METRIC CURVATURE AND CONVERGENCE

by

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We present conditions under which metric curvature bounds and geodesic completeness are preserved in Gromov-Hausdorff limits. In particular we show that if $X_i \to X$ and the "comparison radius" for bounded curvature in each X_i is uniformly bounded below by some $r > 0$, then X has the same curvature bound, with comparison radius at least r. If in addition the X_i are geodesically complete, then X is also geodesically complete, provided the injectivity radii inj (X_i) have a uniform positve lower bound.

As a consequence of these general results we obtain a "Compactness Theorem" (Theorem 9) for geodesically complete inner metric spaces having curvature and injectivity radius bounded below, and diameter bounded above. Since Theorem 9 does not require an upper curvature bound, it is both a weakened generalization of the Convergence Theorem for Riemannian manifolds ([Pe1], [GW], [Ka]) and a generalization (with exceptional dimensions 3 and 4) of Cheeger's Finiteness Theorem ([C], [Pe2]). Our results also allow an entirely "metric" proof of the Convergence Theorem using [N1] (cf. also [N2]).

Our result on the lower curvature bound (Proposition 6) was essentially proved in [GP], §2, *under the additional assumption that each X_i is geodesically complete*. Our removal of this extra assumption is important for the following reason: In [P2] we prove a generalization of Toponogov's Theorem for convex, geodesically complete inner metric spaces of curvature $\geq k$, which essentially states that the (lower curvature bound) comparison radius for such a space is infinite. It is not known if the theorem can be further generalized by weakening geodesic completeness to metric completeness. Theorem 8 c) shows that the class of spaces for which Toponogov's Theorem holds is Gromov-Hausdorff closed, making a counterexample, if one exists, more difficult to find.

A few definitions and will be recalled below. For more details, and examples, see [P1]. All curves are assumed parameterized proportional to arclength. We assume throughout that X is a metrically complete, convex inner metric space. Then by definition every pair of points $x, y \in X$ can be connected by a *minimal curve* whose length realizes $d(x, y)$; a curve which is locally minimizing is called a *geodesic*. The notation γ_{ab} is reserved for a geodesic from a to b. A *geodesic terminal* is a point in X beyond which some geodesic cannot be extended. An open subset U of X is *geodesically complete* if it has no geodesic terminals.

S_k will denote the simply connected, 2-dimensional space form of curvature k. By *monotonicity* we mean the well known fact that the angle between two minimal curves of fixed length in S_k is a monotone increasing function of the distance between the endpoints opposite the angle.

Definition 1. An open set U in X is said to be a *region of curvature* $\geq k$ (resp. $\leq k$) if for every triangle $(\gamma_{ab}, \gamma_{bc}, \gamma_{ca})$ of minimal curves in U,

a) there exists a representative $(\tilde{\gamma}_{AB}, \tilde{\gamma}_{BC}, \tilde{\gamma}_{CA})$ in S_k (i.e., $\tilde{\gamma}_{AB}, \tilde{\gamma}_{BC}, \tilde{\gamma}_{CA}$ are minimal of the same length as their correspondent curves) and

b) for any y on γ_{ab} and Y on $\tilde{\gamma}_{AB}$ such that $d(y, a) = d(Y, A)$, $d(y, c) \geq d(Y, C)$ (resp. $d(y, c) \leq d(Y, C)$).

If x is contained in a region of curvature $\geq k$, let $c_k(x) = \sup\{r : B(x, r)$ is a region of curvature $\geq k\}$, and put $c_k(x) = 0$ otherwise. Then the *comparison radius* c_k is continuous or $c_k = \infty$. If for all $x \in X$ there is a k such that $c_k(x) > 0$ then we say X has *curvature locally bounded below.* If for some fixed $p \in X$ and k, c_k has a positive lower bound on $B(p, r)$ for all r, we say X has *curvature uniformly* $\geq k$. If X is locally compact, curvature uniformly $\geq k$ is equivalent to $c_k > 0$ on X. Using regions of curvature $\leq k$ we can similarly define c^k and curvature locally or uniformly bounded above.

Monotonicity implies that a region of curvature $\geq k$ (resp. $\leq k$) contains a *Gebiet der Riemannscher Krümmung* $\geq k$ (resp. $\leq k$) in the sense of [R] (and conversely, such a *Gebiet* satisfies Definition 1); therefore the *angle* $\alpha(\gamma_1, \gamma_2)$ between two geodesics exists and is a bona fide metric on the *space of* *directions* S_p (unit geodesics) at a point $p \in X$. For $\gamma \in S_p$ we let $C(\gamma) = \sup\{t : \gamma|_{[0,t]}$ is minimal$\}$ and define the *injectivity radius* $inj\ (p) = \inf\{C(\gamma) : \gamma \in S_p\}$. Finally, we let $inj\ (X) = \inf\{inj\ (p) : p \in X\}$.

