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by

Dan Burghelea
Stefan Haller

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Dan Burghelea
Stefan Haller

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Department of Mathematics
The Ohio State University
231 West 18th Avenue
Columbus, OH 43210
USA

Department of Mathematics
University of Vienna
Nordbergstrasse 15
1090 Vienna
Austria

MPIM 13-12

TOPOLOGY OF ANGLE VALUED MAPS, BAR CODES AND JORDAN BLOCKS.

DAN BURGHELEA AND STEFAN HALLER

Abstract. In this paper one presents a collection of results relating the "bar codes" and "Jordan blocks", a new class of invariants for a tame angle valued map, with the topology of underlying space (and map). As a consequence one proposes refinements of Betti numbers and Novikov–Betti numbers provided by a continuous real or angle valued map defined on a compact ANR. These refinements can be interpreted as monic polynomials of degree the Betti numbers or Novikov–Betti numbers. One shows that these polynomials depend continuously on the real or the angle valued map and satisfy a Poincaré duality property in case the underlying space is a closed manifold. Our work offers an alternative perspective on Morse–Novikov theory which can be applied to a considerably larger class of spaces and maps and provides features inexistent in classical Morse–Novikov theory.

Contents

1. The results

In this paper a *nice space* is a friendlier name for a locally compact ANR. In particular a metrizable, locally compact, finite dimensional locally contractible space is *nice*. Finite dimensional simplicial complexes and finite dimensional topological manifolds are nice spaces but the class is considerably larger. A *tame* map is a

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proper continuous map $f : X \to \mathbb{R}$ or $f : X \to \mathbb{S}^1$, defined on a nice space X which satisfies:

(i) each fiber of f is a neighborhood deformation retract,
(ii) away from a discrete set $\Sigma \subset \mathbb{R}$ or $\Sigma \subset \mathbb{S}^1$ the restriction of f to $X \setminus f^{-1}(\Sigma)$ is an Hurewicz fibration, cf. [1].

All proper simplicial maps, and proper smooth generic maps defined on a smooth manifold [1], in particular proper real or angle valued Morse maps, are tame.

The subspace of tame maps is residual in the space of continuous maps when equipped with the compact open topology and weakly homotopy equivalent to the space of all continuous maps (equipped with compact open topology)[2].

Since our invariants are based on homology we fix once for all a field κ and write $H_r(X)$ for the singular homology of X with coefficients in κ. A *vector space* without additional specifications will be *over the field* κ.

We consider a tame map, $f : X \to \mathbb{S}^1$, and as in [1] associate to it:

(i) the critical angles $0 < \theta_1 < \theta_2 < \cdots < \theta_m \leq 2\pi$,
and for any $r = 0, 1, \ldots, \dim X$,
(ii) four type of intervals of real numbers, subsequently called *r-bar codes*, $r = 0, 1, \cdots$ whose ends mod 2π are the critical angles
(1) closed $[a, b]$,
(2) open (a, b),
(3) closed–open $[a, b)$,
(4) open–closed $(a, b]$,
and
(iii) a collection of *Jordan blocks*, i.e. isomorphism classes of indecomposable pairs $J = (V, T)$, V a finite dimensional κ-vector space, T a linear isomorphism.

We will denote by $\mathcal{B}^c_r(f)$, $\mathcal{B}^o_r(f)$, $\mathcal{B}^{co}_r(f)$, $\mathcal{B}^{oc}_r(f)$ the collections (multisets) of closed, open, closed-open and open-closed r-bar codes and by $\mathcal{J}_r(f)$ the collection of r-Jordan blocks. Each bar code or Jordan block appears in its collection with a multiplicity possibly larger than one. For $u \in \kappa \setminus 0$ we denote by $\mathcal{J}_{r,u}(f)$ the sub collection $\{(V, T) \in \mathcal{J}_r(f) \mid u \in \text{spect}(T)\}$.

In the Appendix the reader can see an example. As shown in[1] these invariants are effectively computable.

In this paper the bar codes will be recorded as the finite configurations of points in $\mathbb{C} \setminus 0$, denoted by $C_r(f)$ and $C^m_r(f)$[3]respectively, see below.

A pair (V, T) as in (iii) above is *indecomposable* if not isomorphic to the sum of two nontrivial pairs. Note that if T has $\lambda \in \kappa$ as an eigenvalue all other eigenvalues

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[1]here "generic" means that for any $x \in M$ the quotient algebra of germs of smooth functions at x by the ideal of partial derivatives is a finite dimensional vector space

[2]we are unable to locate a reference in literature for this statement, however in case that the space X is homeomorphic to a finite simplicial complex, it is a straightforward consequence of the approximability of continuous maps by pl-maps

[3]actually $C^m_r(f)$ is a configuration of points in $\mathbb{C} \setminus \{S^1 \sqcup 0\}$

are equal to λ and (V,T) is isomorphic to $(\kappa^k, T(\lambda,k))$ where

$$T(\lambda,k)=\begin{pmatrix}\lambda & 1 & 0 & \cdots & 0\\ 0 & \lambda & 1 & \ddots & \vdots\\ 0 & 0 & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \lambda & 1\\ 0 & \cdots & 0 & 0 & \lambda\end{pmatrix}. \tag{1}$$

In [1] the indecomposable pairs $(\kappa^k, T(\lambda,k))$ were called *Jordan cells.* When κ is algebraically closed all Jordan blocks are Jordan cells.

Each tame map with X compact has finitely many bar codes and Jordan blocks.

These type of invariants, are based on changes in the homology of the fibers and have been introduced in [4] and [1] using graph representations (in [4] only for real valued maps).

Let $\xi_f \in H^1(X;\mathbb{Z})$ be the integral cohomology class represented by f. The first result we prove in this paper is:

Theorem 1.1 (Homotopy invariance)**.** *If $f\colon X\to \mathbb{S}^1$ is a tame map then:*

(1) $\sharp\mathcal{B}^c_r(f)+\sharp\mathcal{B}^o_{r-1}(f)$ is a homotopy invariant of the pair (X,ξ_f), more precisely equal to the Novikov–Betti number $\beta^N_r(X,\xi_f)$ (see the definition in Section 4).

(2) The collection $\mathcal{J}_r(f)$ is a homotopy invariant of the pair (X,ξ_f). More precisely, $\bigoplus_{J\in\mathcal{J}_r}(V(J),T(J))$ is the monodromy of $(X;\xi_f)$ (see the definition in Section 4).

(3) $\sharp\mathcal{B}^c_r(f)+\sharp\mathcal{B}^o_{r-1}(f)+\sharp\mathcal{J}_{r,1}(f)+\sharp\mathcal{J}_{r-1,1}(f)$ is a homotopy invariant of X, more precisely the Betti number $\beta_r(X)$.

Here $\sharp$ denotes cardinality of multi set. Item (3) has been already established in [1] and is included in Theorem 1.1 only for the completeness of the topological information derived from bar codes and Jordan blocks.

In view of Theorem 1.1 it is natural to put together $\mathcal{B}^c_r(f)$ and $\mathcal{B}^o_{r-1}(f)$. For this purpose consider $\mathbb{T}=\mathbb{C}/\mathbb{Z}$ and $\Delta_{\mathbb{T}}=\Delta/\mathbb{Z}$ where the $\mathbb{Z}$-action on $\mathbb{C}$ is given by $(n,z)=z+(2\pi n+i2\pi n)$ and $\Delta=\{z=a+ib\mid a=b\}$. We will record the collections $\mathcal{B}^c_r(f)\sqcup\mathcal{B}^o_{r-1}(f)$ as a finite configuration of points in $\mathbb{T}$, denoted by $C_r(f)$, and the collection $\mathcal{B}^{co}_r(f)\sqcup\mathcal{B}^{oc}_r(f)$ as a finite configuration of points in $\mathbb{T}\setminus\Delta_{\mathbb{T}}$, denoted by $C^m_r(f)$. Precisely in the first case a closed r-bar code $[a,b]$ will be written as the complex number $z=a+ib$ mod the action of $\mathbb{Z}$ and an open $(r-1)$-bar code (α,β) as the complex number $z=\beta+i\alpha$ mod the action of $\mathbb{Z}$. Similarly, in the second case, a closed-open r-bar code $[a,b)$ will be written as the complex number $z=a+ib$ mod the action of $\mathbb{Z}$ and an open-closed r-bar code $(\alpha,\beta]$ as the complex number $z=\beta+i\alpha$ mod the action of $\mathbb{Z}$.

In Section 4 we will provide a direct definition of the configuration $C_r(f)$ of which we derive the $r-$closed and $(r-1)-$open bar codes of f and in Section 7 we will do the same for the configuration $C^m_r(f)$. The direct definition of $C^m_r(f)$ is essentially a reformulation of the definition of persistence diagrams used in [5] but the one for $C_r(f)$ is not closed to anything considered so far. It should be noticed that the configuration $C_r(f)$ makes sense for any continuous map and implicitly the close and open bar codes can be defined for any such map.

In view of Theorem 1.1 if f is in the homotopy class defined by $\xi \in H^1(X;\mathbb{Z})$ then the configuration $C_r(f)$ has the support of cardinality[4] exactly $\beta_r^N(X;\xi)$, see below, and can be regarded as a point in the n-fold symmetric product $S^n(\mathbb{T}), n = \beta^N(X,\xi)$ of $\mathbb{T}$. Note also that $\mathbb{T}$ can be identified to $\mathbb{C}\backslash 0$ via the map $z \to e^{i\overline{z} - \frac{(z+\overline{z})}{2}}$. Therefore each $C_r(f)$, and in fact any element of $S^n(\mathbb{T})$, can be regarded as a monic polynomial $P_r^f(z)$ of degree n with non-vanishing free coefficient, hence $S^n(\mathbb{T})$ identifies to $\mathbb{C}^{n-1} \times (\mathbb{C} \setminus 0)$. We equip $S^n(\mathbb{T})$ with the topology of the symmetric product or equivalently with the topology of $\mathbb{C}^{n-1} \times (\mathbb{C} \setminus 0)$.

Let $C(X,\mathbb{S}^1)$ denote the space of all continuous maps equipped with the compact open topology and let $C_\xi(X,\mathbb{S}^1)$ be the connected component corresponding to ξ. Let $C_{\xi,t}(X,\mathbb{S}^1)$ be the subspace of tame maps in $C_\xi(X,\mathbb{S}^1)$. Our next result and in some sense the least expected is the following theorem.

Theorem 1.2 (Stability). *The assignment $C_{\xi,t}(X,\mathbb{S}^1) \ni f \rightsquigarrow C_r(f) \in S^n(\mathbb{T})$, $n = \beta_r^N(X,\xi)$, is continuous. Moreover, if X is homeomorphic to a simplicial complex, it extends to a continuous assignment $C_\xi(X,\mathbb{S}^1) \ni f \rightsquigarrow C_r(f) \in S^n(\mathbb{T})$.*

The configuration $C_r(f)$, equivalently the polynomial $P_r^f(z)$, can be viewed as a refinement of the r-Novikov–Betti number. The Poincaré duality for closed manifolds extends from Novikov–Betti numbers to these refinements and we have:

Theorem 1.3 (Poincaré duality). *If M^n is a closed κ-orientable[5] topological manifold with $f\colon M \to \mathbb{S}^1$ a tame map then $C_r(f)(z) = C_{n-r}(\overline{f})(z^{-1})$ where $\mathbb{S}^1$ is viewed as the set of complex numbers of absolute value equal to 1, $\overline{f}\colon X \to \mathbb{S}^1 \subset \mathbb{C}$ denotes the composition of f with the complex conjugation and $C_r(f)$ and C_{n-r} are viewed as configurations of points in $\mathbb{C} \setminus 0$.*

The proofs of Theorems 1.2 and 1.3 we provide use an alternative definition of the configuration $C_r(f)$. More precisely, one defines the function δ_r^f on $\mathbb{T}$ with values in $\mathbb{Z}_{\geq 0}$, one checks that it is equal to the configuration $C_r(f)$ and one verifies Theorems 1.2 and 1.3 for δ_r^f instead of $C_r(f)$.

Similarly, the Jordan blocks introduced in [1] via graph representations, can be recovered in a different manner, more precisely, as the regular part of a linear relation, as stated in Theorem 1.4 below.

Recall that a linear relation $R\colon V \rightsquigarrow V$, concept generalizing linear map, discussed in more details in Section 8, has a canonical linear isomorphism $R_{\mathrm{reg}}\colon V_{\mathrm{reg}} \to V_{\mathrm{reg}}$ associated with it, cf. Section 8. We continue to write R_{reg} for the pair $(V_{\mathrm{reg}}, R_{\mathrm{reg}})$.

Given a tame map $f\colon X \to \mathbb{S}^1$ the infinite cyclic covering $\tilde{f}\colon \tilde{X} \to \mathbb{R}$ is defined by the pullback diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \mathbb{R} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & S^1. \end{array}$$

[4]the cardinality of the support of a configuration is the sum of the multiplicities of its points

[5]If κ has characteristic 2 any manifold is κ-orientable if not the manifold should be orientable.

For any $\theta \in \mathbb{S}^1$ regular angle, one obtains a linear relation R^θ_r by passing to homology in the diagram

$$f^{-1}(\theta) = \tilde{f}^{-1}(\tilde{\theta}) \hookrightarrow \tilde{f}^{-1}\big([\tilde{\theta}, \tilde{\theta} + 2\pi]\big) \hookleftarrow \tilde{f}^{-1}(\tilde{\theta} + 2\pi) = f^{-1}(\theta).$$

Here the real number $\tilde{\theta} \in \mathbb{R}$ corresponds to the angle θ. We have the following theorem.

Theorem 1.4. *If f is a tame map then for any angle θ, and any r, nonegative integer, the pair $(R^\theta_r)_{\rm reg}$ is isomorphic to $\bigoplus_{J=(V,T)\in\mathcal{J}_r(f)}(V,T)$.*

Finally we note that the collection $\mathcal{B}^{co}_r(f)$ can be identified to the collection of *persistence intervals* considered in [12] or [5] for the map $\tilde{f} : \tilde{X} \to \mathbb{R}$, made equivalent modulo $2\pi-$translation. Similarly the collection $\mathcal{B}^{oc}_r(f)$, after changing $(a, b]$ into $[-b, -a)$ can be identified to the collection of *persistence intervals* of $-\tilde{f}$. The stability result of [5] can be reformulated as a stability result for the configuration $C^m_r(f)$. The configurations $C^m_r(f)$s do not have the supports of constant cardinality when f varies in a fixed homotopy class. To give meaning to "stability" the set of finite configurations of points in $\mathbb{T} \setminus \Delta_{\mathbb{T}}$ has to be equipped with the topology induced from the bottle neck metric introduced by the authors of [5]. This metric can make arbitrary "close" configurations with supports of different cardinality, provided the difference is caused by points close to $\Delta_{\mathbb{T}}$. A statement of the result in [5] (in a slightly weaker form), in terms of the configuration $C^m_r(f)$ is provided in Section 7, see Theorem 7.1. In this case one can not extend the assignment $f \rightsquigarrow C^m_r(f)$ continuously to the entire space $C_\xi(X; \mathbb{S}^1)$.

Poincaré duality holds for the configuration $C^m_r(f)$ but in analogy with the Poincaré duality for the torsion of the integral homology for closed orientable manifolds. Precisely we have the following result.

Theorem 1.5. (*Poincaré Duality*) *If M^n is a closed κ-orientable topological manifold and $f \colon M \to \mathbb{S}^1$ a tame map then $C^m_r(f)([a, b]) = C^m_{n-1-r}(-f)([-a, -b])$ with $[a, b]$ denotes the image of (a, b) in $\mathbb{T}$.*

When f is real valued $C_r(f)$ and $C^m_r(f)$ can be considered as a finite configuration of points in $\mathbb{R}^2$ without passing to $\mathbb{T}$. The cardinality of the support of $C_r(f)$ is the standard Betti number $\beta_r(X)$, the Poincaré dualities become $C_r(f)(a, b) = C_{n-r}(-f)(-a, -b)$ and $C^m_r(f)(a, b) = C_{n-1-r}(-f)(-a, -b)$ and there are no Jordan blocks. These configurations can be recovered from the information derived via zigzag persistence proposed in [4].

We like to regard the elements (i), (ii), (iii) associated to a tame angle valued map $f \colon X \to \mathbb{S}^1$ in analogy to the rest points, the isolated trajectories between rest points and the closed trajectories (actually Poincaré return maps for closed trajectories) of $\operatorname{grad}_g f$ when (M, g) is a closed Riemannian manifold and $f \colon M \to \mathbb{S}^1$ a Morse map. These are the elements which enter the classical Morse–Novikov theory.

The generality of the class of spaces and maps which our theory can handle, the finiteness of the number of the elements (i), (ii) and (iii), the computability (by implementable algorithms) at least for X simplicial complex and f simplicial map), cf. [1], end especially the robustness of $C_r(f)$ to small perturbations of f, make this theory "computer friendly" and hopefully of some relevance outside mathematics.

The paper contains in addition to the present section, which summarizes the results, seven more sections and one appendix. In Section 2 we review and prove simple results about graph representations of the two relevant graphs for this paper,

G_{2m} and $\mathcal{Z}$. In Sections 3 and 4 we provide the background and intermediate results for the proof of Theorem 1.1 and the verification that δ^f_r and $C_r(f)$ are equal. We also prove Theorem 1.1. In Section 5 we define the function δ^f_r and prove Theorem 1.2. In Sections 6 and 7 we discuss the Poincaré duality for the configurations $C_r(f)$ and $C^m_r(f)$ and establish Theorems 1.3 and 1.5. In Section 8 we discuss some linear algebra of linear relations and prove Theorem 1.4. The appendix provides an example of tame map and describes its bar codes and Jordan cells. The example is taken from [1].

The algebraic topology-minded reader can easily realize that the collection of bar codes described in this paper can be derived from the Leray spectral sequence of the map $f : X \to S^1$ whose E_2- term is the homology of S^1 with coefficients in the constructible sheaf defined by the homology of $f^{-1}(U)$, $U \subset \mathbb{S}^1$. The interpretation of the stability results (Theorems 1.2 and 7.1) in terms of such spectral sequence is an interesting problem.

Prior work: The approach of relating the topology of a space to the homological behavior of the levels of a real or angle valued map expands the ideas of "persistence theory" introduced in [12]. It also owes to the apparently forgotten efforts and ideas of R. Deheuvels to extend Morse theory to all continuous functions (fonctionelles) cf. [8], ideas which preceded persistence theory. The stability phenomena for bar codes in classical persistence theory was first established in [5]. The first use of graph representations in connection with persistence appears first in [4] under the name of zigzag persistence. The definition of bar codes and of Jordan cells for $\mathbb{S}^1$-valued tame maps was first provided in [1] based on graph representations.

2. Graph representations

Let κ be a fixed field and Γ an oriented graph, possibly with infinitely many vertices. A Γ-representation ρ is an assignment which to each vertex x of Γ assigns a finite dimensional vector space V_x and to each oriented arrow from the vertex x to the vertex y a linear map $V_x \to V_y$. The concepts of morphism, isomorphism= equivalence, sum, direct summand, zero and nontrivial representations are obvious.

If $\rho_\alpha,\ \alpha \in \mathcal{A}$, is a family of $\Gamma-$ representations with the property that for any x all but finitely vector spaces V^α_x are zero dimensional, then one considers $\sum_{\alpha\in\mathcal{A}}\ \rho_\alpha$ the $\Gamma-$representation whose vector space for the vertex x is the direct sum $\oplus_\alpha V^\alpha_x$ and for each oriented arrow the linear map is the direct sum $\oplus_\alpha V^\alpha_x \to \oplus V^\alpha_y = \bigoplus_\alpha (V^\alpha_x \to V^\alpha_y)$.

The $\Gamma-$representation ρ is called:

regular, if all the linear maps are isomorphisms,

with *finite support*, if $V_x = 0$ for all but finitely many vertices and

indecomposable, if not the sum of two nontrivial representations.

In this paper the oriented graph Γ of primary concern will be G_{2m} and for technical reasons we will need the infinite oriented graph $\mathcal{Z}$. The graph $\Gamma = G_{2m}$ has vertices $x_1, x_2, \dots, x_{2m}$ and arrows $a_i\colon x_{2i-1} \to x_{2i}$, $1 \le i \le m$, and $b_i\colon x_{2i+1} \to x_{2i}$, $1 \le i \le m-1$ and $b_m\colon x_1 \to x_{2m}$. The graph $\Gamma = \mathcal{Z}$ has vertices $x_i,\ i \in \mathbb{Z}$, and arrows $a_i\colon x_{2i-1} \to x_{2i}$ and $b_i\colon x_{2i+1} \to x_{2i}$.

Both G_{2m} and $\mathcal{Z}-$representations ρ will be recorded as

$$\rho := \{V_r,\ \alpha_i : V_{2i-1} \to V_{2i},\ \beta_i : V_{2i+1} \to V_{2i}\}$$

in the first case with $1 \leq r \leq 2m, 1 \leq i \leq m$, with the convention that $V_{m+1} = V_1$, in the second case with $r, i \in \mathbb{Z}$.

Any regular G_{2m}-representation $\rho = \{V_r, \alpha_i, \beta_i\}$, not necessary indecomposable, is equivalent = isomorphic to the representation

$$\rho(V,T) = \{V'_r = V, \alpha'_1 = T, \alpha'_i = Id\ i \neq 1,\ \beta'_i = Id\}$$

with $T = \beta_m^{-1} \cdot \alpha_m^{-1} \cdots \beta_1^{-1} \cdot \alpha_1$

The $\mathcal{Z}-$representations we consider are either with finite support or periodic. The representation is periodic if for some integer N, $V_r = V_{r+2N}$, $\alpha_i = \alpha_{i+N}, \beta_i = \beta_{i+N}$. Both type of $\mathcal{Z}-$representations, periodic and with finite support, as well as a finite direct sum of of such representations will be referred to as *good* $\mathcal{Z}-$representations.

2.1. The indecomposable G_{2m} and good $\mathcal{Z}-$representations.

The indecomposable $G_{2m}-$ representations are of two types, (cf. [1]).

