A Characterization of "Rapidity" Curve in Chiral Potts Model

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Introduction

Nowadays much attention is paid to the exactly solvable models of statistical mechanics. The integrable N-state chiral Potts model is the first solvable model where the Boltzmann weights lie on a complex curve of genus greater than one. It contains the natural generalization of Ising model and many of the remarkable properties known for the Ising model (N=2) can be extended to the cases N≥3 (see refs. [1, 5, 8, 15] and references therein) . This model gives the solution of Yang-Baxter equation (or the star-triangle equation) which has a very simple product form for the Boltzmann weights

$$\frac{W_{pq}(n)}{W_{pq}(0)} = \prod_{j=1}^{n} \frac{d_p b_q - a_p c_q \omega^j}{b_p d_q - c_p a_q \omega^j} \quad ,$$

$$\frac{\overline{W}_{pq}(n)}{\overline{W}_{pq}(0)} = \prod_{j=1}^{n} \frac{\omega a_p d_q - d_p a_q \omega^j}{c_p b_q - b_p c_q \omega^j} \quad ,$$

depending on the two "rapidities" p,q and a parameter k', where

$$\omega = e^{\frac{2\pi i}{N}} \quad , \quad j : \text{integer mod. N}.$$

The "rapidity" variables p,q are represented by the ratio of 4-vectors [a,b,c,d] satisfying

$$k a^N + k' c^N = d^N \quad ,$$
$$k b^N + k' d^N = c^N \quad ,$$

with

$$k^2 + k'^2 = 1 \quad .$$

These equations describe a complex curve of genus $N^3 - 2N^2 + 1$ as the intersection of two "Fermat surfaces" in $\mathbf{P}^3$, or as the fiber product of two Fermat curves over the rational curve. It has a group of automorphisms of order $4N^3$ and the quotient of some N^2 automorphisms gives a genus (N-1) hyperelliptic curve which is represented by

$$W_{N,k'} \quad : \quad t^N = \frac{(1 - k'\lambda)(1 - k'\lambda^{-1})}{k^2} \quad , \quad (t,\lambda) \, \varepsilon \, \mathbf{C}^2.$$

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(The integer N shall always be assumed to be ≥ 3 unless otherwise specified.) The transfer matrix T(u) of the classical statistical mechanical model now depends on the variables lying on Riemann surface $W_{N,k'}$, and it satisfies the commutating relation for different u, u'. However this is not sufficient to solve the eigenvalue problem of the quantum chiral Potts spin chain associated to T(u). One needs a functional equation for the transfer matrix T(u) which was obtained by [1, 2, 11, 13]. For the superintegrable case, when the "vertical rapidity" p has a special value, some simplifications occur and the eigenvalue problem for T(u) are solved in [1, 2, 9]. For the general case, one can explicitly solve the functional equation for the largest eigenvalue in the large-lattice limit [10, 11], and compute the spectrum of low-lying excitations to discuss the phenomena of level crossing transition to a new ground state [15] from the physical consideration. Here no uniformizing substitutions are used to obtain all those results. It is the belief that these results should be Abelian integrals over the Riemann surface. One difficulty is that it lacks the "different property" as in the eight-vertex model solved by Baxter, where the uniformizing parametrization leads to the elliptic functions. However in [6, 7], the significance of the Abelian functions was indicated in the computation of single-spin expectation values of the general chiral Potts N-state model. Hence the theory of prime form of hyperelliptic curves [17] is expected to immerse into the computation of the interesting physical quantities.

This note deals with two mathematical problems arisen from the above rational family $W_{N,k'}$ of hyperelliptic curves. In this paper, we shall call $W_{N,k'}$ a chiral Potts N-state curve , or simply a CP N-curve. The first problem is to give a geometrical characterization of this family of CP N-curves among the hyperelliptic curves and compare them with the case of ellliptic curves. The other is to describe the relationship between the symmetries of a CP N-curve and its Jacobian, in particular how the general prime form of Riemann surfaces can be applied to the expression of k' in terms of the period of $W_{N,k'}$ which was obtained by Baxter in [6].

First we note that $W_{N,k'}$ for N=2 is the algebraic form of an elliptic curve with a spin 1/2 structure. As an 1–dimension complex torus, the fixed part of the canonical involution θ of an elliptic curve E consists of 4 elements. It is in one-one correspondence with the branched loci S under the natural projection from E to P^1 (=E/<θ>) . The order 2 translation of E corresponding to the spin 1/2 structure induces an automorphism of P^1, and its orbits in S form a partition of it with each member having 2 elements. Such partition of S naturally leads

to the algebraic expression of $W_{N,k'}$ for N=2, also in the same process to the general N≥3. The original Boltzmann weights a,b,c,d are closely connected to the elements in S. The derivation and their relationship will be given in Sections 2 and 4. In section 1, we discuss the parametrization of these partitions of S for a subset S of $\mathbf{P}^1$ with 4 elements. The formulation is guided by the intent of finding the geometrical meaning of the relations between different variables appeared in the physical literatures [1, 6, 10]. In Section 3, we shall derive the original "product Fermat" curve of the variable [a,b,c,d] from the curve $W_{N,k'}$ and the other two related ones, $W_{N,ik'}$, $W_{N,k}$. In Section 5, we shall discuss the uniformizing parametrization of a chiral Potts N-state curve. The special form of the period obtained by Baxter [6] is discussed from the geometrical point of view, that is how does it reflect the symmetries of the Riemann surface. We apply the theory of prime form to obtain the explicit expression of the parameter k' in terms of the hyperelliptic theta functions.

1. *Parametrization of 4 points in* $\mathbf{P}^1$

Let E be a 1–torus (= 1–dimension complex torus), T be the subgroup of Aut(E) consisting of all translations of E, i.e. the automorphisms $t_a : E \to E$, x $\rightsquigarrow$ x+a , x∈ E. T is isomorphic to E. Denote T(2)= the 2–torsion subgroup of T.

Let θ: E ---> E be an involution with 4 fixed points.

<u>LEMMA 1</u>   T(2) = { t ∈ T | t θ = θ t }.

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Proof. Write $E = C/L$, and $\sigma : [z] \to [-z + c]$ for some lattice L, and $c \in C$. Then the conclusion follows easily from the definition of T. q.e.d.

Denote $H = (T(2) \times <\theta>)/<\theta>$. Identifying $E/<\theta>$ with P^1, H can be considered as a subgroup of $Aut(P^1)$ through its action on $E/<\theta>$. The branched locus S of the double cover

$$\Psi (= \Psi_\theta) : E \to E/_{<\theta>} = P^1 \quad . \tag{1}$$

forms a H-orbit with 4 elements. As any two such involutions θ of E are conjugate by a translation, the subset S of P^1 is uniquely determined up to the action of $Aut(P^1)$. It is well-known that every 1-torus can be obtained as a double cover of P^1 with 4 branched points, hence we have the one-one correspondence between the following sets:

$$\{\text{isomorphic class of } 1 - \text{torus}\} \leftrightarrow \{S | S \subset P^1, |S| = 4\} / Aut(P^1) . \tag{2}$$

DEFINITION For a subset S of P^1 consisting of 4 elements, a partition of S with 2 elements in each of its member is called a (2,2)-partition of S.

We are going to characterize the data of S with a (2,2)-partition . First we notice that the above group H is determined by the set S via (i) of the following lemma.

LEMMA 2

(i) For each subset S of P^1 with $|S|=4$, there exists an unique subgroup H of $Aut(P^1)$ such that $H \simeq (Z/2Z)^2$ and S= a H-orbit.

(ii) Let H be a subgroup of $Aut(P^1)$ with $H \simeq (Z/2Z)^2$. Then $(P^1)^h$, h $\in$ H$-\{id.\}$, are mutually disjoint.

Proof: (i) Let H be a subgroup of $Aut(P^1)$ satisfying the condition . The <h>-orbits in S forms a (2,2)-partition of S for a given non-trivial element h of H. Then h is uniquely determined by its values on S. Since S has exact three (2,2)-partitions , H is unique. It remains to show that for any (2,2)-partition of S, there always exists an automorphism h of P^1 such that the <h>-orbits in S form the given (2,2)-partition. By conjugating some element in $Aut(P^1)$, we may assume S=$\{0,\infty,1,\alpha\}$ and the partition = $\{0,\infty\} \coprod \{1, \alpha\}$. Then h is defined by z $\to \alpha/z$, z $\in$ C.

(ii) If H has a common fixed point x in P^1, the homomorphism which assigns each element of H to its differential at x is an injective map. Since the linear

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transformation group of the tangent at x is isomorphic to $\mathbb{C}^*$, H is cyclic. This contradicts the assumption of the structure of H. q.e.d.

PROPOSITION 1 { isomorphic class of (E,t)| E: 1-torus, $t \in$ T(2)-{id.} } is in one-one correspondence with each of the following sets:

(i) {(S with a (2,2)-partition) | $S \subset \mathbf{P}^1$, |S|=4 }/Aut($\mathbf{P}^1$) .

(ii) { $\pi : \mathbf{P}^1 \to \mathbf{P}^1$, a degree 2 morphism with critical value not equal to 0, ∞ }/ $\sim$, here $\pi_1 \sim \pi_2$ iff for some f, f' $\in$ Aut($\mathbf{P}^1$) , f({0,∞})= {0,∞},

$$\begin{array}{ccc} \mathbf{P}^1 & \xrightarrow{\ f'\ } & \mathbf{P}^1 \\ {\scriptstyle \pi_1}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \pi_2} \\ \mathbf{P}^1 & \xrightarrow{\ f\ } & \mathbf{P}^1 \end{array}$$

(iii) { $(\rho , p) \mid p \in \mathbf{P}^1 - \{\infty\}$, ρ : a degree 2 endomorphism of $\mathbf{P}^1$ with critical value not equal to p, ∞ }/ $\approx$, here $\rho_1 \approx \rho_2$ iff for some f,g $\in$ Aut($\mathbf{P}^1$) with g(p) = p, g(∞) = ∞,

$$\begin{array}{ccc} \mathbf{P}^1 & \xrightarrow{\ f\ } & \mathbf{P}^1 \\ {\scriptstyle \rho_1}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \rho_2} \\ \mathbf{P}^1 & \xrightarrow{\ g\ } & \mathbf{P}^1 \end{array}$$

Proof. For $S \subset \mathbf{P}^1$ with |S|=4, let E be the corresponding 1-torus of (2). The group H in Lemma 2 (i) corresponding to the set S can be identified with $(T(2) \times <\sigma>)/<\sigma>$. For t in T(2)-{id.}, let h be the coset of t in H. Then the <h>-orbits in S forms a (2,2)-partition of S. In this way we define a map from { isomorphism class of (E, t)| $t \in$ T(2)- {id.} } to (i), which is bijective by Lemma 2.

For the above automorphism h of $\mathbf{P}^1$, the degree 2 morphism $\pi : \mathbf{P}^1 \to \mathbf{P}^1/<h>$ $=\mathbf{P}^1$ has the critical points disjoint with S. By a suitable coordinate of $\mathbf{P}^1$, we may assume that $S = \pi^{-1}(\{0,\infty\})$. In this way, we have the one-one correspondence between (i) and (ii).

The projection

ρ : $\mathbf{P}^1/<h> \to \mathbf{P}^1$ /H ,

can be considered as a degree 2 endomorphism of $\mathbf{P}^1$. The H-orbit S corresponds to a point of $\mathbf{P}^1$/H = $\mathbf{P}^1$, which equals to ∞ by a suitable coordinate of $\mathbf{P}^1$/H.

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The fixed point set of h corresponds to another H-orbit p. Then p is not equal to ∞ and the critical values of ρ can not be p or ∞ . This gives the correspondence between (ii) and (iii) . q.e.d.

Then by (2), the following correspondence can easily be obtained.

PROPOSITION 2 { isomorphic class of 1–torus } is in one-one correspondence with each of the following sets:

(i) { S | S $\subset \mathbf{P}^1$, |S| = 4 }/ Aut($\mathbf{P}^1$) .

(ii) {$\pi:\mathbf{P}^1 \rightarrow \mathbf{P}^1$ degree 2 morphism with critical value not equal to 0, ∞ } / $\cong$, here $\pi_1 \cong \pi_2$ iff for some f' in Aut($\mathbf{P}^1$) , f' sends $\pi_1^{-1}\{0,\infty\}$ to $\pi_2^{-1}\{0,\infty\}$.

(iii) { (ρ, p) | p $\in \mathbf{P}^1 - \{\infty\}$, ρ: a degree 2 morphism of $\mathbf{P}^1$ onto itself with critical value not equal to p, ∞}/ $\cong$, here $(\rho_1, p_1) \cong (\rho_2, p_2)$ iff there exists some g in Aut($\mathbf{P}^1$) such that g(∞) = ∞, and g({ p_1, critical values of ρ_1 }) = { p_2, critical values of ρ_2 }.

We are going to parametrize of the data in Proposition 1 and 2. The following easy lemma is useful for our derivation, and its proof is omitted.

LEMMA 3 For 4 distinct elements x,y,z,w in $\mathbf{P}^1$, there exists an automorphism g of $\mathbf{P}^1$ with g(x)= 0, g(y)= ∞, g(z)= α, g(w)= α^{-1} for some α in C$-\{0, \pm 1\}$. And α is unique up to sign.

Define

$$\pi_\alpha : \mathbf{P}^1 \rightarrow \mathbf{P}^1 ,\lambda \rightsquigarrow \pi_\alpha(\lambda) = \frac{(1 - \alpha\lambda)(1 - \alpha\lambda^{-1})}{1 - \alpha^2} ,$$

here α is a constant in C-$\{0,\pm 1\}$,

$$\rho_0 : \mathbf{P}^1 \rightarrow \mathbf{P}^1 , \quad \zeta \rightsquigarrow \frac{\zeta + \zeta^{-1}}{2} .$$

$\mathcal{A}=$ C$-\{0,\pm 1\}/ \equiv,$ here $\alpha \equiv \beta$ iff $\alpha^2 = \beta^{\pm 2}$.

