# TOWARDS AN EFFECTIVISATION OF THE RIEMANN THEOREM

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Abstract

The Riemann Theorem states, that for any nontrivial connected and simply connected domain on the Riemann sphere there exists its conformal bijection to the exterior of the unit disk. In this paper we find an explicit form of this map for a broad class of domains with analytic boundaries.

## 1. Introduction.

The Riemann Theorem states, that if connected and simply connected subdomain $Q$ on the Riemann sphere $\bar{\mathbb{C}}$ is not nonequivalent to complex plane, then there exists a *Riemann map.* The Riemann map is a conformal bijection $w_Q$ between $Q$ and the exterior of the unit disk $U = \{u \in \bar{\mathbb{C}} | |u| > 1\}$. In this paper we find an explicit form of the Riemann map for a broad class of domains with analytic boundaries.

Without loss of generally it is possible to consider that $\infty \in Q$ and $\infty \in Q_\circ$, where $Q_\circ = \bar{\mathbb{C}} - (Q \cup \partial Q)$ and $\partial Q$ is the boundary of $Q$. Let $\mathbb{H}$ be the set of all such domains. The harmonic moments

$$t_0 = \frac{1}{\pi} \int\limits_{Q_\circ} dxdy, \quad t_k = -\frac{1}{\pi k} \int\limits_{Q} z^{-k} dxdy \quad (k = 1, 2...) \tag{1.1}$$

form a coordinate system $t = (t_0, t_1, t_2, ...)$ on a considerable part of $\mathbb{H}$ [ 1, 13 ]. Let $Q^t \in \mathbb{H}$ be the domain corresponding to $t$. The conditions $\partial_z w_{Q^t}(\infty) \in \mathbb{R}$, $\partial_z w_{Q^t}(\infty) > 0$ uniquely define the Riemann maps

$$w(z, t) = w_{Q^t}(z) = p(t)z + \sum_{j=0}^{\infty} p_j(t) z^{-j}, \qquad p(t) \in \mathbb{R}, p(t) > 0. \tag{1.2}$$

Our goal is a presentation $p_j(t)$ as Taylor series from $t$ and $\bar{t}$.

Put

$$\partial_i = \frac{\partial}{\partial t_i}, \quad \bar{\partial}_i = \frac{\partial}{\partial \bar{t}_i}, \quad D(z) = \sum_{k \geqslant 1} \frac{z^{-k}}{k} \partial_k, \bar{D}(\bar{z}) = \sum_{k \geqslant 1} \frac{\bar{z}^{-k}}{k} \bar{\partial}_k \tag{1.3}$$

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In §1 we, following by [ 2,14,15 ], reproduce an important result [ 3 ]: the Riemann maps have a potential $F : \mathbb{H} \to \mathbb{C}$ (tau-function of the curves as regards to [10]) suth that

$$w(z,t) = zexp((-\frac{1}{2}\partial_0^2 - \partial_0 D(z))F(t)) \tag{1.4}$$

and moreover $F$ satisfies differential equations

$$(z-\xi)e^{D(z)D(\xi)F} = ze^{-\partial_0 D(z)F} - \xi e^{-\partial_0 D(\xi)F}. \tag{1.5(a)}$$

$$(\bar{z}-\bar{\xi})e^{\bar{D}(\bar{z})\bar{D}(\bar{\xi})F} = \bar{z}e^{-\partial_0 \bar{D}(\bar{z})F} - \bar{\xi} e^{-\partial_0 \bar{D}(\bar{\xi})F}. \tag{1.5(b)}$$

$$1 - e^{-D(z)\bar{D}(\bar{\xi})F} = \frac{1}{z\bar{\xi}} e^{\partial_0(\partial_0 + D(z) + \bar{D}(\bar{\xi}))F}, \tag{1.5(c)}$$

This system of the nonlinear differential equations is well known in mathematical physics and in theory of integrable systems as dispersionless limit of the $2D$ Toda hierarchy [ 4 ]. The solution $F(t)$ satisfies some additional equation, and it appears in string theory as "string solution" [ 5 ]. The string solution of dispersionless limit of $2D$ Toda hierarchy appears also in matrix models and in some other problems of mathematical physics [ 6,13 ]. Thus, a description of it has an independent interest.

In §2 we, follow by [ 7 ], find recurrent formulas for coefficients of the Taylor series for the potential.

$$F = \sum N(i|i_1..., i_k|\bar{i}_1..., \bar{i}_k) t_0^i t_{i_1} \cdots t_{i_k} \bar{t}_{\bar{i}_{\bar{1}}} \cdots \bar{t}_{\bar{i}_{\bar{k}}}. \tag{1.6}$$

These formulas together with (1.4) allow to find the Riemann maps via harmonic moments of domains.

In §3 we follow by [ 8 ] find sufficient condition for convergence of the Taylor series (1.6).

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## 2. Potential for Riemann maps.

The bijection $w : Q \to U$ generates the Dirichlet Green function on $Q \times Q$

$$G_Q(z,\xi) = \log \left| \frac{w(z) - w(\xi)}{w(z)\bar{w}(\xi) - 1} \right|. \tag{2.1}$$

It solves a Dirichlet problem in $Q$: the formula

$$u(z) = -\frac{1}{2\pi} \oint_{\partial Q} u_0(\xi) \partial_n G_Q(z,\xi) |d\xi| \tag{2.2}$$

restores a harmonic function by its boundaries values $u_0 = u|_{\partial D}$. Here $\partial_n$ is the normal derivative inward $Q$ on the boundary $\partial Q$ with respect to the second variable, and $|d\xi|$ is an infinitesimal element of ligth of $\partial Q$.

The Dirichlet Green function is uniquely determined by the following properties [ 9 ].

$G_Q(z,\xi)$ is symmetric and harmonic everywhere in Q in both arguments except $z = \xi$, where $G_Q(z,\xi) = \log|z - \xi| + O(1)$ as $z \to \xi$. (2.3($a$))

$G_Q(z,\xi)=0$ if both of variables $z,\xi$ belong to the boundary. (2.3(b))

Denote by $\mathbb{H}_z$ the set of all domains $Q\in\mathbb{H}$ containing $z$. The infinitesimal shift of the boundary $\partial Q=\{\xi\}$ on $-\frac{\varepsilon}{2}\partial_n G_Q(z,\xi)$ $(\varepsilon\to 0)$ generates a vector field $\delta_z$ on $\mathbb{H}_z$. In particulary,

$$\delta_z(t_k)=-\frac{1}{\varepsilon\pi k}\oint_{\partial Q}\xi^{-k}(-\frac{\varepsilon}{2}\partial_n G_Q(z,\xi))|d\xi|=\frac{z^k}{k}\quad(k>0)\qquad\qquad \delta_z(t_0)=1. \tag{2.4}$$

Thus, for any functional $X:\mathbb{H}_z\to\mathbb{C}$

$$\delta_z X=\frac{\partial X}{\partial t_0}\delta_z t_0+\sum\frac{\partial X}{\partial t_k}\delta_z t_k+\sum\frac{\partial X}{\partial \bar t_k}\delta_z \bar t_k=\nabla(z)X, \tag{2.5}$$

where $\nabla(z)=\partial_0+D(z)+\bar D(\bar z)$.

Put $v_0=\frac{2}{\pi}\int\limits_{Q_\circ}\log|z|dxdy$ and $v_k=\frac{1}{\pi}\int\limits_{Q_\circ}z^k dxdy$ for $k>0$.

**Theorem 2.1.** [3,10,15]. *The function* $F(t)=-\frac{1}{\pi}\iint\limits_{Q^t_\circ Q^t_\circ}\log|\eta^{-1}-\nu^{-1}|dx_\eta dy_\eta dx_\nu dy_\nu$ *(where* $Q^t_\circ=\bar{\mathbb{C}}-Q^t$*) satisfies the equations (1.4) and (1.5). Moreover* $\partial_k F=v_k$.

Proof: It is follow from (2.2) and from the definition of $\delta_z$ that

$$\tilde G_Q(z,\xi)=\log\left|\frac{1}{z}-\frac{1}{\xi}\right|+\frac{1}{2}\delta_z\delta_\xi F \tag{2.6}$$

satisfies (2.3). Thus, $\tilde G_Q(z,\xi)=G_Q(z,\xi)$. Using (2.1) and (2.5), we find that

$$\log\left|\frac{w(z)-w(\xi)}{w(z)\bar w(\xi)-1}\right|=\log\left|\frac{1}{z}-\frac{1}{\xi}\right|+\frac{1}{2}\nabla(z)\nabla(\xi)F \tag{2.7}$$

This equation implies equations (1.4) and (1.5). Really, it implies

$$h=\log\left|\frac{w(z)-w(\xi)}{w(z)\bar w(\xi)-1}\right|^2-\log\left|\frac{1}{z}-\frac{1}{\xi}\right|^2-\nabla(z)\nabla(\xi)F=0. \tag{2.8}$$

Besides $h=h_1+h_2$ where

$$h_1=\log\left(\frac{w(z)-w(\xi)}{w(z)\bar w(\xi)-1}\right)-\log\left(\frac{1}{z}-\frac{1}{\xi}\right)-\left(\frac{1}{2}\partial_0+D(z)\right)\nabla(\xi)F \tag{2.9}$$

is a holomorphic function of $z$ while

$$h_2=\log\left(\frac{\bar w(z)-\bar w(\xi)}{\bar w(z)w(\xi)-1}\right)-\log\left(\frac{1}{\bar z}-\frac{1}{\bar\xi}\right)-\left(\frac{1}{2}\partial_0+\bar D(\bar z)\right)\nabla(\xi)F \tag{2.10}$$

is an antiholomorphic function of $z$. Thus, $h_1$ is independent from $z$. Passing on to the limit $z\to\infty$, we obtain

$$h_1=\log\left(\frac{1}{\bar w(\xi)}\right)-\log\left(-\frac{1}{\xi}\right)-\frac{1}{2}\partial_0\nabla(\xi)F. \tag{2.11}$$

Thus,

$$\log\left(\frac{w(z)-w(\xi)}{z-\xi}\frac{z\bar{w}(\xi)}{w(z)\bar{w}(\xi)-1}\right)=D(z)\nabla(\xi)F. \tag{2.12}$$

Passing on to the limit $\xi\to\infty$ in (2.12), we obtain $\log\left(\frac{pz}{w(z)}\right)=D(z)\partial_0F$. Substituting (1.2) in (2.7) and passing on to the limit $z\to\infty,\xi\to\infty$, we obtain $\log(p)=-\frac{1}{2}\partial_0^2F$. A comparison of these formulas gives

$$\log\left(\frac{z}{w(z)}\right)=D(z)\partial_0F+\frac{1}{2}\partial_0^2F, \tag{2.13}$$

that is equivalent (1.4).

