Combinatorial Identities Related to Representations of $U_q(\widetilde{\mathfrak{gl}_2})$

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Introduction

Recently N. Jing discovered in [J] the following combinatorial identity:

$$(\mathbb{I})\quad \sum_{k=0}^{\ell}\prod_{s=0}^{k-1}\frac{\eta^{\ell}-\eta^{s}}{1-\eta^{s+1}}\ \sum_{\sigma\in\mathbf{S}_\ell}\Bigl(\prod_{1\leqslant a\leqslant k}(t_{\sigma_a}-1)\prod_{k<b\leqslant\ell}(t_{\sigma_b}-\eta^{\ell-1})\prod_{1\leqslant a<b\leqslant\ell}\frac{t_{\sigma_a}-\eta t_{\sigma_b}}{t_{\sigma_a}-t_{\sigma_b}}\Bigr)\,=\,0\,.$$

In his paper the identity comes from validity of the Serre relations in some vertex representations of quantum Kac-Moody algebras.

In this note we are going to generalize this identity, see Theorems 1.1, 4.1. The obtained identities are equivalent to existence of a singular vector in certain tensor products of evaluation representations of the quantum loop algebra $U_q(\widetilde{\mathfrak{gl}_2})$, see Proposition 3.1, or more generally, of the elliptic quantum group $E_{\rho,\gamma}(\mathfrak{sl}_2)$.

There are two directions for generalizing the identity $(\mathbb{I})$. First one can replace linear functions of $t_1,\dots,t_\ell$ by polynomials of larger degree. This is done in Section 1, see (1.3).

Furthermore, it is possible to take elliptic theta-functions instead of polynomials, see Section 4. The resulting identities depend on two extra parameters: the elliptic modulus p and the dynamical parameter α. In the limit $p\to 0$ the elliptic functions degenerate into polynomials and we get a family of polynomial identities depending on α and turning into the identities (1.3) if either $\alpha\to 0$ or $\alpha\to\infty$.

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1. Polynomial identities

Given nonnegative integers ℓ and n let $\mathcal{P}_{\ell,n}$ be a set of partitions $\lambda=(\lambda_1,\dots,\lambda_\ell)$ such that $n\geqslant\lambda_1\geqslant\dots\geqslant\lambda_\ell\geqslant 1$. For a partition λ let $\omega_{k,\lambda}=\#\{j\mid\lambda_j=k\}$.

Introduce indeterminates $t_1,\dots,t_\ell$, $x_1,\dots,x_n$, $y_1,\dots,y_n$, η. In the paper we use the following compact notations:

$$t=(t_1,\dots,t_\ell)\,,\qquad x=(x_1,\dots,x_n)\,,\qquad y=(y_1,\dots,y_n)\,.$$

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For any $m = 1, \ldots, n$ set

$$X_m(u;x;y) \;=\; u \prod_{1 \leqslant j < m} (u - y_j) \prod_{m < k \leqslant n} (u - x_k)\,,$$

$$X'_m(u;x;y) \;=\; \prod_{1 \leqslant j < m} (u - x_j) \prod_{m < k \leqslant n} (u - y_k)\,,$$

and for any $\lambda \in \mathcal{P}_{\ell,n}$ set

$$r_\lambda(\eta) \;=\; \prod_{m=1}^{n} \prod_{s=1}^{\omega_{m,\lambda}} \frac{1-\eta}{1-\eta^s}\,,$$

$$P_\lambda(t;x;y;\eta) \;=\; r_\lambda(\eta) \sum_{\sigma \in \mathbf{S}_\ell} \Bigl(\prod_{a=1}^{\ell} X_{\lambda_a}(t_{\sigma_a};x;y) \prod_{1 \leqslant a < b \leqslant \ell} \frac{t_{\sigma_a} - \eta t_{\sigma_b}}{t_{\sigma_a} - t_{\sigma_b}} \Bigr)\,, \tag{1.1}$$

$$P'_\lambda(t;x;y;\eta) \;=\; r_\lambda(\eta) \sum_{\sigma \in \mathbf{S}_\ell} \Bigl(\prod_{a=1}^{\ell} X'_{\lambda_a}(t_{\sigma_a};x;y) \prod_{1 \leqslant a < b \leqslant \ell} \frac{\eta t_{\sigma_a} - t_{\sigma_b}}{t_{\sigma_a} - t_{\sigma_b}} \Bigr)\,. \tag{1.2}$$

Notice that

$$X_m(u;x;y) \;=\; u\, X'_m(u;y;x)$$

and

$$P_\lambda(t;x;y;\eta) \;=\; \eta^{\,\ell(\ell-1)/2 \,-\, \sum_{m=1}^{n} \omega_{m,\lambda}(\omega_{m,\lambda}-1)/2}\; t_1 \ldots t_\ell\, P'_\lambda(t;y;x;\eta^{-1})\,.$$

Notice also that both P_λ and P'_λ are polynomials in all the indeterminates involved.

Theorem 1.1. *Let* $x_j = \eta^{\ell-1} y_i$ *for some* $i < j$ *. Then*

$$\sum_{\lambda \in \mathcal{P}_\ell^{i,j}} c_\lambda^{(i,j)}(x;y;\eta)\, P_\lambda(t;x;y;\eta) \;=\; 0 \tag{1.3}$$

where $\mathcal{P}_\ell^{i,j} = \{\, \lambda = (\lambda_1, \ldots, \lambda_\ell) \mid j \geqslant \lambda_1 \geqslant \ldots \geqslant \lambda_\ell \geqslant i \,\}$ *and*

$$c_\lambda^{(i,j)}(x;y;\eta) \;=\; (-1)^{\omega_{i,\lambda}}\, \eta^{\omega_{j,\lambda}(\omega_{j,\lambda}-1)/2} \prod_{i<k<j} \prod_{s=0}^{\omega_{k,\lambda}-1} (x_k - \eta^s y_k) \;\times$$

$$\times \prod_{a=1}^{\ell} \Bigl(\prod_{i<k<\lambda_a} (\eta^{\ell-a} y_i - x_k) \prod_{\lambda_a < m < j} (\eta^{\ell-a} y_i - y_m) \Bigr)\,.$$

The theorem is proved in the next section.

