Perturbation theory for quasiperiodic solutions of infinite–dimensional Hamiltonian systems

1. Symplectic structures and Hamiltonian systems in the scales of Hilbert spaces

by

S.B. Kuksin

Max–Planck–Institut
für Mathematik
Gottfried–Claren–Straße 26
D–5300 Bonn 3

Federal Republic of Germany

Institute of Control Sciences
65 Profsoyuznaya Street
117806 Moscow, GSP–7

USSR

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Ball model for Hilbert's twelvth problem

by

R.–P. Holzapfel

Max–Planck–Institut
für Mathematik
Gottfried–Claren–Straße 26
D–5300 Bonn 3

Federal Republic of Germany

MPI/90–95

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This article is the first one in the following series of 3 articles on the complete proofs of the author's theorems on perturbations of quasiperiodic solutions of infinite–dimensional Hamiltonian systems. The articles are based on the author's doctoral thesis "Perturbation theory for families of quasiperiodic solutions of infinite–dimensional Hamiltonian systems and its applications" (Moscow 1989, in Russian).

The aim of the first article is to present basic concepts of Hamiltonian mechanics in a form applicable to nonlinear differential equations of mathematical physics.

The following notations are used: for Hilbert spaces X, Y, Z the norms are denoted by $|\cdot|_X$, $|\cdot|_Y$, $|\cdot|_Z$ and inner products by $\langle\cdot,\cdot\rangle_X$, $\langle\cdot,\cdot\rangle_Y$, $\langle\cdot,\cdot\rangle_Z$; dist_X – distance in the space X ; for domains $O_X \subset X$, $O_Y \subset Y$ the space of k–times Fréchet differentiable mappings $O_X \longrightarrow O_Y$ is denoted by $C^k(O_X;O_Y)$ and $C(O_X;O_Y) = C^0(O_X;O_Y)$, $C^k(O_X;\mathbb{R}) = C^k(O_X)$ $\forall k \geq 0$; for $\phi \in C^1(O_X;O_Y)$ the tangent (cotangent) mapping is denoted by $\phi_*(\phi^*)$ (tangent spaces are identified with X and Y, cotangent spaces $T_x^* O_X$, $T_y^* O_Y$ are identified with X and Y through Riesz's isomorphism). For a mapping $G : O_X \longrightarrow O_Y$ we denote by $\mathrm{Lip}(G) = \mathrm{Lip}(G : O_X \longrightarrow O_X)$ its Lipschitz constant,

$$\mathrm{Lip}(G) = \sup_{x_1 \neq x_2} \frac{|G(x_1)-G(x_2)|_Y}{|x_1-x_2|_X} .$$

1. Symplectic Hilbert scales and Hamitonian equations

Let Z be a Hilbert space with inner product $\langle\cdot,\cdot\rangle_Z$ and $\{Z_s \mid s \in \mathbb{R}\}$ a scale of Hilbert spaces with following properties:

a) the Hilbert space Z_{s_1} is densely inclosed in Z_{s_2} if $s_1 \geq s_2$ and the linear space $Z_\infty = \cap Z_s$ is dense in $Z_s \; \forall s$;

b) $Z_0 = Z$;

c) the spaces Z_s and Z_{-s} are dual with respect to inner product $\langle\cdot,\cdot\rangle_Z$.

The norm (inner product) in Z_s will be denoted by $\|\cdot\|_s = (\langle\cdot,\cdot\rangle_s)$. In particular $\|\cdot\|_0 = |\cdot|_Z$ and $\langle\cdot,\cdot\rangle_0 = \langle\cdot,\cdot\rangle_Z$. The pairing between Z_s and Z_{-s} will be denoted $\langle\cdot,\cdot\rangle_0$ or $\langle\cdot,\cdot\rangle_Z$.

Let $J : Z_\infty \longrightarrow Z_\infty$ be a linear operator such that $J(Z_\infty) = Z_\infty$ and

d) J determines isomorphism of scale $\{Z_s\}$ of order $d_J \geq 0$, i.e. for every $s \in \mathbb{R}$ J may be continued to a continuous linear isomorphism $J : Z_s \xrightarrow[\sim]{} Z_{s-d_J}$;

e) the operator J with domain of definition Z_∞ is antisymmetric in Z , i.e. $\langle Jz_1, z_2\rangle_Z = -\langle z_1 , Jz_2\rangle_Z \quad \forall\, z_1, z_2 \in Z_\infty$.

Let us denote by $\bar{J}$ the isomorphism of order $-d_J$ of scale $\{Z_s\}$:

$$\bar{J} = -(J)^{-1} : Z_s \xrightarrow[\sim]{} Z_{s+d_J} \quad \forall s \in \mathbb{R} \qquad (1.1)$$

Lemma 1.1. The operator $\bar{J} : Z \longrightarrow Z_{d_J} \subset Z$ is anti selfadjoint in Z .

