Kashiwara theorem for hyperbolic algebras.

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Abstract. The class of hyperbolic algebras contains Weyl algebras and their quantized versions, the quantized and classical enveloping algebras of $sl(2)$ and many others. A purpose of this work is to obtain an analog of the Kashiwara theorem for hyperbolic algebras and to study Gelfand-Kirillov dimension of modules over them. As an application, we establish Bernstein's inequality for the GK dimension of modules over certain hyperbolic algebras.

Contents:

Introduction

Let X, Y be smooth algebraic varieties and $\iota : Y \to X$ be a closed embedding. An important theorem by Kashiwara says that the embedding ι induces an equivalence of the category $\mathcal{M}(Y)$ of D-modules on Y and the full subcategory $\mathcal{M}_Y(X)$ of $\mathcal{M}(X)$ formed by D-modules on X with support in Y. Being of local nature, this theorem is equivalent to a certain statement on modules over Weyl algebras which we shall also call Kashiwara theorem. One of the purposes of this work is to extend the Kashiwara theorem from Weyl algebras to a much larger class of so called *hyperbolic algebras*.

Let R be an associative algebra, $\theta : R \to R$ an algebra automorphism, ξ a central element of R. To this data there corresponds an algebra $R\{\theta, \xi\}$ generated by R and the elements x, y subject to the following relations:

$$xy = \xi, \quad yx = \theta^{-1}(\xi); \quad xr = \theta(r)x, \quad ry = y\theta(r) \quad \text{for every } r \in R$$

Due to the first two relations, $R\{\theta, \xi\}$ is called *a hyperbolic algebra of rank one over* R (it is called otherwise *a generalized Weyl algebra* [Bav]). A hyperbolic algebra of arbitrary finite rank over R is obtained by the iteration of this construction.

Hyperbolic algebras are particularly convenient for studying representations (see [R], Chapters 2 and 4). On the other hand, by pure luck, quite a few algebras of interest (such as all Weyl algebras and their quantized versions, the quantized and classical enveloping algebras of $sl(2)$ and many others) are hyperbolic.

The first section contains examples of hyperbolic algebras.

In Section 2 we introduce an analog of the category $\mathcal{O}$ for hyperbolic algebras and prove the corresponding version of the Kashiwara theorem.

In Section 3 we consider some examples (those of Section 1) and apply Kashiwara theorem to estimate the Gelfand-Kirillov (GK) dimension of modules of the category $\mathcal{O}$.

In Section 4 we estimate the GK dimension for arbitrary modules over a hyperbolic algebra. The main tool is the description of the spectrum of hyperbolic algebras obtained in [R1], Ch.4 and Ch.5. For readers' convenience, we include a short discussion of the spectrum.

In Section 5, we deduce from results of Section 4 the Bernstein's inequality when it is available.

Finally in the complementary section we discuss the categorical Kashiwara theorem.

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1. Hyperbolic algebras.

1.1. Definition. Fix a commutative ring k. Let R by an associative k-algebra, θ automorphism of R; and let ξ be a central element of R. *The hyperbolic algebra over R determined by the automorphism θ and the element ξ* is the algebra $R\{\theta,\xi\}$ generated by R and the elements x, y subject to the following relations:

$$xr = \theta(r)x, \quad ry = y\theta(r) \quad \text{for every } r \in R \tag{1}$$

$$xy = \xi, \quad yx = \theta^{-1}(\xi) \tag{2}$$

Let $\mathfrak{R}$ be an associative k-algebra with unit, and let R be a subalgebra of $\mathfrak{R}$. We say that $\mathfrak{R}$ *is a hyperbolic algebra over R*, or *an iterated hyperbolic algebra over R*, if there is an increasing filtration $\{R_m | m \geq 0\}$ of $\mathfrak{R}$ such that each R_m is a subalgebra of $\mathfrak{R}$, $R_0 = R$, $\sum_{m\geq 0} R_m = \mathfrak{R}$, and, for any $m \geq 1$, R_{m+1} is a hyperbolic algebra over R_m, i.e. $R_{m+1} = R_m\{\theta_m, \xi_m\} = R_m\{\theta_m, \xi_m; x_{m+1}, y_{m+1}\}$ for some automorphism θ_m and a central element ξ_m of the algebra R_m and elements x_{m+1}, y_{m+1} of R_{m+1}. If $\mathfrak{R} = R_\nu$, then the number ν will be called *the rank of the algebra $\mathfrak{R}$ over R*. For simplicity in this work we will always assume that all ξ_i are elements of R.

1.1.1. Lemma. *The hyperbolic algebra $R\{\theta,\xi\}$ of rank one over R is a free R-module with the basis formed by the monomials x^m, y^n.*

Proof. This is an easy exercise. See also Lemma II.3.1.6 in [R1]. ∎

1.1.2. Corollary. *Let $\mathfrak{R}$ be a hyperbolic algebra with the hyperbolic structrure $R_{m+1} = R_m\{\theta_m, \xi_m; x_{m+1}, y_{m+1}\}$, $m \geq 1$. An element $r \in R$ is central iff it belongs to the center of R and $\theta_m(r) = r$ for all m.*

Proof. The conditions are sufficient, because if $\theta_m(r) = r$ for all m, the element r commutes with x_{m+1} and y_{m+1} for all m. It follows from Lemma 1.1.1 that they are necessary. ∎

1.1.3 Remark. The hyperbolic algebra $R\{\theta, \xi\}$ admits a Fourier transform. Namely, we replace the hyperbolic data $\{x, y, \xi, \theta\}$ by the data $\{y, x, \theta^{-1}(\xi), \theta^{-1}\}$.

1.1.4. Remark. It follows from the definition of a hyperbolic algebra and the Corollary 1.1.2 that $\theta_m(\xi_j) = \xi_j$ for $m < j$.

1.2. Some examples of hyperbolic algebras.

1.2.1. A general example. Fix a commutative ring k. Let R be an associative k-algebra with unit, $\{\vartheta_i \mid i \in J\}$ a family of pairwise commuting k-algebra automorphisms of R. Let $\{\xi_i \mid i \in J\}$ be a family of central elements of R such that $\vartheta_i(\xi_j) = \xi_j$ for any $i, j \in J$, $i \neq j$. Denote the data $\{\vartheta_i | i \in J\}$ by Θ, and $\{\xi_i \mid i \in J\}$ by ξ. Let $\mathbf{q} = (q_{ij})_{i,j \in J}$ be a matrix with entrees in k^* such that $1 = q_{ij}q_{ji}$ for all $i, j \in J$, $i \neq j$. Let $R\{\Theta, \xi\}$ be the k-algebra generated by R and by $x_i, y_i,\ i \in J$, satisfying the relations:

$$x_i r = \vartheta_i(r) x_i, \quad r y_i = y_i \vartheta_i(r) \quad \text{for any } r \in R \tag{1}$$

$$x_i y_i = \xi_i, \quad y_i x_i = \vartheta_i^{-1}(\xi_i) \quad \text{for any } i \in J \tag{2}$$

$$x_i y_j = q_{ji} y_j x_i, \quad x_i x_j = q_{ij} x_j x_i, \quad y_i y_j = q_{ij} y_j y_i \tag{3}$$

for all $i, j \in J$ such that $i \neq j$.

For any $m \in J$, denote by R_m the algebra generated by R and $\{x_i, y_j | i, j \leq m\}$. We define an automorphism θ_m of the algebra R_m as an extension of ϑ_m by setting $\theta_m(x_i) = q_{mi} x_i, \quad \theta_m(y_i) = q_{im} y_i$. It follows that R_{m+1} is a hyperbolic algebra over R_m defined by the automorphism θ_m and the element ξ_m. Clearly the union of R_m coincides with the algebra $R\{\Theta, \xi\}$.

1.2.1.1. Remark. The special case of the algebras of Example 1.2.1, when $q_{ij} = 1$ for all $i, j \in J$, was introduced independently by Bavula [Bav] (who called them *generalized Weyl algebras*) at the end of 80s, approximately at the same time when hyperbolic algebras were introduced by the second author of this work. Generalized Weyl algebras appeared also in the work by Smith and Bell [BS]. ∎

1.2.2. The algebra of q-differential operators D_q. This is the simplest particular case of Example 1.2.1: $J = 1,\ R = k$. In other words, the algebra D_q is generated by x, y subject to the relation

$$xy - qyx = 1.$$

1.2.3. The Weyl algebras. Recall that the n-th Weyl algebra, A_n, over a field k is generated by the set of elements $\{x_i, y_i \mid 1 \geq i \geq n\}$ subject to the relations:

$$x_i y_i - y_i x_i = 1 \quad \text{for all } i \in J;$$

$$x_i y_j = y_j x_i,\ x_i x_j = x_j x_i,\ y_i y_j = y_j y_i \quad \text{for all } i, j \in J,\ i \neq j$$

Let $R = k[\xi_1, ..., \xi_n]$ be the algebra of polynomials in n variables. And let θ_i denote the automorphism of R determined by $\theta_i(\xi_j) = \xi_j + \delta_{ij}$. The map sending x_i into x_i and

y_i into y_i defines an isomorphism from the Weyl algebra A_n onto the hyperbolic algebra $R\{\Theta,\xi\}$, $\Theta := (\theta_i),\ \xi := (\xi_i)$.

1.2.4. The quantized Heisenberg and Hayashi's Weyl algebras. Recall that the quantized Heisenberg algebra, $H_q(J)$, over a field k is generated by the set of elements $\{x_i, y_i, z_i \mid i \in J\}$ subject to the relations:

$$x_i z_i = q z_i x_i, \quad z_i y_i = q y_i z_i,$$

$$x_i y_i - q^{-1} y_i x_i = z_i$$

$$x_i y_j = y_j x_i,\ x_i x_j = x_j x_i,\ y_i y_j = y_j y_i,\ z_j z_i = z_i z_j$$

for all $i, j \in J,\ i \neq j$.

T.Hayashi [Ha] defined a quantized Weyl algebra $W_q(J)$ as a quotient algebra of the algebra $H_q(J)$ obtained by adding the relations

$$(x_i y_i - q y_i x_i) z_i = 1 = z_i (x_i y_i - q y_i x_i)$$

for all $i \in J$. Clearly $W_1(J)$ is the conventional Weyl algebra.

In $H_q(J)$ we have the relation $(x_i y_i) z_j = z_j (x_i y_i)$ for all $i, j \in J$, and, moreover, the morphism of the algebra $k[(z_i),(t_i)]$ of polynomials in variables $z_i, t_i, i \in J$, to the quantized Heisenberg algebra $H_q(J)$ which sends z_i into z_i and t_i into the product $x_i y_i$ is injective. This implies that the algebra $H_q(J)$ is generated by the commutative (polynomial) algebra $R := k[(z_i),(t_i)]$ and the elements $x_i, y_i,\ i \in J$, subject to the relations:

$$x_i r = \theta_i(r) x_i, \quad r y_i = y_i \theta_i(r)$$

$$x_i y_i = t_i, \quad y_i x_i = q(t_i - z_i)$$

$$x_i y_j = y_j x_i$$

for all for all $r \in R$ and $i, j \in J$, where $j \neq i$.

Here θ_i is an automorphism of the algebra R defined by

$$\theta_i(z_i) = q z_i, \quad \theta_i(t_i) = q^{-1} t_i + q z_i,$$

and

$$\theta_i(z_j) = z_j, \quad \theta_i(t_j) = t_j \quad \text{if } j \neq i$$

Note that $\theta_i^{-1}(t_i) = q(t_i - z_i)$; i.e. $y_i x_i = \theta_i^{-1}(t_i)$. These relations show that the quantized Heisenberg algebra is hyperbolic.