Example. Identity (1.3) for $i = 1$, $j = n = 2$ is equivalent to identity ($\mathbb{I}$).

Remark. Multiplied by a certain polynomial in x, y, η the identity (1.3) becomes more transparent and understandable, see (2.6).

Remark. All over the paper we assume integers ℓ and n to be fixed. There are two natural embeddings of $\mathcal{P}_{\ell,n-1}$ into $\mathcal{P}_{\ell,n}$ given by either $\lambda \mapsto \lambda$ or $\lambda \mapsto \lambda' = (\lambda_1+1, \ldots, \lambda_\ell+1)$. Indicating for a while dependence of polynomials defined by (1.1) on n explicitly, that is, writing $P_{n,\lambda}$ instead of P_λ, we have

$$P_{n,\lambda}(t;x;y;\eta) = P_{n-1,\lambda}(t;x^{(n)};y^{(n)};\eta) \prod_{a=1}^{\ell} (t_a - x_n),$$

$$= P_{n-1,\lambda'}(t;x^{(1)};y^{(1)};\eta) \prod_{a=1}^{\ell} (t_a - y_1),$$

where $x^{(n)} = (x_1, \ldots, x_{n-1})$, $y^{(n)} = (y_1, \ldots, y_{n-1})$, $x^{(1)} = (x_2, \ldots, x_n)$, $y^{(1)} = (y_2, \ldots, y_n)$. Therefore, the claim of Theorem 1.1 for $i = 1$, $j = n$ implies the claim of the theorem for general $i < j$.

2. Proof of Theorem 1.1

In this section we think $x_1, \ldots, x_n$, $y_1, \ldots, y_n$, η being complex variables rather than indeterminates and consider them as parameters which functions of $t_1, \ldots, t_\ell$ can in addition depend on.

Let f, g be polynomials in $t_1, \ldots, t_\ell$. We define below their scalar product $\langle f, g\rangle_S$. Let

$$S(t_1, \ldots, t_\ell) = \prod_{a=1}^{\ell} \prod_{m=1}^{n} (t_a - x_m)(t_a - y_m) \prod_{\substack{a,b=1 \\ a\neq b}}^{\ell} \frac{t_a - \eta t_b}{t_a - t_b}.$$

If $|\eta| > 1$ and $|x_m| < 1$, $|y_m| > 1$ for all $m = 1, \ldots, n$, then we set

$$\langle f, g\rangle_S = \frac{1}{(2\pi i)^\ell \ell!} \int_{\mathbb{T}^\ell} \frac{f(t)\, g(t)}{S(t)} \, (dt/t)^\ell$$

where $(dt/t)^\ell = \prod_{a=1}^{\ell} dt_a/t_a$ and $\mathbb{T}^\ell = \{t \in \mathbb{C}^\ell \mid |t_1| = 1, \ldots, |t_\ell| = 1\}$ each circle oriented counterclockwise. Then one can show that $\langle f, g\rangle_S$ is a rational function of $x_1, \ldots, x_n$, $y_1, \ldots, y_n$, η.

The polynomials defined by (1.1) and (1.2) are biorthogonal with respect to the introduced scalar product.

Theorem 2.1. [TV1, Theorem C.9] $\langle P'_\lambda, P_\mu\rangle_S = N_\lambda^{-1}\, \delta_{\lambda\mu}$ *where*

$$N_\lambda = \prod_{m=1}^{n} \prod_{s=1}^{\omega_{m,\lambda}} \frac{(1-\eta^s)(x_m - \eta^{s-1} y_m)}{1-\eta}. \tag{2.1}$$

Proof. The proof is based on Lemmas 2.2 and 2.3. The first equality in (2.3) implies that $\langle P'_\lambda, P_\mu\rangle_S = 0$ unless $\lambda \leqslant \mu$, while the second equality in (2.3) gives $\langle P'_\lambda, P_\mu\rangle_S = 0$ unless $\lambda \geqslant \mu$ and $\langle P'_\lambda, P_\lambda\rangle_S = P_\lambda(y \triangleleft \lambda)\, P'_\lambda(y \triangleleft \lambda)$. The rest of the proof is straightforward. $\square$

Remark. Notations here and in [TV1] are not always the same though we try to keep them consistent whenever possible. The identification of labels should be mentioned: a partition λ here corresponds to a label $\mathfrak{l} = (\omega_{1,\lambda}, \dots, \omega_{n,\lambda})$ in [TV1], the polynomial P_λ being a numerator of the trigonometric weight function $w_{\mathfrak{l}}$ in [TV1] up to a simple factor.

For any $\lambda \in \mathcal{P}_{\ell,n}$ introduce points $x \triangleright \lambda$, $y \triangleleft \lambda \in \mathbb{C}^\ell$ as follows:

$$x \triangleright \lambda \,=\, (\eta^{1-\omega_{1,\lambda}} x_1, \dots, x_1, \eta^{1-\omega_{2,\lambda}} x_2, \dots, x_2, \ \dots\ , \eta^{1-\omega_{n,\lambda}} x_n, \dots, x_n)\,,$$

$$y \triangleleft \lambda \,=\, (\eta^{\omega_{1,\lambda}-1} y_1, \dots, y_1, \eta^{\omega_{2,\lambda}-1} y_2, \dots, y_2, \ \dots\ , \eta^{\omega_{n,\lambda}-1} y_n, \dots, y_n)\,.$$

For partitions $\lambda, \mu \in \mathcal{P}_{\ell,n}$ say that $\lambda \geqslant \mu$ if $\lambda_a \geqslant \mu_a$ for any $a = 1, \dots, \ell$.

