DIFFERENTIABLE FUNCTIONS DEFINED IN CLOSED SETS. A PROBLEM OF WHITNEY

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Abstract. In 1934, Whitney raised the question of how to recognize whether a function f defined on a closed subset X of $\mathbb{R}^n$ is the restriction of a function of class $\mathcal{C}^p$. A necessary and sufficient criterion was given in the case $n = 1$ by Whitney, using limits of finite differences, and in the case $p = 1$ by Glaeser (1958), using limits of secants. We introduce a necessary geometric criterion, for general n and p, involving limits of finite differences, that we conjecture is sufficient at least if X has a "tame topology". We prove that, if X is a compact subanalytic set, then there exists $q = q_X(p)$ such that the criterion of order q implies that f is $\mathcal{C}^p$. The result gives a new approach to higher-order tangent bundles (or bundles of differentiable operators) on singular spaces.

1. Introduction

In 1934, Hassler Whitney published three pioneering articles on criteria for a function $f : X \to \mathbb{R}$, where X is a closed subset of $\mathbb{R}^n$, to be the restriction of a function of class $\mathcal{C}^p$, [W1], [W2], [W3]. ($\mathcal{C}^p$ means continuously differentiable to order p, where $p \in \mathbb{N}$.) Whitney's extension theorem [W1] gives a necessary and sufficient condition for a field of polynomials $\sum_{|\alpha|\le p} f_\alpha(a)(x-a)^\alpha$, $a \in X$, where $f_0 = f$, to be the field of Taylor polynomials of a $\mathcal{C}^p$ function. (We use multiindex notation: $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. $\mathbb{N}$ denotes the nonnegative integers.) In general, the functions f_α are, of course, not uniquely determined by f. In *Differentiable functions defined in closed sets. I* [W2], Whitney raises the deeper question of a necessary and sufficient criterion involving only the values of f, and he answers the question in the case $n = 1$. Whitney

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proves that, if X is a closed subset of the real line, then f extends to a $\mathcal{C}^p$ function if and only if the limiting values of all p'th divided differences $[x_0, x_1, \dots, x_p]f$, where the $x_i \in X$ and $x_i \neq x_j$ if $i \neq j$, define a continuous function on the diagonal $\{x_0 = x_1 = \cdots = x_p\}$. ($[x_0, x_1, \dots, x_n]f = p!c_p$, where $P(x) = c_0 + c_1 x + \cdots + c_p x^p$ is the *Lagrange interpolating polynomial* for f at the points $x_0, x_1, \dots, x_p$; i.e., the unique polynomial of degree at most p such that $P(x_i) = f(x_i)$, $i = 0, \dots, p$.)

"Differentiable functions defined in closed sets. II" never appeared, and up to now the only significant progress on Whitney's problem following [W3] seems to have been the beautiful theorem of Georges Glaeser ([G], 1958) which solves the problem in the case $p = 1$ (cf. [Br]. See also Remark 2.3.) Glaeser defines a "(linearized) paratangent bundle" $\tau(X)$ using limits of secant lines. (See Section 3 below.) Suppose that f is continuous and let $\tau(f)$ denote the paratangent bundle of the graph of f. Then $\tau(f)$ can be regarded as a bundle over X, and $\tau(f) \subset \tau(X) \times \mathbb{R}$ (but $\tau(f)$ does not necessarily project onto $\tau(X)$). Glaeser proves that f is the restriction of a $\mathcal{C}^1$ function if and only if $\tau(f)$ defines a function $\tau(f) : \tau(X) \to \mathbb{R}$ (i.e., $\tau(f)$ is the graph of a function $\tau(X) \to \mathbb{R}$; it will be convenient to identify a function with its graph).

In this article, we introduce a "(linearized) paratangent bundle of order p" $\tau^p(X)$, for any $p \in \mathbb{N}$, using limits of finitely supported distributions with values in the dual space $\mathcal{P}_p^*$ of the space $\mathcal{P}_p = \mathcal{P}_p(\mathbb{R}^n)$ of polynomial functions on $\mathbb{R}^n$ of degree at most p (Section 4 below). Each fibre $\tau_a^p(X)$, $a \in X$, is a linear subspace of $\mathcal{P}_p^*$. Our construction involves a new interpretation of the remainder condition in Whitney's extension theorem.

To every function $f : X \to \mathbb{R}$, we associate a bundle $\nabla^p f \subset \tau^p(X) \times \mathbb{R}$.

Conjecture. *f is the restriction of a $\mathcal{C}^p$ function if and only if $\nabla^p f$ defines a function $\nabla^p f : \tau^p(X) \to \mathbb{R}$.*

Moreover, if $\nabla^p f : \tau^p(X) \to \mathbb{R}$ and $\nabla_a^p f = 0$, for some $a \in X$, then there exists $F \in \mathcal{C}^p(\mathbb{R}^n)$ such that $F|X = f$ and $T_a^p F = 0$, where $T_a^p F$ denotes the Taylor polynomial of order p of F at a.

Necessity of the criterion $\nabla^p f : \tau^p(X) \to \mathbb{R}$ is not difficult; the following theorem is proved in Section 4.

Theorem 1.1. *If $f : X \to \mathbb{R}$ extends to a $\mathcal{C}^p$ function, then*

$$\nabla^p f :\ \tau^p(X) \to \mathbb{R}\ .$$

Moreover, if $F \in \mathcal{C}^p(\mathbb{R}^n)$ and $F|X = f$, then, for all $a \in X$ and $\xi \in \tau^p_a(X) \subset \mathcal{P}^*_p$,

$$\nabla^p f(\xi)\ =\ \xi(T^p_a F)\ .$$

The converse direction is true if X is a $\mathcal{C}^p$ submanifold. In Section 4, we prove more precisely:

Theorem 1.2. *If $X \subset M$, where M is a $\mathcal{C}^p$ submanifold of $\mathbb{R}^n$ and X is the closure of its interior in M, then $f : X \to \mathbb{R}$ extends to a $\mathcal{C}^p$ function if and only if $\nabla^p f : \tau^p(X) \to \mathbb{R}$.*

In order to make the conjecture tractable in general, it is reasonable to restrict to closed sets X that have a "tame" geometry ("géométrie modérée"); for example, closed subanalytic sets or, more generally, closed sets that are definable in an o-minimal structure (cf. [vdD]). Our main result is the following theorem (proved in Section 5).

Theorem 1.3. *Let X be a compact subanalytic subset of $\mathbb{R}^n$. Then there is a function $q = q_X(p) \geq p$ from $\mathbb{N}$ to itself such that, if $f : X \to \mathbb{R}$, $q \geq q_X(p)$ and*

$$\nabla^q f :\ \tau^q(X) \to \mathbb{R}\ ,$$

then f extends to a $\mathcal{C}^p$ function. If, moreover, $a \in X$ and $\nabla^q_a f = 0$, then f extends to a $\mathcal{C}^p$ function that is p-flat at a.

The novelty of Theorem 1.3 lies in the construction of $\tau^p(X)$ and $\nabla^p f$. Let X be a compact subanalytic subset of $\mathbb{R}^n$. Then there is a compact real analytic manifold M such that $\dim M = \dim X$, and a real analytic mapping $\varphi : M \to \mathbb{R}^n$ such that

$\varphi(M) = X$ (by the Uniformization Theorem [BM1, Thm. 0.1]). Let $g = f \circ \varphi$. We prove that if $\nabla^p f : \tau^p(X) \to \mathbb{R}$, then:

(1) $\nabla^p g : \tau^p(M) \to \mathbb{R}$ (Theorem 5.2); therefore, $g \in \mathcal{C}^p(M)$ by Theorem 1.2. (All notions make sense for manifolds.)

(2) g is *formally a composite with* φ; i.e., for all $a \in X$, there exists $P \in \mathcal{P}_p(\mathbb{R}^n)$ such that $g - P \circ \varphi$ is p-flat at every point $b \in \varphi^{-1}(a)$ (Corollary 5.3).

Theorem 1.3 is then a consequence of the following composite function theorem [BMP]: There is a function $q = q_\varphi(p)$ such that if $g \in \mathcal{C}^q(M)$ is formally a composite with φ, then there exists $F \in \mathcal{C}^p(\mathbb{R}^n)$ such that $g = F \circ \varphi$. (Moreover, if S is a finite subset of X and g is q-flat on $\varphi^{-1}(S)$, then there exists F with the additional property that F is p-flat on S.)

Let $\mathcal{C}^{(\infty)}(X) = \bigcap_{p \in \mathbb{N}} \mathcal{C}^p(X)$, where $\mathcal{C}^p(X)$ denotes the space of restrictions of $\mathcal{C}^p$ functions to X.

Corollary 1.4. *If $X \subset \mathbb{R}^n$ is a closed subanalytic set and $f : X \to \mathbb{R}$, then $f \in \mathcal{C}^{(\infty)}(X)$ if and only if $\nabla^p f : \tau^p(X) \to \mathbb{R}$, for all $p \in \mathbb{N}$.*

Of course $\mathcal{C}^\infty(X) \subset \mathcal{C}^{(\infty)}(X)$, where $\mathcal{C}^\infty(X)$ denotes the restrictions of $\mathcal{C}^\infty$ functions to X, and $\mathcal{C}^{(\infty)}(X) = \mathcal{C}^\infty(X)$ if X is a $\mathcal{C}^\infty$ submanifold (but not in general [P2]). Among closed subanalytic sets, equality characterizes the proper subclass of sets that have a "semicoherent" (or stratified coherent) structure [BM2]; see Remarks 2.7 below. In Section 2, we use Theorem 1.3 to compare the paratangent bundle $\tau^p(X)$ with another natural idea of a bundle of differential operators on a singular space. Surprisingly, uniformity of a "Chevalley estimate" (which also characterizes the class of semicoherent subanalytic sets) is related to important stability properties of these bundles. (See Remarks 2.7, Theorem 2.9 and Corollary 2.10.)