PROPOSITION 3 $\mathcal{A}$ is in one-one correspondence with the data in Proposition 1. In fact, for an element [α] of $\mathcal{A}$ determined by a complex number α, the elements of (i), (ii), (iii) in Proposition 1 corresponding to [α] are as follows:

(i) the class of $\{0, \infty, \alpha, \alpha^{-1} \}$ with the partition $\{0, \infty \} \coprod \{\alpha, \alpha^{-1}\}$.

(ii) the class of π_α.

(iii) the class of $(\rho_0, \frac{1+\alpha^2}{1-\alpha^2})$.

Proof :By Lemma 3, every element in (i) of Proposition 1 is represented by $\{0,\infty,\alpha,\alpha^{-1}\}$ with the partition $\{0,\infty\} \coprod \{\alpha,\alpha^{-1}\}$ for $\alpha \in$ C$-\{0, \pm 1\}$. This

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corresponds to an element of (ii) in Proposition 1, which is represented by some $\pi:\mathbf{P}^1 \to \mathbf{P}^1$. Choose the coordinate λ of C ($\subset\mathbf{P}^1$) such that $\pi^{-1}(\infty)= \{0, \infty\}$, $\pi^{-1}(0)= \left\{\alpha, \alpha^{-1}\right\}$. Then π is equivalent to π_α. Since the critical values of π_α are $\frac{1-\alpha}{1+\alpha}$, $\frac{1+\alpha}{1-\alpha}$, the element in Proposition 1 (iii) is represented by (ρ, p) with $\rho^{-1}(\infty)$ $= \{0,\infty\}$, $\rho^{-1}(\mathrm{p})= \{ \frac{1-\alpha}{1+\alpha}, \frac{1+\alpha}{1-\alpha}\}$. So ρ is equivalent to $(\rho_0, \frac{1+\alpha^2}{1-\alpha^2})$. It remains to show that if $(\rho_0, \frac{1+\alpha^2}{1-\alpha^2})$ is equivalent to $(\rho_0, \frac{1+\beta^2}{1-\beta^2})$, then $[\alpha] = [\beta]$, i.e. $\beta^2= \alpha^{\pm2}$. Let f, g be the elements of $\mathrm{Aut}(\mathbf{P}^1)$ with $\mathrm{f}\rho_0= \rho_0$ g, and $\mathrm{g}(\infty) = \infty$, $\mathrm{g}(\frac{1+\alpha^2}{1-\alpha^2})=$ $\frac{1+\beta^2}{1-\beta^2}$. Then $\mathrm{f}\{0,\infty\} = \{0,\infty\}$ and $\mathrm{f}(\{1,-1\})= \{1,-1\}$, hence f is defined by f (ζ)= $\pm\zeta^{\pm1}$. Therefore g is defined by $\mathrm{g}(\zeta)= \pm\zeta$, which implies $\beta^2= \alpha^{\pm2}$. q.e.d.

LEMMA 4. For a subset S of $\mathbf{P}^1$ with 4 elements, let H be the subgroup of $\mathrm{Aut}(\mathbf{P}^1)$ associated to S in Lemma 2, and $\Phi: \mathbf{P}^1 \to \mathbf{P}^1/\mathrm{H}$ be the natural projection. Then the following data are in one-one correspondence:

$$\{(2,2) - \text{partition of S}\} \leftrightarrow \mathrm{H} - \{\mathrm{id}.\} \leftrightarrow \{\text{critical value of } \Phi\}$$
$$\Delta \qquad\qquad \leftrightarrow \qquad \mathrm{h} \quad \leftrightarrow \qquad \mathrm{c}$$

with the relation Δ = the <h>-orbits decomposition of S, $(\mathbf{P}^1)^\mathrm{h}= \Phi^{-1}(\mathrm{c})$.

Proof. For a critical value c of Φ, $\Phi^{-1}(\mathrm{c})$ is the fixed point set of a non-trival element of H, and vice versa. Then the conclusion follows immediately from Lemma 2. q.e.d.

With the same convention as in [1, 6, 10], we shall always denote k the solution of the equation

$$k^2 + k'^2 = 1$$

for a given $\mathrm{k}'\epsilon \mathrm{C}-\{0, \pm1\}$, and λ (or λ_j) the coordinate of the domain of π_α.

PROPOSITION 4

(i) For $[\mathrm{k}'] \epsilon \mathcal{A}$, the elements of $\mathcal{A}$ having the same image as $[\mathrm{k}']$ in Proposition 2 are : $[\mathrm{k}']$, $[\mathrm{k}]$, $\left[\frac{ik'}{k}\right]$.

(ii) For S = $\{0,\infty, k', k'^{-1}\}$, the elements in (i) of Proposition 1 corresponding to the above three elements of $\mathcal{A}$ are as follows:

$[\mathrm{k}'] \leftrightarrow$ S with the partition $\{0, \infty\} \amalg \{k', k'^{-1}\}$,

$\left[\frac{ik'}{k}\right] \leftrightarrow$ S with the partition $\{0, k'^{-1} \} \amalg \{\infty, k'\}$.

$[\mathrm{k}] \leftrightarrow$ S with the partition $\{0, k'\} \amalg \{\infty, k'^{-1}\}$.

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(iii) Let ζ, ζ_1, ζ_2 be the coordinates of the domains of ρ_0 in (iii) of Proposition 3 such that :

[k'] $\leftrightsquigarrow (\rho_0, \frac{1+k'^2}{1-k'^2})$ with the coordinate ζ ,

$\left[\frac{ik'}{k}\right]$ $\leftrightsquigarrow (\rho_0, 1-2k'^2)$ with the coordinate ζ_1,

[k] $\leftrightsquigarrow (\rho_0, \frac{1+k^2}{1-k^2})$ with the coordinate ζ_2.

Then $\zeta, \zeta_1. \zeta_2$ are related by

$$\zeta_1 + \zeta_1 + 2 = k^2(\zeta + \zeta^{-1} + 2) ,$$
$$\zeta_2 + \zeta_2 - 2 = -\frac{k^2}{k'^2}(\zeta + \zeta^{-1} - 2) .$$

Proof. In this proof, we shall always denote k' an element in $C - \{0, \pm 1\}$, $S = \{0, \infty, k', k'^{-1}\}$, H = the order 4 subgroup of $\text{Aut}(\mathbf{P}^1)$ associated to S in (i) of Lemma 2, ζ, ζ_1, ζ_2 = the three coordinates for $\mathbf{P}^1$ in the above (iii) of this Proposition. Let h, h_1, h_2 be the non-trivial elements of H such that the correspondence in Lemma 4 are as follows:

$h \leftrightsquigarrow \{0, \infty\} \coprod \{k', k'^{-1}\}$,

$h_1 \leftrightsquigarrow \{0, k'^{-1}\} \coprod \{\infty, k'\}$,

$h_2 \leftrightsquigarrow \{0, k'\} \coprod \{\infty, k'^{-1}\}$.

Identify the projection $\mathbf{P}^1 \to \mathbf{P}^1/\langle h\rangle$ with $\pi_{k'}$, $\mathbf{P}^1/\langle h\rangle \to \mathbf{P}^1/H$ with ρ_0, and $\Phi: \mathbf{P}^1 \to \mathbf{P}^1/H$ with $\rho_0\pi_{k'}$. Then {critical values of Φ}=$\{\pm 1, \frac{1+k'^2}{1-k'^2}\}$, and h, h_1, h_2 are defined by

$$h(\lambda) = \lambda^{-1} \quad , \quad h_1(\lambda) = \frac{1-k'\lambda}{k'-\lambda} \quad , \quad h_2(\lambda) = \frac{k'-\lambda}{1-k'\lambda} \quad .$$

The correspondence between the critical values of Φ and the elements of $H - \{id.\}$ in Lemma 4 is described as follows:

$$h \leftrightsquigarrow \frac{1+k'^2}{1-k'^2} \quad , \quad h_1 \leftrightsquigarrow 1 \quad , \quad h_2 \leftrightsquigarrow -1 \quad .$$

Let $[\alpha]$ be an element of $\mathcal{A}$ having the same image as [k'] in Proposition 2 , and assume $[\alpha] \neq [k']$. Then $(\rho_0, \frac{1+k'^2}{1-k'^2})$ and $(\rho_0, \frac{1+\alpha^2}{1-\alpha^2})$ are equivalent under the relation $\approx$ in (iii) of Proposition 2, i.e. for some g in $\text{Aut}(\mathbf{P}^1)$, $g(\infty) = (\infty)$, $g(\{\frac{1+k'^2}{1-k'^2}, \pm 1\}) = \{\frac{1+\alpha^2}{1-\alpha^2}, \pm 1\}$. By (iii) of Proposition 1, we may assume $g(\frac{1+k'^2}{1-k'^2}) = \pm 1$. Then the g is uniquely determined by the value of 1. When $(g(\frac{1+k'^2}{1-k'^2}), g(1)) = (1, \frac{1+\alpha^2}{1-\alpha^2})$ or $(-1, \frac{1+\alpha^2}{1-\alpha^2})$, $g(-1) = -1$ or 1 respectively. It follows $g(\eta) = \pm(k^2\eta - k'^2)$

, which implies $\alpha^2 = \frac{-k'^2}{k^2}$ or $\frac{-k^2}{k'^2}$, hence $[\alpha] = \left[\frac{ik'}{k}\right]$. In this cases, the (2,2)-partition of S is given by the $\langle h_1 \rangle$—orbits. When $\alpha^2 = \frac{-k'^2}{k^2}$, we have $k^2(\frac{\zeta+\zeta^{-1}}{2}) - k'^2 = \frac{\zeta_1+\zeta_1^{-1}}{2}$, hence $\zeta_1 + \zeta_1^{-1} + 2 = k^2(\zeta + \zeta^{-1} + 2)$.

When $(g(\frac{1+k'^2}{1-k'^2}), g(1)) = (1, -1)$ or $(-1, 1)$, the (2,2)-partition of S is given by the $\langle h_2 \rangle$-orbits and $g(\eta) = \pm(k^2\eta - 1)/k'^2$. This implies $\alpha^2 = k^{-2}$ or k^2, hence $[\alpha] = [k]$. When $\alpha^2 = k^2$, we have $\frac{1}{k'^2}[-k^2(\frac{\zeta+\zeta^{-1}}{2})+1] = \frac{\zeta_2+\zeta_2^{-1}}{2}$, hence $\zeta_2 + \zeta_2^{-1} - 2 = -\frac{k^2}{k'^2}(\zeta + \zeta^{-1} - 2)$. This completes the proof of this proposition. q.e.d.

REMARK

(i) By Proposition 1 (ii), there is an automorphism f' of $\mathbf{P}^1$ which sends $\{0, k'^{-1}\} \coprod \{\infty, k'\}$ to $\{0, \infty\} \coprod \left\{-i\frac{k}{k'}, i\frac{k'}{k}\right\}$. In fact, f' is defined by

$$f'(\lambda) = \frac{ik\lambda}{1 - k'\lambda}.$$

And the f' sending $\{0, k'\} \coprod \{\infty, k'^{-1}\}$ to $\{0, \infty\} \coprod \{k, k^{-1}\}$ is defined by $f'(\lambda) = \frac{k\lambda}{\lambda - k'}$.

(ii) By introducing the variables u, $\phi, \overline{\phi}$

$$-\zeta = e^{2iu}, \quad -\zeta_1 = e^{2i\phi}, \quad \zeta_2 = e^{2i\overline{\phi}},$$

the relations of (iii) in the above proposition are equivalent to

$$\sin\phi = \pm k \sin u, \quad k' \sin\overline{\phi} = \pm ik \cos u,$$

which are the relations of the parameters for "p-variables" in [1, 5, 6, 10]. The variables e^{iu}, $e^{i\phi}$, $e^{i\overline{\phi}}$ (as Im(u), Im(ϕ), Im($\overline{\phi}$) → ∞) can be considered as the local coordinates of the 2-torsion points of the elliptic curve corresponding to [k] in Proposition 2.

2. Chiral Potts N − state curves

The algebraic curve where the "rapidity" variables of chiral Potts N-state model (N ≥ 3) lie is defined as follows:

DEFINITION. The chiral Potts N-state curve (N ≥ 3) is a genus (N-1) hyperelliptic curve W with an order N automorphism having exactly 4 fixed points. And we shall call W a CP N-curve.

The order N automorphism of W in the above definition shall always be denoted by θ ($=\theta_W$), and the hyperelliptic involution be denoted by σ ($=\sigma_W$). By the uniqueness of the hyperelliptic involution, θ commutes with σ. First we note that the following properties hold for CP N-curves.

(I) $<\theta>$ acts freely on $W - W^\theta$, and $W/<\theta> \simeq \mathbf{P}^1$.

Let g_0 be the genus of $W/<\theta>$. By the Hurwitz theorem for the projection from W to $W/<\theta>$, $4 - 2N = N(2 - 2g_0) - 4(N-1) -\!\!\star$, here $\star$ = the sum of ramification contribution of $x \in W - W^\theta$. Hence $0 = 2Ng_0 +\star$, and $g_0 = \star = 0$. Then the conclusion follows immediately.

(II) W^θ is disjoint with W^σ.

By the commutativity of θ and σ, W^θ is stable under σ. Hence $|W^\theta \bigcap W^\sigma|$ = 2 or 4 when it is non-zero. As $<\theta>$ acts freely on $W^\sigma - W^\theta$, the order of $W^\sigma - W^\theta$, (= 2N-2 or 2N-4), is divisible by N. By $N \geq 3$, this is only possible for $N = 4$, and $|W^\theta \bigcap W^\sigma| = |W^\theta| = 4$. By Hurwitz's Theorem , the sum of all the ramification contribution of elements in W for the projection $W \to W/<\theta, \sigma> (=\mathbf{P}^1)$ equal to 20, which is greater than the contribution from the 4 elements in W^θ. This gives a contradiction because the ramification contribution of each element in W^θ is 7. Therefore $W^\theta \bigcap W^\sigma = \emptyset$.