Definition 2. We say that a (geodesic) triangle $(\gamma_1, \gamma_2, \gamma_3)$ in X is *T1(k)* (resp. *A1(k)*) if there exists a representative triangle $(\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3)$ in S_k and $\alpha(\bar{\gamma}_i, \bar{\gamma}_2) \le \alpha(\gamma_i, \gamma_2)$ (resp. $\alpha(\bar{\gamma}_i, \bar{\gamma}_2) \ge \alpha(\gamma_i, \gamma_2)$) for $i = 1, 3$. We say that a (geodesic) wedge $(\gamma_{ab}, \beta_{ac})$ is *T2(k)* (resp. *A2(k)*) if there is a representative wedge $(\tilde{\gamma}_{AB}, \tilde{\beta}_{AC})$ in S_k (i.e., whose sides are minimal with $L(\tilde{\gamma}_{AB}) = L(\gamma_{ab})$, $L(\tilde{\beta}_{AC}) = L(\beta_{ac})$, $\alpha(\tilde{\gamma}_{AB}, \tilde{\beta}_{AC}) = \alpha(\gamma_{ab}, \beta_{ac})$) and $d(B, C) \ge d(b, c)$ (resp. $d(B, C) \ge d(b, c)$).

When no confusion is likely to result we omit the "k" from the notation, e.g. writing "T1" for "T1(k)."

A triangle $(\gamma_1, \gamma_2, \gamma_3)$ or wedge (γ_1, γ_2) is $(k\text{-})$*proper* if γ_1 and γ_3 are minimal and $L(\gamma_2) < \pi/\sqrt{k}$. If $c_k(x) = \rho > 0$ (resp. $c^k(x) = \rho > 0$) then every proper triangle in $B(x, \rho)$ is T1(k) (resp. A1(k)) and every proper wedge in $B(x, \rho)$ is T2(k) (resp. A2(k)).

The Gromov-Hausdorff metric is defined in [GLP], [G]. Let X_i be a collection of compact, metrically complete inner metric spaces of diameter $\le D < \infty$, which are convernt in the Gromov-Hausdorff metric to a metrically complete metric space X.

By the results of §6 in [G] we can assume that the spaces X_i and X are all embedded in a fixed compact metric space Z, and convergence is in the classical Hausdorff sense.

A prerequisite to understanding the geometry of X is the ability to express a minimal curve in X as a uniform limit of minimal curves in X_i. We now give a condition on X which guarantees that this is possible.

Definition 3. Let Y be and inner metric space. A *closed branch* in Y is a pair of minimal curves α_{ab}, β_{ab} : [0, 1] -> Y such that for some $T \in (0, 1)$, $\alpha_{ab}(t) = \beta_{ab}(t)$ for all $t \geq T$ and $\alpha_{ab}(s) \neq \beta_{ab}(s)$ for some $s < T$. If T is taken to be the smallest such value, the point $c = \alpha_{ab}(T)$ is called the *branch point* of the closed branch.

The non-existence of closed branches in a space Y is equivalent to the following: if γ_{ab} : [0, 1] -> Y is minimal, then for all $t \in (0, 1)$, $\gamma_{ab}|_{[0,t]}$ is the *unique* minimal curve from a to $\gamma_{ab}(t)$.

Proposition 4. *If X has no closed branches then for every minimal curve γ in X there exist minimal curves γ_i in X_i such that some subsequence of $\{\gamma_i\}$ converges uniformly to γ. Furthermore, if α is minimal in X and has a common endpoint with γ, then each approximating curve α_i can be chosen to have the corresponding endpoint in common with γ_i.*

Proof. Let $\gamma : [0, 1] \to X$ be minimal from a to b. Suppose $b_j = \gamma(t_j)$ with $b_j \to b$. Let a_i, $b_{ji} \in X_i$ be such that $a_i \to a$ and $b_{ji} \to b_j$ for all j, and let γ_j denote the segment of γ from a to b_j. Let $\gamma_{ji} : [0, 1] \to X_i$ be minimal from a_i to b_{ji}. For fixed j, the curves γ_{ji} are arclength parameterized and have lengths uniformly bounded above, so we can apply Ascoli's Theorem to obtain a convergent subsequence of $\{\gamma_{ji}\}$. Such a subsequence converges uniformly to a minimal curve from a to b_j, and since γ_j is the unique such curve, the subsequence must in fact converge uniformly to γ_j. We now choose the desired approximating sequence from $\{\gamma_{ji}\}$ using a standard "diagonal" argument.

