# HIGHER ALGEBRAIC $K$-THEORY FOR TWISTED LAURENT SERIES RINGS OVER ORDERS AND SEMISIMPLE ALGEBRAS

ADEREMI KUKU

Abstract. Let $R$ be the ring of integers in a number field $F$, $\Lambda$ any $R$-order in a semisimple $F$-algebra $\Sigma$, $\alpha$ an $R$-automorphism of $\Lambda$. Denote the extension of $\alpha$ to $\Sigma$ also by $\alpha$. Let $\Lambda_\alpha[T]$ (resp. $\Sigma_\alpha$ be the $\alpha$-twisted Laurent series ring over $\Lambda$ (resp. $\Sigma$). In this paper we prove that

(i) There exist isomorphisms $\mathbb{Q}\otimes K_n(\Lambda_\alpha[T]) \simeq \mathbb{Q}\otimes G_n(\Lambda_\alpha[T]) \simeq \mathbb{Q}\otimes K_n(\Sigma_\alpha[T])$ for all $n\geq 1$.

(ii) $G_n^{\mathrm{pr}}(\Lambda_\alpha[T],\hat{Z}_l)\simeq G_n(\Lambda_\alpha[T],\hat{Z}_l)$ is an $l$-complete profinite Abelian group for all $n\geq 2$.

(iii) $\operatorname{div} G_n^{\mathrm{pr}}(\Lambda_\alpha[T],\hat{Z}_l)=0$ for all $n\geq 2$.

(iv) $G_n(\Lambda_\alpha[T])\longrightarrow G_n^{\mathrm{pr}}(\Lambda_\alpha[T],\hat{Z}_l)$ is injective with uniquely $l$-divisible cokernel (for all $n\geq 2$).

(v) $K_{-1}(\Lambda)$, $K_{-1}(\Lambda_\alpha[T])$ are finitelz generated Abelian groups.

## Contents



## Introduction

Let $R$ be the ring of integers in a number field $F$. The initial motivation for this work was a desire to obtain results on higher $K$-theory of the groupring $RV$ of virtually infinite cyclic group of the form $V=G\rtimes_\alpha T$, where $G$ is a finite group, $\alpha$ an automorphism of $G$ and the action of the infinite cyclic group $T=\langle t\rangle$ on $G$ is given by $\alpha(g)=tgt^{-1}$ for all $g\in G$.

Note that undestading $K$-theory of $RV$ is fundamental to Farrel-Jones conjecture which asserts that $K$-theory an arbitrary discrete group $H$ should have as "building blocks" the $K$-theory of virtually cyclic subgroups of $H$ (see [8]). A group $V$ is virtually cyclic if it is either finite or virtually infinite cyclic (i.e., contains a finite index subgroup that is infinite cyclic). For results on higher $K$-theory of grouprings of finite groups see [15, chapter 7] and associated references. There are two types of virtually infinite cyclic groups — one type of the form $V = G \rtimes_\alpha T$ as described above and the other of the form $V = G_0 *_H G_1$, where the groups $G_0$, $G_1$, $H$ are finite and $[G_0 : H] = [G_1 : H] = 2$. For some results on higher $K$-theory of both types of groups see [15, 7.5] or [16]. In this paper, we obtain results on higher $K$-theory of twisted Laurent series ring that translate into results on grouprings $RV$, $V = G \rtimes_\alpha T$, as we now explain.

If $\alpha$ is an automorphism of a finite group $G$, we also denote by $\alpha$ the automorphism induced on $RG$ by $\alpha$ and observe that for $V = G_\alpha \rtimes T$, $RV = (RG)_\alpha[T] = (RG)_\alpha[t, t^{-1}]$ is the $\alpha$-twisted Laurent series ring over the groupring $RG$. Now, $RG$ is an $R$-order in the semi-simple $F$-algebra $FG$ and so, we endeavour in this paper to obtain general results on higher $K$-theory of $\Lambda_\alpha(T)$ where $\Lambda$ is an arbitrary $R$-order in a semi-simple $F$-algebra $\Sigma$ so that results on $(RG)_\alpha[T]$ become examples and applications of our results.

Note also that an $R$-automorphism of $\Lambda$ extends to an $F$-automorphism of $\Sigma$ which we also denote by $\alpha$. We also study higher $K$-theory of $\Sigma_\alpha[T]$ and prove in 1.1.2(b) that there exist isomorphisms

$$\mathbb{Q} \otimes K_n(\Lambda_\alpha[T]) \simeq \mathbb{Q} \otimes G_n(\Lambda_\alpha[T]) \simeq \mathbb{Q} \otimes K_n(\Sigma_\alpha[T])$$

for all $n \geq 2$. Hence $\mathbb{Q} \otimes K_n(RV) \simeq \mathbb{Q} \otimes G_n(RV) \simeq Q \otimes K_n(FV)$ for all $n \geq 2$. Since we have shown in 1.1.2(a) that $G_n(\Lambda_\alpha[T])$ is finitely generated Abelian group for all $n \geq 1$, it follows that $K_n(\Lambda_\alpha[T])$, $K_n(\Sigma_\alpha[T])$ and hence $K_n(RV)$, $K_n(FV)$ have finite torsion-free ranks for all $n \geq 2$.

We next investigate under what conditions $G_n(\Lambda_\alpha[T])$ could actually be a finite group and show in 1.2.1 that when $F$ is a totally real number field with ring of integers $R$ and $\Lambda$ any $R$-order in a semi-simple $F$-algebra, then $G_{2(m+1)}(\Lambda_\alpha[T])$ is finite for all odd $m \geq 1$. Hence $G_{2(m+1)}(RV)$ is finite.

In section 2, we study profinite higher $K$-theory of $\Lambda_\alpha[T]$ and prove that $G_n^{\mathrm{pr}}(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l) = G_n(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l)$ are $l$-complete profinite Abelian groups; $\operatorname{div} G_n^{\mathrm{pr}}(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l) = 0$; and that the map $G_n(\Lambda_\alpha[T]) \longrightarrow G_n^{\mathrm{pr}}(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l)$ is injective with uniquely $l$-divisible cokernel. Corresponding results follow when we replace $\Lambda_\alpha[T]$ by $RV$.

In a final section, we prove that if $F$ is an algebraic number field with ring of integers $R$ and $\Lambda$ any $R$-order in a semi-simple $F$-algebra $\Sigma$, then $K_{-1}(\Lambda)$ and $K_{-1}(\Lambda_\alpha[T])$ are finitely generated Abelian groups; $NK_{-1}(\Lambda, \alpha) = 0$ and $K_{-1}(\Lambda[t]) \simeq K_{-1}(\Lambda)$. That $K_{-1}(\Lambda)$ and $K_{-1}(\Lambda_\alpha[T])$ are finitely generated for arbitrary $R$-orders $\Lambda$ generalize respectively similar results by D. Carter for $K_{-1}(RG)$ ($G$ a finite group, see [4]) and by Farrell/Jones for $K_{-1}(\mathbb{Z}V)$ (see [9]).