Proof. Let $x,y \in Z_\infty$ and $Jx = x_1$, $Jy = y_1$. Then $\bar{J}x_1 = -x$, $\bar{J}y_1 = -y$ and

$$\langle x_1, \bar{J}y_1\rangle_Z = -\langle Jx, y\rangle_Z = \langle x, Jy\rangle_Z = -\langle \bar{J}x_1, y_1\rangle_Z \;.$$

The operator $J : Z \longrightarrow Z$ is continuous, and the space Z_∞ is dense in Z, so the lemma is proved. ∎

Let us introduce in every space Z_s with $s \geq 0$ a 2–form $\alpha = \langle \bar{J}\, dz, dz \rangle_Z$. Here by definition

$$\langle \bar{J}\, dz, dz \rangle_Z \, [z_1, z_2] = \langle \bar{J}\, z_1, z_2 \rangle_Z \; \forall\, z_1, z_2 \in Z_s \tag{1.2}$$

The form α is closed and nondegenerate [A,Ch–B].

Definition. The triple $\{Z, \{Z_s \mid s \in \mathbb{R}\}, \alpha = \langle \bar{J}\, dz, dz \rangle\}$ is called symplectic Hilbert scale (or SHS for brevity).

Example 1.1. Let $Z = \mathbb{R}^n_p \times \mathbb{R}^n_q$, $Z_s = Z\; \forall s$ and $J : Z \longrightarrow Z$, $(p,q) \longmapsto (-q,p)$. In this case $J^2 = -E$ so $\bar{J} = -J^{-1} = J$, $d_J = 0$ and

$$\alpha = \langle \bar{J}\, dz, dz \rangle_Z = \langle J\, dz, dz \rangle_Z = dp \wedge dq \; .$$

Properties a)–e) are obvious and we obtain the classical symplectic structure for even–dimensional spaces [A].

Example 1.2. Let $Z = L_2(S^1) \times L_2(S^1)$, $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, be a space of pairs of square–summable periodic functions $(p(x), q(x))$. Let $Z_s = H^s(S^1) \times H^s(S^1)$. Here $H^s(S^1)$ is the Sobolev space of periodic functions, $s \in \mathbb{R}$ [Ch–B,RS2]. Let us take

$$J : Z_s \longrightarrow Z_s, \; (p(x), q(x)) \longmapsto (-q(x), p(x)) \; .$$

Then $\mathbf{J} = J$ is an isomorphism of scale $\{Z_s\}$ of order zero. Properties a)–e) are evident.

Example 1.3. Let

$$Z_s = H_0^s(S^1) = \{u(x) \in H^s(S^1) \mid \int_0^{2\pi} u(x)dx = 0\} \ .$$

Let us take $J = \partial/\partial x$. Then J is an isomorphism of the scale of order one and $\mathbf{J} = -(J)^{-1}$ is an isomorphism of order -1. Properties a)–e) are evident again and we have got SHS corresponding to symplectic structure of KdV–equation (see below and [A, Appendix 13; N]).

For $f \in C^1(O_s)$ let $\nabla f \in Z_{-s}$ be the gradient of f with respect to the inner product $\langle \cdot,\cdot \rangle_Z$:

$$\langle \nabla f(u),v \rangle_Z = Df(u)(v) = \frac{\partial}{\partial \epsilon} f(u+\epsilon v) \mid_{\epsilon=0} \forall v \in O_s \ .$$

The mapping $O_s \longrightarrow Z_{-s}$, $u \longmapsto \nabla f(u)$, is continuous.

For $H \in C^1(O_s)$ the Hamiltonian vector–field V_H is the mapping $V_H : V_s \longrightarrow Z_{-\infty} = \bigcup_s Z_s$ defined by the following relation [A, Ch–B]:

$$\alpha(\xi,V_H(u)) = \langle \xi,\nabla H(u) \rangle_Z \ \forall \xi \in Z_\infty$$

or

$$\langle \mathbf{J}\xi,V_H(u) \rangle_Z = \langle \xi,\nabla H(u) \rangle_Z \ \forall \xi \in Z_\infty \ .$$

So $V_H(u) = J\nabla H(u)$ and

$$\dot{u} = J\nabla H(u) \tag{1.3}$$

is the Hamiltonian equation corresponding to the hamiltonian H. Let us denote

$$D_s(V_H) = \{u \in O_s \mid V_H(u) = J\nabla H(u) \in Z_s\} .$$

Definition (cf. [B]). A curve $u(t)$, $0 \leq t \leq T$, is called a strong solution in the space Z_s of the equation (1.3) iff $u \in C^1([0,T];Z_s)$, $u(t) \in D_s(V_H)\ \forall t \in [0,T]$ and $\forall t$ equation (1.3) is satisfied. A curve $u \in C([0,T];Z_s)$ is called a weak solution of (1.3) iff it is the limit in $C([0,T];Z_s)$–norm of some sequence of strong solutions.