Type I (*bar codes*): They are indexed by the four types of intervals I with integer valued ends r and s, $r \leq s$, $1 \leq r \leq m$, namely $[r,s]$ with $r \leq s$, and $(r,s), [r,s), (r,s]$ with $r < s$,

They are denoted by $\rho^I(\{r,s\})$ with "{" notation for either "[" or "(" and "}" for either "]" or")" and described as follows.

Suppose the vertices $x_1, x_2, \cdots x_{2m-1}, x_{2m}$ are located counter-clockwise on the unit circle say at the the angles $0 < t_1 < \theta_1 < t_2 < \theta_2 < \cdots < t_m < \theta_m \leq 2\pi$ with the t_i angle corresponding to an odd vertices and the θ_i to an even vertices.

To describe the representation $\rho^I(\{i, j+mk\}),\ 1 \leq i,j \leq m$, draw the counterclockwise spiral curve from $a = \theta_i$ to $b = \theta_j + 2\pi k$ with the ends a black or an empty circle if the end is closed or open. Black circle indicates that the end is on our spiral empty circle that the end is not.

Let V_i be the vector space generated by the intersection points of the spiral with the radius corresponding to the vertex x_i and let α_i and β_i be defined on bases as follows: a generator e of $V_{2i\pm1}$ is sent to the generator e' of V_{2i} if connected by a piece of spiral and to 0 otherwise. The spiral in Figure 1 below corresponds to $k = 2$.

Type II (*Jordan blocks/cells*): They are indexed by Jordan blocks $J = (V,T)$ and denoted by $\rho^{II}(J)$. Recall that a Jordan block is an isomorphism class of indecomposable pairs (V,T), V a vector space $T : V \to V$ an isomorphism. The representation $\rho^{II}(J)$ has all vector spaces equal to V, $\alpha_1 = T$ and $\beta_1 = \alpha_i = \beta_i = Id$ for $2 \leq i \leq m$.

One refers to both the interval $\{r,s\}$ and the representation $\rho^I(\{r,s\})$ as *bar code* and to the indecomposable pair J and the representation $\rho^{II}(J)$ as *Jordan block*. One denotes by $\mathcal{B}(\rho)$ the collection of all bar codes (with proper multiplicity when appear multiple times as independent summands) and by $\mathcal{B}^c(\rho), \mathcal{B}^o(\rho), \mathcal{B}^{c,o}(\rho)$ and $\mathcal{B}^{o,c}(\rho)$ the sub collections of barcodes with both ends closed, open, the left closed right open and left open right closed. One denotes by $\mathcal{J}(\rho)$ the collection of all Jordan blocks (with proper multiplicity when appear multiple times as independent summands).

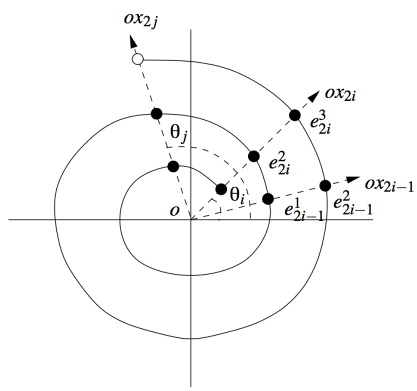


FIGURE 1. The spiral for $[i, j + 2m)$.

For $\lambda \in \kappa \setminus 0$ one denotes by $\mathcal{J}_\lambda(\rho)$ the collection of Jordan blocks $J = (V, T)$ with T having λ as an eigenvalue, hence of the form $(\kappa^k, T(\lambda, k))$.

By Remak-Schmidt theorem any $G_{2m}-$representation ρ can be decomposed as

$$\rho = \bigoplus_{I \in \mathcal{B}(\rho)} \rho^I(I) \oplus \bigoplus_{J \in \mathcal{J}(\rho)} \rho^{II}(J). \tag{2}$$

The indecomposable factors and their multiplicity are unique. The above description is implicit in [13] and [10].

The indecomposable $\mathcal{Z}-$representations with finite support are all *bar codes* indexed by four type of intervals I with ends i and j, $[i, j]$ with $i \leq j$, or $[i, j)$, $(i, j]$, (i, j) with $i < j$ and denoted by $\rho(I)$. The only periodic indecomposable representation is denoted by ρ_∞. The representation denoted by $\rho(I)$ has all vector spaces either $= \kappa$ or 0 and the linear maps α_i, β_j equal to the identity if both the source and the target are nontrivial and zero otherwise. Precisely,

(i) $\rho([i, j]), i \leq j$ has $V_r = \kappa$ for $r = \{2i, 2i + 1, \cdots 2j\}$ and $V_r = 0$ otherwise,
(ii) $\rho([i, j)), i < j$ has $V_r = \kappa$ for $r = \{2i, 2i + 2, \cdots 2j - 1\}$ and $V_r = 0$ otherwise,
(iii) $\rho((i, j]), i < j$ has $V_r = \kappa$ for $r = \{2i + 1, 2i + 2, \cdots 2j\}$ and $V_r = 0$ otherwise,
(iv) $\rho((i, j)), i < j$ has $V_r = \kappa$ for $r = \{2i + 1, 2i + 2, \cdots 2j - 1\}$ and $V_r = 0$ otherwise.

Both the labeling interval I and the representation $\rho(I)$ will be referred to as *bar codes.*

The indecomposable representation ρ_∞, has all vector spaces $V_r = \kappa$ and all linear maps $\alpha_i = \beta_i = Id$.

One denotes by $\mathcal{B}(\rho)$ the collection of all bar codes (with multiplicity) with $\mathcal{B}^c(\rho)$, $\mathcal{B}^o(\rho)$, $\mathcal{B}^{co}(\rho)$ and $\mathcal{B}^{oc}(\rho)$ the sub collections of closed, open, closed-open and open-closed bar codes and by $\mathcal{J}(\rho)$ the collection of all copies of ρ_∞ which can appear as independent direct summands in ρ.

The Remak-Schmidt decomposition for representations with finite support extends to all good $\mathcal{Z}-$representations. Precisely, any such representation ρ is a sum (in the sense described above) of possibly infinitely many indecomposables either with finite support or isomorphic to ρ_∞,

$$\rho = \bigoplus_{I\in\mathcal{B}(\rho)} \rho(I) \oplus \bigoplus_n \rho_\infty, \tag{3}$$

with indecomposable factors and their multiplicity unique up to isomorphism. Here $\bigoplus_n \rho_\infty$ denotes the sum of n copies of ρ_∞. Each indecomposable $\rho(I)$ or ρ_∞ appears with finite multiplicity.

The statements about $G_{2m}-$representations or good $\mathcal{Z}-$representations formulated in this paper will be verified first for the indecomposable representations described above and if hold true, in view of the Remak–Schmidt decomposition theorem, concluded for arbitrary representations.

2.2. **Two basic constructions.**

The *infinite cyclic covering* of a $G_{2m}-$representation $\rho = \{V_r, a_i, b_i, 1 \leq r \leq 2m, 1 \leq i \leq m\}$ is the periodic $\mathcal{Z}-$representation $\tilde\rho := \{\tilde V_r, \tilde a_i, \tilde b_i, r, i \in \mathbb{Z}\}$ defined by $\tilde V_{r+2mk} = V_r, \tilde a_{i+km} = a_i, \tilde b_{i+km} = b_i$. When applied to indecomposable $\rho^I(I)$ or $\rho^{II}(J)$ one obtains :

$$\begin{aligned}\widetilde{\rho^I(I)} &= \bigoplus_{k\in\mathbb{Z}} \rho(I+mk)\\ \widetilde{\rho^{II}(J)} &= \bigoplus_n \rho_\infty, n = \sum_{J\in\mathcal{J}(\rho)} \dim V, J=(V,T).\end{aligned} \tag{4}$$

where $I+a$, $a\in\mathbb{Z}$ denotes translate of the interval I, with a units.

The *truncation* $T_{k,l}$ of a $\mathcal{Z}-$representation is defined for any pair of integers $k,l,\ k\leq l$ and of a $G_{2m}-$representation for a any pair of integers $k,l, 1\leq k\leq l\leq m$.

If $\rho=\{V_r,\alpha_i,\beta_i\}$ and $T_{k,l}(\rho)=\{V'_r,\alpha'_i,\beta'_i\}$ then

$$\begin{aligned}V'_r &= \begin{cases} V_r & 2k\leq r\leq 2l\\ 0 & \text{otherwise}\end{cases}\\ \alpha'_r &= \begin{cases} \alpha_r & k+1\leq r\leq l\\ 0 & \text{otherwise}\end{cases}\\ \beta'_r &= \begin{cases} \beta_r & k\leq r\leq l-1\\ 0 & \text{otherwise.}\end{cases}\end{aligned} \tag{5}$$

More precisely for the indecomposable $\mathcal{Z}-$representations one obtains

$$\begin{aligned}T_{k,l}(\rho_\infty) &= \rho([k,l])\\ T_{k,l}(\rho(I)) &= \rho(I\cap[k,l])\end{aligned} \tag{6}$$

and for the indecomposable $G_{2m}-$representations one obtains

$$\begin{aligned}T_{k,l}(\rho^I(\{i,l\}) &= \rho^I(\{i,l\}\cap[k,l])\\ T_{k,l}(\rho^{II}(J)) &= \bigoplus_n \rho^I([k,l]),\ n=\dim V.\end{aligned} \tag{7}$$

Given a $G_{2m}-$representation ρ we write:

$\tilde{\mathcal{J}}(\rho)$ for the collection which contains with any Jordan block $J \in \mathcal{J}(\rho)$, a number of $n(J) = \dim(V)$ copies of $J = (V, T)$ and

$\tilde{\mathcal{B}}^{\cdots}(\rho) := \{I + 2\pi k \mid I \in \mathcal{B}^{\cdots}(\rho), k \in \mathbb{Z}\}$ with $\tilde{\mathcal{B}}^{\cdots}$ any of $\tilde{\mathcal{B}}, \tilde{\mathcal{B}}^c, \tilde{\mathcal{B}}^o, \tilde{\mathcal{B}}^{co}, \tilde{\mathcal{B}}^{oc}$.

With the above notation one has :

Observation 2.1.

1.If ρ is a $G_{2m}-$representation then

$$\mathcal{B}^{\cdots}(\tilde{\rho}) = \tilde{\mathcal{B}}^{\cdots}(\rho),$$
$$\mathcal{J}(\tilde{\rho}) = \tilde{\mathcal{J}}(\rho).$$

2. If ρ is a good $\mathcal{Z}$ or a $G_{2m}-$representation then

$$\mathcal{B}^c(T_{l,k}(\rho)) = \{I \in \mathcal{B}(\rho), I \cap [k,l] \neq \emptyset \text{ and closed}\} \sqcup \tilde{\mathcal{J}}(\rho),$$
$$\mathcal{B}^o(T_{l,k}(\rho)) = \{I \in \mathcal{B}^o(\rho), I \subset [k,l]\}.$$

2.3. **The matrix $M(\rho)$ and the representation ρ_u.**

For a $G_{2m}-$representation $\rho = \{V_r, \alpha_i, \beta_i\}$ $1 \leq r \leq 2m, 1 \leq i \leq m$, the linear map $M(\rho) : \bigoplus_{1\leq i\leq m} V_{2i-1} \to \bigoplus_{1\leq i\leq m} V_{2i}$ is defined by the block matrix

$$\begin{pmatrix} \alpha_1 & -\beta_1 & 0 & \dots & \dots & 0 \\ 0 & \alpha_2 & -\beta_2 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & \dots & \dots & \dots & \dots\alpha_{m-1} & -\beta_{m-1} \\ -\beta_m & \dots & \dots & \dots & \dots & \alpha_m \end{pmatrix}.$$

and the $G_{2m}-$representation $\rho_u = \{V'_r, \alpha'_i, \beta'_i\}$ by

$V'_r = V_r$, $\alpha'_1 = u\alpha_1$, $\alpha'_i = \alpha_i$ for $i \neq 1$ and $\beta'_i = \beta_i$.

For a $\mathcal{Z}-$representation $\rho = \{V_r, \alpha_i, \beta_i\}$ the linear map $M(\rho) : \bigoplus_{i\in\mathbb{Z}} V_{2i-1} \to \bigoplus_{i\in\mathbb{Z}} V_{2i}$, is defined by the infinite block matrix with entries

$$M(\rho)_{2r-1,2s} = \begin{cases} \alpha_r, & \text{if s = r} \\ \beta_{r-1}, & \text{if s = r} - 1 \\ 0 & \text{otherwise .} \end{cases}$$

Denote by:

(i) $\dim(\rho) : \Gamma \to \mathbb{Z}_{\geq 0}$ with $\dim(\rho)(x_r) = \dim V_r$,
(ii) $d\ker(\rho) := \dim\ker M(\rho)$ and
(iii) $d\,\mathrm{coker}(\rho) := \dim\mathrm{coker}\, M(\rho)$.

As noticed in [1]

Observation 2.2.

(i) $\dim(\rho_u) = \dim(\rho)$,
(ii) $(\rho_1 \oplus \rho_2)_u = (\rho_1)_u \oplus (\rho_2)_u$,
(iii) $\rho^{II}(\lambda, k)_u = \rho^{II}(u\lambda, k)$,
(iv) $\rho^I(\{i,j\}; k))_u \equiv \rho^I(\{i,j\}; k)$,
(v) $\dim(\rho_1 \oplus \rho_2) = \dim(\rho_1) + \dim(\rho_2)$,
(vi) $d\ker(\rho_1 \oplus \rho_2) = d\ker(\rho_1) + d\ker(\rho_2)$,

(vii) $d\,\mathrm{coker}(\rho_1 \oplus \rho_2) = d\,\mathrm{coker}(\rho_1) + d\,\mathrm{coker}(\rho_2)$.

and one has

Proposition 2.3.

1. For indecomposable $G_{2m}-$representations of type I

(i) $d\ker\rho^I([i,j]) = 0$, $d\,\mathrm{coker}\,\rho^I([i,j]) = 1$,
(ii) $d\ker\rho^I([i,j)) = 0$, $d\,\mathrm{coker}\,\rho^I([i,j)) = 0$,
(iii) $d\ker\rho^I((i,j]) = 0$, $d\,\mathrm{coker}\,\rho^I((i,j]) = 0$,
(iv) $d\ker\rho^I((i,j)) = 1$, $d\,\mathrm{coker}\,\rho^I((i,j)) = 0$

and for indecomposable $\mathcal{Z}-$representations with finite support

(i) $d\ker\rho([i,j]) = 0$, $d\,\mathrm{coker}\,\rho([i,j]) = 1$,
(ii) $d\ker\rho([i,j)) = 0$, $d\,\mathrm{coker}\,\rho([i,j)) = 0$,
(iii) $d\ker\rho((i,j]) = 0$, $d\,\mathrm{coker}\,\rho((i,j]) = 0$,
(iv) $d\ker\rho((i,j)) = 1$, $d\,\mathrm{coker}\,\rho((i,j)) = 0$.

2. For indecomposable $G_{2m}-$representations of type II

(i) $d\ker\rho^{II}(J) = 0$ *if* $J \neq (\kappa^k, T(1,k))$; $d\ker\rho^{II}(\kappa^k, T(1,k)) = 1$
(ii) $d\,\mathrm{coker}\,\rho^{II}(J) = 0$ *if* $J \neq (\kappa^k, T(1,k))$; $d\,\mathrm{coker}\,\rho^{II}(\kappa^k, T(1,k)) = 1$

and for the representation ρ_∞

(i) $d\ker(\rho_\infty) = 0$ $d\,\mathrm{coker}(\rho_\infty) = 1$.

To check Proposition 2.3 one notices that the calculation of the kernel of $M(\rho)$ boils down to the description of the space of solutions of the linear system

$$\begin{aligned}\alpha_1(v_1) =&\beta_1(v_3)\\ \alpha_2(v_3) =&\beta_2(v_5)\\ &\cdots\\ \alpha_m(v_{2m-1}) =&\beta_m(v_1)\end{aligned}$$

which in the case of indecomposable are easy to do.

Proposition 2.3 can be refined. For each indecomposable consider the concrete description presented above and specify a nonzero vector in $\ker M(\rho)$ or $\mathrm{coker}(M(\rho)$ when the case. For example for Jordan blocks such choice is needed only for the Jordan cells of form $(1,k)$ since the kernels and cokernels are otherwise zero dimensional. This additional specification will be regarded as part of the concrete realization of the indecomposable representation and referred to as the *model for the indecomposable.*

Recall that for a set S one denotes by $\kappa[S]$ the vector space generated by S, equivalently the vector space of $\kappa-$valued maps on S with finite support.

Proposition 2.4. *1. Let ρ be a $G_{2m}-$representation equipped with a decomposition $\rho = \bigoplus_{I\in\mathcal{B}(\rho)}\rho^I(I) \oplus \bigoplus_{J\in\mathcal{J}(\rho)}\rho^{II}(J)$. The decomposition induces the canonical isomorphisms*

$$\Psi^c : \kappa[\mathcal{B}^c(\rho) \sqcup \mathcal{J}(\rho)(1)] \to \mathrm{coker}\,M(\rho)$$
$$\Psi^o : \kappa[\mathcal{B}^o(\rho) \sqcup \mathcal{J}(\rho)(1)] \to \ker M(\rho).$$

compatible with truncations.

2. Let ρ be a good $\mathcal{Z}-$representation equipped with a decomposition $\rho = \bigoplus_{I\in\mathcal{B}(\rho)}\rho(I) \oplus \bigoplus_n \rho_\infty$, $n = \sharp J(\rho)$. The decomposition induces the canonical isomorphisms

$$\Psi^c : \kappa[\mathcal{B}^c(\rho) \sqcup \mathcal{J}(\rho)] \to \mathrm{coker}\,M(\rho)$$
$$\Psi^o : \kappa[\mathcal{B}^o(\rho)] \to \ker M(\rho).$$

compatible with truncations.

The construction of Ψ^c and Ψ^o is tautological for the model of indecomposables as presented above. For an arbitrary representation the decomposition permits to assemble the tautological Ψ^c's and Ψ^o's into isomorphisms as stated. Note that a specified decomposition of ρ provides, in view of Observation 2.1, a decomposition of $\tilde{\rho}$ and of the truncations $T_{k,l}(\tilde{\rho})$ and $T_{k,l}(\rho)$.

Let us explain in more details what "compatible with the truncations" means.

The inclusions of sets $\{i \mid k \leq i \leq l\} \subseteq \{i \mid k' \leq i \leq l'\} \subset \mathbb{Z}$ for $i' \leq i, l' \geq l$, induce the commutative diagram

$$\begin{array}{ccccc} \oplus_{k\leq i\leq l} V_{2i-1} & \longrightarrow & \oplus_{k'\leq i\leq l'} V_{2i-1} & \longrightarrow & \oplus_i V_{2i-1} \\ \downarrow{\scriptstyle M(T_{k,l}(\rho))} & & \downarrow{\scriptstyle M(T_{k',l'}(\rho))} & & \downarrow{\scriptstyle M((\rho)} \\ \oplus_{k\leq i\leq l} V_{2i} & \longrightarrow & \oplus_{k'\leq i\leq l'} V_{2i} & \longrightarrow & \oplus_i V_{2i} \end{array} \tag{8}$$

and then the linear maps

$$\ker M(T_{k,l}(\rho)) \xrightarrow{i} \ker M(T_{k',l'}(\rho)) \xrightarrow{i'} \ker M(\rho) \tag{9}$$

and

$$\operatorname{coker} M(T_{k,l}(\rho)) \xrightarrow{j} \operatorname{coker} M(T_{k',l'}(\rho)) \xrightarrow{j'} \operatorname{coker} M(\rho). \tag{10}$$

The linear maps i and i' are injective since by Observation 2.1 (2.) we have the inclusions $\mathcal{B}(T_{k,l}(\rho))^o \subseteq \mathcal{B}(T_{k',l'}(\rho))^o \subseteq \mathcal{B}(\rho)^o \subseteq \mathcal{B}(\rho)^o \sqcup \mathcal{J}(1)$, which make the linear maps

$$\kappa[\mathcal{B}^o(T_{k,l}(\rho))] \longrightarrow \kappa[\mathcal{B}^o(T_{k',l'}(\rho))] \longrightarrow \kappa[\mathcal{B}^o(\rho) \sqcup \mathcal{J}(1)] \tag{11}$$

injective.

We also have the linear maps

$$\kappa[\mathcal{B}^c(T_{k,l}(\rho)) \sqcup \mathcal{J}(1)] \longrightarrow \kappa[\mathcal{B}^c(T_{k',l'}(\rho)) \sqcup \mathcal{J}(1)] \longrightarrow \kappa[\mathcal{B}^c(\rho) \sqcup \mathcal{J}(1)] \tag{12}$$

which are not necessary injective, defined as follows. As the elements of $\mathcal{B}^c(T_{k,l}(\rho))$ are elements of $\mathcal{B}(\rho)$, the linear maps in the sequence (12) send a bar code $I \in \mathcal{B}^c(T_{k,l}(\rho))$ to itself if it belongs to the next set and to zero otherwise and any element of $\mathcal{J}(1)$ to itself. The compatibility with truncation means the commutativity of the diagrams.

$$\begin{array}{ccccc} \ker M(T_{k,l}(\rho)) & \xrightarrow{i} & \ker M(T_{k',l'}(\rho)) & \xrightarrow{i'} & \ker M(\rho) \\ \uparrow & & \uparrow & & \uparrow \\ \kappa[\mathcal{B}^o(T_{k,l}(\rho))] & \longrightarrow & \kappa[\mathcal{B}^o(T_{k',l'}(\rho))] & \longrightarrow & \kappa[\mathcal{B}^o(\rho) \sqcup \mathcal{J}(1) \end{array} \tag{13}$$

with the vertical arrows Ψ^os and

$$\begin{array}{ccccc}
\operatorname{coker} M(T_{k,l}(\rho)) & \xrightarrow{j} & \operatorname{coker} M(T_{k',l'}(\rho)) & \xrightarrow{j'} & \operatorname{coker} M(\rho) \\
\uparrow & & \uparrow & & \uparrow \\
\kappa[\mathcal{B}^c(T_{k,l}(\rho)) \sqcup \mathcal{J}(1)] & \longrightarrow & \kappa[\mathcal{B}^c(T_{k',l'}(\rho)) \sqcup \mathcal{J}(1)] & \longrightarrow & \kappa[\mathcal{B}^c(\rho) \sqcup \mathcal{J}(1).
\end{array} \tag{14}$$

with vertical arrows Ψ^cs.