*Notes on notation.* If $\alpha$ is an automorphism of a ring $A$, we shall write $A_\alpha[T] = A_\alpha[t, t^{-1}]$ for the $\alpha$-twisted Laurent series ring over $A$. Note that additively $A_\alpha[T] = A_\alpha[t, t^{-1}]$ with multiplication given by $(at^i) \cdot (bt^j) = a\alpha^{-1}(b)t^{i+j}$ for $a, b \in A$. $A_\alpha[t]$ (resp. $A_\alpha[t^{-1}]$) is the subring of $A_\alpha[T]$ generated by $A$ and $t$ (resp. $A$ and $t^{-1}$). Call $A_\alpha[t]$ the $\alpha$-twisted polynomial ring over $A$. We also have inclusion maps $i : A \longrightarrow A_\alpha[T]$, $i^+ : A \longrightarrow A_\alpha[t]$ and $i^- : A \longrightarrow A_\alpha[t^{-1}]$.

The augmentation map $\varepsilon : A_\alpha[t] \longrightarrow A$ induces a group homomorphism $\varepsilon_* : K_n(A_\alpha[t]) \longrightarrow K_n(A)$ and we put $NK_n(A, \alpha) := \ker \varepsilon_*$. Since $\varepsilon$ is split by $i^+$, we have $K_n(A_\alpha[t]) \simeq K_n(A) \oplus NK_n(A, \alpha)$.

If $B$ is an additive Abelian group and $m$ is a positive integer, we shall write $B/m$ for $B/mB$ and $B[m]$ for the set of elements $x$ of $B$ such that $mx = 0$. We write $\operatorname{div} B$ for the subgroup of divisible elements fo $B$. If $l$ is a rational prime, we write $B_l$ for the $l$-primary subgroup of $B$. Note that $B_l = \bigcup B[l^s] = \varinjlim B[l^s]$.

**Acknowledgements.** Part of the work reported in this article was done while I was visiting University of Bielefeld, Germany and IHES, Paris. It was concluded and written up while I was visiting Max-Planck-Institut für Mathematik, Bonn, Germany. I like to thank the three institutions for hospitality and financial support.

## 1. Higher $K$-theory of $\Lambda_\alpha[T]$, $\Sigma_\alpha[T]$ ($\Lambda$ arbitrary orders)

### 1.1. $K_n(\Lambda_\alpha[T])$, $G_n(\Lambda_\alpha[T])$, $K_n(\Sigma_\alpha[T])$.

1.1.1. Let $R$ be the ring of integers in a number field $F$, $\Lambda$ any $R$-orders in a semi-simple $F$-algebra $\Sigma$, $\alpha$ an $R$-automorphism of $\Lambda$. Then $\alpha$ can be extended to an $F$-automorphism of $\Sigma$ (since $\Sigma = \Lambda \otimes_R F$). The aim of this section is to prove the following theorem.

1.1.2. **Theorem.** *Let $F$ be an algebraic number field with ring of integers $R$, $\Lambda$ any $R$-order in a semi-aimple $F$-algebra $\Sigma$, $\alpha$ an $R$-automorphism of $\Lambda$. Denote the extension of $\alpha$ to $\Sigma$ also by $\alpha$. Let $\Lambda_\alpha[T]$ (resp. $\Sigma_\alpha[T]$) be the $\alpha$-twisted Laurent series ring over $\Lambda$ (resp. $\Sigma$). Then we have*

(a) *$G_n(\Lambda_\alpha[T])$ is a finitely generated Abelian group for all $n \geq 1$.*
(b) *There exist isomorphisms:*

$$\mathbb{Q} \otimes K_n(\Lambda_\alpha[T]) \simeq \mathbb{Q} \otimes G_n(\Lambda_\alpha[T]) \simeq \mathbb{Q} \otimes K_n(\Sigma_\alpha[T])$$

*for $n \geq 2$.*

Before proving 1.1.2 we state the following consequence of the result.

1.1.3. **Corollary.** *Let $V = G \rtimes_\alpha T$ be the virtually infinite cyclic subgroup where $G$ is a finite group, $\alpha \in \mathrm{Aut}(G)$ and the action of $T$ on $G$ is given by $\alpha(g) = tgt^{-1}$. for all $g \in G$. Then,*

(a) $G_n(RV)$ *is a finitely generated Abelian group for al* $n \geq 1$.
(b) $\mathbb{Q} \otimes K_n(RV) \simeq \mathbb{Q} \otimes G_n(RV) \simeq \mathbb{Q} \otimes K_n(FV)$ *for all* $n \geq 2$.

The proof of 1.1.2(b) will proceed in several steps (see theorems 1.1.5, 1.1.6, 1.1.7 below). However, we first recall the following result (1.1.4).

1.1.4. **Theorem** ([15, theorem 7.3.2] or [16])**.** *Let $R$ be the ring of integers in a number field $F$, $\Lambda$ any $R$-order in a semi-simple $F$-algebra $\Sigma$. If $\alpha : \Lambda \longrightarrow \Lambda$ is an $R$-automorphism, then there exists an $R$-order $\Gamma \subset \Sigma$, such that*

(1) $\Lambda \subset \Gamma$,
(2) $\Gamma$ *is $\alpha$-invariant.*
(3) $\Gamma$ *is (right) regular ring. In fact $\Gamma$ is (right) hereditary.*

1.1.5. **Theorem.** *Let $R$ be the ring of integers in a number field $F$, $\Lambda$ any $R$-order in a semi-simple $F$-algebra, $\alpha : \Lambda \longrightarrow \Lambda$ and $R$-automorphism of $\Lambda$, $\Gamma$ an $\alpha$-invariant order containing $\Lambda$ as in 1.1.4, $\Lambda_\alpha[T]$ (resp. $\Gamma_\alpha[T]$) the $\alpha$-twisted Laurent series ring over $\Lambda$ (resp. $\Gamma$). $\varphi : \Lambda_\alpha[T] \longrightarrow \Gamma_\alpha[T]$ the map induced by the inclusion $\Lambda \longrightarrow \Gamma$. Then the induced homomorphisms $\varphi_n : K_n(\Lambda_\alpha[T]) \longrightarrow K_n(\Gamma_\alpha[T])$ has torsion kernel and cokernel. Hence for all $n \geq 2$ we have $\mathbb{Q} \otimes K_n(\Lambda_\alpha[T]) \simeq \mathbb{Q} \otimes K_n(\Gamma_\alpha[T])$.*

*Proof.* There exists a positive integer $s$ such that $s\Gamma \subset \Lambda$ (see [19] or [15]). Put $q = s\Gamma$. Then $q$ is an ideal of $\Gamma$ and $\Lambda$. Put $B = \Lambda/\underline{q}$, $B' = \Gamma/\underline{q}$. Then we have cartesian squares

$$\begin{array}{ccc} \Lambda & \longrightarrow & \Gamma \\ \downarrow & & \downarrow \\ B & \longrightarrow & B' \end{array} \qquad \text{(I)}$$

and

$$\begin{array}{ccc} \Lambda_\alpha[T] & \longrightarrow & \Gamma_\alpha[T] \\ \downarrow & & \downarrow \\ B_\alpha[T] & \longrightarrow & B'_\alpha[T]. \end{array} \qquad \text{(II)}$$

So, by [5] and [19], we have a long exact sequence

$$\cdots \longrightarrow K_{n+1}(B'_\alpha[T])(\tfrac{1}{s}) \longrightarrow K_n(\Lambda_\alpha[T])(\tfrac{1}{s}) \longrightarrow$$
$$K_n(\Gamma_\alpha[T])(\tfrac{1}{s}) \oplus K_n(B_\alpha[T])(\tfrac{1}{s}) \longrightarrow K_n(B'_\alpha[T])(\tfrac{1}{s}) \longrightarrow \cdots \qquad \text{(III)}$$

Now, $\Gamma$, B, B' are quasi-regular rings, so are $\Gamma_\alpha[T]$, $B_\alpha[T]$ and $B'_\alpha[T]$ (see [9]). If we write $A$ for $B_\alpha[T]$ or $B'_\alpha[T]$, $JA$ for the Jacobson's radical of $A$, then by [19] $K_n(A, JA)$ is $s$-torsion since $s$ annihilates $A$ and so from the relative sequence

$$\cdots \longrightarrow K_n(A, JA) \longrightarrow K_n(A) \longrightarrow K_n(A/J) \longrightarrow \cdots$$

we have $K_n(A)(\frac{1}{s}) \simeq K_n(A/JA)(\frac{1}{s})$. We now claim that $K_n(A)(\frac{1}{s}) \simeq K_n(A/JA)(\frac{1}{s})$ is torsion.