Definition. Let $O_s^1 \subset O_s$ be a domain such that for every $u_0 \in O_s^1$ there exist a unique weak solution $u(t) = S^t(u_0)$ $(0 \leq t \leq T)$ of equation (1.3) with initial condition $u(0) = u_0$. The set of mappings

$$S^t : O_s^1 \longrightarrow O_s ,\ u_0 \longmapsto S^t(u_0) \quad (0 \leq t \leq T)$$

is called "local semiflow of equation (1.3)" or "flow of equation (1.3)" for short.

Weak solutions of equations (1.3) are generalized ones in the sense of distributions (see [L] for systematic use of this type of solutions):

Proposition 1.4. Let us suppose that for some $s_1 \in \mathbb{R}$, $\mathrm{Lip}(\nabla H : O_s \longrightarrow Z_{s_1}) < \infty$.

Then a weak solution $u(t) \in O_s$ $(0 \le t \le T)$ of equation (1.3) is a generalised solution and after substitution of $u(t)$ into (1.3) the left and right hand sides of the equation coincide as elements of the space $D'((0,T);Z_{s_2})$ of distributions on $(0,T)$ with values in Z_{s_2}, $s_2 = \min\{s, s_1 - d_J\}$.

Proof. By definition of weak solution there exist a sequence of strong solutions $u_n(t)$ such that $u_n(\cdot) \longrightarrow u(\cdot)$ in $C([0,T];X_s)$. For this sequence

$$\dot{u}_n \longrightarrow \dot{u} \text{ in } D'((0,T);Z_s) \ ,$$

$J\nabla H(u_n) \longrightarrow J\nabla H(u)$ in $C([0,T];Z_{s_1-d_J})$. After transition to limit in equation (1.3) one obtains the result. ■

Example 1.1, again. Let $H \in C^1(\mathbb{R}^n_p \times \mathbb{R}^n_q)$. The Hamiltonian equation takes the classical form:

$$\dot{p} = -\nabla_q H(p,q) \ , \ \dot{q} = \nabla_p H(p,q) \ .$$

If $H \in C^2(\mathbb{R}^{2n})$ then a weak solution is a strong one and it exists for some $T > 0$, $T = T(p(0),q(0))$.

Example 1.2, again. Let us consider the hamiltonian

$$H = \frac{1}{2}\int_0^{2\pi} (p_x(x)^2 + q_x(x)^2 + V(x)(p(x)^2 + q(x)^2) + \chi(p(x)^2 + q(x)^2))dx$$

with analytical function χ and smooth function V. Then $H \in C^1(Z_s)$ for $s \geq 1$ and

$$\nabla H(p,q) = (-p_{xx} + V(x)p + \chi'(p^2+q^2)p\,,\ -q_{xx} + V(x)q(x) + \chi'(p^2 + q^2)q)\ .$$

The equation (1.3) takes now the following form:

$$\dot p = q_{xx} - V(x)q - \chi'(p^2+q^2)q\ ,$$
$$\dot q = -p_{xx} + V(x)p + \chi'(p^2+q^2)p\ .$$

Let us denote $u(t,x) = p(t,x) + iq(t,x)$. The last equations are equivalent to nonlinear Schrödinger equation with real potential $V(x)$ for complex functions $u(t,x)$:

$$\begin{aligned} &\dot u = i(-u_{xx} + V(x)u + \epsilon\,\chi'(|u(x)|^2)u)\ , \\ &u(t,x) \equiv u(t,x+2\pi)\ . \end{aligned} \tag{1.4}$$

The problem (1.4) has an unique strong solution $u(t,x)$, $u(t,\cdot) \in Z_s$, $0 \leq t \leq T = T(u(0,x))$, if $s \geq 1$ and $u(0,x) \in Z_{s+2}$ (we interpret here Z_s as the Sobolev space of periodic complex–valued functions), and (1.4) has an unique weak solution for $0 \leq t \leq T$ if $u(0,x) \in Z_s$. For the simple proof see part 3 below.

