Weighted estimates for the Cauchy-Riemann equation on piecewise strictly pseudoconvex domains

Bert Fischer

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

Germany

MPI / 92-12

Weighted estimates for the Cauchy-Riemann equation on piecewise strictly pseudoconvex domains

Bert Fischer

1 Introduction

In the beginning of the seventies HENKIN, GRAUERT, LIEB and others investigated the Cauchy-Riemann equation in smooth strictly pseudoconvex domains. They considered uniformly bounded $\bar{\partial}$-closed (0,q)-forms f and defined an integral operator R such that $u = Rf$ is a solution of $\bar{\partial}u = f$ in D and the 1/2-Hölder norm of Rf can be estimated by the supremum norm of f. After some simple modifications of such an operator R one can consider also $\bar{\partial}$-closed (0,q)-forms f which are not uniformly bounded but satisfy an inequality like

$$|f(z)| \leq C[\text{dist}(z, bD)]^{-\beta} \quad \text{for} \quad 0 \leq \beta < 1.$$

It is not difficult to show that $u = Rf$ is still a solution of $\bar{\partial}u = f$ in D and that u is (1/2-β)-Hölder continuous for $0 \leq \beta < 1/2$ and admits an estimate like

$$|u(z)| \leq C[\text{dist}(z, bD)]^{1/2-\beta} \quad \text{for} \quad 1/2 < \beta < 1.$$

A result similar to this can be found for instance in the paper of LIEB and RANGE [LR]. In 1973 RANGE and SIU gave an integral operator, denote it again by R, to solve the Cauchy-Riemann equation on domains which are only piecewise smooth strictly pseudoconvex. For uniformly bounded $\bar{\partial}$-closed (0,q)-forms f they proved that the solution $u = Rf$ admits (1/2-ε)-Hölder estimates for any small ε (see [RS]).

In this paper we also assume the domain to be piecewise smooth strictly pseudoconvex. But we consider $\bar{\partial}$-closed (0,q)-forms f which have singularities at the boundary or on some submanifold of the boundary. That is we assume

$$|f(z)| \leq C[\text{dist}(z, N)]^{-\beta}$$

where N is a submanifold of bD that is in general position and has codimension d in bD and β is a real number with $0 \leq \beta < 1 + d$.

First let us give a motivation for this investigation. Let M be a real hyperplane in $\mathbb{C}^n$ and let D be a domain (for example piecewise smooth strictly pseudoconvex). Let M intersect D and denote the parts of D by D_+ and D_-. Now let f be a bounded (0,q)-form on $M \cap \bar{D}$ with $\bar{\partial}_M f = 0$. We are looking for a solution u of the equation $\bar{\partial}_M u = f$ on $M \cap D$. To get such a solution it is possible to go the following way. First there are two operators S and S' defined with the help of the Martinelli-Bochner kernel such that $\bar{\partial}Sf = S'f$ on $D_+ \cup D_-$. Since $S'f$ is $\bar{\partial}$-closed one can solve the equation $\bar{\partial}g = S'f$ on D and set $f_+ = Sf - g$ on D_+ and $f_- = Sf - g$ on D_-. Obviously f_+ and f_- are $\bar{\partial}$-closed in D_+ resp. D_-. So we can solve the equations $\bar{\partial}u_+ = f_+$ in D_+ and $\bar{\partial}u_- = f_-$ in D_-. If u_+ and u_- are continuous up to the boundary of D_+ and D_- we can set $u = u_+|_{M \cap \bar{D}_+} - u_-|_{M \cap \bar{D}_-}$ on $M \cap D$. It is easy to show that $\bar{\partial}_M u = f$ on $M \cap D$. (For a more detailed description of these facts see [LL] or [AH].) But at least it is necessary that we have u_+ and u_- continuous up to the boundary. And therefore we need some good estimates for f_+ and f_-. But this requires a solution of $\bar{\partial}g = S'f$ with uniform estimates. It is known that $S'f$ has a singularity on $M \cap bD$ and that is the reason why we consider forms with singularities on submanifolds of the boundary.

But it is not so easy to solve the Cauchy-Riemann equation for forms with such singularities. The solution operator defined in [RS] can not be used because it contains some integrals over parts of the boundary. And these integrals need not be defined in our case. So we first have to modify at least these parts of the solution operator. The idea is to use Stokes theorem to transform boundary integrals into integrals over some submanifolds in the interior of the domain. But before we can use Stokes theorem we have to modify the kernels too. At last we get a solution operator T that can be used also for unbounded forms if there is an estimate like

$$|f(z)| \leq C[\text{dist}(z, N)]^{-\beta} \quad \text{for} \quad 0 \leq \beta < 1 + d. \tag{1}$$

The next step is to give some estimates for the solutions obtained by using the above mentioned operator. We prove that u is (1/2-β')-Hölder continuous for $0 \leq \beta < \beta' < 1/2$ and that

$$|u(z)| \leq C[\text{dist}(z, bD)]^{1/2-\beta'} \quad \text{for} \quad 1/2 \leq \beta < \beta' < 1 + d$$

if f satisfies the inequality (1). The proof requires a detailed investigation of the kernel of the solution operator and a lot of computations.

In Section 2 we recall some facts about piecewise smooth strictly pseudoconvex domains, define some special submanifolds in D and introduce some weighted norms and Banach spaces. The construction of the solution operator will be done in Section 3. Before we are able to do it we recall some well-known properties of some of the functions we use, without any proof. At the end of Section 3 we state the main theorem of this paper, namely the estimates mentioned above. Section 4 contains only the proof of the main theorem.

2 Preliminaries

According to Range and Siu a piecewise smooth strictly pseudoconvex domain D is given by a frame $\{U_j, \varrho_j\}_{j=1}^k$ with the following properties:

(i) $\{U_j\}_{j=1}^k$ is a finite open covering of an open neighborhood of bD,

(ii) the functions $\varrho_j : U_j \to \mathbb{R}$ are of class C^2, strictly plurisubharmonic and with $d\varrho_j \neq 0$,

(iii) $D \cap U = \{x \in U : \text{ for all } j \text{ either } x \notin U_j \text{ or } \varrho_j(x) < 0\}$ where U is the union of all U_j,

(iv) for $1 \leq i_1 < \ldots < i_l \leq k$ the 1-forms $d\varrho_{i_1}, \ldots, d\varrho_{i_l}$ are lineary independent over $\mathbb{R}$ at every point of $\bigcap_{\nu=1}^{l} U_{i_\nu}$.

The last condition means that the parts of the boundary have to intersect transversally. Sometimes we only write ($\mathbb{R}$) for it. In addition to this condition we will also consider a stronger condition. This condition means that the intersection of the parts of the boundary must be transversal in a complex sense.

($\mathbb{C}$) For $1 \leq i_1 < \ldots < i_l \leq k$ the (1,0)-forms $\partial\varrho_{i_1}, \ldots, \partial\varrho_{i_l}$ are lineary independent over $\mathbb{C}$ at every point of $\bigcap_{\nu=1}^{l} U_{i_\nu}$.

Now let us give some definitions. For every ordered subset $I = \{i_1, \ldots, i_l\}$ of $\{1, \ldots, k\}$ we define

$$S_I := \{x \in bD \cap (\bigcap_{i \in I} U_i) \ : \ \varrho_i(x) = 0 \ \forall i \in I\}$$

and choose the orientation on S_I such that the orientation is skew symmetric in the components of I and the following two equations hold when D is given the natural orientation:

$$bD = \sum_{j=1}^{k} S_j, \quad bS_I = \sum_{j=1}^{k} S_{Ij}.$$

Further let

$$\Delta = \{\lambda = (\lambda_0, \ldots, \lambda_k) \in \mathbb{R}^{k+1} \ : \ \lambda_j \geq 0, \sum_{j=0}^{k} \lambda_j = 1\}$$

be the standard simplex in $\mathbb{R}^{k+1}$ with the canonical orientation. For every ordered subset $J = \{j_1, \ldots, j_m\}$ of $\{0, \ldots, k\}$ with strictly increasing components we set

$$\Delta_J = \{\lambda \in \Delta \ : \ \sum_{j \in J} \lambda_j = 1\}$$

with the orientation of Δ_J chosen so that

$$b\Delta_J = \sum_{\nu=1}^{m} (-1)^{\nu+1} \Delta_{j_1 .. \hat{j_\nu} .. j_m}.$$

Now we can choose an ε so small, that $\{U_j, \ \varrho_j + \delta\}_{j=1}^{k}$ is a frame for D^δ for all $0 \leq \delta \leq \varepsilon$ where D^δ is defined by

$$D^\delta \cap U = \{x \in U : \text{ for all } j \text{ either } x \notin U_j \text{ or } \varrho_j(x) + \delta < 0\}.$$

For $0 \leq \delta \leq \varepsilon$ and $I = \{i_1, \ldots, i_l\} \subset \{1, \ldots, k\}$ we define

$$\begin{aligned} S_I^\delta \ := \ & \{x \in \bigcap_{j \in I} U_j \ : \ \varrho_{i_1}(x) = \ldots = \varrho_{i_l}(x) = -\delta \\ & \text{and } \forall j \notin I \text{ either } x \notin U_j \text{ or } \varrho_j(x) + \delta < 0\} \end{aligned}$$

and

$$R_I := \{x \in \bigcap_{j\in I} U_j : -\varepsilon < \varrho_{i_1}(x) = \ldots = \varrho_{i_l}(x) < 0$$
$$\text{and } \forall j \notin I \text{ either } x \notin U_j \text{ or } \varrho_j(x) < \varrho_{i_l}(x)\}.$$

Obviously we have $R_I = \bigcup_{0<\delta<\varepsilon} S_I^\delta$. We choose the orientation of the R_I such that the orientation is skew symmetric in the components of I and the following equations hold:

$$D\backslash D^\varepsilon = \sum_{j=1}^k R_j, \quad bR_I = S_I - S_I^\varepsilon - \sum_{j=1}^k R_{Ij}$$

where S_I^ε has the same orientation like all S_I^δ including $S_I^0 = S_I$. Observe that the dimension of R_I is $2n - |I| + 1$. Therefore we have

$$b(R_I \times \Delta_J) = bR_I \times \Delta_J - (-1)^{|I|} R_I \times b\Delta_J.$$

Combining all the relations between the different sets we get the following result:

Lemma 2.1

$$b(\sum_I{}'(-1)^{|I|}R_I \times \Delta_{0I}) = -\sum_I{}' R_I \times \Delta_I + (D\backslash D^\varepsilon) \times \Delta_0 + \sum_I{}'(-1)^{|I|}(S_I \times \Delta_{0I} - S_I^\varepsilon \times \Delta_{0I}) \tag{2}$$

where $|I|$ is the length of I and $\sum'$ means the summation over all ordered subsets I of $\{1, \ldots, k\}$ with strictly increasing components.

