Asymptotic Expansions of Solution to Differential Equations on Complex Manifolds Near Singular Points

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Abstract

Singularities of the solution of the Cauchy problem on complex manifold are investigated. The case of differential equation with constant coefficients is considered in detail with the help of integral representations of solutions. In the case of variable coefficients the singularities of the solution are examined by expansion in series with parameter-dependent integrals as their terms. Normal forms of corresponding integral representations are obtained that allowed to construct asymptotic expansions near singular point of the solution for singularity types of generic position.

Keywords: complex manifold, asymptotic expansion, normal form, parameter-dependent integral, Cauchy problem, vanishing cycle, singularities.

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Contents

Introduction

The present paper is written in the framework of analysis of differential equations on complex manifolds. This theory originated by the classical works of Jean Leray [1] — [6] at present have its further development in a set of works by Boris Sternin and Victor Shatalov (see [7] and the bibliography therein). In particular, for differential equation with constant coefficients it allows to obtain explicit solution in form of parameter-dependent integral with the help of an integral transform of ramifying analytic functions introduced by Boris Sternin and Victor Shatalov (see [7] — [10]).

In case of equation with variable coefficients, clearly, there is no possibility to present an exact formula for solutions. However, it is possible to obtain an asymptotics of solutions near its singular[1] points. Such asymptotics have the form of parameter-dependent integrals over some cycles (homology classes) of the corresponding complex spaces (see [7], [11]).

Clearly (this concernes the case of constant coefficients as well), one should like to obtain more explicit expressions for asymptotic expansions of solutions (say, in

[1]Singular points arising in this theory are as a rule ramification points of a solution.

the form of Puiset series). This paper is aimed at writing out such formulas at least in generic position. To this end, since solutions to Cauchy problems are, as a rule, ramifying, such an investigation must include the description of the Riemannian structure of solutions. This can be done in terms of normal forms, and the investigation can be naturally accomplished within the framework of the theory of singularities [12], [13] and given paper is devoted to this theme.

1 Preliminary Results

1.1 Integral representations of the solutions

The main tool of investigation of solutions to differential equations on complex manifolds is the integral representation given in [7]. To make the presentation in this paper self-contained, let us make a short description of corresponding representation for solution to the Cauchy problem with constant coefficients.

1. Let $f(x) = f(x^1, \dots, x^n)$ be a complex-analytic function,

$$X = \{x \in \mathbf{C}^n \mid s(x) = 0\}$$

be a complex algebraic manifold. Consider a linear differential operator of order m with constant coefficients

$$H\left(-\frac{\partial}{\partial x}\right) = \sum_{|\alpha|=m} a_\alpha \left(-\frac{\partial}{\partial x}\right)^\alpha,$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multiindex with integer α_js, $|\alpha| = \alpha_1 + \dots + \alpha_n$, and

$$\left(-\frac{\partial}{\partial x}\right)^\alpha = \left(-\frac{\partial}{\partial x^1}\right)^{\alpha_1} \cdots \left(-\frac{\partial}{\partial x^n}\right)^{\alpha_n}.$$

Consider the Cauchy problem

$$\begin{cases} H\left(-\frac{\partial}{\partial x}\right) u(x) = f(x), \\ u(x) \equiv 0 \pmod{m} \text{ on } X, \end{cases} \tag{1}$$

where the comparizon in (1) means that the function $u(x)$ has zero of order m on the manifold X. The solution to this problem may be represented in integral form (see [7]):

$$u(x) = (n-m-1)! \left(\frac{i}{2\pi}\right)^{n-1} \int_{h(x)} \operatorname{Res} \frac{f(y)\, dy \wedge \omega(p)}{H(p)\,(p(y-x))^{n-m}}, \quad n > m, \tag{2}$$

$$u(x) = \frac{(-1)^{m-n}}{(m-n)!}\left(\frac{i}{2\pi}\right)^{n-1} \int\limits_{h_1(x)} \frac{f(y)\,(p(y-x))^{m-n}\; dy \wedge \omega(p)}{H(p)}, \quad n \leq m, \quad (3)$$

where $\omega(p)$ is Leray's form

$$\omega(p) = \sum_{j=1}^{n} (-1)^j p_j dp_1 \wedge \ldots \wedge \widehat{dp_j} \wedge \ldots \wedge dp_n,$$

and

$$p(y-x) = \sum_{j=1}^{n} p_j(y^j - x^j).$$

The homology classes $h(x)$ and $h_1(x)$ involved in the integrals on the right in (2) and (3) can be described as follows. The class

$$h(x) \in H_{2n-2}(\Sigma_x \setminus \operatorname{char} H, X)$$

is a homology class in the space $\mathbf{C}_y^n \times \mathbf{CP}_{n-1,p}$ coinciding with the vanishing cycle of the complex quadrics $(\Sigma_x, \Sigma_x \cap X)$ for x close to X (one can show that the intersection $\Sigma_x \cap X$ is biholomorhic to a complex quadrics for such values of x). The class

$$h_1(x) \in H_{2n-1}(\mathbf{C}_y^n \times \mathbf{CP}_{n-1,p} \setminus \operatorname{char} H, \Sigma_x \cup X)$$

is now defined as a solution to the equation

$$\partial h_1(x) = h(x),$$

where

$$\partial \; : \; H_{2n-2}(\Sigma_x \setminus \operatorname{char} H, X) \rightarrow H_{2n-1}\left(\mathbf{C}_y^n \times \mathbf{CP}_{n-1,p} \setminus \operatorname{char} H\right)$$

is the Bokstein homomorphism. In the latter formulas we used the following notation:

$$\Sigma_x = \{(y,p) \in \mathbf{C}_y^n \times \mathbf{CP}_{n-1,p} | p(y-x) = 0\},$$
$$\operatorname{char} H = \{(y,p) \in \mathbf{C}_y^n \times \mathbf{CP}_{n-1,p} | H(p) = 0\},$$

and X is the lifting of initial manifold to the space $\mathbf{C}_y^n \times \mathbf{CP}_{n-1,p}$. We remark that near any noncharacteristic point x of the initial data manifold the solution $u(x)$ to problem (1) belongs to function space $\mathcal{A}_m$ (see [7]). The latter means that there exist a constant C such that

$$|u(x)| \leq C|s(x)|^m.$$

Near a noncharacteristic point of X, the solution to Cauchy problem (1) in the integral form (2), (3) can be expanded in a series in integer degrees of x (the Taylor formula). On the contrary, it is clear that in the neighborhood of a characteristic point there exists no such expansion. So, the problem of obtaining an asymptotic expansion for solutions to the Cauchy problem near characteristic points of the manifold X is of great inerest.

The asymptotic expansions for solutions to the Cauchy problem can be solved with the help of classification of singularities of special contour integrals being the representation of the solution. For problem (1) this investigation can be naturally split into the following three subproblems:

- To construct normal forms of sets X, Σ_x and char H included in representations (2), (3).
- To obtain normal forms of integrals (2), (3).
- To investigate singularities of obtained normal forms of integrals and point out possible types of ramification.