Now we are going to describe the algebraic form of CP N-curves. Let Ψ, ψ, Π, π be the morphisms defined by the following commutative diagram:

$$
\begin{array}{ccc}
W & \xrightarrow{\;\;\Psi\;\;} & \mathbf{P}^1 \;\; = W/<\theta> \\
\Big\downarrow{\Pi} & \circlearrowleft & \Big\downarrow{\pi} \\
W/<\sigma> = \;\; \mathbf{P}^1 & \xrightarrow{\;\;\psi\;\;} & \mathbf{P}^1 \;\; = W/<\theta,\sigma> .
\end{array}
\qquad (3)
$$

Let S be the branched locus of Ψ, and h be the automorphism of $\mathbf{P}^1$ (= $W/<\theta>$) induced by σ . By the above (II), the orbits of the automorphism of $W/<\theta>$ induced by σ form a (2,2)-partition of S. By Proposition 3, for some coordinate λ of $\mathbf{P}^1$ and some $k' \in \mathbf{C} - \{0, \pm 1\}$, $S = \{0, \infty\} \coprod \{k', k'^{-1}\}$, and $\pi = \pi_{k'}$. Choose the coordinate t of $\mathbf{P}^1$ (= $W/<\sigma>$) such that ψ is defined by $t \rightsquigarrow t^N$. Then W is isomorphic to

$$
W_{N,k'} : \quad t^N = \frac{(1-k'\lambda)(1-k'\lambda^{-1})}{k^2} \quad . \qquad (4)
$$

In terms of the coordinate (t, λ), θ and σ are defined by

$$\theta : \quad (t, \lambda) \quad \rightsquigarrow \quad (\omega t, \lambda) \quad ,$$

$$\sigma : \quad (t, \lambda) \quad \rightsquigarrow \quad (t, \lambda^{-1}) \quad , \quad here \quad \omega = e^{2\pi i/N} \quad . \tag{5}$$

Since the branched locus of Π is equal to

$$\{\, b \mid \quad b^N = \quad \tfrac{1-k'}{1+k'} \quad or \quad \tfrac{1+k'}{1-k'} \,\} \quad . \tag{6}$$

$W_{N,k'}$ is also birationally equivalent to the curve in $\mathbb{C}^2$ defined by the equation:

$$w^2 = \quad (t^N - \tfrac{1-k'}{1+k'})(t^N - \tfrac{1+k'}{1-k'}) \quad , \quad (t, w) \epsilon \, \mathbb{C}^2 \quad . \tag{7}$$

The above coordinates λ, w are related by

$$w = \frac{k'}{k^2}\left(\lambda - \frac{1}{\lambda}\right) ,$$

$$\lambda = \frac{1}{2k'}\{k^2(w - t^N) + k'^2 + 1\} .$$

For the rest of this note, (t, λ) and (t, w) shall always denote the above coordinates for the curve $W_{N,k'}$. In terms of the coordinate (t, w), θ and σ are defined by

$$\theta : \quad (t, w) \rightsquigarrow (\omega t, w) ,$$

$$\sigma : \quad (t, w) \rightsquigarrow (t, -w) .$$

Let $\iota\left(= \iota_{W_{N,k'}}\right)$ be the order 2 automorphism of $W_{N,k'}$ defined by

$$\iota : \quad W_{N,k'} \rightarrow \quad W_{N,k'}, \quad (t, w) \quad \rightsquigarrow \quad \left(\tfrac{1}{t}, \tfrac{w}{t^N}\right) \quad ,$$

which is the same as

$$\iota : \quad (t, \lambda) \rightsquigarrow \quad \left(\frac{1}{t}, \, \frac{1 - k'\lambda}{k' - \lambda}\right) \tag{8}$$

by the equality

$$\frac{1}{2}\left(\frac{\lambda - \lambda^{-1}}{t^N} - \frac{k^2}{k't^N} + k' + \frac{1}{k'}\right) = \frac{1 - k'\lambda}{k' - \lambda} \quad .$$

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Then $\theta \iota = \iota \theta^{-1}$, and $< \theta, \iota > \simeq$ the subgroup $\overline{D_N}$ of Aut(P^1) with the Klein group D_N as its double cover. Now we can described all the symmetries of a CP N-curve.

__PROPOSITION 5__ For a CP N-curve W, Aut(W) = $<\sigma, \theta, \iota > \simeq (Z/2Z) \times \overline{D_N}$, and $<\theta>$ is the unique order N cyclic subgroup of Aut(W) generated by an automorphism with non-empty fixed point set.

Proof. Since σ commutes with any automorphism of W, the group Aut(W)/$<\sigma>$ acts on P^1 (= W/$<\sigma>$) and preserves the branched locus (6). It is not hard to see $< \theta, \iota > \simeq$ Aut(W) /$<\sigma> \simeq \overline{D_N}$, hence Aut(W) $\simeq (Z/2Z) \times \overline{D_N}$. For an order N automorphism ϕ of W with non-empty fixed part W^ϕ, write $\phi = \sigma^m \iota^n \theta^j$, for some integers $0 \leq m,n \leq 1$, $0 \leq j \leq (N-1)$. We need to show m=n=0. Since $\iota \cdot \theta^j$ is of order 2, n has to be 0. If $\phi = \sigma \cdot \theta^j$, j is greater than 0. The image of the fixed point set of ϕ under the projection Π in (3) is contained in the fixed part of the automorphism of W/$<\sigma>$ induced by θ^j. Hence W^ϕ is contained in W^θ, then also in W^σ. But the fixed parts of θ and σ are disjoint. This contradicts the assumption of the non-emptyness of the fixed point set of ϕ. Therefore j =0, and $<\phi> = <\theta>$. q.e.d.

__COROLLARY.__ The morphism $\Psi : W \rightarrow P^1$ is the unique (up to isomorphism) order N cyclic covering of W over P^1 with exactly 4 distinct critical values.

Proof. Let ϕ be an order N automorphism of W such that the projection W $\rightarrow$ W/$<\phi>$ = P^1 has exactly 4 critical values. By Hurwitz's Theorem, $|W^\phi|$= 4 . Hence $<\phi> = <\theta>$ by the above Proposition, and the conclusion follows immediately. q.e.d.

__COROLLARY__ The curves $W_{N,\alpha}$ and $W_{N,\beta}$ are isomorphic as Riemann surfaces if and only if $[\alpha] = [\beta]$ in $\mathcal{A}$.

Proof. Let ϕ be the biregular isomorphism from $W_{N,\alpha}$ to $W_{N,\beta}$. Then

$$\phi^{-1} \sigma_{W_{N,\beta}} \cdot \phi = \sigma_{W_{N,\alpha}} \quad \text{(by the uniqueness of hyperelliptic involution),}$$

$$\phi^{-1} \theta_{W_{N,\beta}} \cdot \phi = \theta_{W_{N,\alpha}} \quad \text{(by the above Corollary).}$$

Hence the diagrams (3) for the curves $W_{N,\alpha}$ and $W_{N,\beta}$ are equivalent through the morphisms induced by ϕ. This is equivalent to $[\alpha] = [\beta]$ in $\mathcal{A}$ because the (2,2)-partitions of the branched locus of Ψ for $W_{N,\alpha}$ and $W_{N,\beta}$ are equivalent in the relation of Proposition 1 (i). q.e.d.

3. *Three related chiral Potts N − curves*

We continue to study the CP N-curves. By the Corollary of Proposition 5, the CP N-curves are parametrized by $\mathcal{A}$. For a given k'ϵC$-\{0,\pm1\}$, the elements [k'], $\left[\frac{ik'}{k}\right]$, [k] of $\mathcal{A}$ are related in the sense that they determine the same 1–torus in Proposition 2 by the result of Proposition 4. For a given k'ϵC$-\{0,\pm1\}$, we are going to study the relation between the three CP N-curves $W_{N,k'}, W_{N,\frac{ik'}{k}}, W_{N,k}$. The curve $W_{N,k'}$ is defined by the coordinate (t, λ) of (4). By (ii) of Proposition 3, we have

$$W_{N,\frac{ik'}{k}} \simeq W_r \; :$$

$$t_r^N = \frac{1-k'\lambda}{1-k'\lambda^{-1}} \quad , \quad (t_r,\lambda) \quad \epsilon \; \mathbf{C}^2 \quad , \tag{9}$$

and the automorphisms σ, ι for W_r are defined by

$$\sigma_{W_r} : \quad (t_r,\lambda) \quad \rightsquigarrow \quad \left(t_r , \frac{1-k'\lambda}{k'-\lambda}\right) \quad ,$$

$$\iota_{W_r} : \quad (t_r,\lambda) \quad \rightsquigarrow \quad \left(\frac{1}{t_r}, \frac{1}{\lambda}\right) \quad . \tag{10}$$

Similarly, $W_{N,k} \simeq W_\ell$:

$$t_l^N = \frac{k'-\lambda^{-1}}{k'-\lambda} \quad , \quad (t_l,\lambda) \quad \epsilon \; \mathbf{C}^2 \quad , \tag{11}$$

with the automorphisms

$$\sigma_{W_l} : \quad (t_l,\lambda) \quad \rightsquigarrow \quad \left(t_l , \frac{k'-\lambda}{1-k'\lambda}\right) \quad ,$$

$$\iota_{W_l} : \quad (t_l,\lambda) \quad \rightsquigarrow \quad \left(\frac{1}{t_l}, \frac{1}{\lambda}\right) \quad . \tag{12}$$

In this section, we shall write $W=W_{N,k'}$. The morphisms Ψ, Ψ_r, Ψ_ℓ from W, W_r, W_ℓ to $\mathbf{P}^1$ in (3) are now defined by $\Psi(t,\lambda)= \lambda$, $\Psi_r(t_r,\lambda)= \lambda$, $\Psi_\ell(t_\ell,\lambda)= \lambda$.

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Denote the fiber product $W \times_{\mathbf{P}^1} W_r$ of Ψ and Ψ_r by $\mathcal{W}_r$:

$$
\begin{array}{ccccc}
 & & \mathcal{W}_r & & \\
 & \swarrow & & \searrow & \\
W & & \circlearrowleft & & W_r \\
 & \Psi \searrow & & \swarrow \Psi_r & \\
 & & \mathbf{P}^1 & &
\end{array}
\qquad (13)
$$

and $\mathcal{W}_r$ is defined by

$$
\left\{
\begin{array}{rcll}
t^N & = & \dfrac{(1-k'\lambda)(1-k'\lambda^{-1})}{k^2} & \\[2mm]
t_r^{\ N} & = & \dfrac{1-k'\lambda}{1-k'\lambda^{-1}} & , \quad for \quad (t, t_r, \lambda) \varepsilon \quad \mathbf{C}^3 \quad .
\end{array}
\right.
$$

The automorphisms θ_W, θ_{W_r} of W, W_r induce the automorphisms Θ, Θ_r of $\mathcal{W}_r$ which are described by

$$
\Theta : \quad (t, t_r, \lambda) \quad \rightsquigarrow \quad (\omega t, t_r, \lambda) \quad ,
$$

$$
\Theta_r : \quad (t, t_r, \lambda) \quad \rightsquigarrow \quad (t, \omega t_r, \lambda) \quad .
$$

Then the group $< \Theta, \Theta_r >$ isomorphic to $(\mathbf{Z}/N\mathbf{Z})^2$, and acts freely on $\mathcal{W}_r$ outside the 4 critical points of the projection from $\mathcal{W}_r$ to the λ−plane, which has the branched locus $\{0, \infty, k', k'^{-1}\}$. Consider the quotients:

$$
V_1 = \mathcal{W}_r /_{<\Theta\Theta_r>} \quad , \qquad V_2 = \mathcal{W}_r /_{<\Theta\Theta_r^{-1}>} \quad .
$$

Then V_1, V_2 are N-sheet cover of the λ−plane $\mathbf{P}^1$ branched at $\{k', 0\}$ and $\{\frac{1}{k'}, \infty\}$ respectively with the following coordinates:

$$
V_1 \quad : \quad \left(\tfrac{t}{t_r}\right)^N = \dfrac{(1-k'\lambda^{-1})^2}{k^2} \quad , \quad \left(\tfrac{t}{t_r}, \lambda\right) \varepsilon \, \mathbf{C}^2 \quad ;
$$

$$
V_2 \quad : \quad (t t_r)^N = \dfrac{(1-k'\lambda)^2}{k^2} \quad , \quad (t t_r, \lambda) \varepsilon \, \mathbf{C}^2 \quad .
$$

Consider the degree 2 cover of the $(\frac{t}{t_r})$-plane ,

$$
\mathbf{P}^1 \to \mathbf{P}^1 \quad , \quad x \rightsquigarrow \left(\frac{t}{t_r}\right) = x^2 \quad .
$$

The fiber product of x-plane with V_1 over the $(\frac{t}{t_r})$-plane is reducible with two rational curves as its irreducible components. They are described by

$$
x^N = \pm \, \frac{1-k'\lambda^{-1}}{k} \quad , \qquad (x, \lambda) \varepsilon \, \mathbf{C}^2 \quad .
$$

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Let X be the component

$$x^N = \frac{1-k'\lambda^{-1}}{k} \quad , \quad (x,\lambda) \quad \varepsilon\, C^2 \quad . \qquad (14)$$

X is a N-fold cover of the $\lambda-$plane $\mathbf{P}^1$ branched at $\{k' , 0\}$. With the same discussion for V_2 , we have a N-fold cover Y of the $\lambda-$plane $\mathbf{P}^1$ branched at $\{\frac{1}{k'} , \infty\}$ which is defined by

$$y^N = \frac{1-k'\lambda}{k} \quad , \quad (y,\lambda) \quad \varepsilon\, C^2 \quad . \qquad (15)$$