Lemma 2.2. *$P_\lambda(x \triangleright \mu) = 0$ and $P'_\lambda(y \triangleleft \mu) = 0$ unless $\lambda \geqslant \mu$. $P_\lambda(y \triangleleft \mu) = 0$ and $P'_\lambda(x \triangleright \mu) = 0$ unless $\lambda \leqslant \mu$. Besides, only the terms in* (1.1), (1.2) *corresponding to the identity permutation contribute into the values $P_\lambda(x \triangleright \lambda)$, $P_\lambda(y \triangleleft \lambda)$, $P'_\lambda(x \triangleright \lambda)$, $P'_\lambda(y \triangleleft \lambda)$.*

The proof is straightforward.

For a function $f(t_1, \dots, t_\ell)$ and a point $t^\star = (t^\star_1, \dots, t^\star_\ell)$ define a multiple residue $\mathrm{Res}\bigl(f(t)\,(dt/t)^\ell\bigr)\big|_{t=t^\star}$ by

$$\mathrm{Res}\bigl(f(t)\,(dt/t)^\ell\bigr)\big|_{t=t^\star} \,=\, \mathrm{Res}\bigl(\dots \mathrm{Res}\bigl(f(t_1, \dots, t_\ell)\,(dt_\ell/t_\ell)\bigr)\big|_{t_\ell=t^\star_\ell} \cdots\, (dt_1/t_1)\bigr)\big|_{t_1=t^\star_1}$$

and set

$$x \triangleright Res(f) \,=\, \sum_{\lambda \in \mathcal{P}_{\ell,n}} \mathrm{Res}\bigl(f(t_1, \dots, t_\ell)\,(dt/t)^\ell\bigr)\big|_{t=x \triangleright \lambda}\,, \tag{2.2}$$

$$y \triangleleft Res(f) \,=\, \sum_{\lambda \in \mathcal{P}_{\ell,n}} \mathrm{Res}\bigl(f(t_1, \dots, t_\ell)\,(dt/t)^\ell\bigr)\big|_{t=y \triangleleft \lambda}\,.$$

Lemma 2.3. [TV1, Lemma C.8] *Let polynomials f, g be such that their product is a symmetric polynomial in $t_1, \dots, t_\ell$ of degree less than $2n$ in each of the indeterminates and divisible by $t_1 \dots t_n$. Then*

$$\langle f, g\rangle_S \,=\, x \triangleright Res(fg/S) \,=\, (-1)^\ell\, y \triangleleft Res(fg/S)\,. \tag{2.3}$$

Let Q_λ be a monomial symmetric polynomial:

$$Q_\lambda(t_1, \dots, t_\ell) \,=\, \frac{1}{\omega_{1,\lambda}! \dots \omega_{n,\lambda}!}\, \sum_{\sigma \in \mathbf{S}_\ell} t_{\sigma_1}^{\lambda_1} \dots t_{\sigma_\ell}^{\lambda_\ell}\,.$$

Introduce transition coefficients $A_{\lambda\mu}$, $\lambda, \mu \in \mathcal{P}_{\ell,n}$:

$$P_\lambda \,=\, \sum_{\mu \in \mathcal{P}_{\ell,n}} A_{\lambda\mu}\, Q_\mu\,. \tag{2.4}$$

Then by Theorem 2.1 for any $\nu \in \mathcal{P}_{\ell,n}$

$$\sum_{\mu \in \mathcal{P}_{\ell,n}} A_{\lambda\mu} \langle P'_\nu, Q_\mu \rangle \,=\, N_\lambda^{-1} \delta_{\lambda\nu} \,,$$

and inverting this relation we get

$$\sum_{\lambda \in \mathcal{P}_{\ell,n}} N_\lambda \langle P'_\lambda, Q_\mu \rangle \, A_{\lambda\mu} \,=\, \delta_{\mu\nu} \,.$$

Calculating the scalar products by Lemma 2.3 we have

$$\sum_{\kappa,\lambda \in \mathcal{P}_{\ell,n}} M_\kappa^{-1} \, Q_\mu(x \triangleright \kappa) \, P'_\lambda(x \triangleright \kappa) \, N_\lambda \, A_{\lambda\mu} \,=\, \delta_{\mu\nu} \,, \tag{2.5}$$

where $M_\kappa^{-1} = \mathrm{Res}\,\bigl(S(t)^{-1}(dt/t)^\ell\bigr)\big|_{t = x \triangleright \kappa}$. Notice that M_κ is a rational function in x_1, $\dots, x_n$, η and a polynomial in $y_1, \dots, y_n$.

The matrix $\bigl[Q_\lambda(x \triangleright \kappa)\bigr]_{\kappa,\lambda \in \mathcal{P}_{\ell,n}}$ is invertible, see e.g. (2.7), and we denote by B the inverse matrix, which is a rational function of $x_1, \dots, x_n$, η and, trivially, does not depend on $y_1, \dots, y_n$. Finally, the relation (2.5) is transformed to

$$\sum_{\lambda \in \mathcal{P}_{\ell,n}} P'_\lambda(x \triangleright \kappa) \, N_\lambda \, A_{\lambda\mu} \,=\, M_\kappa \, B_{\kappa\mu} \,.$$

Proof of Theorem 1.1. Fix $i < j$. Let $\kappa^{(j)} = (j, \dots, j)$. Assume that $y_i = \eta^{1-\ell} x_j$. Then $M_{\kappa^{(j)}} = 0$, and taking into account the definition (2.4) of $A_{\lambda\mu}$ we have an identity

$$\sum_{\lambda \in \mathcal{P}_{\ell,n}} P'_\lambda(x \triangleright \kappa^{(j)}) \, N_\lambda \, P_\lambda \,=\, 0 \,. \tag{2.6}$$