Now let R be the quotient algebra of the polynomial algebra $k[(z_i),(t_i)]$ by the relations

$$z_i(t_i(1-q^2) + q^2 z_i) = 1, \quad i \in J.$$

The automorphisms θ_i, $i \in J$ are defined by the same formulas as for the Heisenberg quantized algebra $H_q(J)$; i.e.

$$\theta_i(z_i) = qz_i, \quad \theta_i(t_i) = q^{-1}t_i + qz_i \quad \text{for all } i \in J$$

$$\theta_i(z_j) = z_j, \quad \theta_i(t_j) = t_j \quad \text{if } j \neq i$$

The morphism of the hyperbolic algebra $R\{\Theta, \mathbf{t}\}$ determined by the data $(\Theta, \mathbf{t}) = \{\theta_i, t_i \mid i \in J\}$ to the Hayashi's Weyl algebra which sends z_i into z_i and t_i into $x_i y_i$, $i \in J$, is an isomorphism.

1.2.5. The enveloping algebra of $sl(2)$. Recall that the enveloping algebra of $sl(2)$, $U(sl(2))$, is generated by x, y, z subject to the relations

$$xz = zx + \alpha x, \quad yz = zy - \alpha y, \quad xy - yx = z \tag{1}$$

where $\alpha \neq 0$.

Let $R = k[z, \xi]$ and let the automorphism θ be determined by the equalities: $\theta(z) = z + \alpha$, $\theta(\xi) = \xi + z + \alpha$. Then the algebra $U(sl(2))$ defined by the relations (1) is naturally isomorphic to the hyperbolic algebra $R\{\theta, \xi\}$.

1.2.6. Example: the quantum enveloping algebra of $sl(2,k)$. The quantum enveloping algebra of the Lie algebra $sl(2)$, $U_q(sl(2))$ is defined by the relations:

$$xz = qzx, \quad zy = qyz; \quad xy - yx = \frac{z - z^{-1}}{q - q^{-1}} \tag{1}$$

where q is an element of $k - \{0, 1\}$.

Let $R = k[z, z^{-1}, \xi]$ and let the automorphism θ be determined by the equalities: $\theta(z) = qz$, $\theta(\xi) = \xi + u(qz)$, where $u(z) = \frac{z-z^{-1}}{q-q^{-1}}$. Then the algebra $U_q(sl(2))$ defined by the relations (1) is naturally isomorphic to the hyperbolic algebra $R\{\theta, \xi\}$.

1.2.7. Example: the coordinate algebra of $SL_q(2,k)$. The coordinate algebra $A(SL_q(2,k))$ of the *algebraic quantum group* $SL_q(2,k)$ (cf. [M]) is the k-algebra generated by the indeterminates x, y, u, v satisfying the equations:

$$qux = xu, \quad qvx = xv, \quad qyu = uy, \quad qyv = vy, \quad uv = vu, \tag{1}$$

$$xy - quv = 1 = yx - q^{-1}uv \tag{2}$$

Let R be the algebra $k[u, v]$ of polynomials in u, v. Set $\theta f(u, v) := f(qu, qv)$ for any polynomial $f(u, v)$ and denote by ξ the element $1 + quv$. Then the relations (1), (2) become equivalent to the relations determining the algebra $R\{\theta, \xi\}$.

1.3. The algebra of differential operators on a quantum space. Fix a commutative ring k. Let $\mathbf{q} = (q_{ij})_{i,j \in J}$ be a matrix with entries in k^* satisfying the relations $1 = q_{ij}q_{ji}$ for all $i, j \in J$ such that $i \neq j$. A *skew polynomial k-algebra* is the k-algebra $S_{\mathbf{q}}$ generated by indeterminates x_i, $i \in J$, subject to the relations:

$$x_i x_j = q_{ij} x_j x_i \quad , i \neq j. \tag{1}$$

The algebra $S_{\mathbf{q}}$ is regarded as the algebra of functions on a 'quantum space' (more specifically, a $\mathbf{q}$-space).

The algebra $D_{\mathbf{q}}(S_{\mathbf{q}})$ of $\mathbf{q}$-differential operators on $S_{\mathbf{q}}$ is the subalgebra of $End_k(S_{\mathbf{q}})$ generated by $x_i,\ \partial_j,\ i,j \in J$. Here by x_i we mean the endomorphism of multiplication by x_i; and ∂_j is *the partial* $\mathbf{q}$*-derivation of* R determined by

$$\partial_i(k) = \{0\}, \quad \partial_i(x_j r) = \delta_{ij} r + q_{ji} x_j \partial_i(r) \tag{2}$$

for all $r \in S_{\mathbf{q}}$. The relations between different $x_i,\ i \in J$, are given by (1). The relations between x_i and $\partial_j,\ i,j \in J$:

$$\partial_i x_j - q_{ji} x_j \partial_i = \delta_{ij} \quad \text{for all } i,j \in J. \tag{3}$$

The relations between $\partial_i,\ i \in J$, look as follows:

$$\partial_i \partial_j = q_{ij} \partial_j \partial_i, \quad \text{for all } i \neq j. \tag{4}$$

Thus, $D_{\mathbf{q}}(R)$ is generated by $x_i, \partial_j,\ i,j \in J$, subject to the relations (1)–(4).

1.3.1. The structure of a hyperbolic algebra on $D_{\mathbf{q}}(S_{\mathbf{q}})$. For any $i \in J$, set $\xi_i = \partial_i x_i$. It follows from the relations above that $\xi_i \xi_j = \xi_j \xi_i$ for all $i,j \in J$. Denote by R the algebra $k[(\xi_i)]$ of polynomials in $\xi_i,\ i \in J$. Define automorphisms $\theta_i, i \in J$, of the algebra R by the formulas:

$$\theta_i(\xi_j) = \xi_j \text{ if } i \neq j;\ \theta_i(\xi_i) = q_{ii}\xi_i + 1 \tag{1}$$

Then we can regard the algebra $D_{\mathbf{q}}$ of $\mathbf{q}$-differential operators on $S_{\mathbf{q}}$ as a k-algebra generated by R and elements x_i, ∂_i subject to the relations:

$$x_i x_j = q_{ij} x_j x_i, \quad \partial_i \partial_j = q_{ij} \partial_j \partial_i \quad \partial_i x_j = q_{ji} x_j \partial_i \tag{2}$$

for all $i,j \in J$, $i \neq j$;

$$\partial_i x_i = \xi_i, \quad x_i \partial_i = \theta_i^{-1}(\xi_i) \tag{3}$$

$$\partial_i r = \theta_i(r)\partial_i, \quad r x_i = x_i \theta_i(r) \tag{4}$$

for all $i \in J$ and $r \in R$.

The relations (2), (3), (4) show that the algebra $D_{\mathbf{q}}$ is a particular case of Example 1.2.1.

1.4. The Quantum Weyl algebra $A_{\mathbf{q}}$. This algebra appeared naturally in the context of studying differential operators on "quantum spaces" (cf. [LR1]). It can be obtained from the algebra $D_{\mathbf{q}}$ of Example 1.3 as follows.

Note that there is a natural action of the group $\Gamma := \mathbb{Z}^J$ on the algebra $S_{\mathbf{q}}$ of $\mathbf{q}$-polynomials (cf. 1.3): the canonical generator z_i of Γ sends x_j into $(q_{ii}\delta_{ij} + 1 - \delta_{ij})x_j$. This way we define a group homomorphism $\phi : \Gamma \to Aut(S_{\mathbf{q}})$. By definition, *the quantum Weyl algebra* $A_{\mathbf{q}}$ is the subalgebra of $End_k(S_{\mathbf{q}})$ generated by the algebra $D_{\mathbf{q}}$ (of 1.3) and the image of the group Γ in $End_k(S_{\mathbf{q}})$.

1.5. The algebra $A'_{\mathbf{q}}$. The action of Γ on $S_{\mathbf{q}}$ extends naturally to an action of Γ on the algebra of differential operators $D_{\mathbf{q}}$: for any $i, j \in J$, the canonical generator z_i of Γ sends x_j to $(q_{ii}\delta_{ij} + 1 - \delta_{ij})x_j$ and ∂_j to $(q_{ii}^{-1}\delta_{ij} + 1 - \delta_{ij})\partial_j$. This defines a group homomorphism $\phi : \Gamma \to Aut(D_{\mathbf{q}})$. Denote by $A'_{\mathbf{q}}$ the corresponding crossed product of $D_{\mathbf{q}}$ and Γ: $A'_{\mathbf{q}} = D_{\mathbf{q}}\#\Gamma$. It follows from the definition that $A'_{\mathbf{q}}$ is generated by the algebra $D_{\mathbf{q}}$ and elements $(z_i, z_i^{-1}|i \in J)$ satisfying the relations:

$$z_i x_j = b_{ij} x_j z_i, \quad z_i\partial_j = b_{ij}^{-1}\partial_j z_i; \quad z_i z_j = z_j z_i \tag{1}$$

for all $i, j \in J$. Here $b_{ij} := q_{ii}\delta_{ij} + (1 - \delta_{ij})$.

1.5.1. Remark. *The algebra $A_{\mathbf{q}}$ is naturally a homomorphic image of $A'_{\mathbf{q}}$.*

1.5.2. The hyperbolic structure of $A'_{\mathbf{q}}$. We take $R = k[(\xi_i)_{i\in J}] \otimes_k k[(z_i)_{i\in J}]$ and define the automorphrphisms $\theta_i : R \to R,\ i \in J$, by

$$\theta_i(\xi_j) = b_{ij}\xi_j + \delta_{ij}, \quad \theta_i(z_j) = b_{ij}z_j \tag{1}$$

where $b_{ij} := q_{ii}\delta_{ij} + (1 - \delta_{ij})$. Then the algebra $A'_{\mathbf{q}}$ is generated by R, x_i, and ∂_i, $i \in J$, subject to the relations (2)–(4) of 1.3.1.

1.5.3. A generalization. The construction of 1.5 is readily extended to a more general setting of Example 1.2.1. Namely, we take the trivial action of Γ on the "coefficient algebra" R; and define the action of Γ on x_i and y_i as in 1.4; i.e., for any $i, j \in J$, the canonical generator z_i of Γ sends x_j into $b_{ij}x_j$ and y_j into $b_{ij}^{-1}y_j$, where $b_{ij} := (q_{ii}\delta_{ij} + 1 - \delta_{ij})$. This way we define a group homomorphism $\phi : \Gamma \to Aut(R\{\Theta, \xi\})$.

1.5.4. The hyperbolic structure of $A_{\mathbf{q}}$. The hyperbolic structure of $A_{\mathbf{q}}$ is induced by that of $A'_{\mathbf{q}}$ via the canonical epimorphism $A'_{\mathbf{q}} \to A_{\mathbf{q}}$ (cf. 1.5.1). It is possible, however, to give an explicit description of $A_{\mathbf{q}}$.

For any $i \in J$, set $\eta_i := \partial_i x_i - x_i\partial_i = (q_{ii} - 1)\theta_i^{-1}(\xi_i) + 1$. Note that $\theta_i(\eta_j) = b_{ij}\eta_j$; i.e. $\theta_i(\eta_i) = q_{ii}\eta_i$ and $\theta_i(\eta_j) = \eta_j$ if $i \neq j$. This implies that each η_i is a *normal element*, i.e. the left (or right) ideal in $R\{\theta, \xi\}$ generated by η_i is two-sided. Denote by S the multiplicative subset of R generated by η_i, q_{ii} $i \in J$. Note that, for any $i \in J$, the set S is θ_i-stable; hence θ_i induces an automorphism, θ'_i, of the localization $R' = S^{-1}R$ of the algebra R at the multiplicative set S. Denote by ξ'_i the image of ξ_i in R', $i \in J$, and set $\Theta' = (\theta'_i)$, $\xi' := (\xi'_i)$. This data determines a hyperbolic algebra $\mathfrak{R}' := R'\{\Theta', \xi'\}$ generated by R', (x_i), (∂_i).