2. Let us consider now the Cauchy problem with variable coefficients:

$$\begin{cases} H(x, -\partial/\partial x)u(x) = f(x), \\ u(x) \equiv 0 \pmod{m} \text{ on } X, \end{cases} \tag{4}$$

where function f is holomorphic,

$$X = \{x \in \mathbf{C}^n \mid s(x) = 0\}$$

is a complex algebraic manifold, and

$$H\left(x, -\frac{\partial}{\partial x}\right) = \sum_{|\alpha| \le m} a_\alpha(x) \left(-\frac{\partial}{\partial x}\right)^\alpha$$

is a differential operator with variable holomorphic coefficients $a_\alpha(x)$. The asymptotic expansion of the solution to problem (4) can be written down (see [7]) in the form integrals similar to that involved in representations (2), (3), but with different phase function and with some amplitude function not equal to 1, namely

$$u(x) = 2\pi i \left(\frac{i}{2\pi}\right)^{n-1} \int_{h(x)} \left(\frac{\partial}{\partial q_0}\right)^{n-m} G(x, q, t) \mid_{qy=0} f(y) \omega(t, q') \wedge dy, \tag{5}$$

In the latter formula we have used the following notation:

i) The variables q, q', and t belong to the spaces

$$q = (q_0, q_1, \dots, q_n) \in \mathbf{C}_q^{n+1},\ q' = (q_1, \dots, q_n) \in \mathbf{C}_{q'}^n \text{ and } t \in \mathbf{C}_t,$$

and $qy = q_0 + q_1 y^1 + \dots + q_n y^n$.

ii) $\omega(t, q')$ is modified Leray form

$$\omega(t, q') = (1-m)t\, dq_1 \wedge \dots \wedge dq_n - dt \wedge \tilde{\omega}(q'),$$

where

$$\tilde{\omega}(q') = \sum_{j=1}^{n} (-1)^{j-1} q_j dq_1 \wedge \dots \wedge \hat{dq_j} \wedge \dots \wedge dq_n.$$

iii) The function $G(x, q, t)$ is an *elementary solution* to problem 4 (see [7], [14]). This solution has the form

$$G(x, q, t) = \sum_{k=-1}^{\infty} a_k(x, q', t) f_{k+1}(q_0 + S(x, q', t)), \tag{6}$$

where $a_k(x, q, t)$ is some holomorphic amplitude function and $\{f_k(z)\}$ is the following Ludwig sequence[2] ([15]):

$$f_j(z) = \begin{cases} (-1)^{j-1} \dfrac{(-j-1)!}{z^j}, & j < 0, \\ \dfrac{z^j}{j!}(\ln z - c_j), & j \geq 0, \end{cases}$$

where

$$c_j = \begin{cases} 0, & j = 0, \\ \sum_{i=1}^{j} 1/i & j \geq 1. \end{cases} \tag{7}$$

iv) If the operator $(\partial/\partial q_0)^{n-m}$ has negative power $n < m$, then by definition

$$\left(\frac{\partial}{\partial q_0}\right)^{-1} \sum a_k f_k = \sum a_k f_{k+1},$$

and operators $(\partial/\partial q_0)$, $(\partial/\partial q_0)^{-1}$ are obviously inverse to each other on the set of functions representable in the form (6).

[2] A function sequence $f_k(z)$ is called a *Ludwig sequence* if it is closed under differentiation: $f_j'(z) = f_{j-1}(z)$.

v) The function $S(x,q',t)$ is a solution to a special Cauchy problem for the Hamilton-Jacobi equation

$$\begin{cases} \dfrac{\partial S(x,q',t)}{\partial t} + H\left(x, \dfrac{\partial S(x,q',t)}{\partial x}\right) = 0 \\ S(x,q',0) = q_1 x^1 + \ldots + q_n x^n = q'x. \end{cases} \tag{8}$$

To describe the class $h(x)$, we introduce the set of singularities of the integrand in (5):

$$\Sigma_x = \{(q',t,y) | S(x,q',t) - q'y = 0\}. \tag{9}$$

Then $h(x)$ is a relative homology class

$$h(x) \in H_{2n}(\mathbf{C}_t \times \mathbf{CP}^n_{q'} \times \mathbf{C}^n_y \setminus \Sigma_x, X \cup \{t=0\}). \tag{10}$$

More precisely, $h(x)$ is to be a homology class on the Riemannian surface of elementary solution $G(x,q,t)$ ramifying along Σ_x logarithmically. The detailed description of $h(x)$ the reader will find in Subsection 4.2 below.

1.2 Some facts from the theory of normal forms

The notion of monodromy naturally arises in the investigation of ramification of parameter-dependent integrals. Suppose that on a level set $G_z = \{g(y) = z\}$ of some function $g(y)$ a homology class $\sigma(y)$ is defined, and in the plane $\mathbf{C}$ with the coordinate z some path γ is given. Then the analytic continuation of integral

$$\int_{\sigma(y)} \omega$$

along path γ can be found in the form of the integral

$$\int_{M_\gamma \sigma} \omega,$$

where the monodromy operator M_γ is an automorphism of homology groups of the set G_z onto itself along path γ.

Let us now introduce some facts from the theory of normal forms (see, for example, [13]) that will be used in the sequel for investigation of singularities of parameter-dependent integrals in question.

Let

$$g : M^n \to \mathbf{C}$$

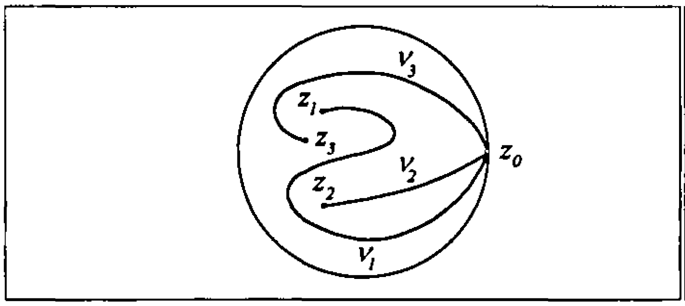

Figure 1. Marked set of cycles

be a holomorphic function defined on n-dimensional complex manifold M, and U be some contractible compact domain in the plane $\mathbf{C}$. As above, we denote the level set of the function g by $G_z = g^{-1}(z)$ and fix some non-critical value $z_0 \in \partial U$ on the boundary of the domain U.

Definition 1 A set of cycles $\Delta_1, \ldots, \Delta_\mu$ from $(n-1)$-dimensional homology group $H_{n-1}(G_{z_0})$ together with the set of paths $v_1, \ldots, v_\mu$ connecting the base point z_0 with critical points $z_1, \ldots, z_\mu$ is called *the marked set of cycles* if it the following three conditions are fulfilled:

- for any $i = 1, \ldots, \mu$ the cycle Δ_i is a vanishing cycle along the path v_i;
- any two different paths v_i and v_j have the only common point $v_i(1) = v_j(1) = z_0$;
- the paths $v_1, \ldots, v_\mu$ are enumerated in accordance with the order, in which they enter into point z_0 counterclockwise, beginning from the boundary ∂U of domain U (see Fig. 1).

The matrix of intersection numbers for marked set of cycles (this set forms a basis in corresponding homology group) defines the monodromy group of the singularity. There exist a number of ways to describe intersection numbers of cycles. We shall use below the so-called *D-diagrams* (*Dynkin's diagrams*, [13]). To define D-diagrams we need the notion of stabilization.

Definition 2 [13]. Let

$$g(x) : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$$

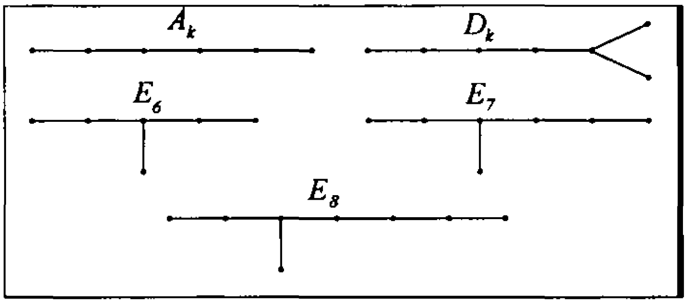

Figure 2. Diagrams of singularities

be a germ of holomorphic function. For any integer m a germ of function

$$g(x) + \sum_{j=1}^{m} t_i^2 : (\mathbf{C}^{n+m}, 0) \to (\mathbf{C}, 0)$$

is called a *stabilization* of germ $g(x)$.