Example 1.3, again. In the situation of example 1.3 let us consider the hamiltonian

$$H = \int_0^{2\pi} (\tfrac{1}{2} u_x{}^2 + u^3)dx\ .$$

Then $H \in C^1(Z_s)$ for $s \geq 1$ and $\nabla H(u(x)) = -u_{xx} + 3u^2$. So now equation (1.3) is the

KdV equation

$$\dot{u}(t,x) = -u_{xxx}+6uu_x \tag{1.5}$$

for periodic on x functions with zero mean value:

$$u(t,x) \equiv u(t,x+2\pi)\ ,\ \int_0^{2\pi} u(t,x)dx \equiv 0 \tag{1.5'}$$

It is well known [K] that for $s \geq 3$ the problem (1.5), (1.5′) has an unique strong solution $u(t,x)$, $u(t,\cdot) \in Z_s \forall t$, for every initial condition $u(0,x) = u_0(x) \in Z_{s+3}$ and has an unique weak solution for every $u_0(x) \in Z_s$. The flow of problem (1.5), (1.5′) defines a homeomorphisms of phase space

$$S^t : Z_s \xrightarrow[\sim]{} Z_s \ \forall\, t \geq 0 \quad \forall\, s \geq 3 \ .$$

It is worth to mention that any Hamiltonian equation (including (1.4) and (1.5), (1.5′)) may be written down in a form (1.3) in many different ways. For this statement see below Corollary 2.3.

2. Canonical transformations

Let $\{X,\{X_s\},\alpha^X\}$ and $\{Y,\{Y_s\},\alpha^Y\}$ be two SHS with 2–forms $\alpha^X = \langle J^X dx,dx\rangle_X$ and $\alpha^Y = \langle J^Y dy,dy\rangle_Y$ respectively; let J^X (J^Y) be an isomorphism of scale $\{X_s\}$ $(\{Y_s\})$ of order $-d_{J^X}$ $(-d_{J^Y})$; $d_{J^X}, d_{J^Y} \geq 0$. A mapping $\phi : O^X_{s_X} \longrightarrow O^Y_{s_Y}$ is a C^1–diffeomorphism of domains $O^X_{s_X} \subset X_{s_X}$ and $O^Y_{s_Y} \subset Y_{s_Y}$ $(s_X \geq 0, s_Y \geq 0)$, if ϕ is one–to–one onto $O^Y_{s_Y}$ and

$$\phi \in C^1(O^X_{s_X};O^Y_{s_Y}) , \ \phi^{-1} \in C^1(O^Y_{s_Y};O^X_{s_X}) \tag{2.1}$$

Definition. A C^1–diffeomorphism $\phi : O^X_{s_X} \longrightarrow O^Y_{s_Y}$ is canonical transformation iff it transforms 2–form α^Y into 2–form α^X :

$$\phi^* \alpha^Y = \alpha^X \ . \tag{2.2}$$

Proposition 2.1. A C^1–diffeomorphism $\phi : O^X_{s_X} \longrightarrow O^Y_{s_Y}$ is canonical iff

$$\phi^* J^Y \phi_* \equiv J^X \tag{2.3}$$

(the identity takes place in the space $L(X_{s_X};X_{-s_X})$) .

Proof. From (2.2) one has for $v \in O^X_{s_X}$ and $\xi_1,\xi_2 \in X_{s_X}$

$$\langle J^Y \phi_*(v)\xi_1, \phi_*(v)\xi_2\rangle_Y = \langle J^X \xi_1,\xi_2\rangle_X \ . \tag{2.4}$$

Therefore

$$\langle \phi^*(v) J^Y \phi_*(v) \xi_1, \xi_2 \rangle_X = \langle J^X \xi_1, \xi_2 \rangle_X$$

for all $\xi_1, \xi_2 \in X_{s_X}$. This identity implies the stated assertion.

■

As in the finite–dimensional case [A] a canonical transformation transforms solutions of Hamiltonian equation into solutions of equation with transformed hamiltonian:

Theorem 2.2. Let $\phi : O^X_{s_X} \longrightarrow O^Y_{s_Y}$ be a canonical transformation and let $y : [0,T] \longrightarrow O^Y_{s_Y}$ be a strong solution of Hamiltonian equation

$$\dot{y} = V_{H^Y}(y) = J^Y \nabla H^Y(y) \, , \ \ H^Y \in C^1(O^Y_{s_Y};\mathbb{R}) \ . \tag{2.5}$$

Then $x(t) = \phi^{-1}(y(t))$ is a strong solution in $O^X_{s_X}$ of equation

$$\dot{x} = V_{H^X}(x) = J^X \nabla H^X(x) \, , \ \ H^X = H^Y \circ \phi \ . \tag{2.6}$$

If the mapping $\phi^{-1} : O^Y_{s_Y} \longrightarrow O^X_{s_X}$ is Lipschitz and y is a weak solution of (2.5) then x is a weak solution of (2.6).