1.5.4.1. Proposition. *The algebra $A_{\mathbf{q}}$ is naturally isomorphic to the hyperbolic algebra $\mathfrak{R}'$ defined above.*

Proof. The assertion follows from the simple observation that the image of the generator z_i of the group Γ in $Aut(S_{\mathbf{q}})$ (cf. 1.4) coincides with η_i. ■

1.5.4.2. Remark. The multiplicative set S is an Ore set in the hyperbolic algebra $\mathfrak{R} = R\{\Theta, \xi\}$ and the localized algebra $S^{-1}\mathfrak{R}$ is naturally isomorphic to the algebra $\mathfrak{R}'$.

1.6. Quantum Weyl algebra of Maltsiniotis. Fix again a matrix $\mathbf{q} = (q_{ij})$ such that $q_{ij}q_{ji} = 1$ and $q_{ii} = 1$ for all $1 \leq i, j \leq n$. Fix an element $\lambda := (\lambda_1, ..., \lambda_n)$ of $(k^*)^n$. The *quantum Weyl algebra of Maltsiniotis*, $A_n^{\mathbf{q},\lambda}$, is generated by $x_i, y_j,\ 1 \le i, j \le n$ subject to the relations:

(a) For any $1 \le i < j \le n$,

$$x_i x_j = \lambda_i q_{ij} x_j x_i, \quad y_i y_j = q_{ij} y_j y_i$$

$$x_i y_j = q_{ji} y_j x_i, \quad y_i x_j = \lambda_i^{-1} q_{ji} x_j y_i$$

(b) For any $1 \le i \le n$,

$$x_i y_i - \lambda_i y_i x_i = 1 + \sum_{1 \le j < i} (\lambda_j - 1) y_j x_j$$

For any $1 \le i \le n$, set $y_i x_i = \xi_i$. It follows from the relations (a) that

$$\xi_i x_j = x_j \xi_i, \quad \xi_i y_j = y_j \xi_i \quad \text{if } i < j$$

$$\xi_i x_j = \lambda_j^{-1} x_j \xi_i, \quad \xi_i y_j = \lambda_j y_j \xi_i \quad \text{if } i > j.$$

These relations imply that $\xi_i \xi_j = \xi_j \xi_i$ for all i, j. Set $R := k[\xi_i, ..., \xi_n]$. For any $0 < m \le n$, denote by R_m the algebra generated by R and the elements $x_i, y_j,\ m < i, j \le n$. Thus $R_n = A_n^{q,\lambda}$, $R_0 = R$, and $R_i \subset R_j$ if $i < j$. Denote by θ_m the k-algebra automorphism of R_m defined by

$$\theta_m(x_i) = \lambda_m q_{mi} x_i, \quad \theta_m(y_i) = \lambda_m^{-1} q_{im} y_i \quad \text{for } m < i \le n \tag{1}$$

$$\theta_m(\xi_i) = \xi_i \quad \text{if } i < m, \quad \theta_m(\xi_i) = \lambda_m^{-1} \xi_i \quad \text{if } i > m, \tag{2}$$

$$\theta_m^{-1}(\xi_m) = \lambda_m \xi_m + 1 + \sum_{1 \le j < m} (\lambda_j - 1) \xi_j \tag{3}$$

It follows that R_{m+1} is a hyperbolic algebra over its subalgebra R_m. Explicitly, $R_{m+1} = R_m\{\theta_m, \xi_m; y_m, x_m\}$.

1.6.1. The algebra $B_n^{\mathbf{q},\lambda}$. Note that the elements $\eta_i := x_i y_i - y_i x_i = \xi_i - \theta_i^{-1}(\xi_i)$ have the property:

$$\theta_j(\eta_i) = \eta_i \quad \text{if } i < j \quad \text{and} \quad \theta_j(\eta_i) = \lambda_j \eta_i \quad \text{if } i \ge j \tag{1}$$

It follows from (1) that the elements $\eta_i,\ i \in J$ are normal; hence the generated by these elements multiplicative subset S of $A_n^{\mathbf{q},\lambda}$ is a left and right Ore set. We denote by $B_n^{\mathbf{q},\lambda}$ the localization of the algebra $A_n^{\mathbf{q},\lambda}$ with respect to S.

2. Kashiwara theorem for hyperbolic algebras.

2.1. The category $\mathcal{O}$ of a hyperbolic algebra. Fix an automorphism θ and a central element ξ of an associative algebra R. Let $\mathfrak{R}$ denote the hyperbolic algebra $R\{\theta, \xi\} = R\{\theta, \xi; x, y\}$. Denote by $\mathcal{O}^-$ be the full subcategory of $\mathfrak{R} - mod$ consisting of modules M such that any element of M is annihilated by some power of y. We denote by $\mathcal{O}$ the full subcategory of $\mathcal{O}^-$ consisting of modules of finite type. This choice of notations is, of course, not accidental (cf. 1.2.5 and 1.2.6).

2.1.1. Lemma. *The category $\mathcal{O}^-$ is a Serre subcategory of $\mathfrak{R} - mod$. The quotient category $\mathfrak{R} - mod/\mathcal{O}^-$ is equivalent to the category $R[y, y^{-1}; \theta^{-1}] - mod$.*

Here $R[y, y^{-1}; \theta^{-1}]$ is the algebra of skew Laurent polynomials in y.

Proof. a) Note that the multiplicative set $S = \{y^n \mid n \geq 0\} \subset \mathfrak{R}$ satisfies left and right Ore conditions. We shall check the left Ore condition (for any $s \in S$ and any $r \in \mathfrak{R}$, there exist $s' \in S$ and $r' \in \mathfrak{R}$ such that $r's = s'r$) leaving the checking of the right one to the reader. One might also observe that the latter follows automatically by duality (the Fourier transform defined in 1.1.3 above).

Clearly y is a normal element of the subalgebra $R[y, \theta^{-1}]$ of $\mathfrak{R}$. So that it suffices to check the Ore property for $s = y^n$ and $r = x^m$. Take $s' = y^{\nu+n+m}$. Then $s'r = y^{\nu+n}(y^m x^m) = (\theta^{-\nu-n}(y^m x^m)y^\nu)y^n$. Here we use the fact that $y^m x^m$ is an element of $R:\ y^m x^m = \prod_{1 \leq i \leq m} \theta^{-i}(\xi)$.

b) Since S is an Ore set, and $\mathcal{O}^-$ is the corresponding to S Serre subcategory of $\mathfrak{R} - mod$, the localization at $\mathcal{O}^-$ is equivalent to the tensoring over $\mathfrak{R}$ with the algebra of fractions $S^{-1}\mathfrak{R}$, and $\mathfrak{R} - mod/\mathcal{O}^- \simeq S^{-1}\mathfrak{R} - mod$. It remains to show that the algebra $S^{-1}\mathfrak{R}$ is isomorphic to the algebra $R[y, y^{-1}; \theta^{-1}]$ of skew Laurent polynomials.

In fact, $S^{-1}\mathfrak{R}$ is obtained from $\mathfrak{R}$ by inverting y. Clearly $S^{-1}\mathfrak{R}$ contains the algebra $R[y, y^{-1}; \theta^{-1}]$. It remains to show that the image of $\mathfrak{R}$ in $S^{-1}\mathfrak{R}$ is contained in the algebra $R[y, y^{-1}; \theta^{-1}]$.

Any element f of $\mathfrak{R}$ is uniquely expressed as $\sum_{i \geq 1} a_i x^i + \sum_{m \geq 0} b_m y^m$ (Lemma 1.1.1). The image of this element in $S^{-1}\mathfrak{R}$ equals to $f' = \sum_{i \geq 1} a_i (x^i y^i) y^{-i} + \sum_{m \geq 0} b_m y^m$. Since $x^i y^i$ is an element of R, $x^i y^i = \prod_{0 \leq \nu < i} \theta^\nu(\xi)$, for any $i \geq 1$, f' is a Laurent polynomial in y. ∎

Set $A := R/R\xi$. We will identify the category $A - mod$ with the corresponding full subcategory of $R - mod$. There is a functor $\Phi : \mathcal{O}^- \to A - mod$ which assigns to any $M \in Ob\mathcal{O}^-$ the set $M_0 := \{z \in M \mid y \cdot z = 0\}$ with the induced action of A.

On the other hand, there is a functor $\Psi : A - mod \to \mathcal{O}^-$ which assigns to any A-module V the corresponding Verma module (with the subalgebra $\mathcal{B} = R[y; \theta^{-1}]$ of $\mathfrak{R}$ of skew polynomials playing the role of a Borel subalgebra); i.e. Ψ is the composition of the embedding $A - mod \to \mathcal{B} - mod$ and the functor the $\mathfrak{R} \otimes_{\mathcal{B}}$. It is convenient to use the following canonical description of the functor Ψ (which is defined uniquely up to isomorphism). Namely, Ψ assigns to any A-module V the direct sum $\oplus_{n \geq 0} V_n$ of copies of V with the following action of $\mathfrak{R}$: The element x acts as the identical map from V_n to V_{n+1} for all $n \geq 0$; y acts from V_{n+1} to V_n as multiplication by $\theta^{-n-1}(\xi)$ for $n \geq 0$, and sends V_0 to zero. For any $r \in R$, r acts on V_n as $\theta^{-n}(r)$.

Thus we get a pair of adjoint functors (Ψ, Φ). We are interested to find the conditions which imply that Ψ (and hence also Φ) is an equivalence of categories.

2.2. Theorem. *The following conditions are equivalent:*

(a) $R\xi + R\theta^n(\xi) = R$ *for any* $n \neq 0$*;*

(b) the functors Φ *and* Ψ *are mutually inverse equivalences of categories.*

Proof. (b)⇒(a) . Suppose that $R\xi + R\theta^n(\xi)$ is a proper ideal for some $n \neq 0$. We can (and will) assume that $n < 0$. And let μ be a maximal left ideal in R containing $R\xi + R\theta^n(\xi)$. Then $V := R/\mu$ is a simple A-module; while $\Psi(V)$ is not a simple object of the category $\mathcal{O}^-$. The latter follows from the observation that $\oplus_{m\geq -n}V_m$ is a nontrivial submodule of $\Psi(V) := \oplus_{m\geq 0}V_m$. But if Ψ were an equivalence of categories, it should send simple objects into simple objects.

(a)⇒(b). Fix an $\mathfrak{R}$-module $M \in Ob\mathcal{O}^-$. And set $M_n := \{z \in M \mid \theta^n(\xi)z = 0\}$.

(i) Note first that, if $n \neq m$, then $M_n \neq M_m$.

In fact, let $z \in M_n \cap M_m$ which means that $\theta^n(\xi)$ and $\theta^m(\xi)$ are elements of the ideal $Ann_R(z)$. Therefore $\theta^{n-m}(\xi)$ and ξ belong to the ideal $\mu := \theta^{-m}(Ann_R(z))$. Clearly $z \neq 0 \Rightarrow Ann_R(z)$ is a proper ideal $\Rightarrow \mu$ is a proper ideal. But, the latter contradicts the condition (a): $R\theta^{n-m}(\xi) + R\xi = R$.

(ii) Clearly, for any n, M_n is an R-submodule of M. One can see that the action of y (resp. x) maps M_n into M_{n-1} (resp. M_{n+1}): if $z \in M_n$, i.e. $\theta^n(\xi)z = 0$, then $\theta^{n+1}(\xi)(xz) = x\theta^n(\xi)z = 0$, $\theta^{n-1}(\xi)(yz) = y\theta^n(\xi)z = 0$.