*Proof of claim.* Note that $A/JA \simeq (A'/JA')_\alpha[T]$ is a regular ring (see [9]) where $A'/JA'$ is a finite semi-simple ring which is a finite direct product of matrix algebras over finite fields. Hence $K_n((A'/JA')_\alpha[T])$ is a finite direct sum of $K$-groups of the form $K_n((F_i)_\alpha[T])$ where $F_i$ is a finite field. Also, $(F_i)_\alpha[T]$ is a regular ring and so $K_n((F_i)_\alpha[T]) \simeq G_n((F_i)_\alpha[T])$.

Now, for each $F_i$, we have by [15, theorem 7.5.3(iii)] or [16], that there exists a long exact sequence

$$\cdots \longrightarrow G_n(F_i) \longrightarrow G_n(F_i) \longrightarrow G_n((F_i)_\alpha[T]) \longrightarrow G_{n-1}(F_i) \longrightarrow G_{n-1}(F_i) \longrightarrow \cdots \quad \text{(IV)}$$

where each $G_n(F_i) \simeq K_n(F_i)$ is a finite Abelian group for $n \geq 2$ — by [15, theorem 7.1.12] or by Quillen's result. So, from (IV) above, $G_n((F_i)_\alpha[T])$ is finite for all $n \geq 2$, i.e. $K_n((F_i)_\alpha[T]) \simeq G_n((F_i)_\alpha[T])$ is a finite Abelian group. Hence $(K_n(A'/JA')_\alpha[T])$, as a finite direct sum of Abelian groups of the form $K_n(F_i)_\alpha[T]$ is a finite group. Hence $K_n((A'/JA')_\alpha[T])(\frac{1}{s})$ is torsion. So, for $A = B_\alpha(T)$ or $B'_\alpha[T]$, $K_n(A)(\frac{1}{s}) \simeq K_n((A/JA)(\frac{1}{s}))$ is torsion and $\mathbb{Q} \otimes K_n(A)(\frac{1}{s}) = 0$.

So, by tensoring the Mayer-Vietoris exact sequence (III) with $\mathbb{Q}$ we get an isomorphism

$$\mathbb{Q} \otimes K_n(\Lambda_\alpha[T]) \simeq \mathbb{Q} \otimes K_n(\Gamma_\alpha[T])$$

for all $n \geq 2$. □

1.1.6. **Theorem.** *Let $R$, $F$, $\Lambda$, $\alpha$; $\Gamma$, $\Lambda_\alpha[T]$, $\Gamma_\alpha[T]$ be as in 1.1.5. Let $\varphi_n : G_n(\Gamma_\alpha[T]) \longrightarrow G_n(\Lambda_\alpha[T])$ be the homomorphism induced by the exact functor $\mathcal{M}(\Gamma_\alpha[T]) \longrightarrow \mathcal{M}(\Lambda_\alpha[T])$ given by 'restriction of scalars'. Then for all $n \geq 2$, $\varphi_n$ has fnite kernel and torsion cokernel and hence induces an isomorphism*

$$\mathbb{Q} \otimes G_n(\Gamma_\alpha[T]) \simeq \mathbb{Q} \otimes G_n(\Lambda_\alpha[T])$$

*Proof.* First note that the exact functor $\mathcal{M}(\Gamma) \longrightarrow \mathcal{M}(\Lambda)$ given by 'restriction of scalars' yields group homomorphisms $\delta_n : G_n(\Gamma) \longrightarrow G_n(\Lambda)$. Now, by replacing the macimal order $\Gamma$ in the proof of [15, theorem 7.2.3, p. 146] or [16] with the $\alpha$-invariant order $\Gamma$ containing $\Lambda$, as in 1.1.4, we have that for all $n \geq 1$, $\delta_n : G_n(\Gamma) \longrightarrow G_n(\Lambda)$ has finite kernel and cokernel. The proof in [15, theorem 7.2.3] works for this $\Gamma$ also. Now from [15, theorem 7.5.3(b)] or [16], we have the following horizontal exact sequence and hence a commutative diagram

$$\begin{array}{ccccccccc}
G_n(\Gamma) & \xrightarrow{1-\alpha_*} & G_n(\Gamma) & \longrightarrow & G_n(\Gamma_\alpha[T]) & \longrightarrow & G_{n-1}(\Gamma) & \xrightarrow{1-\alpha_*} & G_{n-1}(\Gamma) \\
\downarrow{\scriptstyle \delta_n} & & \downarrow{\scriptstyle \delta_n} & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \delta_{n-1}} & & \downarrow{\scriptstyle \delta_{n-1}} \\
G_n(\Lambda) & \xrightarrow{1-\alpha_*} & G_n(\Lambda) & \longrightarrow & G_n(\Lambda_\alpha[T]) & \longrightarrow & G_{n-1}(\Lambda) & \xrightarrow{1-\alpha_*} & G_{n-1}(\Lambda)
\end{array} \quad \text{(V)}$$

By taking kernels and cokernels of vertical arrows in (V), we have a top (resp. bottom) horizontal exact sequence consisting of kernels (resp. cokernels) of the vertical maps. Since we saw above that $\delta_n$ has finite kernels and cokernels, we then have that $\phi_n : G_n(\Gamma_\alpha[T]) \longrightarrow G_n(\Lambda_\alpha[T])$ has finite kernel and cokernel for each $n \geq 2$. Hence $\mathbb{Q} \otimes G_n(\Gamma_\alpha[T]) \simeq \mathbb{Q} \otimes G_n(\Lambda_\alpha[T])$. But $\Gamma_\alpha[T]$ is regular. Hence

$$\mathbb{Q} \otimes K_n(\Gamma_\alpha[T]) \simeq \mathbb{Q} \otimes G_n(\Lambda_\alpha[T]).$$

□

1.1.7. **Theorem.** *Let $R$, $F$, $\Sigma$, $\Lambda$, $\alpha$, $T$ be as in theorem 1.1.2. Then for all $n \geq 2$, the map $\theta_n : G_n(\Lambda_\alpha[T]) \longrightarrow G_n(\Sigma_\alpha[T]) \simeq K_n(\Sigma_\alpha[T])$ indiced by the canonical map $\Lambda_\alpha[T] \longrightarrow \Sigma_\alpha[T]$ has finite kernel and torsion cokernel. Hence*

$$\mathbb{Q} \otimes G_n(\Lambda_\alpha[T]) \simeq \mathbb{Q} \otimes G_n(\Sigma_\alpha[T]) \simeq \mathbb{Q} \otimes K_n(\Sigma_\alpha[T]).$$

*Proof.* Note that the canonical (inclusion) map $\Lambda \xrightarrow{\rho} \Sigma$ induces a group homomorphism $\rho_n : G_n(\Lambda) \longrightarrow G_n(\Sigma) \simeq K_n(\Sigma)$ (note that $G_n(\Sigma) \simeq K_n(\Sigma)$ since $\Sigma$ is regular).