Definition 3 [13]. *D-diagram* is a graph satisfying the following two conditions:

- nodes of graph are in one-to-one correspondence with elements Δ_i of marked base of homology group of level manifold considered for stabilization of the initial function with number of variables equal to $N \equiv 3 \bmod 4$;
- i-th and j-th nodes of graph are connected by edge of order $< \Delta_i, \Delta_j >$, where by $<,>$ we denote the intersection number; edges of negative order are shown by dotted lines.

For simple singularities the following theorem holds (see [13, p. 75]):

Theorem 1 *For simple singularities of the types*

$$A_k\,(x^{k+1} + \sum t_i^2),$$
$$D_k\,(x^2 y + y^{k-1} + \sum t_i^2),$$
$$E_6\,(x^3 + y^4 + \sum t_i^2),$$
$$E_7\,(x^3 + xy^3 + \sum t_i^2),$$
$$E_8\,(x^3 + y^5 + \sum t_i^2)$$

there exist marked bases of vanishing cycles, for that intersection numbers are given by D-diagrams shown on Figure 2.

Having intersection numbers of vanishing cycles for stabilization with some fixed numbers of variables, one can easy calculate intersection numbers for stabilization of any number of variables with the help of the following proposition (see [13, p. 46]):

Theorem 2 *If $\{\Delta_i\}$- marked base of cycles in homology group of non-singular level manifold for singularity $g(x)$, then there exists a marked base $\{\tilde{\Delta}_i\}$ for singularity $g(x)+\sum_{j=1}^{m} t_i^2$ such that the matrix of intersection numbers is defined by the relation*

$$<\tilde{\Delta}_i, \tilde{\Delta}_j> = [\mathrm{sgn}(j-i)]^m (-1)^{nm+\frac{m(m-1)}{2}} <\Delta_i, \Delta_j>,$$

and marked bases $\{\Delta_i\}$, $\{\tilde{\Delta}_i\}$ correspond to the same sets of paths connecting critical and non-critical values of functions $\tilde{g}(x)$, $\tilde{g}(x)+\sum_{j=1}^{m} t_i^2$.

Consider now a holomorphic function

$$h(y) \; : \; \mathbf{C}_y^n \to \mathbf{C}$$

with an isolated critical point and let

$$\tilde{h} : \mathbf{C}_y^n \times \mathbf{C}_x^k \to \mathbf{C}$$

be its deformation, that is $\tilde{h}(y,0) = h(y)$. Let some complex plane be chosen in parameter space $\mathbf{CP}_x^k$ which is not contained in the set of critical values σ of the function $\tilde{h}(x,y)$. Then intersection of the plane and set of critical values is discrete. Denote by t a holomorphic coordinate on the plane with the origin in point of intersection. In a small neighborhood of the selected point let us consider a sector $a \le \arg t \le b$. For any $t \neq 0$ in this sector one can choose a base

$$\Delta_1(t), \ldots, \Delta_\mu(t)$$

in homologies of local non-singular level hypersurface of function $\tilde{h}$. Let a form ω be a holomorphic $(n-1)$-form in $\mathbf{CP}_y^n \times \mathbf{CP}_x^k$ near the origin. In the sector introduced above we consider the vector-function

$$I(t) = \left(\int\limits_{\Delta_1(t)} \omega, \ldots, \int\limits_{\Delta_\mu(t)} \omega \right).$$

The following theorem is valid (see [13], [16]).

Theorem 3 *The function $I(t)$ admits the expansion into the series*

$$\sum_{\alpha,k} a_{k,\alpha}\, t^{\alpha}\, (\ln t)^k$$

convergent for sufficiently small in module value of the parameter. The numbers α are rational and non-negative. For every α involved in the latter expansion the number $\exp(2\pi i\alpha)$ *is an eigenvalue of the classical monodromy operator. The coefficients $a_{k,\alpha}$ are vectors in $\mathbf{C}^{\mu}$ such that $a_{k,0} = 0$ for any $k > 0$, and $a_{k,\alpha} = 0$ when Jordan form of classical monodromy operator has no blocks of dimension more than k with the eigenvalue* $\exp(2\pi i\alpha)$.

2 The Cauchy Problem with Constant Coefficients

2.1 Geometric description

The following two sections are aimed at the consideration of the Cauchy problem (1) with constant coefficients.

As it was mentioned in the previous section, solutions to Cauchy problem in question can be written down in the form of integral representations (2), (3). The integrand in both these representations is a differential form on the space $\mathbf{C}^n_x \times \mathbf{C}^n_y \times \mathbf{CP}_{n-1,p}$. We remark that, since X is an algebraic manifold, it can be lifted into the space $\mathbf{CP}^n_y$ as an analytic set. This allows one to use the Thom theorem (see, e. g., [17]) for investigation of singularities of solutions to (1). More precisely, to apply the Thom theorem, one has only to compactify the space $\mathbf{C}^n_y$.

Denote by

$$\Sigma_x = \{p(y-x) = 0\} \in \mathbf{C}^n_x \times \mathbf{CP}^n_y \times \mathbf{CP}_{n-1,p}$$

and

$$\operatorname{char} H = \{H(p) = 0\} \in \mathbf{C}^n_x \times \mathbf{CP}^n_y \times \mathbf{CP}_{n-1,p}$$

the zero sets of the denominator in the representation (2), and by X the lifting of the initial set to $\mathbf{C}^n_x \times \mathbf{CP}^n_y \times \mathbf{CP}_{n-1,p}$ (we remark that all these sets included in the definition of cycles $h(x)$ and $h_1(x)$ are well-defined in this space).

Consider the natural projection

$$\mathbf{C}^n_x \times \mathbf{CP}^n_y \times \mathbf{CP}_{n-1,p} \to \mathbf{C}^n_x,$$

that determines (in general, not locally trivial) bundle

$$\Sigma_x \setminus (\operatorname{char} H \cup X) \to \mathbf{C}^n_x. \tag{11}$$

Later on, bundle (11) induces the projection of the space $T_x(\Sigma_x \setminus (\mathrm{char}\, H \cup X))$ to the base space $\mathbf{C}^n_x$:

$$T_x(\Sigma_x \setminus (\mathrm{char}\, H \cup X)) \rightarrow \mathbf{C}^n_x.$$

Let $L \subset \mathbf{C}^n_x$ be a set of points in the base of bundle (11) such that the rank of the latter projection degenerates. Then, due to the Thom theorem, bundle (11) is locally trivial in a sufficiently small neighborhood of any point $x \in \mathbf{C}^n_x \setminus L$ and, hence, all singularities of integrals (2), (3) lye in set L.

Geometrically, it means that singularities of the solution to problem (1) occur for values of x such that the manifold Σ_x is tangent to some stratum of the union of initial and characteristic sets $X \cup \mathrm{char}\, H$. We distinguish the following four cases:

- Σ_x *is tangent to* $\mathrm{char}\, H$. The tangency condition has the form

  $$(x-y)dp - p\,dy = \lambda \frac{\partial H(p)}{\partial p} dp$$

  for a constant λ. One can easily see that it holds just for $p = 0$. However, this is impossible since $p \in \mathbf{CP}_{n-1,p}$. So, this case does not contribute to the singularity set of the solution.