Proof. Let $I = \{i_1, \ldots, i_l\}$ and let $J_\nu = \{i_1, ..\hat{i_\nu}.., i_l\}$. Then

$$(-1)^{|I|} b(R_I \times \Delta_{0I}) = (-1)^{|I|}(S_I - S_I^\varepsilon) \times \Delta_{0I} - (-1)^{|I|} \sum_{j=1}^k R_{Ij} \times \Delta_{0I}$$
$$-R_I \times \Delta_I - \sum_{\nu=1}^l (-1)^\nu R_I \times \Delta_{0J_\nu}. \tag{3}$$

We will show that, when we sum over all ordered subsets I of $\{1, \ldots, k\}$ with strictly increasing components, the sum of the second term on the right hand side almost cancels the sum of the fourth term and the net result of the cancellation is $(D\backslash D^\varepsilon) \times \Delta_0$. Obviously it is

$$R_I = (-1)^{\nu - l} R_{J_\nu i_\nu}.$$

It follows that

$$\sum_I{}' \sum_{\nu=1}^l -(-1)^\nu R_I \times \Delta_{0J_\nu} = \sum_I{}' \sum_{\nu=1}^l -(-1)^l R_{J_\nu i_\nu} \times \Delta_{0J_\nu}$$
$$= \sum_{j=1}^k R_j \times \Delta_0 + \sum_J{}' \sum_{j=1}^k (-1)^{|J|} R_{Jj} \times \Delta_{0J}. \tag{4}$$

Using the fact that $(D \backslash D^\epsilon) = \sum_{j=1}^{k} R_j$ the proposition follows from (3) and (4). ■

Now we discuss the relation between N and the S_I. Assume that N is given as $\{\zeta \in bD : \tau_1(\zeta) = ... = \tau_d(\zeta) = 0\}$. For a later use we need some lower estimates for $\text{dist}(\zeta, N)$ in terms which can be more easily computed on R_I and S_I. Fix I and let ζ be a point on R_I. Since $\text{dist}(\zeta, bD) \geq C\text{dist}(\zeta, S_I)$ we also get $\text{dist}(\zeta, N) \geq C\text{dist}(\zeta, S_I)$ for all $\zeta \in R_I$. But in general this is not the best possible estimate. So let M_I be a submanifold of S_I such that $\text{dist}(\zeta, N) \geq C\text{dist}(\zeta, M_I)$ for all $\zeta \in R_I$ and the codimension of M_I in S_I ($d_I := \text{codim}_{S_I} M_I$) is maximal. Since it is not so clear how to find such a submanifold M_I we discuss some special cases. Assume that for some $0 \leq m \leq d$ the submanifold N satisfies the condition

(P_m) There exist m indices $1 \leq j_1 < ... < j_m \leq d$ such that the 1-forms $d\varrho_{i_1}, ..., d\varrho_{i_l}$, $d\tau_{j_1}, ..., d\tau_{j_m}$ are lineary independent over $\mathbb{R}$ at every point in a neighbourhood of $S_I \cap N$.

Then we can set $M_I = \{\zeta \in S_I : \tau_1(\zeta) = ... = \tau_m(\zeta) = 0\}$ and $d_I = m$. Of course there are still some cases where this estimate is not the best one. But notice that we have in the generic case

(G) For every I either $\text{dist}(N, S_I) \geq C > 0$ or the condition (P_d) holds.

That means we can set $M_I = S_I \cap N$ and $d_I = d$ for all I. In this paper we will only consider submanifolds N of the boundary of the domain which are in general position; that is we always assume the generic condition (G).

It remains to define some special Banach spaces. If φ is a nonnegative continuous function on D then by $B_*^\varphi(D)$ we denote the Banach space of differential forms $f \in C_*^0(D)$ with

$$||f||_\varphi := \sup_{z \in D} ||f(z)||\varphi(z) < \infty.$$

For $z \in D$ and $0 \leq \beta < \infty$ we set

$$\begin{aligned} \varphi(\beta, N)(z) &:= [\text{dist}(z, N)]^\beta, \\ \varphi(\beta, bD)(z) &:= [\text{dist}(z, bD)]^\beta. \end{aligned}$$

Further, for $f \in C_*^0(D)$ we set

$$||f||_{C^0} := \sup_{z \in D} ||f(z)||$$

and

$$||f||_{C^\alpha} := ||f||_{C^0} + \sup_{\substack{z,\zeta \in D \\ z \neq \zeta}} \frac{||f(\zeta) - f(z)||}{|\zeta - z|^\alpha} \quad \text{if} \quad 0 < \alpha < 1.$$

Notice that

$$B_*^{\varphi(\beta,N)}(D) \subseteq C_*^0(D) \cap L_*^1(D) \quad \text{if} \quad 0 \leq \beta < d + 1$$

where d was the codimension of N in bD. Since we assume that the submanifold N satisfies the generic condition (G) we have also

$$B_*^{\varphi(\beta,N)}(D) \subseteq C_*^0(D) \cap \bigcap_I L_*^1(R_I) \quad \text{if} \quad 0 \leq \beta < d + 1.$$

3 Construction of the solution operator

As mentioned in the introduction we want to study the Cauchy-Riemann equation for such $(0,q)$-forms f which have singularities at the boundary of the domain D or at some submanifold N of the boundary bD. To do this we can not directly use the operators defined by RANGE and SIU because parts of these operators are integrals over submanifolds S_I of the boundary and are possibly not defined in our case. So we have to modify at least these parts of the operators. Especially we will change the boundary integrals into integrals over some special submanifolds R_I of the interior of D by using Stokes theorem. For this we first define some modified versions of the support functions called $\tilde{\Phi}_j$ which have no singularities inside D. With the help of these functions we later define the kernels $\tilde{K}$ and K'. After using Stokes theorem with some integrals defined by these kernels we get a formula containing seven different integrals. A closer investigation of some of these integrals enables us to combine this formula with a formula stated by RANGE and SIU. At last we get a homotopy formula in terms of operators T_q which contain only integrals over submanifolds lying in the interior of D.

First we recall some well-known facts on some functions which are related with a piecewise smooth strictly pseudoconvex domain that is given by a frame $\{U_j,\ \varrho_j\}_{j=1}^k$.

Proposition 3.1 *There are positiv constants* $c_1, c_2, c_3, c_4 > 0$ *and functions* F_j, H_j, $\Phi_j, \tilde{\Phi}_j(\zeta, z) : U_j \times D \to \mathbb{C}$ *and functions* $P^j(\zeta, z) : U_j \times D \to \mathbb{C}^n$, *with the following properties:*

(i) $F_j(\zeta,z)$ *is the Levi polynomial of* ϱ_j *with a small perturbation of the quadratic terms,* $F_j(\zeta,z)$ *is* C^1 *in* ζ *and holomorphic in* z,

(ii) $\Phi_j(\zeta,z)$ *and* $\tilde{\Phi}_j(\zeta,z)$ *are* C^1 *in* ζ *and holomorphic in* z,

(iii) $\Phi_j(\zeta,\zeta) = 0$ *and* $\Phi_j(\zeta,z) \neq 0$ *for* $|\zeta - z| \geq c_1$,

(iv) $\tilde{\Phi}_j(\zeta,z) \neq 0$ *for* $|\zeta - z| \geq c_1$,

(v) $\tilde{\Phi}_j(\zeta,z) = H_j(\zeta,z)(F_j(\zeta,z) - 2\varrho_j(\zeta))$ *for* $|\zeta - z| < c_1$,

(vi) $c_3 < |H_j(\zeta,z)| < c_4$,

(vii) $\operatorname{Re}\tilde{\Phi}_j(\zeta,z) \geq c_2(|\varrho_j(\zeta)| + |\varrho_j(z)| + |\zeta - z|^2)$ *for* $\zeta,\ z \in U_j$, $|\zeta - z| < c_1$ *and*

$$|\tilde{\Phi}_j(\zeta,z)| \geq c_2(|\varrho_j(\zeta)|+|\varrho_j(z)|+|\operatorname{Im}\tilde{\Phi}_j(\zeta,z)|+|\zeta-z|^2) \quad \text{for} \quad \zeta,\ z \in U_j,\ |\zeta-z| < c_1, \tag{5}$$

(viii) $\tilde{\Phi}_j(\zeta,z) \neq 0$ *and* $\tilde{\Phi}_j(\zeta,z) \geq c_2|\zeta - z|^2$ *for* $\zeta, z \in D$,

(ix) $\tilde{\Phi}_j(\zeta,z) = \Phi_j(\zeta,z)$ *for* $\zeta \in \bar{S}_j$, $z \in D$,

(x) $P^j(\zeta,z)$ *is* C^1 *in* ζ *and holomorphic in* z,

(xi) $\Phi_j(\zeta,z) = \sum_{\nu=1}^{n} P_\nu^j(\zeta,z)(\zeta_\nu - z_\nu)$,

(xii) $P_\nu^j(\zeta,z) = H_j(\zeta,z)\frac{\partial \varrho_j}{\partial \zeta_\nu}(\zeta) + O(|\zeta - z|)$.

The construction of the Φ_j is done by using F_j and solving some $\bar\partial$-equation. The $\tilde\Phi_j$ are constructed in the same way but with F_j replaced by $(F_j - 2\varrho_j)$. For a more detailed description of the construction and a proof of the facts of the proposition see [RS] or [HL].

We further define $P^0(\zeta,z) := \bar\zeta - \bar z$, $\Phi_0 := |\zeta - z|^2$, $\eta^j := P^j/\Phi_j$ and $\tilde\eta^j := P^j/\tilde\Phi_j$. Moreover we choose a C^∞-function χ with $\chi(\zeta) \equiv 1$ for ζ in a neighbourhood of bD and $\chi(\zeta) \equiv 0$ for ζ in $D^{\epsilon/2}$. We set

$$\begin{aligned} \eta(\zeta,z,\lambda) &:= \sum_{j=0}^{k} \lambda_j \eta^j(\zeta,z), \\ \tilde\eta(\zeta,z,\lambda) &:= \lambda_0 \eta^0(\zeta,z) + \chi(\zeta)\sum_{j=1}^{k} \lambda_j \tilde\eta^j(\zeta,z) \end{aligned}$$

and

$$\omega(\zeta) := d\bar\zeta_1 \wedge \ldots \wedge d\bar\zeta_n.$$

In the sequel we want to make use of some determinants whose entries are differential forms. For this purpose we define the determinant of a $n \times n$-matrix $(a_{\alpha\beta})$ of differential forms as follows

$$\det(a_{\alpha\beta}) = \sum_\sigma \operatorname{sgn}(\sigma) a_{\sigma(1),1} \wedge \ldots \wedge a_{\sigma(n),n},$$

where the summation is over all permutations σ of $\{1,\ldots,n\}$ and $\operatorname{sgn}(\sigma)$ is the signature of σ. We will also use the notation

$$\det{}_{m_1,\ldots,m_j}(a_1,\ldots,a_j) := \det(\underbrace{a_1,\ldots,a_1}_{m_1},\ldots,\underbrace{a_j,\ldots,a_j}_{m_1})$$

where the a_i shall be column vectors of differential forms and the sum of the m_j must be equal to n.

Now we are able to give the definitions of some kernels.

$$\begin{aligned} K(\zeta,z,\lambda) &:= \det{}_{1,n-1}(\eta(\zeta,z,\lambda), (\bar\partial_\zeta + \bar\partial_z + d_\lambda)\eta(\zeta,z,\lambda)) \wedge \omega(\zeta) \\ \tilde K(\zeta,z,\lambda) &:= \det{}_{1,n-1}(\tilde\eta(\zeta,z,\lambda), (\bar\partial_\zeta + \bar\partial_z + d_\lambda)\tilde\eta(\zeta,z,\lambda)) \wedge \omega(\zeta) \\ K'(\zeta,z,\lambda) &:= \det{}_n((\bar\partial_\zeta + \bar\partial_z + d_\lambda)\tilde\eta(\zeta,z,\lambda)) \wedge \omega(\zeta) \end{aligned}$$

Observe that the first kernel is the kernel used by RANGE and SIU to define the basic solution operator for piecewise smooth strictly pseudoconvex domains. The Kernels $\tilde K$ and K' which are defined by means of $\tilde\eta$ will be used to define a new solution operator. But first we give some properties of the kernels. We denote by K_q resp. $\tilde K_q$ the sum of all monomials of K resp. $\tilde K$ which are of degree $(0,q)$ in z.