Proof. For $H^X = H^Y \circ \phi$ and $x = \phi^{-1} \circ y$ $\quad \nabla H^X = \phi^* \nabla H^Y$. Then $x : [0,T] \longrightarrow O^X_{s_X}$ is C^1 and for $y = \phi \circ x$

$$\phi_* \dot{x} = \dot{y} = J^Y \nabla H^Y(y) = J^Y(\phi^*)^{-1} \nabla H^X(x) \tag{2.7}$$

or

$$\dot{x} = (\phi_*)^{-1} J^Y (\phi^*)^{-1} \nabla H^X(x) \tag{2.8}$$

(the right–hand side is well defined because $J^Y(\phi^*)^{-1} \nabla H^X(x) \in C([0,T]; O^Y_{s_Y})$ for (2.1)). By (2.3), $J^X = (\phi_*)^{-1} J^Y (\phi^*)^{-1}$, hence

$$\dot{x} = J^X \nabla H^X(x)$$

as stated.

The second statement of the theorem follows from the first one and the definition of a weak solution because the mapping ϕ^{-1} is Lipshitz. ■

Let $\{Y,\{Y_s\},\alpha^Y\}$ be a SHS, let L be an isomorphism of scale $\{Y_s\}$ of order $\Delta \leq \frac{1}{2} d_{J^Y}$, $L : Y_s \xrightarrow{\sim} Y_{s-\Delta}$ $\forall s$. Let us define second SHS $\{X,\{X_s\},\alpha^X\}$ where $X = Y$, $X_s = Y_s$ and $\alpha^X = \langle J^X dx, dx \rangle_X$, $J^X = L^* J^Y L$. Let $O^X_{s_X}$ be a domain in X_{s_Y} and $O^Y_{s_Y} = L(O^X_{s_X}) \subset Y_{s_Y}$, $s^Y = s^X - \Delta$. The mapping $L : O^X_{s_X} \longrightarrow O^Y_{s_Y}$ is canonical due to Proposition 2.1. So we have trivial

Corollary 2.3 (change of symplectic structure). Let $H^Y \in C^1(O^Y_{s_Y})$ and let $y(t) \in O^Y_{s_Y}$ $(0 \leq t \leq T)$ be a solution of equation (2.5) (strong or weak). Then $x(t) = L^{-1} y(t)$ is a solution of Hamiltonian equation

$$\dot{x} = J^X \nabla H^X(x) \,, \; J^X = L^{-1} J^Y (L^*)^{-1} \,,$$

with a hamiltonian $H^X = H^Y \circ L \in C^1(O^X_{sX})$.

Let $\{X, \{X_s\}, \alpha = \langle J\, dx, dx \rangle_X\}$ be a SHS, O^1_s and O_s be domains in X_s, $O^1_s \subset O_s$ and

$$\mathrm{d\,ist}_{X_s}(O^1_s; X_s \backslash O_s) > \delta > 0 \tag{2.10}$$

Let $H \in C^2(O_s)$ and

$$\nabla H \in C^1(O_s; X_{s+d_J}) \,, \; \|J\nabla H(x)\|_s \leq K \,, \; \mathrm{Lip}(J\nabla H : O_s \longrightarrow X_s) \leq K \,, \tag{2.11}$$

Let us consider the Hamiltonian equation

$$\dot{x} = J\nabla H(x) \tag{2.12}$$

From (2.10), (2.11) one can easily obtain that the flow of equation (2.12) defines mappings $S^t \in C^1(O^1_s; O_s)$ $\forall t \in [0,T]$, $T = \delta/K$, and every S^t is a C^1–diffeomorphism onto its image.

Theorem 2.4. For every $0 \leq t \leq T$ the mapping S^t is a canonical transformation.

Proof. One has to prove that

$$(S^t)^* \alpha(x)[\eta_1,\eta_2] = \alpha[\eta_1,\eta_2] \ \forall x \in O_s^1 \ \forall \eta_1,\eta_2 \in X_s$$

Since $S^0 = \mathrm{Id}$ it is sufficient to prove that

$$(S^\tau)^* \alpha(x)[\eta_1,\eta_2] = \mathrm{const}(\tau) \tag{2.13}$$

Let $x(\tau)$ be the solution of equation (2.12) for $x(0) = x$, and $\eta^j(t)$ $(j = 1,2)$ be the solution of Cauchy problem for linearized on $x(\cdot)$ equation:

$$\dot\eta^j(\tau) = J(\nabla H)_*(x(\tau))\eta^j(\tau)\ , \quad \eta^j(0) = \eta_j \ . \tag{2.14}$$

Then $(S^\tau)_*(z)\eta_j = \eta^j(\tau)$, $j = 1,2$ and

$$\begin{aligned}(S^\tau)^* \alpha(x)[\eta_1,\eta_2] &= \alpha[\eta^1(\tau),\eta^2(\tau)] = \\ &= \langle J\eta^1(\tau),\eta^2(\tau)\rangle_X \equiv \ell(\tau)\end{aligned} \tag{2.15}$$