The space $\mathcal{C}^{(\infty)}(X)$ seems to be an interesting function space for a closed set X that is definable in an o-minimal structure. A definable set has a $\mathcal{C}^p$ cell decomposition for every p, but $\mathcal{C}^\infty$ cell decomposition, in general, is unknown and likely untrue. Theorem 1.3 provides strong evidence for the conjecture above in the case of a definable set X. (See Final Remarks 5.5.) The loss of differentiability in Theorem 1.3 is related to the use of [BMP] via a uniformization of X.

2. Geometric and algebraic paratangent bundles

In this section, we introduce a "Zariski paratangent bundle" $\mathcal{T}^p(X)$ – a (higher-order) $\mathcal{C}^p$ analogue of the Zariski tangent bundle studied in algebraic geometry – and we use the results above to compare the paratangent bundle $\tau^p(X)$ with $\mathcal{T}^p(X)$. (Whitney [W4] makes such a comparison in order 1, for various notions of tangent spaces to an analytic variety.) It seems less interesting to use the Zariski paratangent bundle to provide a criterion to recognize whether a function $f : X \to \mathbb{R}$ is the restriction of a $\mathcal{C}^p$ function because $\mathcal{T}^p(X)$ is defined already in terms of the ideal of $\mathcal{C}^p$ functions vanishing on X (hence essentially in terms of the space of restrictions to X of $\mathcal{C}^p$ functions); see Remark 2.3. The interest is rather in the opposite direction – to use the conjecture or the results in Section 1, involving limits of finite differences, to get a better understanding of higher-order tangent bundles (or bundles of differential operators) on singular spaces. This section is not used in the rest of the paper, except in Remarks 4.13(1).

We use the notation of Section 1. If $V \subset \mathcal{P}_p = \mathcal{P}_p(\mathbb{R}^n)$, let $V^\perp$ denote the orthogonal complement of V in the dual space $\mathcal{P}_p^*$. Let X be a closed subset of $\mathbb{R}^n$. Let $I^p(X) \subset \mathcal{C}^p(\mathbb{R}^n)$ denote the ideal of $\mathcal{C}^p$ functions that vanish on X.

Definition 2.1. The *Zariski paratangent bundle* of *order* p, $\mathcal{T}^p(X)$, is the subbundle of $X \times \mathcal{P}_p^*$ with fibre $\mathcal{T}_a^p(X) = (T_a^p I^p(X))^\perp$, for each $a \in X$. (See Definition 3.1 below.)

The bundle $\mathcal{T}^p(X)$ is closed in $X \times \mathcal{P}_p^*$ because, for all $h \in I^p(X)$, $Z(h) := \{(a,\xi) \in X \times \mathcal{P}_p^* :\ \xi(T_a^p h) = 0\}$ is closed, and $\mathcal{T}^p(X) = \bigcap_{h \in I^p(X)} Z(h)$. Moreover, if $a \in X$, then

$$\tau_a^p(X) \ \subset \ (T_a^p I^p(X))^\perp \ \subset \ \mathcal{P}_p^*$$

(by Definition 4.12 or Theorem 1.1); thus $\tau^p(X) \subset \mathcal{T}^p(X)$. Both $\tau^p(\cdot)$ and $\mathcal{T}^p(\cdot)$ are functors on the category of closed (or locally closed) subsets of Euclidean spaces (with morphisms given by the restrictions of $\mathcal{C}^p$ mappings); cf. Theorem 5.2(1) below.

Consider $q \geq p$. Let $a \in \mathbb{R}^n$. Let $\underline{m}_a^{p+1} \subset \mathcal{P}_q$ denote the subspace of polynomials of order at least $p+1$ at a; i.e.,

$$\underline{m}_a^{p+1} \;=\; \{P \in \mathcal{P}_q : \; (D^\alpha P)(a) = 0, \; |\alpha| \leq p\}.$$

There is a projection $\mathcal{P}_q \to \mathcal{P}_p$ defined by truncating terms in $x - a$ of order $> p$ in the expression of any $P \in \mathcal{P}_q$ as a polynomial in $x - a$. Let $\mathcal{P}_p^* \hookrightarrow \mathcal{P}_q^*$ denote the embedding dual to this projection. The projection induces an isomorphism $\mathcal{P}_q/\underline{m}_a^{p+1} \cong \mathcal{P}_p$, and the embedding $\mathcal{P}_p^* \hookrightarrow \mathcal{P}_q^*$ has image $(\underline{m}_a^{p+1})^\perp$.

Let $\tau^q(X)_p$ denote the subbundle of $X \times \mathcal{P}_p^*$ with fibre

$$\tau_a^q(X)_p \;:=\; \tau_a^q(X) \cap (\underline{m}_a^{p+1})^\perp \;,$$

for each $a \in X$, where $(\underline{m}_a^{p+1})^\perp$ is identified with $\mathcal{P}_p^*$ via the embedding above. Of course, $\tau^p(X)_p = \tau^p(X)$.

Lemma 2.2. *$\tau^q(X)_p$ is a closed subbundle of $X \times \mathcal{P}_p^*$.*

Proof. There is a continuous bundle mapping (cf. Definition 4.23) $X \times \mathcal{P}_p^* \to X \times \mathcal{P}_q^*$ with closed image, where, for each $a \in X$, the fibre $\mathcal{P}_p^*$ over a is embedded in $\mathcal{P}_q^*$ as above. Via this mapping, $X \times \mathcal{P}_p^*$ is a closed subbundle of $X \times \mathcal{P}_q^*$. Of course, $\tau^q(X)_p = \tau^q(X) \cap (X \times \mathcal{P}_p^*)$. $\square$

If $F \in \mathcal{C}^q(\mathbb{R}^n)$, let $T_a^p F$ denote the Taylor polynomial of order p of F at a. Let $\mathcal{T}^q(X)_p \subset X \times \mathcal{P}_p^*$ denote the bundle with fibre

$$\mathcal{T}_a^q(X)_p \;:=\; (T_a^p I^q(X))^\perp \;=\; \mathcal{T}_a^q(X) \cap (\underline{m}_a^{p+1})^\perp \;,$$

for each $a \in X$. Then $\mathcal{T}^q(X)_p$ is closed in $X \times \mathcal{P}_p^*$, and

$$\tau^q(X)_p \;\subset\; \mathcal{T}^q(X)_p \;.$$

Remark 2.3. It is possible to formulate various criteria for the existence of a $\mathcal{C}^p$ extension of $f : X \to \mathbb{R}$ involving only the values of f on X. The following is essentially tautological.

Implicit function criterion. Suppose that f is continuous. Then f is the restriction of a $\mathcal{C}^p$ function if and only if, for every $a \in X$, there is a neighbourhood of $(a, f(a))$ in $\mathbb{R}^n \times \mathbb{R}$ in which the graph of f lies in a $\mathcal{C}^p$ submanifold whose tangent space at $(a, f(a))$ contains no vertical vector (i.e., no derivation in the vertical direction).

The result of O'Farrell and Watson [O'FW] is a closely related criterion that can be expressed in terms of the Zariski paratangent bundle $\mathcal{T}^p(X)$. Suppose that f is continuous. Let $\mathcal{T}^p(f)$ denote the Zariski paratangent bundle of order p of the graph of f. Then $\mathcal{T}^p(f)$ can be regarded as a bundle over X, and the projection $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ induces a bundle mapping $\pi : \ \mathcal{T}^p(f) \to \mathcal{T}^p(X)$. The theorem of [O'FW] asserts essentially that f is the restriction of a $\mathcal{C}^p$ function if and only if π is bijective. "Only if" is obvious. On the other hand, if π is bijective, then, for each $a \in X$, $\mathcal{T}^p_a(f)_1$ contains no derivation in the vertical direction. It follows from Definition 2.1 that, in some neighbourhood of $(a, f(a))$, the graph of f lies in a submanifold whose tangent space at $(a, f(a))$ contains no vertical derivation. The result follows from the implicit function criterion.

Our conjecture in Section 1 implies that

$$\tau^p(X) \ = \ \mathcal{T}^p(X) \ :$$

We need only show that $\mathcal{T}^p(X) \subset \tau^p(X)$. Suppose that $P \in \tau^p_a(X)^\perp \subset \mathcal{P}_p$; we have to show that $P \in T^p_a I^p(X)$. Let $f = P|X$. The conjecture asserts that there exists $F \in \mathcal{C}^p(\mathbb{R}^n)$ such that $F|X = f$ and $T^p_a F = 0$. Let $G = P - F$. Then $G \in I^p(X)$, and $T^p_a G = P - T^p_a F = P$.

The following is a corollary of Theorem 1.3.

Corollary 2.4. *Suppose that X is a compact subanalytic subset of $\mathbb{R}^n$. Let $q = q_X(p)$ denote a function satisfying the assertion of Theorem 1.3. If $s \geq p$ and*

$q \geq q_X(s)$, then

$$\mathcal{T}^s(X)_p \;\subset\; \tau^q(X)_p\ .$$

Proof. Let $a \in X$ and let $P \in \tau^q_a(X)^\perp_p \subset \mathcal{P}_q$. We have to show that $P \in T^p_a I^s(X) + \underline{m}_a^{p+1}$. Recall that $\tau^q_a(X)_p = \tau^q_a(X) \cap (\underline{m}_a^{p+1})^\perp$; thus $P \in \tau^q_a(X)^\perp + \underline{m}_a^{p+1}$; i.e., there exists $Q \in \tau^q_a(X)^\perp \subset \mathcal{P}_q$ such that $T^p_a Q - P \in \underline{m}_a^{p+1}$. Let $f = Q|X$. By Theorem 1.1, $\nabla^q f : \tau^q(X) \to \mathbb{R}$ and $\nabla^q_a f = 0$ (since $Q \in \tau^q_a(X)^\perp$). By Theorem 1.3, there exists $F \in \mathcal{C}^s(\mathbb{R}^n)$ such that $F|X = f$ and $T^s_a F = 0$. Set $G = Q - F$. Then $G \in I^s(X)$, and $T^p_a G - P = T^p_a Q - P \in \underline{m}_a^{p+1}$. □

Following the viewpoint of Corollary 1.4 above, we can also introduce

$$\underline{\tau}^p(X) \;:=\; \bigcup_{q \geq p} \tau^q(X)_p\ ,$$
$$\underline{\mathcal{T}}^p(X) \;:=\; \bigcup_{q \geq p} \mathcal{T}^q(X)_p\ .$$

Then $\underline{\tau}^p(X) \subset \underline{\mathcal{T}}^p(X)$.