Notice that $x \triangleright \kappa^{(j)} = (\eta^{1-\ell} x_j, \dots, x_j)$, so only the term in (1.2) corresponding to the identity permutation contribute into the value $P'_\lambda(x \triangleright \kappa^{(j)})$, and $P'_\lambda(x \triangleright \kappa^{(j)}) = 0$ unless $\lambda \in \mathcal{P}_\ell^{i,j}$. Calculating $P'_\lambda(x \triangleright \kappa^{(j)}) \, N_\lambda$ explicitly and removing all factors which does not depend on λ we get the identity (1.3). Theorem 1.1 is proved. $\square$

Remark. Let us mention two determinant formulae though they are not actually used in the proof:

$$\det\bigl[Q_\lambda(x \triangleright \mu)\bigr]_{\lambda,\mu \in \mathcal{P}_{\ell,n}} \,=\, \eta^{-n(n+1)/2 \cdot \binom{n+\ell-1}{n+1}} \prod_{m=1}^{n} x_m^{\binom{n+\ell-1}{n}} \,\times$$
$$\times \prod_{s=1-\ell}^{\ell-1} \prod_{1 \leqslant j < k \leqslant n} (\eta^s x_k - x_j)^{D(n,\ell,s)} \tag{2.7}$$

where $D(n,\ell,s) \,=\, \displaystyle\sum_{\substack{r \in \mathbb{Z}_{\geqslant 0} \\ 2r \leqslant \ell - |s| - 1}} \binom{n + \ell - |s| - 2r - 3}{n-2}$, see formula (A.14) in [TV1], and

$$\det\left[A_{\lambda\mu}\right]_{\lambda,\mu\in\mathcal{P}_{\ell,n}} = \prod_{s=0}^{\ell-1}\prod_{1\leqslant j<k\leqslant n}(\eta^s y_j - x_k)^{\binom{n+\ell-s-2}{n-1}}, \tag{2.8}$$

see formula (A.10) in [TV1]. Formula (2.7) is a deformation of a symmetric power of the Vandermonde determinant. Formula (2.8) implies that the polynomials P_λ, $\lambda\in\mathcal{P}_{\ell,n}$ are linear independent if x, y, η are generic, and there are linear relations between them if $x_k = \eta^s y_j$ for some $j<k$ and $s\in\{0,\dots,\ell-1\}$. For $s=\ell-1$ there is just one linear relation given by (2.6). Relations for $s<\ell-1$ can be obtained in a similar manner. They also can be derived from the relation (2.6) written for polynomials in less number of indeterminates.

3. Tensor products of evaluation modules over $U_q(\widetilde{\mathfrak{gl}_2})$

Let q be a nonzero complex number which is not a root of unity. Consider the quantum group $U_q(\mathfrak{sl}_2)$ with generators E, F, q^H and relations

$$q^H E = qEq^H, \qquad q^H F = q^{-1}Fq^H, \qquad [E,F] = \frac{q^{2H}-q^{-2H}}{q-q^{-1}},$$

and the quantum loop algebra $U_q(\widetilde{\mathfrak{gl}_2})$ with generators $L_{ij}^{(+0)}$, $L_{ji}^{(-0)}$, $1\leqslant j\leqslant i\leqslant 2$, and $L_{ij}^{(s)}$, $i,j=1,2$, $s=\pm1,\pm2,\dots$, subject to relations (3.1).

Let e_{ij}, $i,j=1,2$, be the 2×2 matrix with the only nonzero entry 1 at the intersection of the i-th row and j-th column. Set

$$\begin{aligned}R(u) = {} & (uq-q^{-1})(e_{11}\otimes e_{11}+e_{22}\otimes e_{22}) + \\ & + (u-1)(e_{11}\otimes e_{22}+e_{22}\otimes e_{11}) + u(q-q^{-1})\,e_{12}\otimes e_{21} + (q-q^{-1})\,e_{21}\otimes e_{12}.\end{aligned}$$

Introduce the generating series $L_{ij}^{\pm}(u) = L_{ij}^{(\pm0)} + \sum_{s=1}^{\infty} L_{ij}^{(\pm s)}u^{\pm s}$. The relations in $U_q(\widetilde{\mathfrak{gl}_2})$ have the form:

$$\begin{aligned} R(u/z)\,L^{\nu}_{(1)}(u)\,L^{\nu}_{(2)}(z) &= L^{\nu}_{(2)}(z)\,L^{\nu}_{(1)}(u)\,R(u/z), \qquad \nu=\pm, \\ R(u/z)\,L^{+}_{(1)}(u)\,L^{-}_{(2)}(z) &= L^{-}_{(2)}(z)\,L^{+}_{(1)}(u)\,R(u/z), \\ L_{ii}^{(+0)}L_{ii}^{(-0)} = 1, \qquad & L_{ii}^{(-0)}L_{ii}^{(+0)} = 1, \qquad i=1,2, \end{aligned} \tag{3.1}$$

where $L^{\nu}_{(1)}(u) = \sum_{ij} e_{ij}\otimes 1\otimes L^{\nu}_{ij}(u)$ and $L^{\nu}_{(2)}(u) = \sum_{ij} 1\otimes e_{ij}\otimes L^{\nu}_{ij}(u)$.