There are the morphisms from the fiber product of X and Y over the $\lambda-$plane into W, W_r with the following commutative diagram:

$$
\begin{array}{ccc}
& X \times_{\mathbf{P}^1} Y & \\
\swarrow & & \searrow \\
W \qquad\quad & \circlearrowleft & \qquad W_r \\
\Psi \searrow & & \swarrow \Psi_r \\
& \mathbf{P}^1 &
\end{array}
\qquad (16)
$$

which are described by:

$$
\begin{array}{ccc}
& (x,y,\lambda) & \\
\swarrow & & \searrow \\
(xy,\lambda) \; = (t,\lambda) & & (t_r,\lambda) \; = \left(\frac{y}{x},\lambda\right) \\
\searrow & & \swarrow \\
& \lambda &
\end{array}
$$

Similarly by replacing the above W_r by W_ℓ, we have the rational curves X', Y' with the degree N morphisms over the $\lambda-$plane :

$$X' \to \mathbf{P}^1, \quad x' \rightsquigarrow \lambda, \quad \text{with} \quad x'^N = \frac{\lambda-k'}{k}, \quad \left(x'^2 = \frac{t}{t_\ell}\right),$$

$$Y' \to \mathbf{P}^1, \quad y' \rightsquigarrow \lambda, \quad \text{with} \quad y'^N = \frac{\lambda^{-1}-k'}{k}, \quad (y'^2 = t t_\ell),$$

$$\qquad (17)$$

and also the diagram:

$$
\begin{array}{ccc}
& X' \times_{\mathbf{P}^1} Y' & \\
\swarrow & & \searrow \\
W_\ell \qquad\quad & \circlearrowleft & \qquad W \\
\Psi_\ell \searrow & & \swarrow \Psi \\
& \mathbf{P}^1 &
\end{array}
\qquad (18)
$$

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defined by

$$\left(\tfrac{y'}{x'},\lambda\right) = (t_\ell,\lambda) \quad \begin{matrix} & (x',y',\lambda) & \\ \nearrow & & \searrow \\ & & \\ \searrow & & \nearrow \\ & \lambda & \end{matrix} \quad (t,\lambda) = (x'y',\lambda)$$

The coordinates of X , X' and Y, Y' are related by :

$$\left(\frac{x}{x'}\right)^N = \lambda^{-1} \ , \quad \left(\frac{y}{y'}\right)^N = \lambda \ .$$

It follows that under the base change

$$\mathbf{P}^1_\mu \to \mathbf{P}^1 \ , \quad \mu \rightsquigarrow \lambda = \mu^N \ , \tag{19}$$

we have

$$X \times_{\mathbf{P}^1} \mathbf{P}^1_\mu \simeq X' \times_{\mathbf{P}^1} \mathbf{P}^1_\mu \ , \quad Y \times_{\mathbf{P}^1} \mathbf{P}^1_\mu \simeq Y' \times_{\mathbf{P}^1} \mathbf{P}^1_\mu \ .$$

In fact, these are the Fermat curves in $\mathbf{P}^2$:

$$\mathfrak{X} = X \times_{\mathbf{P}^1} \mathbf{P}^1_\mu \ , \quad ka^N + k'c^N = d^N \ ,$$
$$\mathfrak{Y} = Y \times_{\mathbf{P}^1} \mathbf{P}^1_\mu \ , \quad kb^N + k'd^N = c^N \ ,$$

here [a,c,d], [b,c,d] are the homogenous coordinates of $\mathbf{P}^2$, which are related with
x, y, x',y',μ by

$$\left(\tfrac{a}{d},\tfrac{c}{d}\right) = (x,\mu^{-1}), \quad \left(\tfrac{b}{c},\tfrac{d}{c}\right) = (y,\mu), \quad x' = x\mu, \quad y' = y\mu^{-1}. \tag{20}$$

Consider the fiber product of $\mathfrak{X}$ and $\mathfrak{Y}$ over the $\mu-$plane

$$\mathfrak{W} = \mathfrak{X} \times_{\mathbf{P}^1_\mu} \mathfrak{Y} \quad \left(= (X \times_{\mathbf{P}^1} Y) \times_{\mathbf{P}^1} \mathbf{P}^1_\mu\right) \ .$$

Then $\mathfrak{W}$ is is a curve of genus $N^3 - 2N^2 + 1$, defined by

$$\mathfrak{W} : \left\{ \begin{matrix} ka^N + k'c^N = d^N \\ kb^N + k'd^N = c^N \end{matrix} \right. \quad \text{for } [a,b,c,d] \varepsilon \mathbf{P}^3 \ , \tag{21}$$

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which is also equal to

$$\left\{ \begin{array}{l} a^N + k'b^N = kd^N \\ k'a^N + b^N = kc^N \end{array} \right. \quad \text{for} \quad [a, b, c, d] \varepsilon \mathbf{P}^3 \ . \tag{22}$$

Then the curves W, W_r, W_l, X, X', Y, Y' are the quotients of $\mathfrak{W}$ by various order N^2 subgroups of Aut($\mathfrak{W}$) which are described as follows :

$$X = \mathfrak{W}/< \begin{array}{ccc} [a,b,c,d] & \rightsquigarrow & [a,\omega b,c,d] \\ [a,b,c,d] & \rightsquigarrow & [a,b,\omega c,d] \end{array} > \ ,$$

$$X' = \mathfrak{W}/< \begin{array}{ccc} [a,b,c,d] & \rightsquigarrow & [a,\omega b,c,d] \\ [a,b,c,d] & \rightsquigarrow & [\omega a,b,\omega c,d] \end{array} > \ ,$$

$$Y = \mathfrak{W}/< \begin{array}{ccc} [a,b,c,d] & \rightsquigarrow & [\omega a,b,c,d] \\ [a,b,c,d] & \rightsquigarrow & [a,b,c,\omega d] \end{array} > \ ,$$

$$Y' = \mathfrak{W}/< \begin{array}{ccc} [a,b,c,d] & \rightsquigarrow & [\omega a,b,c,d] \\ [a,b,c,d] & \rightsquigarrow & [a,\omega b,c,\omega d] \end{array} > \ .$$

Define

$$\mathfrak{J}: \quad \mathfrak{W} \to \mathfrak{W} \quad , [a,b,c,d] \rightsquigarrow \quad [b,a,d,c],$$

$$\mathfrak{J}_r: \quad \mathfrak{W} \to \mathfrak{W} \quad , [a,b,c,d] \rightsquigarrow \quad \left[\omega^{\frac{1}{2}}c, d, a, \omega^{\frac{1}{2}}b\right],$$

$$\mathfrak{J}_\ell: \quad \mathfrak{W} \to \mathfrak{W} \quad , [a,b,c,d] \rightsquigarrow \quad \left[\omega^{\frac{1}{2}}d, \omega c, b, \omega^{\frac{1}{2}}a\right], \tag{23}$$

$$\mathfrak{R}: \quad \mathfrak{W} \to \mathfrak{W} \quad , [a,b,c,d] \rightsquigarrow \quad [\omega a,b,\omega c,d],$$

$$\mathfrak{S}: \quad \mathfrak{W} \to \mathfrak{W} \quad , [a,b,c,d] \rightsquigarrow \quad [\omega a,\omega b,c,d],$$

$$\mathfrak{T}: \quad \mathfrak{W} \to \mathfrak{W} \quad , [a,b,c,d] \rightsquigarrow \quad [a,\omega b,\omega c,d],$$

Then it is easy to see

the order of $\mathfrak{R}, \mathfrak{S}, \mathfrak{T} = N$,

the order of $\mathfrak{J}, \mathfrak{J}_\ell, \mathfrak{J}_r = 2$,

and

$$\mathfrak{R}\mathfrak{J} = \mathfrak{J}\mathfrak{R}^{-1}, \quad \mathfrak{S}\mathfrak{J} = \mathfrak{J}\mathfrak{S}, \quad \mathfrak{T}\mathfrak{J} = \mathfrak{J}\mathfrak{T}^{-1},$$
$$\mathfrak{R}\mathfrak{J}_r = \mathfrak{J}_r\mathfrak{R}, \quad \mathfrak{S}\mathfrak{J}_r = \mathfrak{J}_r\mathfrak{S}^{-1}, \quad \mathfrak{T}\mathfrak{J}_r = \mathfrak{J}_r\mathfrak{T}^{-1},$$
$$\mathfrak{R}\mathfrak{J}_\ell = \mathfrak{J}_\ell\mathfrak{R}^{-1}, \quad \mathfrak{S}\mathfrak{J}_\ell = \mathfrak{J}_\ell\mathfrak{S}^{-1}, \quad \mathfrak{T}\mathfrak{J}_\ell = \mathfrak{J}_\ell\mathfrak{T}, \tag{24}$$
$$\mathfrak{R}\mathfrak{J}_\ell\mathfrak{J} = \mathfrak{J}\mathfrak{J}_\ell, \quad \mathfrak{S}\mathfrak{J}_r\mathfrak{J}_\ell = \mathfrak{J}_\ell\mathfrak{J}_r, \quad \mathfrak{T}\mathfrak{J}_r\mathfrak{J} = \mathfrak{J}\mathfrak{J}_r.$$

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Hence $<\mathfrak{R}, \mathfrak{S}, \mathfrak{T}, \mathfrak{J}, \mathfrak{J}_r, \mathfrak{J}_\ell>$ is a group of automorphisms of $\mathfrak{W}$, and is also equal to $<\mathfrak{J}, \mathfrak{J}_r, \mathfrak{J}_\ell>$. It can be shown that the quotients of $\mathfrak{W}$ by various normal subgroups of $<\mathfrak{J}, \mathfrak{J}_r, \mathfrak{J}_\ell>$ also give the curves $X \times_{\mathbf{p}^1} Y$, $X' \times_{\mathbf{p}^1} Y'$, W, W_r, W_ℓ:

$$X \times_{\mathbf{p}^1} Y = \mathfrak{W}/ <\mathfrak{T}>,$$

$$X' \times_{\mathbf{p}^1} Y' = \mathfrak{W}/ <\mathfrak{R}>,$$

$$W = \mathfrak{W}/ <\mathfrak{R}, \mathfrak{T}>,$$

$$W_r = \mathfrak{W}/ <\mathfrak{S}, \mathfrak{T}>, \qquad (25)$$

$$W_\ell = \mathfrak{W}/ <\mathfrak{R}, \mathfrak{S}> .$$

By Hurwitz's Theorem, $\mathfrak{W}$ is the N^2-fold unramified cover of W, W_r, W_ℓ. This implies that with the morphisms in (16) (or (18)), $X \times_{\mathbf{p}^1} Y$, (or $X' \times_{\mathbf{p}^1} Y'$ resp.) is the $N-$fold unramified cover of W, W_r (or W, W_ℓ resp.). (We have demonstrated here that the curves W, W_r, W_ℓ can be constructed from the Fermat curves $\mathfrak{X}$ and $\mathfrak{Y}$. In fact, that is how the CP N-curves had originally been derived in the physical literatures [1, 12]. The variables a,b,c,d are the Boltzmann weights of the statistical model.)

The automorphism group $<\mathfrak{J}, \mathfrak{J}_r, \mathfrak{J}_\ell>$ of $\mathfrak{W}$ induces the groups of automorphisms of W, W_r, W_ℓ. In fact, these are the group of all automorphisms for W, W_r, W_ℓ by the following result.

PROPOSITION 6

(i) Under the canonical homomorphism $<\mathfrak{J}, \mathfrak{J}_r, \mathfrak{J}_\ell> \to$ Aut (W) ,

$$\mathfrak{J} \rightsquigarrow \sigma_W, \quad \mathfrak{J}_r \rightsquigarrow \iota_W, \quad \mathfrak{J}_\ell \rightsquigarrow \theta_W \iota_W \sigma_W \quad .$$

(ii) Under the canonical homomorphism $<\mathfrak{J}, \mathfrak{J}_r, \mathfrak{J}_\ell> \to$ Aut (W_r) ,

$$\mathfrak{J}_r \rightsquigarrow \sigma_{W_r}, \quad \mathfrak{J} \rightsquigarrow \iota_{W_r}, \quad \mathfrak{J}_\ell \rightsquigarrow \theta_{W_r} \iota_{W_r} \sigma_{W_r} \quad .$$

(iii) Under the canonical homomorphism $<\mathfrak{J}, \mathfrak{J}_r, \mathfrak{J}_\ell> \to$ Aut (W_ℓ) ,

$$\mathfrak{J}_\ell \rightsquigarrow \sigma_{W_\ell}, \quad \mathfrak{J} \rightsquigarrow \iota_{W_\ell}, \quad \mathfrak{J}_r \rightsquigarrow \theta_{W_\ell}^{-1} \iota_{W_\ell} \sigma_{W_\ell} \quad .$$

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Proof. By (16) (18) (19) (20) (21), t $= \frac{ab}{cd}$, $t_r = \frac{bd}{ac}$, $t_\ell = \frac{bc}{ad}$, $\lambda = (\frac{d}{c})^N$, $\frac{1-k'\lambda}{k'-\lambda} = -(\frac{b}{a})^N$. Then the results follows from the the definition of $\mathfrak{J},\mathfrak{J}_r,\mathfrak{J}_\ell$ and (5) (8) (10) (12). q.e.d.