Corollary 2.5. *If X is a closed subanalytic subset of $\mathbb{R}^n$, then $\underline{\tau}^p(X) = \underline{\mathcal{T}}^p(X)$.*

Definition 2.6. Let $\mathbb{R}[[x-a]]$ denote the ring of formal power series in $(x_1 - a_1, \dots, x_n - a_n)$. If $F \in \mathbb{R}[[x-a]]$, let $T^p_a F(x)$ denote the Taylor polynomial of order p of F at a; i.e., the polynomial of degree $\leq p$ obtained by truncating the terms of F of order $> p$ in $x - a$. Suppose $a \in X$. We define the *formal local ideal* $\mathcal{F}_a(X)$ of X at a as $\{F \in \mathbb{R}[[x-a]] :\ T^p_a F(x) = o(|x-a|^p)$, where $x \in X$, for all $p \in \mathbb{N}\}$. (See [BM2, Lemma 6.1].)

Remarks 2.7. For each $q \geq p$, $\tau^q(X)_p \subset \tau^{q+1}(X)_p$ (as in Remarks 4.13(1) below) and $\mathcal{T}^q(X)_p \subset \mathcal{T}^{q+1}(X)_p$; i.e., $\{\tau^q(X)_p\}_{q \geq p}$ and $\{\mathcal{T}^q(X)_p\}_{q \geq p}$ are increasing sequences of closed subbundles of $X \times \mathcal{P}^*_p$. If $\{\tau^q(X)_p\}_{q \geq p}$ (respectively, $\{\mathcal{T}^q(X)_p\}_{q \geq p}$) stabilizes, then $\underline{\tau}^p(X)$ (respectively, $\underline{\mathcal{T}}^p(X)$) is closed. Since $\tau^q(X)_p \subset \mathcal{T}^q(X)_p$, for all $q \geq p$, it follows from Corollary 2.4 that, if X is a compact subanalytic set, then, for all $p \in \mathbb{N}$, $\{\tau^q(X)_p\}_{q \geq p}$ stabilizes if and only if $\{\mathcal{T}^q(X)_p\}_{q \geq p}$ stabilizes. We say that a closed subanalytic subset X of $\mathbb{R}^n$ is *semicoherent* if it has a locally finite subanalytic stratification such that the formal local ideals $\mathcal{F}_a(X)$ are generated over each stratum by finitely many subanalytically parametrized formal power series [BM2, Definition 1.2]. In Corollary 2.10 below, we show that a compact subanalytic subset X of $\mathbb{R}^n$ is semicoherent if and only if the sequence $\{\mathcal{T}^q(X)_p\}_{q\geq p}$ (or the sequence $\{\tau^q(X)_p\}_{q\geq p}$) stabilizes, for all $p \in \mathbb{N}$.

For compact subanalytic sets X in general, however, $\underline{\mathcal{T}}^p(X)$ is not necessarily closed, and $\{\mathcal{T}^q(X)_p\}_{q\geq p}$ does not necessarily stabilize even if $\underline{\mathcal{T}}^p(X)$ is closed. If $n \leq 4$, $\dim X \leq 2$, or $\dim X \geq n-1$, then X is semicoherent. In $\mathbb{R}^5$, consider any sequence of distinct points $\{a_j\}$ tending to the origin along some line. By the construction of [P1], there is a compact 3-dimensional subanalytic subset X of $\mathbb{R}^5$ such that X is not semicoherent, X is semicoherent outside 0, and $\mathcal{F}_a(X) = 0$ if and only if $a \in \{a_j\}$ (cf. [BM2, Examples 1.29]). By Corollary 2.10, $\underline{\mathcal{T}}^p(X)\backslash\underline{\mathcal{T}}^p_0(X)$ is closed in $(X\backslash 0)\times\mathcal{P}^*_p$, for all p. For all p and j, $\underline{\mathcal{T}}^p_{a_j}(X) = \mathcal{P}^*_p$, by Lemma 2.8, since $\mathcal{F}_{a_j}(X) = 0$. It follows that, if $0 \in \{a_j\}$, then $\underline{\mathcal{T}}^p(X)$ is closed, for all p. On the other hand, if $0 \notin \{a_j\}$, then $\mathcal{F}_0(X) \neq 0$, so there exists p such that $\underline{\mathcal{T}}^p_0(X) \neq \mathcal{P}^*_p$, and it follows that $\underline{\mathcal{T}}^p(X)$ is not closed.

Let X be a closed subset of $\mathbb{R}^n$ and let $a \in X$. Then $\{T^p_a I^q(X)\}_{q\geq p}$ is a decreasing sequence of linear subspaces of $\mathcal{P}_p$. Let $s_X(a,p)$ denote the smallest integer $s \geq p$ such that $T^p_a I^q(X) = T^p_a I^s(X)$, for all $q \geq s$. Of course, $s_X(a,p)$ is the smallest integer $s \geq p$ such that $\mathcal{T}^s_a(X)_p = \bigcup_{q\geq p}\mathcal{T}^q_a(X)_p$.

Lemma 2.8. *Suppose that X is subanalytic. Let $a \in X$ and let $s \geq p$. Then $s \geq s_X(a,p)$ if and only if*

$$T^p_a I^s(X) \;=\; T^p_a \mathcal{F}_a(X)\ .$$

Proof. First we show that, if $s \geq s_X(a,p)$, then $T^p_a I^s(X) \subset T^p_a\mathcal{F}_a(X)$, *for any* closed $X \subset \mathbb{R}^n$. Let $p_0 = p$ and $p_j = s_X(a,p_{j-1})$, for all $j \geq 1$. Let $P = T^p_a f_0$, where $f_0 \in I^{p_1}(X)$. We have to show that $P = T^p_a F$, where $F \in \mathcal{F}_a(X)$. By the definition of $\{p_j\}$, for all $j \geq 1$, $T^{p_{j-1}}_a I^{p_j}(X) = T^{p_{j-1}}_a I^{p_{j+1}}(X)$, so that we can inductively find $f_j \in I^{p_{j+1}}(X)$, $j \geq 1$, such that $T_a^{p_{j-1}} f_{j-1} = T_a^{p_{j-1}} f_j$. Let $Q_j = T_a^{p_j} f_j$, $j \geq 0$, so that $Q_j(x) = o(|x-a|^{p_j})$, where $x \in X$, and $Q_{j-1} = T_a^{p_{j-1}} Q_j$, $j \geq 1$. Take $F \in \mathbb{R}[[x-a]]$ such that $Q_j = T_a^{p_j} F$, for all j. Then $P = T_a^p F$ and, for all j, $T_a^{p_j} F = Q_j = o(|x-a|^{p_j})$, where $x \in X$, as required.

We can assume that X is a compact subanalytic set. Then $T_a^q \mathcal{F}_a(X) \subset T_a^q I^q(X)$, for all q, as follows: There is a compact real analytic manifold M and a real analytic mapping $\varphi: M \to \mathbb{R}^n$ such that $\varphi(M) = X$. Let $F \in \mathcal{F}_a(X)$. Take $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that F is the formal Taylor series of f at a. Set $g = f \circ \varphi$. Then g is flat on $\varphi^{-1}(a)$. By [BMP], for all $q \in \mathbb{N}$, there exists $f_q \in \mathcal{C}^q(\mathbb{R}^n)$, such that $T_a^q f_q = 0$ and $g = f_q \circ \varphi$. Then $f - f_q \in I^q(X)$ and $T_a^q(f - f_q) = T_a^q f = T_a^q F$.

Thus $T_a^p \mathcal{F}_a(X) \subset T_a^p I^q(X)$, for all $q \geq p$. If $s \geq s_X(a,p)$, then $T_a^p I^s(X) \subset T_a^p \mathcal{F}_a(X)$; hence $T_a^p I^s(X) = T_a^p \mathcal{F}_a(X)$. Conversely, if $T_a^p I^s(X) = T_a^p \mathcal{F}_a(X)$, then, for all $q \geq s$, $T_a^p I^q(X) \subset T_a^p I^s(X) = T_a^p \mathcal{F}_a(X) \subset T_a^p I^q(X)$, so that $T_a^p I^q(X) = T_a^p I^s(X)$, and hence $s \geq s_X(a,p)$. □

Let X denote a compact subanalytic subset of $\mathbb{R}^n$. Let $\varphi: M \to \mathbb{R}^n$ be a real analytic mapping from a compact real analytic manifold M, such that $\varphi(M) = X$. By [BMP], for all $p \in \mathbb{N}$, there exists $q \geq p$ with the following property: if $a \in X$ and $g \in \mathcal{C}^q(M)$ such that g is formally a composite with φ and g is q-flat on $\varphi^{-1}(a)$, then there exists $f \in \mathcal{C}^p(\mathbb{R}^n)$ such that $g = f \circ \varphi$ and f is p-flat at a. Let $q_\varphi(p)$ denote the least such q.

By a lemma of Chevalley (cf. [BM2, Section 6]), for all $k \in \mathbb{N}$, there exists $l \in \mathbb{N}$, $l \geq k$, with the following property: for every polynomial $F(x)$ such that $F(x) = o(|x-a|^l)$, where $x \in X$, there exists $G \in \mathcal{F}_a(X)$ such that $T_a^k F = T_a^k G$. Given k, let $l_X(a,k)$ denote the least such l. We call $l_X(a,k)$ a *Chevalley estimate.*

Theorem 2.9. *Let X be a compact subanalytic subset of $\mathbb{R}^n$ and let $\varphi: M \to \mathbb{R}^n$ be a real analytic mapping as above. Then, for all $a \in X$ and all $p \in \mathbb{N}$,*

$$s_X(a,p) \leq l_X(a,p) \leq q_\varphi(s_X(a,p)) .$$

Proof. For the first inequality, let $s = l_X(a,p)$; by Lemma 2.8, it is enough to show that if $f \in I^s(X)$, then $T^p_a f \in T^p_a \mathcal{F}_a(X)$. Let $P = T^s_a f$. Then $P(x) = o(|x-a|^s)$, where $x \in X$, so the result follows from the definition of $l_X(a,p)$.