The function $\ell(\tau)$ is continuously differentable. So (2.13) is equivalent to relation $d/d\tau\, \ell(\tau) \equiv 0$. One has

$$\frac{d}{d\tau}\ell(\tau) = \langle J\,\dot\eta^1,\eta^2\rangle_X + \langle J\,\eta^1,\dot\eta^2\rangle_X =$$

$$= \langle JJ(\nabla H)_*(x)\eta^1,\eta^2\rangle_X + \langle J\,\eta^1,J(\nabla H)_*(x)\eta^2\rangle_X =$$

$$= -\,\langle(\nabla H)_*(x)\eta^1,\eta^2\rangle_X + \langle\eta^1,(\nabla H)_*(x)\eta^2\rangle_X = 0$$

because operator J is anti selfadjoint (Lemma 1.1) and operator $(\nabla H)_*$ is selfadjoint.

The theorem is proved. ∎

Let $H_j \in C^1(O_s)$, $\nabla H_j \in C(O_s;X_{s_j})$ $(j = 1,2)$.

Definition. Let $s_1 + s_2 \geq d_J$. The Poisson bracket of the functions H_1, H_2 is the function $\{H_1,H_2\} \in C(O_s)$ defined by

$$\{H_1,H_2\} = \langle J\nabla H_1, \nabla H_2\rangle_X .$$

Let $0 < \epsilon \leq 1$ and $H \in C^2(O_s)$, let conditions (2.10), (2.11) be satisfied and $S^t \in C^1(O_s^1;O_s)$, $0 \leq t \leq T = \delta/K$, be the flow of the equation

$$\dot{x} = \epsilon\, J\nabla H(x) .$$

Theorem 2.5. For every $G \in C^1(O_s)$ $G(S^t(x)) = G(x) + t\epsilon\{H,G\}(x) + O((\epsilon t)^2)$ $\forall x \in O_s^1$, $\forall\, 0 \leq t \leq T$.

Proof. From the conditions on H it is easy to see that

$$S^t(x) = x + t\epsilon J\nabla H(x) + O(t\epsilon)^2 \text{ in } X_s .$$

So

$$G(S^t(x)) - G(x) = \langle \nabla G(x), S^t(x) - x\rangle_X + O\|S^t x - x\|_s^2 =$$

$$= t\epsilon \left\langle \nabla G(x), J\nabla H(x) \right\rangle_X + O(\epsilon t)^2$$

and the theorem is proved. ■

3. Local solvability of Hamiltonian equations

Let $\{Y,\{Y_s\},\alpha\}$ be SHS, let O_s be a domain in Y_s and let

$$H \in C^2(O_s)\ ,\ \ H(y) = \frac{1}{2}\langle Ay,y\rangle_Y + H_0(y)\ .$$

Here A is an isomorphism of scale $\{Y_s\}$ of order $d_A \geq 0$;

$$A : Y_s \xrightarrow[\sim]{} Y_{s-d_A} \quad \forall s \in \mathbb{R}\ , \tag{3.1}$$

and the operator

$$A : D(A) \subset Y \longrightarrow Y\ ,\ \ D(A) = Y_{d_A}$$

is selfadjoint. So $\nabla(\frac{1}{2}\langle Ay,y\rangle_Y)(y) = Ay$, and the Hamiltonian equation corresponding to H has the form

$$\dot{y} = J(Ay + \nabla H_0(y)) \tag{3.2}$$

We shall prove a simple theorem on the local solvability of equation (3.2) which will suit well to our aims. To formulate the theorem let us suppose that

$$\mathrm{Lip}(J\nabla H_0 : O_s \longrightarrow Y_s) \leq K \tag{3.3}$$

for some $s \geq 0$ and let $O^2, O^1 \subset Y_s$ be domains with the following properties:

$$O^2 \subset O^1 \subset O_s\,,\ \ \mathrm{dist}_{Y_s}(O^1, Y_s \backslash O_s) \geq \delta > 0\ . \tag{3.4}$$

Theorem 3.1. Let

$$AJy = JAy \quad \forall\, y \in Y_\infty \tag{3.5}$$

$$\langle Ay_1, y_2\rangle_s = \langle y_1, Ay_2\rangle_s\,,\ \ \langle Jy_1, y_2\rangle_s = -\langle y_1, Jy_2\rangle_s \quad \forall\, y_1, y_2 \in Y_\infty\ . \tag{3.6}$$

Suppose that every strong solutions $y(t)$ of equation (3.2) with initial condition $y(0) = y_0 \in O^2$ stays inside O^1 for $0 \leq t \leq T$. Then for $y_0 \in O^2 \cap Y_{s+d_1}$, $d_1 = d_A + d_J$, there exists a unique strong solution $y(t)$ for $0 \leq t \leq T$, and for $y_0 \in O^2$ there exists a unique weak solution $y(t)$ for $0 \leq t \leq T$.