We finish this section with an observation about the $\mathcal{Z}-$representations $\tilde{\rho}$ when ρ is a $G_{2k}-$representation. The shift of indices $r \to r + 2k$ for vector spaces and $i \to i + k$ for linear maps induces the linear endomorphism τ_k on the kernel and cokernel of the associated matrices $M(\tilde{\rho})$. We will need the compositions

$$(\Psi^o)^{-1} \cdot \tau_k \cdot \Psi^o : \kappa[\mathcal{B}^o(\tilde{\rho})] \to \kappa[\mathcal{B}^o(\tilde{\rho})]$$

and

$$(\Psi^c)^{-1} \cdot \tau_k \cdot \Psi^c : \kappa[\mathcal{B}^c(\tilde{\rho}) \sqcup \mathcal{J}(\tilde{\rho})] \to \kappa[\mathcal{B}^c(\tilde{\rho}) \sqcup \mathcal{J}(\tilde{\rho})]$$

to provide a $\kappa[T^{-1}, T]-$module structure (multiplication by T) on $\ker M(\tilde{\rho})$ and $\operatorname{coker} M(\tilde{\rho})$.

It suffices to describe these compositions separately, for $G_{2k}-$representations ρ with $\mathcal{J}(\rho) = \emptyset$ and with $\mathcal{B}(\rho) = \emptyset$. In the second case ρ is regular, hence isomorphic with the representation $\{V_r = V, \alpha_1 = T, \beta_1 = \beta_i = \beta_i = Id, i \geq 2\}$ with $T : V \to V$ isomorphism.

Observation 2.5. *1. If ρ is a $G_{2k}-$representation with $\mathcal{J}(\rho) = \emptyset$ then the compositions above are induced by the map on bar codes which sends the interval $\{r, s\}$ into the interval $\{r + k, l + k\}$.*

2. If ρ is a $G_{2k}-$representation with $\mathcal{B}(\rho) = \emptyset$ then $\mathcal{B}^o(\tilde{\rho}) = \mathcal{B}^c(\tilde{\rho}) = \emptyset$ and the pair (V, T) is isomorphic to $(\kappa[\mathcal{J}(\tilde{\rho})], (\Psi^c)^{-1} \cdot \tau_k \cdot \Psi^c\ )$.

Recall that $\sharp\mathcal{J}(\tilde{\rho}) = \sum_{(V,T)\in\mathcal{J}(\rho)} \dim V$.

3. Bar codes and Jordan blocks via graph representations

Let $f : X \to S^1$ be a tame map and $0 < \theta_1 < \theta_2 < \cdots \theta_m \leq 2\pi$ be the critical angles (the angles of the set Σ in the definition of tameness). Choose the regular values $t_1 < t_2, \cdots < t_m$ with $\theta_{i-1} < t_i < \theta_i$ and $0 < t_1 < \theta_1$. In order to differentiate between regular and singular fibers we write $R_i := f^{-1}(t_i)$ and $X_i := f^{-1}(\theta_i)$.

The tameness of f induces the maps $a_i : R_i \to X_i$ for $1 \leq i \leq m$, $b_i : R_{i+1} \to X_1$ for $i \leq m - 1$ and $b_m : R_1 \to X_m$ which are unique up to homotopy; this means that different choices of the regular values, say t'_i instead of t_i, lead to homotopy equivalences $\omega_i : R_i \to R'_i$ s.t. $a'_i \cdot \omega_i$ is homotopic to a_i and $b'_i \cdot \omega_i$ is homotopic to b_i. Indeed the fiber R_i identifies up to homotopy to regular fiber $f^{-1}(t)$ and $f^{-1}(t')$ with t a regular value closed enough to θ_i and t' a regular value closed enough to θ'_{i-1} to insure that $f^{-1}(t)$ resp. $f^{-1}(t')$ is contained in an open set which retracts to X_i resp. X_{i-1}. The maps a_i or b_{i-1} are the composition of such identifications with these retractions to X_i resp. X_{i-1}. We leave the reader to do the tedious verification that the homotopy classes of a_i and b_{i-1} are independent of the choices made. Passing to $r-$homology one obtains the $G_{2m}-$representation $\rho_r = \rho_r(f)$ whose vector spaces are $V_{2s} = H_r(X_s)$ and $V_{2s-1} = H_r(R_s)$ and linear maps α_i and β_i are induced by the continuous maps a_i and b_i.

The representation $\rho_r(f)$ has bar codes whose ends are $i, j+km$, $1 \leq i, j \leq m$. Denote by $\mathcal{B}_r(f)$, the collections of intervals defined by the bar codes of $\rho_r(f)$ with ends i and $j+km$ replaced by θ_i and $\theta_j + 2\pi k$. Denote by $\mathcal{J}_r(f)$ the collection of Jordan blocks of the representation $\rho_r(f)$.

One can think to these bar codes in a way more consistent with points in the space $\mathbb{T}$. If $\tilde{f} : \tilde{X} \to \mathbb{R}$ is the infinite cyclic covering of f then the real numbers $\theta_i + 2\pi k$ are the critical values and $t_i + 2\pi k$ are regular values (between consecutive critical values) and the tameness of $\tilde{f}$ gives the maps $a_{i+km} : \tilde{X}_{t_{i+1}+2\pi k} \to \tilde{X}_{\theta_i+2\pi k}$ and $b_{i+km} : \tilde{X}_{t_i+2\pi k} \to \tilde{X}_{\theta_i+2\pi k}$. By passing to homology in dimension r one obtains a good $\mathcal{Z}-$representation $\rho_r(\tilde{f})$ which is exactly the infinite cyclic covering $\widetilde{\rho_r(f)}$. The collections $\mathcal{B}_r(\tilde{f}), \mathcal{B}^c_r(\tilde{f})$, $\mathcal{B}^o_r(\tilde{f})$, $\mathcal{B}^{co}_r(\tilde{f})$, $\mathcal{B}^{oc}_r(\tilde{f})$ are invariants w.r to the 2π translation and the collections $\mathcal{B}_r(f), \mathcal{B}^c_r(f)$ $\mathcal{B}^o_r(f)$ $\mathcal{B}^{co}_r(f)$ $\mathcal{B}^{oc}_r(f)$ can be viewed as equivalence (= modulo the 2π translation) classes of elements of $\mathcal{B}^c_r(\tilde{f})$, $\mathcal{B}^o_r(\tilde{f})$, $\mathcal{B}^{co}_r(\tilde{f})$, $\mathcal{B}^{oc}_r(\tilde{f})$.

Given $\xi \in H^1(X;\mathbb{Z})$ and $u \in \kappa \setminus 0$, the pair (ξ, u) denotes the rank one representation $H_1(X;\mathbb{Z}) \to \mathbb{Z} \to \kappa \setminus 0$, where the first arrow is given by ξ and the second by the homomorphism $< u >: \mathbb{Z} \to \kappa \setminus 0$ defined by $< u > (n) = u^n$. One denotes by $H_r(X; (\xi, u))$ the homology of X with coefficients in the local system defined by the representation (ξ, u), which for $u = 1$ satisfies $H_r(X; (\xi, 1)) = H_r(X)$. When restricted to R_i and X_i the local system is the constant one with fiber κ so by passing to homology the $G_{2m}-$representation obtained will have the same vector spaces for all u's but not necessary the same α'_is and β'_is. The $G_{2m}-$representation obtained will be isomorphic to $(\rho_r(f))_u$. More general for $X_{[\theta_1,\theta_2]} = f^{-1}([\theta_1, \theta_2])$ with $\theta_2 - \theta_1 < 2\pi$, the restriction of the local system considered above is isomorphic to the constant local system with fiber κ and the inclusion $X_{[\theta_1,\theta_2]} \subset X$ induces the homomorphism

$$H_r(X_{[\theta_1,\theta_2]}) \to H_r(X; (\xi, u)).$$

3.1. **The relevant exact sequences. (cf. [1]).** The tool which permits the calculation of the homology of X, $\tilde{X}$ and various pieces of these spaces is provided by Proposition 3.1 below.

Observe that for $\theta_i \leq \theta_j$ critical angles of f and $f_{[\theta_i,\theta_j]}$ denoting the restriction of f to $X_{[\theta_i,\theta_j]} = f^{-1}[\theta_i, \theta_j]$ one has

$$\rho_r(f_{[\theta_i,\theta_j]}) = T_{i,j}(\rho_r(f)).$$

Similarly, for $c_i \leq c_j$ critical values of $\tilde{f}$ and $\tilde{f}_{[c_i,c_j]}$ denoting the restriction $\tilde{f}$ to $\tilde{X}_{[c_i,c_j]} = \tilde{f}^{-1}[c_i, c_j]$ one has

$$\rho_r(\tilde{f}_{[c_i,c_j]}) = T_{i,j}(\tilde{\rho}_r(f)).$$

Proposition 3.1. *Let $f : X \to \mathbb{S}^1$ a tame map, $\tilde{f} : \tilde{X} \to \mathbb{R}$ its infinite cyclic covering. Let $\rho_r = \rho_r(f)$ and $\tilde{\rho}_r = \rho_r(\tilde{f}) = \widetilde{\rho}_r(f)$ be the representations associated with f and $\tilde{f}$. One has the following short exact sequences:*

$$0 \to \operatorname{coker} M((\rho_r)_u) \to H_r(X; (\xi_f, u)) \to \ker M((\rho_{r-1})_u) \to 0, \tag{15}$$

which for $u = 1$ becomes

$$0 \to \operatorname{coker} M(\rho_r) \to H_r(X) \to \ker M(\rho_{r-1}) \to 0, \tag{16}$$

and

$$0 \to \operatorname{coker} M(\tilde\rho_r) \to H_r(\tilde X) \to \ker M(\tilde\rho_{r-1}) \to 0. \tag{17}$$

The sequences are compatible with the truncations with respect to the pairs of critical angles (θ_i, θ_j) and $(\theta_{i'}, \theta_{j'})$, $0 < \theta_i \leq \theta_{i'} \leq \theta_{j'} \leq \theta_j \leq 2\pi$ resp. the pairs of critical values (c_i, c_j) and $(c_{i'}, c_{j'})$ with $c_i \leq c_{i'} \leq c_{j'} \leq c_j$.

In the case of $G_{2m}-$ representation $\rho_r(f)$ "compatibility with truncation" means the commutativity of the diagram (18)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{coker} M(T_{i',j'}(\rho_r)) & \longrightarrow & H_r(X_{[\theta_{i'},\theta_{j'}]}) & \xrightarrow{\pi'} & \ker M(T_{i',j'}(\rho_{r-1})) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle v_l} & & \big\downarrow{\scriptstyle v} & & \big\downarrow{\scriptstyle v_r} & & \\
0 & \longrightarrow & \operatorname{coker} M(T_{i,j}(\rho_r)) & \longrightarrow & H_r(X_{[\theta_i,\theta_j]}) & \xrightarrow{\pi"} & \ker M(T_{i,j}(\rho_{r-1})) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle v'_l} & & \big\downarrow{\scriptstyle v'} & & \big\downarrow{\scriptstyle v'_r} & & \\
0 & \longrightarrow & \operatorname{coker} M((\rho_r)_u) & \longrightarrow & H_r(X;(\xi_f,u)) & \xrightarrow{\pi} & \ker M((\rho_{r-1})_u) & \longrightarrow & 0
\end{array} \tag{18}$$

and in the case of the $\mathcal{Z}-$representation $\tilde\rho_r$ the commutativity of the diagram (19)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{coker} M(T_{i',j'}(\tilde\rho_r)) & \longrightarrow & H_r(\tilde X_{[c_{i'},c_{j'}]}) & \xrightarrow{\pi'} & \ker M(T_{i',j'}(\tilde\rho_{r-1})) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle v_l} & & \big\downarrow{\scriptstyle v} & & \big\downarrow{\scriptstyle v_r} & & \\
0 & \longrightarrow & \operatorname{coker} M(T_{i,j}(\tilde\rho_r)) & \longrightarrow & H_r(\tilde X_{[c_i,c_j]}) & \xrightarrow{\pi"} & \ker M(T_{i,j}(\tilde\rho_{r-1})) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle v'_l} & & \big\downarrow{\scriptstyle v'} & & \big\downarrow{\scriptstyle v'_r} & & \\
0 & \longrightarrow & \operatorname{coker} M(\tilde\rho_r) & \longrightarrow & H_r(\tilde X) & \xrightarrow{\pi} & \ker M(\tilde\rho_{r-1}) & \longrightarrow & 0.
\end{array} \tag{19}$$

To establish these diagrams denote by $\mathcal{R} := \sqcup_{1\leq i\leq m} R_i$, $\tilde{\mathcal{R}} := \sqcup_{i\in\mathbb{Z}} R_i$, $\mathcal{X} := \sqcup_{1\leq i\leq m} X_i$ and $\tilde{\mathcal{X}} := \sqcup_{i\in\mathbb{Z}} X_i$.

The short exact sequences (15) and (16) follow from the long exact sequence

$$\cdots \to H_r(\mathcal{R}) \xrightarrow{M((\rho_r)_u)} H_r(\mathcal{X}) \to H_r(X;(\xi,u)) \to H_{r-1}(\mathcal{R}) \xrightarrow{M(\rho_{r-1})} H_{r-1}(\mathcal{X}) \to \cdots . \tag{20}$$

with $H_r(\mathcal{R}) = \bigoplus_{1\leq i\leq m} H_r(R_i)$ and $H_r(\mathcal{X}) = \bigoplus_{1\leq i\leq m} H_r(X_i)$ (16 for $u=1$) and the short exact sequence (17) from the long exact sequence

$$\cdots \to H_r(\tilde{\mathcal{R}}) \xrightarrow{M(\rho_r)} H_r(\tilde{\mathcal{X}}) \to H_r(\tilde X) \to H_{r-1}(\tilde{\mathcal{R}}) \xrightarrow{M(\rho_{r-1})} H_{r-1}(\tilde{\mathcal{X}}) \to \cdots . \tag{21}$$

Since both long exact sequences (20) and (21) are derived in the same way we will work only on (20) and for simplicity only for $u = 1$.

First choose an $\epsilon > 0$ small enough so that $2\epsilon < t_1$ and $\theta_{i-1} + 2\epsilon < t_i < \theta_i - 2\epsilon$. To simplify the writing, since $i \leq m$, introduce $\theta_{m+1} = \theta_1 + 2\pi$ and define $f^{-1}([\theta_m \pm \epsilon, \theta_{m+1} \pm \epsilon) := \tilde{f}^{-1}([\theta_m \pm \epsilon, \theta_1 + 2\pi \pm \epsilon])$.

Define

(i) $\mathcal{P}' = \sqcup_{1 \leq i \leq m} f^{-1}([\theta_i, \theta_{i+1} - \epsilon))$
(ii) $\mathcal{P}'' = \sqcup_{1 \leq i \leq m} f^{-1}((\theta_i + \epsilon, \theta_{i+1}])$

and observe that in view of the choice of ϵ and the tameness of f the inclusions $\mathcal{X} \subset \mathcal{P}'$, $\mathcal{X} \subset \mathcal{P}$" and $\mathcal{X} \sqcup \mathcal{R} \subset \mathcal{P}' \cap \mathcal{P}''$ are homotopy equivalences.

The Mayer Vietoris long exact sequence for $X = \mathcal{P}' \cup \mathcal{P}''$ gives the diagram

$$\begin{array}{ccccccccc}
 & & H_r(\mathcal{R}) & \xrightarrow{M(\rho_r(f))} & H_r(\mathcal{X}) & & & & \\
 & \nearrow & \uparrow{\scriptstyle pr_1} & & \uparrow{\scriptstyle (Id,-Id)} & \searrow & & & \\
\longrightarrow H_{r+1}(\mathcal{T}) & \xrightarrow{\partial_{r+1}} & H_r(\mathcal{R}) \oplus H_r(\mathcal{X}) & \xrightarrow{N} & H_r(\mathcal{X}) \oplus H_r(\mathcal{X}) & \xrightarrow{(i^r,-i^r)} & H_r(\mathcal{T}) \longrightarrow & & \\
 & & \uparrow{\scriptstyle in_2} & & \uparrow{\scriptstyle \Delta} & & & & \\
 & & H_r(\mathcal{X}) & \xrightarrow{Id} & H_r(\mathcal{X}) & & & &
\end{array} \tag{22}$$

where Δ denotes the diagonal, in_2 the inclusion on the second component, pr_1 the projection on the first component, i^r the linear map induced in homology by the inclusion $\mathcal{X} \subset \mathcal{T}$.

The matrix $M(\rho_r(f))$ is defined by

$$\begin{pmatrix}
\alpha_1^r & -\beta_1^r & 0 & \cdots & 0 \\
0 & \alpha_2^r & -\beta_2^r & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \alpha_{m-1}^r & -\beta_{m-1}^r \\
-\beta_m^r & 0 & \cdots & 0 & \alpha_m^r
\end{pmatrix}$$

with $\alpha_i^r \colon H_r(R_i) \to H_r(X_i)$ and $\beta_i^r \colon H_r(R_{i+1}) \to H_r(X_i)$ induced by the maps a_i and b_i and the block matrix N defined by

$$\begin{pmatrix}
\alpha^r & \mathrm{Id} \\
-\beta^r & \mathrm{Id}
\end{pmatrix}$$

where α^r and β^r are the matrices

$$\begin{pmatrix}
\alpha_1^r & 0 & \cdots & 0 \\
0 & \alpha_2^r & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \alpha_{m-1}^r
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
0 & \beta_1^r & 0 & \dots & 0 \\
0 & 0 & \beta_2^r & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 \\
0 & 0 & \dots & 0 & \beta_{m-1}^r \\
\beta_m^r & 0 & \dots & 0 & 0
\end{pmatrix}.$$

The long exact sequence (20) is the top sequence in the diagram (22).

By carefully following the above construction one verifies the commutativity of the diagrams. q.e.d.

4. Proof of Theorem 1.1.

Consider the pair $(X, \xi \in H^1(X;\mathbb{Z}))$ with X a compact ANR and denote by $\tilde{X} \to X$ the infinite cyclic covering associated to ξ. Recall from Section 1 that for $\xi = \xi_f$ the covering $\tilde{X} \to X$ is the pull back by f of the universal covering $\mathbb{R} \to \mathbb{S}^1$

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \mathbb{R} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & S^1. \end{array}$$

The vector space $H_r(\tilde{X})$ is actually a $\kappa[T^{-1}, T]$-module [6] where the multiplication by T is the linear isomorphism induced by the deck transformation $\tau\colon \tilde{X} \to \tilde{X}$.

Let $\kappa[T^{-1}, T]]$ be the field of Laurent power series and define

$$H_r^N(X;\xi) := H_r(\tilde{X}) \otimes_{\kappa[T^{-1},T]} \kappa[T^{-1}, T]].$$

The $\kappa[T^{-1}, T]]$-vector spaces $H_r^N(X;\xi)$ is called the $r-$th Novikov homology[7] and its dimension over the field $\kappa[T^{-1}, T]]$, the *Novikov–Betti number* $\beta_r^N(X;\xi)$.

Consider $H_r(\tilde{X}) \to H_r^N(X;\xi)$ the $\kappa[T^{-1}, T]$-linear map induced by taking the tensor product with $\kappa[T^{-1}, T]]$ over $\kappa[T^{-1}, T]$. The $\kappa[T^{-1}, T]-$ module $V(\xi)$,

$$V(\xi) := \ker(H_r(\tilde{X}) \to H_r^N(X;\xi)),$$

when regarded as a $\kappa-$vector space equipped with the linear isomorphism $T(\xi)$ provided by the multiplication by T is referred to as the $r-$*monodromy* of (X, ξ). As a $\kappa[T^{-1}, T]-$module $V_r(\xi)$ is exactly the torsion of the $\kappa[T^{-1}, T]-$module $H_r(\tilde{X})$.

Since X is a compact ANR all numbers $\dim H_r(X)$, $\beta_r^N, \dim V(\xi)$ are finite.

A nonempty subset K of $\mathbb{S}^1$ or $\mathbb{R}$, will be called a *closed multi-interval* if it is a finite union of disjoint closed intervals $[\theta_1, \theta_2]$ with $0 \leq \theta_1 \leq \theta_2 < 2\pi$ in the case of $\mathbb{S}^1$, and $[a, b]$ with $a \leq b$ or $(-\infty, a]$ or $[b, \infty)$ in the case of $\mathbb{R}$. One denotes by $X_K := f^{-1}(K)$ if $K \subset \mathbb{S}^1$ and by $\tilde{X}_K = f^{-1}(K)$ if $K \subset \mathbb{R}$.

In case $K \subset \mathbb{S}^1$ one considers

(i) $\mathcal{B}^c_{r,K}(f) = \{I \in \mathcal{B}^c_r(f) \mid I \cap K \neq \emptyset\}$
(ii) $\mathcal{B}^o_{r-1,K}(f) = \{I \in \mathcal{B}^o_{r-1}(f) \mid I \subset K\}$
and for $u \in \kappa \setminus 0$ the sets:
(iii) $S_{r,K,u}(f) = \mathcal{B}^c_{r,K}(f) \sqcup \mathcal{B}^o_{r-1,K}(f) \sqcup \mathcal{J}_{r,u}(f)$
(iv) $S_{r,u}(f) = \mathcal{B}^c_r(f) \sqcup \mathcal{B}^o_{r-1}(f) \sqcup \mathcal{J}_{r,u}(f) \sqcup \mathcal{J}_{r-1,u}(f)$.