For the second inequality, let $s = s_X(a,p)$ and $q = q_\varphi(s)$. Let F denote a polynomial such that $F(x) = o(|x-a|^q)$, where $x \in X$. By Lemma 2.8, it is enough to show that $T^p_a F \in T^p_a I^s(X)$. Let $g = F \circ \varphi$. Then $g \in \mathcal{C}^\infty(M)$ and g is q-flat on $\varphi^{-1}(a)$. By [BMP], there exists $f \in \mathcal{C}^s(\mathbb{R}^n)$ such that $T^s_a f = 0$ and $g = f \circ \varphi$. Then $f = F$ on X and $T^s_a f = 0$, so that $T^p_a F = T^p_a(F - f) \in T^p_a I^s(X)$. □

Corollary 2.10. *Let X be a compact subanalytic subset of $\mathbb{R}^n$.*

(1) Let $p \in \mathbb{N}$. Then the increasing union $\bigcup_{q \ge p} \mathcal{T}^q(X)_p$ stabilizes if and only if there exists $l \in \mathbb{N}$ such that $l_X(a,p) \le l$, for all $a \in X$.

(2) The increasing union $\bigcup_{q \ge p} \mathcal{T}^q(X)_p$ (or the increasing union $\bigcup_{q \ge p} \tau^q(X)_p$) stabilizes, for all $p \in \mathbb{N}$, if and only if X is semicoherent.

Proof. The first assertion is immediate from Theorem 2.9. By [BM2, Theorem 1.13], X is semicoherent if and only if there exists a *uniform Chevalley estimate*; i.e., a function $l = l_X(k)$ such that $l_X(a,k) \le l_X(k)$, for all $k \in \mathbb{N}$ and $a \in X$. So the second statement also follows. □

3. Glaeser's construction

Let X be a metric space and let V be a real vector space of finite dimension r.

Definition 3.1. A *bundle* (*of linear subspaces of* V) *over* X is a subset E of $X \times V$ such that, for all $a \in X$, the *fibre* $E_a := \{v \in V : (a,v) \in E\}$ is a linear subspace of V.

Definition 3.2. A *Glaeser operation* (on bundles of linear subspaces of V over X) is an operation ρ that associates to each bundle E a bundle $\rho(E)$ such that:

(1) $\overline{E} \subseteq \rho(E)$;

(2) ρ is *local*; i.e., if E, F are bundles over X and $E_a = F_a$ for all $a \in U$, where $U \subset X$ is open, then $\rho(E)_a = \rho(F)_a$ for all $a \in U$.

We include a proof of the following lemma of Glaeser [G] because we use it in Sections 4 and 5.

Lemma 3.3. *Let E be a bundle of linear subspaces of V over X, and let ρ be a Glaeser operation. Write $\rho^i := \rho \circ \cdots \circ \rho$ (i times). Then:*

(1) $\rho^i = \rho^{2r}$ if $i \geq 2r$;

(2) $\widehat{E} := \rho^{2r}(E)$ is a closed bundle;

(3) $\dim \widehat{E}_a$ is an upper-semicontinuous function of $a \in X$.

Proof. Set

$$d_i(a) \;=\; \dim \rho^i(E)_a \;, \qquad \lambda_i(a) \;=\; \inf_{\delta>0} \sup_{\sigma(a,x)<\delta} d_i(x) \;,$$

for all $a \in X$ and $i = 0, 1, \ldots$, where $\sigma(\cdot,\cdot)$ denotes the metric on X. Then, for all i, λ_i is upper-semicontinuous and $d_i \leq \lambda_i \leq d_{i+1}$ (the latter inequality since $\overline{\rho^i(E)} \subset \rho^{i+1}(E)$). Let

$$G_i \;:=\; \operatorname{int}\{a \in X :\; d_i(a) = d_{i+1}(a)\}$$

and let $Z_i := X \backslash G_i$. Then $G_i \subset G_{i+1}$ for all i, since, for all $a \in G_i$, $\rho^i(E)_a = \rho^{i+1}(E)_a$ and therefore $\rho^{i+1}(E)_a = \rho^{i+2}(E)_a$ by locality (Definition 3.2, property (2)). Thus $Z_i \supset Z_{i+1}$ for all i.

We claim that $d_{i+2}(a) > i/2$, for all $a \in Z_i$, $i \in \mathbb{N}$. First, this holds for $i = 0$: Otherwise, there exists $a \in Z_0$ such that $d_2(a) = 0$. Then $\lambda_1(a) = 0$, hence $d_0(x) = d_1(x) = 0$ in a neighbourhood of a, so that $a \in G_0$ (a contradiction). The claim is true for $i = 1$ because, if $a \in Z_1$, then $a \in Z_0$ so that $d_3(a) \geq d_2(a) \geq 1$. Consider $i \geq 2$ and suppose that $d_{j+2}(a) > j/2$ for all $a \in Z_j$, when $j < i$. Set $\Sigma_i := \{a \in X : d_i(a) < d_{i+1}(a)\}$. Then $Z_i = \overline{\Sigma}_i$, so $\Sigma_i \subset Z_i \subset Z_{i-2}$. Therefore, if $a \in \Sigma_i$, then $d_i(a) > (i-2)/2$, so that $d_{i+1}(a) > i/2$. It follows that, for all $a \in Z_i$, $d_{i+2}(a) \geq \lambda_{i+1}(a) > i/2$. This proves the claim, by induction.

It follows that $Z_{2r} = \emptyset$, so (1) holds. (2) follows because $\overline{\widehat{E}} \subset \rho(\widehat{E}) \subset \widehat{E}$, and (3) because $d_{2r}(a) \leq \lambda_{2r}(a) \leq d_{2r+1}(a)$, $a \in X$. □

Example 3.4. [G]. For any bundle $E \subset X \times V$, define

$$\widetilde{E} \; := \; \bigcup_{a \in X} \{a\} \times \operatorname{Span} E_a$$

where Span denotes the linear span). Set $\lambda(E) := \overline{\widetilde{E}}$. Then λ is a Glaeser operation and $\lambda(E) \subset \rho(E)$ for any Glaeser operation ρ.

Definition 3.5. [G]. Let X be a closed subset of $\mathbb{R}^n$. Define

$$\begin{aligned}\operatorname{ptg}(X) \; &:= \; \{(a, \sigma u) \in X \times \mathbb{R}^n \, : \; \sigma \in \mathbb{R} \; , \; u = \lim_{j \to \infty} \frac{x_j - y_j}{|x_j - y_j|} \; , \\ &\qquad\qquad \text{where } (x_j), (y_j) \subset X, \; x_j \neq y_j, \text{ for all } j \; , \\ &\qquad\qquad \text{and } \lim_{j \to \infty} x_j = a = \lim_{j \to \infty} y_j \} \; ; \\ \tau(X) \; &:= \; \widehat{\widetilde{\operatorname{ptg}(X)}} \; ,\end{aligned}$$

where the "saturation" $\widehat{\;\;}$ is with respect to the Glaeser operation λ of Example 3.4. We call $\tau(X)$ the *(linearized) paratangent bundle* of X. If $a \in X$, the fibre $\tau_a(X)$ is called the *paratangent space* of X at a.

Let $f : X \to \mathbb{R}$ be a continuous function. (We identify a function with its graph and therefore write $\tau(f) = \tau(\operatorname{graph} f)$.) We can consider $\tau(f)$ as a bundle over X, so that $\tau(f) \subset \tau(X) \times \mathbb{R}$.

Theorem 3.6. [G], [Br]. *Let X be a closed subset of $\mathbb{R}^n$ and let $f : X \to \mathbb{R}$ be a continuous function. Then there exists $F \in \mathcal{C}^1(\mathbb{R}^n)$ such that $F|X = f$ if and only if $\tau(f) : \tau(X) \to \mathbb{R}$ (i.e., $\tau(f)$ is the graph of a function $\tau(X) \to \mathbb{R}$). In this case, each $\tau_a(f) : \tau_a(X) \to \mathbb{R}$, $a \in X$, is the restriction to the paratangent space $\tau_a(X)$ of the derivative of F.*

4. Higher-order paratangent spaces

The remainder term in Taylor's theorem. Let $X \subset U \subset \mathbb{R}^n$, where U is open and X is closed in U. Let $p \in \mathbb{N}$. Let $F = (F^\alpha)_{\alpha \in \mathbb{N}^n, |\alpha| \le p}$, where each $F^\alpha : X \to \mathbb{R}$. If $a \in X$, define

$$(T_a^p F)(x) \; := \; \sum_{|\alpha| \le p} \frac{1}{\alpha!} F^\alpha(a)(x - a)^\alpha \; .$$

Let $D^\alpha := \partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$, $\alpha \in \mathbb{N}^n$. If $|\alpha| \le p$ and $b \in X$, set

$$
\begin{aligned}
(4.1) \qquad (R_a^p F)^\alpha(b) &:= F^\alpha(b) - D^\alpha(T_a^p F)(b) \\
&= F^\alpha(b) - \sum_{|\beta| \le p - |\alpha|} \frac{1}{\beta!} F^{\alpha+\beta}(a)(b-a)^\beta \ , \\
(4.2) \qquad \delta_\alpha(a,b) &:= \frac{(R_a^p F)^\alpha(b)}{|b-a|^{p-|\alpha|}} \ .
\end{aligned}
$$

We recall Whitney's extension theorem [W1]:

Theorem 4.3. *Let $F^\alpha : X \to \mathbb{R}$, $|\alpha| \le p$. Then there exists $f \in \mathcal{C}^p(U)$ such that $(D^\alpha f)|X = F^\alpha$, for all $|\alpha| \le p$, if and only if $\delta_\alpha(a,b) \to 0$ if a, $b \in X$ and $|\alpha| \le p$, as $|a-b| \to 0$.*

We say that $F = (F^\alpha)$ is a $\mathcal{C}^p$ *Whitney field* on X if it satisfies the conditions of Theorem 4.3.