The quantum loop algebra $U_q(\widetilde{\mathfrak{gl}_2})$ is a Hopf algebra with a coproduct

$$\Delta : L^{\pm}_{ij}(u) \mapsto \sum_k L^{\pm}_{ik}(u)\otimes L^{\pm}_{kj}(u).$$

There is a one-parametric family of automorphisms $\rho_z : L^{\pm}_{ij}(u) \mapsto L^{\pm}_{ij}(u/z)$ and the evaluation homomorphism $\epsilon : U_q(\widetilde{\mathfrak{gl}_2}) \to U_q(\mathfrak{sl}_2)$:

$$\epsilon : L^{\pm}(u) \mapsto \mp\, u^{\vartheta_\pm} \begin{pmatrix} uq^H - q^{-H} & uF(q-q^{-1}) \\ E(q-q^{-1}) & uq^{-H} - q^H \end{pmatrix}$$

where $\vartheta_+ = 0$, $\vartheta_- = -1$. For any $U_q(\mathfrak{sl}_2)$-module V denote by $V(z)$ the $U_q(\widetilde{\mathfrak{gl}_2})$ module obtained from V via the homomorphism $\epsilon \circ \rho_z$. The module $V(z)$ is called the evaluation module. In a tensor product $V_1(z_1) \otimes \ldots \otimes V_n(z_n)$ of evaluation modules actions of the series $L^+(u)$ and $(-u)^n L^-(u)$ coincide, so it is enough to look at the action of only one of them. Besides, this implies that the series $L^+(u)$ acts as a polynomial in u, so it can be evaluated for u being a complex number.

Let $V_1, \ldots, V_n$ be $U_q(\mathfrak{sl}_2)$ Verma modules with highest weights $q^{\Lambda_1}, \ldots, q^{\Lambda_n}$ and generating vectors $v_1, \ldots, v_n$, respectively. Using commutation relations in $U_q(\widetilde{\mathfrak{gl}_2})$ it is rather straightforward to obtain the following statements.

Proposition 3.1. *Consider $U_q(\widetilde{\mathfrak{gl}_2})$ evaluation modules $V_1(z_1), \ldots, V_n(z_n)$. Let $z_i = q^{2\Lambda_i + 2\Lambda_j - 2\ell} z_j$ for some $i < j$ and $\ell \in \mathbb{Z}_{\geqslant 0}$. Then*

a) *The vector $v_1 \otimes \ldots \otimes v_n \in V_1(z_1) \otimes \ldots \otimes V_n(z_n)$ generates a $U_q(\widetilde{\mathfrak{gl}_2})$-submodule, which is annihilated by the action of a product $L^+_{21}(q^{2\Lambda_j} z_j) \ldots L^+_{21}(q^{2\Lambda_j - 2\ell} z_j)$.*

b) *A vector $\tilde v = L^+_{12}(q^{2\Lambda_j} z_j) \ldots L^+_{12}(q^{2\Lambda_j - 2\ell} z_j)\, v_n \otimes \ldots \otimes v_1 \in V_n(z_n) \otimes \ldots \otimes V_1(z_1)$ is singular with respect to the $U_q(\widetilde{\mathfrak{gl}_2})$-action, that is, $L^{\pm}_{21}(u)\, \tilde v = 0$.*

Corollary 3.2. *Let $z_i = q^{2\Lambda_i + 2\Lambda_j - 2\ell} z_j$ for some $i < j$ and $\ell \in \mathbb{Z}_{\geqslant 0}$. Then for arbitrary $t_1, \ldots, t_\ell$*

$$L^+_{21}(q^{2\Lambda_j} z_j) \ldots L^+_{21}(q^{2\Lambda_j - 2\ell} z_j)\, L^+_{12}(t_1) \ldots L^+_{12}(t_\ell)\, v_1 \otimes \ldots \otimes v_n = 0 \tag{3.2}$$

in $V_1(z_1) \otimes \ldots \otimes V_n(z_n)$ and

$$L^+_{21}(t_1) \ldots L^+_{21}(t_\ell)\, L^+_{21}(q^{2\Lambda_j} z_j) \ldots L^+_{12}(q^{2\Lambda_j - 2\ell} z_j)\, v_n \otimes \ldots \otimes v_1 = 0 \tag{3.3}$$

in $V_n(z_n) \otimes \ldots \otimes V_1(z_1)$.

Both relations (3.2) and (3.3) are equivalent to the identitiy (2.6) and, hence, the identity (1.3). For (3.2) it follows from Proposition 3.3 and formula (2.1), while for (3.3) an analogue of Proposition 3.3 for the tensor product $V_n(z_n) \otimes \ldots \otimes V_1(z_1)$ is required.

Remark. It is rather easy to see that Corollary 3.2, Proposition 3.3 and the determinant formula (2.8) together imply Proposition 3.1.

Let parameters q, $q^{\Lambda_1}, \ldots, q^{\Lambda_n}$, $z_1, \ldots, z_n$ used in this section be related to the previously used parameters η, $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ as follows:

$$\eta = q^2, \qquad x_m = q^{2\Lambda_m} z_m, \qquad y_m = q^{-2\Lambda_m} z_m.$$

Proposition 3.3. [KBI], [TV1, Lemma 4.3], [TV2, Lemma 4.18]

$$L^+_{12}(t_1)\dots L^+_{12}(t_\ell)\, v_1\otimes\dots\otimes v_n \;=\; (q-q^{-1})^\ell \prod_{m=1}^n (-z_m)^{-\ell} \sum_{\lambda\in\mathcal{P}_{\ell,n}} P_\lambda(t_1,\dots,t_\ell) \;\times$$

$$\times \prod_{1\leqslant j<k\leqslant n} q^{\Lambda_j\omega_{k,\lambda}-\Lambda_k\omega_{j,\lambda}-\omega_{j,\lambda}\omega_{k,\lambda}}\, F^{\omega_{1,\lambda}}v_1\otimes\dots\otimes F^{\omega_{n,\lambda}}v_n\,.$$

$$L^+_{21}(t_1)\dots L^+_{21}(t_\ell)\, F^{\omega_{1,\lambda}}v_1\otimes\dots\otimes F^{\omega_{n,\lambda}}v_n \;=$$

$$=\prod_{m=1}^n (-z_m)^{\omega_{m,\lambda}-\ell}\, P'_\lambda(t_1,\dots,t_\ell) \prod_{m=1}^n \prod_{s=1}^{\omega_{m,\lambda}} \frac{(q^s-q^{-s})\,(q^{2\Lambda_m-s+1}-q^{-2\Lambda_m+s-1})}{q-q^{-1}} \;\times$$

$$\times \prod_{1\leqslant j<k\leqslant n} q^{\Lambda_k\omega_{j,\lambda}-\Lambda_j\omega_{k,\lambda}-\omega_{j,\lambda}\omega_{k,\lambda}}\, v_1\otimes\dots\otimes v_n\,.$$