When N is even, $W_{N,k'}/<\theta^{\frac{N}{2}}>$ is the 1–torus , and it is the double cover of $\mathbf{P}^1$ branched at $\{0,\infty,k',k'^{-1}\}$. By Proposition 4, the 1–tori determined by $W_{N,k'},W_{N,\frac{ik'}{k}},W_{N,k}$ are isomorphic. Hence we have the following diagram:

$$
\begin{array}{ccccc}
W_{N,k} & & W_{N,k'} & & W_{N,\frac{ik'}{k}} \\
 & \searrow & \downarrow & \swarrow & \\
 & & 1-\text{torus} & & \\
 & & \downarrow & & \\
 & & \mathbf{P}^1 & &
\end{array}
$$
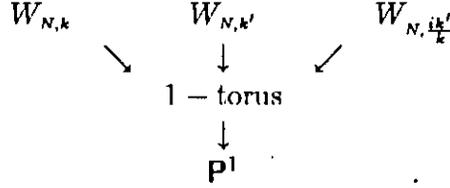
4. $N-torsion$ $line$ $bundles$ of $chiral$ $Potts$ $N-curves$

As before, W $= W_{N,k'}$, $\mathfrak{W}$ = the fiber product of Fermat curves with the homogenous coordinate [a,b,c,d] defined in (21), $\mathfrak{R},\mathfrak{S},\mathfrak{T},\mathfrak{J},\mathfrak{J}_r$,= the automorphisms of $\mathfrak{W}$ in (23). Let $\mathbf{u},\mathbf{u}',\mathbf{v},\mathbf{v}'$ be the elements in W whose λ–values are $0,\infty,k',k'^{-1}$ respectively, and

$$\mathfrak{p}:\quad \mathfrak{W} \quad \to \quad W \quad (=\mathfrak{W}/_{<\mathfrak{R},\mathfrak{T}>})$$

be the unramified N^2–fold cover in (25). By (14) (15) (16) (19) (20) , the divisors for the sections a,b,c,d in $\mathfrak{W}$ are

$$div(a) = \mathfrak{p}^*(\mathbf{v}), \quad div(b) = \mathfrak{p}^*(\mathbf{v}'), \quad div(c) = \mathfrak{p}^*(\mathbf{u}'), \quad div(d) = \mathfrak{p}^*(\mathbf{u}).$$

For (m,n)ϵ $(\mathbf{Z}/N\mathbf{Z})^2$, let $\kappa_{(m,n)}$: $<\mathfrak{T},\mathfrak{R}> \to \mathbf{C}^*$ be the character with $\kappa_{(m,n)}(\mathfrak{T}) = \omega^m$, $\kappa_{(m,n)}(\mathfrak{R}) = \omega^n$. Then

$$(\mathfrak{W} \times \mathbf{C})/(\mathrm{w} \times \zeta \sim \mathrm{wg} \times \kappa_{(m,n)}(\mathrm{g})\zeta)$$

is a line bundle over W, and its associated $\mathcal{O}_W$–sheaf is denoted by $\mathcal{L}_{(m,n)}$.

PROPOSITION 7

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(i) $p_* \mathcal{O}_\mathfrak{W} = \displaystyle\bigoplus_{(m,n)\epsilon(\mathbb{Z}/N\mathbb{Z})^2} \mathcal{L}_{(m,n)}.$

(ii) $\mathcal{L}_{(1,0)} = \mathcal{O}_W[v' - u]$, $\mathcal{L}_{(0,1)} = \mathcal{O}_W[v - u]$, $\mathcal{L}_{(1,1)} = \mathcal{O}_W[u' - u]$.

Proof. For an open set U of W, any function f of $p^{-1}(U)$ can be uniquely written as $f = \displaystyle\sum_{(m,n)} f_{(m,n)}$, where $f_{(m,n)}$ is an eigenfunction of $<\mathfrak{T}, \mathfrak{R}>$ with eigenvalue $\mathcal{L}_{(m,n)}$. Hence we obtain (i).

With the homogeneous coordinate [a,b,c,d] of $\mathfrak{W}$, the divisors of the rational functions a/d , b/d , c/d of $\mathfrak{W}$ equal to $p^*(v - u)$, $p^*(v' - u)$, $p^*(u' - u)$ respectively. Since the rational function a/d (b/d , c/d) corresponds to a meromorphic section of $\mathcal{L}_{(0,1)}$ ($\mathcal{L}_{(1,0)}$, $\mathcal{L}_{(1,1)}$ resp.) over W, (ii) follows immediately. q.e.d.

By the same argument, we have the similar conclusions for the curves W_r, W_ℓ.

<u>PROPOSITION 8</u> Let

$$p_r : \mathfrak{W} \;\;\to\; W_r \quad (= \mathfrak{W}/_{<\mathfrak{S},\mathfrak{T}>})$$

and

$$p_\ell : \mathfrak{W} \;\;\to\; W_\ell \quad (= \mathfrak{W}/_{<\mathfrak{R},\mathfrak{S}>})$$

be the unramified N^2-fold covers in (25). Let $u_r, u_r' v_r, v_r'$, ($u_\ell, u_\ell', v_\ell, v_\ell'$), be the elements of W_r, (W_ℓ resp.), such that as the divisors of $\mathfrak{W}$,

$$div(a) = p_r^*(v_r), div(b) = p_r^*(u_r'), div(c) = p_r^*(v_r'), \; div(d) = p_r^*(u_r) ,$$

$$div(a) = p_\ell^*(u_\ell'), \; div(b) = p_\ell^*(v_\ell), \; div(c) = p_\ell^*(v_\ell'), \; div(d) = p_\ell^*(u_\ell).$$

Denote

$\mathcal{L}_{r(m,n)}$ = the $\mathcal{O}_{W_r}$-sheaf associated the character $< \quad \mathfrak{S}, \mathfrak{T} >\to \mathbf{C}^*$, $\mathfrak{T} \rightsquigarrow \omega^m , \mathfrak{S} \rightsquigarrow \omega^n$.

$\mathcal{L}_{\ell(m,n)}$ = the $\mathcal{O}_{W_\ell}$-sheaf associated to the character $< \mathfrak{R}, \mathfrak{S} >\to \mathbf{C}^*$, $\mathfrak{R} \rightsquigarrow \omega^m$, $\mathfrak{S} \rightsquigarrow \omega^n$.

Then

(i) $p_{r*} \mathcal{O}_\mathfrak{W} = \displaystyle\bigoplus_{(m,n)\epsilon(\mathbb{Z}/N\mathbb{Z})^2} \mathcal{L}_{r(m,n)},$ $\quad p_{\ell*}\mathcal{O}_\mathfrak{W} = \displaystyle\bigoplus_{(m,n)\epsilon(\mathbb{Z}/N\mathbb{Z})^2} \mathcal{L}_{\ell(m,n)}.$

(ii) Over the curves W_r, W_ℓ, we have
$\mathcal{L}_{r(1,0)} = \mathcal{O}_{W_r}[v_r' - u_r]$, $\mathcal{L}_{r(0,1)} = \mathcal{O}_{W_r}[v_r - u_r]$, $\mathcal{L}_{r(1,1)} = \mathcal{O}_{W_r}[u_r' - u_r]$.
$\mathcal{L}_{\ell(1,0)} = \mathcal{O}_{W_\ell}[v_\ell' - u_\ell]$, $\mathcal{L}_{\ell(0,1)} = \mathcal{O}_{W_\ell}[v_\ell - u_\ell]$, $\mathcal{L}_{\ell(1,1)} = \mathcal{O}_{W_\ell}[u_\ell' - u_\ell]$.

5. *Jacobian variety of a chiral Potts N − curve*

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Let r be an element in $\mathbb{C}-\{0, \pm 1\}$, $\omega = e^{2\pi i/N}$, $g = N-1$. Assume that $|\frac{1-r}{1+r}| \le 1$, and denote $(\frac{1-r}{1+r})^{1/N} = |\frac{1-r}{1+r}|^{1/N} e^{i\vartheta/N}$, here $0 \le \vartheta = \arg(\frac{1-r}{1+r}) < 2\pi$. In this section, we shall denote

$W = W_{N,r}$,

Ψ, ψ, Π, π = the morphisms of (3) for W,

$\{\theta, \sigma, \iota\}$ = the generators of Aut(W) in Proposition 5.

W is a genus g CP N-curve defined by the coordinate (t, λ) of (4), or (t, w) of (7). Denote

$$\text{Jac}(W) = \text{the Jacobian variety of } W.$$

The critical points of Π are the elements b_j, b_j', $1 \le j \le N$, of W with $\Pi(b_j) = \omega^{-j}(\frac{1+r}{1-r})^{1/N}$ and $\Pi(b_j') = \omega^{-j}(\frac{1-r}{1+r})^{1/N}$. We also denote $o = b_N$, $o' = b_N'$. The critical points of Ψ are the elements p, p', q, q' of W with the λ—values $0, \infty, r, r^{-1}$ respectively. Then p and q are on the "same sheet" of the covering of Π. (Here we consider W as the double cover over the t-plane with cuts on the segments from $\omega^j(\frac{1-r}{1+r})^{1/N}$ to $\omega^j(\frac{1+r}{1-r})^{1/N}$, $1 \le j \le N$. The value of the coordinate w determine the "sheet" of W.) We are going to describe Jac(W) using the data of $\{b_j, b_j' | 1 \le j \le N\}$, and then $\{p, p', q, q'\}$.

It is well-known that the abelian differentials of first kind for W has the following expression:

<u>LEMMA 5</u> $\Gamma(W, \Omega^1)$ = the $\mathbb{C}$ —space consisting of

$$p(t)\frac{dt}{w}, \quad p(t) = \text{a polynomial of degree} \le (N-2).$$

Denote $\varphi_\alpha = t^{\alpha-1}\frac{dt}{w}$ for $1 \le \alpha \le g$. Then $\{\varphi_\alpha, 1 \le \alpha \le g\}$ forms a base of $\Gamma(W, \Omega^1)$ with

$$\theta^*(\varphi_\alpha) = \omega^\alpha \varphi_\alpha \quad , \quad \sigma^*(\varphi_\alpha) = -\varphi_\alpha \quad , \quad \iota^*(\varphi_\alpha) = -\varphi_{N-\alpha} \quad . \quad (26)$$

By a path in W we shall always mean an oriented one. It is easy to see the following lemma holds.

<u>LEMMA 6</u> For a path γ in W, $\displaystyle\int_{\theta^m(\gamma)} \varphi_\alpha = \omega^{\alpha m} \int_\gamma \varphi_\alpha$.

We shall denote

$$
\int_\gamma \varphi = \begin{pmatrix} \int_\gamma \varphi_1 \\ \vdots \\ \int_\gamma \varphi_g \end{pmatrix}.
$$

Let A and B be the σ–invariant 1–cycles on W such that by the morphism Π A lies over the segment from $(\frac{1-r}{1+r})^{\frac{1}{N}}$ to $(\frac{1+r}{1-r})^{\frac{1}{N}}$, and B lies over a path from $\omega(\frac{1-r}{1+r})^{\frac{1}{N}}$ to $(\frac{1+r}{1-r})^{\frac{1}{N}}$ with A intersecting B only at o having the intersection number I(A, B) = 1. Define

$$
A_j = \theta^{-j}(A), \quad B_j = \theta^{-j}(B) \quad \text{for integer } j.
$$
$$
C_\alpha = \sum_{1 \leq j \leq \alpha} B_j \quad \text{for} \quad 1 \leq \alpha \leq g.
$$

Then $A_{j+N} = A_j$, $B_{j+N} = B_j$, and $\sum\limits_{1 \leq j \leq N} A_j$, $\sum\limits_{1 \leq j \leq N} B_j$ are homologous to zero. (The cycles A_α, B_α, C_α are shown in Fig. 1.)

The intersection numbers of the cycles A_α, B_α, C_α , $1 \leq \alpha \leq g$, are given by

$$
I(A_\alpha, A_\beta) = I(B_\alpha, B_\beta) = 0 \ ,
$$
$$
I(A_\alpha, B_\beta) = 0, \quad \text{except} \quad I(A_\alpha, B_\alpha) = 1, I(A_\alpha, B_{\alpha+1}) = -1.
$$

Hence

$$
I(A_\alpha, A_\beta) = I(C_\alpha, C_\beta) = 0, \quad I(A_\alpha, C_\beta) = \delta_{\alpha\beta} \ .
$$

Let

$$
l = \int_A \varphi = \begin{pmatrix} l_1 \\ \vdots \\ l_g \end{pmatrix}, \quad l' = \int_B \varphi = \begin{pmatrix} l'_1 \\ \vdots \\ l'_g \end{pmatrix}. \tag{27}
$$

By Lemma 6,

$$
\left(\int_{A_1} \varphi, \dots, \int_{A_g} \varphi \right) = (\delta_{\alpha\beta} l_\alpha)(\omega^{-\alpha\beta})_{1 \leq \alpha,\beta \leq g} \quad ,
$$

$$
\left(\int_{B_1} \varphi, \dots, \int_{B_g} \varphi \right) = (\delta_{\alpha\beta} l'_\alpha)(\omega^{-\alpha\beta})_{1 \leq \alpha,\beta \leq g} \quad .
$$

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Hence

$$\left(\int_{C_1} \varphi, ..., \int_{C_\varrho} \varphi \right) = (\delta_{\alpha\beta} l'_\alpha) (k'_{\alpha\beta})_{1 \le \alpha,\beta \le \varrho} .$$

here $k'_{\alpha\beta} = \sum_{1 \le m \le \beta} \omega^{-m\alpha} = i e^{\frac{\pi i}{N}} (\omega^{-\alpha\beta} - 1)/2\sin \frac{\pi\alpha}{N}.$

Let $\phi_1, .., \phi_\varrho$ the base of $\Gamma(W, \Omega^1)$ defined by

$$\phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_\varrho \end{pmatrix} = (\omega^{-\alpha\beta})^{-1} (\delta_{\alpha\beta} l_\alpha)^{-1} \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_\varrho \end{pmatrix} \qquad (28)$$

then

$$\left(\int_{A_\alpha} \phi , \int_{B_\alpha} \phi \right) = (I_\varrho , (\tau_{\alpha\beta})) ,$$

Here

$$\tau_{\alpha\beta} = \frac{i}{2N} \sum_\gamma \frac{(\omega^{\alpha\gamma} - 1)(\omega^{-\gamma\beta} - 1)\omega^{-\gamma/2}}{\sin \frac{\pi\gamma}{N}} \chi_\gamma$$

$$= \frac{2i}{N} \sum_\gamma \frac{\cos \frac{\pi\gamma(\alpha-\beta)}{N} \sin \frac{\pi\alpha\gamma}{N} \sin \frac{\pi\beta\gamma}{N} \omega^{-\gamma/2}}{\sin \frac{\pi\gamma}{N}} \chi_\gamma ,$$

$$\text{with } \chi_\gamma = \frac{\ell'_\gamma}{\ell_\gamma} .$$

(The above $\omega^{-\gamma/2}\chi_\gamma$ for $W = W_{N, \frac{ik'}{k}}$ is equal to the $\frac{l'_\gamma}{l_\gamma}$ in [6]).
Hence the period of the Jacobian of W is equal to $\tau = {}^t\tau = (\tau_{\alpha\beta})_{1 \le \alpha,\beta \le \varrho}$ and

$$Jac(W) = \text{the torus } \mathbb{C}^g/(\mathbb{Z}^g + \tau\mathbb{Z}^g) .$$

For a fixed base point $\star$ of W, we have the canonical embedding of W into Jac(W) :

$$W \longrightarrow Jac(W) \qquad (29)$$

$$\mathbf{w} \rightsquigarrow [\int_\star^{\mathbf{w}} \phi]$$

The period of Jac(W) is related to the elements b_j, b'_j by the following proposition.