Using (2.13) we transform the holomorphic parts of (2.12) to

$$\log\left(\frac{w(z)-w(\xi)}{z-\xi}\right)=-\frac{1}{2}\partial_0^2F+D(z)D(\xi)F. \tag{2.14}$$

A substituting (1.4) to (2.14) gives (1.5($a$)). Changing $(z,\xi)$ to $(\bar{z},\bar{\xi})$ in the proof we obtain (1.5($b$)).

The antiholomorphic parts of (2.12) is

$$-\log\left(1-\frac{1}{w(z)\bar{w}(\xi)}\right)=D(z)\bar{D}(\bar{\xi})F. \tag{2.15}$$

A substituting (1.4) to (2.15) gives (1.5($c$)).

Moreover if $z\in\partial Q$ then

$$\nabla(z)F=\delta_zF=-\frac{2}{\pi}\int\limits_{Q_\circ^t(t)}\log|\nu^{-1}-z^{-1}|d\nu d\bar{\nu}=v_0+2Re\sum_{k\geqslant1}\frac{v_k}{k}z^{-k}.$$

Thus, $\partial_kF=v_k$. $\square$

## 3. Taylor series for the potential.

In this section we follow by [ 7 ] find recurrent formulas for coefficients $N$ of the Taylor series $F=\sum N(i|i_1...,i_k|\bar{i}_1...,\bar{i}_k)t_0^it_{i_1}\cdots t_{i_k}\bar{t}_{\bar{i}_{\bar{1}}}\cdots\bar{t}_{\bar{i}_{\bar{k}}}$ for the potential $F$.

The formulas for $N$ are found by the following scheme. At first, using some combinatorial calculations, we transform the equation (1.5(a)) to an infinite system of equations

$$\partial_{i_1}\partial_{i_2}\cdots\partial_{i_k}F\ =$$

$$=\sum_{m=1}^{\infty}\left(\sum_{\substack{s_1+\dots+s_m=i_1+\dots+i_k\\ \ell_1+\dots+\ell_m=m+k-2\\ s_j,\ell_j\geqslant1}}\frac{i_1\cdots i_k}{s_1\cdots s_m}T_{i_1\cdots i_k}\begin{pmatrix}s_1\cdots s_m\\ \ell_1\cdots\ell_m\end{pmatrix}\partial_0^{\ell_1}\partial_{s_1}F\cdots\partial_0^{\ell_m}\partial_{s_m}F\right). \tag{3.1}$$

In passing we find some recurrent formulas for calculation of $T$.

Later, using the definition of $F$ as a function on the space of analytical curves, we find, that

$\partial_0 F\,|_{t_0}\, = -t_0 + t_0 \ln t_0 \,+\, const$ and $\partial_k F\,|_{t_0}\, = 0$, if $k > 0$,
where here and later $|_{t_0}$ means the restriction of a function on a straight line $t_1 = \bar{t}_1 = t_2 = \bar{t}_2 = \cdots = 0$.

For this formula and from the equation (1.5(c)) follow, that

$$\partial_i \bar{\partial}_j F\,|_{t_0}\, = \begin{cases} 0, \text{ if } i \neq j, \\ i t_0^i, \text{ if } i = j\ . \end{cases}$$

Later, using (3.1) and the symmetry of the equations (1.5) we find, that

$$\partial_i \bar{\partial}_{i_1} \cdots \bar{\partial}_{i_k} F\,|_{t_0}\, = \bar{\partial}_i \partial_{i_1} \cdots \partial_{i_k} F\,|_{t=0}\, = \begin{cases} 0, \text{ if } i_1 + \cdots + i_k \neq i, \\ i_1 \cdots i_k \frac{i!}{(i-k+1)!} t_0^{i-k+1}, \text{ if } i_1 + \cdots + i_k = i. \end{cases}$$

This condition and the equation (3.1) give some recurrent formulas for coefficients $N$. As the final result we get

**Theorem 3.1.** *Let the formal series* (1.6) *satisfies in its convergence domain the equations* (1.4) *and* (1.5). *Then up to linear functions it has the form*

$$F \;=\; \frac{1}{2}\, t_0^2 \log t_0 \;-\; \frac{3}{4}\, t_0^2 + \sum_{\substack{k,\bar{k},n_r,\bar{n}_r \geqslant 1 \\ 0<i_1<\cdots<i_k \\ 0<\bar{i}_1<\cdots<\bar{i}_{\bar{k}} \\ i-(n_1+\cdots+n_k+\bar{n}_1+\cdots+\bar{n}_{\bar{k}})+2 \geqslant 0}} \frac{i_1^{n_1} \ldots i_k^{n_k}}{n_1! \ldots n_k!}\, \frac{\bar{i}_1^{\bar{n}_1} \ldots \bar{i}_{\bar{k}}^{\bar{n}_{\bar{k}}}}{\bar{n}_1! \ldots \bar{n}_{\bar{k}}!} \cdot$$

$$\cdot N_i^2 \left( \begin{array}{ccc|ccc} i_1, & \ldots, & i_k & \bar{i}_1, & \ldots, & \bar{i}_{\bar{k}} \\ n_1, & \ldots, & n_k & \bar{n}_1, & \ldots, & \bar{n}_{\bar{k}} \end{array} \right) t_0^{i-(n_1+\cdots+n_k+\bar{n}_1+\cdots+\bar{n}_{\bar{k}})+2} t_{i_1}^{n_1} \ldots t_{i_k}^{n_k} \bar{t}_{\bar{i}_1}^{\bar{n}_1} \ldots \bar{t}_{\bar{i}_{\bar{k}}}^{\bar{n}_{\bar{k}}}$$

*In this formula the coefficients $N^2$ are found by follow recurrent rules:*

*1)* $P_{i,j}(s_1, \ldots, s_m) \;=\; \#\,\{(i_1, \ldots, i_m) \mid i = i_1 + \cdots + i_m,\; 1 \leqslant i_r \leqslant s_r - 1\}$, *where* $\#V$ *is the cardinality of the set* $V$*;*

*2)* $T^1_{i,j}(s_1, \ldots, s_m) \;=$

$$\sum_{\substack{k \geqslant 1 \\ n_1+\cdots+n_k=m \\ n_r \geqslant 1}} \frac{1}{k n_1! \ldots n_k!}\, P_{i,j} \left( \underbrace{s_1 + \cdots + s_{n_1}}_{n_1}, \ldots, \underbrace{s_{n_1+\cdots+n_{k-1}+1} + \cdots + s_{n_1+\cdots+n_k}}_{n_k} \right);$$

$$T^2_{i_1,i_2} \begin{pmatrix} s_1, & \ldots, & s_m \\ 1, & \ldots, & 1 \end{pmatrix} \;=\; T^1_{i_1,i_2}(s_1, \ldots, s_m);$$

$$T^2_{i_1,\ldots,i_k} \begin{pmatrix} s_1, & \ldots, & s_m \\ l_1, & \ldots, & l_m \end{pmatrix} \;=$$

$$= \sum_{\substack{1\leqslant i\leqslant j\leqslant m\\ s,l\geqslant 1}} l\; T^1_{s,i_k}(s_i,\dots,s_j)\; T^2_{i_1,\dots,i_{k-1}}\begin{pmatrix} s_1, & \dots, & s_{i-1}, & s, & s_{j+1}, & \dots, & s_m\\ l_1, & \dots, & l_{i-1}, & l, & l_{j+1}, & \dots, & l_m\end{pmatrix},$$

*where* $s = s_i + \dots + s_j - i_k,\; l = (l_i - 1) + \dots + (l_j - 1)$ ;

*3)* $S_{\bar{i}_1,\dots,\bar{i}_{\bar{k}}}\begin{pmatrix} s_1, & \dots, & s_m\\ l_1, & \dots, & l_m\end{pmatrix} =$

$$= \sum_{\substack{\{\bar{i}^1_1,\dots,\bar{i}^{n_1}_1\}\sqcup\dots\sqcup\{\bar{i}^1_m,\dots,\bar{i}^{n_m}_m\}=\{\bar{i}_1,\dots,\bar{i}_{\bar{k}}\}\\ \bar{i}^1_r+\dots+\bar{i}^{n_r}_r=s_r\\ s_r-n_r-\ell_r+1\geqslant 0}} \tfrac{(s_1-1)!}{(s_1-n_1-l_1+1)!(l_1-1)!}\times\dots\times\tfrac{(s_m-1)!}{(s_m-n_m-l_m+1)!(l_m-1)!}\;;$$

*4)* $N^1_i(i_1,\dots,i_k|\bar{i}_1,\dots,\bar{i}_{\bar{k}}) \;=\; 0,\quad$ *if* $i\neq i_1+\dots+i_k$ *or* $i\neq \bar{i}_1+\dots+\bar{i}_{\bar{k}}$ ;

*in the other cases*

$N^1_i(i|\bar{i}_1,\dots,\bar{i}_{\bar{k}}) \;=\; \frac{(i-1)!}{(i-\bar{k}+1)!}\;;\quad N^1_i(i_1,\dots,i_k|\bar{i}) \;=\; \frac{(i-1)!}{(i-k+1)!}\;;$