Let $\mathcal{P}_p = \mathcal{P}_p(\mathbb{R}^n)$ denote the real vector space of polynomial functions on $\mathbb{R}^n$ of degree at most p. Let $\xi \in \mathcal{P}_p^*$, where $\mathcal{P}_p^*$ denotes the dual of $\mathcal{P}_p$. Set

$$
(4.4) \qquad \xi(F,a) := \xi(T_a^p F) = \sum_{|\alpha| \le p} F^\alpha(a) \xi_\alpha(a) \ ,
$$

where

$$
(4.5) \qquad \xi_\alpha(a) := \xi\left(\frac{1}{\alpha!}(x-a)^\alpha\right) \ .
$$

If $\eta \in \mathcal{P}_p^*$ and $b \in X$, then

$$\begin{aligned}
\eta(F,b) &= \sum_{|\alpha|\le p} F^\alpha(b)\eta_\alpha(b) \\
&= \sum_{|\alpha|\le p}\left(\sum_{|\beta|\le p-|\alpha|}\frac{1}{\beta!}F^{\alpha+\beta}(a)(b-a)^\beta + \delta_\alpha(a,b)|b-a|^{p-|\alpha|}\right)\eta_\alpha(b) \\
&= \sum_{|\alpha|\le p}\sum_{\beta\le\alpha}\frac{1}{\beta!}F^\alpha(a)(b-a)^\beta\eta_{\alpha-\beta}(b) + \sum_{|\alpha|\le p}\delta_\alpha(a,b)|b-a|^{p-|\alpha|}\eta_\alpha(b) \\
&= \sum_{|\alpha|\le p}\sum_{\beta\le\alpha}\frac{1}{\beta!}F^\alpha(a)(b-a)^\beta\eta\left(\frac{1}{(\alpha-\beta)!}(x-b)^{\alpha-\beta}\right) + \sum_{|\alpha|\le p}\delta_\alpha(a,b)|b-a|^{p-|\alpha|}\eta_\alpha(b) \\
&= \sum_{|\alpha|\le p}\frac{1}{\alpha!}F^\alpha(a)\eta\left(\sum_{\beta\le\alpha}\binom{\alpha}{\beta}(b-a)^\beta(x-b)^{\alpha-\rho}\right) + \sum_{|\alpha|\le p}\delta_\alpha(a,b)|b-a|^{p-|\alpha|}\eta_\alpha(b) \\
&= \sum_{|\alpha|\le p}F^\alpha(a)\eta\left(\frac{1}{\alpha!}(x-a)^\alpha\right) + \sum_{|\alpha|\le p}\delta_\alpha(a,b)|b-a|^{p-|\alpha|}\eta_\alpha(b)\ .
\end{aligned}$$

(If $\alpha = (\alpha_1,\dots,\alpha_n)$ and $\beta = (\beta_1,\dots,\beta_n)$, then $\beta \le \alpha$ means $\beta_i \le \alpha_i$, $i = 1,\dots,n$.) Therefore,

$$\eta(F,b) \;=\; \eta(F,a) + \sum_{|\alpha|\le p}\delta_\alpha(a,b)|b-a|^{p-|\alpha|}\eta_\alpha(b); \tag{4.6}$$

$$\xi(F,a) + \eta(F,b) \;=\; (\xi+\eta)(F,a) + \sum_{|\alpha|\le p}\delta_\alpha(a,b)|b-a|^{p-|\alpha|}\eta_\alpha(b)\ . \tag{4.7}$$

We will use the following lemma only in the case $k = 1$ (but see Remarks 4.13(2) and Final Remarks 5.5).

Lemma 4.8. *Let $X \subset U \subset \mathbb{R}^n$, where U is open and X is closed in U. Let $(a_{ij}) = (a_{i1}, a_{i2}, \dots)$ and $(\xi_{ij}) = (\xi_{i1}, \xi_{i2}, \dots)$ denote sequences in X and $\mathcal{P}_p^*$, respectively, for $i = 0, 1, \dots, k$, such that:*

(1) The sequences (a_{ij}), $i = 0, 1, \dots, k$, converge to a common point $a \in X$, and $\sum_{i=0}^k \xi_{ij}$ converges to $\xi \in \mathcal{P}_p^*$.

(2) $|a_{ij} - a_{0j}|^{p-|\alpha|}|\xi_{ij,\alpha}(a_{ij})| \le c$, for all i, j and $|\alpha| \le p$ (where c is a constant).

If $F = (F^\alpha)_{|\alpha|\le p}$ is a $\mathcal{C}^p$ Whitney field on X, then

$$\xi(F,a) \;=\; \lim_{j\to\infty} \sum_{i=0}^{k} \xi_{ij}(F,a_{ij}) \;.$$

Proof. For each $j = 1, 2, \ldots,$

$$\begin{aligned}
&\xi(F,a) - \sum_{i=0}^{k} \xi_{ij}(F,a_{ij}) \\
&= \xi(F,a) - \sum_{i=0}^{k} \xi_{ij}(F,a) + \sum_{i=0}^{k} \big(\xi_{ij}(F,a) - \xi_{ij}(F,a_{0j})\big) + \sum_{i=0}^{k} \big(\xi_{ij}(F,a_{0j}) - \xi_{ij}(F,a_{ij})\big) \\
&= \left(\xi - \sum_{i=0}^{k} \xi_{ij}\right)(T_a^p F) + \sum_{i=0}^{k} \xi_{ij}(T_a^p F - T_{a_{0j}}^p F) + \sum_{i=0}^{k} \xi_{ij}(T_{a_{0j}}^p F - T_{a_{ij}}^p F) \\
&= \left(\xi - \sum_{i=0}^{k} \xi_{ij}\right)(T_a^p F) + \sum_{i=0}^{k} \sum_{|\alpha|\le p} \delta_\alpha(a_{0j},a)|a-a_{0j}|^{p-|\alpha|}\xi_{ij,\alpha}(a) \\
&\qquad\qquad - \sum_{i=0}^{k} \sum_{|\alpha|\le p} \delta_\alpha(a_{0j},a_{ij})|a_{ij}-a_{0j}|^{p-|\alpha|}\xi_{ij,\alpha}(a_{ij}) \;.
\end{aligned}$$

Each of the three terms tends to 0 as $j \to \infty$. □

The paratangent bundle of order p. We consider Glaeser operations on bundles of subspaces of $\mathcal{P}_p^*$. Let $X \subset U \subset \mathbb{R}^n$, where U is open and X is closed in U. Let $E \subset X \times \mathcal{P}_p^*$ be any bundle of linear subspaces of $\mathcal{P}_p^*$ over X. Define

$$\begin{aligned}
\text{(4.9)}\qquad \Delta E \;:=\; \{(a,b,\xi+\eta) :\, & a,b \in X,\ \xi \in E_a,\ \eta \in E_b, \\
& |a-b|^{p-|\alpha|}|\eta_\alpha(b)| \le 1 \ \text{ for all } |\alpha| \le p\} \;.
\end{aligned}$$

Let $\pi : X^2 \times \mathcal{P}_p^* \to X \times \mathcal{P}_p^*$ denote the projection $\pi(a,b,\xi) = (a,\xi)$. Define

$$\text{(4.10)}\qquad E' \;:=\; \pi(\overline{\Delta E} \cap \{(a,a,\xi) :\ a \in X, \xi \in \mathcal{P}_p^*\}) \;.$$

Clearly, $\overline{E} \subset E'$. We define a Glaeser operation

$$\text{(4.11)}\qquad \rho(E) \;:=\; \widetilde{E'} \;:=\; \bigcup_{a\in X} \{a\} \times \operatorname{Span} E'_a \;.$$

(Recall Example 3.4.)

Definition 4.12. Let $X \subset U \subset \mathbb{R}^n$ be as above, and set

$$E \ := \ \{(a, \lambda\delta_a) : \ a \in X, \lambda \in \mathbb{R}\} \ ,$$

where $\delta_a \in \mathcal{P}_p^*$ denotes the delta-function $\delta_a(P) := P(a)$, $P \in \mathcal{P}_p$. Define

$$\tau^p(X) \ := \ \widehat{E} \ ,$$

where $\widehat{E}$ denotes the saturation of E with respect to the Glaeser operation (4.11) (cf. Lemma 3.3). We call $\tau^p(X)$ the *(linearized) paratangent bundle* of X of *order* p.

Remarks 4.13. (1) Consider $q \geq p$. Recall that $X \times \mathcal{P}_p^*$ embeds in $X \times \mathcal{P}_q^*$ as a closed subbundle, where, for each $a \in X$, the fibre $\mathcal{P}_p^*$ over a is identified as in Section 2 with $(\underline{m}_a^{p+1})^\perp \subset \mathcal{P}_q^*$. If $b \in X$ and $\eta \in (\underline{m}_b^{p+1})^\perp$, then $\eta_\alpha(b) = 0$, for all $p < |\alpha| \leq q$. It follows from the definition above that $\tau^p(X)$ is a closed subbundle of $\tau^q(X)$.

(2) The definition above involves distributions with values in $\mathcal{P}_p^*$ supported at pairs of points $a, b \in X$, according to (4.9), and suffices for all results in this paper. But a more general definition of $\tau^p(X)$ involving distributions supported at $k+1$ points (where $k \geq p$) is necessary for our main conjecture in Section 1. See Final Remarks 5.5. We have stated Lemma 4.8 and Lemma 5.1 below for distributions supported at $k+1$ points in order that they be available more generally.