Proof. Let us continue the mapping $J\nabla H_0 : O^1 \longrightarrow Y_s$ to a Lipschitz one $V : Y_s \longrightarrow Y_s$. One may take for example

$$V(y) = \begin{cases} \chi(y) J \nabla H_0(y)\,, & y \in O_s \\ 0, y \notin O_s\,, & \end{cases}$$

where $\chi(y) = \delta^{-1} \max\,(0, \delta - \mathrm{dist}_{Y_s}(y; O^1))$ (see (3.4)). The function χ is Lipschitz, it is equal to 1 in O^1 and to 0 out of O_s. So $\mathrm{Lip}(V) \leq K^1$ and $V|_{O^1} = J\nabla H_0$.

Let us consider the equation

$$\dot{y} = JAy + V(y) \tag{3.7}$$

Its solution $y(t)$ is a solution of equation (3.2) until $y(t) \in O^1$.. Let us consider the linear equation

$$\dot{y} = JAy \ , \tag{3.8}$$

too. From (3.5), (3.6) it follows that

$$\langle AJy_1, y_2 \rangle_s = -\langle y_1, AJy_2 \rangle_s \quad \forall y_1, y_2 \in Y_\infty$$

so by repeating the proof of Lemma 1.1 one can obtain that operator $(AJ)^{-1} : Y_s \longrightarrow Y_s$ is anti selfadjoint. So the operator

$$AJ : D(AJ) = Y_{s+d_1} \subset Y_s \longrightarrow Y_s$$

is anti selfadjoint, too. Due to Stone's theorem [RS1] for $y(0) = y_0 \in Y_{s+d_1}$ equation (3.8) has a unique strong solution and the mapping

$$S^T : Y_{s+d_1} \longrightarrow Y_{s+d_1} \ , \ y(0) \longmapsto y(T) \ , \ T > 0 \ ,$$

is isometric with respect to the Y_s–norm. Equation (3.7) is a Lipschitz perturbation of (3.8). So it has the unique strong solution $y(t)$, $t \geq 0$, for every $y(0) \in Y_{s+d_1}$ and the unique weak solution for every $y(0) \in Y_s$ (see [B]) . If $y(0) = y_0 \in O^2$ then due to the theorem's hypotheses such a solution does not leave domain O^1 for $0 \leq t \leq T$ and for such a " t " it is the unqiue solution of equation (3.7).

■

The theorem above reduces the problem of solving equation (3.2) to the problem of finding a *priori* estimate for its solutions.

4. Toroidal phase space

Let us consider a toroidal phase space of the form $\mathcal{Y} = \mathbb{T}^n \times \mathbb{R}^n \times Y$. Here $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$ is the n–dimensional torus, $Y = Y_0$, $\{Y_s \mid s \in \mathbb{R}\}$ is a scale of Hilbert spaces which satisfies properties a)–c) (see above). Let us denote $\mathcal{Y}_s = \mathbb{T}^n \times \mathbb{R}^n \times Y_s$. Every space $\mathcal{Y}_s$ has a natural metric dist_s and a natural structure of a Hilbert manifold with local charts

$$K(q^0) \times \mathbb{R}^n \times Y_s\,,\ K(q^0) = \{q \in \mathbb{R}^n \,\big|\, |q_j - q_j^0| < \pi\ \forall j\}$$

(see [Ch–B]). So

$$T_u\,\mathcal{Y}_s \cong \mathbb{R}^n \times \mathbb{R}^n \times Y_s \equiv Z_s \quad \forall u \in \mathcal{Y}_s\,,$$

Let J^Y be an isomorphism of the scale $\{Y_s\}$ with properties d), e) and

$$J^T : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \times \mathbb{R}^n\,,\ (q,p) \longmapsto (-p,q)\ .$$

Let us denote by $J\mathcal{Y}$ the operator

$$J\mathcal{Y} = J^T \times J^Y : Z_s = (\mathbb{R}^n \times \mathbb{R}^n) \times Y_s \longrightarrow Z_{s-d_J} = (\mathbb{R}^n \times \mathbb{R}^n) \times Y_{s-d_J}$$

and introduce in $\mathcal{Y}_s$, $s \geq 0$, a 2–form

$$\alpha\mathcal{Y} = \langle \bar{J}\mathcal{Y}\,du,du \rangle_Z\,,\ \bar{J}\mathcal{Y} = -(J\mathcal{Y})^{-1}\,,\ T_u\mathcal{Y}_s \cong Z_s\ .$$

Definition. The triple $\{\mathcal{Y},\{\mathcal{Y}_s\},\alpha^{\mathcal{Y}}\}$ is called toroidal symplectic Hilbert scale (TSHS).