$N^1_i(i_1,\dots,i_k|\bar{i}_1,\dots,\bar{i}_{\bar{k}}) \;=$

$$= \sum_{\substack{m\geqslant 1\\ s_1+\dots+s_m=i_1+\dots+i_k\\ l_1+\dots+l_m=m+k-2\\ s_r,l_r\geqslant 1}} (-1)^{m+1}\; S_{\bar{i}_1,\dots,\bar{i}_{\bar{k}}}\begin{pmatrix} s_1, & \dots, & s_m\\ l_1, & \dots, & l_m\end{pmatrix}\quad T^2_{i_1,\dots,i_k}\begin{pmatrix} s_1, & \dots, & s_m\\ l_1, & \dots, & l_m\end{pmatrix}\;,\; \textit{if}$$

$k,\bar{k} > 1$ ;

$$N^2_i\left(\begin{matrix} i_1, & \dots, & i_k\\ n_1, & \dots, & n_k\end{matrix}\,\middle|\,\begin{matrix}\bar{i}_1, & \dots, & \bar{i}_{\bar{k}}\\ \bar{n}_1, & \dots, & \bar{n}_{\bar{k}}\end{matrix}\right) \;=$$

$$=\; N^1_i\left(\underbrace{i_1,\dots,i_1}_{n_1},\dots,\underbrace{i_k,\dots,i_k}_{n_k}\,\middle|\,\underbrace{\bar{i}_1,\dots,\bar{i}_1}_{\bar{n}_1},\dots,\underbrace{\bar{i}_{\bar{k}},\dots,\bar{i}_{\bar{k}}}_{\bar{n}\bar{k}}\right)\;.\;\square$$

A proof follow by next lemmas

**Lemma 3.1.** *The following relations hold*

$$z - \sum_{j=1}^{\infty}\frac{1}{j}z^{-j}\partial_1\partial_j F = ze^{-\partial_0 D(z)}F$$

$$\partial_1\partial_j F = \sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m!}\sum_{\substack{k_1+\dots+k_m=j+1\\ k_i>0}}\frac{j}{k_1\cdots k_m}\partial_0\partial_{k_1}F\cdots\partial_0\partial_{k_m}F.$$

Proof: According (1.5(a)), $(z-\xi)e^{D(z)D(\xi)F} = (z-\xi)(1+(D(z)D(\xi)F)+\frac{1}{2}(D(z)D(\xi)F)^2+\cdots) = (z-\xi)(1+z^{-1}\xi^{-1}\partial_1^2F + z^{-1}\sum_{j=2}^{\infty}\frac{1}{j}\xi^{-j}\partial_1\partial_jF + \xi^{-1}\sum_{j=2}^{\infty}\frac{1}{j}z^{-j}\partial_1\partial_jF + z^{-2}\xi^{-2}f) = (z-\xi)+\xi^{-1}\partial_1^2F - z^{-1}\partial_1^2F + \sum_{j=2}^{\infty}\frac{1}{j}\xi^{-j}\partial_1\partial_jF - \sum_{j=2}^{\infty}\frac{1}{j}z^{-j}\partial_1\partial_jF + z^{-1}\xi^{-1}f.$

On the other hand, according (1.5(a)), the function $(z-\xi)e^{D(z)D(\xi)F}$ is a sum of two functions $f_1(z)+f_2(\xi)$. Thus, $f=0$ and $ze^{-\partial_0D(z)F} = z - \sum_{j=1}^{\infty}\frac{1}{j}z^{-j}\partial_1\partial_jF$. Therefore,

$$\sum_{j=1}^{\infty}\frac{1}{j}z^{-(j+1)}\partial_1\partial_jF = 1-e^{-\partial_0D(z)F} = 1-\big(1+\sum_{m=1}^{\infty}\frac{(-\partial_0D(z)F)^m}{m!}\big) =$$

$$-\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}\Big(\sum_{k=1}^{\infty}\frac{z^{-k}}{k}\partial_0\partial_kF\Big)^m =$$

$$= -\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}\Big(\sum_{n=1}^{\infty}z^{-n}\sum_{k_1+\cdots+k_m=n}\frac{1}{k_1\dots k_m}\partial_0\partial_{k_1}F\cdots\partial_0\partial_{k_m}F\Big) =$$

$$= -\sum_{n=1}^{\infty}z^{-n}\Big(\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}\sum_{k_1+\cdots+k_m=n}\frac{1}{k_1\dots k_m}\partial_0\partial_{k_1}F\cdots\partial_0\partial_{k_m}F\Big).$$

Thus,

$$\frac{1}{j}\partial_1\partial_jF = -\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}\sum_{k_1+\cdots+k_m=j+1}\frac{1}{k_1\dots k_m}\partial_0\partial_{k_1}F\cdots\partial_0\partial_{k_m}F.\square$$

**Lemma 3.2.** *The following relation holds*

$$\partial_i\partial_jF = \sum_{m=1}^{\infty}\sum_{\substack{s_1+\cdots+s_m=j+1\\ s_i>1}}\frac{(-1)^{m+1}}{m}\frac{ij}{(s_1-1)\cdots(s_m-1)}P_{ij}(s_1-1,...,s_m-1)\cdot$$

$$\cdot\partial_1\partial_{s_1-1}F\cdots\partial_1\partial_{s_m-1}F,$$

*where* $P_{ij}(s_1-1,...,s_m-1)$ *is the number of representations* $\{i=i_1+\cdots+i_m | 1\leqslant i_k\leqslant s_k-1,\ k=1,\cdots,m\}$ *for number* $i$.

Proof: According to lemma 3.1 and equation (1.5(a)), $(z-\xi)e^{D(z)D(\xi)F} =$

$$= z-\sum_{j=1}^{\infty}\frac{1}{j}z^{-j}\partial_1\partial_jF - (\xi-\sum_{j=1}^{\infty}\frac{1}{j}\xi^{-j}\partial_1\partial_jF) = (z-\xi)-\sum_{j=1}^{\infty}\frac{1}{j}(z^{-j}-\xi^{-j})\partial_1\partial_jF.$$

Thus,

$$e^{D(z)D(\xi)} = 1+z^{-1}\xi^{-1}\sum_{j=1}^{\infty}\frac{1}{j}\frac{(z^{-j}-\xi^{-j})}{(z^{-1}-\xi^{-1})}\partial_1\partial_jF =$$

$$= 1 + z^{-1}\xi^{-1}\sum_{j=1}^{\infty}\frac{1}{j}(\sum_{\substack{s+t=j-1\\ s,t\geqslant 0}} z^{-s}\xi^{-t})\partial_1\partial_j F =$$

$$= 1 + \sum_{j=1}^{\infty}\frac{1}{j}(\sum_{\substack{s+t=j+1\\ s,t\geqslant 1}} z^{-s}\xi^{-t})\partial_1\partial_j F.$$

Therefore,

$$D(z)D(\xi)F = \sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\Big(\sum_{n=1}^{\infty}\big(\sum_{\substack{s+t=n+1\\ s,t\geqslant 1}} z^{-s}\xi^{-t}\big)\frac{1}{n}\partial_1\partial_n F\Big)^m =$$

$$= \sum_{j=1}^{\infty}\frac{(-1)^{m+1}}{m}\sum_{i,j\geqslant 1} z^{-i}\xi^{-j}\cdot$$

$$\Big(\sum_{\substack{i_1+\cdots+i_m=i\\ j_1+\cdots j_m=j\\ i_k,j_k\geqslant 1}} \frac{1}{i_1+j_1-1}\partial_1\partial_{i_1+j_1-1}F\cdots\frac{1}{i_m+j_m-1}\partial_1\partial_{i_m+j_m-1}F\Big),$$

that is

$$\partial_i\partial_j F = \sum_{m=1}^{\infty}\sum_{s_1+\cdots+s_m=i+j}\frac{(-1)^{m+1}}{m}\frac{ij}{(s_1-1)\cdots(s_m-1)}\cdot$$

$$\cdot P_{ij}(s_1-1,...,s_m-1)\partial_1\partial_{s_1-1}F\cdots\partial_1\partial_{s_m-1}F.\square$$

**Remark.** The equations

$$\partial_i\partial_j F = \sum_{m=1}^{\infty}\sum_{s_1+\cdots+s_m=i+j}\frac{(-1)^{m+1}}{m}\frac{ij}{(s_1-1)\cdots(s_m-1)}\cdot$$

$$\cdot P_{ij}(s_1-1,...,s_m-1)\partial_1\partial_{s_1-1}F\cdots\partial_1\partial_{s_m-1}F$$

describe the dispersionless limit of KP equation. Some other description of this hierarchy is presented in [ 11 ]. A comparison of these descriptions gives some nontrivial combinatorial identity $P_{ij}$.

**Lemma 3.3.** *The following relation holds*

$$\partial_i\partial_j F = \sum_{m=1}^{\infty}\sum_{p_1+\cdots+p_m=j+i}\frac{ij}{p_1\cdots p_m}T_{ij}(p_1\cdots p_m)\partial_0\partial_{p_1}F\cdots\partial_0\partial_{p_m}F,$$

*where*
$T_{ij}(p_1...p_m) = \sum_{\substack{n_1+\cdots+n_k=m \\ n_i>0}} \frac{(-1)^{m+1}}{k}\frac{1}{n_1!\cdots n_k!}P_{ij}(p_1+\cdots+p_{q_1}-1,...,p_{q_{k-1}+1}+\cdots+p_{q_k}-1)$, *and* $q_j=\sum_{i=1}^{j} n_i$.