Now, by [15, theorem 7.5.3(b)] or [16], we have the following horizontal exact sequences and hence a commutative diagram

$$\begin{array}{ccccccccc}
G_n(\Lambda) & \xrightarrow{1-\alpha_*} & G_n(\Lambda) & \longrightarrow & G_n(\Lambda_\alpha[T]) & \longrightarrow & G_{n-1}(\Lambda) & \longrightarrow & G_{n-1}(\Lambda) \\
\downarrow{\rho_n} & & \downarrow{\rho_n} & & \downarrow{\theta_n} & & \downarrow{\rho_{n-1}} & & \downarrow{\rho_{n-1}} \\
G_n(\Sigma) & \xrightarrow{1-\alpha_*} & G_n(\Sigma) & \longrightarrow & G_n(\Sigma_\alpha[T]) & \longrightarrow & G_{n-1}(\Sigma) & \longrightarrow & G_{n-1}(\Sigma)
\end{array}$$

(VI)

Now, from the commutative diagram

$$\begin{array}{ccc}
G_n(\Lambda) & \xrightarrow{\rho_n} & G_n(\Sigma) \simeq K_n(\Sigma) \\
\nwarrow{\delta_n} & & \nearrow{\beta_n} \\
& K_n(\Gamma) &
\end{array}$$

(VII)

we have

$$0 \longrightarrow \ker \delta_n \longrightarrow \ker \beta_n \longrightarrow \ker \rho_n \longrightarrow \operatorname{coker} \delta_n \longrightarrow \operatorname{coker} \beta_n \longrightarrow \operatorname{coker} \rho_n \longrightarrow 0$$

Now, by the proof of 1.1.6, $\ker \delta_n$ and $\operatorname{coker} \delta_n$ are finite. Also by [15, theorem 7.2.2] or [12], $\ker \beta_n$ is finite and $\operatorname{coker} \beta_n$ is torsion for all $n \geq 2$. Hence from fiagram (VII) above, $\ker \rho_n$ is finite and $\operatorname{coker} \rho_n$ is torsion for all $n \geq 2$. It then follows from the diagram (VI) above that $\ker \theta_n$ is finite and $\operatorname{coker} \theta_n$ is torsion. □

*Proof of 1.1.2.* (a) From [15, theorem 7.5.3(b)] or [16], we have an exact sequence

$$G_n(\Lambda) \xrightarrow{1-\alpha_*} G_n(\Lambda) \longrightarrow G_n(\Lambda_\alpha[T]) \longrightarrow G_(\Lambda) \xrightarrow{1-\alpha_*} G_n(\Lambda)$$

Also by [15, theorem 7.1.13] or [10] $G_n(\Lambda)$ is a finitely generated Abelian group for all $n \geq 1$. Hence $G_n(\Lambda_\alpha[T])$ is finitely generated for all $n \geq 2$. (b) That $\mathbb{Q} \otimes K_n(\Lambda_\alpha[T]) \simeq \mathbb{Q} \otimes G_n(\Lambda_\alpha[T])$ follow from theorem 1.1.4 i.e. $\mathbb{Q} \otimes K_n(\Lambda_\alpha[T]) \simeq \mathbb{Q} \otimes K_n(\Gamma_\alpha[T])$ and 1.1.5 i.e. $\mathbb{Q} \otimes G_n(\Lambda_\alpha[T]) \simeq \mathbb{Q} \otimes K_n(\Sigma_\alpha[T])$. □

1.1.8. *Remarks.* Since by 1.1.2(a), $G_n(\Lambda_\alpha[T])$ is finitely generated Abelian group for all $n \geq 2$, it follows that $K_n(\Lambda_\alpha[T])$ and $K_n(\Sigma_\alpha[T])$ have finite torsion free rank just like $G_n(\Lambda_\alpha[T])$.

Hence if $V = G \rtimes_\alpha T$ is a vistually infinite cyclic group, then $K_n(RV)$, $K_n(FV)$ have finite torsion-free rank for $n \geq 2$.

1.2. **Finiteness of $\boldsymbol{G_{2(m+1)}(\Lambda_\alpha[T])}$.** In this subsection, we investigate under what circumstances $G_n(\Lambda_\alpha[T])$ could actually be a finite group. We prove below (see theorem 1.2.1) that if $F$ is a totally real field, then the group $G_{2(m+1)}(\Lambda_\alpha[T])$ is finite for all odd positive integers $m$. We state this formally:

1.2.1. **Theorem.** *Let $R$ be the ring of integers in a totally real number field $F$, $\Lambda$ an $R$-order in a semi-simple $F$-algebra, $\alpha : \Lambda \longrightarrow \Lambda$ and $R$-automorsphim. Then for all odd positive integers $m$, $G_{2(m+1)}(\Lambda_\alpha[T])$ is a finite group. Hence in the notation of 1.1.2, $G_{2(m+1)}(RV)$ is finite.*

The proof of 1.2.1 will make use of the following:

1.2.2. **Theorem.** *Let $F$ be a number field with ring of integers $R$, $\Lambda$ and $R$-order in a semi-simple $F$-algebra $\Sigma$. Then (a) For all $n \geq 1$, $G_{2n}(\Lambda)$ is a finite group. (b) If $F$ is totally real, then $G_{2m+1}(\Lambda)$ is also finite for all odd $m \geq 1$.*

*Proof.* Part (a) is proved in [15] and [14]. See [15, theorem 7.2.7].

If $F$ is a totally real number field with ring of integers $O_F$, a similar proof works. We only have to show that $K_{2m+1}(\Gamma)$ is finite if $\Gamma$ is a maximal order in a central division algebra $D$ over a totally real number field $F$ with ring of integer $O_F$. Let the dimension of $D$ over $F$ be $s^2$. We know from [15, theorem 7.1.11] or [11] that $K_{2m+1}(\Gamma)$ is finitely generated. We only need to show that $K_{2m+1}(\Gamma)$ is torsion. Let $\mathrm{tr} : K_{2m+1}(\Gamma) \longrightarrow K_{2m+1}(O_F)$ be the tranfer map and $i : K_{2m+1}(O_F) \longrightarrow K_{2m+1}(\Gamma)$ the map induced by the inclusion map $O_F \longrightarrow \Gamma$. Let $x \in K_{2m+1}(\Gamma)$. Then $i \circ \mathrm{tr}(x) = x^{s^2}$. But $K_{2m+1}(\Gamma)$ is finite since it is also finitely generated. □

*Proof of 1.2.1.* Assume that $m$ is an odd positive integer. The we have an exact sequence

$$\cdots \longrightarrow G_{2m+2}(\Lambda) \xrightarrow{1-\alpha_n} G_{2m+2}(\Lambda) \xrightarrow{\beta} G_{2m+2}(\Lambda_\alpha[T]) \xrightarrow{\gamma} G_{2m+1}(\Lambda) \longrightarrow \cdots$$

where $G_{2m+2}(\Lambda)$ is finite by 1.2.2(a) and $G_{2m+1}(\Lambda)$ is finite by 1.2.2(b). So $G_{2m+2}(\Lambda_\alpha[T])/\operatorname{Im}\beta \simeq \operatorname{Im}\gamma$.