4. Elliptic identities

In this section we extend Theorem 1.1 to the elliptic case, see Theorem 4.1. The obtained identities have the same relation to representations of the elliptic quantum group $E_{\rho,\gamma}(\mathfrak{sl}_2)$ as the identities (1.3) have to representations of the quantum loop algebra $U_q(\widetilde{\mathfrak{gl}_2})$.

Fix a nonzero complex number p such that $|p|<1$. Let

$$(u)_\infty \;=\; (u;p)_\infty \;=\; \prod_{s=0}^\infty (1-p^s u)$$

and let $\theta(u)=\theta(u;p)=(u)_\infty(pu^{-1})_\infty(p)_\infty$ be the Jacobi theta-function.

In addition to $t_1,\dots,t_\ell$, $x_1,\dots,x_n$, $y_1,\dots,y_n$ and η introduce one more variable α called the *dynamical parameter*.

Remark. The parameter α is related to the dynamical variable in the elliptic quantum group $E_{\rho,\gamma}(\mathfrak{sl}_2)$.

For any $m=1,\dots,n$ set

$$Z_m(u;x;y;\alpha) \;=\; \theta\bigl(\alpha_m^{-1}u/x_m\bigr) \prod_{1\leqslant j<m} \theta(u/y_j) \prod_{m<k\leqslant n} \theta(u/x_k)\,,$$

$$Z'_m(u;x;y;\alpha) \;=\; \theta\bigl(\alpha_m u/y_m\bigr) \prod_{1\leqslant j<m} \theta(u/x_j) \prod_{m<k\leqslant n} \theta(u/y_k)\,,$$

where $\alpha_m=\alpha\prod_{1\leqslant j<m} x_j/y_j$, and for any $\lambda\in\mathcal{P}_{\ell,n}$ set

$$\rho_\lambda(\eta) \;=\; \prod_{m=1}^n \prod_{s=1}^{\omega_{m,\lambda}} \frac{\theta(\eta)}{\theta(\eta^s)}\,,$$

$$(4.1)\quad \varXi_\lambda(t;x;y;\eta;\alpha) \,=\, \rho_\lambda(\eta) \sum_{\sigma\in\mathbf{S}_\ell} \Bigl(\prod_{a=1}^{\ell} Z_{\lambda_a}(t_{\sigma_a};x;y;\alpha\eta^{2a-2\ell}) \prod_{1\leqslant a<b\leqslant\ell} \frac{\theta(\eta t_{\sigma_b}/t_{\sigma_a})}{\theta(t_{\sigma_b}/t_{\sigma_a})} \Bigr)\,,$$

$$(4.2)\quad \varXi'_\lambda(t;x;y;\eta;\alpha) \,=\, \rho_\lambda(\eta) \sum_{\sigma\in\mathbf{S}_\ell} \Bigl(\prod_{a=1}^{\ell} Z'_{\lambda_a}(t_{\sigma_a};x;y;\alpha\eta^{2\ell-2a}) \prod_{1\leqslant a<b\leqslant\ell} \frac{\theta(\eta t_{\sigma_a}/t_{\sigma_b})}{\theta(t_{\sigma_a}/t_{\sigma_b})} \Bigr)\,.$$

Notice that

$$Z_m(u;x;y;\alpha) \,=\, Z'_m(u;y;x;\alpha^{-1})$$

and

$$\varXi_\lambda(t;x;y;\eta;\alpha) \,=\, \eta^{\,\ell(\ell-1)/2\,-\sum_{m=1}^{n}\omega_{m,\lambda}(\omega_{m,\lambda}-1)/2}\, \varXi'_\lambda(t;y;x;\eta^{-1};\alpha^{-1})\,.$$

Theorem 4.1. *Let* $x_j=\eta^{\ell-1}y_i$ *for some* $i<j$ *. Then*

$$(4.3)\qquad \sum_{\lambda\in\mathcal{P}_\ell^{i,j}} C_\lambda^{(i,j)}(x;y;\eta;\alpha)\,\varXi_\lambda(t;x;y;\eta;\alpha) \,=\, 0$$

where $\mathcal{P}_\ell^{i,j}=\{\,\lambda=(\lambda_1,\dots,\lambda_\ell)\mid j\geqslant\lambda_1\geqslant\ldots\geqslant\lambda_\ell\geqslant i\,\}$ *,*