Proof: According to lemmas 3.1 and 3.2,

$$\partial_i\partial_j F = \sum_{m=1}^{\infty}\sum_{s_1+\cdots+s_m=j+i}\frac{(-1)^{m+1}}{m}\frac{ij}{(s_1-1)\cdots(s_m-1)}P_{ij}(s_1-1,\cdots,s_m-1)\cdot$$

$$\cdot\partial_1\partial_{s_1-1}F\cdots\partial_1\partial_{s_m-1}F) = \sum_{m=1}^{\infty}\sum_{s_1+\cdots+s_m=j+i}\frac{(-1)^{m+1}}{m}\frac{ij}{(s_1-1)\cdots(s_m-1)}\cdot$$

$$\cdot P_{ij}(s_1-1,\cdots,s_m-1)\Big(\sum_{n_1=1}^{\infty}\sum_{p_1+\cdots+p_{n_1}=s_1}\frac{(-1)^{n_1+1}}{n_1!}\frac{s_1-1}{p_1\cdots p_{n_1}}\partial_0\partial_{p_1}F\cdots\partial_0\partial_{p_{n_1}}F\Big)\cdots$$

$$\cdots\Big(\sum_{n_m=1}^{\infty}\sum_{p_1+\cdots+p_{n_m}=s_m}\frac{(-1)^{n_m+1}}{n_m!}\frac{s_m-1}{p_1\cdots p_{n_m}}\partial_0\partial_{p_1}F\cdots\partial_0\partial_{p_{n_m}}F\Big)=$$

$$=\sum_{m=1}^{\infty}\sum_{p_1+\cdots+p_m=j+i}\frac{ij}{p_1\cdots p_m}T_{ij}(p_1\cdots p_m)\partial_0\partial_{p_1}F\cdots\partial_0\partial_{p_m}F.\square$$

Using the induction, we get from lemma 3.3

**Lemma 3.4.** *The following relation holds*

$$\partial_{i_1}\partial_{i_2}\cdots\partial_{i_k}F = \sum_{m=1}^{\infty}\Big(\sum_{\substack{s_1+\cdots+s_m=i_1+\cdots+i_k \\ \ell_1+\cdots+\ell_m=m+k-2 \\ s_j,\ell_j\geqslant 1}}\frac{i_1\cdots i_k}{s_1\cdots s_m}\cdot$$

$$\cdot T_{i_1\cdots i_k}\begin{pmatrix} s_1\cdots s_m \\ \ell_1\cdots\ell_m\end{pmatrix}\partial_0^{\ell_1}\partial_{s_1}F\cdots\partial_0^{\ell_m}\partial_{s_m}F\Big),$$

$$\text{where}\quad T_{i_1,i_2}\begin{pmatrix} s_1\cdots s_m \\ \ell_1\cdots\ell_m\end{pmatrix} = \begin{cases} T_{i_1i_2}(s_1\cdots s_m), & \text{if } \ell_1=\cdots=\ell_m=1 \\ 0 & \text{in another cases}\end{cases},$$

$$T_{i_1\cdots i_k}\begin{pmatrix} s_1\cdots s_m \\ \ell_1\cdots\ell_m\end{pmatrix} = \sum_{\substack{1\leqslant i\leqslant j\leqslant m \\ s,\ell>0}} T_{i_1\cdots i_{k-1}}\begin{pmatrix} s_1\cdots s_{i-1} & \begin{pmatrix} s \\ \ell\end{pmatrix} & s_{j+1}\cdots s_m \\ \ell_1\cdots\ell_{i-1} & & \ell_{j+1}\cdots\ell_m\end{pmatrix}.$$

$$\cdot T_{s,i_k}(s_i, s_{i+1}, ..., s_j)\frac{\ell!}{(\ell_i-1)!\cdots(\ell_j-1)!},$$

$$\textit{and}\quad s = s_i + s_{i+1} + \cdots + s_j - i_k, \quad \ell = (\ell_i - 1) + \cdots + (\ell_j - 1).\square$$

Define now the Cauchy data for $F$.

**Lemma 3.5.** *The following relations hold*

$$\partial_0 F|_{t_0} = -t_0 + t_0 \ln t_0 \quad + \textit{const} \quad \textit{and} \quad \partial_k F|_{t_0} = 0 \quad \textit{for} \quad k > 0.$$

Proof: If $t_1 = t_2 = \cdots = 0$, then $Q^t_\circ = \{w \in \mathbb{C} | |w| < t_0^{\frac{1}{2}}\}$ and thus, $w(z,t)|_{t_0} = Rz = t_0^{\frac{1}{2}} z$. It is follow from theorem 2.1 that

$$\partial_0 F\Big|_{t_0} = \frac{2}{\pi}\int\limits_{|z|\leqslant R} \ln|z| dxdy = \frac{2}{\pi}\int\limits_0^R dr \int\limits_0^{2\pi} d\varphi \cdot r \cdot \ln|r| =$$

$$= \frac{2\cdot 2\pi}{2\pi}\int\limits_0^R \ln|r| dr^2 = 2(r^2 \ln r\Big|_0^R - \int\limits_0^R r dr) =$$

$$= r^2 \ln r^2\Big|_0^R - 2\frac{1}{2}r^2\Big|_0^R = R^2 \ln R^2 - R^2 = -t_0 + t_0 \ln t_0.$$

$\partial_k F|_{t_0} = \frac{1}{\pi}\int\limits_{Q^t_\circ} z^k dxdy = 0$ for $k > 0$. $\square$

**Lemma 3.6.** *The following relations hold*

$$\partial_i \bar{\partial}_j F|_{t_0} = \begin{cases} 0 & \textit{for } i \neq j, \\ i t_0^i & \textit{for } i = j. \end{cases}$$

Proof: It is follow from lemma 3.5 that $\partial_0 \partial_k F\Big|_{t_0} = 0$ for $k > 0$ and $\partial_0^2 F|_{t_0} = \ln t_0$. Thus,

$$e^{\partial_0(\partial_0 + D(z) + \bar{D}(\bar{\xi}))F}\Big|_{t_0} = t_0.$$

Moreover, according to (1.5(c)),

$$1 - e^{-D(z)\bar{D}(\bar{\xi})F} = z^{-1}\bar{\xi}^{-1} e^{\partial_0(\partial_0 + D(z) + \bar{D}(\bar{\xi}))F},$$

and thus,

$$-D(z)\bar{D}(\bar{\xi})F\Big|_{t_0} = \ln(1 - z^{-1}\bar{\xi}^{-1}t_0) = -\sum_{k=1}^{\infty} k z^{-k}\xi^{-k} t_0^k.$$

Therefore $\partial_i\bar\partial_j F\big|_{t_0}=0$ for $i\neq j$ and $\partial_i\bar\partial_i F\big|_{t_0}=it_0^i$ □.

**Lemma 3.7.** *The following relations hold*

$$\partial_i\bar\partial_{i_1}\cdots\bar\partial_{i_k}F\Big|_{t_0}=\bar\partial_i\partial_{i_1}\cdots\partial_{i_k}F\big|_{t_0}=\begin{cases}0, & \text{if}\quad i_1+\cdots+i_k\neq i\\ i_1\cdots i_k\frac{i!}{(i-k+1)!}t_0^{i-k+1}, & \text{if}\ i=i_1+\cdots+i_k\end{cases}.$$

Proof: The differentials $\partial$ and $\bar\partial$ occur in (1.5) symmetrically. This gives the first equality. Moreover, according to lemmas 3.3 and 3.4,

$$\partial_{i_1}\partial_{i_2}\cdots\partial_{i_k}F=\frac{i_1\cdots i_k}{i}T_{i_1\cdots i_k}\binom{i}{k-1}\partial_0^{k-1}\partial_iF+$$

$$+\sum_{m=2}^{\infty}\Bigg(\sum_{\substack{s_1+\cdots+s_m=i_1+\cdots+i_k\\ \ell_1+\cdots\ell_m=m+k-2\\ s_j,t_j\geqslant1}}\frac{i_1\cdots i_k}{s_1\cdots s_m}T_{i_1\cdots i_k}\begin{pmatrix}s_1\cdots s_m\\ \ell_1\cdots\ell_m\end{pmatrix}\cdot$$

$$\cdot\partial_0^{\ell_1}\partial_{s_1}F\cdots\partial_0^{\ell_m}\partial_{s_m}F\Bigg)=\frac{i_1\cdots i_k}{i}\partial_0^{k-1}\partial_iF+$$

$$+\sum_{m=2}^{\infty}\Bigg(\sum_{\substack{s_1+\cdots+s_m=i_1+\cdots+i_k\\ \ell_1+\cdots\ell_m=m+k-2\\ s_j,t_j\geqslant1}}\frac{i_1\cdots i_k}{s_1\cdots s_m}T_{i_1\cdots i_k}\begin{pmatrix}s_1\cdots s_m\\ \ell_1\cdots\ell_m\end{pmatrix}\cdot$$

$$\cdot\partial_0^{\ell_1}\partial_{s_1}F\cdots\partial_0^{\ell_m}\partial_{s_m}F\Bigg),$$

where $i=i_1+\cdots+i_k$. This equality and lemmas 3.5 and 3.6 give the second equality in the affirmation of lemma 3.7. □

**Lemma 3.8.** *The following relation holds*

$$\partial_{i_1}\cdots\partial_{i_k}\bar\partial_{\bar i_1}\cdots\bar\partial_{\bar i_{\bar k}}F\Big|_{t_0}=\sum_{i=1}^{\infty}\tilde N_i(i_1\cdots i_k\big|\bar i_1\cdots\bar i_{\bar k})t_0^{i-(k+\bar k)+2},\ \textit{where}\ \tilde N_i\big(i_1\cdots i_k\big|\bar i_1\cdots\bar i_{\bar k}\big)=0$$

*if* $i-(k+\bar k)+2<0,\ \sum\limits_{j=1}^{k}i_j\neq i$ *or* $\sum\limits_{j=1}^{\bar k}\bar i_j\neq i$. *In opposite case,* $\tilde N_i\big(i_1\cdots i_{i_k}\big|\bar i_1,\cdots\bar i_{\bar k}\big)=$

$$=\sum_{m=1}^{\infty}\Bigg(\sum_{\substack{s_1+\cdots+s_m=i_1+\cdots+i_k\\ \ell_1+\cdots\ell_m=m+k-2\\ s_j,\ell_j\geqslant1}}i_1\cdots i_k\bar i_k\cdots\bar i_{\bar k}\cdot$$

$$\cdot T_{i_1\cdots i_k}\begin{pmatrix} s_1\cdots s_m\\ \ell_1\cdots\ell_m\end{pmatrix} S_{\bar i_1\cdots\bar i_{\bar k}}\begin{pmatrix} s_1\cdots s_m\\ \ell_1\cdots\ell_m\end{pmatrix}\Big)\Big),$$