Lemma 3.2 *(i)* $\eta(\zeta,z,\lambda)=\tilde\eta(\zeta,z,\lambda)$ *for* $\zeta\in bD$ *or* $\lambda\in\Delta_0$

(ii) $(\bar\partial_\zeta+\bar\partial_z+d_\lambda)K=0$

(iii) $(\bar\partial_\zeta+\bar\partial_z+d_\lambda)\tilde K=K'$

(iv) $K_q=\binom{n-1}{q}\det_{1,q,n-q-1}(\eta,\bar\partial_z\eta,(\bar\partial_\zeta+d_\lambda)\eta)\wedge\omega(\zeta)$

(v) $\tilde K_q=\binom{n-1}{q}\det_{1,q,n-q-1}(\tilde\eta,\bar\partial_z\tilde\eta,(\bar\partial_\zeta+d_\lambda)\tilde\eta)\wedge\omega(\zeta)$

Proof. Using Proposition 3.1 (ix) and the fact that $\chi(\zeta)\equiv 1$ for $\zeta\in bD$ part (i) of the lemma follows from the definitions of η and $\tilde\eta$. The propositions (iv) and (v) follow by using the linerarity of the determinant in each column and the fact that two columns of 1-forms can be interchanged without changing the sign. To prove (iii) we only have to use the fact that the differential of a determinant is a sum of determinants where the differential is applied to the different columns. When the differential is applied to the first column the sign is plus. And if the differential is applied to one of the other columns this column is zero and therefore the whole determinant vanishes. Proving (ii) we can go the same way and get a determinant similar to K' but $\tilde\eta$ replaced by η. Because of Proposition 3.1 (xi) we now have

$$\sum_{\nu=1}^{n}\eta_\nu(\zeta,z,\lambda)(\zeta_\nu-z_\nu)\equiv 1.$$

Applying $(\bar\partial_\zeta+\bar\partial_z+d_\lambda)$ we get

$$\sum_{\nu=1}^{n}(\bar\partial_\zeta+\bar\partial_z+d_\lambda)\eta_\nu(\zeta,z,\lambda)(\zeta_\nu-z_\nu)=0.$$

It follows that the row vectors

$$((\bar\partial_\zeta+\bar\partial_z+d_\lambda)\eta_\nu,\ldots,(\bar\partial_\zeta+\bar\partial_z+d_\lambda)\eta_\nu)$$

are lineary dependent over $\mathbb{C}$ and therefore the determinant vanishes. ∎

As a corollary of the lemma we get for $(0,q)$-forms f the following formula

$$\begin{aligned}(d_\zeta+d_\lambda)(f\wedge\tilde K) &= \bar\partial_\zeta f\wedge\tilde K+(-1)^q f\wedge(\bar\partial_\zeta+d_\lambda)\tilde K\\ &= \bar\partial_\zeta f\wedge\tilde K+(-1)^q(f\wedge K'-f\wedge\bar\partial_z\tilde K)\\ &= (-1)^q f\wedge K'+\bar\partial_\zeta f\wedge\tilde K-\bar\partial_z(f\wedge\tilde K).\end{aligned}\tag{6}$$

Before we come to the construction of our own operator we state one more result from Range and Siu [RS].

Theorem 3.3 (Range and Siu) *Let f be a (0,q)-form such that f, $\bar\partial f\in C^0_*(\bar D)$ and let $c_n=1/(2\pi i)^n$. Then we have*

$$
\begin{aligned}
(-1)^q c_n f(z) &= \bar\partial_z\Big(\int\limits_{\sum_I'(-1)^{|I|}S_I\times\Delta_{0I}} f\wedge K_{q-1} + \int\limits_{D\times\Delta_0} f\wedge K_{q-1}\Big) \\
&\quad -\Big(\int\limits_{\sum_I'(-1)^{|I|}S_I\times\Delta_{0I}} \bar\partial f\wedge K_q + \int\limits_{D\times\Delta_0} \bar\partial f\wedge K_q\Big) \\
&\quad + \int\limits_{\sum_I' S_I\times\Delta_I} f\wedge K_q. \qquad (7)
\end{aligned}
$$

Moreover the last integral vanishes except for $q = 0$.

For simplicity we introduce the notations $R := \sum_I'(-1)^{|I|}R_I\times\Delta_{0I}$, $S := \sum_I'(-1)^{|I|}S_I\times\Delta_{0I}$ and $S^\epsilon := \sum_I'(-1)^{|I|}S_I^\epsilon\times\Delta_{0I}$. Now we want to apply Stokes theorem. Keep in mind (2), (6) and the fact that $\int_R \bar\partial_z(f\wedge\tilde K) = -\bar\partial_z\int_R f\wedge\tilde K$ because the dimension of R is odd. We get

$$
\begin{aligned}
&\int\limits_R (-1)^q f\wedge K' + \int\limits_R \bar\partial f\wedge\tilde K + \bar\partial_z\int\limits_R f\wedge\tilde K \\
&= -\int\limits_{\sum_I' R_I\times\Delta_I} f\wedge\tilde K_{q-1} + \int\limits_{(D\setminus D^\epsilon)\times\Delta_0} f\wedge\tilde K_{q-1} + \int\limits_S f\wedge\tilde K_{q-1} - \int\limits_{S^\epsilon} f\wedge\tilde K_{q-1}. \qquad (8)
\end{aligned}
$$

Let us consider the integrals on the right hand side. Since $\tilde\eta$ is holomorphic in z for $\lambda\in\Delta_I$ it follows from Lemma 3.2 (v) that $\tilde K_q = 0$ on $R_I\times\Delta_I$ for $q\neq 0$. So we get

$$
\begin{aligned}
\int\limits_{\sum_I' R_I\times\Delta_I} f\wedge\tilde K_{q-1} &= 0 \quad\text{for}\quad q\neq 1, \\
\bar\partial_z \int\limits_{\sum_I' R_I\times\Delta_I} f\wedge\tilde K_{q-1} &= 0 \quad\text{for}\quad q = 1. \qquad (9)
\end{aligned}
$$

From Lemma 3.2 (i) we obtain

$$
\int\limits_{(D\setminus D^\epsilon)\times\Delta_0} f\wedge\tilde K_{q-1} = \int\limits_{(D\setminus D^\epsilon)\times\Delta_0} f\wedge K_{q-1} \qquad (10)
$$

and

$$
\int\limits_S f\wedge\tilde K_{q-1} = \int\limits_S f\wedge K_{q-1}. \qquad (11)
$$

It remains to investigate the last integral at the right hand side of (8). Since $\chi(\zeta)\equiv 0$ in $D^{\epsilon/2}$ we have $\tilde\eta = \lambda_0\eta^0$ for $\zeta\in S^\epsilon$, so the first column in the determinant of $\tilde K$ is $\lambda_0\eta^0$. Further $\dim\Delta_{0I} = |I|\geq 1$ and consequently at least one $d\lambda_j$ is needed for the integration. After expanding the determinant into a sum of determinants let us assume that each term has a $d\lambda_j$ in the second column. This $d\lambda_j$ must be either $\eta^0 d\lambda_0$ or $0\cdot d\lambda_j$ for some $j>0$. In both cases the determinant vanishes. The result is

$$
\int\limits_{S^\epsilon} f\wedge\tilde K_{q-1} = 0. \qquad (12)
$$

Keeping in mind (10), (11) and (12) we can combine (8) with (7).

$$\begin{aligned}(-1)^q c_n f(z) \;=\; & \bar\partial_z\Big(\int_{\sum_I{}' R_I\times\Delta_I} f\wedge \tilde K_{q-1} - \int_{(D\setminus D^\epsilon)\times\Delta_0} f\wedge K_{q-1} + \int_{D\times\Delta_0} f\wedge K_{q-1} \\ & - \int_R (-1)^q f\wedge K' - \int_R \bar\partial f\wedge \tilde K - \bar\partial_z \int_R f\wedge \tilde K\Big) \\ & -\Big(\int_{\sum_I{}' R_I\times\Delta_I} \bar\partial f\wedge \tilde K_q - \int_{(D\setminus D^\epsilon)\times\Delta_0} \bar\partial f\wedge K_q + \int_{D\times\Delta_0} \bar\partial f\wedge K_q \\ & - \int_R (-1)^{q+1}\bar\partial f\wedge K' - \int_R \bar\partial\bar\partial f\wedge \tilde K - \bar\partial_z \int_R \bar\partial f\wedge \tilde K\Big) \\ & + \int_{\sum_I{}' S_I\times\Delta_I} f\wedge K_q\end{aligned}$$

Obviously the 6th and the 11th integral on the right hand side vanish. And the sum of the 5th and the 12th integral vanish too. From (9) and the remark in Theorem 3.3 it follows that the first integral vanishes and that the 7th and the last integral are nonzero only for $q = 0$. Let us define the operators

$$T_0 f := \int_{\sum_I{}' S_I\times\Delta_I} f\wedge K_0 - \int_{\sum_I{}' R_I\times\Delta_I} \bar\partial f\wedge \tilde K_0$$

and

$$T_q f := -\int_R f\wedge K' - (-1)^{q-1} \int_{D^\epsilon\times\Delta_0} f\wedge K_{q-1} \quad \text{for} \quad 1\le q\le n.$$

Consider the integral $\int_R f\wedge K' = \int_{\sum_I{}'(-1)^{|I|}R_I\times\Delta_{0I}} f\wedge K'$. Observe that if f is a $(0,q)$-form each summand of $f\wedge K'$ is a form of degree at least $n+q$ in ζ. Thus

$$\int_{R_I\times\Delta_{0I}} f\wedge K' = 0 \quad \text{if} \quad \dim R_I < n+q.$$

It follows that we only need to sum over the I with $|I|\le n-q+1$.

Theorem 3.4 *Let f be a (0,q)-form on D such that $f\in C^0_{(0,q)}(D)\cap\bigcap_{|I|\le n-q+1} L^1_{(0,q)}(R_I)$ and $\bar\partial f\in C^0_{(0,q+1)}(D)\cap\bigcap_{|I|\le n-q} L^1_{(0,q+1)}(R_I)$ and let $c_n = 1/(2\pi i)^n$. Then we have*

$$c_n f = T_0 f + T_1\bar\partial f \quad \textit{for} \quad q=0$$

and

$$c_n f = \bar\partial T_q f + T_{q+1}\bar\partial f \quad \textit{for} \quad 1\le q\le n.$$

Proof. It is shown above that the theorem is true for $f, \bar\partial f \in C^0_*(\bar D)$. A short investigation of the kernels shows that the operators can be used also if f and $\bar\partial f$ are only continuous in the interior of the domain and integrable on all of the R_I to be considered. According to the above remark it is sufficient to have $f\in C^0_{(0,q)}(D)\cap\bigcap_{|I|\le n-q+1} L^1_{(0,q)}(R_I)$ and $\bar\partial f\in C^0_{(0,q+1)}(D)\cap\bigcap_{|I|\le n-q} L^1_{(0,q+1)}(R_I)$. The fact that the equation still holds, follows by some simple approximation arguments. ∎

Remark. In the smooth case when we have only ϱ_1 we get $R_1 = D$ and $D^\epsilon = \emptyset$. For $q \geq 1$ we have $T_q f = -\int_{-D\times[0,1]} f \wedge K'$. This is exactly the operator which is given for instance in Section 3 in [HL]. For some weighted estimates of this operator see [F].