Now let $\Phi \subset X \times (\mathcal{P}_p^* \times \mathbb{R})$ be a bundle of linear subspaces of $\mathcal{P}_p^* \times \mathbb{R}$ over X. Define

(4.14)
$$\begin{aligned}\Delta\Phi \ := \ \{(a, b, \xi + \eta, \lambda + \mu) : \ & a, b \in X, \ (\xi, \lambda) \in \Phi_a \ , \ (\eta, \mu) \in \Phi_b \ , \\ & |a - b|^{p-|\alpha|} |\eta_\alpha(b)| \leq 1 \text{ for all } |\alpha| \leq p\} \ ;\end{aligned}$$

(4.15)
$$\Phi' \ := \ \pi(\overline{\Delta\Phi} \cap \{(a, a, \xi, \lambda) : \ a \in X, \xi \in \mathcal{P}_p^* \ , \lambda \in \mathbb{R}\}) \ ,$$

where $\pi : X^2 \times \mathcal{P}_p^* \times \mathbb{R} \to X \times \mathcal{P}_p^* \times \mathbb{R}$ is the projection $\pi(a, b, \xi, \lambda) = (a, \xi, \lambda)$. As before, $\overline{\Phi} \subset \Phi'$.

Definition 4.16. Define

$$\nabla^p f \ := \ \widehat{\Phi} \ ,$$

where

$$\Phi \ := \ \{(a, \lambda\delta_a, \lambda f(a)) : \ a \in X, \lambda \in \mathbb{R}\}$$

and $\widehat{\Phi}$ denotes the saturation with respect to the Glaeser operation $\rho(\Phi) := \widetilde{\Phi'}$.

Clearly, $\nabla^p f \subset \tau^p(X) \times \mathbb{R}$. Theorem 1.1 is a restatement of Theorem 4.18 below.

Lemma 4.17. *Let $X \subset U \subset \mathbb{R}^n$, where U is open and X is closed in U. Let $f : X \to \mathbb{R}$. Let $p \in \mathbb{N}$ and suppose there is a $\mathcal{C}^p$ Whitney field $F = (F^\alpha)_{|\alpha| \le p}$ on X such that $F^0 = f$. Consider the bundles E, Φ over X and the Glaeser operations ρ of Definitions 4.12 and 4.16. Then, for each $i \in \mathbb{N}$,*

$$\rho^i(\Phi) : \ \rho^i(E) \to \mathbb{R} \ ;$$

moreover, if $a \in X$ and $\xi \in \rho^i(E)_a$, then

$$\rho^i(\Phi)(\xi) \ = \ \xi(T^p_a F) \ = \ \xi(F, a) \ .$$

Proof. First consider $i = 1$. Any element of Φ' can be expressed

$$\lim_{j \to \infty} (a_{0j}, a_{1j}, \xi_{0j} + \xi_{1j}, \lambda_{0j} f(a_{0j}) + \lambda_{1j} f(a_{1j})) \ ,$$

where $\xi_{ij} = \lambda_{ij}\delta_{a_{ij}}$, $i = 0, 1$, $j = 1, 2, \ldots$, and (a_{ij}), (ξ_{ij}) satisfy the hypotheses of Lemma 4.8 (case $k = 1$). Of course , $\lambda_{ij} f(a_{ij}) = \xi_{ij}(F, a_{ij})$, for all i, j. By Lemma 4.8,

$$\lim_{j \to \infty} (\lambda_{0j} f(a_{0j}) + \lambda_{1j} f(a_{1j})) \ = \ \lim_{j \to \infty} (\xi_{0j} + \xi_{1j})(F, a) \ .$$

Therefore, $\Phi' : E' \to \mathbb{R}$ and, for all $\xi \in E'_a$, $a \in X$, $\Phi'(\xi) = \xi(F, a) = \xi(T^p_a F)$. It follows that $\rho(\Phi) : \rho(E) \to \mathbb{R}$ and, for all $\xi \in \rho(E)_a$, $\rho(\Phi)(\xi) = \xi(T^p_a F) = \xi(F, a)$.

The result then follows from Lemma 4.8, by induction on i. $\square$

Theorem 4.18. *Let $X \subset U \subset \mathbb{R}^n$, where U is open and X is closed in U. Let $f : X \to \mathbb{R}$. Let $p \in \mathbb{N}$ and suppose there is a $\mathcal{C}^p$ Whitney field $F = (F^\alpha)_{|\alpha|\le p}$ on X such that $F^0 = f$. Then*

$$\nabla^p f : \ \tau^p(X) \to \mathbb{R} \ ;$$

moreover, if $a \in X$ and $\xi \in \tau^p_a(X) \subset \mathcal{P}^*_p$, then

$$\nabla^p f(\xi) \ = \ \xi(T^p_a F) \ = \ \xi(F, a) \ .$$

This is an immediate consequence of Lemma 4.17.

Remarks 4.19. Let $f : X \to \mathbb{R}$. If $\nabla^p f : \tau^p(X) \to \mathbb{R}$, then $\nabla^p f$ (as well as f) is necessarily continuous and, for all $a \in X$, the induced function on the fibre $\nabla^p_a f : \tau^p_a(X) \to \mathbb{R}$ is necessarily linear. Consider $q \ge p$. Then $\nabla^p f$ is a closed subbundle of $\nabla^q f$ (cf. Remarks 4.13(1)); if $\nabla^q f : \tau^q(X) \to \mathbb{R}$, then $\nabla^p f : \tau^p(X) \to \mathbb{R}$ and $\nabla^p f$ is the restriction of $\nabla^q f$.

We will show that Theorem 1.2 is a consequence of Theorem 4.21 below.

Lemma 4.20. *Suppose that $X \subset U \subset \mathbb{R}^n$, where U is open and X is the closure of* $\operatorname{int} X$ *in U. Then $\tau^p(X) = X \times \mathcal{P}^*_p$, for all $p \in \mathbb{N}$.*

Proof. It is enough to show that $\tau^p_a(X) = \mathcal{P}^*_p$, where $a \in \operatorname{int} X$. If $\alpha \in \mathbb{N}^n$, $|\alpha| \le p$, and $b \in X$, define $D^\alpha(b) \in \mathcal{P}^*_p$ by

$$D^\alpha(b)(P) \ := \ (D^\alpha P)(b) \ , \quad P \in \mathcal{P}_p \ .$$

If $|\alpha| < p$, then

$$\frac{D^\alpha(b) - D^\alpha(a)}{|b-a|} \ \longrightarrow \ \sum_{i=1}^n u_i D^{\alpha+(i)}(a)$$

if $b \to a$ and $\dfrac{b-a}{|b-a|} \to u = (u_1, \ldots, u_n)$, where (i) denotes the multiindex with 1 in the i'th place and 0 elsewhere. Let $\eta = \dfrac{D^\alpha(b)}{|b-a|}$ (where $|\alpha| < p$). Then, for all $\gamma \in \mathbb{N}^n$, $|\gamma| \le p$,

$$\begin{aligned} |a-b|^{p-|\gamma|}\eta_\gamma(b) \ &= \ |a-b|^{p-|\gamma|-1} D^\alpha\left(\frac{1}{\gamma!}(x-b)^\gamma\right)(b) \\ &= \ \begin{cases} 0 & , \quad \gamma \ne \alpha \\ |a-b|^{p-|\alpha|-1} & , \quad \gamma = \alpha \ . \end{cases} \end{aligned}$$

By Definition 4.12, it follows by induction on $|\gamma|$ that $D^\gamma(a) \in \tau_a^p(X)$, for all $|\gamma| \le p$; i.e., $\tau_a^p(X) = \mathcal{P}_p^*$. □

Theorem 4.21. *Let $X \subset U \subset \mathbb{R}^n$, where U is open and X is the closure of* $\mathrm{int} X$ *in U. Let $f : X \to \mathbb{R}$. Let $p \in \mathbb{N}$. Suppose that*

$$\nabla^p f : \ \tau^p(X) \to \mathbb{R} \ .$$

Then there is a $\mathcal{C}^p$ Whitney field $F = (F^\alpha)_{|\alpha| \le p}$ on X such that $F^0 = f$.

Proof. By Lemma 4.20, we have

$$\nabla^p f : \ X \times \mathcal{P}_p^* \to \mathbb{R} \ .$$

Define

$$F^\alpha(a) \ := \ (\nabla^p f)(a, D^\alpha(a)) \ , \quad a \in X \ ,$$

for all $\alpha \in \mathbb{N}^n$, $|\alpha| \le p$. (We use the notation of the proof of Lemma 4.20.) Let $c \in X$ and $\alpha \in \mathbb{N}^n$, $|\alpha| \le p$. Then

$$\frac{D^\alpha(b) - \sum_{|\beta| \le p - |\alpha|} \frac{1}{\beta!}(b-a)^\beta D^{\alpha+\rho}(a)}{|b-a|^{p-|\alpha|}} \longrightarrow 0$$

as a, $b \to c$, where a, $b \in X$, $a \ne b$. (In fact, this element of $\mathcal{P}_p^*$ equals zero since, for all $P \in \mathcal{P}_p$, $(D^\alpha P)(b) = (T_a^{p-|\alpha|} D^\alpha P)(b)$.) If $\eta := \dfrac{D^\alpha(b)}{|b-a|^{p-|\alpha|}}$, then

$$|a-b|^{p-|\gamma|}\eta_\gamma(b) \ = \ \begin{cases} 0 \ , & \gamma \ne \alpha \\ 1 \ , & \gamma = \alpha \ . \end{cases}$$

Hence

$$\frac{(\nabla^p f)(b, D^\alpha(b)) - \sum_{|\beta| \le p-|\alpha|} \frac{1}{\beta!}(b-a)^\beta (\nabla^p f)(a, D^{\alpha+\beta}(a))}{|b-a|^{p-|\alpha|}} \longrightarrow 0$$

as a, $b \to c$ in X, $a \ne b$; in other words,

$$F^\alpha(b) - \sum_{|\beta| \le p-|\alpha|} \frac{1}{\beta!} F^{\alpha+\beta}(a)(b-a)^\beta \ = \ o(|b-a|^{p-|\alpha|}) \ ,$$

as required. □

Remark 4.22. The following is a simple generalization of Theorem 4.21: Consider $1 \leq m \leq n$ and $X \subset U \times \{0\} \subset \mathbb{R}^m \times \mathbb{R}^{n-m} = \mathbb{R}^n$, where U is open in $\mathbb{R}^m$ and $X = \overline{\operatorname{int} X}$ as a subset of U. Let $f : X \to \mathbb{R}$. Suppose that $\nabla^p f : \tau^p(X) \to \mathbb{R}$. Then there is a $\mathcal{C}^p$ Whitney field $F = (F^\alpha)_{\alpha \in \mathbb{N}^n, |\alpha| \leq p}$ on X such that $F^0 = f$ and $F^\alpha = 0$ when $\alpha_{m+1} + \cdots + \alpha_n > 0$.