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PROPOSITION 8 For $1 \le \beta \le g$, let

$\tau_\beta=$ the $\beta-$th column of $(\tau_{\alpha\beta})$,

$e_\beta=$ the $\beta-$the element of the standard base of $\mathbf{Z}^g$.

Denote $\frac{1}{2} = \frac{1}{2} \sum_{1 \le \beta \le g} e_\beta$.

(i) For an element w of W, let $[s]$, $[z]$ be the images of w under the maps (29) for $\star = o$, o' respectively. Then $[s - \frac{1}{2}] = [z]$.

(ii) Under the map (29) for $\star = o'$,

$$o \rightsquigarrow \quad \left[\tfrac{1}{2}\right] \quad ,$$

$$b_\beta \rightsquigarrow \quad \left[\tfrac{1}{2}\tau_\beta + \tfrac{1}{2} \sum_{\alpha \le (\beta-1)} e_\alpha\right] \quad ,$$

$$b'_\beta \rightsquigarrow \quad \left[\tfrac{1}{2}\tau_\beta + \tfrac{1}{2} \sum_{\alpha \le \beta} e_\alpha\right] \quad , \quad for \quad 1 \le \beta \le g.$$

Proof. Let $\overline{oo'}$ be the path from o to o' along the curve A. Then

$$\int_{\overline{oo'}} \phi = \tfrac{1}{2} \left(\int_{\overline{oo'}} \phi + \int_{-\sigma(\overline{oo'})} \phi \right)$$

$$= \tfrac{1}{2} \int_A \phi = \tfrac{1}{2} \int_{-\sum_{1 \le \alpha \le g} A_\alpha} \phi = -\tfrac{1}{2}.$$

Hence

$$[z] = \left[\int_{o'}^{w} \phi\right] = \left[\int_{o}^{w} \phi\right] - \left[\int_{\overline{oo'}} \phi\right] = \left[s - \tfrac{1}{2}\right] \quad .$$

So we obtain (i), hence the first correspondence of (ii) . Let $\overline{b'_{j-1}b_j}$, $(\overline{b_jb'_j})$, $1 \le j \le N$, be the path from b'_{j-1} to b_j (b_j to b'_j resp.) along B_j (A_j resp.) . Let $\overline{o'b_\beta}$, $\overline{o'b'_\beta}$, $1 \le \beta \le g$, be the curves from o' to b_β , b'_β defined by

$$\overline{o'b_\beta} = \sum_{1 \le j \le \beta} \overline{b'_{j-1}b_j} + \sum_{1 \le j \le (\beta-1)} \overline{b_jb'_j} \quad ,$$

$$\overline{o'b'_\beta} = \overline{o'b_\beta} + \overline{b_\beta b'_\beta} \quad .$$

Then

$$\int_{\overline{b_jb'_j}} \phi = \frac{-1}{2} \int_{A_j} \phi = \frac{-1}{2} e_j \quad ,$$

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$$\int_{\substack{\sum \overline{b'_{j-1}b_j} \\ 1\leq j\leq\beta}} \phi = \frac{1}{2}\int_{\substack{\sum B_j \\ 1\leq j\leq\beta}} \phi = \frac{1}{2}\int_{C_\beta} \phi = \frac{1}{2}\tau_\beta \quad,$$

$$\int_{\overline{o'b_\beta}} \phi = \frac{1}{2}\tau_\beta - \frac{1}{2}\sum_{j\leq(\beta-1)} e_j \quad,$$

$$\int_{\overline{o'b'_\beta}} \phi = \frac{1}{2}\tau_\beta - \frac{1}{2}\sum_{j\leq\beta} e_j \quad.$$

Hence we obtain (ii). q.e.d.

We are going to express the period of Jac(W) in terms of the data { p, p', q, q' }. Define

$$\epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_g \end{pmatrix} = \frac{1}{N}\begin{pmatrix} 1 \\ 2 \\ \vdots \\ g \end{pmatrix},$$

$$\rho = \begin{pmatrix} \rho_1 \\ \vdots \\ \rho_g \end{pmatrix} = \frac{1}{N}\begin{pmatrix} \sum_{1\leq\beta\leq g} \tau_{1\beta} \\ \vdots \\ \sum_{1\leq\beta\leq g} \tau_{g\beta} \end{pmatrix}.$$

Consider the following paths in W:

$\overline{oq'}\left(\overline{o'q'}\right)$ = the path from o(o' resp.) to q' with t-values in the segment from $(\frac{1+r}{1-r})^{1/N}$ ($(\frac{1-r}{1+r})^{1/N}$ resp.) to 0 ,

$$\overline{po} = \iota\left(-\overline{o'q'}\right) \quad,$$

$$\overline{pp'} = \overline{po} - \sigma(\overline{po}) \quad, \qquad \overline{pq'} = \overline{po} + \overline{oq'} \quad.$$

$$\overline{pq} = \overline{po} + \sigma\left(\overline{oq'}\right) \quad, \qquad \overline{p'q'} = \sigma(\overline{pq}) \quad.$$

(The above paths are shown in Fig. 2 .) Then
$\theta^{j-1}\left(\overline{p'q'}\right) - \theta^j\left(\overline{p'q'}\right)$ is homologous to $-A_{N-j}$,
$\theta^{-j+1}\left(\overline{pp'}\right) - \theta^{-j}\left(\overline{pp'}\right) = -A_{j-1} + B_j$.

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<u>LEMMA 7</u> $\int\limits_{\overline{p'q'}} \phi = -\epsilon$, $\int\limits_{\overline{pp'}} \phi = \rho + \epsilon$, $\int\limits_{\overline{pq'}} \phi = \rho$, $\int\limits_{\overline{pq}} \phi = \epsilon$.

Proof. For $0 \le j \le$ N-1, we have

$$(\omega^{-\alpha\beta})^{-1}(\delta_{\alpha\beta}l_\alpha)^{-1} \int\limits_{\theta^j(\overline{p'q'})} \varphi = \int\limits_{\theta^j(\overline{p'q'})} \phi = \int\limits_{\overline{p'q'}} \phi + \int\limits_{\sum\limits_{(N-1)\ge m\ge(N-j)} A_m} \phi ,$$

$$(\omega^{-\alpha\beta})^{-1}(\delta_{\alpha\beta}l_\alpha)^{-1} \int\limits_{\theta^{-j}(\overline{pp'})} \varphi = \int\limits_{\theta^{-j}(\overline{pp'})} \phi = \int\limits_{\overline{pp'}} \phi - \int\limits_{C_j} \phi + \int\limits_{\sum\limits_{(j-1)\ge m\ge 0} A_m} \phi,$$

(here $C_0 = \sum\limits_{N\ge m\ge 1} B_m$ which is homologous to 0).

By Lemma 6,

$$0 = \sum\limits_{(N-1)\ge j\ge 0} [(\omega^{-\alpha\beta})^{-1}(\delta_{\alpha\beta}l_\alpha)^{-1} \int\limits_{\theta^j(\overline{p'q'})} \varphi] = N \int\limits_{\overline{p'q'}} \phi + \int\limits_{\sum\limits_{(N-1)\ge m\ge 1} mA_m} \phi ,$$

$$0 = \sum\limits_{(N-1)\ge j\ge 0} [(\omega^{-\alpha\beta})^{-1}(\delta_{\alpha\beta}l_\alpha)^{-1} \int\limits_{\theta^{-j}(\overline{pp'})} \varphi]$$

$$= N \int\limits_{\overline{pp'}} \phi - \int\limits_{\sum\limits_{(N-1)\ge j\ge 1} C_j} \phi - \int\limits_{\sum\limits_{(N-1)\ge m\ge 1} mA_m} \phi ,$$

which imply the first two equalities, then the rest follows immediately. q.e.d.

<u>PROPOSITION 9</u> Under the map (29) for $\star = o$,

$$\mathbf{p} \rightsquigarrow \left[\frac{-1}{2}\rho - \frac{1}{2}\epsilon\right] , \quad \mathbf{p}' \rightsquigarrow \left[\frac{1}{2}\rho + \frac{1}{2}\epsilon\right] ,$$

$$\mathbf{q} \rightsquigarrow \left[\frac{-1}{2}\rho + \frac{1}{2}\epsilon\right] , \quad \mathbf{q}' \rightsquigarrow \left[\frac{1}{2}\rho - \frac{1}{2}\epsilon\right] .$$

Proof. We have

$$\int\limits_o^p \phi = \frac{-1}{2}\left(\int\limits_{\overline{po}} \phi + \int\limits_{\sigma(\overline{po})} \sigma^*\phi\right)$$

$$= \frac{-1}{2} \int\limits_{\overline{po}-\sigma(\overline{po})} \phi = \frac{-1}{2} \int\limits_{\overline{pp'}} \phi$$

$$= \frac{-1}{2}\rho - \frac{1}{2}\epsilon , \quad \text{(by Lemma 7)}.$$

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Hence we obtain the value of p of the map (29) for $\star = 0$. The other values of this map follows from Lemma 7 and the following identities:

$$\int_0^{p'} \phi = \int_0^p \phi + \int_{\overline{pp'}} \phi \ ,$$

$$\int_0^q \phi = \int_0^p \phi + \int_{\overline{pq}} \phi \ ,$$

$$\int_0^{q'} \phi = \int_0^p \phi + \int_{\overline{pq'}} \phi \ .$$

q.e.d.

LEMMA 8 For a path γ in W,

$$\int_{\theta^{-1}(\gamma)} \phi \ = \ (r_{\alpha\beta}) \ \int_\gamma \phi \quad ,$$

here

$$r_{\alpha\beta} = 0 \quad \text{except} \quad r_{(\beta+1)\beta} = 1 \quad (1 \le \beta \le g), \quad r_{\alpha g} = -1 \quad (1 \le \alpha \le g).$$

Proof. By Lemma 6, we have

$$\int_{\theta^{-1}(\gamma)} \phi = (\omega^{-\alpha\beta})^{-1}(\delta_{\alpha\beta}l_\alpha)^{-1} \int_{\theta^{-1}(\gamma)} \varphi$$

$$= (\omega^{-\alpha\beta})^{-1}(\delta_{\alpha\beta}l_\alpha)^{-1}(\delta_{\alpha\beta}\omega^{-\alpha})\int_\gamma \varphi$$

$$= (\omega^{-\alpha\beta})^{-1}(\delta_{\alpha\beta}l_\alpha)^{-1}(\delta_{\alpha\beta}\omega^{-\alpha})(\delta_{\alpha\beta}l_\alpha)(\omega^{-\alpha\beta})\int_\gamma \phi$$

$$= (\omega^{-\alpha\beta})^{-1}(\delta_{\alpha\beta}\omega^{-\alpha}) \ (\omega^{-\alpha\beta})\int_\gamma \phi,$$

which implies the result. q.e.d.

The period of Jac(W) is expressed by the vector ρ via the following proposition.

PROPOSITION 10 Define $\rho_0 = 0$. Then for $\alpha, \beta = 1, ..., g$,

$$\rho_\alpha = \rho_{N-\alpha} \quad ,$$

$$r_{\alpha\beta} = \rho_\alpha + \rho_\beta - \rho_{|\alpha-\beta|} \quad .$$

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Proof. It is easy to see that $-\iota(A)$ is homologous to A. Hence the l_α's in (27) have the relation:

$$l_\alpha = l_{N-\alpha} \quad \text{for } 1 \le \alpha \le g .\tag{30}$$

By the definition of the paths,

$-\iota\left(\overline{\mathbf{pq}'}\right)$ is homologous to $\overline{\mathbf{pq}'}$.

Hence

$$\rho = \int_{\overline{\mathbf{pq}'}} \phi = -\int_{\iota\left(\overline{\mathbf{pq}'}\right)} \phi = -\int_{\overline{\mathbf{pq}'}} \iota^*(\phi)$$

$$= -\left(\omega^{-\alpha\beta}\right)^{-1}\left(\delta_{\alpha\beta}\ell_\alpha\right)^{-1} \int_{\overline{\mathbf{pq}'}} \iota^*(\varphi)$$

$$= \left(\omega^{-\alpha\beta}\right)^{-1}\left(\delta_{\alpha\beta}\ell_\alpha\right)^{-1}\left(\delta_{\alpha(N-\alpha)}\right) \int_{\overline{\mathbf{pq}'}} \varphi$$

$$= \left(\omega^{-\alpha\beta}\right)^{-1}\left(\delta_{\alpha\beta}\ell_\alpha\right)^{-1}\left(\delta_{\alpha(N-\alpha)}\right)\left(\delta_{\alpha\beta}\ell_\alpha\right)\left(\omega^{-\alpha\beta}\right) \int_{\overline{\mathbf{pq}'}} \phi$$

$$= \left(\omega^{-\alpha\beta}\right)^{-1}\left(\delta_{\alpha\beta}\ell_\alpha\right)^{-1}\left(\delta_{\alpha\beta}\ell_\alpha\right)\left(\delta_{\alpha(N-\alpha)}\right)\left(\omega^{-\alpha\beta}\right) \rho \qquad \text{(by (30))}$$

$$= \left(\omega^{-\alpha\beta}\right)^{-1}\left(\delta_{\alpha(N-\alpha)}\right)\left(\omega^{-\alpha\beta}\right)\left(\delta_{\alpha(N-\alpha)}\right)^2 \rho = \left(\delta_{\alpha(N-\alpha)}\right)\rho \quad .$$

which implies $\rho_\alpha = \rho_{N-\alpha}$.