In case $K \subset \mathbb{R}$ one considers

(i) $\tilde{\mathcal{B}}^c_{r,K}(f) = \{I \in \tilde{\mathcal{B}}^c_r(f) \mid I \cap K \neq \emptyset\}$
(ii) $\tilde{\mathcal{B}}^o_{r-1,K}(f) = \{I \in \tilde{\mathcal{B}}^o_{r-1}(f) \mid I \subset K\}$ and the sets:
(iii) $\tilde{S}_{r,K}(f) = \tilde{\mathcal{B}}^c_{r,K}(f) \sqcup \tilde{\mathcal{B}}^o_{r-1,K}(f) \sqcup \tilde{\mathcal{J}}_r(f)$
(iv) $\tilde{S}_r(f) = \tilde{\mathcal{B}}^c_r(f) \sqcup \tilde{\mathcal{B}}^o_{r-1}(f) \sqcup \tilde{\mathcal{J}}_r(f)$.

---

[6] $\kappa[T^{-1}, T]$ denotes the ring of Laurent polynomials with coefficients in κ

[7] instead of $\kappa[T^{-1}, T]]$ one can consider the field $\kappa[[T^{-1}, T]$ of Laurent power series in T^{-1}, which is isomorphic to $\kappa[T^{-1}, T]]$ by an isomorphism induced by $T \to T^{-1}$. The (Novikov) homology defined using this field has the same Novikov–Betti numbers as the the one defined using $\kappa[T^{-1}, T]]$.

These sets have the following properties:

(i) If K_1, K_2, K are closed multi-intervals in $\mathbb{S}^1$ or $\mathbb{R}$ with $K_1 \cap K_2 = \emptyset$ and $K = K_1 \cup K_2$ then $S_{r,K,u} = S_{r,K_1,u} \cup S_{r,K_2,u}$ and $\tilde{S}_{r,K} = \tilde{S}_{r,K_1} \cup \tilde{S}_{r,K_2}$

(ii) If K_1, K_2, K are closed multi-intervals in $\mathbb{S}^1$ or $\mathbb{R}$ with $K_1 \cap K_2 = K$ then $S_{r,K,u} = S_{r,K_1,u} \cap S_{r,K_2,u}$ and $\tilde{S}_{r,K} = \tilde{S}_{r,K_1} \cap \tilde{S}_{r,K_2}$,

(iii) If K_1, K_2 closed multi-intervals with $K_1 \subset K_2$ then $S_{r,K_1,u} \subseteq S_{r,K_2,u}$ and $\tilde{S}_{r,K_1} \subseteq \tilde{S}_{r,K_2}$.

For K a multi-interval in $\mathbb{S}^1$ or $\mathbb{R}$ denote by:
$\mathbb{I}_r(f; K, u) := \mathrm{img}(H_r(X_K) \to H_r(X; (\xi, u)))$, and
$\mathbb{I}_r(\tilde{f}; K) := \mathrm{img}(H_r(\tilde{X}_K) \to H_r(\tilde{X}))$.

With the notations and definitions above we have the following result which calculates the homologies of X and $\tilde{X}$.

Proposition 4.1. *Let $f : X \to \mathbb{S}^1$ be a tame map and suppose that for each r a decomposition of the representation $\rho_r(f)$ as a sum of bar code representations and Jordan block representations is given. Let $u \in \kappa \setminus 0$.*

1. One can provide the isomorphism

$$\omega_{r,u} : \kappa[S_{r,u}(f)] \to H_r(X; (\xi_f, u))$$

and for any closed multi interval $K \subset \mathbb{S}^1$ the isomorphism

$$\omega_{r,K,u} : \kappa[S_{r,K,u}(f)] \to \mathbb{I}_r(f; K, u)$$

such that for K', K closed multi-intervals in $\mathbb{S}^1$ with $K' \subset K$, the diagram

$$\begin{array}{ccccc} \mathbb{I}_r(f; K', u) & \xrightarrow{\subseteq} & \mathbb{I}_r(f; K, u) & \xrightarrow{\subseteq} & H_r(X; (\xi_f, u)) \\ \uparrow{\scriptstyle \omega_{r,K',u}} & & \uparrow{\scriptstyle \omega_{r,K,u}} & & \uparrow{\scriptstyle \omega_{r,u}} \\ \kappa[S_{r,K',u}(f)] & \longrightarrow & \kappa[S_{r,K,u}(f)] & \longrightarrow & \kappa[S_{r,u}(f)] \end{array} \tag{23}$$

is commutative. The horizontal arrows of the bottom line in the diagram are induced by the inclusions of the sets in brackets.

2. One can provide the isomorphism

$$\tilde{\omega}_r : \kappa[\tilde{S}_r(f)] \to H_r(\tilde{X})$$

and for any closed multi interval $K \subset \mathbb{R}$ the isomorphism

$$\tilde{\omega}_{r,K} : \kappa[\tilde{S}_{r,K}(f)] \to \mathbb{I}_r(\tilde{f}; K)$$

such that for K', K closed multi-intervals in $\mathbb{R}$ with $K' \subset K$, the diagram

$$\begin{array}{ccccc} \mathbb{I}_r(\tilde{f}; K') & \xrightarrow{\subseteq} & \mathbb{I}_r(\tilde{f}; K) & \xrightarrow{\subseteq} & H_r(\tilde{X}) \\ \uparrow{\scriptstyle \tilde{\omega}_{r,K'}} & & \uparrow{\scriptstyle \tilde{\omega}_{r;K}} & & \uparrow{\scriptstyle \tilde{\omega}_{r}} \\ \kappa[\tilde{S}_{r,K'}(f)] & \longrightarrow & \kappa[\tilde{S}_{r,K}(f)] & \longrightarrow & \kappa[\tilde{S}_{r}(f)] \end{array} \tag{24}$$

is commutative. The horizontal arrows in the bottom line are induced by the inclusions of the sets in brackets.

3. One can provide an isomorphism $\omega_r^N : \kappa[T^{-1}, T]][S_r] \to H_r^N(X; \xi_f)$.

It is also possible to calculate $H_r(X_K)$ for $K \subset \mathbb{S}^1$ and $H_r(\tilde{X}_K)$ for $K \subset \mathbb{R}$. In this case, in addition to closed and open bar codes and to Jordan blocks, mixed bar codes will appear. In this case it suffices to state the result for K consisting of only one interval say $[\theta_1, \theta_2],\ 0 \leq \theta_1 \leq \theta_2 < 2\pi$ in case of $\mathbb{S}^1$ and $[a, b], -\infty < a \leq b < \infty$ in case of $\mathbb{R}$.

To formulate the result one extends the sets $S_{r,K}(f), \tilde{S}_{r,K}(f)$ to $S'_{r,K}(f), \tilde{S}'_{r,K}(f)$, K a closed interval in $\mathcal{S}^1$ or $\mathbb{R}$ as follows.

For $K \subset \mathbb{S}^1$ define

$$S'_{r,K}(f) = \mathcal{B}'{}_{r,K}(f) \sqcup \mathcal{B}^o_{r-1,K}(f) \sqcup \mathcal{J}_r(f)$$

where $\mathcal{B}'_{r,K}(f) = \{I \in \mathcal{B}_r \mid I \cap K \neq \emptyset, \text{and closed}\}$ and for $K \subset \mathbb{R}$ define

$$\tilde{S}'_{r,K}(f) = \tilde{\mathcal{B}}'_{r,K}(f) \sqcup \tilde{\mathcal{B}}^o_{r-1,K}(f) \sqcup \mathcal{J}_r(f)$$

where $\tilde{\mathcal{B}}'_{r,K}(f) = \{I \in \tilde{\mathcal{B}}_r \mid I \cap K \text{ closed and} \neq \emptyset, \}$.

Proposition 4.2. *Under the same hypothesis as in Proposition 4.1 one has:*

1. For any pair of angles θ', θ'', $0 < \theta' \leq \theta'' < 2\pi$ one can provide the isomorphisms $\omega'_{r,[\theta',\theta'']} : \kappa[S'_{r,[\theta',\theta'']}(f)] \to H_r(X_{\theta',\theta''})$ so that for $0 < \theta_1 \leq \theta_2 \leq \theta_3 \leq \theta_4 < 2\pi$ the following diagram

$$\begin{array}{ccccc}
H_r(X_{[\theta_2,\theta_3]}) & \xrightarrow{v_r} & H_r(X_{[\theta_1,\theta_4]}) & \xrightarrow{v'_r} & H_r(X;(\xi_f,u)) \\
\uparrow {\scriptstyle \omega'_{r,[\theta_2,\theta_3]}} & & \uparrow {\scriptstyle \omega'_{r,[\theta_1,\theta_4]}} & & \uparrow {\scriptstyle \omega_{r,u}} \\
\kappa[S'_{r,[\theta_2,\theta_3]}(f)] & \longrightarrow & \kappa[S'_{r,[\theta_1,\theta_4]}(f)] & \longrightarrow & \kappa[S_{r,u}(f)].
\end{array} \tag{25}$$

is commutative.

2. For any pair of numbers a', b', $a' \leq b'$ or $a' = -\infty$ or $b' = \infty$ one can provide the isomorphisms $\tilde{\omega}'_{r,[a,b]} : \kappa[\tilde{S}'_{r,[a,b]}(f)] \to H_r(\tilde{X}_{[a,b]})$ so that for $a \leq b \leq c \leq d$ the following diagram

$$\begin{array}{ccccc}
H_r(\tilde{X}_{[b,c]}) & \xrightarrow{v_r} & H_r(\tilde{X}_{[a,d]} & \xrightarrow{v'_r} & H_r(\tilde{X}) \\
\uparrow {\scriptstyle \tilde{\omega}_{c,d}} & & \uparrow {\scriptstyle \tilde{\omega}_{a,d}} & & \uparrow {\scriptstyle \tilde{\omega}_{r}} \\
\kappa[\tilde{S}'_{r,[b,c]}(f)] & \longrightarrow & \kappa[\tilde{S}'_{r,[a,d]}(f)] & \longrightarrow & \kappa[\tilde{S}_r(f)]
\end{array} \tag{26}$$

is commutative.

In both cases the horizontal arrows in the top line are inclusion induced linear maps in homology, while in the bottom line are defined as follows: a bar code in the set $S'_{r,\cdots}$ or in $\tilde{S}'_{r,\cdots}$ is sent to itself if continues to belong to the next set or if not to the zero vector in the next vector space.

The isomorphisms claimed in these propositions are uniquely determined by the decomposition of ρ'_rs and by the choice of a splitting in the short exact sequences (16), (17), (15).

Let $\alpha \leq a \leq b \leq \beta$, $i(a, b; \alpha, \beta) : \tilde{X}_{[a,b]} \subseteq \tilde{X}_{[\alpha,\beta]}$ be the inclusion and $i_r(a, b : \alpha, \beta) : H_r(X_{[a,b]}) \to H_r(X_{[\alpha,\beta]})$ be the inclusion induced linear maps. The following corollary of Proposition 4.2 will be of use later.

Proposition 4.3. *Under the same hypothesis as in Proposition 4.1 one has:*

$$\begin{aligned}
\dim H_r(\tilde X_{[a,b]}) =& \sharp\{I \in \tilde{\mathcal{B}}_r(f) \mid I\cap [a,b]\neq\emptyset \text{ and closed}\}\\
&+\sharp\{I\in \tilde{\mathcal{B}}^o_{r-1}(f)\mid I\subset [a,b]\}+\sharp\tilde{\mathcal{J}}_r(f)\\
\dim \operatorname{img} i_r(a,b;\alpha,\beta) =& \sharp\{I \in \tilde{\mathcal{B}}_r(f) \mid I\cap [\alpha,\beta]\neq\emptyset \text{ and closed}, \mathrm{I}\cap[\mathrm{a},\mathrm{b}]\neq\emptyset\}+\\
&+\sharp\{I\in \tilde{\mathcal{B}}^o_{r-1}(f)\mid I\subset [a,b]\}+\sharp\tilde{\mathcal{J}}_r(f)\\
\dim \operatorname{coker} i_r(a,b;\alpha,\beta) =& \sharp\{I \in \tilde{\mathcal{B}}_r \mid I\cap [\alpha,\beta]\neq\emptyset \text{ and closed}, \mathrm{I}\cap[\mathrm{a},\mathrm{b}]=\emptyset\}\\
&+\sharp\{I\in \tilde{\mathcal{B}}^o_{r-1}\mid I\subset [\alpha,\beta], I\nsubseteq [a,b]\}\\
\dim \ker i_r(a,b;\alpha,\beta) =& \sharp\{I \in \tilde{\mathcal{B}}_r \mid I\cap [a,b]\neq\emptyset \text{ and closed}, \mathrm{I}\cap[\alpha,\beta] \text{ not closed}\}\\
\dim H_r(\tilde X_{[\alpha,\beta]},\tilde X_{[a,b]}) =& \dim\operatorname{coker} i_r(a,b;\alpha,\beta)+\dim\ker i_{r-1}(a,b;\alpha,\beta)
\end{aligned}$$

Proof of Propositions 4.1 and 4.2.

Proof. In view the properties of the sets $S_{K,\cdots}$ and $\tilde S_{K,\cdots}$ it suffices to prove the statements for K consisting of one single interval and in view the tameness of f one can suppose that θ_1,θ_2 are critical angles and a,b critical values.

For each r choose a decomposition of ρ_r in bar codes and Jordan blocks, which implies decompositions of $T_{k,l}(\rho_r)$s and choose a linear splitting $s:\ker(M((\rho_{r-1})_u)\to H_r(X;(\xi_f,u))$ of π in diagram (18).

We treat first the item (1.) in both propositions.

In view of the injectivity of v_r and v'_r, in diagrams (18) and (19) in Proposition 3.1, the splitting s provides by restriction the compatible splittings

$$s_{[\theta_1,\theta_4]}:\ker(M((T_{\theta_1,\theta_4}(\rho_{r-1}))\to H_r(X;_{[\theta_1,\theta_4]})$$

and

$$s_{[\theta_2,\theta_3]}:\ker(M((T_{\theta_2,\theta_3}(\rho_{r-1}))\to H_r(X;_{[\theta_2,\theta_3]}).$$

This leads to the commutative diagram (27) with horizontal arrows isomorphisms

$$\begin{array}{ccc}
\operatorname{coker} M(T_{\theta_2,\theta_3}(\rho_r))\oplus\ker M(T_{\theta_2,\theta_3}(\rho_{r-1})) & \longrightarrow & H_r(X_{[\theta_2,\theta_3]})\\
\downarrow{\scriptstyle v_l\oplus v_r} & & \downarrow{\scriptstyle v}\\
\operatorname{coker} M(T_{\theta_1,\theta_4}(\rho_r))\oplus\ker M(T_{\theta_1,\theta_4}(\rho_{r-1})) & \longrightarrow & H_r(X_{[\theta_1,\theta_4]})\\
\downarrow{\scriptstyle v'_l\oplus v'_r} & & \downarrow{\scriptstyle v'}\\
\operatorname{coker} M((\rho_r)_u)\oplus\ker M((\rho_{r-1})_u) & \longrightarrow & H_r(X;(\xi_f,u)).
\end{array}\tag{27}$$

Proposition 2.4 combined with Observation 2.1 gives the commutative diagram

$$\begin{array}{ccc}
\kappa[S'_{r,\theta_1,\theta_4}] & \longrightarrow & \operatorname{coker} M(T_{\theta_2,\theta_3}(\rho_r))\oplus\ker M(T_{\theta_2,\theta_3}(\rho_{r-1}))\\
\downarrow & & \downarrow{\scriptstyle v_l\oplus v_r}\\
\kappa[S'_{r,\theta_2,\theta_3}] & \longrightarrow & \operatorname{coker} M(T_{\theta_1,\theta_4}(\rho_r))\oplus\ker M(T_{\theta_1,\theta_4}(\rho_{r-1}))\\
\downarrow & & \downarrow{\scriptstyle v'_l\oplus v'_r}\\
\kappa[S_{r,u}] & \longrightarrow & \operatorname{coker} M((\rho_r)_u)\oplus\ker M((\rho_{r-1})_u).
\end{array}\tag{28}$$

The isomorphism ω_u (in Proposition 4.1) is the composition of horizontal arrows in the last line of diagrams (27) (28) while the isomorphism $\omega'_{r,[\theta_2,\theta_3]}$ and $\omega'_{r,[\theta_1,\theta_4]}$ (in Proposition 4.2) are the compositions of the horizontal arrows in the first and second lines of the same diagrams. The isomorphisms $\omega_{r,[\theta_2,\theta_3],u}$and $\omega_{r,[\theta_1,\theta_4],u}$ are restrictions of $\omega_{r,u}$. The commutativity of the diagrams claimed in Proposition 4.1 and 4.2 is the consequence of the commutativity of the diagrams (27), (28). This establishes item (1.) in both Propositions 4.1 and 4.2.

Item (2.) is verified essentially in the same way. More precisely:

The decompositions of $\rho'_r s$ imply decompositions of $\tilde\rho'_r s$ and $T_{k,l}(\tilde\rho_r)'s$. Observe that the commutative diagrams (27), (28) remain valid when we replace X by $\tilde X$, the representation ρ_r by $\tilde\rho_r$, and $\theta_1, \theta_2, \theta_3, \theta_4$ by a, b, c, d. In this case $\tilde\omega$ is defined as ω_u was, namely as the composition of the horizontal arrows of the last lines in the replaced diagrams (27), (28).

To check item (3.) in Proposition 4.1 one observes that $\omega^N = \omega \otimes \kappa[T^{-1}, T]]$.
$\square$

Proof of Theorem 1.1.

Proof. Item (1.) and item (3.) follow from Proposition 4.1 (3.) and (1.) To check item (2.) we first observe that the sequence (17)

$$0 \longrightarrow \operatorname{coker} M((\tilde\rho_r)) \longrightarrow H_r(\tilde X) \xrightarrow{\pi} \ker M((\tilde\rho_{r-1})) \longrightarrow 0$$

is actually a sequence of $\kappa[T^{-1}, T]-$modules where the multiplication by T on the first and third term in given by the $m-$shift described in the end of Section 2.

Next we consider the diagram (29), whose horizontal arrows on the second line are induced by inclusion and projection (cf. the definitions of the sets $\tilde S_r(f)$ and $\widetilde{\mathcal{J}}_r(f)$). Observe that the diagram is actually a commutative diagram of $\kappa[T^{-1}, T]-$ modules, with the module structure on the vector spaces located on the last two horizontal lines of the diagram (29) as described in Observation 2.5.

$$\begin{array}{ccccccccc}
 & & & & H_r(\tilde X)\otimes_{\kappa[T^{-1},T]}\kappa[T^{-1},T]] & & & & \\
 & & & & \uparrow & & & & \\
0 & \longrightarrow & \operatorname{coker} M((\tilde\rho_r)) & \longrightarrow & H_r(\tilde X) & \longrightarrow & \ker M(\tilde\rho_{r-1})) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \Psi^c} & & \uparrow{\scriptstyle \tilde\omega_r} & & \uparrow{\scriptstyle \Psi^o} & & \\
0 & \longrightarrow & \kappa[\mathcal{B}^c(\tilde\rho_r)\sqcup\widetilde{\mathcal{J}(\rho_r)}] & \longrightarrow & \kappa[\tilde S_r(f)] & \longrightarrow & \kappa[\mathcal{B}^o(\tilde\rho_{r-1})] & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & \kappa[\widetilde{\mathcal{J}(\rho_r)}] & \xrightarrow{=} & \kappa[\widetilde{\mathcal{J}}_r(f)] & & & &
\end{array} \tag{29}$$

In view of Observation 2.5 the $\kappa[T^{-1}, T]-$module $\kappa[\widetilde{\mathcal{J}(\rho_r)}] = \kappa[\widetilde{\mathcal{J}}_r(f)]$ is the $\kappa-$vector space $\bigoplus_{J\in\mathcal{J}_r} V(J)$ with the multiplication by T given by the linear isomorphism $\bigoplus_{J\in\mathcal{J}_r} T(J)$. This is exactly the torsion of the $\kappa[T^{-1}, T]-$module

$\kappa[\tilde{S}_r(f)]$ isomorphic to $H_r(\tilde{X})$ hence $V(\xi_f)$. This verifies item (2.) and then finishes the proof. □

5. Stability for configurations $C_r(f)$. Proof of Theorem 1.2

The proof of Theorems 1.2 and 1.3 will require an alternative definition of the configurations $C_r(f)$. This will be provided by the integer valued functions δ^f_r which will be defined for an arbitrary real valued tame map and then, via the infinite cyclic covering for an angle valued tame map.

5.1. **Real valued maps.** For $f : X \to \mathbb{R}$ a map and $a, b \in \mathbb{R}$ denote by:

(i) $X(a) = f^{-1}(a)$, $X^f_a = f^{-1}(-\infty, a])$, $X^b_f = f^{-1}([b, \infty))$, $X^b_a = X^a \cap X_b$ and $i_a : X_a \to X$, $i^b : X^b \to X$ the obvious inclusions,

(ii) $\mathbb{I}^f_a(r) := \mathrm{img}(i_a(r) : H_r(X_a) \to H_r(X))$, $\mathbb{I}^b_f(r) := \mathrm{img}(i^b(r) : H_r(X^b) \to H_r(X))$, and then

(iii) $F^f_r(a, b) := \dim(\mathbb{I}^f_a(r) \cap \mathbb{I}^b_f(r))$ and $G^f_r(a, b) := \dim H_r(X)/(\mathbb{I}^f_a(r) + \mathbb{I}^b_f(r))$.

and observe that:

Observation 5.1.

1. For $a \leq a'$ $b' \leq b$ $F^f_r(a, b) \leq F^f_r(a', b')$ and $G^f_r(a, b) \geq G^f(a', b')$

2. If $|f - g| < \epsilon$ and $a \leq b$ then $F^f_r(a - \epsilon, b + \epsilon) \leq F^g(a, b)$ and $G^f_r(a, b) \leq G^g(a - \epsilon, b + \epsilon)$

3. $F^f_r(a, b) = F^{-f}_r(-b, -a)$ and $G^f_r(a, b) = G^{-f}_r(-b, -a)$

To check (1.) notice that $X^f_a \subseteq X^f_{a'}$ and $X^{b'}_f \supseteq X^b_f$ which imply $\mathbb{I}^f_a \subseteq \mathbb{I}^f_{a'}$ and $\mathbb{I}^{b'}_f \subseteq \mathbb{I}^b_f$ hence $\mathbb{I}^f_a \cap \mathbb{I}^b_f \subseteq \mathbb{I}^f_{a'} \cap \mathbb{I}^{b'}_f$ and then the statement.