(iii) If $z \in M$ is such that $y^m z = 0$, $m \geq 1$, then $z \in \oplus_{0\leq n<m}M_n$.

In fact, if $yz = 0$, then $\xi z = 0$, i.e. $z \in M_0$. Suppose that the assertion is true for m; and let $y^{m+1}z = 0$. Then, by the induction hypothesis, $yz \in_{0\leq n<m} M_n$. This and *(ii)* imply that $\xi z := (xy)z \in \oplus_{0\leq n\leq m}M_n$. We claim that this inclusion implies that $z \in \oplus_{0\leq n\leq m}M_n$.

Again, this is true (by definition) if $m = 0$.

Note now that $\theta^m(\xi)(\xi z) = \xi\theta^m(\xi)z \in \oplus_{0\leq n<m}M_n$ which implies, by the induction hypothesis, that $\theta^m(\xi)z \in \oplus_{0\leq n<m}M_n$. Therefore $(R\xi + R\theta^m(\xi))z \subseteq \oplus_{0\leq n\leq m}M_n$, and the equality $R\xi + R\theta^m(\xi) = R$ implies the desired inclusion: $z \in \oplus_{0\leq n\leq m}M_n$.

(iv) It follows from *(i)*, *(ii)*, and *(iii)* that any $M \in Ob\mathcal{O}^-$ is a $\mathbb{Z}_+$-graded module: $M =_{0\leq n} M_n$.

(v) Note that the actions $x : M_n \to M_{n+1}$, $y : M_{n+1} \to M_n$ are isomorphisms for any $n \geq 0$. In fact, $xy = \xi : M_{n+1} \to M_{n+1}$. By assumption, there exist $r, s \in R$ such that the element $s\xi + r\theta^{n+1}(\xi) = 1$. But $s\xi + r\theta^{n+1}(\xi)$ acts on M_{n+1} as $s\xi$. Therefore the action of ξ on M_{n+1} is invertible. Similarly, one can show that $yx : M_n \to M_n$ is invertible.

(vi) It follows from *(v)* that for an A-module V $\Phi\Psi(V) = V$ and that the adjunction morphism $\Psi\Phi \to Id$ is an isomorphism, i.e. the functors Φ and Ψ are mutually inverse equivalences of categories. ∎

2.3. Kashiwara theorem for general hyperbolic algebras. Let $\mathfrak{R}$ be a hyperbolic algebra over R with a hyperbolic structure $(R_m)_{\nu\geq m\geq 0}$, $R_m = R_{m-1}\{\theta_m, \xi_m; x_m, y_m\}$. We assume that the following condition holds:

(#) $\theta_j(y_i) = y_i$ for all $i < j$.

This condition implies that $y_i x_j = x_j y_i$ for $i < j$ and that $y_i y_j = y_j y_i$ for all i, j.

Denote by $\mathcal{O}^-$ the full subcategory of $\mathfrak{R} - mod$ consisting of $\mathfrak{R}$-modules M such that, for any i, any element of M is annihilated by some power of (y_i). The subcategory $\mathcal{O}^-$ is a Serre subcategory of $\mathfrak{R} - mod$.

Let A be the quotient of the algebra R by the ideal generated by all ξ_i. We will identify the category $A - mod$ with the corresponding full subcategory of $R - mod$. Let $\Phi : \mathcal{O}^- \to A - mod$ denote the functor which assigns to any $M \in Ob\mathcal{O}^-$ the subspace $M_0 := \{z \in M \mid y_i \cdot z = 0 \text{ for all } i\}$ with the induced action of A.

The functor Φ has a left adjoint, Ψ, which assigns to any A-module V the corresponding Verma module (with the subalgebra $\mathcal{B} = R[(y_i); (\theta_i^{-1})]$ of $\mathfrak{R}$ of skew polynomials playing the role of a Borel subalgebra); i.e. Ψ is the composition of the embedding $A - mod \to \mathcal{B} - mod$ and the functor the $\mathfrak{R}\otimes_{\mathcal{B}}$.

2.3.1. Proposition. *Suppose the condition (#) holds and* $R_{i-1}\xi_i + R_{i-1}\theta_i^N(\xi_i) = R_{i-1}$ *for all* $i \geq 0$ *and for all* $N \neq 0$*. Then the functor* $\Phi : \mathcal{O}^- \to A - mod$ *is an equivalence of categories with the quasi-inverse functor* Ψ*.*

Proof. The assertion is true by trivial reasons if one of ξ_i is invertible in $\mathfrak{R}$. In fact, this implies that $A = 0$ and that y_i cannot annihilate a nonzero element; hence both the categories $A - mod$ and $\mathcal{O}^-$ consist of zero modules only. For the rest of the argument we assume that none of ξ_i, $1 \leq i \leq \nu$, is invertible. In particular, all the ideals $R_{i-1}\xi_i$ (in R_{i-1}) are proper.

Note that, for any $0 \leq i < \nu$, the ideal $R_i\xi_\nu$ is stable under the action of the automorphism θ_i (since $\theta_i(\xi_\nu) = \xi_\nu$; cf. 1.1.4). Therefore the structure of a hyperbolic algebra on $\mathfrak{R}$ induces on $\mathfrak{R}' := \mathfrak{R}/(\xi_\nu)$ a structure of a hyperbolic algebra over $R' := R/R\xi_\nu$: $R'_m = R'_{m-1}\{\theta'_m, \xi'_m; x_m, y_m\}$, $0 \leq m < \nu$, where $R'_m := R_m/R_m\xi_\nu$, ξ'_m is the image of ξ_m in R'_m, θ'_m is the induced by θ_m automorphism of R'_m. Clearly the equalities $R_{i-1}\xi_i + R_{i-1}\theta_i^N(\xi_i) = R_{i-1}$ imply the equalities $R'_{i-1}\xi_i + R'_{i-1}\theta_i'^N(\xi'_i) = R'_{i-1}$.

Denote by $\Phi_\nu : \mathcal{O}^- \to \mathfrak{R}' - mod$ the functor

$$\Phi_\nu(M) = \{m \in M | y_\nu m = 0\}.$$

If $M \in Ob\mathcal{O}^-$, then $\Phi_\nu(M) \in Ob\mathcal{O}'^-$, where $\mathcal{O}'^-$ is the corresponding category of $y_1, y_2, ..., y_{\nu-1}$ - locally finite $\mathfrak{R}'$-modules (since y_ν commutes with the y_i's). We have $x_\nu y_j = y_j x_\nu$ for all $j < \nu$. Therefore the corresponding induction functor $\Psi_\nu : \mathfrak{R}' - mod \to \mathfrak{R} - mod$ restricts to a functor

$$\Psi_\nu : \mathcal{O}'^- \to \mathcal{O}^-.$$

By Theorem 2.2 the functors (Ψ_ν, Φ_ν) are mutually inverse equivalences of categories $\mathcal{O}'^-$ and $\mathcal{O}^-$. Now apply induction on ν.

3. Examples. The GK dimension of modules of the category $\mathcal{O}$.

3.1. Examples.

3.1.1. The algebras $D_{\mathbf{q}}$ and $A_{\mathbf{q}}$. Consider the algebra $D_{\mathbf{q}}$ (cf. 1.3). It follows from 1.3.1(1) that $\theta_i^n(\xi_i) = q_{ii}^n \xi_i + \sum_{0\le j<n} q_{ii}^j$. Hence the ideal $R\xi_i + R\theta_i^n(\xi_i)$ contains the element $\sum_{0\le j<n} q_{ii}^j$. Suppose that one of the following conditions holds:

(a) $q_{ii} = 1$ and R is a $\mathbb{Q}$-algebra.

(b) $1 - q_{ii}^n$ is invertible for any positive integer n.

Then $\sum_{0\le j<n} q_{ii}^j$ is invertible for any $n \ge 1$; hence the condition of Proposition 2.3.1 holds. The same for the algebra $A_{\mathbf{q}}$ (cf. 1.4 and 1.5.4).

3.1.2. The Weyl algebra by Maltsiniotis. It follows from 1.6(3) that

$$\theta_m^{-N}(\xi_m) = \lambda_m^N \xi_m + (\sum_{0\le i<N} \lambda_m^i)(1 + \sum_{1\le j<m} (\lambda_j - 1)\xi_j) \tag{1}$$

It follows that if $N \ge 1$, the ideal in $\mathfrak{R}$ generated by the elements ξ_m and $\theta_m^N(\xi_m)$ coincides with the ideal generated by ξ_m and $v_{m,N} := (\sum_{0\le i<N} \lambda_m^i)(1+\sum_{1\le j<m}(\lambda_j - 1)\xi_j)$. Suppose that one of the following conditions holds:

(a) $\lambda_1 = 1$ and R is a $\mathbb{Q}$-algebra

(b) $1 - \lambda_1^N$ is invertible for any positive integer N.

Then the element $v_{1,N} = \sum_{0\le i<N} \lambda_m^i$ is invertible for any $N \ge 1$. But if $m \ge 2$, the elements ξ_m and $v_{m,N}$ generate a proper ideal in R and, therefore, in $\mathfrak{R}$; i.e. the conditions of Proposition 2.3.1 do not hold.

3.1.2.1. Note. Suppose that, for any $1 \le i < n$, one of the following conditions holds:

(a') $\lambda_i = 1$ and R is a $\mathbb{Q}$-algebra

(b') $1 - \lambda_i^N$ is invertible for any positive integer N.

Then the algebra $B_n^{\mathbf{q},\lambda}$ of 1.6.1 — the localization of $A_n^{\mathbf{q},\lambda}$ at the multiplicative set generated by the elements $\eta_i := x_i y_i - y_i x_i = \xi_i - \theta_i^{-1}(\xi_i)$, $1 \le i \le n$, satisfies the conditions of Proposition 2.3.1, since it is isomorphic to the algebra $A'_{\mathbf{q}}$ of 1.5 (cf. 1.6.1 and 3.1.1). To see this directly, one might use the fact that $v_{m,N} := (\sum_{0\le i<N} \lambda_m^i)\eta_{m-1}$. Each of the conditions (a'), (b') garantees that the ideal generated by ξ_m and $v_{m,N}$ is generated by ξ_m and η_{m-1}. And the element η_{m-1} becomes invertible in the algebra $B_n^{\mathbf{q},\lambda}$. ∎

3.1.3. The quantized Heisenberg and the Hayashi's Weyl algebras. In the case of the quantized Heisenberg algebra (cf. 1.2.4), we have:

$$\theta_i^{-N}(\xi_i) = q^N \xi_i - q^N (\sum_{1\le i\le N} q^{-2i}) z_i \tag{1}$$

for all nonnegative integers N. This implies that the ideal $R\xi_i + R\theta_i^N(\xi_i)$ is generated by ξ_i and $(\sum_{1\le i\le N} q^{-2i}) z_i$. It is a proper ideal of R for all N; that is the Kashiwara theorem fails for the Heisenberg algebras.

Note that, since z_i is a normal element (cf. 1.2.4), $S = (z_i^m | m \geq 1, \ i \in J)$ is a left and right Ore set, and the localization of $H_{\mathbf{q}}(J)$ at S is a hyperbolic algebra $R'\{\Theta', \xi'\}$, where $R' = S^{-1}R = k[(z_i, z_i^{-1}), (\xi_i)]$ and Θ' and ξ' are induced by Θ and ξ. Suppose one of the following conditions holds for all $i \in J$:

(a) $q_i = 1$ and R is a $\mathbb{Q}$-algebra

(b) $1 - q_i^{2N-1}$ is invertible for any positive integer N.

Then it follows from (1) that $R'\xi_i' + R'\theta_i^N(\xi_i') = R'$ for all $i \in J$; i.e. the conditions of the Kashiwara theorem hold. By the same reason they hold for the Hayashi's Weyl algebra $W_{\mathbf{q}}(J)$ (cf. 1.2.4).