Definition 4.23. Let X, Y denote metric spaces and V, W finite-dimensional real vector spaces. Let $E \subset X \times V$ and $F \subset Y \times W$ be bundles (of linear subspaces of V and W, respectively). A *morphism* $E \to F$ is a continuous mapping $\psi : E \to F$ of the form $\psi(a, v) = (\varphi(a), \psi_1(a, v))$, where $(a, v) \in E$, such that, for all $a \in X$, $\psi_1(a, \cdot) : E_a \to F_{\varphi(a)}$ is linear. An *isomorphism* is a morphism with a continuous inverse (which is necessarily a morphism).

Suppose that U_1, U_2 are open subsets of $\mathbb{R}^n$ and that X_1, X_2 are closed subsets of U_1, U_2, respectively. Let $\sigma : U_1 \to U_2$ be a $\mathcal{C}^p$ diffeomorphism ($p \in \mathbb{N}$) such that $\sigma(X_1) = X_2$. Clearly, σ induces an isomorphism $\sigma_* : \tau^p(X_1) \to \tau^p(X_2)$. (See Theorem 5.2 below.) If $f_1 : X_1 \to \mathbb{R}$ and $f_2 : X_2 \to \mathbb{R}$ are functions such that $f_1 = f_2 \circ \sigma$, then σ induces an isomorphism $\sigma^* : \nabla^p f_1 \to \nabla^p f_2$. These observations can be used to generalize the results above to manifolds. Theorem 1.2 is a special case of the following.

Theorem 4.24. *Let $X \subset M \subset U \subset \mathbb{R}$, where U is open, M is a closed $\mathcal{C}^p$ submanifold of U, and $X = \overline{\operatorname{int} X}$ as a subset of M. Let $f : X \to \mathbb{R}$. If $\nabla^p f : \tau^p(X) \to \mathbb{R}$, then f is the restriction of an element of $\mathcal{C}^p(U)$.*

5. Composite functions

Let U, V be open subsets of $\mathbb{R}^n$, $\mathbb{R}^m$ (respectively) and let $\varphi : V \to U$ be a $\mathcal{C}^p$ mapping. Let $b \in V$, $a = \varphi(b)$. Then φ induces a linear mapping

$$\begin{aligned} \varphi_b^* :\ & \mathcal{P}_p(\mathbb{R}^n) \to \mathcal{P}_p(\mathbb{R}^m) \\ & P \mapsto T_b^p(P \circ \varphi)\ ; \end{aligned}$$

i.e., if $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_m)$ denote the coordinates of $\mathbb{R}^n$, $\mathbb{R}^m$ (respectively) and $\varphi = (\varphi_1, \dots, \varphi_n)$, then $\varphi_b^*(P)$ is given by substituting $T_b^p \varphi = (T_b^p \varphi_1, \dots, T_b^p \varphi_n)$ into $P(x) = (T_a^p P)(x)$ and truncating terms involving $(y-b)^\beta$ where $|\beta| > p$. By duality, there is a linear mapping

$$\varphi_{*b} : \mathcal{P}_p(\mathbb{R}^m)^* \to \mathcal{P}_p(\mathbb{R}^n)^* ;$$

i.e., $\varphi_{*b}(\eta)(P) = \eta(\varphi_b^*(P))$, where $\eta \in \mathcal{P}_p(\mathbb{R}^m)^*$ and $P \in \mathcal{P}_p(\mathbb{R}^n)$.

Note that $\varphi_{*b}(\delta_b) = \delta_a$. We will need the following lemma only in the case $k = 1$ (cf. Lemma 4.8 and Remarks 4.13(2)).

Lemma 5.1. *Let X, Y be closed subsets of U, V (respectively), where $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ are open. Let $\varphi : V \to U$ be a $\mathcal{C}^p$ mapping such that $\varphi(Y) \subset X$. Let $(b_{ij}) = (b_{i1}, b_{i2}, \dots)$ and $(\eta_{ij}) = (\eta_{i1}, \eta_{i2}, \dots)$ denote sequences in Y and $\mathcal{P}_p(\mathbb{R}^m)^*$, respectively, for $i = 0, 1, \dots, k$ such that:*

(1) The sequences (b_{ij}), $i = 0, 1, \dots, k$, converge to a common point $b \in Y$, and $\left(\sum_{i=0}^k \eta_{ij}\right)$ converges to $\eta \in \mathcal{P}_p(\mathbb{R}^m)^*$.

(2) $|b_{ij} - b_{0j}|^{p-|\beta|} |\eta_{ij,\beta}(b_{ij})| \le c$, for all i, j and $\beta \in \mathbb{N}^m$, $|\beta| \le p$, where c is a constant.

Set $a_{ij} = \varphi(b_{ij}) \in X$ and $\xi_{ij} = \varphi_{*b_{ij}}(\eta_{ij})$, for all i, j, and set $a = \varphi(b)$, $\xi = \varphi_{*b}(\eta)$. Then:

(1′) (a_{ij}) converges to $a \in X$, for all $i = 0, 1, \dots, k$, and $\left(\sum_{i=0}^k \xi_{ij}\right)$ converges to ξ.

(2′) $|a_{ij} - a_{0j}|^{p-|\alpha|} |\xi_{ij,\alpha}(a_{ij})| \le c'$, for all i, j and $\alpha \in \mathbb{N}^n$, $|\alpha| \le p$, where c' is a constant.

Proof. Obviously, each (a_{ij}) converges to a. Let $P \in \mathcal{P}_p(\mathbb{R}^n)$. Then, for each j,

$$\begin{aligned} \xi(P) - \sum_{i=0}^k \xi_{ij}(P) &= \eta(\varphi_b^*(P)) - \sum_{i=0}^k \eta_{ij}(\varphi_{b_{ij}}^*(P)) \\ &= \eta(G, b) - \sum_{i=0}^k \eta_{ij}(G, b_{ij}) \ , \end{aligned}$$

where G denotes the $\mathcal{C}^p$ Whitney field on Y induced by $P\circ\varphi$. Therefore $(1')$ follows from Lemma 4.8.

There is a constant C such that, for all i, j and $\alpha\in\mathbb{N}^n$, $|\alpha|\le p$,

$$\begin{aligned}
&|a_{ij}-a_{0j}|^{p-|\alpha|}|\xi_{ij,\alpha}(a_{ij})|\\
&\quad\le\; C|b_{ij}-b_{0j}|^{p-|\alpha|}\left|\eta_{ij}\left(\varphi^*_{b_{ij}}\left(\frac{1}{\alpha!}(x-a_{ij})^\alpha\right)\right)\right|\\
&\quad\le\; C|b_{ij}-b_{0j}|^{p-|\alpha|}\sum_{\substack{\beta\in\mathbb{N}^m\\ |\alpha|\le|\beta|\le p}}\Lambda_\beta\Big(\big(D^\gamma\varphi(b_{ij})\big)_{1\le|\gamma|\le p}\Big)\big|\eta_{ij,\beta}(b_{ij})\big|\;,
\end{aligned}$$

where each Λ_β is a polynomial function. Therefore, $(2')$ follows from (2). □

Theorem 5.2. *Suppose $X\subset U\subset\mathbb{R}^n$ and $Y\subset V\subset\mathbb{R}^m$, where U, V are open and X, Y are closed in U, V (respectively). Let $\varphi: V\to U$ be a $\mathcal{C}^p$ mapping such that $\varphi(Y)\subset X$. Then:*

(1) φ induces a bundle morphism

$$\varphi_*:\;\tau^p(Y)\to\tau^p(X)$$

such that , if $b\in Y$ and $\eta\in\tau^p_b(Y)\subset\mathcal{P}_p(\mathbb{R}^m)^*$, then

$$\varphi_*(\eta)\;=\;\varphi_{*b}(\eta)\;.$$

Moreover, let $f: X\to\mathbb{R}$ and set $g=f\circ\varphi: Y\to\mathbb{R}$. Suppose that $\nabla^p f:\tau^p(X)\to\mathbb{R}$. Then:

(2) $\nabla^p g:\tau^p(Y)\to\mathbb{R}$ and, if $b\in Y$ and $\eta\in\tau^p_b(Y)$, then

$$\nabla^p g(\eta)\;=\;\nabla^p f(\varphi_*\eta)\;.$$

(3) Let $b\in Y$ and $a=\varphi(b)$. Choose $P\in\mathcal{P}_p(\mathbb{R}^n)$ such that $P|\tau^p_a(X):\tau^p_a(X)\to\mathbb{R}$ coincides with $\nabla^p_a f$ (where we have identified $\mathcal{P}_p(\mathbb{R}^n)$ with $\mathcal{P}_p(\mathbb{R}^n)^{**}$). Then, for all $\eta\in\tau^p_b(Y)\subset\mathcal{P}_p(\mathbb{R}^m)^*$,

$$\nabla^p g(\eta)\;=\;\eta\big(\varphi^*_b(P)\big)\;.$$

Proof. (1) follows from Lemma 5.1 and the definition of the paratangent bundle in the same way that Lemma 4.17 is proved above using Lemma 4.8.