To check (2.) notice that $|f - g| < \epsilon$ implies $f - \epsilon < g < f + \epsilon$ which implies to $X^f_{a-\epsilon} \subseteq X^g_a$ and $X^f_{b+\epsilon} \subseteq X^g_b$. These inclusions imply $\mathbb{I}^f_{a-\epsilon} \subseteq \mathbb{I}^g_a$ and $\mathbb{I}^{b+\epsilon}_f \subseteq \mathbb{I}^b_g$ hence $F^f(a - \epsilon, b + \epsilon) \leq F^g(a, b)$. The arguments for G are similar.

To check (3.) one uses the fact that $f^{-1}((-\infty, a]) = (-f)^{-1}([-a, \infty))$ q.e.d

If X is a compact ANR it is immediate that both $F^f_r(a, b)$ and $G^f_r(a, b)$ are finite since $\dim H_r(X)$ is finite. The same remains true for $f : X \to \mathbb{R}$ a tame map with X not compact but this statement requires arguments since $\dim H_r(X)$ is not necessary finite. We have the following:

Proposition 5.2. *For $f : X \to \mathbb{R}$ a tame map then:*

1. $F^f_r(a, b) < \infty$,

2. $G^f_r(a, b) < \infty$,

3. If $a \geq b$ then $F^f_r(a, b) = \mathrm{img}(H_r(X^b_a) \to H_r(X))$

Proof. (1.) : In view of Observation 5.1 it suffices to check the statements for $a > b$. Consider

$$i_a(r) - i^b(r) : H_r(X_a) \oplus H_r(X^b) \to H_r(X)$$

and

$$i_a(r) + i^b(r) : H_r(X_a) \oplus H_r(X^b) \to H_r(X)$$

and observe that $\mathbb{I}^f_a(r) \cap \mathbb{I}^b_f(r)) = (i_a(r) + i^b(r))(\ker((i_a(r) - i^b(r)))$. Then

$$\dim(\mathbb{I}^f_a(r) \cap \mathbb{I}^b_f(r)) \leq \dim(\ker((i_a(r) - i^b(r))).$$

Since $a \geq b$ we have $X = X_a \cup X^b$. In view of the Mayer-Vietoris long exact sequence associated with $X = X_a \cup X^b$

$$\ker(i_a(r) - i^b(r)) = \mathrm{img}(: H_r(X_a^b) \to H_r(X_a) \oplus H_r(X^b))$$

has finite dimension since $\dim H_r(X_a^b)$ is finite.

(2.): If $a < b$ one uses the exact sequence of the pair $(X, X_a \sqcup X^b)$ to conclude that $H_r(X)/(\mathbb{I}_a^f(r) + \mathbb{I}_f^b(r))$ is isomorphic to a subspace of $H_r(X, X_a \sqcup X^b) = H_r(X_a^b, X(a) \sqcup X(b))$ which is of finite dimension. Indeed f tame implies $X(a), X(b)$ and X_a^b, compact ANRs, hence with finite dimensional homology.

If $a \geq b$ one use the Mayer-Vietoris exact sequence associated with X_a, X^b to conclude that $H_r(X)/(\mathbb{I}_a^f(r) + \mathbb{I}_f^b(r))$ is isomorphic to a subspace of $H_r(X_a^b)$ which is of finite dimension. This long exact sequence implies item (3.) as well.

□

Let $a < b, c < d$. We refer to the set

$$B(a, b : c, d) = (a, b] \times [c, d) \subset \mathbb{R}^2,\ a < b,\ c < d$$

as a "box " and define

$$\begin{aligned} \mu_r^{F,f}(B) =& F_r^f(a, d) + F_r^f(b, c) - F_r^f(a, c) - F_r^f(b, d) \\ \mu_r^{G,f}(B) =& -G_r^f(a, d) - G_r^f(b, c) + G_r^f(a, c) + G_r^f(b, d). \end{aligned} \tag{30}$$

One has

Proposition 5.3. *If X is compact or f is a tame map then:*

1. $\mu_r^{F,f}(B) = \mu_r^{G,f}(B)$.

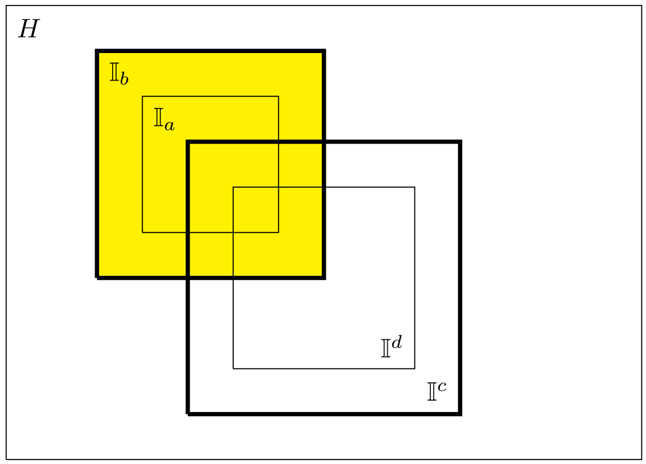
Let $\mu_r^f(B) := \mu_r^{F,f}(B) = \mu_r^{G,f}(B)$.

2. $\mu_r^f(B)$ is a nonnegative integer number.

3. If $B = B_1 \cup B_2, B_1 \cap B_2 = \emptyset$ with B_1, B_2, B_3 boxes then $\mu^f(B) = \mu^f(B_1) + \mu^f(B_2)$, in particular if the B' and B'' are two boxes with $B' \subset B''$ one has $\mu^f(B') \leq \mu^f(B'')$.

Proof. To ease the writing, we drop f and r from the notations involving $\mathbb{I}$ and f and introduce:

(i) $I_1 := \dim(\mathbb{I}_a \cap \mathbb{I}^d)$
(ii) $I_2 := \dim(\mathbb{I}_a \cap \mathbb{I}^c / \mathbb{I}_a \cap \mathbb{I}^d)$
(iii) $I_3 := \dim(\mathbb{I}_b \cap \mathbb{I}^d / \mathbb{I}_a \cap \mathbb{I}^d)$
(iv) $I_4 := \dim(\mathbb{I}_b \cap \mathbb{I}^c / \mathbb{I}_a \cap \mathbb{I}^c + \mathbb{I}_b \cap \mathbb{I}^d)$
(v) $I_5 := \dim \mathbb{I}_b / \mathbb{I}_a + \mathbb{I}_b \cap \mathbb{I}^c$
(vi) $I_6 := \dim \mathbb{I}^c / \mathbb{I}_a \cup \mathbb{I}^d + \mathbb{I}^d$
(vii) $I_7 := \dim H / \mathbb{I}_b + \mathbb{I}^c$ with $H = H_r(X)$.



Using the picture above is not hard to notice that:
$F(a,d) = I_1$
$F(b,c) = (I_1 + I_2 + I_3 + I_4)$
$F(a,c) = (I_1 + I_2)$
$F(b,d) = (I_1 + I_3)$
and
$G(a,d) = (I_7 + I_6 + I_5 + I_4)$
$G(b,c) = I_7$
$G(a,c) = (I_7 + I_5)$
$G(b,d) = (I_7 + I_6)$
Then we have:
$F(a,d)+F(b,c)-F(a,c)-F(b,d) = I_1+(I_1+I_2+I_3+I_4)-(I_1+I_2)-(I_1+I_3) = I_4$
and
$G(a,d)+G(b,c)-G(a,c)-G(b,d) = (I_7+I_6+I_5+I_4)+I_7-(I_7+I_5)-(I_7+I_6) = I_4$.
These equalities establish items (1.) and (2.). Item (3.) follows from definition by inspecting the relative positions of B_1 and B_2.

□

Define the *jump function*

$$\delta^f_r(a,b) := \lim_{\epsilon\to 0} \mu^f((a-\epsilon, a+\epsilon] \times [b-\epsilon, b+\epsilon)), \tag{31}$$

The limit exists since by Proposition 5.3 the right side decreases when ϵ decreases.

This function has values in $\mathbb{Z}_{\geq 0}$, since the critical values of a tame map are discrete, has discrete support and satisfies the following proposition.

Proposition 5.4. *If X compact or f is a tame map then:*

1. *For $a < b, c < d$ one has $\mu^f_r((a,b] \times [c,d)) = \sum_{a<x\leq b, c\leq y<d} \delta^f_r(x,y)$,*
2. *$F^f_r(b,c) = \sum_{-\infty, x\leq b; c\leq y,\infty} \delta^f_r(x,y)$,*
3. *$G^f_r(a,d) = \sum_{a\leq x<\infty; -\infty<y\leq c} \delta^f_r(x,y)$.*

Proof. Item (1.) follows from Proposition 5.3 (3.)
Item (2.) follows from item (1.) by making a goes to $-\infty$ and d to ∞ and item (3.) follows from item (1.) by making b goes to ∞ and c to $-\infty$. □

For a tame map f the set of critical values is discrete so they can be written as $\cdots c_i < c_{i+1} < \cdots$. Define

$$\epsilon(f) = \inf_{i\in\mathbb{Z}}(c_{i+1} - c_i).$$

Clearly if $f : X \to \mathbb{R}$ is tame with X compact then $\epsilon(f) > 0$ and if $f : X \to \mathbb{S}^1$ is tame then the infinite cyclic covering $\tilde{f} : \tilde{X} \to \mathbb{R}$ is tame and $\epsilon(\tilde{f}) > 0$.

Proposition 5.5. *Let $f : X \to \mathbb{R}$ be a tame map with $\epsilon(f) > 0$.*
1. For any $\epsilon, \epsilon' < \epsilon(f)$ one has:
$F^f_r(c_i, c_j) = F^f_r(c_i + \epsilon, c_j - \epsilon') = F^f_r(c_{i+1} - \epsilon, c_{j-1} + \epsilon')$,
2. $\delta^f_r(c_i, c_j) = F^f_r(c_{i-1}, c_{j+1}) + F^f_r(c_i, c_j) - F^f_r(c_{i-}, c_j) - F^f_r(c_i, c_{j+1})$.

Proof. The tameness of f and of the hypothesis the inclusions $X^f_{c_i} \subseteq X^f_{c_i+\epsilon}$, $X^f_{c_i} \subseteq X^f_{c_{i+1}-\epsilon'}$ and $X_f^{c_j-\epsilon} \supseteq X_f^{c_j}$, $X_f^{c_{j-1}+\epsilon'} \supseteq X_f^{c_j}$ induce isomorphisms in homology. These facts imply that $\mathbb{I}^f_{c_i} = \mathbb{I}^f_{c_i+\epsilon} = \mathbb{I}^f_{c_{i+1}-\epsilon'}$ and $\mathbb{I}_f^{c_{j-1}+\epsilon} = \mathbb{I}_f^{c_j-\epsilon'} = \mathbb{I}_f^{c_j}$ which imply item (1.). To check item (2.) recall that in view of the definition, for ϵ very small, one has $\delta^f(c_i, c_j) = F(c_i-\epsilon, c_j+\epsilon)+F(c_i+\epsilon, c_j-\epsilon)-F(c_i-\epsilon, c_j-\epsilon)-F(c_i+\epsilon, c_j+\epsilon)$. Item (2.) follows then from item (1.) by taking $\epsilon < \epsilon(f)$. □

For a pair $(a, b) \in \mathbb{R}^2$ and $\epsilon > 0$ consider the box $B(a, b; 2\epsilon) = (a - 2\epsilon, a + 2\epsilon] \times [b - 2\epsilon, b + 2\epsilon)$.

Proposition 5.6. *Let $f : X \to \mathbb{R}$ be a tame map. For any $\epsilon < \epsilon(f)/6$, g an tame map with $|f - g| < \epsilon$ and $(a, b) \in \mathrm{supp}\ \delta^{\mathrm{f}}_{\mathrm{r}}$ one has:*
1. $\mathrm{supp}\ \delta^{\mathrm{f}}_{\mathrm{r}} \cap \mathrm{B(a, b; 2\epsilon)} \equiv \mathrm{(a, b)}$
2. $\sharp(\mathrm{supp}\ \delta^{\mathrm{g}} \cap (\sqcup_{(\mathrm{a,b})\in \mathrm{supp}\ \delta^{\mathrm{f}}} \mathrm{B(a, b; 2\epsilon)})) = \sharp(\mathrm{supp}\ \delta^{\mathrm{f}}_{\mathrm{r}})$.
In particular if the cardinality of the supports[8] of δ^f_r and δ^g_r are equal and $|g-f| < \epsilon$, then the support of δ^g_r lies in an $\epsilon-$neighborhood [9] of the support of δ^f_r.

Proof. To simplify the writing the index r will be omitted from the notation.

Item (1.) follows from definition of δ^f.

To prove item (2.) observe that if $(a, b) \in \mathrm{supp}\ \delta^{\mathrm{f}}$ both numbers have to be critical values, hence the $a = c_i, b = c_j$. In view of Proposition 5.5, for any $\epsilon', \epsilon'' < \epsilon(f)/2$ one has

$$\begin{aligned} F^f(c_{i-1}, c_{j+1}) =& F^f(a - \epsilon', b + \epsilon'') \\ F^f(c_i, c_j) =& F^f(a + \epsilon', b - \epsilon'') \\ F^f(c_i, c_{j+1}) =& F^f(a + \epsilon', b + \epsilon'') \\ F^f(c_{i-1}, c_j) =& F^f(a - \epsilon', b - \epsilon''). \end{aligned} \tag{32}$$

Since $|f - g| < \epsilon$, in view of Observation 5.1 one has

[8]recall that the cardinality of the support is the sum of multiplicity of the elements in the support

[9]here $\epsilon-$neighborhood of (a, b) means the domain $(a - \epsilon, a + \epsilon) \times (b - \epsilon, b + \epsilon)$

$$\begin{aligned} F^f(a-3\epsilon,b+3\epsilon) \leq F^g(a-2\epsilon,b+2\epsilon) &\leq F^f(a-\epsilon,b+\epsilon),\\ F^f(a+\epsilon,b-\epsilon) \leq F^g(a+2\epsilon,b-2\epsilon) &\leq F^f(a+3\epsilon,b-3\epsilon),\\ F^f(a+\epsilon,b+3\epsilon) \leq F^g(a+2\epsilon,b+2\epsilon) &\leq F^f(a+3\epsilon,b+\epsilon),\\ F^f(a-3\epsilon,b-\epsilon) \leq F^g(a-2\epsilon,b-2\epsilon) &\leq F^f(a-\epsilon,b-3\epsilon). \end{aligned} \tag{33}$$

Since $\epsilon < \epsilon(f)/6$, (32) and (33) imply that

$$\begin{aligned} F^g(a-2\epsilon,b+2\epsilon) &= F^f(c_{i-1},c_{j+1})\\ F^g(a+2\epsilon,b-2\epsilon) &= F^f(c_i,c_j)\\ F^g(a+2\epsilon,b+2\epsilon) &= F^f(c_i,c_{j+1})\\ F^g(a-2\epsilon,b-2\epsilon) &= F^f(c_{i-1},c_j). \end{aligned} \tag{34}$$

In view of Proposition 5.4

$$\begin{aligned} \sharp(\mathrm{supp}\ \delta^{\mathrm{g}} \cap \mathrm{B(a,b:2\epsilon)}) = \mu^{\mathrm{g}}(\mathrm{B(a,b:2\epsilon)}) = \\ F^g(a-2\epsilon,b+2\epsilon)+F^g(a+2\epsilon,b-2\epsilon)\\ -F^g(a-2\epsilon,b-2\epsilon)-F^g(a+2\epsilon,b+2\epsilon) \end{aligned}$$

which in view of (33) and (34) and Proposition 5.5 (2.) leads to

$$\sharp(\mathrm{supp}\ \delta^{\mathrm{g}} \cap \mathrm{B(a,b:2\epsilon)}) = \sharp(\mathrm{supp}\ \delta^{\mathrm{f}} \cap \mathrm{B(a,b:2\epsilon)}) = \delta^{\mathrm{f}}(\mathrm{a,b}).$$

□

5.2. **Angle valued maps.** Let $f : X \to \mathbb{S}^1$ be a tame map and $\tilde{f} : \tilde{X} \to \mathbb{R}$ its infinite cyclic covering. Recall that $\epsilon(\tilde{f}) > 0$ and observe that

$$\delta_r^{\tilde{f}}(a,b) = \delta_r^{\tilde{f}}(a+2\pi, b+2\pi). \tag{35}$$

Consider the projection Let $p : \mathbb{R}^2 \to \mathbb{T} = \mathbb{R}^2/\mathbb{Z}$, with $\mathbb{T}$ the quotient space of $\mathbb{R}^2$ by the action $\mu : \mathbb{Z} \times \mathbb{R}^2 \to \mathbb{R}^2$ given by $\mu(n,(a,b)) = (a+2\pi n, b+2\pi n)$.

Define

$$\epsilon(f) := \epsilon(\tilde{f})$$

and

$$\delta_r^f(p(a,b)) := \delta_r^{\tilde{f}}(a,b). \tag{36}$$

In view of (35) δ_r^f is a well defined integer valued function with finite support. and Proposition 5.6 holds for $f : X \to \mathbb{S}^1$ with exactly the same conclusion.

Proposition 5.6 equally implies that the cardinality of the support of δ_r^g with g closed enough to f in C^0 topology is larger or equal to the cardinality of the support of δ_r^f and therefore the cardinality of the supports of tame maps in the same connected components is constant, a fact we already knew by Theorem 1.1.

For the proof of Theorem 1.2 we also need to show that δ_r^f and $C_r(f)$ when viewed as functions on $\mathbb{T}$ are equal.

Proposition 5.7. *If f is a tame real or angle valued map defined on X, a compact ANR, then δ_r^f and $C_r(f)$ are equal as functions.*

Proof. We check the case of an angle valued map $f : X \to \mathbb{S}^1$ only. The real valued case can be regarded as a particular case of this one.First note that $\epsilon(f) > 0$. In view of the definition of $\delta_r^{\tilde f}$ it suffices to check that:

(i) If at least one, a or b, is not a critical value then we have $\delta_r^{\tilde f}(a, b) = 0$.
(ii) If $a = c_i$ $b = c_j$ are critical value with $c_i \geq c_j$

$$\delta_r^{\tilde f}(c_i, c_j) = \sharp\{I \in \tilde{\mathcal{B}}^c_r(f) \mid I = [c_j, c_i]\}.$$

(iii) If $a = c_i$ $b = c_j$ are critical value with $c_i < c_j$

$$\delta_r^{\tilde f}(c_i, c_j) = \sharp\{I \in \tilde{\mathcal{B}}^o_{r-1}(f) \mid I = (c_j, c_j)\}.$$

Recall that $\delta_r(a, b) := \lim_{\epsilon\to 0}(-F_r(a - \epsilon, b - \epsilon) - F_r(a + \epsilon, b + \epsilon) + F_r(a - \epsilon, b + \epsilon) + F_r(a + \epsilon, b - \epsilon)$.

In view of Proposition 5.5 if a is not critical value, for ϵ sufficiently small $F_r^{\tilde f}(a - \epsilon, \cdots) = F_r^{\tilde f}(a + \epsilon, \cdots)$ which implies $\delta_r^{\tilde f}(a, \cdots) = 0$, and if b is not critical value for ϵ sufficiently small $F_r^{\tilde f}(\cdots, b - \epsilon) = F_r^{\tilde f} \cdots, b + \epsilon)$ which implies $\delta_r^{\tilde f}(\cdots, b) = 0$. This establishes statement (i)

Suppose that $a = c_i$ and $b = c_j$ critical values. In view of Proposition 5.5 and of the definition of $\delta^{\tilde f}$ one obtains

$$\delta_r^{\tilde f}(c_i, c_j) = -F_r^{\tilde f}(c_{i-1}, c_j) - F_r^{\tilde f}(c_i, c_{j+1}) + F_r^{\tilde f}(c_{i-1}, c_{j+1}) + F_r^{\tilde f}(c_i, c_j) \qquad (37)$$

By Propositions 5.2 and 4.3, when $c_i \geq c_j$, one has

$$F_r^{\tilde f}(c_i, c_j) = \sharp \begin{cases} \{I \in \tilde{\mathcal{B}}^c_r(f) \mid I \cap [c_j, c_i] \neq \emptyset\} \sqcup \\ \{I \in \tilde{\mathcal{B}}^o_{r-1}(f) \mid I \subset (c_j, c_i) \sqcup \\ \qquad \tilde{\mathcal{J}}_r(f) \end{cases} \qquad (38)$$

and when $c_i > c_j$, in view of Proposition 4.1 one has

$$F_r^{\tilde f}(c_i, c_j) = \sharp \begin{cases} \{I \in \tilde{\mathcal{B}}^c_r(f) \mid I \supset [c_i, c_j] \sqcup \\ \qquad \tilde{\mathcal{J}}_r(f) \end{cases} \qquad (39)$$

Comparing the collections of bar codes whose cardinality are given by $F_r^{\tilde f}(c_{i-1}, c_j)$, $F_r^{\tilde f}(c_i, c_{j+1})$, $F_r^{\tilde f}(c_{i-1}, c_{j+1})$ and $F_r^{\tilde f}(c_i, c_j)$ and using (37) and (38) one derives the statement ii), and using (37) and 39) one derives the statement iii).

□

5.3. **Proof of Theorem 1.2.** We begin with a few observations.