3.1.4. The quantum coordinate algebra of SL(2). Let $R\{\theta, \xi\}$ be the quantum coordinate algebra of $SL(2, k)$; i.e.

$$R = k[u, v], \quad \theta f(u, v) = f(qu, qv) \quad \text{for any } f \in R, \quad \xi = 1 + quv$$

(cf. Example 1.2.7). Note that

$$\theta^n(\xi) = 1 + q^{2n+1}uv \quad \text{for any } n \geq 1$$

Since $\xi = 1 + quv$, this implies that the conditions of Theorem 2.2 hold, if q is not a root of one.

3.1.5. The quantum and classical enveloping algebras of $sl(2)$. Let now $R\{\theta, \xi\} = U(sl(2))$; i.e. $R = k[z, \xi]$, and $\theta f(z, \xi) = f(z + \alpha, \xi + z)$ (see 1.2.5). Then, for any $n \geq 1$,

$$\theta^{-n}(\xi) = \xi - \sum_{1 \leq i \leq n} \theta^{i-n}(u) = \xi - nz - \frac{(n-1)n}{2}\alpha \tag{1}$$

This implies that the ideal $R\xi + R\theta^{-n}(\xi)$ is generated by ξ and $z - \frac{n-1}{2}\alpha$. In particular, this ideal is proper for any n.

In the quantum case, $U_q(sl(2)) = R\{\theta, \xi\}$, where $R = k[z, z^{-1}\xi]$, and $\theta f(z, \xi) = f(qz, \xi + qz + \frac{z - z^{-1}}{q - q^{-1}})$ (see 1.2.5). Then, for any $n \geq 1$,

$$\theta^{-n}(\xi) = \xi + z^{-1}q^2(1 - q^{-n})((q^2 - 1)(1 - q))^{-1}(z^2 - q^{-n+1}) \tag{1}$$

This implies that the ideal $R\xi + R\theta^{-n}(\xi)$ is proper for any n.

3.2. The GK dimension of modules of the category $\mathcal{O}^-$. For any k-algebra A, we denote by $GK_l(A)$ the minimal value of GK-dimension of nonzero modules:

$$GK_l(A) = inf\{GK(M) | M \text{ is a nonzero } A\text{-module}\}$$

It follows that $GK_l(A) = 0$ iff there are nonzero A-modules of finite type over k. Otherwise $GK_l(A) \geq 1$ (cf. [McR], Proposition 8.1.17).

Let $\mathfrak{R} = R\{\theta, \xi; x, y\}$ be a hyperbolic algebra of the rank one over R. Set $S = \{y^n \mid n \geq 1\}$. For any $\mathfrak{R}$-module M, $t_S M$ denotes the S-torsion of M.

3.2.1. Proposition. *With the notations above, suppose that $R\xi + R\theta^N(\xi) = R$ for all $N \neq 0$. Then, for any $\mathfrak{R}$-module M having nonzero S-torsion, $GK(M) \geq 1 + GK_l(R/R\xi)$.*

Proof. Let $t_S M \neq 0$; or, equivalently, the $R/R\xi$-module $M_0 := \{v \in M | y \cdot v = 0\}$ is not equal to zero. By (the argument of) Theorem 2.2, $t_S M \simeq \mathfrak{R} \otimes_{\mathfrak{B}} M_0$, where $\mathfrak{B}$ is the subalgebra $R[y, y^{-1}; \theta^{-1}]$ of $\mathfrak{R}$. Since $\mathfrak{R} \otimes_{\mathfrak{B}} M_0 \simeq (\oplus_{n \geq 0} x^n M_0, \mu)$ (here μ denotes the action of $\mathfrak{R}$),

$$GK_{\mathfrak{R}}(t_S M) = GK_{R/R\xi}(M_0) + 1 \geq GK_l(R/R\xi) + 1$$

■

Let $\mathfrak{R}$ be a hyperbolic algebra of rank ν over R with the hyperbolic structure, $R_{m+1} = R_m\{\theta_m, \xi_m; x_m, y_m\}$. Denote by R_{-1} the quotient of the algebra R by the ideal generated by the elements ξ_m, $1 \leq m \leq \nu$. Let $\mathcal{O}^-$ be the Serre subcategory of $\mathfrak{R} - mod$ defined in 2.3: an $\mathfrak{R}$-module M belongs to $\mathcal{O}^-$ iff, for any $1 \leq i \leq \nu$, any element of M is annihilated by some power of y_i.

3.2.2. Proposition. *In the above notation suppose that the assumptions of the Proposition 2.3.1 are satisfied. Then*

$$inf\{GK(M) | M \in Ob\mathcal{O}^-\} = \nu + GK_l(R_{-1})$$

In particular, for any $\mathfrak{R}$-module M with the nonzero submodule in $\mathcal{O}^-$, $GK(M) \geq \nu + GK_l(R_{-1})$.

Proof. The assertion follows from Proposition 2.3.1 (see the argument of Proposition 3.2.1). ■

4. The GK-dimension of modules over hyperbolic algebras.

Theorem 2.2 and Proposition 2.3.1 allow to estimate, under certain conditions, the GK dimension of nonzero modules of the category $\mathcal{O}^-$. To get estimates for the GK-dimension of arbitrary modules over a hyperbolic algebra, we need some more subtle tools, those of the noncommutative local algebra developed in [R1], [R2]; in particular we need results on the spectrum of category of modules over a hyperbolic algebra ([R1], Ch.4). For the reader's convenience, we sketch below the necessary preliminaries on the spectrum. Then we apply the spectrum to obtain estimates on GK-dimension of modules which allow to establish easily the Bernstein's inequality for all known examples of hyperbolic algebras having this property.

4.0. Preliminaries on the spectrum. Fix an abelian category $\mathcal{A}$. For any two objects X, Y of $\mathcal{A}$, we write $X \succ Y$ if Y is a subquotient of a finite direct sum of copies of X (cf. Note 2.5.1). For any $X \in Ob\mathcal{A}$, denote by $\langle X \rangle$ the full subcategory of $\mathcal{A}$ such that $Ob\langle X \rangle = Ob\mathcal{A} - \{Y \in Ob\mathcal{A} \mid Y \succ X\}$. It is easy to check that $X \succ Y$ iff $\langle Y \rangle \subseteq \langle X \rangle$. This observation provides a convenient realization of the quotient of $(Ob\mathcal{A}, \succ)$ with respect to the equivalence relation induced by $\succ$: $X \approx Y$ if $X \succ Y \succ X$. Namely, $(Ob\mathcal{A}, \succ)/\approx$ is isomorphic to $(\{\langle X \rangle \mid X \in Ob\mathcal{A}\}, \supseteq)$.

Set $Spec\mathcal{A} = \{P \in Ob\mathcal{A} \mid P \neq 0$, and for any nonzero subobject X of $P, X \succ P\}$. *The spectrum*, **Spec**$\mathcal{A}$, *of the category* $\mathcal{A}$ is the preordered set of equivalence (with respect to $\succ$) classes of objects of $Spec\mathcal{A}$. The canonical realization of $(Ob\mathcal{A}, \succ)/\approx$ induces a canonical realization of $\mathbf{Spec}\mathcal{A}$: $(\mathbf{Spec}\mathcal{A} = \{\langle P\rangle \mid P \in Spec\mathcal{A}\}, \supseteq)$.

4.0.1. Proposition. *For any $P \in Spec\mathcal{A}$, the subcategory $\langle P\rangle$ is a Serre subcategory of $\mathcal{A}$. If $\mathcal{A}$ is a category with the property (sup), then the converse is true: if X is an object of $\mathcal{A}$ such that $\langle X\rangle$ is a Serre subcategory of $\mathcal{A}$, then X is equivalent (in the sense of $\succ$) to* a *$P \in Spec\mathcal{A}$; i.e. $\langle X\rangle = \langle P\rangle$.*

Proof. See Proposition 2.3.3 and 2.4.7 in [R1]. ∎

A nonzero object X of a category $\mathcal{A}$ is called *quasifinal* if, for any nonzero object Y of $\mathcal{A}$, $Y \succ X$. The category $\mathcal{A}$ having a quasifinal objects is called *local*.

One can check that all simple objects of a local category (if any) are isomorphic to each other. In particular, the category of left modules over a commutative ring R is local iff the ring R is local.

4.0.2. Proposition. *And the quotient category $\mathcal{A}/\langle P\rangle$ is local.*

Proof. See Proposition 3.3.1 and Corollary 3.3.2 in [R1]. ∎

4.0.3. Proposition. *(a) For any topologizing (i.e. full and closed with respect to taking direct sums and subquotients) subcategory $\mathbb{T}$ of $\mathcal{A}$, the inclusion functor $\mathbb{T} \longrightarrow \mathcal{A}$ induces an embedding* $\mathbf{Spec}\mathbb{T} \longrightarrow \mathbf{Spec}\mathcal{A}$.

(b) For any exact localization $Q : \mathcal{A} \longrightarrow \mathcal{A}/\mathbb{S}$ and for any $P \in Spec\mathcal{A}$, either $P \in Ob\mathbb{S}$, or $Q(P) \in Spec\mathcal{A}/\mathbb{S}$; hence Q induces an injective map from $\mathbf{Spec}\mathcal{A} - \mathbf{Spec}\mathbb{S}$ *to* $\mathbf{Spec}\mathcal{A}/\mathbb{S}$.

4.0.4. Localizations at subsets of the spectrum. For any subset U of $\mathbf{Spec}\mathcal{A}$, denote by $\langle\langle U\rangle\rangle$ the intersection $\bigcap_{\langle P\rangle \in U}\langle P\rangle$. Being the intersection of a set of Serre subcategories, $\langle\langle U\rangle\rangle$ is a Serre subcategory. *A localization at the subset U* is a localization at the Serre subcategory $\langle\langle U\rangle\rangle$.

4.0.5. The support of an object. For any $M \in Ob\mathcal{A}$, *the support of M*, **Supp**(M), consists of all $\langle P\rangle \in \mathbf{Spec}\mathcal{A}$ such that $M \notin Ob\langle P\rangle$.

4.0.6. Associated points. Let $M \in Ob\mathcal{A}$. An element $\langle P\rangle \in \mathbf{Spec}\mathcal{A}$ is called *an associated point of M* if there exists a monomorphism $P \to M$. The set of all associated points of M is denoted by $Ass(M)$. It follows that $Ass(M) \subset Supp(M)$.

4.0.7. The spectrum and the GK-dimension. Suppose that $\mathcal{A} = R - mod$ for a finitely generated k-algebra R. The GK-dimension (as well as any other known dimension function) has the property:

(#) For any $M, M' \in Ob\mathcal{A}$ such that $M \succ M'$, $GK(M) \geq GK(M')$.

The immediate consequences of this properties:

(a) $GK(M)$ depends only on the equivalence class of M, i.e. $GK(M) = GK(\langle M\rangle)$

(b) $GK(M) \geq sup(GK(\langle P\rangle)|\langle P\rangle \in Supp(M)) \geq sup(GK(\langle P\rangle)|\langle P\rangle \in Ass(M))$.

Moreover, there is the following fact:

(c) Suppose that the module M has a property: for any nonzero submodule M' of M, $Ass(M') \neq 0$. Then $sup(GK(\langle P\rangle)|\langle P\rangle \in Supp(M)) = sup(GK(\langle P\rangle)|\langle P\rangle \in Ass(M))$. ∎

4.1. The case of the rank one.

4.1.1. Lemma. *Let $\mathfrak{R} = R\{\theta, \xi; x, y\}$ be a hyperbolic algebra over R. And let M be an S-torsion free $\mathfrak{R}$-module, $S = \{y^n \mid n \geq 1\}$.*

If there is $\langle P \rangle \in Ass({}_R M)$ such that the orbit $\{\theta^n(\langle P \rangle) | n \in \mathbb{Z}\}$ is infinite, then $GK(M) \geq GK_R(P) + 1$.