Let O_s be a domain in $\mathcal{Y}_s$ and $H \in C^1(O_s)$. Then the Hamiltonian equations corresponding to H have the form

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j} \ (1 \le j \le n), \quad \dot{y} = J^Y \nabla_y H \tag{4.1}$$

The definitions of strong and weak solutions for equations (4.1) are analogous to those for equation (1.3).

The Poisson bracket of two functions H_1, H_2 with $H_j \in C^1(O_s)$, $\nabla_y H_j \in C(O_s;Y_{s_j})$ $(j = 1,2)$, $s_1 + s_2 \ge d_J$, takes the form

$$\{H_1,H_2\}(q,p,y) = \sum_{j=1}^{n} \left[-\frac{\partial H_1}{\partial q_j}\frac{\partial H_2}{\partial p_j} + \frac{\partial H_1}{\partial p_j}\frac{\partial H_2}{\partial q_j}\right] + \langle J^Y \nabla_y H_1, \nabla_y H_2 \rangle_Y .$$

The results of section 1–3 readily extend to canonical transformations and Hamiltonian equations in TSHS. We'll formulate analogs of Theorems 2.2, 2.4, 2.5 and 3.1 only.

Proposition 4.1. The statements of Theorem 2.2 remain true if anyone of the spaces X, Y is replaced by a toroidal symplectic Hilbert space (with equations of motion replaced accordingly).

Let O_s^1, O_s be domains in $\mathcal{Y}_s$, $O_s^1 \subset O_s$ and

$$\operatorname{dist}_{\mathscr{Y}_s}(O_s^1; \mathscr{Y}_s \setminus O_s) > \delta > 0 \ . \tag{4.2}$$

Let $H \in C^2(O_s)$ and $V_H = (\nabla_p H, -\nabla_q H, J^Y \nabla_y H)$ be corresponding Hamitonian vector–field. Let us suppose that $V_H \in C^1(O_s; Z_s)$ and

$$|V_H(q,p,y)| \leq K \ \ \forall (q,p,y), \ \ \operatorname{Lip}(V_H : O_s \longrightarrow Z_s) \leq K \tag{4.3}$$

Then the flow mappings $S^t : O_s^1 \longrightarrow O_s$ exist for $0 \leq t \leq T = \delta/K$ and every S^t is C^1–diffeomorphism on its image.

Proposition 4.2. For every $0 \leq t \leq \delta/K$ the mapping S^t is a canonical transformation.

Let conditions (4.2), (4.3) be fulfilled and $S^t \in C^1(O_S^1; O_S)$ be the flow of equation $\frac{d}{dt}(q,p,y) = \epsilon \, V_H(q,p,y)$.

Proposition 4.3. For every $G \in C^1(O_s)$ $G(S^t(\mathfrak{h})) = G(\mathfrak{h}) + t\epsilon\{H,G\}(\mathfrak{h}) + O(\epsilon t)^2$ $\forall \, \mathfrak{h} = (q,p,y) \in O_s^1$, $\forall \, 0 \leq t \leq T = \delta/K$.

Let in (4.1) $H = \frac{1}{2} \langle Ay,y \rangle_Y + H_0(p,q,y)$ and let the linear operator A be the same as in part 3. Let O_s^1 , O_s^2 , O_s be domains in $\mathscr{Y}_s$, $O_s^2 \subset O_s^1 \subset O_s$ and suppose inequality (4.2) is fulfilled. Let us suppose that $\operatorname{Lip}(V_{H_0} : O_s \longrightarrow Z_s) \leq K$.

Proposition 4.4. Let us suppose that relations (3.5), (3.6) are fulfilled and that every strong solution of (4.1) with initial point $\mathfrak{h}_0 = (q_0,p_0,y_0) \in O_s^2$ stays in domain O_s^1 for

$0 \leq t \leq T$. Then for $\mathfrak{h}_0 \in O_s^2 \cap \mathscr{Y}_{s+d_1}$, $d_1 = d_A + d_J$, and for $0 \leq t \leq T$ there exists a unique strong solution of (4.1); for $\mathfrak{h}_0 \in O_s^2$, $0 \leq t \leq T$, there exists a unique weak solution of (4.1).

The proofs of Propositions 4.1–4.3 are the same as the proofs of the corresponding theorems.

5. A version of the former constructions

All construction of the sections 1–4 have natural analogs for the scales of Hilbert spaces depending on the integer index, i.e. for the scales $\{Z_s \mid s \in \mathbb{Z}\}$. SHS and TSHS with discrete scales $\{Z_s\}$ are sometimes more convenient to study Hamiltonian equations of form (3.2) with integer d_A, d_J . For example, KdV equation (1.5), (1.5′) $(d_J = 1,\ d_A = 2)$ and nonlinear Schrödinger equation (1.4) $(d_J = 0,\ d_A = 2)$.

All the statements of sections 1–4 have natural analogs for discrete scales. The proofs are the same.

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