At the end of this section we state the main result of this paper. The proof of this theorem will be the subject of the next section.

Theorem 3.5 *Let $0 \leq \beta < 1 + d$, $1 \leq q \leq n$. And assume that the submanifold N satisfies the generic condition (G). That is for every I with $|I| \leq n - q + 1$ we have either $\operatorname{dist}(N, S_I) \geq C > 0$ or the condition (P_d). Further assume the condition ($\mathbb{C}$) in a neighbourhood of the submanifold N. Then there is a positiv constant C such that for each $f \in B^{\varphi(\beta,N)}_{(0,q)}(D)$ we have*

(i) $||T_q f||_{C^{1/2-\beta'}} \leq C||f||_{\varphi(\beta,N)}$ *for* $0 \leq \beta < \beta' < 1/2$,

(ii) $||T_q f||_{\varphi(\beta'-1/2,bD)} \leq C||f||_{\varphi(\beta,N)}$ *for* $1/2 \leq \beta < \beta' < 1 + d$.

4 Estimates for the operator T_q

This section contains the proof of the main theorem of this paper. To do this proof we first give some lemmas.

Lemma 4.1 *Let $I = \{i_1, ..., i_l\}$ be a fixed ordered subset of $\{1, ..., k\}$ with strictly increasing components. Then the integral $\int_{R_I\times\Delta_{0I}} f \wedge K'$ can be written as a linear combination of integrals of the type $\int_{R_I} f \wedge A$ where A is one of the kernels*

$$\det{}_{1,1,...,1,s-1,1,...,1}(P^0, P^{i_1}, ..., P^{i_l}, d\bar\zeta - d\bar z, \bar\partial_\zeta P^{j_1}, ..., \bar\partial_\zeta P^{j_{n-l-s}}) \wedge \\ \wedge \frac{\chi^{n-s}\omega(\zeta) \wedge (\bar\partial_\zeta + \bar\partial_z)|\zeta - z|}{\tilde\Phi_{i_1} \cdot ... \cdot \tilde\Phi_{i_l}|\zeta - z|^{2(s)+1}\tilde\Phi_{j_1} \cdot ... \cdot \tilde\Phi_{j_{2n-l-s}}}$$

for $1 \leq s \leq n - l$ and $j_1, ..., j_{n-l-s} \in I$,

$$\det{}_{1,1,...,1,s,1,...,1}(P^0, P^{i_1}, ..., P^{i_l}, d\bar\zeta - d\bar z, \bar\partial_\zeta P^{j_1}, ..., \bar\partial_\zeta P^{j_{n-l-s-1}}) \wedge \\ \wedge \frac{\chi^{n-s}\omega(\zeta) \wedge (\bar\partial_\zeta\chi - \chi\frac{\bar\partial_\zeta\tilde\Phi_h}{\tilde\Phi_h})}{\tilde\Phi_{i_1} \cdot ... \cdot \tilde\Phi_{i_l}|\zeta - z|^{2(s+1)}\tilde\Phi_{j_1} \cdot ... \cdot \tilde\Phi_{j_{2n-l-s-1}}}$$

for $0 \leq s \leq n - l - 1$ and $j_1, ..., j_{n-l-s-1}, h \in I$,

$$\det{}_{1,...,1,s,1,...,1}(P^{i_1}, ..., P^{i_l}, d\bar\zeta - d\bar z, \bar\partial_\zeta P^{j_1}, ..., \bar\partial_\zeta P^{j_{n-l-s}}) \wedge \\ \wedge \frac{\chi^{n-s}\omega(\zeta)}{\tilde\Phi_{i_1} \cdot ... \cdot \tilde\Phi_{i_l}|\zeta - z|^{2(s)}\tilde\Phi_{j_1} \cdot ... \cdot \tilde\Phi_{j_{2n-l-s}}}$$

for $0 \leq s \leq n - l$ and $j_1, ..., j_{n-l-s} \in I$,

$$\det{}_{1,1,...,1,s,1,...,1}(P^0, P^{i_1}, ..\hat{h}.., P^{i_l}, d\bar\zeta - d\bar z, \bar\partial_\zeta P^{j_1}, ..., \bar\partial_\zeta P^{j_{n-l-s}}) \wedge \\ \wedge \frac{\chi^{n-s-1}\omega(\zeta)}{\tilde\Phi_{i_1} \cdot ..\hat{h}.. \cdot \tilde\Phi_{i_l}|\zeta - z|^{2(s+1)}\tilde\Phi_{j_1} \cdot ... \cdot \tilde\Phi_{j_{2n-l-s}}}$$

for $0 \leq s \leq n - l$ and $j_1, ..., j_{n-l-s}, h \in I$ and $\tilde\Phi_{i_1} \cdot ..\hat{h}.. \cdot \tilde\Phi_{i_l}$ means that $\tilde\Phi_h$ has to be omitted.

Remark. Note that all the kernels contain at least one χ except for $l = 1$ and $s = n-1$ where the last kernel becomes the Martinelli-Bochner kernel. But this kernel is well known. So in the sequel we only have to consider the other kernels and A shall denote any of the kernels of the lemma but not the Martinelli-Bochner kernel.

Remark. Since $f = \sum'_K f_K d\bar{\zeta}^K$ we have $\int_{R_I} f \wedge A = \sum'_K \int_{R_I} f_K (d\bar{\zeta}^K \wedge A)$. By A' we denote any of the kernels $(d\bar{\zeta}^K \wedge A)$. Instead of the integrals $\int_{R_I} f \wedge A$ we can now investigate integrals of the form $\int_{R_I}(f)A'$ where (f) denotes any of the coefficients of f. The advantage of this representation is that f gives only something of function type to the integral and A' has the right degree to be integrated over R_I.

Proof. We have to investigate the kernel K'. Because of the integration over Δ_{0I} we only have to consider monomials of degree l in $d\lambda_j$. That means instead of K' we have to compute

$$\det{}_{l,n-l}(d_\lambda\tilde{\eta},\ (\bar{\partial}_\zeta + \bar{\partial}_z)\tilde{\eta}) \wedge \omega(\zeta). \tag{13}$$

Observe that on Δ_{0I} we have $\tilde{\eta} = \lambda_0\eta^0 + \chi\sum_{j\in I}\lambda_j\tilde{\eta}^j$ and $\lambda_0 = 1 - \sum_{j\in I}\lambda_j$. From Proposition 3.1 (ii) and (x) we have that all $\tilde{\eta}^j$ are holomorphic in z. Thus (13) is equal to

$$\det{}_{l,n-l}\Big(\sum_{j\in I}(\chi\tilde{\eta}^j - \eta^0)d\lambda_j,\ \lambda_0(\bar{\partial}_\zeta + \bar{\partial}_z)\eta^0 + \sum_{j\in I}\lambda_j\bar{\partial}_\zeta(\chi\tilde{\eta}^j)\Big) \wedge \omega(\zeta)$$

which is a linear combination of

$$\det{}_{l,s,n-l-s}\Big(\sum_{j\in I}(\chi\tilde{\eta}^j - \eta^0)d\lambda_j,\ \lambda_0(\bar{\partial}_\zeta + \bar{\partial}_z)\eta^0,\ \sum_{j\in I}\lambda_j\bar{\partial}_\zeta(\chi\tilde{\eta}^j)\Big) \wedge \omega(\zeta) \quad \text{for} \quad 0 \le s \le n-l.$$

In the s columns in the middle we have $\lambda_0(\bar{\partial}_\zeta + \bar{\partial}_z)\eta^0 = \lambda_0(\frac{d\bar{\zeta}-d\bar{z}}{|\zeta-z|^2} - \eta^0\frac{2(\bar{\partial}_\zeta+\bar{\partial}_z)|\zeta-z|}{|\zeta-z|})$ and the last $n-l-s$ columns are $\sum_{j\in I}\lambda_j\bar{\partial}_\zeta(\chi\tilde{\eta}^j) = \sum_{j\in I}\chi\lambda_j\frac{\bar{\partial}_\zeta P^j}{\tilde{\Phi}_j} + \sum_{j\in I}\lambda_j(\bar{\partial}_\zeta\chi - \chi\frac{\bar{\partial}_\zeta\tilde{\Phi}_j}{\tilde{\Phi}_j})\tilde{\eta}^j$. Now we expand the sums in all columns. Keeping in mind that the determinant vanishes if there are forms b_i, b_j and functions c_k such that for two different columns $a_{ki} = c_k b_i$ and $a_{kj} = c_k b_j$, we compute that (13) is a linear combination of some terms

$$\det{}_{1,1,\ldots,1,s-1,n-l-s}\Big(\eta^0, \tilde{\eta}^{i_1}, \ldots, \tilde{\eta}^{i_l}, \frac{d\bar{\zeta} - d\bar{z}}{|\zeta - z|^2}, \sum_{j\in I}\chi\lambda_j\frac{\bar{\partial}_\zeta P^j}{\tilde{\Phi}_j}\Big) \wedge$$

$$\wedge\omega(\zeta) \wedge d\Lambda_I \wedge \frac{(\bar{\partial}_\zeta + \bar{\partial}_z)|\zeta - z|}{|\zeta - z|}\chi^l\lambda_0^s \quad \text{for} \quad 1 \le s \le n-l,$$

$$\det{}_{1,1,\ldots,1,s,n-l-s-1}\Big(\eta^0, \tilde{\eta}^{i_1}, \ldots, \tilde{\eta}^{i_l}, \frac{d\bar{\zeta} - d\bar{z}}{|\zeta - z|^2}, \sum_{j\in I}\chi\lambda_j\frac{\bar{\partial}_\zeta P^j}{\tilde{\Phi}_j}\Big) \wedge$$

$$\wedge\omega(\zeta) \wedge d\Lambda_I \wedge \Big(\bar{\partial}_\zeta\chi - \chi\frac{\bar{\partial}_\zeta\tilde{\Phi}_h}{\tilde{\Phi}_h}\Big)\chi^{l-1}\lambda_0^s\lambda_h \quad \text{for} \quad 0 \le s \le n-l-1,\ h \in I,$$

$$\det{}_{1,\ldots,1,s,n-l-s}\Big(\tilde{\eta}^{i_1}, \ldots, \tilde{\eta}^{i_l}, \frac{d\bar{\zeta} - d\bar{z}}{|\zeta - z|^2}, \sum_{j\in I}\chi\lambda_j\frac{\bar{\partial}_\zeta P^j}{\tilde{\Phi}_j}\Big) \wedge$$

$$\wedge\omega(\zeta)\wedge d\Lambda_I\,\chi^l\lambda_0^s \quad \text{for} \quad 0\le s\le n-l,$$

$$\det_{1,1,\dots,1,s,n-l-s}(\eta^0,\tilde\eta^{i_1},..\hat{h}..,\tilde\eta^{i_l},\frac{d\bar\zeta-d\bar z}{|\zeta-z|^2},\sum_{j\in I}\chi\lambda_j\frac{\bar\partial_\zeta P^j}{\tilde\Phi_j})\wedge$$