- Σ_x *is tangent to* X *and not to* $X \cap \mathrm{char}\, H$. The direct computation shows that this is possible iff x is a noncharacteristic point of X. However, as it follows from the Cauchy-Kowalewski theorem, integrals (2) and (3) are holomorphic for such values of x (see, for instance, [18]). On the other hand one can give a direct proof of the assertion that integrals (2) and (3) are holomorphic at such a point. Namely, the tangency condition

  $$(x-y)dp - p\,dy = \lambda \frac{\partial s(y)}{\partial y} dy$$

  gives

  $$x = y \text{ and } p = -\lambda \frac{\partial s(y)}{\partial y},$$

  that is, $x \in X$ and p is a covector of X in the tangency point. Later on, the integration cycle $h(x)$ is the vanishing cycle of the quadrics $\Sigma_x \cap X$ of dimension $2n-2$. The self-intersection number $< h, h >$ equals zero for given dimension of $h(x)$. According to Picard-Lefschetz theory [19] for quadric singularity the vanishing cycle variation vanishes:

  $$\mathrm{var}\, h(x) = (-1)^{(2n-2)(2n-1)/2} < h, h > h = 0,$$

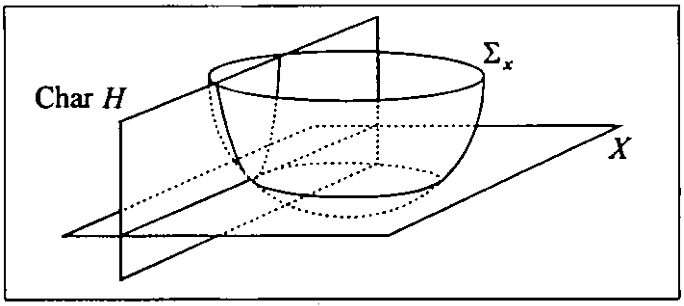

Figure 3. Σ_x contacts $X \cap \operatorname{char} H$ and not contacts X

which means that there is no ramification at the point in question. Since $u(x) \in A_m$ with $m > -1$, and (2) is a single-valued function, integral (2) is holomorphic due to the removed singularity theorem. As it will be shown below integral (3) can be evaluated via derivatives of integral (2) with respect to x, whence it follows that integral (3) is also holomorphic.

- Σ_x *is tangent both to* X *and* $X \cap \operatorname{char} H$. The straightforward computations show that this is possible iff x is a characteristic point of the inital manifold X. This case was investigated by Jean Leray [1], [6] with the help of the uniformization of the Cauchy problem.

- Σ_x *is tangent to* $X \cap \operatorname{char} H$ *and not to* X. Integrals (2),(3) do have singularities for such type of tangency. The geometric explanation of this fact is that cycles of integration $h(x)$ and $h_1(x)$ are covanishing cycles and their variation does not vanish. The simplified representation of this case is shown on Figure 3.

2.2 General form of integrals

Let us show that for investigation of singularities of the solution of the problem (1) it is sufficient to examine integral (2).

Let x be a point where the solution $u(x)$ is regular and let $m = n$. One can easily check that derivatives of function $u(x)$ are given by

$$\frac{\partial u(x)}{\partial x^j} = \left(\frac{i}{2\pi}\right)^{n-1} \int\limits_{h(x)} \operatorname{Res} \frac{p_j f(y)\, dy \wedge \omega(p)}{H(p)\,(p(y-x))},$$

due to representation (3) and the identity $\partial h_1(x) = h(x)$. If $m > n$, then the equality

$$\frac{\partial^{m-n} u(x)}{\partial x^{j_1} \dots \partial x^{j_{m-n}}} =$$
$$= \left(\frac{i}{2\pi}\right)^{n-1} \int\limits_{h_1(x)} \frac{p_{j_1} \cdots p_{j_{m-n}} f(y) \, (p(y-x))^{m-n} \, dy \wedge \omega(p)}{H(p)},$$

holds, which allows to write down integral representation for derivatives of the solution

$$\frac{\partial^{m-n+1} u(x)}{\partial x^{j_1} \dots \partial x^{j_{m-n+1}}} = \left(\frac{i}{2\pi}\right)^{n-1} \int\limits_{h(x)} \operatorname{Res} \frac{p_{j_1} \cdots p_{j_{m-n+1}} f(y) \, dy \wedge \omega(p)}{H(p) \, (p(y-x))},$$

for any $m \geq n$. Next, if function $u(x)$ is regular along some path connecting points a and x then

$$\frac{\partial^k u}{\partial x^{j_1} \dots \partial x^{j_k}}(x) = \frac{\partial^k u}{\partial x^{j_1} \dots \partial x^{j_k}}(a) + \int\limits_a^x \sum_{i=1}^n \frac{\partial^{k+1} u}{\partial x^{j_1} \dots \partial x^{j_k} \partial x^i}(y) \, dy^i.$$

By that, singularities of the solution are expressed by singularities of its first derivatives, singularities of first derivatives are expressed by singularities of second derivatives, and so on. Hence, the investigation of singularities of the solution of problem (1) is reduced to the investigation of integrals of the form

$$\int\limits_{h(x)} \operatorname{Res} \frac{\tilde{f}(y) \, dy \wedge \omega(p)}{H(p) \, (p(y-x))^N}, \quad N = 1, 2, 3, \dots \tag{12}$$

with some holomorphic function $\tilde{f}(y)$.

In space $\mathbf{CP}_y^n \times \mathbf{CP}_{n-1,p}$, we have three manifolds Σ_x, X and $\operatorname{char} H$ defining the integratiion cycle $h(x)$. On the other hand, as it follows from definition of representation (2), one can see that $h(x) \subset \Sigma_x$. Let us use this fact and restrict integral (12) to Σ_x for its investigation. Evidently, the class $h(x)$ on Σ_x is now defined by two manifolds $\Sigma_x \cap X$ and $\Sigma_x \cap \operatorname{char} H$, and the singularities of integral (12) are originated by their tangency. Under the requirement that X and $\operatorname{char} H$ are regular manifolds, the intersections $\Sigma_x \cap X$ and $\Sigma_x \cap \operatorname{char} H$ are also regular (supposing $\Sigma_x \cap X$ and $\Sigma_x \cap \operatorname{char} H$ are not tangent to each other), and one can use coordinates

$z = (z_0, \dots, z_{2n-2})$ such that

$$\Sigma_x \cap X = \{z_0 = 0\},$$
$$\Sigma_x \cap \operatorname{char} H = \{z_0 = \Phi(z_1, \dots, z_{2n-2}, x)\}.$$

Now critical points of function Φ can be classified according to standard methods (see [12]) and normal form of Φ can be obtained for every concrete critical point type.

The calculation of the residue in formula (12) gives

$$\operatorname{Res} \frac{\tilde{f}(y)\, dy \wedge \omega(p)}{H(p)\,(p(y-x))^N} = \frac{\Omega(y,p)}{H(p)^N},$$

with some regular form $\Omega(y,p)$. The latter formula together with the Taylor expansion of the form Ω allows us to expand integral (12) in a series

$$\sum_j \sum_\alpha C_j^\alpha \int\limits_{h(x)} \frac{z^j}{(z_0 - \Phi(z', x))^l} \tilde{\omega}_\alpha(z).$$

Here and below we use the following notation:

$$z = (z_0, z') \text{ where } z' = (z_1, \dots, z_{2n-2}),$$
$$z^j = (z_0)^{j_0} \dots (z_{2n-2})^{j_{2n-2}},$$

and

$$\tilde{\omega}_\alpha(z) = dz_0 \wedge \dots \wedge \hat{dz_\alpha} \wedge \dots \wedge dz_{2n-2}.$$

The obtained expansion leads us to the investigation of singularities of the integrals

$$I_j^{\alpha,l} = \int\limits_{h(x)} \frac{z^j}{(z_0 - \Phi(z', x))^l} \tilde{\omega}_\alpha(z). \tag{13}$$

After the change of variables that brings function $\Phi(z', x)$ to the normal form, the equation of the manifold $\Sigma_x \cap \operatorname{char} H$ takes the form $z_0 = g(z', \lambda)$, where $g(z', \lambda)$ is a *versal deformation* (see [13, p. 67]) for a normal form of the corresponding singularity type, and $\lambda = \lambda(x)$ is new parameters of integral (13). Here and below we use the same notation z for variables of integration assuming that integral (13) is already normalized. Since a versal deformation function $g(z', \lambda)$ has an absolute term λ_0, the equality holds:

$$\left(\frac{\partial}{\partial \lambda_0}\right)^{l-1} \int\limits_{h(\lambda)} \frac{z^j}{z_0 - g(z', \lambda)} \tilde{\omega}_\alpha(z) = (-1)^{l-1}(l-1)!\, I_j^{\alpha,l}(x).$$