But $\operatorname{Im}\beta$ is finite and $\operatorname{Im}\gamma$ is also finite as a subgroup of the finite group $G_{2m+1}(\Lambda)$. Note that $\operatorname{Im}\beta$ is finite as a homomorphic image of the finite group $G_{2m+2}(\Lambda)$. Hence $G_{2m+2}(\Lambda_\alpha[T])$ is finite for all odd positive integers $m$. □

## 2. Mod-$l^s$ and profinite higher $K$-theory of $\Lambda_\alpha(T)$

### 2.1. Mod-$l^s$ theory.

2.1.1. Let $\mathcal{C}$ be an exact category, $l$ a rational prime, $s$ a positive integer, $M^{n+1}_{l^s}$ the $(n+1)$-dimensional mod-$l^s$-space, i.e. the space obtained from $S^n$ by attaching and $(n+1)$-cell via a map of degree $l^s$ (see [3], [17], [15]).

If $X$ is an $H$-space, let $[M^{n+1}_{l^s}, X]$ be the set of homotopy classes of maps from $M^{n+1}_{l^s}$ to $X$.We shall write $\pi_{n+1}(X, \mathbb{Z}/l^s)$ for $[M^{n+1}_{l^s}, X]$. If $\mathcal{C}$ is an exact category and we put $X = BQ\mathcal{C}$, we write $K_n(\mathcal{C}, \mathbb{Z}/l^s)$ for $\pi_{n+1}(BQ\mathcal{C})$, we write $K_n(\mathcal{C}, \mathbb{Z}/l^s)$ for $\pi_{n+1}(\mathcal{C}, \mathbb{Z}/l^s)$ and $K_0\mathcal{C}, \mathbb{Z}/l^s$ for $K_0(\mathcal{C}) \otimes \mathbb{Z}/l^s$. We shall refer to $K_n(\mathcal{C}, \mathbb{Z}/l^s)$ as mod-$l^s$ $K$-theory of $\mathcal{C}$.

2.1.2. From [15, 8.1.2] or [13], we have an exact sequence

$$K_n(\mathcal{C}) \xrightarrow{l^s} K_n(\mathcal{C}) \xrightarrow{\rho} K_n(\mathcal{C}, \mathbb{Z}/l^s) \xrightarrow{\beta} K_{n-1}(\mathcal{C}) \longrightarrow K_{n-1}(\mathcal{C})$$

and hence a short exact sequence for all $n \geq 2$

$$0 \longrightarrow K_n(\mathcal{C})/l^s \longrightarrow K_n(\mathcal{C}, \mathbb{Z}/l^s) \longrightarrow K_n(\mathcal{C})[l^s] \longrightarrow 0$$

where $K_n(\mathcal{C})[l^s] = \{x \in K_n(\mathcal{C}) \mid l^s x = 0\}$.

2.1.3. *Examples.*

(i) Let $A$ be a ring with identity and $\mathcal{P}(A)$ the category of finitely generated projective $A$-modules. We write $K_n(A, \mathbb{Z}/l^s)$ for $K_n(\mathcal{P}(A), \mathbb{Z}/l^s)$. We are interested in $A = \Lambda_\alpha(T)$. Note that $K_n(A, \mathbb{Z}/l^s)$ is also $\pi_n(BGL(A)^+, \mathbb{Z}/l^s)$.

(ii) Let $A$ be a Noetherian ring and $\mathcal{M}(A)$ the category of finitely generated $A$-modules. We write $G_n(A, \mathbb{Z}/l^s)$ for $K_n(\mathcal{A}, \mathbb{Z}/l^s)$.

(iii) Let $Y$ be a scheme, $\mathcal{C} = \mathcal{P}(Y)$ the category of locally free sheaves of $O_X$-modules of finite rank. We write $K_n(X, \mathbb{Z}/l^s)$ for $K_n(\mathcal{P}(Y), \mathbb{Z}/l^s)$ and observe that for $Y = \operatorname{Spec}(A)$, $A$ a commutative ring, we recover $K_n(A, \mathbb{Z}/l^s)$ as in (i).

(iv) Let $Y$ be a Noetherian scheme and $\mathcal{M}(Y)$ the category of coherent sheaves of $O_Y$-modules. We write $G_n(Y, \mathbb{Z}/l^s)$ for $K_n(\mathcal{M}(Y'), \mathbb{Z}/l^s)$ and when $Y = \operatorname{Spec}(A)$, where $A$ is commutative, then we recover $G_n(A, \mathbb{Z}/l^s)$ as in (ii) above.

(v) It follows from 2.1.2 that we have exact sequences

$$0 \longrightarrow K_n(\Lambda_\alpha[T])/l^s \longrightarrow K_n(\Lambda_\alpha[T], \mathbb{Z}/l^s) \longrightarrow K_n(\Lambda_\alpha[T])[l^s] \longrightarrow 0$$

and

$$0 \longrightarrow G_n(\Lambda_\alpha[T])/l^s \longrightarrow G_n(\Lambda_\alpha[T], \mathbb{Z}/l^s) \longrightarrow G_n(\Lambda_\alpha[T])[l^s] \longrightarrow 0$$

2.2. **Profinite higher $K$-theory.**

2.2.1. Let $\mathcal{C}$ be an exact category, $l$ a rational prime, $s$ a positive integer $M^{n+1}_{l^\infty} = \varprojlim M^{n+1}_{l^s}$. We define the profinite $K$-theory of $\mathcal{C}$ by $K^{\mathrm{pr}}_n(\mathcal{C}, \hat{\mathbb{Z}}_l) := [M^{n+1}_{l^\infty}, BQ\mathcal{C}]$. We write $K_n(\mathcal{C}, \hat{\mathbb{Z}}_l)$ for $\varprojlim_s K_n(\mathcal{C}, \mathbb{Z}/l^s)$.

For more details on these constructions and their properties, see [15, chapter 8] or [13].

2.2.2. *Examples.*

(i) For $\mathcal{C} = \mathcal{P}(A)$ as in 2.1.3(i), we shall write $K^{\mathrm{pr}}_n(A, \hat{\mathbb{Z}}_l)$ for $K^{\mathrm{pr}}_n(\mathcal{P}(A), \hat{\mathbb{Z}}_l)$ and $K_n(A, \hat{\mathbb{Z}}_l)$ for $K_n(\mathcal{P}(A), \hat{\mathbb{Z}}_l)$.
(ii) For $\mathcal{C} = \mathcal{M}(A)$ as in 2.1.3(ii), we shall write $G^{\mathrm{pr}}_n(A, \hat{\mathbb{Z}}_l)$ for $K^{\mathrm{pr}}_n(\mathcal{M}(A), \hat{\mathbb{Z}}_l)$ and $G_n(A, \hat{\mathbb{Z}}_l)$ for $K_n(\mathcal{M}(A), \hat{\mathbb{Z}}_l)$.
(iii) For $\mathcal{C} = \mathcal{P}(Y)$ as in 2.1.3(iii) we shall write $K^{\mathrm{pr}}_n(Y, \hat{\mathbb{Z}}_l)$ for $K^{\mathrm{pr}}_n(\mathcal{P}(Y), \hat{\mathbb{Z}}_l)$ and $K_n(Y, \hat{\mathbb{Z}}_l)$ for $K_n(\mathcal{P}(Y), \hat{\mathbb{Z}}_l)$.
(iv) For $\mathcal{C} = \mathcal{M}(Y)$ as in 2.1.3(iv), we shall write $G^{\mathrm{pr}}_n(Y, \hat{\mathbb{Z}}_l)$ for $K^{\mathrm{pr}}_n(Y, \hat{\mathbb{Z}}_l)$ and $G_n(Y, \hat{\mathbb{Z}}_l) = K_n(\mathcal{M}(Y), \hat{\mathbb{Z}}_l)$.