$$\wedge\omega(\zeta)\wedge d\Lambda_I\,\chi^{l-1}\lambda_0^s \quad \text{for} \quad 0\le s\le n-l,\ h\in I,$$

where $\tilde\eta^{i_1},..\hat{h}..,\tilde\eta^{i_l}$ means that the $\tilde\eta^h$ must be omitted and $d\Lambda_I$ denotes $d\lambda_{i_1}\wedge\ldots\wedge d\lambda_{i_l}$. Now we still have to expand the sum in the last columns and to collect all λ_j, $\tilde\Phi_j$ and $|\zeta-z|^2$ outside the determinant. We get

$$\det_{1,1,\dots,1,s-1,1,\dots,1}(P^0,P^{i_1},\dots,P^{i_l},d\bar\zeta-d\bar z,\bar\partial_\zeta P^{j_1},\dots,\bar\partial_\zeta P^{j_{n-l-s}})\wedge$$
$$\wedge\frac{\chi^{n-s}\mathrm{Poly}(\lambda)\omega(\zeta)\wedge d\Lambda_I\wedge(\bar\partial_\zeta+\bar\partial_z)|\zeta-z|}{\tilde\Phi_{i_1}\cdot\ldots\cdot\tilde\Phi_{i_l}|\zeta-z|^{2(s)+1}\tilde\Phi_{j_1}\cdot\ldots\cdot\tilde\Phi_{j_{2n-l-s}}}$$

for $1\le s\le n-l$ and $j_1,\dots,j_{n-l-s}\in I$,

$$\det_{1,1,\dots,1,s,1,\dots,1}(P^0,P^{i_1},\dots,P^{i_l},d\bar\zeta-d\bar z,\bar\partial_\zeta P^{j_1},\dots,\bar\partial_\zeta P^{j_{n-l-s-1}})\wedge$$
$$\wedge\frac{\chi^{n-s}\mathrm{Poly}(\lambda)\omega(\zeta)\wedge d\Lambda_I\wedge(\bar\partial_\zeta\chi-\chi\frac{\bar\partial_\zeta\tilde\Phi_h}{\tilde\Phi_h})}{\tilde\Phi_{i_1}\cdot\ldots\cdot\tilde\Phi_{i_l}|\zeta-z|^{2(s+1)}\tilde\Phi_{j_1}\cdot\ldots\cdot\tilde\Phi_{j_{2n-l-s-1}}}$$

for $0\le s\le n-l-1$ and $j_1,\dots,j_{n-l-s-1},h\in I$,

$$\det_{1,\dots,1,s,1,\dots,1}(P^{i_1},\dots,P^{i_l},d\bar\zeta-d\bar z,\bar\partial_\zeta P^{j_1},\dots,\bar\partial_\zeta P^{j_{n-l-s}})\wedge$$
$$\wedge\frac{\chi^{n-s}\mathrm{Poly}(\lambda)\omega(\zeta)\wedge d\Lambda_I}{\tilde\Phi_{i_1}\cdot\ldots\cdot\tilde\Phi_{i_l}|\zeta-z|^{2(s)}\tilde\Phi_{j_1}\cdot\ldots\cdot\tilde\Phi_{j_{2n-l-s}}}$$

for $0\le s\le n-l$ and $j_1,\dots,j_{n-l-s}\in I$,

$$\det_{1,1,\dots,1,s,1,\dots,1}(P^0,P^{i_1},..\hat{i}..,P^{i_l},d\bar\zeta-d\bar z,\bar\partial_\zeta P^{j_1},\dots,\bar\partial_\zeta P^{j_{n-l-s}})\wedge$$
$$\wedge\frac{\chi^{n-s-1}\mathrm{Poly}(\lambda)\omega(\zeta)\wedge d\Lambda_I}{\tilde\Phi_{i_1}\cdot..\hat{h}..\cdot\tilde\Phi_{i_l}|\zeta-z|^{2(s+1)}\tilde\Phi_{j_1}\cdot\ldots\cdot\tilde\Phi_{j_{2n-l-s}}}$$

for $0\le s\le n-l$ and $j_1,\dots,j_{n-l-s},h\in I$ and $\mathrm{Poly}(\lambda)$ means a polynomial in the λ_j for $j\in I$. Now we can integrate with respect to $\lambda\in\Delta_{0I}$ and the proposition follows. ■

Let $I=\{i_1,\dots,i_l\}$ be a fixed ordered subset of $\{1,\dots,k\}$, let ζ_0 be a fixed point on S_I and let $U(\zeta_0)$ be a sufficiently small open neighbourhood of ζ_0 in $\mathbb{C}^n$. In the sequel it might be necessary to shrink $U(\zeta_0)$ even if we do not explicitly mention it. We also assume $z\in U(\zeta_0)$ and $U(\zeta_0)$ so small that for all $z,\zeta\in U(\zeta_0)$ we have $|\zeta-z|<c_1$ where c_1 is the constant from Proposition 3.1. For a fixed z we choose real coordinates on $U(\zeta_0)$.

$$x=(x_1,\dots,x_{2n})=(x_1,\dots,x_{2n-l},\varrho_1(\zeta)-\varrho_1(z),\varrho_2(\zeta)-\varrho_1(\zeta),\dots,\varrho_l(\zeta)-\varrho_1(\zeta))$$

We use the notations $x'=(x_1,\dots,x_{2n-l})$, $t=\varrho_1(\zeta)$ and $x''=(x_{2n-l+2},\dots,x_{2n})$. Then $R_I\cap U(\zeta_0)$ lies in the set defined by $x''=0$ and $t\le 0$. Moreover we may assume $x'(z)=$

$x_{2n-l+1}(z) = 0$. Set $u_j(\zeta) = u_j(\zeta, z) = \operatorname{Im} F_j(\zeta, z)$ and take the Taylor expansion of $u_j(\zeta, z)$ as a function in ζ at the fixed point z.

$$u_j(\zeta) = \sum_{\nu=1}^{2n} c_{j\nu}(z)(x_\nu - x_\nu(z)) + \sum_{\nu,\mu=1}^{2n} c_{j\nu\mu}(z)(x_\nu - x_\nu(z))(x_\mu - x_\mu(z)) + O(|\zeta - z|^3)$$

We define

$$p_j(\zeta, z) := \sum_{\substack{\nu=1 \\ \nu\neq 2n-l+1}}^{2n} c_{j\nu}(z)(x_\nu - x_\nu(z)) + \sum_{\substack{\nu,\mu=1 \\ \nu,\mu\neq 2n-l+1}}^{2n} c_{j\nu\mu}(z)(x_\nu - x_\nu(z))(x_\mu - x_\mu(z))$$

and

$$q_j(\zeta, z) := p_j(\zeta, z) + c_{j,2n-l+1}(z)(t - \varrho_1(z)).$$

Notice that

$$(|\operatorname{Im} \tilde{\Phi}_j(\zeta, z)| + |\zeta - z|^2) \geq C(|q_j(\zeta, z)| + |\zeta - z|^2). \qquad (14)$$

In the following we want to use the notation $O(|\zeta - z|)$ also for forms. A form $f(\zeta, z)$ is said to be $O(|\zeta - z|)$ if the function $|f(\zeta, z)|$ is $O(|\zeta - z|)$. Here $|\cdot|$ is any of the norms of a form at a point. On R_I we have $dt = d\varrho_{i_1} = \ldots = d\varrho_{i_l}$. Using this fact we can derive from Proposition 3.1 (i)

$$\begin{aligned} d_\zeta F_j(\zeta, z) &= 2\partial\varrho_j(\zeta) + O(|\zeta - z|), \\ d_\zeta \overline{F_j(\zeta, z)} &= 2\bar{\partial}\varrho_j(\zeta) + O(|\zeta - z|) = 2dt - 2\partial\varrho_j(\zeta) + O(|\zeta - z|) \end{aligned}$$

and consequently

$$d_\zeta u_j(\zeta, z) = -i(2\partial\varrho_j(\zeta) - dt) + O(|\zeta - z|)$$

or

$$\begin{aligned} \partial\varrho_j(\zeta) &= i/2 d_\zeta u_j(\zeta) + 1/2 dt + O(|\zeta - z|) \\ &= i/2 d_\zeta p_j(\zeta, z) + (1/2 + c_{j,2n-l+1}(z))dt + O(|\zeta - z|). \end{aligned} \qquad (15)$$

Using again that $dt = d\varrho_h$ on R_I we derive

$$\bar{\partial}_\zeta(F_h - 2\varrho_h) = O(|\zeta - z|) - 2\bar{\partial}\varrho_h = 2\partial\varrho_h - 2dt + O(|\zeta - z|). \qquad (16)$$

Now we can state the following lemma.

Lemma 4.2 *Let I be a fixed ordered subset of $\{1, \ldots, k\}$ and let ζ_0 be a fixed point on S_I, let $U(\zeta_0)$ be a sufficiently small neighbourhood of ζ_0 and let $z \in U(\zeta_0)$ be fixed. Then the kernels A' given in Lemma 4.1 can be estimated on $R_I \cap U(\zeta_0)$ by kernels of the following type:*

$$\frac{dq_{j_1} \wedge \ldots \wedge dq_{j_s} \wedge dt \wedge \omega_{2n-l-s}}{|\tilde{\Phi}_{j_1}| \cdot \ldots \cdot |\tilde{\Phi}_{j_s}| \cdot |\tilde{\Phi}_h| \cdot |\zeta - z|^{2n-l-s-1}} \quad \textit{for} \quad 0 \leq s \leq l, \quad h, j_1, \ldots, j_s \in I.$$

The ω_{2n-l-s} denote some uniformly bounded forms of degree $2n - l - s$ in ζ.

Remark. Note that the q_i are quadratic polynomials in ζ. So they define a finitely sheeted branched covering and we may use them as independent coordinates.