From this, it follows that the singularities of integral (13) coincide with singularities of the integral

$$\int_{h(\lambda)} \frac{z^j}{z_0 - g(z', \lambda)} \tilde{\omega}_\alpha(z). \tag{14}$$

The latter integral is an integral over a relative cycle homeomorphic to the covanishing cycle from the group $H^{(f)}_{2n-2}(\Sigma_x \setminus (\mathrm{char}\, H \cap \Sigma_x))$ in sufficiently small neighborhood of the tangency point of manifolds $\Sigma_x \cap \mathrm{char}\, H$ and $\Sigma_x \cap X$. It is easier to use integrals over an absolute cycle, therefore we reduce integral (14) to the form

$$I_j^\alpha(\lambda) = \int_{\Delta(\lambda)} \frac{z^j \ln z_0}{z_0 - g(z', \lambda)} \tilde{\omega}_\alpha(z), \tag{15}$$

in small sector up to the factor $1/(2\pi i)$. Here the cycle $\Delta(\lambda)$ is constructed as the union of two copies of a representative of the class $h(\lambda)$ lying on different sheets of the Riemannian surface of the integrand and having opposite orientation. This cycle can be treated as a representative of an absolute homology class in the space

$$\Delta(\lambda) \in H_{2n-2}(\Sigma_x \setminus (\mathrm{char}\, H \cup X) \cap \Sigma_x)$$

(the similar construction was used in [20, p. 154].)

3 Further investigation of singularities

3.1 Non-degenerated singular point

To begin with, let us consider the non-degenerated singularity A_1, for which the manifolds $\Sigma_x \cap \mathrm{char}\, H$ and $\Sigma_x \cap X$ have the quadratic tangency. This means that the function $g(z', \lambda)$ from integral (15) has a non-degenerated second differential and the normal forms for the manifolds $\Sigma_x \cap \mathrm{char}\, H$ and $\Sigma_x \cap X$ are

$$\begin{aligned} \Sigma_x \cap X &= \{z_0 = 0\}, \\ \Sigma_x \cap \mathrm{char}\, H &= \{z_0 = (z_1)^2 + \ldots + (z_{2n-2})^2 + \lambda_0\}. \end{aligned}$$

In this case integral (15) can be rewritten as

$$I_j^\alpha(\lambda) = \int_{\Delta(\lambda)} \frac{z^j \ln z_0}{z_0 - (z')^2 - \lambda_0} \tilde{\omega}_\alpha(z),$$

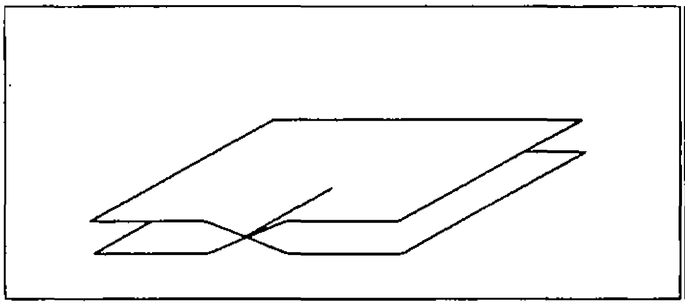

Figure 4. Singularity A_1

where

$$(z')^2 = (z_1)^2 + \ldots + (z_{2n-2})^2.$$

This integral has singularity for $\lambda_0 = 0$ only. The monodromy operator for the integration cycle $\Delta(\lambda)$ (and, hence, for integral (15)) is completely defined by its self-intersection number. Non-critical level manifold is diffeomorphic to fibre space of disks of tangent bundle for $(2n-2)$-dimensional sphere. Therefore, if this manifold is oriented in accordance with structure of tangent bundle to a sphere, the self-intersection number of vanishing cycle in this manifold equals the Euler characteristic of the sphere

$$\chi(S^{2n-2}) = 1 + (-1)^{2n-2} = 2,$$

and, in considered dimension,

$$< \Delta, \Delta > = (-1)^{(2n-2)(2n-3)/2} \chi(S^{2n-2}) = (-1)^{n-1} 2.$$

As it follows from Picard-Lefschetz theorem [13], the action of the monodromy operator corresponding to encircling the critical value $\lambda_0 = 0$ is given by

$$\begin{aligned} M\Delta &= \Delta + (-1)^n < \Delta, \Delta > \Delta = -\Delta, \\ MM\Delta &= -\Delta - (-1)^n < \Delta, \Delta > \Delta = \Delta, \end{aligned}$$

and monodromy operator corresponds to a one-cell matrix $M = -1$ having a single eigenvalue $\exp(2\pi i)$. Thus, non-degenerated singularity A_1 is a singularity of square root type, the Riemannian surface of integral (15) is that of square root (Figure 4). In accordance with theorem 2 from section 1.2 on asymptotic expansion of an integral, integral (15) can be expanded in a series

$$\sum_j a_j \lambda_0^{(2j+1)/2}$$

in the neighborhood of such singular point. In the book [13, p. 205] it is shown that the summation is carried only over non-negative j for singularity of square root type.

3.2 Singular point of A_k type

The singularity A_1 examined in previous section is one of the series A_k (see [12], [13]). For any $k > 1$ not only first but also the second differential of the function $g(z', \lambda)$ from (15) is degenerated, so the situation is a more complicated one. Let us consider singularities A_k in detail.

The manifolds $\Sigma_x \cap \operatorname{char} H$ and $\Sigma_x \cap X$ have the following normal forms:

$$\begin{aligned}\Sigma_x \cap X &= \{z_0 = 0\},\\ \Sigma_x \cap \operatorname{char} H &= \{z_0 = (z_1)^{k+1} + \lambda_{k-1}(z_1)^{k-1} + \ldots + \lambda_1 z_1 +\\ &+ (z_2)^2 + \ldots + z_{2n-2}^2 + \lambda_0\}.\end{aligned}$$

The function

$$(z_1)^{k+1} + \lambda_{k-1}(z_1)^{k-1} + \ldots + \lambda_1 z_1 + (z_2)^2 + \ldots + z_{2n-2}^2$$

has k critical points and critical values, which can be easily computed for every concrete k. For instance, when $k = 2$ one can obtain two critical points

$$z_1 = \pm\sqrt{-\frac{\lambda_1}{3}}, \quad z_2 = \ldots = z_{2n-2} = 0,$$

with the critical values

$$\lambda_0 = \pm\frac{2\lambda_1}{3}\sqrt{-\frac{\lambda_1}{3}}.$$

The critical values of the parameters λ_0 and λ_1 for $k = 2$ are shown on Figure 5. Every critical value w_j is attached with the vanishing cycle

$$\Delta_j \in H_{2n-2}(\Sigma_x \setminus (\operatorname{char} H \cap \Sigma_x)),$$

that vanishes along the path from critical value w_j to some non-critical value. From the theorem about matrix of intersection numbers (see section 1.2) follows that there exists a marked base of vanishing cycles $\Delta_1, \ldots, \Delta_k$ for which the intersection numbers in our dimension are given by the formulas

$$\begin{aligned}< \Delta_i, \Delta_i > &= (-1)^{n-1}2, \quad i = 1, \ldots, k\\ < \Delta_i, \Delta_{i+1} > &= (-1)^n, \quad i = 1, \ldots, k-1,\\ < \Delta_i, \Delta_j > &= 0, \quad |i - j| > 1.\end{aligned}$$