(i) Consider the space of continuous maps $C(X, \mathbb{S}^1)$, X a compact ANR, with the compact open topology. This topology is induced from the metric $D(f, g) := \sup_{x\in\mathbb{X}} d(f(x), g(x))$, with $''d''$ the geodesic distance on $\mathbb{S}^1$ given by $d(\theta_1, \theta_2) = \inf(|\theta_1 - \theta_2|, 2\pi - |\theta_1 - \theta_2|)$, $0 \leq \theta_1, \theta_2 < 2\pi$. With this metric $(C(X, \mathbb{S}^1), D)$ is complete.
(ii) Consider $S^N\mathbb{T} = (\mathbb{T} \times \mathbb{T} \cdots \mathbb{T})/\Sigma_N$, with Σ_N is the $N-$symmetric group acting on the $N-$fold cartesian product of $\mathbb{T}$ by permutations equipped equipped with the induced metric $\underline{D}$ induced from the complete metric on $\mathbb{T}/\mathbb{Z}$. With this metric $(S^N(\mathbb{T}), \underline{D})$ is complete.

(iii) Observe that if f, g are in a connected component $C_\xi(X, \mathbb{S}^1)$ of $C(X, \mathbb{S}^1)$ and $D(f,g) < \pi$ then for any $t \in [0,1]$ the map $h_t := h_t(f,g) \in C(X; \mathbb{S}^1)$ defined by the formulae

$$h_t(x) = \begin{cases} tf(x) + (1-t)g(x) \text{ if } 0 \leq \text{g(x), f(x)} < 2\pi, \text{f(x)} \leq \text{g(x)} \\ (1-t)f(x) + tg(x) \text{ if } 0 \leq \text{g(x), f(x)} < 2\pi, \text{f(x)} \geq \text{g(x)} \end{cases}$$

is continuous and lies in the connected component of $C_\xi(X, \mathbb{S}^1)$ and for any $0 = t_0 < t_1 \cdots t_{N-1} < t_N = 1$ one has

$$D(f,g) = \sum_{0 \leq i < N} D(h_{t_{i+1}}, h_{t_i}). \tag{40}$$

(iv) If X is a simplicial complex and $\mathcal{U} \subset C_\xi(X, \mathbb{S}^1)$ denotes the subset of p.l-maps then:
1. $\mathcal{U}$ is a dense subset
2. $f, g \in \mathcal{U}$ implies $h_t \in \mathcal{U}$ hence $\epsilon(h_t) > 0$ hence for any $t \in [0,1]$ there exists $\delta(t) > 0$ so that $|t' - t| < \delta(t)$ implies $D(h_{t'}, h_t) < \epsilon(h_t)/6$.

Recall that f is p.l on X if with respect to some subdivision is simplicial (i.e. the liftings to $\mathbb{R}$ of the restriction of f to simplexes are linear) and for any two p.l maps f, g there exists a common subdivision of X which makes f and g simultaneously simplicial, hence any h_t is a simplicial map. Item (1.) follows from approximability of continuous maps by p.l maps and item (2.) from the continuity in t of the family h_t and of the compacity of X.

(v) Proposition 5.6 states that $f, g \in C(X, \mathbb{S}^1)_{t,\xi}$ and $D(f,g) < \epsilon(f)/6$ implies

$$\underline{D}(\delta^f_r, \delta^g_r) < 2D(f,g). \tag{41}$$

The above observations combined imply Theorem 1.2. Indeed, Item (v.) makes $\delta : C(X; \mathbb{S}^1)_{t,\xi} \to S^N(\mathbb{T})$ a continuous map and establishes the continuity of the assignment $C(X, \mathbb{S}^1)_{t,\xi} \ni f \to \delta^f_r \in S^N(\mathbb{T})$ $N = \beta^N_r(X, \xi)$. To conclude the existence of a continuous extension of δ_r to the entire $C(X, \mathbb{S}^1)$, in view of item (i) and (ii) and (iv), it suffices to show that for a Cauchy sequence $\{f_n\}$, $f_n \in \mathcal{U}$, $\delta^{f_n}_r$ is a Cauchy sequence in $S^N(\mathbb{T})$. This will follow once we can show that for any two $f, g \in \mathcal{U}$ with $d(f,g) < \pi$ we have $\underline{D}(\delta^f_r, \delta^g) \leq 2D(f,g)$. To establish this last fact we proceed as follows.

Start with $f, g \in \mathcal{U}$ with $D(f,g) < \pi$ and consider h_t, $t \in [o, 1]$ defined above.

Choose a sequence $0 = t_0 < t_2 < t_4, \cdots t_{2N-2} < t_{2N} = 1$ so that the open intervals $I_{2i} = (t_{2i} - \delta(t_{2i}), t_{2i} + \delta(t_{2i}))$ cover $[0,1]$. The compacity of $[0,1]$ makes this possible.

By possibly removing some of the points t_{2i}s and decreasing $\delta(t_{2i})$ one can make $I_{2i} \cap I_{2i+2} \neq \emptyset$ and $t_{2t\_2}, t_{2i+2} \notin I_{2i}$. Choose $t_1 < t_3 < \cdots t_{2N-1}$ with $t_{2i} < t_{2i+1} < t_{2i}$ and $t_{2i+1} \in I_{2i} \cap I_{2i+2}$. We have then $|t_{2i+1} - t_{2i}| < \delta(t_{2i})$ and $|t_{2i+2} - t_{2i+1}| < \delta(t_{2i+2})$.

In view of item (iv) $|t_{2i+1} - t_{2i}| < \delta(t_{2i})$ implies $D(h_{t_{2i}}, h_{t_{2i+1}}) < \epsilon(h_{t_{2i}})/6$ and $|t_{2i+2} - t_{2i+1}| < \delta(t_{2i+2})$ implies $D(h_{t_{2i+2}}, h_{t_{2i+1}}) < \epsilon(h_{t_{2i+2}})/6$. In view of item (v) the last inequalities imply $\underline{D}(\delta_r^{h_{t_{2i+1}}}, \delta_r^{h_{t_{2i}}}) < 2D(h_{t_{2i}}, h_{t_{2i+1}})$ and $\underline{D}(\delta_r^{h_{t_{2i+2}}}, \delta_r^{h_{t_{2i+1}}}) < 2D(h_{t_{2i+2}}, h_{t_{2i+1}})$. Therefore, for any $0 \leq k \leq 2N-1$ one has $\underline{D}(\delta_r^{h_{t_{k+1}}}, \delta_r^{h_{t_k}}) <$

$2D(h_{t_{k+1}}, h_{t_k})$. Then

$$\underline{D}(\delta^f, \delta^g) \leq \sum_{0\leq i<2N-1} D(\delta^h(t_{i+1}, \delta^h(t_i)) \leq 2 \sum_{0\leq i<2N-1} D(h_{t_{i+1}}, h_{t_i}).$$

which by item (iii) is exactly $D(d, g)$.

This finishes the proof of Theorem 1.2.

6. Poincaré duality for configurations $C_r(f)$. Proof of Theorem 1.3

For an $n-$dimensional manifold Y, not necessary compact, Poincaré Duality can be better formulated using Borel–Moore homology, cf. [3], especially tailored for locally compact spaces Y and pairs (Y, K), K closed subset of Y. Borel Moore homology coincides with the standard homology when Y is compact. In general, for a locally compact space Y can be described as the inverse limit of the homology $H_r(Y, Y\setminus U)$ for all U open sets with compact closure. One denotes the Borel–Moore homology in dimension r by H_r^{BM}. For Y a $n-$dimensional topological $\kappa-$orientable manifold, $g : Y \to \mathbb{R}$ a tame map and a a regular value of g,[10] Poincaré Duality provides the commutative diagrams

$$\begin{array}{ccccc} H_r^{BM}(Y_a) & \longrightarrow & H_r^{BM}(Y) & \longrightarrow & H_r^{BM}(Y, Y_a) \\ \downarrow & & \downarrow & & \downarrow \\ H^{n-r}(Y, Y^a) & \longrightarrow & H^{n-r}(Y) & \longrightarrow & H^{n-r}(Y^a) \\ \downarrow & & \downarrow & & \downarrow \\ (H_{n-r}(Y, Y^a))^* & \longrightarrow & (H_{n-r}(Y))^* & \longrightarrow & (H_{n-r}(Y^a))^* \end{array} \quad . \tag{42}$$

$$\begin{array}{ccccc} H_r^{BM}(Y^a) & \longrightarrow & H_r^{BM}(Y) & \longrightarrow & H_r^{BM}(Y, Y^a) \\ \downarrow & & \downarrow & & \downarrow \\ H^{n-r}(Y, Y_a) & \longrightarrow & H^{n-r}(Y) & \longrightarrow & H^{n-r}(Y_a) \\ \downarrow & & \downarrow & & \downarrow \\ (H_{n-r}(Y, Y_a))^* & \longrightarrow & (H_{n-r}(Y))^* & \longrightarrow & (H_{n-r}(Y_a))^* \end{array} \quad . \tag{43}$$

The first vertical arrow in each column of both diagrams is the Poincaré Duality isomorphism, the second is the the isomorphism between cohomology and the dual of homology with coefficients in a field. The horizontal arrows are induced by the inclusions of Y_a or Y^a in Y and the inclusion of pairs $(Y, \emptyset)$ in (Y, Y_a) or (Y, Y^a).

We apply diagrams (42) and (43) to $Y = \tilde{M}^n$ and $g = \tilde{f}$, with $\tilde{f} : \tilde{M} \to \mathbb{R}$ the infinite cyclic covering of $f : M^n \to \mathbb{S}^1$, a tame map defined on a closed $\kappa-$

[10] i.e. $f : f^{-1}(a - \epsilon, a + \epsilon) \to (a - \epsilon, a + \epsilon)$ is a fibration

orientable topological manifold and obtain

$$\begin{array}{ccccc}
H_r^{BM}(\tilde M_a) & \xrightarrow{i_a(r)} & H_r^{BM}(\tilde M) & \xrightarrow{j_a(r)} & H_r^{BM}(\tilde M, \tilde M_a) \\
\downarrow & & \downarrow & & \downarrow \\
H^{n-r}(\tilde M, \tilde M^a) & \xrightarrow{s^a(n-r)} & H^{n-r}(\tilde M) & \xrightarrow{r^a(n-r))} & H^{n-r}(\tilde M^a) \\
\downarrow & & \downarrow & & \downarrow \\
(H_{n-r}(\tilde M, \tilde M^a))^* & \xrightarrow{(j^a(n-r))^*} & (H_{n-r}(\tilde M))^* & \xrightarrow{(i^a(n-r))^*} & (H_{n-r}(\tilde M^a))^*
\end{array} \tag{44}$$

$$\begin{array}{ccccc}
H_r^{BM}(\tilde M^b) & \xrightarrow{i^b(r)} & H_r^{BM}(\tilde M) & \xrightarrow{j^b(r)} & H_r^{BM}(\tilde M, \tilde M^b) \\
\downarrow & & \downarrow & & \downarrow \\
H^{n-r}(\tilde M, \tilde M_b) & \xrightarrow{(s_b(n-r))^*} & H^{n-r}(\tilde M)) & \xrightarrow{r_b(n-r)} & H^{n-r}(\tilde M_b) \\
\downarrow & & \downarrow & & \downarrow \\
(H_{n-r}(\tilde M, \tilde M_b))^* & \xrightarrow{(j_b(n-r))^*} & (H_{n-r}(\tilde M))^* & \xrightarrow{(i_b(n-r))^*} & (H_{n-r}(\tilde M_b))^*
\end{array} . \tag{45}$$

For $\tilde M, \tilde M_a, \tilde M^a$ the Borel–Moore homology can be described as the following inverse limits :

$$\begin{aligned}
H_r^{BM}(\tilde M) &= \varprojlim_{0<l\to\infty} H_r(\tilde M, \tilde M_{-l} \sqcup \tilde M^l), \\
H_r^{BM}(\tilde M_a) &= \varprojlim_{0<l\to\infty} H_r(\tilde M_a, \tilde M_{a-l}), \\
H_r^{BM}(\tilde M^a) &= \varprojlim_{0<l\to\infty} H_r(\tilde M, \tilde M^{a+l}), \\
H_r^{BM}(\tilde M, \tilde M_a) &= \varprojlim_{0<l\to\infty} H_r(\tilde M, \tilde M_a \sqcup \tilde M^{a+l}), \\
H_r^{BM}(\tilde M, \tilde M^a) &= \varprojlim_{0<l\to\infty} H_r(\tilde M, \tilde M^a \sqcup \tilde M_{a-l}).
\end{aligned} \tag{46}$$

The inclusion of pairs $(\tilde M, \tilde M_{-l'} \sqcup \tilde M^{l'}) \subseteq (\tilde M, \tilde M_{-l} \sqcup \tilde M^l)$ for $l' > l$ induces in homology an invers system whose limit is $H_r^{BM}(\tilde M)$. Similar inclusions of pairs associated with $l' > l$ induce inverse systems whose limits are the remaining Borel–Moore homology vector spaces considered above.

The horizontal arrows in both diagrams are inclusion (possibly of pairs) induced linear maps in homology when denoted by $i(\cdots)$s and $j(\cdots)$s or cohomology when denoted by $r(\cdots)$s and $s(\cdots)$s .

In view of the above involvement of Borel–Moore homology, in addition to $\mathbb{I}_a^{\tilde f}(r)$ and $\mathbb{I}_{\tilde f}^a(r)$, consider

$$\begin{aligned}
\mathbb{I}_a^{BM,\tilde f}(r) &= \mathrm{img}(H_r^{BM}(\tilde X_a) \to H_r^{BM}(\tilde X)), \\
\mathbb{I}_{\tilde f}^{BM,a}(r) &= \mathrm{img}(H_r^{BM}(\tilde X^a) \to H_r^{BM}(\tilde X)),
\end{aligned}$$

and $F_r^{BM,f}(a,b) = \dim(\mathbb{I}_a^{BM,\tilde f}(r) \cap \mathbb{I}_{\tilde f}^{BM,b}(r))$.

Note that the exact sequences in Borel–Moore homology of the pairs $(\tilde M, \tilde M_a)$ or $(\tilde M, \tilde M^b)$, the top lines of the two diagrams, give

$$F^{BM,\tilde f}(a,b) = \mathbb{I}_a^{BM,\tilde f}(r) \cap \mathbb{I}_{\tilde f}^{BM,b}(r) = \ker(j_a^{BM}(r), j^{BM,b}(r)). \tag{47}$$

Looking to the right side corners of the diagrams (44) and (45) one concludes that

$$\ker(j_a^{BM}(r), j^{BM,b}(r)) \equiv \ker(r^a(n-r), r_b(n-r)). \tag{48}$$

In view of the canonical isomorphism between cohomology the dual of homology one obtains

$$\ker((r^a(n-r), r_b(n-r)) \equiv (\text{coker}(i^a(n-r) + i_b(n-r)))^*. \tag{49}$$

In view of the definition and of the finite dimensionality of $G^{\tilde f}(a,b)$ one obtains

$$G^{\tilde f}_{n-r}(b,a) := \dim(\text{coker}(i_b(n-r) + i^a(n-r)) = \dim(\text{coker}(i_b(n-r) + i^a(n-r)))^*. \tag{50}$$

Note also that

$$G^{\tilde f}(a,b) = G^{-\tilde f}(-b,-a). \tag{51}$$

Consequently $F_r^{BM,\tilde f}(a,b) = G^{-\tilde f}_{n-r}(-a,-b)$.

In order to conclude that

$$\delta^{\tilde f}_r(a,b) = \delta^{-\tilde f}_{n-r}(-a,-b). \tag{52}$$

it suffices to show that the function $\delta_r^{BM,\tilde f}$ calculated from $F_r^{BM,\tilde f}$ using (31) is the same as the function $\delta^{\tilde f}_r$. If so we obtain

$$\delta^f_r(z) = \delta^{\overline{f}}_{n-r}(z^{-1}) \tag{53}$$

for $z = e^{ia+(b-a)}$,which establishes Theorem 1.3.

For this purpose we need the following proposition.

Proposition 6.1. *$F_r^{BM,\tilde f}(a,b) + \sharp\tilde{\mathcal{J}}_r(f) = F_r^{\tilde f}(a,b)$ with $\sharp$ meaning " cardinality".*

Proposition 6.1 is proved by comparing $F_r^{BM,\tilde f}(a,b)$ and $F_r^{\tilde f}(a,b)$ calculated in terms of number of bar codes with the help of Propositions 4.1 and 4.2.

The final outcome of the calculation can be summarized as follows: $F_r^{BM,\tilde f}(a,b) = \sharp S'$ and $F_r^{\tilde f}(a,b) = \sharp\{S' \sqcup S''\}$ where

when $a \leq b$ $S' = \{I \in \tilde{\mathcal{B}}^c_r \mid I \supseteq [a,b]\}$ and

when $a > b$ $S' = \{I \in \tilde{\mathcal{B}}^c_r \mid I \cap [b,a] \neq \emptyset\} \sqcup \{I \in \mathcal{B}^o_{r-1} \mid I \subset (b,a]\}$

and for any $a, b \in \mathbb{R}$, $S'' = \tilde{\mathcal{J}}_r$.

$F_r^{\tilde f}(a,b)$ can be read off from Proposition 4.1 directly. To calculate $F_r^{BM,\tilde f}(a,b)$ one has to describe $H_r^{BM}(\tilde X_a) \to H_r^{BM}(\tilde X)$ and $H_r^{BM}(\tilde X^b) \to H_r^{BM}(\tilde X)$.

Recall that for an interval I we denote by $\tilde X_I := \tilde f^{-1}(I)$.

Notice that the long exact sequence of the pair $(\tilde X, \tilde X \setminus \tilde X_{(-a,a)})$ and the inclusion of pairs $(\tilde X, \tilde X \setminus \tilde X_{(-a',a')}) \subset (\tilde X, \tilde X \setminus \tilde X_{(-a,a)})$ for $a' > a$, gives rise to the commutative diagram whose lines are short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{coker}_r(-a,a) & \longrightarrow & H_r(\tilde{X},\tilde{X}\setminus\tilde{X}_{(-a,a)}) & \longrightarrow & \ker_{r-1}(-a,a) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{coker}_r(-a',a') & \longrightarrow & H_r(\tilde{X},\tilde{X}\setminus\tilde{X}_{(-a',a')}) & \longrightarrow & \ker_{r-1}(-a',a') & \longrightarrow & 0 \end{array}$$

where

$$\mathrm{coker}_r(-a,a) = \mathrm{coker}(H_r(\tilde{X}\setminus\tilde{X}_{(-a,a)}) \to H_r(\tilde{X}))$$
$$\ker_{r-1}(-a,a) = \ker((H_{r-1}(\tilde{X}\setminus\tilde{X}_{(-a,a)}) \to H_{r-1}(\tilde{X}))$$

In view of Proposition 4.1 one has

$$\varprojlim_{a\to\infty} \ker(H_{r-1}(\tilde{X}\setminus\tilde{X}_{(-a,a)}) \to H_{r-1}(\tilde{X})) = 0$$

and then

$$H_r^{BM}(\tilde{X}) = \varprojlim_{a\to\infty} \mathrm{coker}(H_r(\tilde{X}\setminus\tilde{X}_{(-a,a)}) \to H_r(\tilde{X})). \tag{54}$$

By similar arguments one derives

$$\begin{aligned} H_r^{BM}(\tilde{X}_a) &= \varprojlim_{a'\to-\infty} \mathrm{coker}(H_r(\tilde{X}_a\setminus\tilde{X}_{(a',a]}) \to H_r(\tilde{X}_a)), \\ H_r^{BM}(\tilde{X}^b) &= \varprojlim_{b'\to\infty} \mathrm{coker}(H_r(\tilde{X}^b\setminus\tilde{X}_{[b,b')}) \to H_r(\tilde{X})). \end{aligned} \tag{55}$$

From Proposition (4.1) for $a<b$ one derives that

$$\mathrm{coker}(H_r(\tilde{X}\setminus\tilde{X}_{(a,b)}) \to H_r(\tilde{X})) = H_r(\tilde{X})/\mathbb{I}_a^f(r) + \mathbb{I}_f^b(r)) = \kappa[\overline{S_{r,[a,b]}}]$$

where

$$\overline{S_{r,[a,b]}} = \{I\in\tilde{\mathcal{B}}_r^c(f) \mid I\subset(a,b)\} \sqcup \{I\in\tilde{\mathcal{B}}_{r-1}^o(f) \mid I\cap(a,b)\neq\emptyset\}.$$

which implies

$$H_r^{BM}(\tilde{X}) = Maps(\tilde{\mathcal{B}}_r^c(f)\sqcup\tilde{\mathcal{B}}_{r-1}^o(f),\kappa) \tag{56}$$

and identifies the canonical homomorphism $H_r(\tilde{X})\to H_r^{BM}(\tilde{X})$ to

$$\kappa[\tilde{\mathcal{B}}_r^c(f)\sqcup\tilde{\mathcal{B}}_{r-1}^o(f)\sqcup\tilde{J}_r(f)] \to Maps(\tilde{\mathcal{B}}_r^c(f)\sqcup\tilde{\mathcal{B}}_{r-1}^o(f),\kappa) \tag{57}$$

induced by sending the elements of $\tilde{\mathcal{J}}_r(f)$ to zero and the other to their characteristic map.