Proof. First we apply the inequality $|\tilde{\Phi}_j(\zeta,z)| \geq C|\zeta - z|^2$ to the products $\tilde{\Phi}_{j_1} \cdot \ldots \cdot \tilde{\Phi}_{j_{2n-l-s}}$. Then we can assume $\chi \equiv 1$, $\bar{\partial}_\zeta \chi \equiv 0$ since ζ is in a small neighbourhood of bD. Now we consider the columns of the determinants which contain the P^j, $j \in I$. From Proposition 3.1 (xii) we get $P^j(\zeta,z) = H_j(\zeta,z)\nabla\varrho_j(\zeta) + \gamma_j(\zeta,z)$ with $c \leq |H_j(\zeta,z)| \leq C$, $\gamma_j(\zeta,z) = O(|\zeta - z|)$ and $\nabla\varrho_j(\zeta)$ denotes the column $(\partial\varrho_j/\partial\zeta_1, \ldots, \partial\varrho_j/\partial\zeta_n)^t$. Further let $(j_1,\ldots,j_L)$ be any permutation of I respectivly $I\backslash\{t\}$ if we consider the fourth of the kernels. Now the determinants can be estimated by a sum of determinants of the following type:

$$\det(\nabla\varrho_{j_1},\ldots,\nabla\varrho_{j_r},\gamma_{j_{r+1}},\ldots,\gamma_{j_L},\ldots)$$

with $0 \leq r \leq l$ resp. $0 \leq r \leq l-1$ in the fourth case and the last columns contain some P^0, $d\bar{\zeta} - d\bar{z}$ and some $\bar{\partial}_\zeta P^j$, $j \in I$. Since the forms $d\bar{\zeta} - d\bar{z}$ and $\bar{\partial}_\zeta P^j$, $j \in I$ are uniformly bounded and since P^0 and γ_j can be estimated by $C|\zeta - z|$ it remains to consider the first r columns of the determinants. Using the equation

$$\det\left(\frac{\partial\varrho_{j_i}}{\partial\zeta_{k_l}}\right)^r_{i,l=1} \cdot \omega(\zeta) = \pm \bigwedge_{i=1}^{r} \partial\varrho_{j_i} \wedge \bigwedge_{i\neq k_1,\ldots,k_r} d\zeta_i$$

we conclude, that the kernels A' can be estimated by a sum of terms like

$$\frac{\bar{\partial}_\zeta\tilde{\Phi}_h \wedge \partial\varrho_{j_1} \wedge \ldots \wedge \partial\varrho_{j_r} \wedge \omega_{2n-l-r}|\zeta - z|^{l-r+1}}{|\tilde{\Phi}_{i_1}| \cdot \ldots \cdot |\tilde{\Phi}_{i_l}| \cdot |\zeta - z|^{2(n-l)}} \quad \text{for} \quad 0 \leq r \leq l, \quad h, j_1, \ldots, j_r \in I, \tag{17}$$

$$\frac{\partial\varrho_{j_1} \wedge \ldots \wedge \partial\varrho_{j_r} \wedge \omega_{2n-l-r+1}|\zeta - z|^{l-r}}{|\tilde{\Phi}_{i_1}| \cdot \ldots \cdot |\tilde{\Phi}_{i_l}| \cdot |\zeta - z|^{2(n-l)}} \quad \text{for} \quad 0 \leq r \leq l, \quad j_1, \ldots, j_r \in I, \tag{18}$$

$$\frac{\partial\varrho_{j_1} \wedge \ldots \wedge \partial\varrho_{j_r} \wedge \omega_{2n-l-r+1}|\zeta - z|^{l-r}}{|\tilde{\Phi}_{i_1}| \cdot \cdot \hat{h} \cdot \cdot |\tilde{\Phi}_{i_l}| \cdot |\zeta - z|^{2(n-l+1)}} \quad \text{for} \quad 0 \leq r \leq l-1, \quad h \in I, j_1, \ldots, j_r \in I\backslash\{h\} \tag{19}$$

where $\omega_{2n-l-r+1}$ denotes some uniformly bounded form of degree $2n - l - r + 1$ in ζ.

Because of Proposition 3.1 (v) and the uniform estimates for H_h we can replace $\bar{\partial}_\zeta\tilde{\Phi}_h$ by $\bar{\partial}_\zeta(F_h - 2\varrho_h)$. Now we consider the forms $\partial\varrho_{j_1} \wedge \ldots \wedge \partial\varrho_{j_r} \wedge \omega_{2n-l-r+1}$ and $\bar{\partial}_\zeta(F_h - 2\varrho_h) \wedge \partial\varrho_{j_1} \wedge \ldots \wedge \partial\varrho_{j_r} \wedge \omega_{2n-l-r}$. Using (15) and (16) we get

$$\begin{aligned}
&\partial\varrho_{j_1} \wedge \ldots \wedge \partial\varrho_{j_r} \wedge \omega_{2n-l-r+1} \\
&= \sum_{s=0}^{r}\sum_{\tau} dp_{\tau(1)} \wedge \ldots \wedge dp_{\tau(s)} \wedge \omega_{2n-l-s+1}|\zeta - z|^{r-s} \\
&\quad + \sum_{s=0}^{r-1}\sum_{\tau} dp_{\tau(1)} \wedge \ldots \wedge dp_{\tau(s)} \wedge dt \wedge \omega_{2n-l-s}|\zeta - z|^{r-s-1}
\end{aligned}$$

and

$$
\begin{aligned}
&\bar\partial_\zeta(F_h - 2\varrho_h) \wedge \partial\varrho_{j_1} \wedge \dots \wedge \partial\varrho_{j_r} \wedge \omega_{2n-l-r} \\
= \;& dp_h \wedge \Big(\sum_{s=0}^{r}\sum_\tau dp_{\tau(1)} \wedge \dots \wedge dp_{\tau(s)} \wedge \omega_{2n-l-s}|\zeta - z|^{r-s} \\
&+\sum_{s=0}^{r-1}\sum_\tau dp_{\tau(1)} \wedge \dots \wedge dp_{\tau(s)} \wedge dt \wedge \omega_{2n-l-s-1}|\zeta - z|^{r-s-1}\Big) \\
&+\sum_{s=0}^{r}\sum_\tau dp_{\tau(1)} \wedge \dots \wedge dp_{\tau(s)} \wedge dt \wedge \omega_{2n-l-s}|\zeta - z|^{r-s} \\
&+\Big(\sum_{s=0}^{r}\sum_\tau dp_{\tau(1)} \wedge \dots \wedge dp_{\tau(s)} \wedge \omega_{2n-l-s+1}|\zeta - z|^{r-s+1} \\
&+\sum_{s=0}^{r-1}\sum_\tau dp_{\tau(1)} \wedge \dots \wedge dp_{\tau(s)} \wedge dt \wedge \omega_{2n-l-s}|\zeta - z|^{r-s}\Big)
\end{aligned}
$$

where τ is a permutation of $\{j_1, ..., j_r\}$ and $\sum_\tau$ means the summation over all such permutations and ω_m denotes any uniformly bounded form of degree m in ζ (ω_m may denote different forms only of the same degree even in the same sum). Now let us look at the terms which do not contain a dt. Since the dp_j have by definition no component in the direction dt there must be a dt contained in the corresponding ω_m. Because otherwise the whole form would vanish on R_I due to degree reasons considering the other coordinates. So we have a dt in all of the terms and we can replace all the dp_j by the corresponding dq_j. Some of the terms obviously can be estimated by other ones. We get two inequalities which we can combine with (17)-(19). After some simple changes of the indices we conclude that the kernels A' can be estimated by a sum of the following terms

$$\frac{dq_{j_1} \wedge \dots \wedge dq_{j_s} \wedge dt \wedge \omega_{2n-l-s}|\zeta - z|^{l-s+1}}{|\tilde\Phi_h| \cdot |\tilde\Phi_{i_1}| \cdot \dots \cdot |\tilde\Phi_{i_l}| \cdot |\zeta - z|^{2(n-l)}} \quad \text{for} \quad 0 \le s \le l, \quad h, j_1, .., j_s \in I,$$

$$\frac{dq_{j_1} \wedge \dots \wedge dq_{j_s} \wedge dt \wedge \omega_{2n-l-s}|\zeta - z|^{l-s-1}}{|\tilde\Phi_{i_1}| \cdot \dots \cdot |\tilde\Phi_{i_l}| \cdot |\zeta - z|^{2(n-l)}} \quad \text{for} \quad 0 \le s \le l-1, \quad j_1, .., j_s \in I,$$

$$\frac{dq_{j_1} \wedge \dots \wedge dq_{j_s} \wedge dt \wedge \omega_{2n-l-s}|\zeta - z|^{l-s-1}}{|\tilde\Phi_{i_1}| \cdot \cdot\cdot_{\hat h} \cdot\cdot \cdot |\tilde\Phi_{i_l}| \cdot |\zeta - z|^{2(n-l+1)}} \quad \text{for} \quad 0 \le s \le l-2, \; h \in I, \, j_1, .., j_s \in I\backslash\{h\}.$$

The three kernels still have a lot of $\tilde\Phi_j$ in the denominators. We want to keep there only the $\tilde\Phi_{j_1}, ..., \tilde\Phi_{j_s}$ and one more that will be denoted by $\tilde\Phi_h$. To the rest of the $\tilde\Phi_{j_i}$ we apply the inequality $|\tilde\Phi_j(\zeta, z)| \ge C|\zeta - z|^2$ and the proposition of the lemma follows. ■

In addition to the estimates given in Lemma 4.2 we also need some estimates for the derivatives of $T_q f$ with respect to z. we need these estimates because we will prove some Hölder estimates with the help of the Hardy-Littlewood lemma.[1] Let δ denote any of the $\partial/\partial z_j$ or $\partial/\partial \bar z_j$. We can compute $\delta T_q f$ by differentiating under the integral sign. And by Lemma 4.1 it remains to investigate the terms $\delta A'$. We get the following result.

[1]This lemma shows that a function is α-Hölder continuous if the gradient of the function can be estimated by $\mathrm{dist}(z, bD)^{\alpha-1}$.

Lemma 4.3 *Let I be a fixed ordered subset of $\{1,...,k\}$ and let ζ_0 be a fixed point on S_I, let $U(\zeta_0)$ be a sufficiently small neighbourhood of ζ_0 and let $z \in U(\zeta_0)$ be fixed. Then the terms $\delta A'$ can be estimated on $R_I \cap U(\zeta_0)$ by terms of the following types:*

$$\frac{dq_{j_1} \wedge ... \wedge dq_{j_s} \wedge dt \wedge \omega_{2n-l-s}}{|\tilde{\Phi}_{j_1}| \cdot ... \cdot |\tilde{\Phi}_{j_s}| \cdot |\tilde{\Phi}_h| \cdot |\zeta - z|^{2n-l-s}} \quad \textit{for} \quad 0 \leq s \leq l, \quad h, j_1, ..., j_s \in I,$$

$$\frac{dq_{j_1} \wedge ... \wedge dq_{j_s} \wedge dt \wedge \omega_{2n-l-s}}{|\tilde{\Phi}_{j_1}| \cdot ... \cdot |\tilde{\Phi}_{j_s}| \cdot |\tilde{\Phi}_h| \cdot |\tilde{\Phi}_{h'}| \cdot |\zeta - z|^{2n-l-s-1}} \quad \textit{for} \quad 0 \leq s \leq l, \quad h, h', j_1, ..., j_s \in I,$$

$$\frac{dq_{j_1} \wedge ... \wedge dq_{j_{s-1}} \wedge dt \wedge \omega_{2n-l-s+1}}{|\tilde{\Phi}_{j_1}| \cdot ... \cdot |\tilde{\Phi}_{j_{s-1}}| \cdot |\tilde{\Phi}_{j_s}| \cdot |\tilde{\Phi}_h| \cdot |\zeta - z|^{2n-l-s-1}} \quad \textit{for} \quad 0 \leq s \leq l, \quad h, j_1, ..., j_s \in I.$$

The ω_{2n-l-s} denote some uniformly bounded forms of degree $2n - l - s$ in ζ.