The second part of the proposition is obvious from the above construction. □

We will show that the hypothesis to Proposition 4 is satisfied in any region which is a limit of regions having curvature bounded either above or below: in the first case, minimal curves do not "close," and in the second they do not "branch."

Proposition 5. *Suppose $x \in X$ and there exist $x_i \in X_i$ such that $x_i \to x$ and for some fixed $0 < r \le \pi/2\sqrt{k}$ and all i, $c_k(x_i) \ge r$. Then $B(x, r)$ contains no closed branches, and $c^k(x) \ge r$.*

Proof. Suppose $\alpha, \beta : [0, 1] \to X$ are minimal curves from a to b. Let a_i, $b_i \in X_i$ with $a_i \to a$ and $b_i \to b$, and α_i be minimal from a_i to b_i. Passing to a subsequence we can assume α_i is uniformly convergent to a minimal curve from a to b; changeing curves if necessary we can assume that $\alpha_i \to \alpha$. Suppose there exists some $t \in [0, 1]$ such that $c = \alpha(t) \neq \beta(t) = c'$. Let $c_i = \alpha_i(t)$ and $c'_i \in X_i$ be such that $c'_i \to c'$. Let η_i be minimal from a_i to c'_i and ν_i be minimal from c'_i to b_i. Then since $L(\eta_i) + L(\nu_i) \to L(\alpha_i)$, if $(\bar{\eta}_i, \bar{\alpha}_i, \bar{\nu}_i)$ represents $(\eta_i, \alpha_i, \nu_i)$ in S_k, $\lim_{i\to\infty} \alpha(\bar{\eta}_i, \bar{n}_i) = \pi$. Since $L(\alpha) < \pi/\sqrt{k}$, Definition 1 implies that $\lim_{i\to\infty} d(c_i, c'_i) = 0$, a contradiction. Therefore α and β are identical, and cannot form a closed branch.

The second part of the proposition follows immediately from Proposition 4 and Definition 1. □

Proposition 6. *Suppose $x \in X$ and there exist $x_i \in X_i$ such that $x_i \to x$ and for some fixed $r > 0$ and all i, $c_k(x_i) \geq r$. Then $B(x, r)$ contains no closed branches and $c_k(x) \geq r$.*

Proof. Let $\alpha, \beta : [0, 1] \to X$ be minimal such that $\alpha(t) = \beta(t)$ for all $T \leq t \leq 1$ for some $T \in (0, 1)$. For some $s < T$, let $a = \alpha(s)$, $b = \beta(s)$, $c = \alpha(T)$, and $d = \alpha(1)$, and choose a_i, b_i, c_i, $d_i \in X_i$ converging to the respective points in X. Let η_i, ν_i be minimal from b_i and a_i, respectively, to c_i, and ζ_i be minimal from c_i to d_i. Now $\lim_{i\to\infty} [L(\eta_i) + L(\zeta_i)] = \lim_{i\to\infty} [L(\nu_i) + L(\zeta_i)] = \lim_{i\to\infty} d(a_i, d_i) = \lim_{i\to\infty} d(b_i, d_i)$, and T1 implies that $\lim_{i\to\infty} \alpha(\eta_i, \zeta_i) = \lim_{i\to\infty} \alpha(\nu_i, \zeta_i) = \pi$. By Lemma 2.3, [P2], $\lim_{i\to\infty} \alpha(\eta_i, \nu_i) = 0$. T2 now implies that $\lim_{i\to\infty} d(a_i, b_i) = 0$; in other words $\alpha(s) = \beta(s)$. Since this holds for all $s < T$, α and β coincide, and cannot form a closed branch.