We have

$$\theta^{-\beta+1}\left(\overline{\mathbf{pq}'}\right) - \theta^{-\beta}\left(\overline{\mathbf{pq}'}\right) = B_\beta \qquad \qquad ,$$
$$\overline{\mathbf{pq}'} - \theta^{-\beta}\left(\overline{\mathbf{pq}'}\right) = C_\beta , \qquad \text{for } 1 \le \beta \le g.$$

Hence

$$\int_{C_\beta} \phi = \int_{\overline{\mathbf{pq}'}} \phi - \int_{\theta^{-\beta}\left(\overline{\mathbf{pq}'}\right)} \phi$$

$$= \rho - \left(r_{\alpha\gamma}\right)^\beta \rho , \quad \text{(by Lemma 7),}$$

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which implies

$$\tau_{\alpha\beta} = \rho_\alpha + \rho_{N-\beta} - \rho_{|\alpha-\beta|} = \rho_\alpha + \rho_\beta - \rho_{|\alpha-\beta|} .\quad \text{q.e.d.}$$

REMARK In [6], the ρ_α's are solved from the expression of $\tau_{\alpha\beta}$, and described by

$$\rho_\alpha = \frac{i}{N} \sum_\gamma \frac{\sin^2\left(\frac{\pi\alpha\gamma}{N}\right)}{\sin\frac{\pi\gamma}{N}} \left(\omega^{-\gamma/2} \frac{\ell'_\gamma}{\ell_\gamma} \right).$$

Then the identities in the above proposition can be verified through the explicit expressions of ρ_α and $\tau_{\alpha\beta}$. The proof we give here indicates these identities follows from the symmetries of the CP N-curve W.

By the uniformizing coordinate of Jac(W)

$$s = \begin{pmatrix} s_1 \\ \vdots \\ s_g \end{pmatrix} \varepsilon \, \mathbf{C}^g ,$$

the hyperelliptic theta function of Jac(W) is

$$\Theta(s) \, (= \vartheta(s,\tau)) = \sum_{m\varepsilon \mathbf{Z}^g} e^{2\pi i^t ms + \pi i^t m\tau m} ,$$

and by Proposition 10 it can also be written as

$$\Theta(s) = \sum_{m\varepsilon \mathbf{Z}^g} exp\left\{ 2\pi i \left({}^t ms + \sum_{1\le\alpha,\beta\le g} m_\alpha m_\beta \rho_\beta \right) - \pi i \sum_{1\le\alpha,\beta\le g} m_\alpha \rho_{|\alpha-\beta|} m_\beta \right\} .$$

Then $\Theta(s)$ satisfies the quasi-periodicity and evenness relations

$$\Theta(s+m) = \Theta(s), \qquad \text{for } m \,\varepsilon\, \mathbf{Z}^g ,$$
$$\Theta(s+\tau_\beta) = e^{-2\pi i(s_\beta + \rho_\beta)}\Theta(s), \text{ for } 1 \le \beta \le g ,$$
$$\Theta(-s) = \Theta(s) \quad .$$

The theta function $\vartheta\begin{bmatrix} \delta \\ \nu \end{bmatrix}$ with rational characteristics δ, ν of $\mathbf{Q}^g$ are defined by

$$\vartheta\begin{bmatrix} \delta \\ \nu \end{bmatrix}(s,\tau) = e^{2\pi i^t \delta(s+\nu)+\pi i^t \delta\tau\delta}\vartheta(s+\tau\delta+\nu,\tau) ,$$

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with the quasi-periodicity relation

$$\vartheta\begin{bmatrix}\delta\\\nu\end{bmatrix}(s+m,\tau) = e^{2\pi i^t \delta m}\vartheta\begin{bmatrix}\delta\\\nu\end{bmatrix}(s,\tau),$$

$$\vartheta\begin{bmatrix}\delta\\\nu\end{bmatrix}(s+\tau m,\tau) = e^{-2\pi i^t \nu m - 2\pi i^t ms - \pi i^t m\tau m}\vartheta\begin{bmatrix}\delta\\\nu\end{bmatrix}(s,\tau), \quad \text{for } m \varepsilon \mathbb{Z}^g.$$

We are going to describe the prime form of the Riemann surface W. Consider the divisor D= $\sum_{2\le\beta\le g} b_\beta$. Then the canonical bundle of W equals to $\mathcal{O}_W[2D]$. By Riemann-Roch Theorem and Lemma 5, |D|= a single divisor D. By the general theory of prime forms of Riemann surfaces (see e.g. Ref. [17]) implies the existence of a non-singular, odd, theta characteristic $\vartheta\begin{bmatrix}\bar\delta\\\nu\end{bmatrix}$, $\bar\delta, \bar\nu \varepsilon (\frac{1}{2}\mathbb{Z})^g/\mathbb{Z}^g$, such that

$$\vartheta\begin{bmatrix}\bar\delta\\\nu\end{bmatrix}\left(\int_{w'}^{w}\phi\right) = 0 \quad \Leftrightarrow \quad \begin{cases} w' = w, \\ w' \text{ or } w = b_\beta, \text{for some } 2 \le \beta \le g. \end{cases} \tag{31}$$

Then

$$\zeta = \sum_{1\le\beta\le g} \frac{\partial\vartheta\begin{bmatrix}\bar\delta\\\nu\end{bmatrix}}{\partial s_\beta}(0)\,\phi_\beta$$

is the unique holomorphic 1–form with (ζ)= 2D. So $\sqrt{\zeta}$ is the section of $\mathcal{O}_W[D]$ and the prime form is given by

$$E(w_0, w) = \frac{\vartheta\begin{bmatrix}\bar\delta\\\nu\end{bmatrix}\left(\int_{w_0}^{w}\phi\right)}{\sqrt{\zeta(w)}\sqrt{\zeta(w_0)}}, \quad \text{for } w_0, w \varepsilon W. \tag{32}$$

For a fixed w_0 point of W, $E(w_0, w)$ defines a global section of $\mathcal{O}_W[w_0]$ of the Riemann surface W. Therefore

$$\frac{E(p', w)}{E(p, w)} = \frac{\vartheta\begin{bmatrix}\bar\delta\\\nu\end{bmatrix}\left(\int_{p'}^{w}\phi\right)}{\vartheta\begin{bmatrix}\bar\delta\\\nu\end{bmatrix}\left(\int_{p}^{w}\phi\right)}, \quad w \varepsilon W,$$

corresponds to a section of $\mathcal{O}_W[p'-p]$. Consider the function of $\mathbf{C}^g$:

$$f(s) = \frac{\vartheta\left[\frac{\bar{\delta}}{\nu}\right]\left(s - \frac{1}{2}\rho - \frac{1}{2}\epsilon, \tau\right)}{\vartheta\left[\frac{\bar{\delta}}{\nu}\right]\left(s + \frac{1}{2}\rho + \frac{1}{2}\epsilon, \tau\right)} \quad.$$

By Proposition 9, $\frac{E(p',\mathbf{w})}{E(p,\mathbf{w})} = f(s(\mathbf{w}))$, here $s(\mathbf{w}) = \int_0^{\mathbf{w}} \phi$. By the quasi-periodicity property of $\vartheta\left[\frac{\bar{\delta}}{\nu}\right]$, we have

$$f(s+m) = f(s) \ ,$$
$$f(s+\tau m) = e^{2\pi i^t m(\rho+\epsilon)}f(s), \quad \text{for } s\epsilon\mathbf{C}^g, m\epsilon\mathbf{Z}^g.$$

Since

$$exp\left\{\frac{2\pi i}{N}\sum_{1\le\alpha\le g}\left(s_\alpha + \sum_{1\le\beta\le g}\tau_{\alpha\beta}m_\beta\right)\right\} = e^{2\pi i^t m\rho}exp\left\{\frac{2\pi i}{N}\sum_{1\le\alpha\le g}s_\alpha\right\},$$

the function

$$f_{p',p}(s) = exp\left\{\frac{-2\pi i}{N}\sum_{\alpha=1}^{g}s_\alpha\right\}f(s)$$

$$= \frac{\vartheta\left[\frac{\bar{\delta}}{\nu}\right]\left(s - \frac{1}{2}\rho - \frac{1}{2}\epsilon, \tau\right)}{e^{\frac{2\pi i}{N}\sum_\alpha s_\alpha}\vartheta\left[\frac{\bar{\delta}}{\nu}\right]\left(s + \frac{1}{2}\rho + \frac{1}{2}\epsilon, \tau\right)}$$

satisfies the relation

$$f_{p',p}(s+m) = e^{-2\pi i^t m\frac{1}{N}}f_{p',p}(s) \quad,$$

$$f_{p',p}(s+\tau m) = e^{2\pi i^t m\epsilon}f_{p',p}(s)$$

for $s\epsilon\mathbf{C}^g, m\epsilon\mathbf{Z}^g$, here $\frac{1}{N} = \sum_{1\le\beta\le g} e_\beta$. The restriction of $f_{p',p}$ on W (via (29) for $\star = o$) can be regarded as a section in $\Gamma(W, \mathcal{O}_W(p'-p))$ with the divisor $\text{div}(f_{p',p}) = p'-p$. Similarly, the functions of $\mathbf{C}^g$ defined by

$$f_{q,p}(s) = \frac{\vartheta\left[\frac{\bar{\delta}}{\nu}\right]\left(s + \frac{1}{2}\rho - \frac{1}{2}\epsilon, \tau\right)}{\vartheta\left[\frac{\bar{\delta}}{\nu}\right]\left(s + \frac{1}{2}\rho + \frac{1}{2}\epsilon, \tau\right)} \ ,$$

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$$f_{q',p}(s) = \frac{\vartheta\left[\frac{\bar{\delta}}{\nu}\right]\left(s - \frac{1}{2}\rho + \frac{1}{2}\epsilon, \tau\right)}{e^{\frac{2\pi i}{N}\sum_{1 \le \alpha \le \rho} s_\alpha} \vartheta\left[\frac{\bar{\delta}}{\nu}\right]\left(s + \frac{1}{2}\rho + \frac{1}{2}\epsilon, \tau\right)} \ ,$$

satisfy the relations

$$f_{q,p}(s + m) = f_{q,p}(s) \ , \quad f_{q,p}(s + \tau m) = e^{2\pi i^t m \epsilon} f_{q,p}(s) \ ,$$

$$f_{q',p}(s + m) = e^{-2\pi i \frac{1}{N}} f_{q',p}(s) \ , \quad f_{q',p}(s + \tau m) = f_{q',p}(s) \ ,$$

for $s \varepsilon C^g, m \varepsilon Z^g$. The restrictions of $f_{q,p}, f_{q',p}$ on W are the section in $\Gamma(W, \mathcal{O}_W(q - p))$, $\Gamma(W, \mathcal{O}_W(q' - p))$ respectively, with $div(f_{q,p}) = q - p$, $div(f_{q',p}) = q' - p$. Note that the phase factors in the quasi-periodicity relations of $f_{p',p}, f_{q,p}, f_{q',p}$ are the N-th roots of unity. These functions are closely connected to the variables a,b,c,d of the N^2−cover $\mathfrak{W}$ of W in Section 4. Since the N-th power of $f_{p',p}, f_{q,p}, f_{q',p}$ gives the rational function on W, the situation is much simpler for the purpose of the expression of the parameter r of W.

Consider first the case when r equals to k'. As before, let $\mathfrak{W}$ be the curve defined in (21) with the homogenous coordinates a,b,c,d . The above p, p', q, q' are now the u, u', v, v' in Proposition 7. As the rational functions of W, $f_{v,u}^N, f_{v',u}^N, f_{u',u}^N$ equal to $\frac{a^N}{d^N}, \frac{b^N}{d^N}, \frac{c^N}{d^N}$ up to some constants , i.e.

$$a^N : b^N : c^N : d^N = Ae^{2\pi i \sum_\alpha s_\alpha} \vartheta\left[\frac{\bar{\delta}}{\nu}\right]\left(s + \frac{1}{2}\rho - \frac{1}{2}\epsilon, \tau\right)^N : B\vartheta\left[\frac{\bar{\delta}}{\nu}\right]\left(s - \frac{1}{2}\rho + \frac{1}{2}\epsilon, \tau\right)^N$$

$$: C\vartheta\left[\frac{\bar{\delta}}{\nu}\right]\left(s - \frac{1}{2}\rho - \frac{1}{2}\epsilon, \tau\right)^N : e^{2\pi i \sum_\alpha s_\alpha} \vartheta\left[\frac{\bar{\delta}}{\nu}\right]\left(s + \frac{1}{2}\rho + \frac{1}{2}\epsilon, \tau\right)^N \ ,$$

here s is the uniformizing coordinate of elements of W, and A,B,C are constants. The hyperelliptic involution σ of W induces the automorphism $\mathfrak{J}$ of $\mathfrak{W}$ (defined in (23)), which corresponds the map of $C^g, s \rightsquigarrow -s$. The relations

$$\mathfrak{J}^*\left(\frac{a^N}{d^N}\right) = \frac{b^N}{c^N} \ , \quad \mathfrak{J}^*\left(\frac{c^N}{d^N}\right) = \frac{d^N}{c^N} \ ,$$

imply B=AC, $C^2 = 1$. For s= the origin, it corresponds to the element o of W with the coordinate (t, λ) (defined in (4))= ($\left(\frac{1+k'}{1-k'}\right)^{\frac{1}{N}}$, -1). By (19) (20), the constants A, B, C must satisfy

$$A = B, \quad C = 1.$$

By (21), we have the relation

$$kAe^{2\pi i\sum_\alpha s_\alpha}\vartheta\left[\frac{\bar{\delta}}{\bar{\nu}}\right]\left(s+\frac{1}{2}\rho-\frac{1}{2}\epsilon,\tau\right)^N + k'\vartheta\left[\frac{\bar{\delta}}{\bar{\nu}}\right]\left(s-\frac{1}{2}\rho-\frac{1}{2}\epsilon,\tau\right)^N$$

$$= e^{2\pi i\sum_\alpha s_\alpha}\vartheta\left[\frac{\bar{\delta}}{\bar{\nu}}\right]\left(s+\frac{1}{2}\rho+\frac{1}{2}\epsilon,\tau\right)^N \quad .$$