Proof. First we use the product rule to get sums of terms where the δ is applied to only one of the columns of the determinants ore one of the other factors of the kernels. Since the P^j and $\tilde{\Phi}_j$ are holomorphic in z the arising factors (δP^j and so on) are again uniformly bounded. Now we can trace all the estimates given in the proof of Lemma 4.2 to conclude the proposition of this lemma. ■

Besides the Martinelli-Bochner operator there are now four types of integrals we have to investigate. From Lemma 4.2 arises the integral

$$J_0 := \int\limits_{R_I \cap U(\zeta_0)} \frac{[\operatorname{dist}(\zeta, N)]^{-\beta} dq_{j_1} \wedge ... \wedge dq_{j_s} \wedge dt \wedge \omega_{2n-l-s}}{|\tilde{\Phi}_{j_1}| \cdot ... \cdot |\tilde{\Phi}_{j_s}| \cdot |\tilde{\Phi}_h| \cdot |\zeta - z|^{2n-l-s-1}}$$

for $0 \leq s \leq l$ and $h, j_1, ..., j_s \in I$. And the estimates of the derivatives of the kernel given in Lemma 4.3 lead to the integrals

$$J_1 := \int\limits_{R_I \cap U(\zeta_0)} \frac{[\operatorname{dist}(\zeta, N)]^{-\beta} dq_{j_1} \wedge ... \wedge dq_{j_s} \wedge dt \wedge \omega_{2n-l-s}}{|\tilde{\Phi}_{j_1}| \cdot ... \cdot |\tilde{\Phi}_{j_s}| \cdot |\tilde{\Phi}_h| \cdot |\zeta - z|^{2n-l-s}},$$

$$J_2 := \int\limits_{R_I \cap U(\zeta_0)} \frac{[\operatorname{dist}(\zeta, N)]^{-\beta} dq_{j_1} \wedge ... \wedge dq_{j_s} \wedge dt \wedge \omega_{2n-l-s}}{|\tilde{\Phi}_{j_1}| \cdot ... \cdot |\tilde{\Phi}_{j_s}| \cdot |\tilde{\Phi}_h| \cdot |\tilde{\Phi}_{h'}| \cdot |\zeta - z|^{2n-l-s-1}},$$

and

$$J_3 := \int\limits_{R_I \cap U(\zeta_0)} \frac{[\operatorname{dist}(\zeta, N)]^{-\beta} dq_{j_1} \wedge ... \wedge dq_{j_{s-1}} \wedge dt \wedge \omega_{2n-l-s+1}}{|\tilde{\Phi}_{j_1}| \cdot ... \cdot |\tilde{\Phi}_{j_{s-1}}| \cdot |\tilde{\Phi}_{j_s}| \cdot |\tilde{\Phi}_h| \cdot |\zeta - z|^{2n-l-s-1}}$$

for $0 \leq s \leq l$ and $h, h', j_1, ..., j_s \in I$.

Remember that we suppose that the submanifold N satisfies the condition (G). Indeed we need this only if β is greater or equal to 1. We prove the following lemma.

Lemma 4.4 *There are constants C depending only on β such that*

(i) $J_0 \leq C$ *for* $0 \leq \beta < 1/2$,

(ii) $J_1, J_2, J_3 \leq C[\operatorname{dist}(z, bD)]^{-1/2-\beta'}$ *for* $0 \leq \beta < \beta' < 1/2$,

(iii) $J_0 \leq C[\mathrm{dist}(z, bD)]^{1/2-\beta'}$ *for* $1/2 \leq \beta < \beta' < 1$.

Let the submanifold N satisfy the condition (P_d) and suppose that $(\mathbb{C})$ holds in a neighbourhood of $M_I = N \cap S_I$ then we have

(iv) $J_0 \leq C[\mathrm{dist}(z, bD)]^{1/2-\beta'}$ *for* $1 \leq \beta < \beta' < 1 + d$

where d was the codimension of N in bD and is the codimension of $M_I = N \cap S_I$ in S_I because of (P_d).

Proof. Remember that we have $|t| \leq C\mathrm{dist}(\zeta, bD) \leq C\mathrm{dist}(\zeta, N)$ and that $|\varrho_j(z)| \geq C\mathrm{dist}(z, bD)$ for all j. For simplicity let us assume that we have on R_I coordinates $x_1, ..., x_{2n-l+1}$ with $x'' = (x_{s+1}, ..., x_{2n-l+1})$ such that $\omega_{2n-l-s} = dx_{s+1} \wedge ... \wedge dx_{2n-l+1}$.

We consider the case $0 \leq \beta < \beta' < 1/2$. Using (5) and (14) we get

$$J_0 \leq$$

$$C \int\limits_{R_I \cap U} \frac{|t|^{-\beta} dq_{j_1} \wedge ... \wedge dq_{j_s} \wedge dt \wedge d\sigma_{2n-l-s}}{(|\varrho_{j_1}(\zeta)| + |q_{j_1}|) \cdot ... \cdot (|\varrho_{j_s}(\zeta)| + |q_{j_s}|)(|\varrho_h(z)| + |t| + |\zeta - z|^2)|\zeta - z|^{2n-l-s-1}}.$$

Integrating with respect to the q_{j_i} we obtain

$$J_0 \leq C \int\limits_{\substack{0<|t|<C \\ 0<|x''|<C}} \frac{|t|^{-\beta} \prod_{i=1}^{s} (1 + \ln |\varrho_{j_i}(\zeta)|) dt \wedge d\sigma_{2n-l-s}}{(|\varrho_h(z)| + |t| + |x''(\zeta - z)|^2)|x''(\zeta - z)|^{2n-l-s-1}}$$

and because $|t|^{-\beta} \prod_{j=1}^{s} (1 + \ln |t|) \leq C|t|^{-\beta'}$ this gives

$$\begin{aligned} J_0 &\leq C \int\limits_{\substack{0<|t|<C \\ 0<|x''|<C}} \frac{|t|^{-\beta'} dt \wedge d\sigma_{2n-l-s}}{(|\varrho_h(z)| + |t| + |x''(\zeta - z)|^2)|x''(\zeta - z)|^{2n-l-s-1}} \\ &\leq C \int\limits_{0<|x''|<C} \frac{d\sigma_{2n-l-s}}{(|\varrho_h(z)| + |x''(\zeta - z)|^2)^{\beta'} |x''(\zeta - z)|^{2n-l-s-1}}. \end{aligned}$$

Omitting $|\varrho_h(z)|$ we find further

$$\begin{aligned} J_0 &\leq C \int_0^C \frac{r^{2n-l-s-1} dr}{r^{2n-l-s-1+2\beta'}} \\ &\leq C. \end{aligned}$$

By the same arguments as in the case $0 \leq \beta < 1/2$ we obtain for $1/2 \leq \beta < \beta' < 1$

$$\begin{aligned} J_0 &\leq C \int\limits_{\substack{0<|t|<C \\ 0<|x''|<C}} \frac{|t|^{-\beta'} dt \wedge d\sigma_{2n-l-s}}{(|\varrho_h(z) + |t| + |x''(\zeta - z)|^2)|x''(\zeta - z)|^{2n-l-s-1}} \\ &\leq C \int\limits_{0<|x''|<C} \frac{d\sigma_{2n-l-s}}{(|\varrho_h(z)| + |x''(\zeta - z)|^2)^{\beta'} |x''(\zeta - z)|^{2n-l-s-1}}. \end{aligned}$$

And therefore we get

$$\begin{aligned} J_0 &\leq C\int_0^C \frac{r^{2n-l-s-1}dr}{(|\varrho_h(z)|+r^2)^{\beta'}r^{2n-l-s-1}} \\ &\leq C[\mathrm{dist}(z,bD)]^{1/2-\beta'}. \end{aligned}$$

Now (i) and (iii) are proved.

To prove (iv) we introduce some special coordinates related with N. We fix $\zeta_0 \in M_I = N \cap S_I$. First we consider the coordinates $q_{j_1},...,q_{j_s}, t = x_{s+1}, x_{s+2},...,x_{2n-l+1}$ from above. These are coordinates only if $dq_{j_1} \wedge ... \wedge dq_{j_s}$ does not vanish. But this holds since we assume (C) in a neighbourhood of the ζ_0. After a simple translation we may assume $q_{j_i}(\zeta_0) = x_{s+1}(\zeta_0) = x''(\zeta_0) = 0$. In $U(\zeta_0)$ there exists, maybe after shrinking U, also an other coordinate system $y_1,...,y_{[\beta]+1},...,y_{2n-l+1}$ such that with $y' = (y_1,...,y_{[\beta]+1})$

$$C\mathrm{dist}(\zeta, N) \geq \mathrm{dist}(\zeta, M_I) \geq |y'| \tag{20}$$

and

$$t \in \mathrm{span}\,\{y_1,...,y_{[\beta]+1}\}. \tag{21}$$

Now we can take the $y_1,...,y_{[\beta]+1}$ and choose some q_{j_i} and some x_i so that they all together form coordinates on R_I. Without loss of generality let us assume that the coordinates are $y_1,...,y_{[\beta]+1}, q_{j_{(\nu+1)}},...,q_{j_s}, x_{s+\mu+1},...,x_{2n-l+1}$ with $\nu+\mu = [\beta]+1$ and $\mu \geq 1$ because of (21). We introduce the abbreviations $q' = (q_{j_{(\nu+1)}},...,q_{j_s})$ and $x''' = (x_{s+\mu+1},...,x_{2n-l+1})$ and $\varepsilon = 1/2\,\mathrm{dist}(z,bD)$. Further we set

$$\begin{aligned} J_0' &= \int\limits_{\substack{\zeta\in R_I\cap U(\zeta_0)\\ \mathrm{dist}(\zeta,N)>\varepsilon}} \frac{[\mathrm{dist}(\zeta,N)]^{-\beta}dq_{j_1}\wedge...\wedge dq_{j_s}\wedge dt\wedge d\sigma_{2n-l-s}}{|\tilde\Phi_{j_1}|\cdot...\cdot|\tilde\Phi_{j_s}|\cdot|\tilde\Phi_h|\cdot|\zeta-z|^{2n-l-s-1}}, \\ J_0'' &= \int\limits_{\substack{\zeta\in R_I\cap U(\zeta_0)\\ \mathrm{dist}(\zeta,N)<\varepsilon}} \frac{[\mathrm{dist}(\zeta,N)]^{-\beta}dq_{j_1}\wedge...\wedge dq_{j_s}\wedge dt\wedge d\sigma_{2n-l-s}}{|\tilde\Phi_{j_1}|\cdot...\cdot|\tilde\Phi_{j_s}|\cdot|\tilde\Phi_h|\cdot|\zeta-z|^{2n-l-s-1}}. \end{aligned}$$

From (5) and (14) we get for some λ with $1/2 < \lambda < \lambda' < 1$

$$J_0' \leq C\varepsilon^{\lambda-\beta} \int\limits_{R_I\cap U} \frac{|t|^{-\lambda}dq_{j_1}\wedge...\wedge dq_{j_s}\wedge dt\wedge d\sigma_{2n-l-s}}{(|\varrho_{j_1}(\zeta)|+|q_{j_1}|)\cdot...\cdot(|\varrho_{j_s}(\zeta)|+|q_{j_s}|)(|\varrho_h(z)|+|t|+|\zeta-z|^2)|\zeta-z|^{2n-l-s-1}}.$$

Using the same arguments as in the case $1/2 \leq \beta < 1$ we obtain that

$$\begin{aligned} J_0' &\leq C\varepsilon^{\lambda-\beta}\varepsilon^{1/2-\lambda'} \leq C\varepsilon^{1/2-\beta'} \\ &\leq C[\mathrm{dist}(z,bD)]^{1/2-\beta'} \end{aligned} \tag{22}$$

with $1 \leq \beta < \beta' < 1+d$.

