-1}$ do not belong to $\Gamma_0(p^r)$. There are $p^r + p^{r-1} - 1 + 1 = p^r + p^{r-1} = |\mathbb{P}^1(\mathbb{Z}/p^r\mathbb{Z})|$ coset representatives. □

Lemma 20. *The cusps of $\Gamma_0(p^r)$ can be identified with the set $\{0, \infty, \frac{1}{kp}, ..., \frac{1}{lp^{r-1}} | 1 \leq k \leq p-1, .., 1 \leq l \leq (p-1)\}$.*

Proof. Let $P = \{\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, n \in \mathbb{Z}\}$ be the parabolic subgroup inside the modular group $\mathrm{SL}_2(\mathbb{Z})$. There is a well known bijection between the set of cusps $\Gamma_0(p^r)\backslash\mathbb{P}^1(\mathbb{Q})$ and the double coset space $\Gamma_0(p^r)\backslash\mathrm{SL}_2(\mathbb{Z})/P$. The statement is obvious from the previous lemma and this bijection. □

Proposition 21. *The winding element for $N = p^r$ is determined by the Eisenstein elements corresponding to r Eisenstein series in the basis $E_2(\Gamma_0(p^r))$ with non-zero constant Fourier co-efficients at ∞.*

Proof. We consider the basis of $E_2(\Gamma_0(p^r))$ as in [[4], Theorem 4.6.2]. There are r Eisenstein series with non-zero constant Fourier co-efficients at ∞. Since the map $\zeta : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{H}_1(X_0(p^r), \text{cusps}, \mathbb{Z})$ is surjective, we write the Eisenstein element corresponding to $E \in E_2(\Gamma_0(p^r))$ as $\mathcal{E} = \sum_{g \in \Gamma_0(p^r)\backslash\mathrm{SL}_2(\mathbb{Z})} F_E(g)\zeta(g)$ for some $F_E : \mathbb{P}^1(\mathbb{Z}/p^r\mathbb{Z}) \to K$. By Lemma 19, we have

$$\mathcal{E} = \sum_{g \in \mathbb{P}^1(\mathbb{Z}/p^r\mathbb{Z})} F_E(g)\{g.0, g.\infty\} = F_E(0,1)\{0\ ,\infty\} + F_E(1,0)\{\infty, 0\} + \sum_{x_a \in V} F_E(x_a)\{0, \frac{1}{a}\}$$

$$+\sum_{k=1}^{p-1} F_E(1,kp)\{0,\frac{-1}{kp}\}+\sum_{l=1}^{p-1} F_E(1,lp^{r-1})\{0,\frac{-1}{lp^{r-1}}\}..+\sum_{k=1}^{p-1} F_E(kp,1)\{\frac{1}{kp},0\}+\sum_{l=1}^{p-1} F_E(lp^{r-1},1)\{\frac{1}{lp^{r-1}},0\}.$$

By Corollary 11, the Eisenstein element is determined by the numbers $F_E(h) - F_E(hS) = \int_{h\rho}^{h\rho^*} E(z)dz$. Hence, we have $[F_E(0,1) - F_E(1,0)] = -\int_{\rho^*}^{T\rho^*} E(z)dz = -a_0(E)$. Finally, we deduce that

$$\mathcal{E} = -a_0(E)\{0,\infty\} + \sum_{x_a \in V} F_E(x_a)\{0,\frac{1}{a}\} + \sum_{k=1}^{p-1}[F_E(kp,1) - F_E(1,-kp)]\{\frac{1}{kp},0\}$$

$$... + \sum_{l=1}^{p-1}[F_E(lp^{r-1},1) - F_E(1,-lp^{r-1})]\{\frac{1}{lp^{r-1}},0\}.$$

Since $\int_{\mathcal{E}} \omega = 0$ for all holomorphic differentials ω, we get r equations in $\epsilon_\infty(p^r) - 1 > r$ variables with the right hand sides entries in $\mathrm{H}_1(X_0(p^r),\mathbb{Q})$. The solutions give explicit expressions of the winding elements inside the space of modular symbols for $N = p^r$. □

7. Cuspidal and Shimura subgroups

We recall that the natural map $X_1(N) \to X_0(N)$, induces by functoriality a map between the corresponding Jacobian groups. The Shimura subgroup is the kernel of the above map. The order X of the Shimura subgroups are being calculated in [9]. We calculate the order Y of the divisor $(0) - (\infty)$ from [8]. We summarize the orders in the following table:

N	X	Y
p	n	n
p^2	n	$\frac{p^2-1}{24}$
p^r, $r \geq 3$	$p^{r-1-[\frac{r}{2}]}n$	$p^{r-1}a.b$
pq, $p \neq q$	$\frac{(p-1)(q-1)}{2r}$	$\frac{(p^2-1)(q^2-1)}{m}$

If $N = p$ the Mazur invariant is completely understood by the fundamental work of Merel using the explicit expression of the winding element for $N = p$. From the above table, $X \nmid Y$ for non-prime level in general. If $X \mid Y$, then it makes sense to talk about the Mazur invariant. We expect that the explicit expression of the winding element will help us to define an analogue of the *Mazur invariant* and hence a necessary and sufficient condition of monogenity of the completed Hecke algebra at the Eisenstein ideal. Mazur invariant can be written in terms of the winding element for the general level also if $X \mid Y$[[15], Prop. 12].

References

[1] A. Agashe, The Birch and Swinnerton-Dyer Conjecture for modular abelian varities of analytic rank zero, Ph.D. thesis, University of California at Berkeley (2000).

[2] D. Banerjee, *A note on the Eisenstein elements of prime square level*, To appear in the Proceedings of the American Mathematical Society.

[3] Y. Bilu and P. Parent, *Runge's method and modular curves*, Int. Math. Res. Not. IMRN (2011), no. 9, 1997–2027.

[4] F. Diamond and J. Shurman, A first course in modular forms, Vol. 228 of *Graduate Texts in Mathematics*, Springer-Verlag, New York (2005), ISBN 0-387-23229-X.

[5] A. Dold, Lectures on algebraic topology, Classics in Mathematics, Springer-Verlag, Berlin (1995), ISBN 3-540-58660-1. Reprint of the 1972 edition.

[6] S. Kamienny and G. Stevens, *Special values of L-functions attached to $X_1(N)$*, Compositio Math. **49** (1983), no. 1, 121–142.

[7] D. S. Kubert and S. Lang, Modular units, Vol. 244 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science]*, Springer-Verlag, New York (1981), ISBN 0-387-90517-0.

[8] S. Ling, *On the $\mathbb{Q}$-rational cuspidal subgroup and the component group of $J_0(p^r)$*, Israel J. Math. **99** (1997) 29–54.

[9] S. Ling and J. Oesterlé, *The Shimura subgroup of $J_0(N)$*, Astérisque (1991), no. 196-197, 6, 171–203 (1992). Courbes modulaires et courbes de Shimura (Orsay, 1987/1988).

[10] J. I. Manin, *Parabolic points and zeta functions of modular curves*, Izv. Akad. Nauk SSSR Ser. Mat. **36** (1972) 19–66.

[11] B. Mazur, *Modular curves and the Eisenstein ideal*, Inst. Hautes Études Sci. Publ. Math. (1977), no. 47, 33–186 (1978).

[12] L. Merel, *Intersections sur des courbes modulaires*, Manuscripta Math. **80** (1993), no. 3, 283–289.

[13] ———, *Homologie des courbes modulaires affines et paramétrisations modulaires*, in Elliptic curves, modular forms, & Fermat's last theorem (Hong Kong, 1993), Ser. Number Theory, I, 110–130, Int. Press, Cambridge, MA (1995).

[14] ———, *Bornes pour la torsion des courbes elliptiques sur les corps de nombres*, Invent. Math. **124** (1996), no. 1-3, 437–449.

[15] ———, *L'accouplement de Weil entre le sous-groupe de Shimura et le sous-groupe cuspidal de $J_0(p)$*, J. Reine Angew. Math. **477** (1996) 71–115.

[16] M. Rebolledo, *Module supersingulier et homologie des courbes modulaires*, J. Number Theory **121** (2006), no. 2, 234–264.

[17] G. Stevens, Arithmetic on modular curves, Vol. 20 of *Progress in Mathematics*, Birkhäuser Boston Inc., Boston, MA (1982), ISBN 3-7643-3088-0.

[18] ———, *The cuspidal group and special values of L-functions*, Trans. Amer. Math. Soc. **291** (1985), no. 2, 519–550.

[19] E. T. Whittaker and G. N. Watson, A course of modern analysis. An introduction to the general theory of infinite processes and of analytic functions: with an account of the principal transcendental functions, Fourth edition. Reprinted, Cambridge University Press, New York (1962).

IMSC, Chennai, India