*where*

$$S_{\bar i_1\cdots\bar i_{\bar k}}\begin{pmatrix} s_1\cdots s_m\\ \ell_1\cdots\ell_m\end{pmatrix}=\sum_{\substack{n_1+\cdots+n_m=k\\ s_i-n_i+1-\ell_i\geqslant 0}}\sum\frac{(s_1-1)!\cdots(s_m-1)!}{(s_1-N_1+1-\ell_1)!\cdots(s_m-n_m+1-\ell_m)!}$$

*and second sum is the sum by all partitions of the set* $\{\bar i_1,...,\bar i_{\bar k}\}$ *on subsets* $\{j^p_1,...,j^p_{n_p}\}$ $(p=1,...,m)$ *such that* $\sum\limits_{\alpha=1}^{n_p} j^p_\alpha=s_p$. $\square$

Proof: According to lemma 3.4,

$$\partial_{i_1}\cdots\partial_{i_k}\bar\partial_{\bar i_1}\cdots\bar\partial_{\bar i_{\bar k}}F=\bar\partial_{\bar i_{\bar i}}\cdots\bar\partial_{\bar i_{\bar k}}\Big(\sum_{m=1}^{\infty}\Big(\sum_{\substack{s_1+\cdots+s_m=i_1+\cdots+i_k\\ \ell_1+\cdots+\ell_m=m+k-2\\ s_j,\ell_j\geqslant 1}}\frac{i_1\cdots i_k}{s_1\cdots s_m}\cdot$$

$$\cdot\, T_{i_1\cdots i_k}\begin{pmatrix} s_1\cdots s_m\\ \ell_1\cdots\ell_m\end{pmatrix}\partial_0^{\ell_1}\partial_{s_1}F\cdots\partial_0^{\ell_m}\partial_{s_m}F\Big)\Big)=$$

$$=\sum\Big(\sum_{m=1}^{\infty}\Big(\sum_{\substack{s_1+\cdots+s_m=i_1+\cdots+i_k\\ \ell_1+\cdots+\ell_m=m+k-2\\ s_j,\ell_j\geqslant 1}}\frac{i_1\cdots i_k}{s_1\cdots s_m}\,T_{i_1\cdots i_k}\begin{pmatrix} s_1\cdots s_m\\ \ell_1\cdots\ell_m\end{pmatrix}\cdot$$

$$\cdot\,\partial_0^{\ell_1}\partial_{s_1}\bar\partial^1_{\bar j_1}\cdots\bar\partial^1_{\bar j_{n_1}}F\cdots\partial_0^{\ell_m}\partial_{s_m}\bar\partial^m_{\bar j_1}\cdots\bar\partial^m_{\bar j_{n_m}}F\Big)\Big),$$

where the first sum is taken by all partitions of $\{\bar i_1,...,\bar i_{\bar k}\}$ on subset $\{(j^1_i,...,j^1_{n_1})...(j^m_1,...,j^m_{n_m})\}$. According to lemma 3.7, this gives the affirmation of lemma 3.8 $\square$.

Lemma 3.8 gives

**Lemma 3.9** *The following relation holds*

$$F=\frac{1}{2}t_0^2\log t_0-\frac{3}{4}t_0^2+\sum_{\substack{i_1<\cdots<i_k\\ \bar i_1<\cdots\bar i_{\bar k}}}\sum_{n_j,\bar n_j=1}^{\infty}\frac{1}{n_1!\cdots n_k!\bar n_1!\cdots\bar n_{\bar k}!}\tilde N_i\Big(\begin{matrix} i_1...i_k\\ n_1...n_k\end{matrix}\Big|\begin{matrix}\bar i_1...\bar i_{\bar k}\\ \bar n_1...\bar n_{\bar k}\end{matrix}\Big)\cdot$$

$$\cdot t_0^{\,i-(\sum\limits_{i=1}^{k} n_i+\sum\limits_{i=1}^{\bar k}\bar n_i)+2}\,t_{i_1}^{n_1}\cdots t_{i_k}^{n_k}\bar t_{\bar i_1}^{\bar n_1}\cdots\bar t_{\bar i_{\bar k}}^{\bar n_{\bar k}},$$

*where*

$$\tilde N_i\Big(\begin{matrix} i_1...i_k\\ n_1...n_k\end{matrix}\Big|\begin{matrix}\bar i_1...\bar i_{\bar k}\\ \bar n_1...\bar n_{\bar k}\end{matrix}\Big)=0,$$

*if* $i-(\sum\limits_{i=1}^{k} n_i+\sum\limits_{i=1}^{\bar k}\bar n_i)+2<0,\ i\neq\sum\limits_{j=1}^{k} n_j i_j$ *or* $i\neq\sum\limits_{j=1}^{\bar k}\bar n_j\bar i_j$. *In opposite case*

$$\tilde N_i\left(\begin{matrix} i_1...i_k \\ n_1...n_k\end{matrix}\,\middle|\,\begin{matrix}\bar i_1...\bar i_{\bar k} \\ |n_1...\bar n_{\bar k}\end{matrix}\right)=\tilde N_i(i_1...i_1i_2...i_2...i_k...i_k|\bar i_1...\bar i_1\bar i_2...\bar i_2...\bar i_{\bar k}...\bar i_{\bar k}),$$

*where in the last parentheses any* $i_j$ *(respectively any* $\bar i_j$ *) occurs* $n_j$ *(respectively* $\bar n_j$*) cases.*

The theorem 3.1 is equivalent to lemma 3.9.

Using the algorithm of the present paper, Yu.Klimov and A.Korzh created a computer program for calculations of the coefficients of the Taylor series $F$.

## 4. Convergence conditions of the Taylor series for the potential.

The combinatorial coefficients $N_i^2(...)$ have some remarkable properties. For example,

**Theorem 4.1.** $N_i^2\left(\begin{matrix} i_1\cdots i_k \\ n_1\cdots n_k\end{matrix}\,\middle|\,\begin{matrix}1 \\ \bar n_1\end{matrix}\right)=\begin{cases}(i-1)!, & \textit{if } k=n_1=1,\ i=i_1=\bar n_1,\\ 0 \textit{ in the other cases.}\end{cases}$

*Proof.* According to our definition

$$S_{\underbrace{1,\dots,1}_{\bar k}}\begin{pmatrix} s_1, & \dots, & s_m \\ l_1, & \dots, & l_m\end{pmatrix}=$$

$$=\sum_{\substack{\{\bar i_1^1,\dots,\bar i_1^{n_1}\}\sqcup\cdots\sqcup\{\bar i_m^1,\dots,\bar i_m^{n_m}\}=\\ =\{\underbrace{1,\dots,1}\}\\ \bar i_r^1+\cdots+\bar i_r^{n_r}=s_r\\ s_r-n_r-\ell_r+1\geqslant 0}}\frac{(s_1-1)!}{(s_1-n_1-l_1+1)!(l_1-1)!}\times\cdots\times$$

$$\times\cdots\times\frac{(s_m-1)!}{(s_m-n_m-l_m+1)!(l_m-1)!}=\delta_{\ell_1,1}\dots\delta_{\ell_m,1}\frac{\bar k!}{s_1\dots s_m}.$$

Thus, if $k>2$, then

$$N_i^1(i_1,\dots,i_k|1,\dots,1)=$$

$$=\sum_{\substack{m\geqslant 1\\ s_1+\cdots+s_m=i_1+\cdots+i_k\\ l_1+\cdots+l_m=m+k-2\\ s_r,l_r\geqslant 1}}(-1)^{m+1}\,S_{1,\dots,1}\begin{pmatrix} s_1, & \dots, & s_m \\ l_1, & \dots, & l_m\end{pmatrix}\times$$

$$\times\,T^2_{i_1,\dots,i_k}\begin{pmatrix} s_1, & \dots, & s_m \\ l_1, & \dots, & l_m\end{pmatrix}=0\ .$$

Let now $k=2$ $(i_1,i_2\geqslant 1)$. Then

$$N_i^1(i_1,i_2|\underbrace{1,\dots,1}_{\bar k})\ =$$

$$= \sum_{\substack{m\geqslant 1\\ s_1+\dots+s_m=i_1+i_2\\ s_r\geqslant 1}} (-1)^{m+1}\ S_{\underbrace{1,\dots,1}_{\bar{k}}}\begin{pmatrix} s_1, & \dots, & s_m\\ 1, & \dots, & 1\end{pmatrix}\times$$

$$\times\ T^2_{i_1,i_2}\begin{pmatrix} s_1, & \dots, & s_m\\ 1, & \dots, & 1\end{pmatrix}\ =$$

$$= \sum_{\substack{m\geqslant 1\\ s_1+\dots+s_m=i_1+i_2\\ s_r\geqslant 1}} (-1)^{m+1}\frac{\bar{k}!}{s_1\dots s_m}\ T^2_{i_1,i_2}\begin{pmatrix} s_1, & \dots, & s_m\\ 1, & \dots, & 1\end{pmatrix}\ =$$

$$= \sum_{\substack{m\geqslant 1\\ s_1+\dots+s_m=i_1+i_2\\ s_r\geqslant 1}} (-1)^{m+1}\frac{\bar{k}!}{s_1\dots s_m}\ T^1_{i_1,i_2}(s_1,\dots,s_m)\ =$$

$$= \sum_{\substack{m\geqslant 1\\ s_1+\dots+s_m=i_1+i_2\\ s_r\geqslant 1}} (-1)^{m+1}\frac{\bar{k}!}{s_1\dots s_m}\sum_{\substack{k\geqslant 1\\ n_1+\dots+n_k=m\\ n_r\geqslant 1}}\frac{1}{kn_1!\dots n_k!}\times$$

$$\times\ P_{i,j}\left(\underbrace{s_1+\dots+s_{n_1}}_{n_1},\dots,\underbrace{s_{n_1+\dots+n_{k-1}+1}+\dots+s_{n_1+\dots+n_k}}_{n_k}\right)\ =$$