To estimate J_0'' we use the coordinates $y_1,...,y_{[\beta]+1}, q_{j_{(\nu+1)}},...,q_{j_s}, x_{s+\mu+1},...,x_{2n-l+1}$ introduced above. Note that $|\zeta - z| > \varepsilon$ if $\mathrm{dist}(\zeta, N) < \varepsilon$ and therefore $|\zeta - z| >$

$1/2(|\zeta - z| + \varepsilon)$. Together with (20) and $|\tilde{\Phi}_j(\zeta, z)| \geq C(|\varrho_j(z)| + |q_j| + |\zeta - z|^2) \geq C(\varepsilon + |q_j| + |\zeta - z|^2)$ this implies

$$
\begin{aligned}
J_0'' &\leq C \int\limits_{\substack{|y'|<\varepsilon,\, |q'|<C \\ \varepsilon<|x'''|<C}} \frac{|y'|^{-\beta} d\sigma_{[\beta]+1} dq_{j_{\nu+1}} \ldots dq_{j_s} d\sigma_{2n-l-s-\mu+1}}{|\tilde{\Phi}_{j_{\nu+1}}| \cdot \ldots \cdot |\tilde{\Phi}_{j_s}| (\varepsilon + |x'''(\zeta - z)|^2)^{\nu+1} |x'''(\zeta - z)|^{2n-l-s-1}} \\
&\leq C \int\limits_{|y'|<\varepsilon} |y'|^{-\beta} d\sigma_{[\beta]+1} \\
&\quad \cdot \int\limits_{|q'|<C} \frac{dq_{j_{\nu+1}} \ldots dq_{j_s}}{(\varepsilon + |q_{j_{\nu+1}}|) \ldots (\varepsilon + |q_{j_s}|)} \\
&\quad \cdot \int\limits_{\varepsilon<|x'''|<C} \frac{d\sigma_{2n-l-s-\mu+1}}{(\varepsilon + |x'''(\zeta - z)|^2)^{\nu+1} |x'''(\zeta - z)|^{2n-l-s-1}}.
\end{aligned}
\tag{23}
$$

For the first integral we get

$$
\begin{aligned}
\int\limits_{|y'|<\varepsilon} |y'|^{-\beta} d\sigma_{[\beta]+1} &\leq C \int\limits_0^\varepsilon r^{-\beta} r^{[\beta]} dr \\
&\leq C\varepsilon^{1+[\beta]-\beta}.
\end{aligned}
\tag{24}
$$

Further it is

$$
\int\limits_{|q'|<C} \frac{dq_{j_{\nu+1}} \ldots dq_{j_s}}{(\varepsilon + |q_{j_{\nu+1}}|) \ldots (\varepsilon + |q_{j_s}|)} \leq C(1 + \ln \varepsilon)^{s-\nu}. \tag{25}
$$

So the product of the first two integrals is less than $C\varepsilon^{1+[\beta]-\beta'}$. During the investigation of the third integral we have to consider different values of μ. Remember that we have $\mu \geq 1$. Thus we obtain

$$
\begin{aligned}
\int\limits_{\varepsilon<|x'''|<C} \frac{d\sigma_{2n-l-s-\mu+1}}{(\varepsilon + |x'''(\zeta - z)|^2)^{\nu+1} |x'''(\zeta - z)|^{2n-l-s-1}} &\leq C \int\limits_\varepsilon^C \frac{r^{2n-l-s-\mu} dr}{(\varepsilon + r^2)^{\nu+1} r^{2n-l-s-1}} \\
&\leq C \int\limits_\varepsilon^C \frac{r^{1-\mu} dr}{(\varepsilon + r^2)^{\nu+1}} \\
&\leq C \begin{cases} \varepsilon^{-\nu-1} \varepsilon^{2-\mu} & \text{for} \quad \mu > 2 \\ \varepsilon^{-\nu-1} \ln \varepsilon & \text{for} \quad \mu = 2 \\ \varepsilon^{-\nu-1+(2-\mu)/2} & \text{for} \quad \mu = 1. \end{cases}
\end{aligned}
\tag{26}
$$

Combining (24), (25) and (26) it follows that

$$
J_0'' \leq \begin{cases} C\varepsilon^{1-\beta'} & \text{for} \quad \mu \geq 2 \\ C\varepsilon^{1/2-\beta'} & \text{for} \quad \mu = 1. \end{cases} \tag{27}
$$

Together with (22) this proves part (iv) of the lemma.

Now we consider the integrals J_1, J_2 and J_3. Like in the proof of (i) we can integrate with respect to the q_j. With $\varepsilon := 1/2\text{dist}(z, bD)$ and $0 \leq \beta < \beta' < 1/2$ we find that all the integrals J_1, J_2 and J_3 can be estimated by

$$J_4 := \int\limits_{\substack{0<t<C \\ 0<|x''|<C}} \frac{|t|^{-\beta'} dt \wedge d\sigma_{2n-l-s}}{(\varepsilon + |t| + |x''(\zeta - z)|^2)^\alpha |x''(\zeta - z)|^{2n-l-s+1-\alpha}} \quad \text{for} \quad 0 \leq s \leq l, \ \alpha = 1, 2.$$

Again we use the fact that $|\zeta - z| > \varepsilon$ if $|t| < \varepsilon$ and set

$$J_4' := \int\limits_{\substack{|t|>\varepsilon \\ 0<|x''|<C}} \frac{|t|^{-\beta'} dt \wedge d\sigma_{2n-l-s}}{(\varepsilon + |t| + |x''(\zeta - z)|^2)^\alpha |x''(\zeta - z)|^{2n-l-s+1-\alpha}} \quad \text{for} \quad \alpha = 1, 2,$$

$$J_4'' := \int\limits_{\substack{|t|<\varepsilon \\ \varepsilon<|x''|<C}} \frac{|t|^{-\beta'} dt \wedge d\sigma_{2n-l-s}}{(\varepsilon + |t| + |x''(\zeta - z)|^2)^\alpha |x''(\zeta - z)|^{2n-l-s+1-\alpha}} \quad \text{for} \quad \alpha = 1, 2.$$

We integrate with respect to t only if $\alpha = 2$ and get

$$\begin{aligned} J_4' &\leq C\varepsilon^{-\beta} \int\limits_{0<|x''|<C} \frac{d\sigma_{2n-l-s+2-\alpha}}{(\varepsilon + |x''(\zeta - z)|^2)|x''(\zeta - z)|^{2n-l-s+1-\alpha}} \\ &\leq C\varepsilon^{-\beta} \int_0^C \frac{r^{2n-l-s+1-\alpha} dr}{(\varepsilon + r^2) r^{2n-l-s+1-\alpha}} \\ &\leq C\varepsilon^{-\beta} \int_0^C \frac{dr}{\varepsilon + r^2} \\ &\leq C\varepsilon^{-\beta-1/2}. \end{aligned} \tag{28}$$

And for J_4'' we obtain

$$J_4'' \leq C \int\limits_{\varepsilon<|x''|<C} \frac{d\sigma_{2n-l-s}}{(\varepsilon + |x''(\zeta - z)|^2)^{\alpha-1+\beta'} |x''(\zeta - z)|^{2n-l-s+1-\alpha}}.$$

Considering the two different values of α we find

$$\begin{aligned} J_4'' &\leq C \int_\varepsilon^C \frac{r^{2n-l-s-1} dr}{(\varepsilon + r^2)^{\alpha-1+\beta'} r^{2n-l-s+1-\alpha}} \\ &\leq C \int_\varepsilon^C \frac{r^{\alpha-2} dr}{(\varepsilon + r^2)^{\alpha-1+\beta'}} \\ &\leq C \begin{cases} \varepsilon^{1-\alpha-\beta'} \ln \varepsilon & \text{for} \quad \alpha = 1 \\ \varepsilon^{1-\alpha-\beta'+(\alpha-1)/2} & \text{for} \quad \alpha = 2 \end{cases} \\ &\leq C\varepsilon^{-\beta-1/2}. \end{aligned} \tag{29}$$

Together with (28) and the fact that J_1, J_2 and J_3 can be estimated by J_4 this proves (ii) and the proof of the lemma is complete. ■

Now we are able to proof the estimates given in the main theorem of this paper.

Proof of Theorem 3.5. By definition we have

$$T_q f = -\int_R f \wedge K' - (-1)^{q-1} \int_{D^\epsilon \times \Delta_0} f \wedge K_{q-1}$$

and it is also quite clear that

$$\int_R f \wedge K' \leq ||f||_{\varphi(\beta,N)} \int_R [\mathrm{dist}(\zeta, N)]^{-\beta} \sum_{|I|=q} d\bar{\zeta}^I \wedge K'.$$

On Δ_0 K_{q-1} is exactly the Martinelli-Bochner kernel. Using Lemma 4.1 the first integral can be written as a sum of integrals over R_I. For $l = 1$ there arises one Martinelli-Bochner kernel. Denoting the Martinelli-Bochner operator by B the estimates

$$||Bf||_{C^\alpha} \leq C||f||_{\varphi(\beta,N)} \quad \text{for} \quad 0 \leq \beta < 1, 0 < \alpha < 1 - \beta,$$

$$||Bf||_{\varphi(\beta'-1,N)} \leq C||f||_{\varphi(\beta,N)} \quad \text{for} \quad 1 \leq \beta < \beta' < 1 + d$$

follow by well-known arguments which we omit (see, e.g., [HL] for the case $\beta = 0$). It remains to consider the other integrals which all contain some χ. So we only have to integrate over a small neighbourhood of bD. It is even enough to fix a $\zeta_0 \in N \cap S_I$ and to consider a small neighbourhood $U(\zeta_0)$.[2] Because K' and $\delta K'$ are bounded for $|\zeta - z| > C$ and $[\mathrm{dist}(\zeta, N)]^{-\beta}$ is integrable over R_I we may further assume z to be in $U(\zeta_0)$. Using Lemma 4.2 and Lemma 4.4 we get part (ii) of Theorem 3.5. And the Hölder estimates in part (i) of the Theorem 3.5 follow from Lemma 4.3, Lemma 4.4 and the Hardy-Littlewood lemma. ■

5 References

[AH] A. Andreotti, C.D. Hill:
E. E. Levi convexity and the Hans Lewy problem. Part I: Reduction to vanishing theorems. Ann. Scuola Norm. Sup. Pisa, 26 (1972), 325-363.

[F] B. Fischer:
Cauchy-Riemann equation in spaces with uniform weights. to appear.

[HL] G.M. Henkin, J. Leiterer:
Theory of functions on complex manifolds. Akademie-Verlag, Berlin, 1984.

[LL] C. Laurent-Thiébaut, J. Leiterer: Uniform estimates for the Cauchy-Riemann equation on q-convex edges. Prépublication de l'Institut Fourier, no. 186, 1991.

[LR] I. Lieb, R.M. Range:
Estimates for a class of integral operators and applications to the $\bar{\partial}$-Neumann problem. Invent. math., 85 (1986), 415-438.

[2] For $\zeta_0 \in S_I \backslash N$ we can choose U so small that $\mathrm{dist}(U, N) > C$. So f is bounded in U and that is the same like the case $N = bD$ and $\beta = 0$.

[M] J. Michel:
Randregularität des $\bar{\partial}$-Problems für stückweise streng pseudokonvexe Gebiete in $\mathbb{C}^n$. Math. Ann., 280 (1988), 45-68.

[RS] R.M. Range, Y.T. Siu:
Uniform estimates for the $\bar{\partial}$-equation on domains with piecewise smooth strictly pseudoconvex boundaries. Math. Ann., 206 (1973), 325-354.

Fachbereich Mathematik der
Humboldt-Universität
O-1056 Berlin
and
Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
W-5300 Bonn 3