Similarly one obtains

$$\begin{aligned} H_r^{BM}(\tilde{X}_a) =& Maps(\overline{S}_{r,(-\infty,a]},\kappa) \\ H_r^{BM}(\tilde{X}^b) =& Maps(\overline{S}_{r,[b,\infty)},\kappa) \end{aligned} \tag{58}$$

where

$$\overline{S}_{r,(-\infty,a]} = \{I\in\tilde{\mathcal{B}}_r(f) \mid I\cap(-\infty,a] \text{ closed end } \neq\emptyset\} \sqcup \{\mathrm{I}\in\tilde{\mathcal{B}}^{\mathrm{o}}_{\mathrm{r-1}}(\mathrm{f}) \mid \mathrm{I}\subset(-\infty,\mathrm{a})\}$$
$$\overline{S}_{r,[b,\infty)} = \{I\in\tilde{\mathcal{B}}_r(f) \mid I\cap[b,\infty) \text{ closed end } \neq\emptyset\} \sqcup \{\mathrm{I}\in\tilde{\mathcal{B}}^{\mathrm{o}}_{\mathrm{r-1}}(\mathrm{f}) \mid \mathrm{I}\subset(\mathrm{b},\infty)\}.$$

with $H_r^{BM}(\tilde{X}_a)\to H_r^{BM}(\tilde{X}))$ and $H_r^{BM}(\tilde{X}^b)\to H_r^{BM}(\tilde{X}))$ identified to

$$Maps(\overline{S}_{r,(-\infty,a]},\kappa) \to Maps(\tilde{\mathcal{B}}_r^c(f)\sqcup\tilde{\mathcal{B}}_{r-1}^o(f),\kappa)$$
$$Maps(\overline{S}_{r,[b,\infty)},\kappa) \to Maps(\tilde{\mathcal{B}}_r^c(f)\sqcup\tilde{\mathcal{B}}_{r-1}^o(f),\kappa)$$

defined as follows: If $l \in Maps(\overline{S}_{r,\dots}, \kappa)$ its image $\hat{l} \in Maps((\tilde{\mathcal{B}}^c_r(f) \sqcup \tilde{\mathcal{B}}^o_{r-1}(f), \kappa))$ takes the same value as l on any barcode in $\overline{S}_{r,\dots}$ which belongs to $\tilde{\mathcal{B}}^c_r(f) \sqcup \tilde{\mathcal{B}}^o_{r-1}(f)$ and zero on all others. Using the definition of $F^{BM,\tilde{f}}_r(a,b)$ one obtains $F^{BM}_r(a,b) = \sharp S'$. q.e.d

7. The mixed bar codes. Proof of Theorem 1.5

As pointed out in Section 1 for a tame map $f : X \to \mathbb{S}^1$ the set $\tilde{\mathcal{B}}^{co}_r(f)$ and the collection $\tilde{\mathcal{B}}^{oc}_r(f)$ coincides with the collection of finite persistence bar codes associated to the filtration by the sub-levels and sup-levels of $\tilde{f}$ respectively, as defined in [12]. Precisely the multiplicity of the $r-$persistence barcode (a,b) of the map $\tilde{f}$ is the multiplicity of the closed-open bar code $[a,b)$ in the collection $\tilde{\mathcal{B}}^{co}_r(f)$ and the multiplicity of the $r-$persistence bar code $(-b,-a)$ for $-\tilde{f}$ is the multiplicity of the open-closed bar code $(a,b]$ in the collection $\tilde{\mathcal{B}}^{oc}_r(f)$. This can be easily derived from Proposition4.3 and the relationship between persistence bar codes and persistent homology.

As indicated in Section 1 one can record the closed open $r-$bar code $[a,b)$ as the point $(a,b) \in \mathbb{R}^2 \setminus \Delta$ (above the diagonal) and to open closed $r-$bar code$(c,d]$ as the point $(d,c) \in \mathbb{R}^2 \setminus \Delta$ (below diagonal), equivalently we put together the $r-$persistence diagrams of $\tilde{f}$ and of $-\tilde{f}$. We obtain in this way a configuration $C^m_r(\tilde{f})$ of points in $\mathbb{R}^2 \setminus \Delta$, which defines the configuration $C^m_r(f)$ of points in $\mathbb{T} \setminus \Delta_{\mathbb{T}}$. There is no interaction between points above diagonal and below diagonal when the map f varies, so associating closed-open $r-$bar codes with open-closed $r-$barcodes is only an issue of economy rather than meaning.

One can derive the configuration $C^m_r(f)$ as the "jump function" of the two variable function $T^{\tilde{f}}_r : \mathbb{R}^2 \setminus \Delta \to \mathbb{Z}_{\geq 0}$ in the manner described in section 5 for the configuration $C_r(f)$. The function $T^{\tilde{f}}_r$ is defined by:

$$T^{\tilde{f}}(a,b) = \begin{cases} \dim\ker(H_r(\tilde{X}_a) \to H_r(\tilde{X}_b)) \text{ if } a < b \\ \dim\ker(H_r(\tilde{X}^b) \to H_r(\tilde{X}^a)) \text{ if } a > b \end{cases}$$

If f is tame then so is $\tilde{f}$ and the limit

$$\delta^{m,\tilde{f}}_r(a,b) = \lim_{\epsilon\to 0}(-T^{\tilde{f}}_r(a-\epsilon,b+\epsilon) - T^{\tilde{f}}_r(a+\epsilon,b-\epsilon) + T^{\tilde{f}}_r(a-\epsilon,b-\epsilon) + T^{\tilde{f}}_r(a+\epsilon,b+\epsilon)$$

exists and defines a function which satisfies $\delta^{m,\tilde{f}}_r(a+2\pi,b+2\pi) = \delta^{m,\tilde{f}}_r(a+2\pi,b+2\pi)$ and then, as in section 5, the function $\delta^{m,f}_r : \mathbb{T} \setminus \Delta_T \to \mathbb{Z}_{\geq 0}$. Using Proposition 4.3 on can show that $\delta^{m,f}_r$ and $C^m_r(f)$ are equal. The definition above is essentially the description of the *persistence diagrams* of $\tilde{f}$ and $-\tilde{f}$, cf [11], and will not be pursued further in this paper.

The stability phenomena discovered in [5] can be formulated in terms of configuration $C^m_r(f)$ when one equips the set of finite configurations of points in $\mathbb{T} \setminus \Delta_{\mathbb{T}}$ with the topology induced by the bottle neck distance defined [5].Note that in this case the configurations do not have the same cardinality and, in this topology, the definition of " proximity" largely ignores the points points near the diagonal $\Delta_{\mathbb{T}}$.

Here is the definition for such topology on the space $\mathcal{C}onfg(X \setminus K)$ of finite configurations of points in $X \setminus K$, X locally compact space and K a closed subset of X. Recall that a configuration is a map with finite support, $\delta : X \setminus K \to \mathbb{Z}_{\geq 0}$.

Define a base for the topology by specifying a collection of open sets indexed by systems $S=\{(U_1,k_1),\cdots(U_r,k_r),V\}$ with:
(1) $U_i, i=1\cdots r$ open subsets of $X\setminus K$, V open neighborhood of K,
(2) $k_1,k_2,\cdots k_r$ positive integers.
The "open set" of configurations corresponding to S is $\mathcal{U}(S):=\{\delta\in \mathcal{C}onfg(X\setminus K)\mid$ support $(\delta)\subset \mathrm{U_1\cup U_2\cdots\cup U_r\cup V},\ \sum_{\mathrm{x\in U_i}}\delta(\mathrm{x})=\mathrm{k_i}\}$.
The MAIN THEOREM in [5] implies

Theorem 7.1. *The assignment $f\rightsquigarrow C^m_r(f)$ is a continuous map from the space $C_t(X,\mathbb{S}^1)$ of tame maps to $\mathcal{C}onfg(\mathbb{T}\setminus\Delta)$ when the first space is equipped with the compact open topology and the second with the topology described above in case $(X,K)=(\mathbb{T},\Delta)$.*

Poincaré duality also holds for the configuration $C^m_r(f)$. Theorem 1.5 formulates this duality. We understand that for f a real valued function it is implicit in the work of Edelsbrunner and others. We treat however the angle valued maps rather than real valued maps and derive its proof as a corollary to Proposition 4.2. We provide below the arguments.

7.1. **Proof of Theorem 1.5.** In consistency with the notation in previous sections for $\tilde f:\tilde X\to\mathbb{R}$ the infinite cyclic covering of the tame map $f:X\to\mathbb{S}^1$ we denote by

(i) $i_a(r):H_r(\tilde X_a)\to H_r(\tilde X)$ and $i^{BM}_a(r):H^{BM}_r(\tilde X_a)\to H_r(\tilde X)$,

(ii) $i^a(r):H_r(\tilde X^a)\to H_r(\tilde X)$ and $i^{BM,a}(r):H^{BM}_r(\tilde X_a)\to H_r(\tilde X)$,

and for $a\leq b$

(iii) $i_{a,b}(r):H_r(\tilde X_a)\to H_r(\tilde X_b)$ and $i^{BM}_{a,b}(r):H^{BM}_r(\tilde X_a)\to H_r(\tilde X_b)$,

(iv) $i^{b,a}(r):H_r(\tilde X^b)\to H_r(\tilde X^a)$ and $i^{BM,b,a}(r):H^{BM}_r(\tilde X^b)\to H_r(\tilde X^a)$

the inclusion induced linear maps in homology and Borel-Moore homology.
We introduce

(i) $\mathbb{K}_a(r):=\ker i_a(r)$ and $\mathbb{K}^{BM}_a(r):=\ker i^{BM}_a(r)$,

(ii) $\mathbb{K}^a(r):=\ker i^a(r)$ and $\mathbb{K}^{BM,a}(r):=\ker i^{BM,a}(r)$

and denote by
$'i_{a,b}(r):\mathbb{K}_a(r)\to\mathbb{K}_b(r)$ and $'i^{BM}_{a,b}(r):\mathbb{K}^{BM}_a(r)\to\mathbb{K}^{BM}_b(r)$,

$'i^{b,a}(r):\mathbb{K}^b(r)\to\mathbb{K}^a(r)$ and $'i^{BM,b,a}(r):\mathbb{K}^{BM,b}(r)\to\mathbb{K}^{BM,a}(r)$
the restrictions of of $i_{a,b}(r)$, $i^{BM}_{a,b}(r)$ and of $i^{b,a}(r)$, $i^{BM,b,a}(r)$ to the respective kernels $\mathbb{K}^{\cdots}_{\cdots}(r)$.

Note that in view of the calculations of Borel–Moore homology of $\tilde X^a,\tilde X_a,\tilde X$ and of the canonical homomorphism $H_r(\tilde M\cdots)\to H^{BM}_r(\tilde M\cdots)$ one concludes that

$$\mathbb{K}(r)=\mathbb{K}^{BM}(r)\text{ and }'\mathrm{i(r)}=' \mathrm{i^{BM}(r)}.$$

Proposition 4.2 permits to describe the vector spaces $\mathbb{K}_a(r),\mathbb{K}^a(r)$, ker $'i_{a,b}(r)$, coker $'i_{a,b}(r)$, ker $'i^{b,a}(r)$, coker $'i^{b,a}(r)$ in terms of mixed bar codes as summarized in the next proposition.

Proposition 7.2. *Suppose $f : X \to \mathbb{S}^1$ is a tame map with $\tilde f : \tilde X \to \mathbb{R}$ its infinite cyclic covering, and a, b real numbers with $a \leq b$. Then*

1. $\mathbb{K}^{\tilde f}_a(r) = \kappa[\{I \in \tilde{\mathcal{B}}^{co}_r(f) \mid I \ni a\}]$
2. $\mathbb{K}^a_{\tilde f}(r) = \kappa[\{I \in \tilde{\mathcal{B}}^{oc}_r(f) \mid I \ni a\}]$
3. $\ker\ 'i_{a,b}(r) = \kappa[\{I \in \tilde{\mathcal{B}}^{co}_r(f) \mid I \ni a,\ b \notin I\}]$
   $\operatorname{coker}\ 'i_{a,b}(r) = \kappa[\{I \in \tilde{\mathcal{B}}^{co}_r(f) \mid I \ni b,\ a \notin I\}]$
4. $\ker\ 'i^{b,a}(r) = \kappa[\{I \in \tilde{\mathcal{B}}^{oc}_r(f) \mid I \ni b,\ a \notin I\}]$
   $\operatorname{coker}\ 'i^{b,a}(r) = \kappa[\{I \in \tilde{\mathcal{B}}^{oc}_r(f) \mid I \ni a,\ b \notin I\}]$

The long exact sequence for the pair $(\tilde X, \tilde X_a)$

$$\longrightarrow H_{n-r}(\tilde X) \xrightarrow{j^a(n-r)} H_{n-r}(\tilde X, \tilde X^a) \xrightarrow{\delta^a(n-r)} H_{n-r-1}(\tilde X^a) \xrightarrow{i^a(n-r-1)} H_{n-1-r}(\tilde X) \longrightarrow \tag{59}$$

gives rise to the canonical isomorphism

$$\delta^a(n-r) : \operatorname{coker} j^a(n-r) \to \ker i^a(n-r) = \mathbb{K}^a(n-r-1) \tag{60}$$

which being "natural" w.r. to the inclusion of pairs $(\tilde X, \tilde X^b) \subseteq (\tilde X, \tilde X^a)$ for $a \leq b$ implies the commutativity of the diagram

$$\begin{array}{ccc}
\operatorname{coker} j^b(n-r) & \xrightarrow{\delta^b_{n-r}} & \mathbb{K}^b(n-r-1) \\
\downarrow & & \downarrow{\scriptstyle i^{b,a}(n-r-1)} \\
\operatorname{coker} j^a(n-r) & \xrightarrow{\delta^a(n-r)} & \mathbb{K}^a(n-r-1)
\end{array} \tag{61}$$

Suppose that $X = M^n$ is a closed $\kappa-$orientable manifold and a is a regular value of $\tilde f$. Poincaré Duality for the manifold $\tilde M^n$ and for the pairs $(\tilde M, \tilde M_a)$ and $(\tilde M, \tilde M^a)$ provides the commutative diagram

$$\begin{array}{ccccc}
\mathbb{K}_a(r) & \longrightarrow & H_r(\tilde M_a) & \xrightarrow{i_a(r)} & H_r(\tilde M) \\
\downarrow{\scriptstyle =} & & \downarrow & & \downarrow \\
\mathbb{K}^{BM}_a(r) & \longrightarrow & H^{BM}_r(\tilde M_a) & \xrightarrow{i^{BM}_a(r)} & H^{BM}_r(\tilde M) \\
\downarrow{\scriptstyle PD} & & \downarrow{\scriptstyle PD} & & \downarrow{\scriptstyle PD} \\
\ker(j^a(n-r))^*) & \longrightarrow & (H_{n-r}(\tilde M, \tilde M^a))^* & \longrightarrow & (H_{n-r}(\tilde M))^*
\end{array} \tag{62}$$

with the bottom vertical arrows the Poincaré Duality isomorphisms considered in Section 6. The diagram is natural w.r. to the inclusion of pairs $(X, X_a) \subseteq (X, X_b)$, provided a and b are regular values, and leads to the commutative diagram (63) whose vertical arrows are all isomorphisms.

$$\begin{array}{ccc}
\mathbb{K}_a(r) & \xrightarrow{\ {}'i_{a,b}(r)\ } & \mathbb{K}_b(r) \\
\downarrow & & \downarrow \\
\ker(j^a(n-r))^* & \longrightarrow & \ker(j^b(n-r))^* \\
\downarrow & & \downarrow \\
\operatorname{coker} j^a(n-r))^* & \longrightarrow & \operatorname{coker} j^b(n-r)^* \\
\uparrow & & \uparrow{\scriptstyle \delta^b(n-r)^*} \\
\mathbb{K}^a(n-r-1)^* & \xrightarrow{({}'i^{b,a}(n-r-1))^*} & \mathbb{K}^b(n-r-1))^*.
\end{array} \tag{63}$$

Let us review the information we have:

(i) The tameness of f implies that for $a < b$, a, b critical values and $\epsilon < \epsilon(f)$ the inclusions $\tilde{X}_a \subseteq \tilde{X}_{a+\epsilon}$ and $\tilde{X}^{a-\epsilon} \subset \tilde{X}^a$ are homotopy equivalences,

(ii) Poincaré Duality above and item (i) imply that for $0 \leq \epsilon, \epsilon' < \epsilon(f)$ and $a < b$ critical values one has

$$\ker\ {}'i_{a+\epsilon,b+\epsilon'}(r) \equiv \ker\ {}'i_{a,b}(r) = \operatorname{coker}' i^{b,a}(n-1-r) \equiv \operatorname{coker}\ {}'i^{b-\epsilon,a-\epsilon'}(n-1-r) \tag{64}$$

(iii) Proposition 7.2 implies that for $a < b$ critical values and $0 < \epsilon < \epsilon(f)$

$$C^m_k(f)(a,b) = \begin{cases} \dim\ker\ {}'i_{a,b}(k) - \dim\ker\ {}'i_{a-\epsilon,b}(k) - \dim\ker\ {}'i_{a,b-\epsilon}(k) + \\ + \dim\ker\ {}'i_{a-\epsilon,b-\epsilon}(k) \\ = \\ \dim\operatorname{coker}\ {}'i_{a,b}(k) - \dim\operatorname{coker}\ {}'i_{a-\epsilon,b}(k) - \dim\operatorname{coker}\ {}'i_{a,b-\epsilon}(k) + \\ \dim\operatorname{coker}\ {}'i_{a-\epsilon,b-\epsilon}(k) \end{cases} \tag{65}$$

and

$$C^m_k(f)(b,a) = \begin{cases} + \dim\ker\ {}'i^{b,a}(k) - \dim\ker\ {}'i^{b,a-\epsilon}(k) - \dim\ker\ {}'i^{b-\epsilon,a}(k) + \\ + \dim\ker\ {}'i^{a-\epsilon,b-\epsilon}(k) \\ = \\ \dim\operatorname{coker}\ {}'i^{b,a}(k) - \dim\operatorname{coker}\ {}'i^{b,a-\epsilon}(k) - \dim\operatorname{coker}\ {}'i^{b-\epsilon,a}(k) + \\ + \dim\operatorname{coker}\ {}'i^{a-\epsilon,b-\epsilon}(k) \end{cases} \tag{66}$$

Item (iii) comes down to expressing the number of closed open or open closed bar codes with end a and b critical values in terms of the number of bar codes which contain a but not b and using Proposition 7.2

Putting together items (ii) to (iii) one derives that $C^m_r(\tilde{f})(a,b) = C^m_{n-1-r}(\tilde{f})(b,a)$ and then $C^m_r(f)(a,b) = C^m_{n-1-r}(-\tilde{f})(-a,-b)$ which is what Theorem 1.5 states.

q.e.d

8. Linear relations and monodromy. Proof of Theorem 1.4

This section can be read independently on the rest of the paper. For additional future use we describe this piece of linear algebra in a larger generality, of modules over a commutative ring rather than vector spaces over a field.

8.1. **Linear relations.** Suppose V and W are two modules over a fixed commutative ring in particular field.. Recall that a linear relation from V to W can be considered as a submodule $R \subseteq V \times W$. Notationally, we indicate this situation by $R\colon V \rightsquigarrow W$. For $v \in V$ and $w \in W$ we write vRw iff v is in relation with w, i.e. $(v, w) \in R$. Every module homomorphism $V \to W$ can be regarded as a linear relation $V \rightsquigarrow W$ in a natural way. If U is another module, and $S\colon W \rightsquigarrow U$ is a linear relation, then the composition $SR\colon V \rightsquigarrow U$ is the linear relation defined by $v(SR)u$ iff there exists $w \in W$ such that vRw and wSu. Clearly, this is an associative composition generalizing the ordinary composition of module homomorphisms. For the identical relations we have $R\,\mathrm{id}_V = R$ and $\mathrm{id}_W R = R$. Modules over a fixed commutative ring and linear relations thus constitute a category. If $R\colon V \rightsquigarrow W$ is a linear relation we define a linear relation $R^\dagger\colon W \rightsquigarrow V$ by $wR^\dagger v$ iff vRw. Clearly, $R^{\dagger\dagger} = R$ and $(SR)^\dagger = R^\dagger S^\dagger$.

A linear relation $R\colon V \rightsquigarrow W$ gives rise to the following submodules:

$$\begin{aligned}
\mathrm{dom}(R) &:= \{v \in V \mid \exists w \in W : vRw\}\\
\mathrm{img}(R) &:= \{w \in W \mid \exists v \in V : vRw\}\\
\ker(R) &:= \{v \in V \mid vR0\}\\
\mathrm{mul}(R) &:= \{w \in W \mid 0Rw\}
\end{aligned}$$

Clearly, $\ker(R) \subseteq \mathrm{dom}(R) \subseteq V$, and $W \supseteq \mathrm{img}(R) \supseteq \mathrm{mul}(R)$. Note that R is a homomorphism (map) iff $\mathrm{dom}(R) = V$ and $\mathrm{mul}(R) = 0$. One readily verifies:

Lemma 8.1. *For a linear relation $R\colon V \rightsquigarrow W$ the following are equivalent:*
(a) R is an isomorphism in the category of modules and linear relations.
(b) $\mathrm{dom}(R) = V$, $\mathrm{img}(R) = W$, $\ker(R) = 0$, and $\mathrm{mul}(R) = 0$.
(c) R is an isomorphism of modules.
In this case $R^{-1} = R^\dagger$.

For a linear relation $R\colon V \rightsquigarrow V$, we introduce the following submodules:

$$\begin{aligned}
K_+ &:= \{v \in V \mid \exists k\, \exists v_i \in V : vRv_1Rv_2R\cdots Rv_kR0\}\\
K_- &:= \{v \in V \mid \exists k\, \exists v_i \in V : 0Rv_{-k}R\cdots Rv_{-2}Rv_{-1}Rv\}\\
D_+ &:= \{v \in V \mid \exists v_i \in V : vRv_1Rv_2Rv_3R\cdots\}\\
D_- &:= \{v \in V \mid \exists v_i \in V : \cdots Rv_{-3}Rv_{-2}Rv_{-1}Rv\}\\
D := D_- \cap D_+ &= \{v \in V \mid \exists v_i \in V : \cdots Rv_{-2}Rv_{-1}RvRv_1Rv_2R\cdots\},
\end{aligned}$$

Clearly, $K_- \subseteq D_- \subseteq V \supseteq D_+ \supseteq K_+$. Also note that passing from R to $R^\dagger$, the roles of $+$ and $-$ get interchanged. Moreover, we introduce a linear relation on the quotient module

$$V_{\mathrm{reg}} := \frac{D}{(K_- + K_+) \cap D}$$

defined as the composition

$$V_{\mathrm{reg}} = \frac{D}{(K_- + K_+) \cap D} \overset{\pi^\dagger}{\rightsquigarrow} D \overset{\iota}{\rightsquigarrow} V \overset{R}{\rightsquigarrow} V \overset{\iota^\dagger}{\rightsquigarrow} D \overset{\pi}{\rightsquigarrow} \frac{D}{(K_- + K_+) \cap D} = V_{\mathrm{reg}},$$

where ι and π denote the canonical inclusion and projection, respectively. In other words, two elements in V_{reg} are related by R_{reg} iff they admit representatives in D which are in related by R. We refer to R_{reg} as the *regular part* of R.