Proof. If the orbit $\{\theta^n(\langle P \rangle) | n \in \mathbb{Z}\}$ is infinite, then by Theorem IV.6.4 in [R1], the canonical morphism of the $R[y, y^{-1}; \theta^{-1}]$-module $\oplus_{n \in \mathbb{Z}} \theta^n(P)$ to M is a monomorphism. Therefore $GK(M) \geq GK_B(\oplus_{n \in \mathbb{Z}} \theta^n(P)) = GK_R(P) + 1$. ■

4.1.2. Proposition. *Let $\mathfrak{R} = R\{\theta, \xi; x, y\}$ be a hyperbolic algebra over R such that $R\xi + R\theta^N(\xi) = R$ for all $N \neq 0$. Suppose that, for any nonzero R-module V, $Ass(V) \neq \emptyset$. Then*

$$GK_l(\mathfrak{R}) = min\{1 + GK_l(R_{-1}), 1 + GK_\infty, GK_f\} \tag{1}$$

where

$$GK_\infty := inf\{GK(P) | P \in Spec(R - mod) \text{ and the orbit } (\theta^n(P) | n \in \mathbb{Z}) \text{ is infinite }\}$$

$$GK_f := inf\{GK(P) | P \in Spec(R - mod) \text{ and the orbit } (\theta^n(P) | n \in \mathbb{Z}) \text{ is finite }\}$$

Proof. (a) Suppose $P \in Spec(R - mod)$ and the orbit $(\theta^n(P) | n \in \mathbb{Z})$ is infinite. Then $GK(\mathfrak{R} \otimes_R P) = GK(P) + 1$.

(b) If the orbit $(\theta^n(P) | n \in \mathbb{Z})$ is finite. Then $GK(\mathfrak{R} \otimes_R P) = GK(P)$.

It follows from (a) and (b) that $GK_l(\mathfrak{R}) \leq min\{1 + GK_l(R_{-1}), 1 + GK_\infty, GK_f\}$.

Let (M, m) be an $\mathfrak{R}$-module. By assumption, $Ass(M) \neq \emptyset$; i.e. there exists a monomorphism $P \to M$, $P \in Spec(R - mod)$. Clearly $GK(M) \geq GK(P)$. If the orbit $\{\theta^n(\langle P \rangle) | n \in \mathbb{Z}\}$ is infinite, then by Theorem IV.6.4 in [R1], the canonical morphism of the $R[y; \theta^{-1}]$-module $\oplus_{n \in \mathbb{Z}} \theta^n(P)$ to M is a monomorphism. Therefore, in this case, $GK(M) \geq GK_B(\oplus_{n \in \mathbb{Z}} \theta^n(P)) = GK_R(P) + 1$. ■

4.1.3. Corollary. *Let $\mathfrak{R} = R\{\theta, \xi; x, y\}$ be a hyperbolic algebra over R such that $R\xi + R\theta^N(\xi) = R$ for all $N \neq 0$. Suppose that the following conditions hold:*

(a) For any nonzero R-module V, $Ass(V) \neq \emptyset$.

(b) For any $\langle P \rangle \in Spec(R - mod)$ such that the orbit $\{\theta^n(\langle P \rangle) | n \in \mathbb{Z}\}$ is finite, $GK(P) \geq 1$.

Then, for any nonzero $\mathfrak{R}$-module M, $GK(M) \geq 1$.

4.2. The general case. Let $\mathfrak{R}$ be a hyperbolic algebra over R with the hyperbolic structure $(R_m)_{0 \leq m \leq \nu}$, $R_0 = R$, $R_{m+1} = R_m\{\theta_m, \xi_m; x_{m+1}, y_{m+1}\}$. We assume that R contains all ξ_m (cf. 1.1.3). We assume in addition that the algebra R is θ_m-stable for all m. Denote by ϑ_m the automorphism of the algebra R induced by θ_m. Let Γ denote the group $\mathbb{Z}^\nu$ and Θ the group homomorphism $\Gamma \to Aut(R)$ which sends the m-th canonical generator of Γ into ϑ_m. For any $P \in Spec(R - mod)$, denote by Γ_P the stabilizer of P; i.e. $\Gamma_P := \{\gamma \in \Gamma | \Theta_\gamma(P) = P\}$. Finally, we define the $\mathbb{Q}$-*rank of an abelian group* H by $rk_{\mathbb{Q}}(H) := dim_{\mathbb{Q}} \mathbb{Q} \otimes H$.

4.2.1. Proposition. *Let $\mathfrak{R}$ be a hyperbolic algebra over R satisfying the assumptions above. Suppose that $R_i\xi_i + R_i\theta_i^N(\xi_i) = R_i$ for all $i \geq 0$ and for all $N \neq 0$. Let $\mathfrak{M} = (M, m)$ be an $\mathfrak{R}$-module, and let $P \in Ass(M)$. Then $GK(\mathfrak{M}) \geq rk_{\mathbb{Q}}(\Gamma/\Gamma_P) + GK(P)$.*

Proof. Since $P \in Spec(R - mod)$, either P is annihilated by all ξ_i, or there is i, $1 \leq i \leq N$, such that ξ_i acts injectively on P.

(a) In the first case, the stabilizer Γ_P of P is trivial, hence $rk_{\mathbb{Q}}(\Gamma/\Gamma_P) = rk_{\mathbb{Q}}(\Gamma) = \nu$. And it follows from Proposition 2.3.1 that $GK(M) \geq \nu + GK(P) = rk_{\mathbb{Q}}(\Gamma/\Gamma_P) + GK(P)$.

(b) Consider the second case. Take the maximal i such that ξ_i acts injectively on P.

(b1) Suppose that $i < \nu$. Let S_i denote the multiplicative set of monomials in y_j, where $i < j \leq \nu$. This is a (right and left) Ore set. Denote by $\mathfrak{M}' = (M', m')$ the S_i-torsion of M. Clearly P is a subobject of M'. Let $\mathfrak{M}'_i$ denote the image of $\mathfrak{M}'$ under the canonical functor $\mathfrak{R} - mod \to R_i - mod$. By induction hypothesis, there exists a subobject $P' \to \mathfrak{M}'_i$ such that $P' \in Spec(R_i - mod)$ and P is a subobject of P'_0. The fact that ξ_j annihilate P for $j > i$ implies that they annihilate the R_i-submodule P'. Hence $GK(\mathfrak{M}) \geq \nu - i + GK(P')$. By induction hypothesis, $GK(P') \geq rk_{\mathbb{Q}}(\Gamma'/\Gamma_P) + GK(P)$. Here $\Gamma' = \mathbb{Z}^i$. It follows that $\nu - i + rk_{\mathbb{Q}}(\Gamma'/\Gamma_P) = rk_{\mathbb{Q}}(\Gamma/\Gamma_P)$.

(b2) Suppose that $i = \nu$; i.e. the action of ξ_ν is injective. We have two cases: the orbit $\Omega_{\nu,P} = (\vartheta_\nu^n(P) | n \in \mathbb{Z})$ might be either finite, or infinite.

(b2.1) Suppose that the orbit $\Omega_{\nu,P}$ is finite. By induction hypothesis, we have $GK(\mathfrak{M}) \geq GK(\mathfrak{M}_{\nu-1}) \geq rk_{\mathbb{Q}}(\Gamma''/\Gamma''_P) + GK(P)$, where $\Gamma'' = \mathbb{Z}^{\nu-1}$. Since the orbit $\Omega_{\nu,P}$ is finite, $rk_{\mathbb{Q}}(\Gamma''/\Gamma''_P) = rk_{\mathbb{Q}}(\Gamma/\Gamma_P)$.

(b2.2) Suppose that the orbit $\Omega_{\nu,P}$ is infinite. We can assume that it contributes to the rank $rk_{\mathbb{Q}}(\Gamma/\Gamma_P)$. By induction hypothesis, there exists a subobject $P' \to \mathfrak{M}'_{\nu-1}$ such that $P' \in Spec(R_{\nu-1} - mod)$ and P is a subobject of P'_0. The fact that $\Omega_{\nu,P}$ contributes to the rank means that the orbit $(\theta_\nu^n(P) | n \in \mathbb{Z})$ is infinite. This implies that the canonical morphism $\oplus_{n\in\mathbb{Z}}\theta^n(P') \to \mathfrak{M}$ induced by the embedding $P' \to \mathfrak{M}'_{\nu-1}$ is a monomorphism. Therefore $GK(\mathfrak{M}) \geq 1 + GK(P')$. By induction hypothesis, $GK(P') \geq rk_{\mathbb{Q}}(\Gamma/\Gamma_P) - 1 + GK(P)$; hence the required inequality: $GK(\mathfrak{M}) \geq rk_{\mathbb{Q}}(\Gamma/\Gamma_P) + GK(P)$. ∎

4.2.2. Proposition. *Let $\mathfrak{R}$ be a hyperbolic algebra over R satisfying the assumptions above. Suppose that the following conditions hold:*

(a) $R_i\xi_i + R_i\theta_i^N(\xi_i) = R_i$ for all $i \geq 0$ and for all $N \neq 0$.

(b) For any nonzero R-module V, $Ass(V) \neq \emptyset$.

Then

$$GK_l(\mathfrak{R}) = min\{rk_{\mathbb{Q}}(\Gamma/\Gamma_P) + GK(P) | P \in Spec(R - mod)\} \tag{1}$$

or, what is the same

$$GK_l(\mathfrak{R}) = min\{\nu + GK_l(R_{-1}), rk_{\mathbb{Q}}(\Gamma/\Gamma_P) + GK(P) | P \in U(\xi)\} \tag{1'}$$

where $U(\xi)$ is the complement to the Zariski closed set defined by $\xi = (\xi_i | 1 \leq i \leq \nu)$; i.e. $U(\xi) = Spec(R - mod) - Spec(R_{-1} - mod)$.

Proof. The inequality

$$GK_l(\mathfrak{R}) \geq min\{rk_{\mathbb{Q}}(\Gamma/\Gamma_P) + GK(P) | P \in Spec(R - mod)\}$$

follows from Proposition 4.2.1. It remains to show the inverse inequality. The latter follows from the description of the spectrum of a hyperbolic algebra (cf. [R1], Ch. IV and Ch. V). ■

4.2.2.1. Corollary. *Under the assumptions (a), (b) of Proposition 4.2.2, the following conditions are equivalent:*

(i) $GK_l(\mathfrak{R}) = 0$

(ii) There exists $P \in Spec(R - mod)$ *of finite type over* k *and such that the orbit of* P *under the action of* Γ *is finite.*

Proof. It follows from 4.2.2 that $GK_l(\mathfrak{R}) = 0$ iff there exists $P \in Spec(R - mod)$ such that $rk_{\mathbb{Q}}(\Gamma/\Gamma_P) = 0$ and $GK(P) = 0$. The latter equality implies that P is of finite type over k. The equality $rk_{\mathbb{Q}}(\Gamma/\Gamma_P) = 0$ means that the orbit of P under the action of Γ is finite. ■

5. The Bernstein's inequality. Examples.

5.1. The Bernstein's inequality. Let $\mathfrak{R}$ be a hyperbolic k-algebra over R with the hyperbolic structure $(R_m)_{0 \le m \le \nu}$, $R_0 = R$, $R_{m+1} = R_m\{\theta_m, \xi_m; x_{m+1}, y_{m+1}\}$. We shall say that *the Bernstein's inequality holds for an* $\mathfrak{R}$*-module* M if $GK(M) \ge \nu$. We shall say that *the Bernstein's inequality holds for* $\mathfrak{R}$ if it holds for any nonzero $\mathfrak{R}$-module.