The second part of the proposition again follows from Proposition 4 and Definition 1. □

Proposition 7. *Suppose that for some $r, \epsilon > 0$, $B(x_i, r)$ is geodesically complete and $inj \geq \epsilon$ on $B(x_i, r)$. Then if $B(x, r)$ has no closed branches, $B(x, r)$ is geodesically complete and every geodesic of length $\leq \epsilon$ in $B(x, r)$ is minimal.*

Proof. It suffices to prove that every minimal curve $\gamma : [0, 1] \to B(x, r)$ with $L(\gamma) < \epsilon$ can be extended as a minimal curve beyond $\gamma(1)$. Let $\gamma_i : [0, 1] \to B(x_i, r)$ converge uniformly to γ (passing to a subsequence, if necessary). For all sufficiently large i we can extend γ_i past $\gamma_i(1)$ as a minimal curve γ'_i of length min $\{\epsilon, (r - d(\gamma(1), x) / 2\}$. Then $\{\gamma'_i\}$ converges uniformly to a minimal curve which extends γ. □

The next theorem follows immediately from the above three propositions. Note that if $c^k(Y) \geq r$ then inj $(Y) \geq$ min $\{r, \pi/\sqrt{k}\}$.

Theorem 8. *For fixed* ϵ, $r > 0$ *and* k, *the following classes of locally compact, metrically complete, inner metric spaces are Gromov-Hausdorff closed:*

a) $\{Y : c^k(Y) \geq r,\ r \leq \pi/2\sqrt{k}\}$,

b) $\{Y : c^k(Y) \geq r,\ r \leq \pi/\sqrt{k},\ Y$ *geodesically complete*$\}$,

c) $\{Y : c_k(Y) \geq r\}$,

d) $\{Y : c_k(Y) \geq r,\ inj\ (Y) \geq \epsilon,\ Y$ *geodesically complete*$\}$.

The Convergence Theorem is an immediate consequence corollary of Theorem 8 and [N1], where it is proved that a geodesically complete inner metric space with curvature bounded above and below is a smooth manifold with $C^{1,\alpha}$ Riemannian metric.

A locally compact, metrically complete inner metric space Y is called *almost Riemannian* ([P2]) if Y is finite dimensional, geodesically complete, and has curvature locally bounded below. We let A(n, k, ϵ, D) denote the class of almost Riemannian spaces having dimension n, curvature $\geq$ k, injectivity radius $\geq \epsilon$, and diameter $\leq$ D, and endow A(n, k, ϵ, D) with the Gromov-Hausdorff metric.

Theorem 9. *A(n, k,* ϵ*, D) is compact space of topological manifolds, which, except possibly for* $n = 4$, *are smooth with continuous Riemannian metric. A(n, k,* ϵ*, D) has finitely many homotopy types for any n, finitely many homeomorphism types for* $n \neq 3$, *and finitely many diffeomorphism types for* $n \neq 3, 4$.

Proof. By Theorem G, [P2], any $Y \in A(n, k, \epsilon, D)$ is a topological manifold with continuous fiber metric on its topological tangent bundle TY. In particular, if $n \neq 4$ Y is smooth and the metric is a continuous Riemannian metric. $A(n, k, \epsilon, D)$ is precompact by Theorem H, [P2]. By Theorem 8, to prove compactness we need only show that if $X \in \bar{A}(n, k, \epsilon, D)$ then $\dim X = n$. Choose $X_i \in A(n, k, \epsilon, D)$ with $X_i \to X$ and $p_i \in X_i$ with $p_i \to p$. Let $\alpha_{ji} \in S_{p_i}$ be an orthonormal basis for T_{p_i}; choosing a subsequence if necessary we can assume $\{\alpha_{ij}\}$ converges uniformly to some $\alpha_i \in S_p$ for all j. Then by T1 and the uniform convergence, $\alpha(\alpha_j, \alpha_k) \leq \lim_{i\to\infty} \alpha(\alpha_{ij}, \alpha_{ik}) = \pi/2$. Since X_i and X are geodesically complete, we can apply the same argument to complementary angles and obtain $\pi - \alpha(\alpha_j, \alpha_k) \leq \lim_{i\to\infty} \pi - \alpha(\alpha_{ij}, \alpha_{ik}) = \pi/2$. In other words $\alpha_1, \ldots, \alpha_n$ spans an n-dimensional subspace of T_p and $\dim X \geq n$. To show $\dim X \leq n$ we note that given independent $\alpha_1, \ldots, \alpha_m \in S_p$ we can approximate each α_j uniformly by $\alpha_{ji} \in X_i$, and repeat the above argument to show $\dim X_i \geq m$ for sufficiently large i.

The finiteness parts of the theorem follow from the fact that elements of $A(n, k, \epsilon, D)$ have injectivity radius $\geq \epsilon$ (and so are all $LGC(\rho)$ with $\rho(r) = r$ on $[0, \epsilon/2]$), and the general finiteness theorems of [PV] and [GPW]. Note that $\dim X \leq n$ also follows from [PV]. □

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