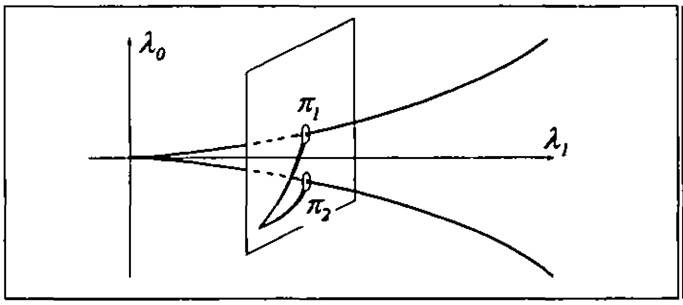

Figure 5. Critical values of parameters for A_2

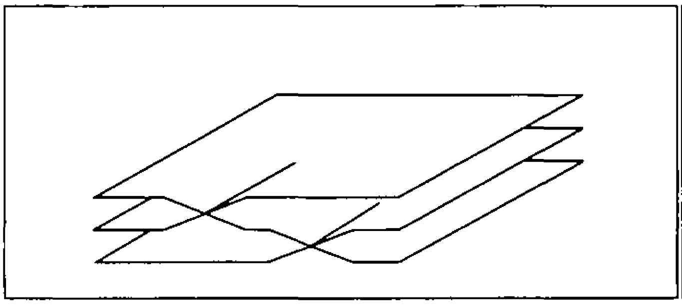

Figure 6. Singularity A_2

Therefore, one can obtain the monodromy operators M_j corresponding to encircling the critical values w_j. For instance, for $k = 2$ we have

$$M_1 = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix},$$

and the loops π_1 and π_2, encircling the corresponding critical points are shown on Figure 5. Further, for $k = 3$

$$M_1 = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, M_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

In general case we obtain k matrices defined by

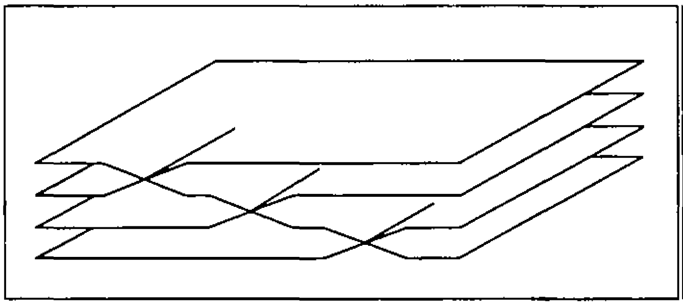

Figure 7. Singularity A_3

$$M_j = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}, \quad j = 1, \dots, k,$$

where -1 is placed in j -th row and j -th column of matrix M_j. Clearly, for investigating the action of the monodromy on the cycle Δ, that is, on the vector $(c_1, \dots, c_k)$ of coefficients from expansion

$$\Delta = \sum_{j=1}^{k} c_j \Delta_j,$$

one have to use transposed matrices. Keeping this fact in mind, for singularity A_2 one can obtain

$$(M_1^T)^2 = (M_2^T)^2 = (M_1^T M_2^T)^3 = (M_2^T M_1^T)^3 = E,$$

where E is the identity matrix. The latter relations for the monodromy matrices describe also the ramification of integral (15) in the case when the parameter λ_0 encircles a critical value, and corresponding Riemannian surface is shown on Figure 6. For asymptotics of integral (15), it is also easy to see that operator of classical monodromy $M = M_1 M_2$ has two different eigenvalues $\exp(\pm\pi i/3)$. Now one can use theorem 2 from section 1.2.

The ramification of integral (15) for other k can be carried out in the similar way. For instance, the Riemannian surface for $k = 3$ is shown on Figure 7.

4 The Cauchy Problem with Variable Coefficients

4.1 Geometric description

In the case when the Cauchy problem with variable coefficients is considered, its solution can be written down in the form of integral representation. So, to describe possible singularities of solutions to the Cuchy problem, one have to investigate the ramification of integrals of the form (5):

$$u(x) = 2\pi i \left(\frac{i}{2\pi}\right)^{n-1} \int\limits_{h(x)} \left(\frac{\partial}{\partial q_0}\right)^{n-m} G(x,q,t) \mid_{qy=0} f(y)\omega(t,q') \wedge dy,$$

with

$$h(x) \in H_{2n}(\mathbf{C}_t \times \mathbf{CP}^n_{q'} \times \mathbf{C}^n_y \setminus \Sigma_x, X \cup \{t=0\}). \tag{16}$$

Similar to the case of constant coefficients, we consider the following three manifolds:

$$X = \{(y,q',t) \mid s(x) = 0\},$$
$$T = \{(y,q',t) \mid t = 0\},$$
$$\Sigma_x = \{(y,q',t) \mid S(x,q',t) - q'y = 0\}$$

involved in (16) (see formula (9) above) in the compactified space $\mathbf{C}^n_x \times \mathbf{CP}^n_y \times \mathbf{CP}_{n-1,q'} \times \mathbf{C}_t$. Consider the natural projection

$$\mathbf{C}^n_x \times \mathbf{CP}^n_y \times \mathbf{CP}_{n-1,q'} \times \mathbf{C}_t \to \mathbf{C}^n_x,$$

that induces a bundle

$$\Sigma_x \setminus (X \cup T) \to \mathbf{C}^n_x. \tag{17}$$

Applying the Thom theorem to bundle (17) one can obtain a simple geometric sense of singular points: they are posited at points where the manifold Σ_x is tangent to a stratum from the union $X \cup T$.

There exist the same four types of tangency:

- Σ_x is tangent to T. From the tangency conditions

$$\lambda dt = \frac{\partial S(x,q',t)}{\partial q'}dq' - ydq' + \frac{\partial S(x,q',t)}{\partial t}dt - q'dy,$$

  it follows that $q' = 0$. Since $q' \in \mathbf{CP}_{n-1,q'}$, this case is exluded.

- Σ_x is tangent to X and not to $X \cap T$. Due to the Cauchy-Kowalewski theorem the solution is holomorphic.

- Σ_x is tangent to X and $X \cap T$. This case was investigated in terms of Leray's uniformisation (see [1], [6], [7]).

- Σ_x is tangent to $X \cap T$ and not to X. At such points x the solution to (4) has singularities, since these points lye in the characteristic conoid of initial manifold X.

So, geometrically there are no essential differences in investigation of singularities in cases of constant and variable coefficients.

4.2 Investigation of integrals

For the case of variable coefficients, the main difficulty is the definition of the homology class since the integrand in (5) is a ramifying function. Let us describe representation (5) in more detail. First of all, class (10) should be defined as a solution of equation

$$\partial h(x) = \delta h_1(x),$$

where δ is the Leray coboundary homomorphism, ∂ is the Bokstein homomorphism and

$$h_1(x) \in H_{2n-2}(\Sigma_x \cap T, X)$$

is a relative vanishing class of the quadrics $\Sigma_x \cap T \cup X$. One has to take into account that the class $h(x)$ must be a homology class on the Rimmanien surface of the elementary solution $G(x,q,t)$ which ramifies around Σ_x logarithmically. However, the class $\delta h_1(x)$ can not be lifted to this surface. Consider the homology class $\delta \tilde{h}_1(x)$ defined by the cycle

$$\delta\gamma_1(x) + \gamma_2(x) - \gamma_3(x)$$

(see Fig. 8, where $\gamma_1(x)$ is representative of $h_1(x)$) as a homology class on universal covering of $R \setminus \Sigma_x$, whose projection to Riemannian surface R coincides with $\delta h_1(x)$. Now the equation

$$\partial h(x) = \delta \tilde{h}_1(x)$$

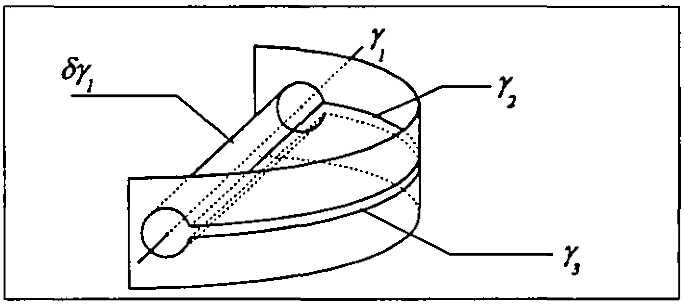

Figure 8. Construction of homology class $h(x)$

is uniquely solvable with respect to $h(x)$ and this solution is used in the integral (5).