2.2.3. *Remarks.* From the results earlier obtained by this author for general exact categories, (see [15, chapter 8] or [13]) we can already deduce the following for $\mathcal{P}(\Lambda_\alpha[T])$ and $\mathcal{M}(\Lambda_\alpha[T])$.

(i) From [15, lemma 8.2.1], we have the following exact sequences for $n \geq 1$.

$$(a)\quad 0 \longrightarrow \varprojlim_s{}^1 K_{n+1}(\Lambda_\alpha[T], \mathbb{Z}/l^s) \longrightarrow K^{\mathrm{pr}}_n(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l) \longrightarrow K_n(\Lambda_\alpha[T], \hat{\mathbb{Z}}/l) \longrightarrow 0$$

$$(b)\quad 0 \longrightarrow \varprojlim_s{}^1 G_{n+1}(\Lambda_\alpha[T], \mathbb{Z}/l^s) \longrightarrow G^{\mathrm{pr}}_n(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l) \longrightarrow G_n(\Lambda_\alpha[T], \hat{\mathbb{Z}}/l) \longrightarrow 0.$$

(ii) From [15, theorem 8.2.2] we have for all $n \geq 2$,

$$(a)\quad \varprojlim_s K^{\mathrm{pr}}_n(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l)[l^s] = 0; \quad \varprojlim_s{}^1 K_{n+1}(\Lambda_\alpha[T], \mathbb{Z}/l^s) = \operatorname{div} K^{\mathrm{pr}}_n(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l);$$

$$(b)\quad \varprojlim_s G^{\mathrm{pr}}_n(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l)[l^s] = 0; \quad \varprojlim_s{}^1 G_{n+1}(\Lambda_\alpha[T], \mathbb{Z}/l^s) = \operatorname{div} G^{\mathrm{pr}}_n(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l).$$

(iii) Form [15, lemma 8.2.2] or [13], we have

$$(a)\quad \varprojlim_s K^{\mathrm{pr}}_n(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l)/l^s \simeq K_n(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l);$$

$$(b)\quad \varprojlim_s G^{\mathrm{pr}}_n(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l)/l^s \simeq G_n(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l).$$

2.3. **Some computations.**

2.3.1. The aim of this subsection is to prove theorem 3.3.2 below. Before stating the result, we first explain the construction of map $\varphi$ in 2.3.2(c) below.

Note that for any exact category $\mathcal{C}$, the natural map $M^{n+1}_{l^\infty} \longrightarrow S^{n+1}$ induces a map

$$[S^{n+1}, BQ\mathcal{C}] \xrightarrow{\varphi} [M^{n+1}_{l^\infty}, BQ\mathcal{C}], \quad \text{i.e.,}$$
$$K_n(\mathcal{C}) \xrightarrow{\varphi} K^{\text{pr}}_n(\mathcal{C}, \hat{\mathbb{Z}}_l).$$

So when $\mathcal{C} = \mathcal{M}(\Lambda_\alpha[T])$ we have a map

$$\varphi : G_n(\Lambda_\alpha[T]) \longrightarrow G^{\text{pr}}_n(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l).$$

2.3.2. **Theorem.** *Let $R$ be the ring of integers in anumber field $F$, $\Lambda$ any $R$-order in a semi-simple $F$-algebra $\Sigma$, $\alpha : \Lambda \longrightarrow \Lambda$ an $R$-automorphism of $\Lambda$, $\Lambda_\alpha[T]$ the $\alpha$-twisted Laurent series ring over $\Lambda$. Then, for all $n \geq 2$:*

(a) $\operatorname{div} G^{\text{pr}}_n(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l) = 0$.
(b) $G^{\text{pr}}_n(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l) \simeq G_n(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l)$ *is an $l$-complete profinite Abelian group.*
(c) *The map $G_n(\Lambda_\alpha[T]) \longrightarrow G^{\text{pr}}_n(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l)$ is injective with uniquely $l$-divisible cokernel.*

*Proof.* (a) From 2.2.3(ii)(b), we have

$$\varprojlim_s{}^1 G_{n+1}(\Lambda_\alpha[T], \mathbb{Z}/l^s) = \operatorname{div} G^{\text{pr}}_n(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l), \tag{I}$$

for all $n \geq 2$. Now, by theorem 1.1.1(a) $G_n(\Lambda_\alpha[T])$ is finitely generated for all $n \geq 1$. Hence $G_n(\Lambda_\alpha[T], \mathbb{Z}/l^s)$ is finite for all $n \geq 1$. In particular, $G_{n+1}(\Lambda_\alpha[T], \mathbb{Z}/l^s)$ is finite for all $n \geq 2$ and so $\varprojlim_s G_{n+1}(\Lambda_\alpha[T], \mathbb{Z}/l^s) = 0$ for all $n \geq 2$. Hence from (I), $\operatorname{div} G^{\text{pr}}_n(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l) = 0$ for all $n \geq 2$.

(b) We saw in (a) above that $G_n(\Lambda_\alpha[T], \mathbb{Z}/l^s)$ is a finite group for all $n \geq 1$. Hence in the exact sequence

$$0 \longrightarrow \varprojlim_s{}^1 G_{n+1}(\Lambda_\alpha[T], \mathbb{Z}/l^s) \longrightarrow G^{\text{pr}}_n(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l) \longrightarrow G_n(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l) \longrightarrow 0$$

we have $\varprojlim{}^1 G_{n+1}(\Lambda_\alpha[T], \mathbb{Z}/l^s) = 0$. Hence,

$$G^{\text{pr}}_n(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l) \simeq G_n(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l). \tag{II}$$

Now, by 2.2.3(ii)(b),

$$G^{\text{pr}}_n(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l)/l^s \simeq G_n(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l). \tag{III}$$

So, from (II) and (III) $G^{\text{pr}}_n(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l)/l^s \simeq G^{\text{pr}}_n(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l)$ i.e. $G^{\text{pr}}_n(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l) \simeq G_n(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l)$ is $l$-complete. It is profinite since $G_n(\Lambda_\alpha[T], \hat{\mathbb{Z}}_l) = \varprojlim G_n(\Lambda_\alpha[T], \mathbb{Z}/l^s)$ where each $G_n(\Lambda_\alpha[T], \mathbb{Z}/l^s)$ is a finite group.