$$\begin{aligned} &C_\lambda^{(i,j)}(x;y;\eta;\alpha) \,=\\ &=\, (\alpha_i x_i/y_i)^{\omega_{i,\lambda}}\,\eta^{-\omega_{i,\lambda}(\omega_{i,\lambda}-1)} \prod_{i<k<j}\ \prod_{s=0}^{\omega_{k,\lambda}-1} \frac{\theta(\eta^{-s}x_k/y_k)}{\theta(\eta^s\alpha_{k,\lambda}^{-1})\,\theta(\eta^{1-s-\omega_{k,\lambda}}\alpha_{k,\lambda}\,x_k/y_k)}\ \times\\ &\times \prod_{s=0}^{\omega_{i,\lambda}-1} \frac{1}{\theta(\eta^{1-s-\omega_{i,\lambda}}\alpha_{i,\lambda}\,x_i/y_i)} \prod_{s=0}^{\omega_{j,\lambda}-1} \frac{1}{\theta(\eta^s\alpha_{j,\lambda}^{-1})} \prod_{a=\omega_{j,\lambda}+1}^{\ell-\omega_{i,\lambda}} \theta(\alpha_{\lambda_a}\eta^{a-\ell}y_i/y_{\lambda_a})\ \times\\ &\qquad\qquad \prod_{a=1}^{\ell}\Bigl(\prod_{i<k<\lambda_a}\theta(\eta^{\ell-a}y_i/x_k) \prod_{\lambda_a<m<j}\theta(\eta^{\ell-a}y_i/y_m)\Bigr)\,, \end{aligned}$$

$\alpha_{k,\lambda}=\alpha\prod_{1\leqslant j<k}\eta^{-2\omega_{j,\lambda}}x_j/y_j$ *and* $\alpha_k=\alpha\prod_{1\leqslant j<k}x_j/y_j$ *.*

Example. Identity (4.3) for $i=1$, $j=n=2$ takes the form

$$\begin{aligned} (4.4)\qquad &\sum_{k=0}^{\ell}(-1)^k\,\theta(\eta^{2k}\beta) \prod_{s=0}^{k-1} \frac{\eta^s\,\theta(\eta^{\ell-s})\,\theta(\eta^s\beta)}{\theta(\eta^{s+1})\,\theta(\eta^{s+\ell+1}\beta)}\ \times\\ &\times \sum_{\sigma\in\mathbf{S}_\ell}\Bigl(\prod_{1\leqslant a\leqslant k}\theta(t_{\sigma_a})\,\theta(\eta^{2-2a-\ell}t_{\sigma_a}/\beta) \prod_{k<b\leqslant\ell}\theta(\eta^{1-\ell}t_{\sigma_b})\,\theta(\eta^{1-2b}t_{\sigma_b}/\beta)\ \times\\ &\qquad\times \prod_{1\leqslant a<b\leqslant\ell}\frac{\theta(\eta t_{\sigma_b}/t_{\sigma_a})}{\theta(t_{\sigma_b}/t_{\sigma_a})}\Bigr) \,=\, 0\,. \end{aligned}$$

Here $\beta = \eta^{1-2\ell}\alpha x_1/y_1$. If $p \to 0$ and then either $\beta \to 0$ or $\beta \to \infty$, then (4.4) transforms into $(\mathbb{I})$.

Proof of Theorem 4.1. The proof of Theorem 4.1 is very similar to the proof of Theorem 1.1. We descripe the main steps below. We do not indicate dependence on $x_1, \ldots, x_n$, $y_1, \ldots, y_n$, η, α explicitly.

Denote by $\mathbb{E}$ the space of functions $f(t_1, \ldots, t_\ell)$ holomorphic outside the coordinate hyperplanes $t_a = 0$, $a = 1, \ldots, \ell$, and such that

$$f(t_1, \ldots, pt_a, \ldots, t_\ell) = t_a^{-2n} \prod_{m=1}^{n} (x_m y_m)\, f(t_1, \ldots, t_\ell).$$

Let

$$\Omega(t_1, \ldots, t_\ell) = \prod_{a=1}^{\ell} \prod_{m=1}^{n} \theta(t_a/x_m)\, \theta(t_a/y_m) \prod_{\substack{a,b=1 \\ a \neq b}}^{\ell} \frac{\theta(\eta t_a/t_b)}{\theta(t_a/t_b)}.$$

There is an analogue of Lemma 2.3.

Lemma 4.2. [TV1, Lemma C.11] *For any function* $f \in \mathbb{E}$ *we have*

$$x \triangleright Res(f/\Omega) = (-1)^\ell \, y \triangleleft Res(f/\Omega).$$

Define a scalar product $\langle f, g \rangle_\Omega = x \triangleright Res(fg/\Omega)$. The functions (4.1) and (4.2) are biorthogonal with respect to the introduced scalar product.

Theorem 4.3. [TV1, Theorem C.9] $\langle \Xi'_\lambda, \Xi_\mu \rangle_\Omega = D_\lambda^{-1} \delta_{\lambda\mu}$ *where*

$$D_\lambda = (-1)^\ell \prod_{m=1}^{n} \prod_{s=0}^{\omega_{m,\lambda}-1} \frac{(p)_\infty^3 \, \theta(\eta^{s+1})\, \theta(\eta^{-s} x_m/y_m)}{\theta(\eta)\, \theta(\eta^s \alpha_{m,\lambda}^{-1})\, \theta(\eta^{1-s-\omega_{m,\lambda}} \alpha_{m,\lambda} x_m/y_m)} \tag{4.5}$$

and $\alpha_{m,\lambda} = \alpha \prod_{1 \leqslant j < m} \eta^{-2\omega_{\lambda,j}} x_j/y_j$.

Let $\vartheta_1, \ldots, \vartheta_n$ be the following functions in one variable:

$$\vartheta_m(u) = u^{m-1} \theta\bigl(-p^{m-1} \eta^{\ell-1} \alpha^{-1} \prod_{m=1}^{n} x_m (-u)^n; p^n\bigr)\, (p^n; p^n)_\infty^{-1}\, (p;p)_\infty^n. \tag{4.6}$$

They are linearly independent, since $\vartheta_m(\varepsilon u) = \varepsilon^{m-1} \vartheta(u)$ where $\varepsilon = e^{2\pi i/n}$. For any $\lambda \in \mathcal{P}_{\ell,n}$ set

$$\Theta_\lambda(t_1, \ldots, t_\ell) = \frac{1}{\omega_{1,\lambda}! \ldots \omega_{n,\lambda}!} \sum_{\sigma \in \mathbf{S}_\ell} \vartheta_{\lambda_1}(t_{\sigma_1}) \ldots \vartheta_{\lambda_n}(t_{\sigma_\ell}). \tag{4.7}$$

The functions Θ_λ, $\lambda \in \mathcal{P}_{\ell,n}$, are linearly independent, and the functions Ξ_μ are linear combinations of the functions Θ_λ, see Proposition 4.4.