$$= \sum_{\substack{m\geqslant 1\\ s_1+\dots+s_m=i_1+i_2\\ k\geqslant 1\\ n_1+\dots+n_k=m}} \frac{(-1)^{m+1}}{kn_1!\dots n_k!}\frac{\bar{k}!}{s_1\dots s_m}\times$$

$$\times\ P_{i,j}\left(\underbrace{s_1+\dots+s_{n_1}}_{n_1},\dots,\underbrace{s_{n_1+\dots+n_{k-1}+1}+\dots+s_{n_1+\dots+n_k}}_{n_k}\right)\ =$$

$$= \sum_{\substack{k\geqslant 1\\ \tilde{s}_1+\dots+\tilde{s}_k=i_1+i_2}} P_{i,j}(\tilde{s}_1,\dots,\tilde{s}_k)\times$$

$$\times \sum_{\substack{n_r\geqslant 1,\ n_1+\dots+n_k=m\\ \underbrace{s_{n_1+\dots+n_{r-1}+1}+\dots+s_{n_1+\dots+n_r}}_{n_r}=\tilde{s}_r}} \frac{(-1)^{m+1}}{kn_1!\dots n_k!}\frac{\bar{k}!}{s_1\dots s_m}\ =$$

$$= \sum_{\substack{k\geqslant 1\\ \tilde{s}_1+\dots+\tilde{s}_k=i_1+i_2}} -\frac{\bar{k}!}{k}\, P_{i,j}(\tilde{s}_1,\dots,\tilde{s}_k)\times$$

$$\times \prod_{1\leqslant r\leqslant k}\ \sum_{\substack{n_r\geqslant 1\\ s_1+\dots+s_{n_r}=\tilde{s}_r}} \frac{(-1)^{n_r}}{n_r!s_1\dots s_{n_r}}\ .$$

In addition if $s>1$, then

$$\sum_{\substack{n\geqslant 1\\ s_1+\dots+s_n=s}} \frac{(-1)^n}{n!s_1\dots s_n} = \frac{1}{s!}\frac{\partial^s}{\partial x^s}\sum_{n\geqslant 1}\frac{(-1)^n}{n!}\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\dots\right)^n\Bigg|_{x=0} =$$

$$= \frac{1}{s!}\frac{\partial^s}{\partial x^s}\sum_{n\geqslant 1}\frac{(-1)^n}{n!}\left(-\log(1-x)\right)^n\big|_{x=0} =$$

$$= \frac{1}{s!}\frac{\partial^s}{\partial x^s}\sum_{n\geqslant 1}\frac{(\log(1-x))^n}{n!}\Bigg|_{x-0} = \frac{1}{s!}\frac{\partial^s}{\partial x^s}\ \left(\exp(\log(1-x))-1\right)\big|_{x=0} = 0.$$

Thus,

$$N_i^1(i_1,i_2|\underbrace{1,\dots,1}_{\bar{k}}) = \sum_{\substack{k\geqslant 1\\ \tilde{s}_1+\dots+\tilde{s}_k=i_1+i_2}} -\frac{\bar{k}!}{k}\, P_{i,j}(\tilde{s}_1,\dots,\tilde{s}_k)\times$$

$$\times \prod_{1\leqslant r\leqslant k}\ \sum_{\substack{n_r\geqslant 1\\ s_1+\dots+s_{n_r}=\tilde{s}_r}} \frac{(-1)^{n_r}}{n_r!s_1\dots s_{n_r}} = 0\ ,$$

because $P_{i,j}(1,\dots,1)\ = 0$.

If $k=1$, then from our definition it follows that

$$N_i^2\left(\begin{array}{c|c} i_1 & 1\\ n_1 & \bar{n}_1\end{array}\right) = \begin{cases}(i-1)!, \text{ if } n_1=1,\ i=i_1=\bar{n}_1,\\ 0 \text{ in the other cases.}\end{cases} \quad . \qquad \square$$

If $t_i,\bar{t}_i=0$ for $i>2$ , then $\partial Q$ is ellipse. In this case theorem 3.1 gives

$$F = -\frac{3}{4}t_0^2+\frac{1}{2}t_0^2\log\Big(\frac{t_0}{1-4|t_2|^2}\Big)+\frac{t_0}{1-4|t_2|^2}\big(|t_1|^2+t_1^2\bar{t}_2+\bar{t}_1^2t_2\big).$$

This formula was at first obtained in [ 3 ], using formulas for conforming maps from ellipse to circle.

The recurrent formulas for coefficients of the Taylor series $F$ give possibility to estimate the coefficients and to find sufficient convergence conditions for $F$, if $t_i,\bar{t}_i=0$ for $i>n$.

**Theorem 4.2.** *Let $\tilde{t} = (t_0, t_1, \bar{t}_1, t_2, \bar{t}_2, ...)$ be such that $t_i, \bar{t}_i = 0$ for $i > n, 0 < t_0 < 1$ and $|t_i|,\ |\bar{t}_i| \leqslant (4n^3 2^n e^n)^{-1}$. Then the series $F(\tilde{t})$ is convergence.*

A proof is based on a sequential of estimations of all values, utilised in the definition of $N^2$. Present these estimations:

**Lemma 4.1.** *Let $i + j = s_1 + \cdots + s_m$. Then $P_{ij}(s_1..., s_m) \leqslant \min(C_{i-1}^{m-1}, C_{j-1}^{m-1})$.*

*Proof.*

$$P_{j,i}(s_1, \dots, s_m) = P_{i,j}(s_1, \dots, s_m) = \#\{(i_1, \dots, i_m) \mid i = $$
$$= i_1 + \cdots + i_m,\ 1 \leqslant i_r \leqslant s_r - 1\} \leqslant$$
$$\leqslant \#\{(i_1, \dots, i_m) | i = i_1 + \cdots + i_m, 1 \leqslant i_r\} = C_{i-1}^{m-1}\ . \quad \square$$

**Lemma 4.2.** *Let $i + j = s_1 + \cdots + s_m$. Then $T^1_{ij}(s_1, ..., s_m) \leqslant \frac{\ell^{m-1}}{m!}$, where $\ell = \min(i, j)$.*

*Proof.*

$$T^1_{j,i}(s_1, \dots, s_m) = T^1_{i,j}(s_1, \dots, s_m) = \sum_{\substack{k \geqslant 1 \\ n_1 + \cdots + n_k = m \\ n_r \geqslant 1}} \frac{1}{kn_1! \dots n_k!} \times$$

$$\times\ P_{i,j}\left(\underbrace{s_1 + \cdots + s_{n_1}}_{n_1}, \dots, \underbrace{s_{n_1+\cdots+n_{k-1}+1} + \cdots + s_{n_1+\cdots+n_k}}_{n_k}\right) \leqslant$$

$$\leqslant \sum_{\substack{k \geqslant 1 \\ n_1 + \cdots + n_k = m \\ n_r \geqslant 1}} \frac{C_{i-1}^{k-1}}{kn_1! \dots n_k!} =$$

$$= \frac{1}{m!} \frac{\partial^m}{\partial x^m} \frac{1}{i} \sum_{k \geqslant 1} C_i^k \sum_{n_1 + \cdots + n_k = m} \frac{1}{n_1! \dots n_k!} =$$

$$\frac{1}{m!} \frac{\partial^m}{\partial x^m} \frac{1}{i} \sum_{k \geqslant 1} C_i^k \left(x + \frac{x^2}{2!} + \dots\right)^k \Bigg|_{x=0} =$$

$$= \frac{1}{m!} \frac{\partial^m}{\partial x^m} \frac{1}{i} \left(\sum_{k \geqslant 1} C_i^k (e^x - 1)^k\right) \Bigg|_{x=0} =$$

$$= \frac{1}{m!} \frac{\partial^m}{\partial x^m} \frac{1}{i} \left((1 + (e^x - 1))^i - 1\right)\Big|_{x=0} = \frac{1}{m!} \frac{\partial^m}{\partial x^m} \frac{1}{i} \left(e^{ix} - 1\right)\Big|_{x=0} = \frac{i^{m-1}}{m!}\ . \quad \square$$

**Lemma 4.3.** *Let* $i_1+\dots+i_k = s_1+\dots+s_m$ *and* $(\ell_1-1)+\dots+(\ell_m-1)=k-2$. *Then* $T^2_{i_1\dots,i_k}\begin{pmatrix} s_1\cdots s_m\\ \ell_1\cdots\ell_m\end{pmatrix} \leqslant \frac{I^{m-1}(k-1)^m(k-2)!}{m!}$ , *where* $I=\max(i_r)$.

*Proof.* We use an induction by $k$. If $k=2$, then

$$T^2_{i_1,i_2}\begin{pmatrix} s_1, & \dots, & s_m\\ 1, & \dots, & 1\end{pmatrix} = T^1_{i_1,i_2}(s_1,\dots,s_m) \leqslant$$

$$\leqslant \frac{I^{m-1}}{m!} = \left.\frac{I^{m-1}(k-1)^m(k-2)!}{m!}\right|_{k\,=\,2}.$$

Let $k>2$. Note at first that, if $l=(l_i-1)+\dots+(l_j-1)$, then

$$\sum_{1\leqslant i\leqslant j\leqslant m,\ j-i=d}\frac{l}{(k-2)(d+1)} = \frac{\sum\limits_{1\leqslant i\leqslant j\leqslant m,\ j-i=d}(l_i-1)+\dots+(l_j-1)}{(k-2)(d+1)} \leqslant$$

$$\leqslant \frac{\sum\limits_{1\leqslant t\leqslant m}(l_t-1)(d+1)}{(k-2)(d+1)} = 1.$$