(c) Since for all $n \geq 1$, $G_n(\Lambda_\alpha[T])$ is a finitely generated Abelian group (see 1.1.1(a)), it follows that $G_n(\lambda_\alpha[T])_l$ is a finite group for each $n$. Hence

$G_n(\Lambda_\alpha[T])_l$ has no non-trivial divisible subgroups. Hence by [15, corollary 8.2.1] or [13], kernel and cokernel of $\varphi$ are uniquely $l$-divisible. But $G_n(\Lambda_\alpha[T])$ is finitely generated and so, $\ker\phi = \operatorname{div}\ker\phi = 0$, as subgroups of $G_n(\Lambda_\alpha[T])$. □

## 3. $K_{-1}(\Lambda)$, $K_{-1}(\Lambda_\alpha[T])$, $\Lambda$ ARBITRARY ORDERS

3.1. **Finite generation of $K_{-1}(\Lambda)$, $K_{-1}(\Lambda_\alpha[T])$.** Let $R$ be the ring of integers in a number field $F$, $\Lambda$ any $R$-order in a semi-simple $F$-algebra $\Sigma$, $\alpha : \Lambda \longrightarrow \Lambda$ and $R$-automorphism of $\Lambda$, $\Lambda_\alpha[T]$, the $\alpha$-twisted Laurent polynomial ring over $\Lambda$. We prove in this section that $K_{-1}(\Lambda)$ and $K_{-1}(\Lambda_\alpha[T])$ are finitely generated Abelian groups for arbitrary $R$-orders $\Lambda$ in semi-simpe $F$-algebras. Note that the proof in [9] by Farrel/Jones is for $\Lambda = \mathbb{Z}G$, $G$ a finite group. Also D. Carter shows in [4] that $K_{-1}(RG)$ is finitely generated and here we show that this result also holds more generally for arbitrary orders.

Finally we prove also that $NK_{-1}(\Lambda, \alpha) = 0$ and so, $K_{-1}(\Lambda_\alpha[t]) \simeq K_{-1}(\Lambda)$.

3.1.1. **Theorem.** *Let $F$ be an algebraic number field with ring of integers $R$, $\Lambda$ any $R$-order in a semi-simple $F$-algebra $\Sigma$, $\alpha : \Lambda \longrightarrow \Lambda$ an $R$-automorphism of $\Lambda$, $\Lambda_\alpha[T]$ the $\alpha$-twisted Laurent series ring over $\Lambda$. Then*

(a) $K_{-1}(\Lambda)$ *is a finitely generated Abelian group.*
(b) $K_{-1}(\Lambda_\alpha[T])$ *is a finitely generated Abelian group.*
(c) $K_{-1}(\Lambda) \simeq K_{-1}(\Lambda_\alpha[t])$.

*Proof.* (a) Let $\Gamma$ be a maximal $R$-order containing $\lambda$. Then, there exists a positive integer $s$ such that $s\Gamma \subset \Lambda$. Then $\underline{q} = s\Gamma$ is an ideal of $\Lambda$ and $\Gamma$. Put $B = \Lambda/\underline{q}$, $B' = \Gamma/\underline{q}$. Then we have a cartesian square

$$\begin{array}{ccc} \Lambda & \longrightarrow & \Gamma \\ \big\downarrow & & \big\downarrow \\ B & \longrightarrow & B' \end{array}$$

and hence a Mayer-Vietoris sequence

$$\cdots \longrightarrow K_1(B') \longrightarrow K_0(\Lambda) \longrightarrow K_0(\Gamma) \oplus K_0(B) \longrightarrow K_0(B') \longrightarrow$$
$$K_{-1}(\Lambda) \longrightarrow K_{-1}(\Gamma) \oplus K_{-1}(B) \longrightarrow \cdots \quad \text{(I)}$$

Now by [1, prop. 10.1, p. 685], $K_{-i}(A) = 0$ for $i \geq 1$ and any quasi-regular ring $A$. Note that $B$, $B'$ are finite rings and hence quasi-regular. Also $\Gamma$ is quasi-regular. Hence for $A = B$, $B'$ or $\Gamma$, $K_{-i}(A) = 0$ for $i \geq 1$. So the sequence (I) becomes

$$\cdots \longrightarrow K_0(\Lambda) \longrightarrow K_0(\Gamma) \oplus K_0(B) \longrightarrow K_0(B') \longrightarrow K_{-1}(\Lambda) \longrightarrow 0. \quad \text{(II)}$$

To show that $K_{-1}(\Lambda)$ is finitely generated it suffices from (II) to show that $K_0(B')$ is finitely generated. Now $B'$ is a finite Artinian ring and so, by [1, p. 465], $K_0(B') \simeq K_0(B'/JB')$ where $JB' =$ radical of $B'$. But $B'/JB'$ is a finite semi-simple ring ans so, $K_0(B') \simeq K_0(B'/JB')$ is a finite direct sum of $K_0$ of (finite) fields each of which is isomorphic to $\mathbb{Z}$. Hence $K_0(B')$ is a (free) Abelian group of finite rank and hence is finitely generated. Hence $K_{-1}(\Lambda)$ is finitely generated.

(b) Let $\Gamma$ be an $\alpha$-invariant order containing $\Lambda$ as in 1.1.3. Let $s$ be a positive integer such that $s\Gamma \subset \Lambda$ and put $\underline{q} = s\Gamma$, $B = \Lambda/\underline{q}$, $B' = \Gamma/\underline{q}$. Then we have cartesian squares

$$\begin{array}{ccc} \Lambda & \longrightarrow & \Gamma \\ \downarrow & & \downarrow \\ B & \longrightarrow & B' \end{array} \qquad \text{(III)}$$

and

$$\begin{array}{ccc} \Lambda_\alpha[T] & \longrightarrow & \Gamma_\alpha[T] \\ \downarrow & & \downarrow \\ B_\alpha[T] & \longrightarrow & B'_\alpha[T] \end{array} \qquad \text{(IV)}$$

and hence a Mayer-Vietoris sequence

$$\cdots \longrightarrow K_0(\Lambda_\alpha[T]) \longrightarrow K_0(\Gamma_\alpha[T]) \oplus K_0(B_\alpha[T]) \longrightarrow$$
$$K_0(B'_\alpha[T]) \longrightarrow K_{-1}(\Lambda_\alpha[T]) \longrightarrow 0. \quad \text{(V)}$$

where $\Gamma_\alpha[T]$, $B_\alpha[T]$ and $B'_\alpha[T]$ are quasi-regular (see [9]). If $A = \Gamma_\alpha[T]$, $B_\alpha[T]$ or $B'_\alpha[T]$ and $T^n$ is the free Abelian group of rank $n$ Then by [1, prop. 10.1], $K_{-i}(A) = 0$ for $i \geq 1$.

Also, by Serre's theorem $K_0(A) \to K_0(A[T^n])$ is an epimorphism (see [7]). Since $K_{-n}(A)$ is a direct summand of the cokernel of $K_0(A) \to K_0(A[T^n])$ we have $K_{-n}(A) = 0$ for $n \geq 1$. So from the exact sequence (I), we have $K_{-n}(\Lambda_\alpha[T]) = 0$ for $n \geq 2$ and $K_0(B'_\alpha[T]) \longrightarrow K_{-1}(\Lambda_\alpha[T])$ is an epimorphism.