Further, differentiating $n-m$ times in q_0 and substituting $q_0 = -q'y$ one obtains

$$\frac{1}{(2\pi i)^2}\left(\frac{i}{2\pi}\right)^{n-1}\int\limits_{h(x)}\left(\frac{\partial}{\partial q_0}\right)^{n-m} G(x,q,t)\,|_{qy=0}\, f(y)\omega(t,q')\wedge dy$$

$$= \left(\frac{i}{2\pi}\right)^{n-1}\left\{\int\limits_{h(x)}\frac{G_1(x,q,t)}{[S(x,q',t)-q'y]^{n-m+1}}f(y)\omega(t,q')\wedge dy\right. \tag{18}$$

$$\left. + \int\limits_{h(x)} G_0(x,q,t)\ln(S(x,q',t)-q'y)f(y)\omega(t,q')\wedge dy\right\}$$

(all manipulations with integral of the type (5) such as differentiation under the integral sign, integration by parts and so on can be performed formally.) Denote

$$\begin{aligned} G_2(x,q,t) &= \int\limits_a^t G_0(x,q,\tau)d\tau, \\ G_3(x,q,t) &= \int\limits_b^{q_1} G_0(x,q_0,\tau,q_2,\ldots,q_n,t)d\tau, \end{aligned}$$

where integrations are performed over regular paths. Then second term in right side of (18) can be rewritten as

$$\int\limits_{h(x)} \left\{ \frac{\partial G_3}{\partial q_1}(1-m)tdq' - \frac{\partial G_2}{\partial t}dt \wedge \tilde{\omega}(q') \right\} \wedge \ln(S(x,q',t) - q'y) f(y) dy$$
$$= - \int\limits_{h(x)} \left\{ G_3(x,q,t) \left(\frac{\partial S(x,q',t)}{\partial q_1} - y_1 \right) (1-m)tdq' - \right.$$
$$\left. G_2(x,q,t) \frac{\partial S(x,q',t)}{\partial t} dt \wedge \tilde{\omega}(q') \right\} \wedge \frac{f(y)}{S(x,q',y) - q'y} dy$$

via integration by parts. The latter formula reduces the investigation of solution to the problem with variable coefficients to the investigation of integrals

$$\int\limits_{h'(x)} \operatorname{Res} \frac{\phi(x,y,q,t)\,\omega}{[S(x,q',t) - q'y]^l}, \quad l = 1, 2, \ldots, \tag{19}$$

where the class $\delta h'(x)$ is a part of $h(x)$ encircling Σ_x, and ω is some regular form.

Since $h'(x) \in H_{2n-2}(\Sigma_x \cap T, X)$, we can restrict the integrand in (19) on Σ_x. It is clear that the cycle $h'(x)$ on Σ_x is now defined by two manifolds $\Sigma_x \cap X$ and $\Sigma_x \cap T$, and the singularities of integral (12) arise when these two manifolds are tangent to each other. If X and $\operatorname{char} H$ are regular manifolds, the intersections $\Sigma_x \cap X$ and $\Sigma_x \cap T$ are also regular (if $\Sigma_x \cap X$ is not tangent to $\Sigma_x \cap T$), and one can choose coordinates $z = (z_0, \ldots, z_{2n-1})$ such that

$$\begin{aligned} \Sigma_x \cap X &= \{z_0 = 0\}, \\ \Sigma_x \cap T &= \{z_0 = \Phi(z_1, \ldots, z_{2n-1}, x)\}. \end{aligned}$$

Now there are no essential differences from the case of constant coefficients. Critical points of function Φ can be classified according to standard methods and a normal form of Φ can be obtained.

Computing the residue in integral (19) and using the Taylor expansion of the amplitude function, we arrive at the formula

$$\int\limits_{h'(x)} \operatorname{Res} \frac{\phi(x,y,q,t)\,\omega}{[S(x,q',t) - q'y]^l} = \int\limits_{h'(x)} \Omega(x,y,q,t) =$$
$$\sum_j \sum_\alpha C_j^\alpha(x) \int\limits_{h'(x)} z^j \tilde{\omega}_\alpha(z),$$

where

$$\begin{aligned} &z = (z_0, z'), \quad z' = (z_1, \ldots, z_{2n-1}), \\ &j = (j_0, \ldots, j_{2n-1}), \\ &z^j = (z_0)^(j_0) \ldots (z_{2n-1})^{j_{2n-1}}, \end{aligned}$$

and

$$\tilde{\omega}_\alpha(z) = dz_0 \wedge \ldots \wedge \widehat{dz_\alpha} \wedge \ldots \wedge dz_{2n-1}.$$

By the latter expansion, we come to investigation of the integral

$$I_j^\alpha = \int\limits_{h'(x)} z^j \tilde{\omega}_\alpha(z). \tag{20}$$

The change of variables transforming function $\Phi(z', x)$ to normal form reduces the equation of the manifold $\Sigma_x \cup T$ to the form $z_0 = g(z', \lambda)$, where $g(z', \lambda)$ is a versal deformation for the normal form of the corresponding singularity type, and $\lambda = \lambda(x)$ are new parameters of integral (20). Similar to Section 2 above, we can represent integral (20) in the form

$$I_j^\alpha(\lambda) = \int\limits_{\Delta(\lambda)} z^j \ln z_0 \, \tilde{\omega}_\alpha(z), \tag{21}$$

in a small sector up to the factor $1/(2\pi i)$ (see the end of Section 2.) Now the integration is carried over the cycle

$$\Delta(\lambda) \in H_{2n-1}(\Sigma_x \setminus (T \cup X)),$$

which determines a homology class in a non-critical level set of the function $g(z', \lambda)$.

All further investigation of integral (21) is quite similar to that for constant coefficients, as described in section 3 but one has to take into account different dimension of the class $\Delta(\lambda)$.

5 Examples

1. Consider the Cauchy problem

$$\begin{cases} \dfrac{\partial^2 u}{(\partial x^1)^2} + \dfrac{\partial^2 u}{(\partial x^2)^2} = f(x^1, x^2), \\ u(x) \equiv 0 \pmod 2 \text{ on } X \end{cases} \tag{22}$$

with a holomorphic function $f(x)$. The initial manifold X is given by

$$(x^1/a)^2 + (x^2/b)^2 = 1,$$

where $a \neq b$. The integral representations of the solution and its derivatives have the form

$$\begin{aligned} u(x^1,x^2) &= \frac{i}{2\pi}\int\limits_{h_1(x)} \frac{f(y)\,dy^1\wedge dy^2\wedge(p_1dp_2-p_2dp_1)}{(p_1)^2+(p_2)^2}, \\ \frac{\partial u}{\partial x^1}(x^1,x^2) &= \frac{i}{2\pi}\int\limits_{h(x)} \operatorname{Res}\frac{p_1f(y)\,dy^1\wedge dy^2\wedge(p_1dp_2-p_2dp_1)}{((p_1)^2+(p_2)^2)(p_1(y^1-x^1)+p_2(y^2-x^2))}, \\ \frac{\partial u}{\partial x^2}(x^1,x^2) &= \frac{i}{2\pi}\int\limits_{h(x)} \operatorname{Res}\frac{p_2f(y)\,dy^1\wedge dy^2\wedge(p_1dp_2-p_2dp_1)}{((p_1)^2+(p_2)^2)(p_1(y^1-x^1)+p_2(y^2-x^2))}. \end{aligned}$$

The characteristic set char H has three strata.