Thus,

$$T^2_{i_1,\dots,i_k}\begin{pmatrix} s_1, & \dots, & s_m\\ l_1, & \dots, & l_m\end{pmatrix} = \sum_{\substack{1\leqslant i\leqslant j\leqslant m\\ s,\ l\geqslant 1}} l\, T^1_{s,i_k}(s_i,\dots,s_j)\ \times$$

$$\times\ T^2_{i_1,\dots,i_{k-1}}\begin{pmatrix} s_1, & \dots, & s_{i-1}, & s, & s_{j+1}, & \dots, & s_m\\ l_1, & \dots, & l_{i-1}, & l, & l_{j+1}, & \dots, & l_m\end{pmatrix} \leqslant$$

$$\leqslant \sum_{1\leqslant i\leqslant j\leqslant m} l\frac{I^{j-i}}{(j-i+1)!}\frac{I^{m-(j-i+1)+1-1}(k-2)^{m-(j-i+1)+1}(k-3)!}{(m-(j-i+1)+1)!} =$$

$$= I^{m-1}(k-3)!\sum_{1\leqslant i\leqslant j\leqslant m} l\frac{(k-2)^{m-(j-i)}}{(j-i+1)!(m-(j-i))!} =$$

$$= \frac{I^{m-1}(k-2)!}{m!}\sum_{1\leqslant i\leqslant j\leqslant m,\ d=j-i}\frac{l}{(k-2)(d+1)}\frac{m!}{d!(m-d)!}(k-2)^{m-d} =$$

$$= \frac{I^{m-1}(k-2)!}{m!}\sum_{d=0}^{m-1}\frac{m!}{d!(m-d)!}(k-2)^{m-d}\left(\sum_{1\leqslant i\leqslant j\leqslant m,\ j-i=d}\frac{l}{(k-2)(d+1)}\right) \leqslant$$

$$\leqslant \frac{I^{m-1}(k-2)!}{m!}\sum_{d=0}^{m-1}\frac{m!}{d!(m-d)!}(k-2)^{m-d} \leqslant \frac{I^{m-1}(k-2)!}{m!}\sum_{t=0,t=m-d}^{m} C^t_m(k-2)^t =$$

$$= \frac{I^{m-1}(k-2)!}{m!}((k-2)+1)^m = \frac{I^{m-1}(k-1)^m(k-2)!}{m!}. \quad \square$$

**Lemma 4.4.** *Let $\bar{I} = max(\bar{i}_r)$ and*

$$\tilde{S}_{\bar{i}_1,\dots,\bar{i}_{\bar{k}}}(m,k) = \sum_{\substack{\{\bar{i}_1^1,\dots,\bar{i}_1^{n_1}\}\sqcup\dots\sqcup\{\bar{i}_m^1,\dots,\bar{i}_m^{n_m}\}= \\ =\{\bar{i}_1,\dots,\bar{i}_{\bar{k}}\} \\ (l_1-1)+\dots+(l_m-1)=k-2 \\ n_r,\ l_r\geqslant 1 \\ s_r=\bar{i}_r^1+\dots+\bar{i}_r^{n_r} \\ s_r-n_r-\ell_r+1\geqslant 0}} \frac{(s_1-1)!}{(s_1-n_1-l_1+1)!(l_1-1)!}\times\dots\times$$

$$\times \frac{(s_m-1)!}{(s_m-n_m-l_m+1)!(l_m-1)!}.$$

*Then* $\tilde{S}_{\bar{i}_1,\dots,\bar{i}_{\bar{k}}}(m,k) \leqslant m(\bar{k}-1)!C^{k-2}_{\bar{I}\bar{k}-\bar{k}}C^{\bar{k}-m}_{\bar{I}\bar{k}}$.

*Proof.* We use the equality

$$\sum_{\tilde{n}_1+\dots+\tilde{n}_m=\bar{k}-m} C^{\tilde{n}_1}_{\bar{I}\tilde{n}_1+\bar{I}}\times\dots\times C^{\tilde{n}_m}_{\bar{I}\tilde{n}_m+\bar{I}}\frac{\bar{I}}{\bar{I}\tilde{n}_1+\bar{I}}\times\dots\times\frac{\bar{I}}{\bar{I}\tilde{n}_m+\bar{I}} =$$

$$= C^{\bar{k}-m}_{\bar{I}(\bar{k}-m)+m\bar{I}}\frac{m\bar{I}}{\bar{I}(\bar{k}-m)+m\bar{I}}$$

that follows from [ 12 ]. Then

$$\tilde{S}_{\bar{i}_1,\dots,\bar{i}_{\bar{k}}}(m,k) =$$

$$= \sum_{\substack{\{\bar{i}_1^1,\dots,\bar{i}_1^{n_1}\}\sqcup\dots\sqcup\{\bar{i}_m^1,\dots,\bar{i}_m^{n_m}\}= \\ =\{\bar{i}_1,\dots,\bar{i}_{\bar{k}}\} \\ (l_1-1)+\dots+(l_m-1)=k-2 \\ n_r,\ l_r\geqslant 1 \\ s_r=\bar{i}_r^1+\dots+\bar{i}_r^{n_r} \\ s_r-n_r-\ell_r+1\geqslant 0}} \frac{(s_1-1)!}{(s_1-n_1-l_1+1)!(l_1-1)!}\times\dots\times$$

$$\times\frac{(s_m-1)!}{(s_m-n_m-l_m+1)!(l_m-1)!} =$$

$$= \sum_{\substack{\{\bar{i}_1^1,\dots,\bar{i}_1^{n_1}\}\sqcup\dots\sqcup\{\bar{i}_m^1,\dots,\bar{i}_m^{n_m}\}= \\ =\{\bar{i}_1,\dots,\bar{i}_{\bar{k}}\} \\ s_r=\bar{i}_r^1+\dots+\bar{i}_r^{n_r}\geqslant n_r\geqslant 1}} \frac{(s_1-1)!}{(s_1-n_1)!}\times\dots\times\frac{(s_m-1)!}{(s_m-n_m)!}\times$$

$$\times \sum_{\substack{(l_1-1)+\dots+(l_m-1)=k-2 \\ s_r-n_r-\ell_r+1\geqslant 0}} \frac{(s_1-n_1)!}{(s_1-n_1-l_1+1)!(l_1-1)!}\times\dots\times$$

$$\times \frac{(s_m - n_m)!}{(s_m - n_m - l_m + 1)!(l_m - 1)!} =$$

$$= \sum_{\substack{\{\bar{i}_1^1,\dots,\bar{i}_1^{n_1}\}\sqcup\cdots\sqcup\{\bar{i}_m^1,\dots,\bar{i}_m^{n_m}\}= \\ =\{\bar{i}_1,\dots,\bar{i}_{\bar{k}}\} \\ s_r=\bar{i}_r^1+\cdots+\bar{i}_r^{n_r}\geqslant n_r\geqslant 1}} \frac{(s_1-1)!}{(s_1-n_1)!} \times \cdots \times$$

$$\times \frac{(s_m-1)!}{(s_m-n_m)!} \sum_{\tilde{l}_1+\cdots+\tilde{l}_m=k-2} C_{s_1-n_1}^{\tilde{l}_1} \dots C_{s_m-n_m}^{\tilde{l}_m} =$$

$$= \sum_{\substack{\{\bar{i}_1^1,\dots,\bar{i}_1^{n_1}\}\sqcup\cdots\sqcup\{\bar{i}_m^1,\dots,\bar{i}_m^{n_m}\}= \\ =\{\bar{i}_1,\dots,\bar{i}_{\bar{k}}\} \\ s_r=\bar{i}_r^1+\cdots+\bar{i}_r^{n_r}\geqslant n_r\geqslant 1}} \frac{(s_1-1)!}{(s_1-n_1)!} \times \cdots \times$$

$$\times \frac{(s_m-1)!}{(s_m-n_m)!} C^{k-2}_{(s_1+\cdots+s_m)-(n_1+\cdots+n_m)} =$$

$$= C^{k-2}_{\bar{i}_1+\cdots+\bar{i}_{\bar{k}}-\bar{k}} \sum_{\substack{\{\bar{i}_1^1,\dots,\bar{i}_1^{n_1}\}\sqcup\cdots\sqcup\{\bar{i}_m^1,\dots,\bar{i}_m^{n_m}\}= \\ =\{\bar{i}_1,\dots,\bar{i}_{\bar{k}}\} \\ s_r=\bar{i}_r^1+\cdots+\bar{i}_r^{n_r}\geqslant n_r\geqslant 1}} \frac{(s_1-1)!}{(s_1-n_1)!} \times \cdots \times \frac{(s_m-1)!}{(s_m-n_m)!} \leqslant$$

$$\leqslant \tilde{S}_{\underbrace{\bar{I},\dots,\bar{I}}_{\bar{k}}}(m,k) = C^{k-2}_{\bar{I}\bar{k}-\bar{k}} \sum_{\substack{\{\bar{i}_1^1,\dots,\bar{i}_1^{n_1}\}\sqcup\cdots\sqcup \\ \sqcup\{\bar{i}_m^1,\dots,\bar{i}_m^{n_m}\}= \\ =\{\bar{I},\dots,\bar{I}\},\ n_r\geqslant 1 \\ s_r=\bar{i}_r^1+\cdots+\bar{i}_r^{n_r}=\bar{I}n_r}} \frac{(s_1-1)!}{(s_1-n_1)!} \times \cdots \times \frac{(s_m-1)!}{(s_m-n_m)!} =$$

$$= C^{k-2}_{\bar{I}\bar{k}-\bar{k}} \sum_{\substack{n_1+\cdots+n_m=\bar{k} \\ n_r\geqslant 1}} \frac{\bar{k}!}{n_1!\dots n_m!} \frac{(\bar{I}n_1-1)!}{(\bar{I}n_1-n_1)!} \times \cdots \times \frac{(\bar{I}n_m-1)!}{(\bar{I}n_m-n_m)!} =$$

$$= \bar{k}! C^{k-2}_{\bar{I}\bar{k}-\bar{k}} \sum_{\substack{\tilde{n}_1+\cdots+\tilde{n}_m=\bar{k}-m \\ \tilde{n}_r=n_r-1\geqslant 0}} \frac{(\bar{I}\tilde{n}_1+\bar{I}-1)!}{(\tilde{n}_1+1)!(\bar{I}\tilde{n}_1+\bar{I}-\tilde{n}_1-1)!} \times \cdots \times$$

$$\times \frac{(\bar{I}\tilde{n}_m+\bar{I}-1)!}{(\tilde{n}_m+1)!(\bar{I}\tilde{n}_m+\bar{I}-\tilde{n}_m-1)!} =$$