By mapping the Mayer-Vietoris sequence associated with cartesian square (I) to the Mayer-Vietoris sequence associated with square (II), we have a commutative square

$$\begin{array}{ccc} K_0(B') & \longrightarrow & K_{-1}(\Lambda) \\ \downarrow & & \downarrow \\ K_0(B'_\alpha[T]) & \longrightarrow & K_{-1}(\Lambda_\alpha[T]). \end{array} \qquad \text{(VI)}$$

To prove that $K_{-1}(\Lambda) \longrightarrow K_{-1}(\Lambda_\alpha[T])$ is an epimorphism, it suffices to prove that $K_0(B') \longrightarrow K_0(B'_\alpha[T])$ is an epimorphism in the commutative diagram

$$\begin{array}{ccc} K_0(B') & \longrightarrow & K_0(B'_\alpha[T]) \\ \downarrow & & \downarrow \\ K_0(B'/JB') & \longrightarrow & K_0((B'/JB')_\alpha[T]) \end{array}$$

where the vertical maps are isomorphisms. Also by [7, theorem 27], the map $K_0(B'/JB') \longrightarrow K_0((B'/JB')_\alpha[T])$ is an epimorphism. Hence $K_0(B') \longrightarrow K_0(B'_\alpha[T])$ is an epimorphism. So $K_{-1}(\Lambda) \longrightarrow K_{-1}(\Lambda_\alpha[T])$ is an epimorphism. Since by (a), $K_{-1}(\Lambda)$ is finitely generated, then $K_{-1}(\Lambda_\alpha[T])$ is also finitely generated.

(c) By definition, $K_{-1}(\Lambda_\alpha[t]) \simeq K_{-1}(\Lambda) \oplus NK_{-1}(\Lambda, \alpha)$. So it suffices to show that $NK_{-1}(\Lambda, \alpha) = 0$.

Let $\Lambda, \Gamma, B = \Lambda/\underline{q}, B' = \Gamma/\underline{q}$ be as in the proof of (a) (b). Then we have two cartesian squares

$$\begin{array}{ccc} \Lambda & \longrightarrow & \Gamma \\ \downarrow & & \downarrow \\ B & \longrightarrow & B' \end{array} \qquad \text{(VII)}$$

and

$$\begin{array}{ccc} \Lambda_\alpha[t] & \longrightarrow & \Gamma_\alpha[t] \\ \downarrow & & \downarrow \\ B_\alpha[t] & \longrightarrow & B'_\alpha[t] \end{array} \qquad \text{(VIII)}$$

where $\Gamma_\alpha[t]$, $B_\alpha[t]$ and $B'_\alpha[t]$ are quasi-regular as well as $\Gamma, B, B'$. Hence we have Mayer-Vietoris sequences

$$\cdots \longrightarrow K_0(\Lambda_\alpha[t]) \longrightarrow K_0(\Gamma_\alpha[t]) \oplus K_0(B_\alpha[t]) \longrightarrow K_0(B'_\alpha[t]) \longrightarrow K_{-1}(\Lambda_\alpha[t]) \longrightarrow \dots \qquad \text{(IX)}$$

and

$$\cdots \longrightarrow K_0(\Lambda) \longrightarrow K_0(\Gamma) \longrightarrow K_0(B) \longrightarrow K_0(B') \longrightarrow K_{-1}(\Lambda) \longrightarrow \cdots \qquad \text{(X)}$$

where for $A = \Gamma$, $B$, $B'$, $\Gamma_\alpha[t]$, $B_\alpha[t]$, $B'_\alpha[t]$, $K_{-i}(A) = 0$ for $i \geq 1$ (see [1, prop. 10.1]). By mapping (IX) to (X) and taking kernels, we have that

$$NK_{-1}(\Lambda, \alpha) = \operatorname{coker}(NK_0(\Gamma, \alpha) \oplus NK_0(B, \alpha) \longrightarrow NK_0(B', \alpha)).$$

So it suffices to show that $NK_0(B', \alpha) = 0$. Since $B'$, $B'_\alpha[t]$ are quasi-regular, the result follows from [6, lemma 2.4]. So $NK_{-1}(\Lambda, \alpha) = 0$ and hence $K_{-1}(\Lambda[t]) \simeq K_{-1}(\Lambda)$. □

3.1.2. **Corollary.** *let $R$ be the ring of integers in a number field $F$, $V = G \rtimes_\alpha T$ a virtually infinite cyclic group where $G$ is a finite group and the action of the infinite cyclic group $T$ on $G$ is given by $\alpha(g) = tgt^{-1}$ for all $g \in G$. Then $K_{-1}(RV)$ is a finitely generated Abelian group.*

3.1.3. **Corollary.** *Let $\alpha$ be an automorphism of a finite group $G$, $R$ the ring of integers in a number field $F$. Denote the induced automorphism on $RG$ also by $\alpha$. Then $K_{-1}(RG) \simeq K_{-1}((RG)_\alpha[t])$ is a finitely generated Abelian group.*

## References


[1] H. Bass. *Algebraic K-theory*. W.A. Benjamin (1968).

[2] A. Borel. *Stable real cohomology of arithmetic groups*. Ann. Sci. École Norm. Sup. (4) **7** (1984), 235–272.

[3] W. Browder. *Algebraic K-theory with coefficients* $\mathbb{Z}/p$. Lecture Notes in Math. **657**. Springer, 40–84.

[4] D.W. Carter. *Localization in lower algebraic K-theory*. Comm. in Algebra (7)**8** (1980), 603–622.

[5] R. Charrey. *A note on excision in K-theory*. Lecture Notes in Math. **1046**. Springer (1984), 47–48.

[6] F. Conolly and S. Prassidis. *On the exponents of NK groups of virtually infinite cyclic groups*. Car.Math. Bull. **45** No.2 (2002), 180–195.

[7] F. T. Farrell and W.C. Hsiang. *A formula for* $K_1(R_\alpha[T])$. Applications of Categorical Algebra. Proc. Symp. Pure Math. **17**. Amer. Math. Soc. (1970), 192–218.

[8] F. T. Farrell and L.E. Jones. *Isomorphisms conjectures in algebraic K-theory*. J. Amer. Math. Soc. **6** (1993), 249–297.

[9] F. T. Farrell and L.E. Jones. *The lowwer algebra K-theory of virtually infinite cyclic groups*. K-Theory **9** (1995), 13–30.

[10] A. O. Kuku. *K-theory of grouprings of finite groups over maximal orders in division algebras*. J. Algebra (1) **91** (1985), 18–31.

[11] A. O. Kuku. $K_n$, $SK_n$ *of integral group rings and orders*. Contemporary Math. AMS **55** (1986), 333–338.

[12] A. O. kuku. *Ranks of* $K_n$ *and* $G_n$ *of orders and grouprings of finite groups over integers in number fields*. J. Pure Appl. Alg. **138** (1999), 39–44.

[13] A. O. Kuku. *Profinite and continuous higher K-theory of exact categories, orders and group rings*. K-theory **22** (2001), 367–392.

[14] A. O. Kuku. *Finiteness of higher K-groups of orders and group rings*. K-Theory **36** (2005), 51–58.

[15] A. O. Kuku. *Representation theory and higher algebraic K-theory*. Chapman and Hall (2007).

[16] A. O. Kuku and G. Tang. *Higher K-theory of grouprings of virtually infinite cyclic groups*. Math. Ann. **323** (2003), 711-725.

[17] J. Neisendorfer. *Primary homotopy theory*. Mem. Amer. Math. Soc. **232** (1980).

[18] I. Reiner. *Maximal orders*. Academic Press. London (1975).

[19] C. Weibel. *Mayer-Vietoris sequence and* mod-*p*-*K-Theory*. Lecture Notes in Math. **966** Springer (1982), 390–407.



Max-Planck-Institut für Mathematik, Bonn, Germany

*E-mail address*: kuku@mpim-bonn.mpg.de