Consider $b_{ij} \in Y$ and $\eta_{ij} \in \tau^p_{b_{ij}}(Y) \subset \mathcal{P}_p(\mathbb{R}^m)^*$, $i = 0, 1$, $j = 1, 2, \ldots$, satisfying the hypotheses of Lemma 5.1 (case $k = 1$). Let a_{ij}, ξ_{ij}, a and ξ be as in Lemma 5.1. Then, by Lemma 5.1 and Remark 4.19,

$$\begin{aligned}
\nabla^p f(\varphi_{*b}(\eta)) &= \nabla^p f(\xi) \\
&= \lim_{j\to\infty} \big(\nabla^p f(\xi_{0j}) + \nabla^p f(\xi_{1j})\big) \\
&= \lim_{j\to\infty} \big(\nabla^p f\big(\varphi_{*b_{0j}}(\eta_{0j})\big) + \nabla^p f\big(\varphi_{*b_{1j}}(\eta_{1j})\big)\big) \ .
\end{aligned}$$

(2) then follows in the same way that (1) is proved.

To prove (3): Let $\eta \in \tau^p_b(Y)$ and let $\xi = \varphi_{*b}(\eta) \in \tau^p_a(X) \subset \mathcal{P}_p(\mathbb{R}^n)^*$. Then $\nabla^p f(\xi) = \xi(P)$, by the choice of P, so that

$$\begin{aligned}
\nabla^p f(\varphi_{*b}(\eta)) &= \nabla^p f(\xi) \\
&= \varphi_{*b}(\eta)(P) \\
&= \eta\big(\varphi^*_b(P)\big) \ ,
\end{aligned}$$

and the result follows from (2). □

Corollary 5.3. *Suppose $X \subset U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$, where U, V are open and X is closed in U. Let $\varphi : V \to U$ be a $\mathcal{C}^p$ mapping such that $\varphi(V) \subset X$. Let $f : X \to \mathbb{R}$ and set $g = f \circ \varphi$. Suppose that $\nabla^p f : \tau^p(X) \to \mathbb{R}$. Then:*

(1) $g \in \mathcal{C}^p(V)$.

(2) g is formally a composite with φ; i.e., for all $a \in X$, there exists $P \in \mathcal{P}_p(\mathbb{R}^n)$ such that $g - P \circ \varphi$ is p-flat at every point $b \in \varphi^{-1}(a)$.

Proof. (1) follows from Theorem 5.2 (2) and Theorem 4.21.

Let $a \in X$. Choose $P \in \mathcal{P}_p(\mathbb{R}^n)$ as in Theorem 5.2 (3). Let $b \in \varphi^{-1}(a)$. We will show that

$$T^p_b g = T^p_b(P \circ \varphi) \ . \tag{5.4}$$

Since $T^p_b(P\circ\varphi) = \varphi^*_b(P)$, (5.4) means that, for all $\eta \in \tau^p_b(V) = \mathcal{P}_p(\mathbb{R}^m)^*$,

$$\eta(T^p_b g) \;=\; \eta\big(\varphi^*_b(P)\big)\ .$$

But $\eta(T^p_b g) = (\nabla^p g)(\eta)$, by Theorem 4.18, so the result follows from Theorem 5.2 (3). □

Differentiable functions on closed subanalytic sets.

Proof of Theorem 1.3. Let X be a compact subanalytic subset of $\mathbb{R}^n$. By [BM1, Thm. 0.1], there is a compact real analytic manifold M and a real analytic mapping $\varphi : M \to \mathbb{R}^n$ such that $\varphi(M) = X$. By [BMP], there is a function $q = q_\varphi(p)$ from $\mathbb{N}$ to itself such that, if $g \in \mathcal{C}^q(M)$ and g is formally a composite with φ, then there exists $F \in \mathcal{C}^p(\mathbb{R}^n)$ such that $g = F\circ\varphi$; moreover, if S is a finite subset of X and g is q-flat on $\varphi^{-1}(S)$, then there exists F with the additional property that F is p-flat on S.

Let $f : X \to \mathbb{R}$. Let $p \in \mathbb{N}$ and suppose that $\nabla^q f : \tau^q(X) \to \mathbb{R}$, where $q = q_\varphi(p)$. Let $g = f\circ\varphi$. By Corollary 5.3 (generalized to a manifold V), $g \in \mathcal{C}^q(M)$ and g is formally a composite with φ. Therefore, $f \in \mathcal{C}^p(X)$. □

Remark 5.4. If X is a closed subanalytic subset of $\mathbb{R}^n$, then $\tau^p(X)$ is a closed subanalytic subset of $\mathbb{R}^n \times \mathcal{P}_p(\mathbb{R}^n)^*$.

Final Remarks 5.5. (1) Let $X \subset U \subset \mathbb{R}^n$, where U is open and X is closed in U. Let $f : X \to \mathbb{R}$. Our definitions of $\tau^p(X)$ and $\nabla^p f$ involve limits of distributions with values in $\mathcal{P}_p(\mathbb{R}^n)^*$ supported at two points. We can generalize the definitions (and all constructions in the article) by using distributions supported at $k+1$ points, for any $k = 1, 2, \ldots$. We simply modify (4.9) and (4.10) in the following way: Let $E \subset X \times \mathcal{P}_p(\mathbb{R}^n)^*$ be any bundle of linear subspaces of $\mathcal{P}_p(\mathbb{R}^n)^*$ over X. Define

(5.6)
$$\begin{aligned}\Delta_{k+1}E \;:=\; \{(a_0, a_1, \ldots, a_k, \xi_0+\xi_1+\cdots+\xi_k) :\ & a_i \in X\ ,\ \xi_i \in E_{a_i}\ ,\\ & |a_i - a_0|^{p-|\alpha|}|\xi_{i\alpha}(a_i)| \le 1\ ,\ \text{for all } |\alpha| \le p\ ,\ i = 0, \ldots, k\}\ ;\end{aligned}$$

(5.7)
$$E'_{k+1} \;:=\; \pi(\overline{\Delta_{k+1}E} \cap \{(a, a, \ldots, a, \xi) :\ a \in X\ ,\ \xi \in \mathcal{P}_p(\mathbb{R}^n)^*)\ ,$$

where π is the projection $\pi(a_0, a_1, \ldots, a_k, \xi) = (a_0, \xi)$. We define $\tau^p_{k+1}(X) = \widehat{E}$ as before, and $\nabla^p_{k+1} f \subset \tau^p_{k+1}(X) \times \mathbb{R}$ also in a similar way.

Of course $\tau^p_k(X) \subset \tau^p_{k+1}(X)$ and $\nabla^p_k f \subset \nabla^p_{k+1} f$, for all $k \geq 2$; in particular, if $\nabla^p_{k+1} f : \tau^p_{k+1}(X) \to \mathbb{R}$, then $\nabla^p_k f : \tau^p_k(X) \to \mathbb{R}$. We have used only $\tau^p(X) = \tau^p_2(X)$ in this article because it suffices for all the results. Our main conjecture in Section 1 should be understood as requiring $\tau^p_{k+1}(X)$, where $k \geq p$. (For example, if

$$X \;=\; \bigcup_{i=0}^{p} \{(x,y) \in \mathbb{R}^2 :\; y = ix^2\} \;,$$

then $k = p$ is necessary and sufficient.)

Questions. Does there exist $r = r(X, p) \in \mathbb{N}$ such that $\tau^p_k(X) = \tau^p_r(X)$ if $k \geq r$? If X is subanalytic, can we take $r = p + 1$?

(2) It is not difficult to see that the definition of $\Delta_{k+1} E$ above is equivalent to that given by replacing the condition

$$|a_i - a_0|^{p-|\alpha|} |\xi_{i\alpha}(a_i)| \;\leq\; 1 \;, \quad i = 0, \ldots, k \;,$$

by the condition

$$|a_i - a_0|^{p-|\alpha|} |\xi_{i\alpha}(a_0)| \;\leq\; 1 \;, \quad i = 0, \ldots, k \;.$$

(Likewise in Lemma 4.8). It is not possible, however, to define $\tau^p_{k+1}(X)$ using limits

$$\xi \;=\; \lim_{j \to \infty} \sum_{i=0}^{k} \xi_{ij}$$

(in the notation of Lemma 4.8) where condition (2) of Lemma 4.8 is replaced by the symmetric condition

$$|a_{ij} - a|^{p-|\alpha|} |\xi_{ij,\alpha}(a_{ij})| \;\leq\; c \;, \tag{5.8}$$

for all i, j and $|\alpha| \leq p$.

For example, let $(x_1, y_1) = (1, 1)$ and, for each $j = 1, 2, \dots$, define (x_{j+1}, y_{j+1}) inductively as follows: If j is odd (respectively, even), let (x_{j+1}, y_{j+1}) be the intersection point of the line through (x_j, y_j) with slope 2 (respectively, -2) and the arc $y = -x^2$, $x > 0$ (respectively $y = x^2$, $x > 0$). Let $X = \{0\} \cup \{x_j : j \geq 1\} \subset \mathbb{R}$. Define $F^0(0) = 0$, $F^0(x_j) = y_j$, for all j, and $F^1(a) = 0$, for all $a \in X$. Then

$$\lim_{j\to\infty} \frac{(R_0^1 F)^0(x_j)}{|x_j - 0|} = 0 ,$$

but

$$\frac{(R_0^1 F)^0(x_j)}{|x_j - x_{j+1}|}$$

does not tend to zero as $j \to \infty$, so that F is not a Whitney field. Take $a_{0j} = x_j$, $a_{1j} = x_{j+1}$, $\xi_{0j} = \dfrac{\delta_{x_j}}{x_j - x_{j+1}}$ and $\xi_{1j} = \dfrac{\delta_{x_{j+1}}}{x_j - x_{j+1}}$, for all j. Then the condition (2) of Lemma 4.8 (case $k = 1$) is satisfied, but not the symmetric condition (5.8).

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