Thus Theorem 2.2 and Proposition 2.3.1 assert that, under certain conditions, the Bernstein's inequality holds for all nonzero modules of the category $\mathcal{O}^-$.

5.1.1 Proposition. *Let* $\mathfrak{R}$ *be a hyperbolic algebra over* R *satisfying the assumptions above. Suppose that the following conditions hold:*

(a) $R_i\xi_i + R_i\theta_i^N(\xi_i) = R_i$ *for all* $i \ge 0$ *and for all* $N \ne 0$.

(b) For any nonzero R*-module* V, $Ass(V) \ne \emptyset$.

(c) For any $P \in Spec(R - mod)$, $GK(P) \ge rk_{\mathbb{Q}}(\Gamma_P)$.

Then the Bernstein's inequality holds for any nonzero $\mathfrak{R}$*-module* M.

Proof. The assertion follows from Proposition 4.2.2. ■

5.1.2. Note. Any of the conditions (a) and (b) of Proposition 5.1.1 cannot be dropped. For instance, the condition (a) (i.e. that of Theorem 2.2) holds for the algebra D_q of q-differential operators (cf. 1.3 and 3.1.1) if q is not a root of one, but this algebra has a family of one dimensional modules. In fact, $D_q = R\{\theta, \xi\}$, where $R = k[\xi]$, $\theta(\xi) = q\xi + 1$. One can check that $\eta := (1 - q)\xi - 1$ is an eigen vector of θ: $\theta(\eta) = q\eta$. Therefore $D_q\eta$ is a two-sided ideal, and $D_q/D_q\eta \simeq k[x, y]/((q - 1)xy - 1) \simeq k[x, x^{-1}]$.

Similarly, the quantum coordinate algebra of $SL(2)$, $A(SL_q(2))$ (cf. 3.1.4), satisfies the condition (a) if q is not a root of one, but it also has a family of one dimensional representations.

On the other hand, the enveloping algebra of $sl(2)$ over the field of zero characteristic and the quantized enveloping algebra $U_q(sl(2))$ in the case when q is not a root of one (cf. 3.1.5) satisfy both the condition (b) and have finite dimensional representations. ■

5.2. The 'classical' Bernstein's inequality. Consider the n-th Weyl algebra A_n over a field k of zero characteristic as a hyperbolic algebra over the algebra $R = k[\xi_1, ..., \xi_n]$,

$\theta_i(\xi_j) = \xi_j + \delta_i j$ (cf. example 1.2.3). Then $\Gamma = \mathbb{Z}^n$, and, for any $P \in Spec(R - mod)$, $GK(P) \geq rk_{\mathbb{Q}}(\Gamma_P)$; i.e. the conditions of Proposition 5.1.1 hold. Therefore we have obtained, among other things, another proof of the Bernstein's inequality: $GK(M) \geq n$ for any nonzero module M.

5.3. The Bernstein's inequality for Hayashi's Weyl algebras. Consider the n-th Hayashi's Weyl algebra with the hyperbolic structure of 1.2.4; i.e. R is the quotient of the algebra $k[(z_i),(\xi_i)]$ of polynomials by the relations $z_i(\xi_i(1-q^2)+q^2 z_i) = 1, \quad i \in J.$, and the automorphisms ϑ_i, $i \in J$ are defined by the same formulas as for the Heisenberg quantized algebra $H_q(J)$; i.e.

$$\vartheta_i(z_i) = qz_i, \quad \vartheta_i(\xi_i) = q^{-1}\xi_i + qz_i \quad \text{for all } i \in J \tag{1}$$

$$\vartheta_i(z_j) = z_j, \quad \vartheta_i(\xi_j) = \xi_j \quad \text{if } j \neq i$$

It follows from (1) that, for any $P \in Spec(R - mod)$, $GK(P) \geq rk_{\mathbb{Q}}(\Gamma_P)$; i.e. the conditions of Proposition 5.1.1 hold. Therefore in the case of Hayashi's algebras we also have the Bernstein's inequality: $GK(M) \geq n$ for any nonzero module M.

5.4. The algebra of differential operators on a quantum space. Consider the algebra $D_{\mathbf{q}}$ of differential operators on a quantum space of dimension n (cf. 1.3) regarded as a hyperbolic algebra over the polynomial algebra $R = k[(\xi_i)]$ (see 1.3.1). The restriction of the automorphisms θ_i to the algebra R is determined by

$$\theta_i(\xi_j) = \xi_j \text{ if } i \neq j;\ \theta_i(\xi_i) = q_{ii}\xi_i + 1 \tag{1}$$

If $q_{ii} \neq 1$ for all i, then R has one dimensional (hence simple) Γ-stable modules. This modules induce one-dimensional (in particular of the GK dimension zero) modules over the algebra $D_{\mathbf{q}}$.

5.5. The quantum Weyl algebra. In the case of the quantum Weyl algebra $A_{\mathbf{q}}$ (cf. 1.5, in particular 1.5.4), one can check that, for any $P \in Spec(R - mod)$, $GK(P) \geq rk_{\mathbb{Q}}(\Gamma_P)$. Hence the Bernstein's inequality holds: $GK(M) \geq n$ for any nonzero module M.

5.6. The quantum Weyl algebra by Maltsiniotis. Consider the quantum Weyl algebra by Maltsiniotis $A_n^{\mathbf{q},\lambda}$ (cf. 1.6). In this case, like in the case of the algebra $D_{\mathbf{q}}$ of 1.3, there are, in general, finite dimensional nonzero modules. An appropriate canonical localization (the one explained in 3.1.2.1) of $A_n^{\mathbf{q},\lambda}$ is isomorphic to the algebra $A_{\mathbf{q}}$. Therefore the Bernstein's inequality holds for all modules over $A_n^{\mathbf{q},\lambda}$ which are not in the kernel of this localization.

5.7. Note. The Bernstein's inequality for the Hayashi's Weyl algebras and the localized Weyl algebra by Maltsiniotis (the algebra $A_{\mathbf{q}}$) was obtained by L. Rigal [Ri1], [Ri2] by different methods.

Complements: Kashiwara theorem for hyperbolic categories.

Although Theorem 2.2 suffices for our immediate needs, it is more convenient to have its relative analog which allows to work, for example, with categories of sheaves of hyperbolic algebras and modules over them. With more reason that both assumptions and the argument look more naturally in the case of hyperbolic categories than in the case of hyperbolic rings.

C.1. Hyperbolic categories. Let θ be an auto-equivalence of an additive category $\mathcal{A}$; and let ξ be an endomorphism of the identical functor of $\mathcal{A}$.

Denote by $\mathcal{A}\{\theta,\xi\}$ the category objects of which are triples (γ, M, η), where $M \in Ob\mathcal{A}$ and $\gamma : M \longrightarrow \theta(M), \eta : \theta(M) \longrightarrow M$ are arrows such that

$$\eta \circ \gamma = \xi(M) \text{ and } \gamma \circ \eta = \xi\theta(M).$$

Morphisms from (γ, M, η) to (γ', M', η') are those morphisms f from M to M' for which the diagram

$$\begin{array}{ccccc} M & \xrightarrow{\gamma} & \theta(M) & \xrightarrow{\eta} & M \\ f\downarrow & & \downarrow\theta(f) & & \downarrow f \\ M' & \xrightarrow{\gamma'} & \theta(M') & \xrightarrow{\eta'} & M' \end{array}$$

is commutative.

The category $\mathcal{A}\{\theta,\xi\}$ will be called *hyperbolic.*

C.1.1. Example. Let R be an associative ring, ϑ an automorphism of R, $\xi' \in R$ a central element, and $R\{\vartheta,\xi'\}$ the related to this data hyperbolic ring.

The category $R\{\vartheta,\xi'\} - mod$ is hyperbolic.

Namely, $R\{\vartheta,\xi'\}$-*mod* is equivalent to the category $\mathcal{A}\{\theta,\xi\}$, where $\mathcal{A} = R$-*mod*, θ is an auto-equivalence of the category $\mathcal{A}$ induced by the automorphism ϑ (cf. 2.1), ξ is the endomorphism of the identical functor, $Id_{\mathcal{A}}$, which assigns to every R-module M the action of the element ξ' on M; i.e. $\xi(w) := \xi' \cdot w$ for each $w \in M$. ■

C.2. Kashiwara theorem for hyperbolic categories. Fix a hyperbolic category $\mathcal{A}\{\theta,\xi\}$ over an abelian category $\mathcal{A}$. Fix an object (s, M, t) in $\mathcal{A}\{\theta,\xi\}$; and denote by $M^{(n)}$ the kernel of the morphism $s^{(n)} := \theta^n s \circ \ldots \circ \theta s \circ s : M \to \theta^{n+1}(M)$. Set $M^{(\infty)} := \sup\{M^{(n)} \mid n \geq 0\}$.

C.2.1. Lemma. *The subobject $M^{(\infty)}$ of M has (necessarily unique) structure of a subobject of (s, M, t).*

Proof. 1) Note that t sends $M^{(n)}$ into $M^{(n+1)}$.

In fact,

$$s^{(n+1)} \circ t = \theta(s^{(n)}) \circ s \circ t = \theta(s^{(n)}) \circ \xi\theta(M) = \xi\theta^{n+1}(M) \circ \theta(s^{(n)}).$$

Therefore, if $\iota^{(n)}$ is the monomorphism $M^{(n)} \to M$, then

$$s^{(n+1)} \circ t \circ \theta\iota^{(n)} = \xi\theta^{n+1}(M) \circ \theta(s^{(n)}) \circ \theta\iota^{(n)} = \xi\theta^{n+1}(M) \circ \theta(s^{(n)} \circ \iota^{(n)}) = 0$$

which is required to show.

2) Even more transparent is the fact that the morphism s sends the subobject $M^{(n)}$ into $\theta(M^{(n-1)})$, $n \geq 1$.

Indeed, since θ is left exact, $\theta\iota^{(n-1)}) : \theta(M^{(n-1)}) \to \theta(M)$ is the kernel of $\theta(s^{(n-1)})$. But we have: $\theta(s^{(n-1)}) \circ s \circ \iota^{(n)} = s^{(n)} \circ \iota^{(n)} = 0$.

2) Since the category $\mathcal{A}$ has the property (sup), it follows that t sends the subobject $\theta(M^{(\infty)})$ to $M^{(\infty)}$, and s sends $M^{(\infty)}$ to $\theta(M^{(\infty)})$. ■

Consider the full subcategory $\mathbb{S}$ of $\mathcal{A}\{\theta,\xi\}$ generated by objects (s, M, t) such that the natural monomorphism $M^{(\infty)} \to M$ is an isomorphism. One can check that $\mathbb{S}$ is a thick subcategory of $\mathcal{A}\{\theta,\xi\}$.

C.2.2. Corollary. *The subcategory $\mathbb{S}$ is a coreflective (hence Serre) subcategory of $\mathcal{A}\{\theta,\xi\}$.*

Proof. One can see that the map assigning to any object (s, M, t) of $\mathcal{A}\{\theta,\xi\}$ the object $(s^{(\infty)}, M^{(\infty)}, t^{(\infty)})$ defined in Lemma C.2.1 extends naturally to a functor from $\mathcal{A}\{\theta,\xi\}$ to $\mathbb{S}$ which is right adjoint to the embedding $\mathbb{S} \to \mathcal{A}\{\theta,\xi\}$. ■

Consider the Verma functor $\mathcal{M}_+ : \mathcal{A}_\xi \to \mathcal{A}\{\theta,\xi\}$ defined in [R4], 5.8. Here $\mathcal{A}_\xi$ is the full (actually, topologizing) subcategory of $\mathcal{A}$ generated by $V \in Ob\mathcal{A}$ such that $\xi(V) = 0$. The functor $\mathcal{M}$ assigns to any object V the triple $(g_+, \theta_+(V), h_+)$, where $\theta_+ :=_{n\geq 0} \theta^n$, $h_+ : \theta \circ \theta_+ \to \theta_+$ and $g_+ : \theta_+ \to \theta \circ \theta_+$ are defined by:

$$h_{+i} = id : \theta \circ \theta^{i-1} \to \theta^i, \quad g_{+i} = \xi\theta^i : \theta^i \to \theta \circ \theta^{i-1}$$

for $i \geq 1$.