We proceed further like in Section 2 and under the assumption of Theorem 4.1: $y_i = \eta^{1-\ell}x_j$ for some $i < j$, we get an identity

$$\sum_{\lambda\in\mathcal{P}_{\ell,n}} \Xi'_\lambda(x \triangleright \kappa^{(j)})\, D_\lambda\, \Xi_\lambda \,=\, 0\,,$$

and then the identity (4.3). Theorem 4.1 is proved. □

Remark. There are elliptic analogues of determinant formulae (2.7) and (2.8), see formulae (B.11) and (B.8) in [TV1]:

$$(4.8)\quad \det\bigl[\Theta_\lambda(x\triangleright\mu)\bigr]_{\lambda,\mu\in\mathcal{P}_{\ell,n}} \,=\, K\,\eta^{n(1-n)/2\cdot\binom{n+\ell-1}{n+1}} \prod_{m=1}^{n} (-x_m)^{(m-1)\binom{n+\ell-1}{n}} \times$$

$$\times \prod_{s=0}^{\ell-1} \theta(\eta^s\alpha^{-1})^{\binom{n+s-1}{n-1}} \prod_{s=1-\ell}^{\ell-1} \prod_{1\leqslant j<k\leqslant n} \theta(\eta^s x_j/x_k)^{D(n,\ell,s)}$$

where $K \,=\, \Bigl[(p)_\infty^{n^2-1} \prod_{m=1}^{n-1} \Bigl(\dfrac{\theta(e^{2\pi i m/n})}{e^{2\pi i m/n}-1}\Bigr)^{m-n}\,\Bigr]^{\binom{n+\ell-1}{n}}$, and

$$(4.9)\qquad \det\,[A^{ell}_{\lambda\mu}]_{\lambda,\mu\in\mathcal{P}_{\ell,n}} \,=\, K^{-1} \prod_{s=1-\ell}^{\ell-1}\prod_{m=1}^{n-1} \theta(\eta^{s+\ell-1}\alpha^{-1}\prod_{1\leqslant j\leqslant m} y_j/x_j)^{d(n,m,\ell,s)} \times$$

$$\times \prod_{m=1}^{n} y_m^{(m-n)\binom{n+\ell-1}{n}} \prod_{s=0}^{\ell-1} \prod_{1\leqslant j<k\leqslant n} \theta(\eta^s y_j/x_k)^{\binom{n+\ell-s-2}{n-1}}$$

where $\Xi_\lambda \,=\, \sum_{\mu\in\mathcal{P}_{\ell,n}} A^{ell}_{\lambda\mu}\,\Theta_\mu$ and $d(n,m,\ell,s) \,=\, \sum_{\substack{i,j\geqslant 0\\ i+j<\ell\\ i-j=s}} \binom{m-1+i}{m-1}\binom{n-m-1+j}{n-m-1}$.

Remark. Consider a limit $p\to 0$ in all formulae of this section, which essentially amounts to replacing the theta-function $\theta(u)$ by the linear function $1-u$. The functions $\vartheta_1,\dots,\vartheta_n$ tend to power functions as $p\to 0$:

$$\vartheta_1(u) \to 1 + \eta^{\ell-1}\alpha^{-1}\prod_{m=1}^{n} x_m\,(-u)^n\,, \qquad \vartheta_m(u)\to u^{m-1}\,, \qquad m=2,\dots,n\,.$$

The limit $p\to 0$ of the identities (4.3) and the determinant formulae (4.8) and (4.9) deliver one-parametric deformations of the identities (1.3) and the determinant formulae (2.7) and (2.8), which can be recovered in the limit either $\alpha\to 0$ or $\alpha\to\infty$. The limit $\alpha\to 0$ is straightforward while the limit $\alpha\to\infty$ produces formulae (1.3), (2.7), (2.8) up to a certain change of variables.

Proposition 4.4. *For any* $\lambda \in \mathcal{P}_{\ell,n}$ *the function* Ξ_λ *, cf.* (4.1), *is a linear combination of the functions* Θ_μ *,* $\mu \in \mathcal{P}_{\ell,n}$ *, cf.* (4.7).

Proof. Let $\mathcal{E}$ be a space of functions $f(u)$ holomorphic for $u \neq 0$ and such that

$$f(pu) \,=\, \alpha\eta^{1-\ell} \prod_{m=1}^{n} x_m \,(-u)^n \, f(u)\,.$$

Let $\mathcal{E}_\ell$ be a space of symmetric functions $f(t_1,\dots,t_\ell)$ which considered as functions of one variable t_a belong to $\mathcal{E}$ for any $a = 1,\dots,\ell$. In particular, $\mathcal{E}_1 = \mathcal{E}$.

It is clear that $\dim \mathcal{E} \,=\, n$, say by Fourier series. A basis in $\mathcal{E}$ is given by the functions $\vartheta_1,\dots,\vartheta_n$. Hence, the functions Θ_μ, $\mu \in \mathcal{P}_{\ell,n}$, form a basis in the space $\mathcal{E}_\ell$.

Since for any $\lambda \in \mathcal{P}_{\ell,n}$ the function Ξ_λ belongs to the space $\mathcal{E}_\ell$, the proposition follows. □

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