$$= \bar{k}! C^{k-2}_{\bar{I}\bar{k}-\bar{k}} \sum_{\tilde{n}_1+\cdots+\tilde{n}_m=\bar{k}-m} \frac{\bar{I}\tilde{n}_1+\bar{I}-\tilde{n}_1}{(\bar{I}\tilde{n}_1+\bar{I})(\tilde{n}_1+1)} \frac{(\bar{I}\tilde{n}_1+\bar{I})!}{\tilde{n}_1!(\bar{I}\tilde{n}_1+\bar{I}-\tilde{n}_1)!} \times \cdots \times$$

$$\times \frac{\bar{I}\tilde{n}_m+\bar{I}-\tilde{n}_m}{(\bar{I}\tilde{n}_m+\bar{I})(\tilde{n}_m+1)} \frac{(\bar{I}\tilde{n}_m+\bar{I})!}{\tilde{n}_m!(\bar{I}\tilde{n}_m+\bar{I}-\tilde{n}_m)!} \leqslant$$

$$\leqslant\ \bar{k}!C^{k-2}_{\bar{I}\bar{k}-\bar{k}} \sum_{\tilde{n}_1+\dots+\tilde{n}_m=\bar{k}-m} \frac{\bar{I}}{\bar{I}\tilde{n}_1+\bar{I}} C^{\tilde{n}_1}_{\bar{I}\tilde{n}_1+\bar{I}} \times \dots \times \frac{\bar{I}}{\bar{I}\tilde{n}_m+\bar{I}} C^{\tilde{n}_m}_{\bar{I}\tilde{n}_m+\bar{I}}\ =$$

$$=\ \bar{k}!C^{k-2}_{\bar{I}\bar{k}-\bar{k}} C^{\bar{k}-m}_{\bar{I}(\bar{k}-m)+m\bar{I}} \frac{m\bar{I}}{\bar{I}(\bar{k}-m)+m\bar{I}}\ \leqslant\ m(\bar{k}-1)!C^{k-2}_{\bar{I}\bar{k}-\bar{k}} C^{\bar{k}-m}_{\bar{I}\bar{k}}\ . \quad \square$$

**Lemma 4.5.** $N^1_i(i_1...,i_k|\bar{i}_1...,\bar{i}_{\bar{k}})\ \leqslant\ (k-1)!(\bar{k}-1)!e^{I(k-1)}2^{\bar{I}\bar{k}-\bar{k}}2^{\bar{I}\bar{k}}$.

*Proof.*

$$N^1_i(i_1,\dots,i_k|\bar{i}_1,\dots,\bar{i}_{\bar{k}})\ =$$

$$= \sum_{\substack{m\geqslant 1\\ s_1+\dots+s_m=i_1+\dots+i_k\\ l_1+\dots+l_m=m+k-2\\ s_r,l_r\geqslant 1}} (-1)^{m+1}\ S_{\bar{i}_1,\dots,\bar{i}_{\bar{k}}}\begin{pmatrix} s_1, & \dots, & s_m\\ l_1, & \dots, & l_m\end{pmatrix}\times$$

$$\times\ T^2_{i_1,\dots,i_k}\begin{pmatrix} s_1, & \dots, & s_m\\ l_1, & \dots, & l_m\end{pmatrix}\ \leqslant$$

$$\leqslant \sum_{\substack{m\geqslant 1\\ s_1+\dots+s_m=i_1+\dots+i_k\\ l_1+\dots+l_m=m+k-2\\ s_r,l_r\geqslant 1}} \frac{I^{m-1}(k-1)^m(k-2)!}{m!}\times$$

$$\times \sum_{\substack{\{\bar{i}^1_1,\dots,\bar{i}^{n_1}_1\}\sqcup\dots\sqcup\{\bar{i}^1_m,\dots,\bar{i}^{n_m}_m\}=\\ =\{\bar{i}_1,\dots,\bar{i}_{\bar{k}}\}\\ \bar{i}^1_r+\dots+\bar{i}^{n_r}_r=s_r\\ s_r-n_r-\ell_r+1\geqslant 0}} \frac{(s_1-1)!}{(s_1-n_1-l_1+1)!(l_1-1)!}\times\dots\times$$

$$\times\ \frac{(s_m-1)!}{(s_m-n_m-l_m+1)!(l_m-1)!}\ =$$

$$=\ \sum_{m\geqslant 1}\frac{I^{m-1}(k-1)^{m-1}(k-2)!}{m!}\times$$

$$\times \sum_{\substack{\{\bar{i}^1_1,\dots,\bar{i}^{n_1}_1\}\sqcup\dots\sqcup\{\bar{i}^1_m,\dots,\bar{i}^{n_m}_m\}=\\ =\{\bar{i}_1,\dots,\bar{i}_{\bar{k}}\}\\ (l_1-1)+\dots+(l_m-1)=k-2\\ n_r,\ l_r\geqslant 1\\ s_r=\bar{i}^1_r+\dots+\bar{i}^{n_r}_r\\ s_r-n_r-\ell_r+1\geqslant 0}} \frac{(s_1-1)!}{(s_1-n_1-l_1+1)!(l_1-1)!}\times\dots\times$$

$$\times\frac{(s_m-1)!}{(s_m-n_m-l_m+1)!(l_m-1)!}\ =$$

$$= \sum_{m\geqslant 1} \frac{I^{m-1}(k-1)^{m-1}(k-2)!}{m!}\ \tilde{S}_{\bar{i}_1,\dots,\bar{i}_{\bar{k}}}(m,k) \ \leqslant$$

$$\leqslant \sum_{m\geqslant 1} \frac{I^{m-1}(k-1)^{m}(k-2)!}{m!}\ \tilde{S}_{\bar{i}_1,\dots,\bar{i}_{\bar{k}}}(m,k) \ \leqslant$$

$$\leqslant \sum_{m\geqslant 1} \frac{I^{m-1}(k-1)^{m}(k-2)!}{m!} m(\bar{k}-1)! C^{k-2}_{\bar{I}\bar{k}-\bar{k}} C^{\bar{k}-m}_{\bar{I}\bar{k}} \ =$$

$$= (k-1)!(\bar{k}-1)! \sum_{m\geqslant 1} \frac{I^{m-1}(k-1)^{m-1}}{(m-1)!}\ C^{k-2}_{\bar{I}\bar{k}-\bar{k}}\ C^{\bar{k}-m}_{\bar{I}\bar{k}} \ \leqslant$$

$$\leqslant (k-1)!(\bar{k}-1)! e^{I(k-1)} 2^{\bar{I}\bar{k}-\bar{k}} 2^{\bar{I}\bar{k}}\ . \quad \Box$$

*Proof of theorem 4.2.* The coefficient for $t_0^{\cdot\cdot} t_{i_1}^{n_1} \dots t_{i_I}^{n_I} \bar{t}_{\bar{i}_1}^{\bar{n}_1} \dots \bar{t}_{\bar{i}_{\bar{I}}}^{\bar{n}_{\bar{I}}}$ is equal

$$\frac{i_1^{n_1}\dots i_I^{n_I}}{n_1!\dots n_I!}\ \frac{\bar{i}_1^{\bar{n}_1}\dots \bar{i}_{\bar{I}}^{\bar{n}_{\bar{I}}}}{\bar{n}_1!\dots \bar{n}_{\bar{I}}!}\ N_i^2 \left( \begin{matrix} i_1, & \dots, & i_I \\ n_1, & \dots, & n_I \end{matrix} \middle| \begin{matrix} \bar{i}_1, & \dots, & \bar{i}_{\bar{I}} \\ \bar{n}_1, & \dots, & \bar{n}_{\bar{I}} \end{matrix} \right) =$$

$$= \frac{i_1^{n_1}\dots i_I^{n_I}}{n_1!\dots n_I!}\ \frac{\bar{i}_1^{\bar{n}_1}\dots \bar{i}_{\bar{I}}^{\bar{n}_{\bar{I}}}}{\bar{n}_1!\dots \bar{n}_{\bar{I}}!}\ N_i^1 \left( \underbrace{i_1,\dots,i_1}_{n_1},\dots,\underbrace{i_I,\dots,i_I}_{n_I} \middle| \underbrace{\bar{i}_1,\dots,\bar{i}_1}_{\bar{n}_1},\dots,\underbrace{\bar{i}_{\bar{k}},\dots,\bar{i}_{\bar{I}}}_{\bar{n}_{\bar{I}}} \right) \ \leqslant$$

$$\leqslant \frac{i_1^{n_1}\dots i_I^{n_I}}{n_1!\dots n_I!}\ \frac{\bar{i}_1^{\bar{n}_1}\dots \bar{i}_{\bar{I}}^{\bar{n}_{\bar{I}}}}{\bar{n}_1!\dots \bar{n}_{\bar{I}}!} k!\bar{k}! e^{\tilde{I}(k-1)} 2^{2\tilde{I}\bar{k}} \ \leqslant\ \tilde{I}^K e^{\tilde{I}K} 2^{\tilde{I}K} \frac{k!\bar{k}!}{n_1!\dots n_I!\bar{n}_1!\dots \bar{n}_{\bar{I}}!} \ \leqslant$$

$$\leqslant\ \tilde{I}^K e^{\tilde{I}K} 2^{\tilde{I}K} \tilde{I}^K \ \leqslant\ (\tilde{I}^2 2^{\tilde{I}} e^{\tilde{I}})^K\ ,$$

where $k = n_1 + \dots + n_I$, $\bar{k} = \bar{n}_1 + \dots + \bar{n}_I$, $K = k + \bar{k}$ and $\tilde{I} = max(I, \bar{I})$.

Consider now monomials from $t_0, t_1, \bar{t}_1, \dots, t_n, \bar{t}_n$ of degree $K$. The number of such monomial is $(2n)^K$. Thus, its sum in the series is not more that

$$(n^2 2^n e^n)^K (2n)^K (4n^3 2^n e^n)^{-K} \ \leqslant 2^{-K}\ .$$

This implies the convergence of the series $F(\tilde{t})$. $\Box$

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