Substituting $s=\frac{-1}{2}\rho+\frac{1}{2}\epsilon$, by (31) we obtain

$$k' = \frac{-e^{-\pi i(\rho_1+...+\rho_g)}\vartheta\left[\frac{\bar{\delta}}{\bar{\nu}}\right](\epsilon,\tau)^N}{\vartheta\left[\frac{\bar{\delta}}{\bar{\nu}}\right](\rho,\tau)^N} \quad . \tag{33}$$

When $r=i\frac{k'}{k}$, the CP N-curve is the curve W_r of (9) in section 3. Its hyperelliptic involution is induced by the automorphism $\mathfrak{J}_r$ of $\mathfrak{W}$ by Proposition 6 (ii). Let τ_r be the period for W_r. By Proposition 8,

$$a^N : b^N : c^N : d^N =$$

$$A_r e^{2\pi i\sum_\alpha s_\alpha}\vartheta\left[\frac{\bar{\delta}}{\bar{\nu}}\right]\left(s+\frac{1}{2}\rho_r-\frac{1}{2}\epsilon,\tau_r\right)^N : B_r\vartheta\left[\frac{\bar{\delta}}{\bar{\nu}}\right]\left(s-\frac{1}{2}\rho_r-\frac{1}{2}\epsilon,\tau_r\right)^N :$$

$$C_r\vartheta\left[\frac{\bar{\delta}}{\bar{\nu}}\right]\left(s-\frac{1}{2}\rho_r+\frac{1}{2}\epsilon,\tau_r\right)^N : e^{2\pi i\sum_\alpha s_\alpha}\vartheta\left[\frac{\bar{\delta}}{\bar{\nu}}\right]\left(s+\frac{1}{2}\rho_r+\frac{1}{2}\epsilon,\tau_r\right)^N ,$$

with $B_r = A_r C_r$, $(B_r)^2 = -1$. By the definition of the element o of W_r, together with the description of the function f' in the remark (i) of Proposition 4, the λ—value of o equals to k'+ik. By the relations (17) (20) , $\frac{b^N}{d^N} = \frac{\lambda^{-1}-k'}{k}$, hence $B_r = (-1)^g i$. By (22),

$$A_r e^{2\pi i\sum_\alpha s_\alpha}\vartheta\left[\frac{\bar{\delta}}{\bar{\nu}}\right]\left(s+\frac{1}{2}\rho_r-\frac{1}{2}\epsilon,\tau_r\right)^N + k'i(-1)^g\vartheta\left[\frac{\bar{\delta}}{\bar{\nu}}\right]\left(s-\frac{1}{2}\rho_r-\frac{1}{2}\epsilon,\tau_r\right)^N$$

$$= ke^{2\pi i\sum_\alpha s_\alpha}\vartheta\left[\frac{\bar{\delta}}{\bar{\nu}}\right]\left(s+\frac{1}{2}\rho_r+\frac{1}{2}\epsilon,\tau_r\right)^N \quad .$$

Hence

$$\frac{ik'}{k} = \frac{(-1)^N e^{-\pi i(\rho_{r1}+...+\rho_{rg})}\vartheta\left[\frac{\bar{\delta}}{\bar{\nu}}\right](\epsilon,\tau_r)^N}{\vartheta\left[\frac{\bar{\delta}}{\bar{\nu}}\right](\rho_r,\tau_r)^N} \quad . \tag{34}$$

For the case r=k, let τ_ℓ be the period of $W_{N,k}(\simeq W_\ell$ in (11)). Similarly, we have

$$a^N : b^N : c^N : d^N =$$

$$(-1)^g \vartheta\left[\frac{\overline{\delta}}{\overline{\nu}}\right]\left(s - \frac{1}{2}\rho_\ell - \frac{1}{2}\epsilon, \tau_\ell\right)^N : B_\ell e^{2\pi i \sum_\alpha s_\alpha} \vartheta\left[\frac{\overline{\delta}}{\overline{\nu}}\right]\left(s + \frac{1}{2}\rho_\ell - \frac{1}{2}\epsilon, \tau_\ell\right)^N$$

$$: (-1)^g B_\ell \vartheta\left[\frac{\overline{\delta}}{\overline{\nu}}\right]\left(s - \frac{1}{2}\rho_\ell + \frac{1}{2}\epsilon, \tau_\ell\right)^N : e^{2\pi i \sum_\alpha s_\alpha} \vartheta\left[\frac{\overline{\delta}}{\overline{\nu}}\right]\left(s + \frac{1}{2}\rho_\ell + \frac{1}{2}\epsilon, \tau_\ell\right)^N ,$$

and

$$(-1)^g \vartheta\left[\frac{\overline{\delta}}{\overline{\nu}}\right]\left(s - \frac{1}{2}\rho_\ell - \frac{1}{2}\epsilon, \tau_\ell\right)^N + k' B_\ell e^{2\pi i \sum_\alpha s_\alpha} \vartheta\left[\frac{\overline{\delta}}{\overline{\nu}}\right]\left(s + \frac{1}{2}\rho_\ell - \frac{1}{2}\epsilon, \tau_\ell\right)^N$$

$$= k e^{2\pi i \sum_\alpha s_\alpha} \vartheta\left[\frac{\overline{\delta}}{\overline{\nu}}\right]\left(s + \frac{1}{2}\rho_\ell + \frac{1}{2}\epsilon, \tau_\ell\right)^N ,$$

hence

$$k = \frac{(-1)^N \vartheta\left[\frac{\overline{\delta}}{\overline{\nu}}\right](\rho_\ell, \tau_\ell)^N}{e^{-\pi i(\rho_{\ell 1} + \dots + \rho_{\ell g})} \vartheta\left[\frac{\overline{\delta}}{\overline{\nu}}\right](\epsilon, \tau_\ell)^N} . \tag{35}$$

Using the identities obtained by Sonya Kowalevski [14], Baxter wrote down the explicit expression of the above $\overline{\delta}$, $\overline{\nu}$, and they are given by

$$\overline{\delta} = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \overline{\nu} = -\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}. \tag{36}$$

(The above $\vartheta\left[\frac{\overline{\delta}}{\overline{\nu}}\right](s,\tau)$ is the function $\Theta\{s\}_1$ of [6]). Therefore we have obtained the following conclusion.

THEOREM Let τ, τ_r, τ_ℓ be the period for the CP N-curves $W_{N,k'}, W_{N,i\frac{k'}{k}}, W_{N,k}$ respectively. Then the relations between k', $i\frac{k'}{k}$, k and τ, τ_r, τ_ℓ are given by (33) (34) (35) and (36).

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Reference

[1] G. Albertini, B. M. McCoy, J. H. H. Perk , Eigenvalue spectrum of the superintegrable chiral Potts model, Advanced Studies in Pure Mathematics 19 (1989) 1–55.

[2] G. Albertini, B. M. McCoy, J. H. H. Perk , Commensurate-incommensurate transition in the ground state of the superintegrable chiral Potts model, Physics Letters A 135 (1989) 159–166.

[3] G. Albertini, B. M. McCoy, J. H. H. Perk , Level crossing transitions and the massless phases of the superintegrable chiral Potts chain, Physics Letters A 139 (1989), 204–212.

[4] G. Albertini, B. M. McCoy, J. H. H. Perk, S. Tang , Excitation spectrum and order parameter for the integrable N-state chiral Potts model, Nuclear Physics B 314 (1989) 741–763

[5] H. Au-Yang, J. H. H. Perk , Onsager's star-triangle equation: Master key to integrability, Advanced Studies in Pure Mathematics 19 (1989) 57–94.

[6] R. J. Baxter , Hyperelliptic function parametrization for the chiral Potts model, preprint.

[7] R. J. Baxter , Corner transfer matrices of the chiral Potts model, preprint

[8] R. J. Baxter , Solving models in statistical mechanics, Advanced Studies in Pure Mathematics 19 (1989) 95–116.

[9] R. J. Baxter , The superintegrable chiral Potts model, Physics Letters A 133 (1988) 185–189.

[10] R. J. Baxter , Chiral Potts model: eigenvalues of the transfer matrix, Physics Letters A 146 (1990) 110–114

[11] R. J. Baxter, V. V. Bazhanov, J. H. H. Perk , Functional relations for transfer matrices of the chiral Potts model, International Journal of Modern Physics B 4 (1990) 803.

[12] R. J. Baxter, J. H. H. Perk, H. Au-Yang , New solutions of the star-triangle relations for the chiral Potts model, Physics Letters A 128 (1988) 138–142.

[13] V. V. Bazhanov, Yu. G. Stroganov , Chiral Potts model as a decendant of the six-vertex model, J. Stat. Phys. 59 (1990) 799–817.

[14] S. Kowalevski , Sur le probleme de la rotation d'un corps solide autour d'un

point fixe, Acta Mathematica 12 (1889) 177–232

[15] B. M. McCoy , The chiral Potts model : from physics to mathematics and back, Lecture in Okayama conference, Japan (1990).

[16] B. M. McCoy, S. S. Roan , Excitation spectrum and phase structure of the chiral Potts model, RIMS-707, to appear in Physics Letters A.

[17] D. Mumford , Tata lectures on theta I, II , Progress in Mathematics, Vol. 28 and Vol. 43 , Birkhauser

r: real
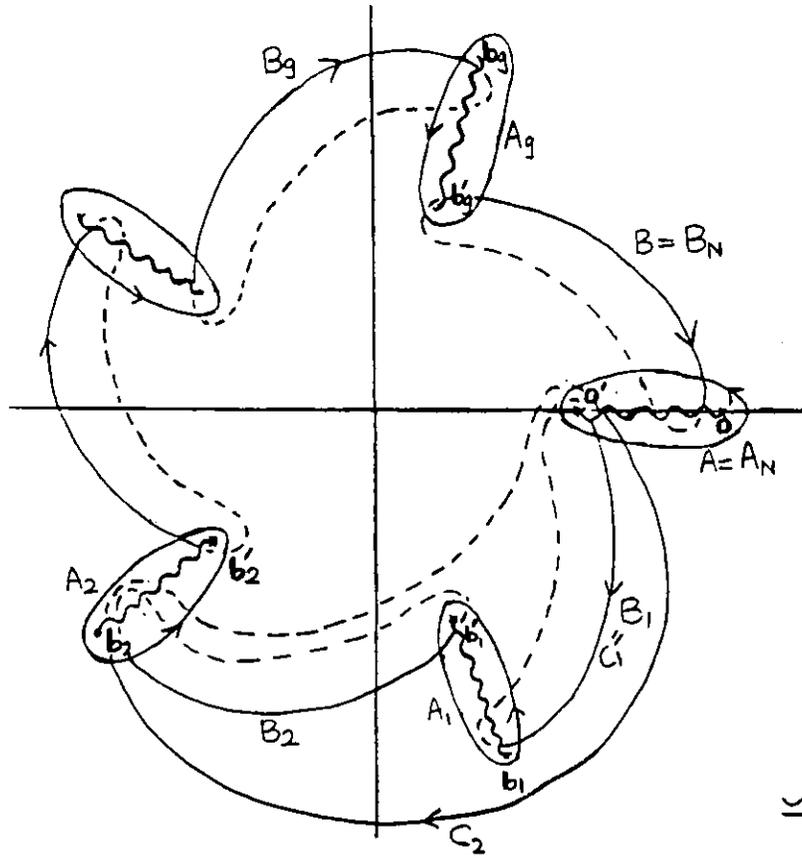
r = $e^{i\theta}$, $\theta \in \mathbb{R}$.

curves on upper sheet

curves on lower sheet
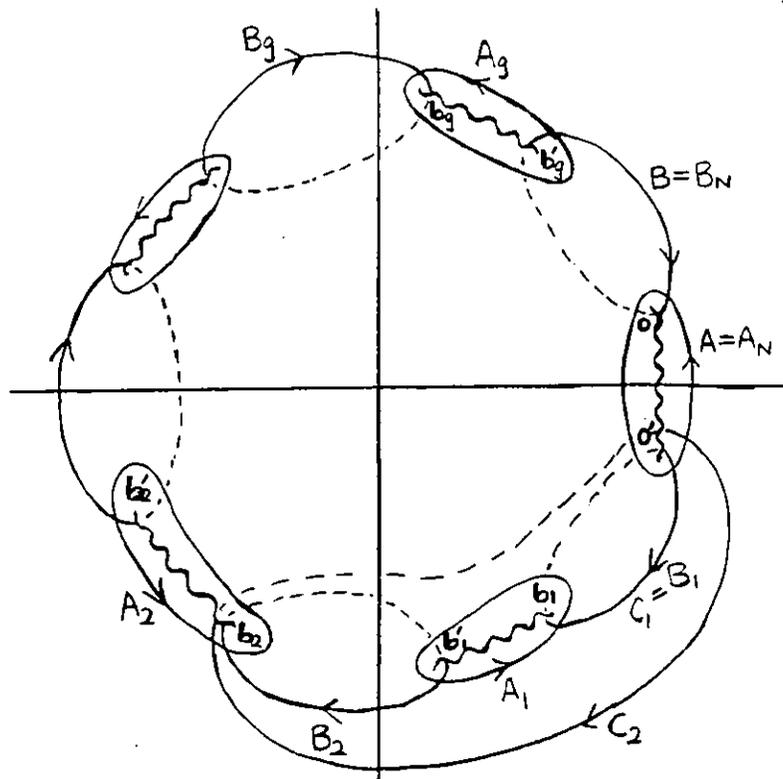
Fig.1. The curves homologous to A_j, B_j, C_j for N=5

Fig 2. The paths pq, pq', pp' for $N=5$, $r=e^{i\theta}$ ($\theta\in\mathbb{R}$).