The stratum

$$A^1 = \{p_1 = p_2 = 0\}$$

is tangent to the initial manifold, so that we do not consider it.

Let us examine the stratum

$$A^2 = \{p_1 + ip_2 = 0 \setminus p_1 = p_2 = 0\}.$$

The tangency condition for the manifolds $A^2|_{\Sigma_x}$ and $X|_{\Sigma_x}$ at some point (y,p) is

$$\begin{cases} (y^2-x^2)-i(y^1-x^1)=0, \\ (y^1/a)^2+(y^2/b)^2=1, \\ y^2a^2-iy^1b^2=0, \end{cases}$$

where the last equality expresses the proportionality of differentials. Resolving this system, one can find the singularity manifold

$$x^2 - ix^1 = \sqrt{b^2-a^2}.$$

Later on, one can choose new variables $z=(z_0,z_1,z_2)$ such that the manifold $X|_{\Sigma_x}$ has the equation $z_0=0$ and the manifold $A^2|_{\Sigma_x}$ has the equation $z_0=(z_1)^2+(z_2)^2+\lambda(x)$ (in the case of square root type ramification). Such symmetric change does exist and does not lead to any essential modification in the investigation. Actually, the substitution $z_0=y^2-iy^1-x^2+ix^1$ leads to the equation $z_0=0$ for the manifold $X|_{\Sigma_x}$ and, moreover, we obtain

$$A^2|_{\Sigma_x} = \left\{z_0 = (y^1/a)^2+(y^2/b)^2-1+y^2-iy^1-\lambda(x)\right\},$$

where $\lambda(x) = x^2 - ix^1$. Since the second differential of the function on the right in the latter equation is non-degenerated, the considered singularity is of A_1 (square root) type at point $\lambda(x) = \sqrt{b^2 - a^2}$ and the following expansions take place

$$\begin{aligned} \frac{\partial u}{\partial x^1} &= \sum_{j=0}^{\infty} a_j^{(1)} (x^2 - ix^1)^{(2j+1)/2}, \\ \frac{\partial u}{\partial x^2} &= \sum_{j=0}^{\infty} a_j^{(2)} (x^2 - ix^1)^{(2j+1)/2}. \end{aligned}$$

Therefore, the solution of the Cauchy problem (22) in the neighborhood of the point

$$\lambda(x) = \sqrt{b^2 - a^2}$$

can be expanded into the series

$$u(x) = u(x_0) + \int_{x_0}^{x} \frac{\partial u}{\partial x^1}(y)\, dy^1 + \frac{\partial u}{\partial x^2}(y)\, dy^2 \cong \sum_{j=1}^{\infty} c_j (x^2 - ix^1)^{(2j+1)/2}.$$

Singularities originated by the stratum

$$A^3 = \{p_1 - ip_2 = 0 \setminus p_1 = p_2 = 0\}$$

can be investigated in the similar way, and one obtains the asymptotics near this singularity in the form

$$u(x) \cong \sum_{j=1}^{\infty} c_j (x^2 + ix^1)^{(2j+1)/2}.$$

2. Let us examine the problem

$$\begin{cases} \dfrac{\partial^2 u}{(\partial x^1)^2} - \dfrac{\partial^2 u}{(\partial x^2)^2} = f(x^1, x^2), \\ u(x) \equiv 0 \pmod 2 \text{ on } X, \end{cases} \tag{23}$$

where a function $f(x)$ is holomorphic and the initial manifold is defined by the equation

$$(x^1)^2 + (x^2)^2 = 1.$$

The corresponding integral representations for the solution and its derivatives are

$$
\begin{aligned}
u(x^1, x^2) &= \frac{i}{2\pi} \int\limits_{h_1(x)} \frac{f(y)\, dy^1 \wedge dy^2 \wedge (p_1 dp_2 - p_2 dp_1)}{(p_1)^2 - (p_2)^2}, \\
\frac{\partial u}{\partial x^1}(x^1, x^2) &= \frac{i}{2\pi} \int\limits_{h(x)} \operatorname{Res} \frac{p_1 f(y)\, dy^1 \wedge dy^2 \wedge (p_1 dp_2 - p_2 dp_1)}{((p_1)^2 - (p_2)^2)(p_1(y^1 - x^1) + p_2(y^2 - x^2))}, \\
\frac{\partial u}{\partial x^2}(x^1, x^2) &= \frac{i}{2\pi} \int\limits_{h(x)} \operatorname{Res} \frac{p_2 f(y)\, dy^1 \wedge dy^2 \wedge (p_1 dp_2 - p_2 dp_1)}{((p_1)^2 - (p_2)^2)(p_1(y^1 - x^1) + p_2(y^2 - x^2))}.
\end{aligned}
$$

The characteristic set $\operatorname{char} H$ has three strata A^1, A^2 and A^3.

The stratum

$$A^1 = \{p_1 = p_2 = 0\},$$

similar to the previous example, is tangent to the initial manifold X.

Let us consider the stratum

$$A^2 = \{p_1 - p_2 = 0 \setminus p_1 = p_2 = 0\}.$$

The tangency condition for manifolds $A^2|_{\Sigma_x}$ and $X|_{\Sigma_x}$ at some point (y, p) is given by the system

$$
\begin{cases}
(y^2 - x^2) + (y^1 - x^1) = 0, \\
(y^1)^2 + (y^2)^2 = 1, \\
y^2 - y^1 = 0,
\end{cases}
$$

where the the latter equality expreses the condition of proportionality of differentials. The equation of singular manifold can be found in the form

$$(x^2 + x^1)^2 = 2.$$

Similar to the previous section, we can choose a variable change such that the equation of manifold $X|_{\Sigma_x}$ becomes $z_0 = 0$, and the equation of $A^2|_{\Sigma_x}$ —

$$z_0 = (z_1)^2 + (z_2)^2 + \lambda(x)$$

(for ramification of square root type). Actually, if $z_0 = y^2 + y^1 - x^2 - x^1$, then the equation of $X|_{\Sigma_x}$ becomes $z_0 = 0$, and the equation of $A^2|_{\Sigma_x}$ is

$$z_0 = (y^1)^2 + (y^2)^2 - 1 + y^2 + y^1 - \lambda(x),$$

where $\lambda(x) = x^2 + x^1$. Since second differential is not degenerated for the function on the right in the latter equation, there is ramification of square root type at point $\lambda^2(x) = 2$. The derivatives of the solution $u(x)$ of the problem (23) are

$$\frac{\partial u}{\partial x^1} = \sum_{j=0}^{\infty} a_j^{(1)}(x^2 + ix^1)^{(2j+1)/2},$$

$$\frac{\partial u}{\partial x^2} = \sum_{j=0}^{\infty} a_j^{(2)}(x^2 + ix^1)^{(2j+1)/2}.$$

Hence, the solution $u(x)$ can be expanded near the point $\lambda^2(x) = 2$ in a series

$$u(x) = u(x_0) + \int_{x_0}^{x} \frac{\partial u}{\partial x^1}(y)\, dy^1 + \frac{\partial u}{\partial x^2}(y)\, dy^2 \cong \sum_{j=1}^{\infty} c_j(x^2 + ix^1)^{(2j+1)/2}.$$

Singularities qriginated by the stratum

$$A^3 = \{p_1 + p_2 = 0 \setminus p_1 = p_2 = 0\}$$

are examined similarly, and the expansion in a neighborhood of the corresponding singular point are

$$u(x) \cong \sum_{j=1}^{\infty} c_j(x^2 - ix^1)^{(2j+1)/2}.$$

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