One can check that the functor $\mathcal{M}_+$ takes values in the Serre subcategory $\mathbb{S}$; so that it induces a functor $\mathcal{M}$ from $\mathcal{A}_\xi$ to $\mathbb{S}$.

On the other hand, there is a functor $\Phi : \mathbb{S} \to \mathcal{A}_\xi$ which assigns to any object (s, M, t) the kernel of s.

We leave to a reader to check that the functor Φ is right adjoint to the functor $\mathcal{M}$. It follows from the definition of $\mathcal{M}$ that $\mathcal{M}$ is exact and the adjunction arrow $Id_{\mathcal{A}_\xi} \to \Phi \circ \mathcal{M}$ is an isomorphism. The latter means exactly that the functor $\mathcal{M}$ is fully faithful.

Dually, define the full subcategory $\mathbb{S}_-$ of $\mathcal{A}\{\theta,\xi\}$ generated by all those objects (s, M, t) for which the canonical arrow from $M_{(\infty)} := \sup_{n\geq 0}(Ker(t \circ \theta t \circ \ldots \circ \theta^n t))$ to M is an isomorphism. We have a functor $\mathcal{M}_-$ from $\mathcal{A}_{\xi\theta}$ to $\mathcal{A}\{\theta,\xi\}$ which takes values in $\mathbb{S}_-$, hence induces a functor $\mathcal{M}' : \mathcal{A}_{\xi\theta} \to \mathbb{S}_-$. Formally, all this can be obtained by switching to the adjoint hyperbolic category $\mathcal{A}\{\theta^\wedge, \xi^\wedge\}$ and using the canonical equivalence of categories $\mathcal{A}\{\theta,\xi\} \to \mathcal{A}\{\theta^\wedge, \xi^\wedge\}$ (cf. [R4], 5.8). Recall that $\theta^\wedge$ is an adjoint to θ functor, and $\xi^\wedge := \sigma \circ \theta^\wedge \xi\theta \circ \sigma^{-1}$, where $\sigma : \theta^\wedge \circ \theta \to Id_\mathcal{A}$ is an adjunction isomorphism.

C.2.3. Theorem. *The following conditions are equivalent:*

(a) If $V \in Ob\mathcal{A}$ is such that $\xi(V) = 0$ and $\xi\theta^n(V) = 0$ for some $n > 0$, then $V = 0$;

(b) the functor $\mathcal{M} : \mathcal{A}_\xi \to \mathbb{S}$ is an equivalence of the categories.

(c) the functor $\mathcal{M}' : \mathcal{A}_{\xi\theta} \to \mathbb{S}_-$ is an equivalence of the categories.

Proof. $(a) \Leftrightarrow (b)$. Suppose there exists a nonzero object V of the category $\mathcal{A}$ such that $\xi(V) = 0$ and $\xi\theta^n(V) = 0$ for some $n \neq 0$. By assumption on $\mathcal{A}$, Supp(V)$\neq$; i.e. there exists $P \in Spec\mathcal{A}$ such that $V \succ P$. The latter implies that the object P satisfies the same conditions: $\xi(P) = 0$, $\xi\theta^n(P) = 0$; in particular, $P \in Spec\mathcal{A} \cap Ob\mathcal{A}_\xi = Spec\mathcal{A}_\xi$. It follows from Theorem 6.2 in [R4] that $\mathcal{M}_+(P) \notin Spec\mathcal{A}\{\theta, \xi\}$. Since $\mathbb{S}$ is a topologizing subcategory of $\mathcal{A}\{\theta, \xi\}$ and $\mathcal{M}(P)$ belongs to $\mathbb{S}$, $\mathcal{M}(P) \notin Spec\mathbb{S}$(=Spec$\mathcal{A}\{\theta, \xi\} \cap Ob\mathbb{S}$). But, if the functor $\mathcal{M}$ were an equivalence of categories, it should send objects of the spectrum to objects of the spectrum.

(b) $\Leftrightarrow$ *(a)*. (i) Note first that the group $Aut(\mathcal{A})$ of isomorphy classes of auto-equivalences of $\mathcal{A}$ acts (by conjugations) on the center $\mathcal{C}(\mathcal{A}) := End(Id_\mathcal{A})$ of $\mathcal{A}$. This action is defined as follows.

Given an element $\lambda \in \mathcal{C}(\mathcal{A})$ and an auto-equivalence Θ of $\mathcal{A}$ set $ad\Theta(\lambda)$ equal to the composition $\Theta\lambda\Theta^\wedge$ with the adjunction isomorphisms $Id \to \Theta \circ \Theta^\wedge$ and $\Theta \circ \Theta^\wedge \to Id$. One can see that $ad\Theta$ coincides with $ad\Theta'$ if Θ is isomorphic to Θ'.

We need this here for the case $\lambda = \xi$ and $\Theta = \theta^n, n \in \mathbb{Z}_+$. To simplify the notations, we shall write ξ_n instead of $ad\theta^n(\xi)$.

Fix $(s, M, t) \in Ob\mathbb{S}$. For any $n \in \mathbb{Z}_+$, set $M_n := Ker\xi_n(M)$.

(ii) Note that $M_n \cap M_m = 0$ if $n \neq$m.

In fact, the object $W := M_n \cap M_m$ has the property $\xi_n(W) = 0 = \xi_m(W)$; or, equivalently, $\xi\theta^{\wedge n}(W) = 0$ and $\xi\theta^{\wedge m}(W) = 0$. Suppose that $n > m$. The equalities above are equivalent to $\xi(\theta^{\wedge n}(W)) = 0, \xi\theta^{m-n}(\theta^{\wedge n}(W)) = 0$. By the condition (a), they imply that the object $\theta^{\wedge n}(W)$ equals to zero, or, which is the same, $W = 0$.

(iii) For any $n \geq 1, \xi(M_n)$ is an isomorphism.

Indeed, the condition *(a)* is equivalent to the following one: $Ker\xi_n \cap Ker\xi = 0$ for all $n > 0$. Since $\xi_n(M_n) = 0$, we have: $0 = Ker\xi_n(M_n) \cap Ker\xi(M_n) = Ker\xi(M_n)$; i.e. $\xi(M_n)$ is a monomorphism.

On the other hand, the object $W := Cok\xi(M_n)$, being a quotient object of M_n, has the property $\xi_n(W) = 0 = \xi(W)$. Therefore $W = 0$; i.e. $\xi(M_n)$ is also an epimorphism.

(iii′) Similarly, one can see that the condition *(a)* is equivalent to the requirement $Ker\xi\theta \cap Ker\xi_n = 0$ for all $n \geq 0$. This implies, by the same argument as in *(iii)*, that $\xi\theta(M_n)$ is an isomorphism for any $n \geq 0$.

(iv) The arrow $t : \theta(M) \to M$ induces, for any $n \geq 0$, an isomorphism from $\theta(M_n)$ to M_{n+1}; and $s : M \to \theta(M)$ induces, for any $n \geq 1$, an isomorphism from M_n to $\theta(M_{n-1})$.

Denote by ι_n the monomorphism $M_n \to M$. We have:

$$\xi_{n+1}(M) \circ t \circ \theta\iota_n = t \circ \xi_{n+1}\theta(M) \circ \theta\iota_n = 0$$

because $\xi_{n+1}\theta = \theta\xi_n$; so that $\xi_{n+1}\theta(M) \circ \theta\iota_n = \theta(\xi_n(M) \circ \iota_n) = 0$. This shows that t maps $\theta(M_n)$ to M_{n+1}. Similarly, the equalities

$$0 = s \circ \xi_n(M) \circ \iota_n = \xi_n\theta \circ s \circ \iota_n = \theta\xi_{n-1} \circ s \circ \iota_n$$

imply that s maps M_n to $\theta(M_{n-1})$, $n > 0$.

(v) For any $n \geq 0$, the induced by s and t morphisms

$$s_n : M_{n+1} \to \theta(M_n) \quad \text{and} \quad t_n : \theta(M_n) \to M_{n+1}$$

are isomorphisms.

In fact, $s_n \circ t_n = \xi\theta(M_n)$, and $t_n \circ s_n = \xi(M_{n+1})$. But, by *(iii)* and *(iii')*, both $\xi\theta(M_n)$ and $\xi(M_{n+1})$ are isomorphisms for any $n \geq 0$.

(vi) For any $n \geq 0, M^{(n)} := Ker(s^{(n)})$, where $s^{(n)} := \theta^n s \circ \ldots \circ \theta s \circ s : M \to \theta^{n+1}(M)$, is contained in the sum of the subobjects $M_m,\ 0 \leq m \leq$n.

We have: $M^{(0)}$:=Ker(s)$\subseteq Ker(t \circ s) = Ker(\xi(M)) := M_0$; i.e. the assertion is true for $n = 0$. For any $n \geq 1$,

$$Ker(s^{(n)}) = Ker(\theta^n s \circ s^{n-1}) \subseteq Ker(\theta^n \xi(M) \circ s^{n-1})$$

$$= Ker(\xi_n \theta^n(M) \circ s^{n-1}) = Ker(s^{n-1} \circ \xi_n(M)).$$

Continuing this way, we obtain that $Ker(s^{(n)}) \subseteq Ker(\xi \circ \xi_1 \circ \ldots \circ \xi_n)(M)$. Thus, it suffices to show that the $Ker((\xi \circ \xi_1 \circ \ldots \circ \xi_n))(M) = \sum_{0 \leq m \leq n} M_m := \sum_{0 \leq m \leq n} Ker(\xi_m(M))$.

This follows (by induction argument) from the fact that, for any $m > 0, \xi(M_m)$ is an isomorphism; hence $\xi(\sum_{1 \leq m \leq n} M_m) = \xi(Ker(\xi_1 \circ \ldots \circ \xi_n)(M)))$ is an isomorphism.

Note that, since the inclusion $\sum_{0 \leq m \leq n} M_m \subseteq Ker(s^{(n)})$ holds, we have, actually, showed that $Ker(s^{(n)}) = \sum_{0 \leq m \leq n} M_m = Ker((\xi \circ \xi_1 \circ \ldots \circ \xi_n)(M))$.

(vii) Combining all above, we see that the natural morphism from $\mathcal{M}(M_0)$ to (s, M, t) is an isomorphism. Thanks to the arbitrarity of $(s, M, t) \in Ob\mathbb{S}$, this means that the adjunction arrow $\eta : \mathcal{M} \circ \Phi \to Id_{\mathbb{S}}$ is an isomorphism; i.e. the functor $\mathcal{M}$ is an equivalence of categories.

(a)$\Rightarrow$ (c). Note that the condition *(a)* is equivalent to the same condition in the adjoint hyperbolic category $\mathcal{A}\{\theta^\wedge, \xi^\wedge\}$. Since *(c)* in $\mathcal{A}\{\theta, \xi\}$ is the same as *(b)* in $\mathcal{A}\{\theta^\wedge, \xi^\wedge\}$, the implications *(a)*$\Rightarrow$ (c) follow from those *(a)*$\Rightarrow$ (b). ∎

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