Proposition 8.2. *The relation $R_{\mathrm{reg}}\colon V_{\mathrm{reg}} \rightsquigarrow V_{\mathrm{reg}}$ is an isomorphism of modules. Moreover, the natural inclusion induces a canonical isomorphism*

$$V_{\mathrm{reg}} = \frac{D}{(K_- + K_+) \cap D} \xrightarrow{\cong} \frac{(K_- + D_+) \cap (D_- + K_+)}{K_- + K_+} \tag{67}$$

which intertwines R_{reg} with the relation induced on the right hand side quotient.

Proof. Clearly, (67) is well defined and injective. To see that it is onto let

$$x = k_- + d_+ = d_- + k_+ \in (K_- + D_+) \cap (D_- + K_+),$$

where $k_\pm \in K_\pm$ and $d_\pm \in D_\pm$. Thus

$$x - k_- - k_+ = d_+ - k_+ = d_- - k_- \in D_- \cap D_+ = D.$$

We conclude $x \in D + K_- + K_+$, whence (67) is onto. We will next show that this isomorphism intertwines R_{reg} with the relation induced on the right hand side. To do so, suppose $xR\tilde{x}$ where

$$x = k_- + d_+ = d_- + k_+ \in (K_- + D_+) \cap (D_- + K_+),$$
$$\tilde{x} = \tilde{k}_- + \tilde{d}_+ = \tilde{d}_- + \tilde{k}_+ \in (K_- + D_+) \cap (D_- + K_+),$$

and $k_\pm, \tilde{k}_\pm \in K_\pm$ and $d_\pm, \tilde{d}_\pm \in D_\pm$. Note that there exist $k'_+ \in K_+$ and $\tilde{k}'_- \in K_-$ such that $k_+ R k'_+$ and $\tilde{k}'_- R \tilde{k}_-$. By linearity of R we obtain

$$\underbrace{(x - k_+ - \tilde{k}'_-)}_{\in D_-} R \underbrace{(\tilde{x} - k'_+ - \tilde{k}_-)}_{\in D_+}.$$

We conclude $d := x - k_+ - \tilde{k}'_- \in D$, $\tilde{d} := \tilde{x} - k'_+ - \tilde{k}_- \in D$, and $dR\tilde{d}$. This shows that the relations induced on the two quotients in (67) coincide. We complete the proof by showing that R_{reg} is an isomorphism. Clearly, $\mathrm{dom}(R_{\mathrm{reg}}) = V_{\mathrm{reg}} = \mathrm{img}(R_{\mathrm{reg}})$. We will next show $\ker(R_{\mathrm{reg}}) = 0$. To this end suppose $dR\tilde{d}$, where

$$d \in D \quad \text{and} \quad \tilde{d} = \tilde{k}_- + \tilde{k}_+ \in (K_- + K_+) \cap D$$

with $\tilde{k}_\pm \in K_\pm$. Note that $\tilde{k}_- = \tilde{d} - \tilde{k}_+ \in K_- \cap D_+$. Thus there exists $k_- \in K_- \cap D_+$ such that $k_- R \tilde{k}_-$. By linearity of R, we get $(d - k_-) R \tilde{k}_+$, whence $d - k_- \in K_+$ and thus $d \in K_- + K_+$. This shows $\ker(R_{\mathrm{reg}}) = 0$. Analogously, we have $\mathrm{mul}(R_{\mathrm{reg}}) = 0$. In view of Lemma 8.1 we conclude that R_{reg} is an isomorphism of modules. □

We will now specialize to linear relations on finite dimensional vector spaces and provide another description of V_{reg} in this case. Consider the category whose objects are finite dimensional vector spaces V equipped with a linear relation $R\colon V \rightsquigarrow V$ and whose morphisms are linear maps $\psi\colon V \to W$ such that for all $x, y \in V$ with xRy we also have $\psi(x)Q\psi(y)$, where W is another finite dimensional vector space with linear relation $Q\colon W \rightsquigarrow W$. It is readily checked that this is an abelian category. By the Remak–Schmidt theorem, every linear relation on a finite dimensional vector space can therefore be decomposed into a direct sum of indecomposable ones, $R \cong R_1 \oplus \cdots \oplus R_N$, where the factors are unique up to permutation and isomorphism. The decomposition itself, however, is not canonical.

Proposition 8.3. *Let $R\colon V \rightsquigarrow V$ be a linear relation on a finite dimensional vector space over an algebraic closed field , and let $R \cong R_1 \oplus \cdots \oplus R_N$ denote a decomposition into indecomposable linear relations. Then R_{reg} is isomorphic to the direct sum of factors R_i whose relations are linear isomorphisms.*

Proof. Since the definition of R_{reg} is a natural one, we clearly have

$$R_{\text{reg}} \cong (R_1)_{\text{reg}} \oplus \cdots \oplus (R_N)_{\text{reg}}.$$

Consequently, it suffices to show the following two assertions:

(a) If $R\colon V \rightsquigarrow V$ is an isomorphism of vector spaces, then $V_{\text{reg}} = V$ and $R_{\text{reg}} = R$.
(b) If $R\colon V \rightsquigarrow V$ is an indecomposable linear relation on a finite dimensional vector space which is not a linear isomorphism, then $V_{\text{reg}} = 0$.

The first statement is obvious, in this case we have $K_- = K_+ = 0$ and $D = D_- = D_+ = V$. To see the second assertion, note that an indecomposable linear relation $R \subseteq V \times V$ gives rise to an indecomposable representation $R \rightrightarrows V$ of the quiver G_2. Since R is not an isomorphism, the quiver representation has to be of the bar code type. Using the explicit descriptions of the bar code representations, it is straight forward to conclude $V_{\text{reg}} = 0$. □

In the subsequent discussion we will also make use of the following result:

Proposition 8.4. *Suppose $R\colon V \rightsquigarrow V$ is a linear relation on a finite dimensional vector space. Then:*

$$D_+ = D + K_+, \quad D_- = K_- + D, \quad \text{and} \tag{68}$$

$$K_- \cap D_+ = K_- \cap K_+ = D_- \cap K_+. \tag{69}$$

For the proof we first establish two lemmas.

Lemma 8.5. *Suppose $R\colon V \rightsquigarrow W$ is a linear relation between vector spaces such that $\dim V = \dim W < \infty$. Then the following are equivalent:*

(a) *R is an isomorphism.*
(b) $\operatorname{dom}(R) = V$ *and* $\ker(R) = 0$.
(c) $\operatorname{img}(R) = W$ *and* $\operatorname{mul}(R) = 0$.

Proof. This follows immediately from the dimension formula

$$\dim\operatorname{dom}(R) + \dim\operatorname{mul}(R) = \dim(R) = \dim\operatorname{img}(R) + \dim\ker(R)$$

and Lemma 8.1. □

Lemma 8.6. *If V is finite dimensional, then the composition of relations*

$$D_+/K_+ \overset{\pi^\dagger}{\rightsquigarrow} D_+ \overset{\iota}{\rightsquigarrow} V \overset{R^k}{\rightsquigarrow} V \overset{\iota^\dagger}{\rightsquigarrow} D_+ \overset{\pi}{\rightsquigarrow} D_+/K_+,$$

is a linear isomorphism, for every $k \geq 0$, where ι and π denote the canonical inclusion and projection, respectively. Analogously, the relation induced by R^k on D_-/K_- is an isomorphism, for all $k \geq 0$. Moreover, for sufficiently large k,

$$D_- = \operatorname{img}(R^k) \quad \text{and} \quad D_+ = \operatorname{dom}(R^k).$$

Proof. One readily verifies $\operatorname{dom}(\pi\iota^\dagger R^k \iota\pi^\dagger) = D_+/K_+$ and $\ker(\pi\iota^\dagger R^k\iota\pi^\dagger) = 0$. The first assertion thus follows from Lemma 8.5 above. Considering $R^\dagger$ we obtain the second statement. Clearly, $\operatorname{dom}(R^k) \supseteq \operatorname{dom}(R^{k+1})$, for all $k \geq 0$. Since V is finite dimensional, we must have $\operatorname{dom}(R^k) = \operatorname{dom}(R^{k+1})$, for sufficiently large k. Given

$v \in \operatorname{dom}(R^k)$, we thus find $v_1 \in \operatorname{dom}(R^k)$ such that vRv_1. Proceeding inductively, we construct $v_i \in \operatorname{img}(R^k)$ such that $vRv_1Rv_2R\cdots$, whence $v \in D_+$. This shows $\operatorname{dom}(R^k) \subseteq D_+$, for sufficiently large k. As the converse inclusion is obvious we get $D_+ = \operatorname{dom}(R^k)$. Considering $R^\dagger$, we obtain the last statement. □

Proof of Proposition 8.4. From Lemma 8.6 we get $\operatorname{img}(\pi\iota^\dagger R^k) = D_+/K_+$, whence $D_+ \subseteq \operatorname{img}(R^k)+K_+$, for every $k \geq 0$, and thus $D_+ \subseteq D_- + K_+$. This implies $D_+ = D + K_+$. Considering $R^\dagger$ we obtain the other equality in (68). From Lemma 8.6 we also get $\operatorname{mul}(\pi\iota^\dagger R^k) = 0$, whence $\operatorname{mul}(R^k) \cap D_+ \subseteq K_+$, for every $k \geq 0$. This gives $K_- \cap D_+ = K_- \cap K_+$. Considering $R^\dagger$ we get the other equality in (69). □

$G_{2m}-$**representations and the associated relations.** For a $G_{2m}-$ representation $\rho = \{V_r, \alpha_i, \beta_j\}$ we have m relations $R_i\colon V_{2i-1} \rightsquigarrow V_{2i+1}$ (considering $V_{2m+k} = V_k$) given by the pair of linear maps $alpha_i : V_{2i-1} \to V_{2i}$ and $\beta_i : V_{2i+1} \to V_{2i}$. One can consider the compositions $R^i\colon V_{2i-1} \rightsquigarrow V_{2i-1}$ $R^i := V_{2i-1} \rightsquigarrow V_{2i+1} \rightsquigarrow \cdots V_{2m-1} \rightsquigarrow V_1 \rightsquigarrow V_3 \rightsquigarrow \cdots V_{2i-3} \rightsquigarrow V_{2i-1}$.

Proposition 8.7. $R^i_{\mathrm{reg}} = R^j_{\mathrm{reg}}$ *for any* i, j *and is conjugate to* $\oplus_{J\in\mathcal{J}} T(J)$.

Proof. The statement is immediate for indecomposable representations for a general representation implied by Proposition 8.3.

8.2. **Monodromy, Proof of Theorem 1.4.** The purpose of this subsection is to establish Theorem 1.4

Suppose $f : X \to S^1$ is a continuous map and let

$$\begin{array}{ccc} \tilde X & \xrightarrow{\tilde f} & \mathbb{R} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & S^1 \end{array}$$

denote the associated infinite cyclic covering. For $r \in \mathbb{R}$ we put $\tilde X_r = \tilde f^{-1}(r)$ and let $H_*(\tilde X_r)$ denote its singular homology with coefficients in any fixed module. If $r_1 \leq r_2$ we define a linear relation

$$B^{r_2}_{r_1}\colon H_*(\tilde X_{r_1}) \rightsquigarrow H_*(\tilde X_{r_2})$$

by declaring $a_1 \in H_*(\tilde X_{r_1})$ to be in relation with $a_2 \in H_*(\tilde X_{r_2})$ iff their images in $H_*(\tilde X_{[r_1,r_2]})$ coincide, where $\tilde X_{[r_1,r_2]} = f^{-1}([r_1, r_2])$. If $r_1 \leq r_2 \leq r_3$ we clearly have $B^{r_3}_{r_2} B^{r_2}_{r_1} \subseteq B^{r_3}_{r_1}$. If r_2 is a tame value this becomes an equality of relations:

Lemma 8.8. *Suppose* $r_1 \leq r_2 \leq r_3$ *and assume* r_2 *is a tame value. Then, as linear relations,* $B^{r_3}_{r_2} B^{r_2}_{r_1} = B^{r_3}_{r_1}$.

Proof. Since r_2 is a tame value, we have an exact Mayer–Vietoris sequence,

$$H_*(\tilde X_{r_2}) \to H_*(\tilde X_{[r_1,r_2]}) \oplus H_*(\tilde X_{[r_2,r_3]}) \to H_*(\tilde X_{[r_1,r_3]}),$$

which immediately implies the statement. □

Fix a tame value $\theta \in S^1$ of f and a lift $\tilde\theta \in \mathbb{R}$, $e^{\mathbf{i}\tilde\theta} = \theta$. Using the projection $\tilde X \to X$, we may canonically identify $\tilde X_{\tilde\theta} = X_\theta = f^{-1}(\theta)$. Moreover, let $\tau\colon \tilde X \to \tilde X$ denote the fundamental deck transformation, i.e. $\tilde f \circ \tau = \tilde f + 2\pi$. Note that τ induces homeomorphisms between levels, $\tau\colon \tilde X_r \to \tilde X_{r+2\pi}$, and define a linear relation

$$R\colon H_*(X_\theta) \rightsquigarrow H_*(X_\theta)$$

as the composition

$$H_*(X_\theta) = H_*(\tilde X_{\tilde\theta}) \overset{B_{\tilde\theta}^{\tilde\theta+2\pi}}{\rightsquigarrow} H_*(\tilde X_{\tilde\theta+2\pi}) \overset{\tau_*^\dagger}{\rightsquigarrow} H_*(\tilde X_{\tilde\theta}) = H_*(X_\theta). \tag{70}$$

In other words, for $a, b \in H_*(X_\theta)$ we have aRb iff $aB_{\tilde\theta}^{\tilde\theta+2\pi}(\tau_* b)$, i.e. iff a and $\tau_* b$ coincide in $H_*(\tilde X_{[\tilde\theta,\tilde\theta+2\pi]})$. Particularly:

Lemma 8.9. *If $a, b \in H_*(X_\theta)$ and aRb, then $a = \tau_* b$ in $H_*(\tilde X)$.*

We will continue to use the notation $K_\pm$, $D_\pm$, and R_{reg} introduced in the previous section for this relation R on $H_*(X_\theta)$. Particularly, its regular part,

$$R_{\mathrm{reg}} \colon H_*(X_\theta)_{\mathrm{reg}} \to H_*(X_\theta)_{\mathrm{reg}},$$

is a module automorphism.

Lemma 8.10. *We have:*

$$K_+ = \ker\bigl(H_*(X_\theta) \to H_*(\tilde X_{[\tilde\theta,\infty)})\bigr)$$
$$K_- = \ker\bigl(H_*(X_\theta) \to H_*(\tilde X_{(-\infty,\tilde\theta]})\bigr)$$

Both maps are induced by the canonical inclusion $X_\theta = \tilde X_{\tilde\theta} \to \tilde X$.

Proof. We will only show the first equality, the other one can be proved along the same lines. To see the inclusion $K_+ \subseteq \ker(H_*(X_\theta) \to H_*(\tilde X_{[\tilde\theta,\infty)}))$, let $a \in K_+$. Hence, there exist $a_k \in H_*(X_\theta)$, almost all of which vanish, such that $aRa_1Ra_2R\cdots$. In $H_*(\tilde X_{[\tilde\theta,\tilde\theta+2\pi]})$, we thus have:

$$a = \tau_* a_1, \quad a_1 = \tau_* a_2, \quad a_2 = \tau_* a_3, \quad \dots$$

In $H_*(\tilde X_{[\tilde\theta,\infty)})$, we obtain:

$$a = \tau_* a_1 = \tau_*^2 a_2 = \tau_*^3 a_3 = \cdots$$

Since some a_k have to be zero, we conclude that a vanishes in $H_*(\tilde X_{[\tilde\theta,\infty)})$.

To see the converse inclusion, $K_+ \supseteq \ker(H_*(\tilde X_\theta) \to H_*(\tilde X_{[\tilde\theta,\infty)}))$, set

$$U := \bigsqcup_{0 \le k \text{ even}} \tilde X_{[\tilde\theta+2\pi k,\tilde\theta+2\pi(k+1)]}, \qquad V := \bigsqcup_{1 \le k \text{ odd}} \tilde X_{[\tilde\theta+2\pi k,\tilde\theta+2\pi(k+1)]}$$

and note that $U \cup V = \tilde X_{[\tilde\theta,\infty)}$, as well as $U \cap V = \bigsqcup_{k\in\mathbb N} \tilde X_{\tilde\theta+2\pi k}$. Since θ is a tame value, we have an exact Mayer–Vietoris sequence

$$\bigoplus_{k\in\mathbb N} H_*(\tilde X_{\tilde\theta+2\pi k}) = H_*\Bigl(\bigsqcup_{k\in\mathbb N} \tilde X_{\tilde\theta+2\pi k}\Bigr) \to H_*(U) \oplus H_*(V) \to H_*(\tilde X_{[\tilde\theta,\infty)}).$$

For $b \in \ker(H_*(X_\theta) \to H_*(\tilde X_{[\tilde\theta,\infty)}))$ we thus find $b_k \in H_*(\tilde X_{\tilde\theta+2\pi k})$, almost all of which vanish, such that:

$$b = b_1 \in H_*(\tilde X_{[\tilde\theta,\tilde\theta+2\pi]}), \quad b_1+b_2 = 0 \in H_*(\tilde X_{[\tilde\theta+2\pi,\tilde\theta+4\pi]}), \quad b_2+b_3 = 0 \in H_*(\tilde X_{[\tilde\theta+4\pi,\tilde\theta+6\pi]}), \quad \dots$$

Putting $c_k := (-1)^{k-1}\tau_*^{-k} b_k \in H_*(\tilde X_{\tilde\theta})$, we obtain the following equalities in $H_*(\tilde X_{[\tilde\theta,\tilde\theta+2\pi]})$:

$$b = \tau_* c_1, \quad c_1 = \tau_* c_2, \quad c_2 = \tau_* c_3, \quad \dots$$

In other words, we have the relations $bRc_1Rc_2Rc_3R\cdots$. Since some c_k has to be zero, we conclude $b \in K_+$, whence the lemma. □

Introduce the upwards Novikov complex as a projective limit of relative singular chain complexes,

$$C_*^{\mathrm{Nov},+}(\tilde X) := \varprojlim_r C_*(\tilde X, \tilde X_{[r,\infty)}),$$

and let $H_*^{\mathrm{Nov},+}(\tilde X)$ denote its homology. Analogously, we define a downwards Novikov complex $C_*^{\mathrm{Nov},-}(\tilde X) = \varprojlim_r C_*(\tilde X, \tilde X_{(-\infty,r]})$ and the corresponding homology, $H_*^{\mathrm{Nov},-}(\tilde X)$. We will also use similar notation for subsets of $\tilde X$.

Lemma 8.11. *We have:*

$$D_+ = \ker\bigl(H_*(X_\theta) \to H_*^{\mathrm{Nov},+}(\tilde X_{[\tilde\theta,\infty)})\bigr)$$
$$D_- = \ker\bigl(H_*(X_\theta) \to H_*^{\mathrm{Nov},-}(\tilde X_{(-\infty,\tilde\theta]})\bigr)$$

Both maps are induced by the canonical inclusion $X_\theta = \tilde X_{\tilde\theta} \to \tilde X$.

Proof. Using the exact Mayer–Vietoris sequence

$$\prod_{k\in\mathbb{N}} H_*(\tilde X_{\tilde\theta+2\pi k}) = H_*^{\mathrm{Nov},+}\Bigl(\bigsqcup_{k\in\mathbb{N}} \tilde X_{\tilde\theta+2\pi k}\Bigr) \to H_*^{\mathrm{Nov},+}(U) \oplus H_*^{\mathrm{Nov},+}(V) \to H_*^{\mathrm{Nov},+}(\tilde X_{[\tilde\theta,\infty)}),$$

this can be proved along the same lines as Lemma 8.10. □

Let us introduce a complex

$$C_*^{\mathrm{l.f.}}(\tilde X) := \varprojlim_r C_*(\tilde X, \tilde X_{(-\infty,-r]} \cup \tilde X_{[r,\infty)})$$

and denote its homology by $H_*^{\mathrm{l.f.}}(\tilde X)$. If f is proper, this is the complex of locally finite singular chains.

Lemma 8.12. *We have:*

$$K_- + K_+ = \ker\bigl(H_*(X_\theta) \to H_*(\tilde X)\bigr)$$
$$K_- + D_+ = \ker\bigl(H_*(X_\theta) \to H_*^{\mathrm{Nov},+}(\tilde X)\bigr)$$
$$D_- + K_+ = \ker\bigl(H_*(X_\theta) \to H_*^{\mathrm{Nov},-}(\tilde X)\bigr)$$
$$D_- + D_+ = \ker\bigl(H_*(X_\theta) \to H_*^{\mathrm{l.f.}}(\tilde X)\bigr)$$

All maps are induced by the canonical inclusion $X_\theta = \tilde X_{\tilde\theta} \to \tilde X$.

Proof. The first statement follows from the exact Mayer–Vietoris sequence

$$H_*(\tilde X_{\tilde\theta}) \to H_*(\tilde X_{(-\infty,\tilde\theta]}) \oplus H_*(\tilde X_{[\tilde\theta,\infty)}) \to H_*(\tilde X)$$

and Lemma 8.10. The second assertion follows from the exact Mayer–Vietoris sequence

$$H_*(\tilde X_{\tilde\theta}) \to H_*(\tilde X_{(-\infty,\tilde\theta]}) \oplus H_*^{\mathrm{Nov},+}(\tilde X_{[\tilde\theta,\infty)}) \to H_*^{\mathrm{Nov},+}(\tilde X)$$

and Lemma 8.10 and 8.11. Similarly, one can check the third equality. To see the last statement we use the exact Mayer–Vietoris sequence

$$H_*(\tilde X_{\tilde\theta}) \to H_*^{\mathrm{Nov},-}(\tilde X_{(-\infty,\tilde\theta]}) \oplus H_*^{\mathrm{Nov},+}(\tilde X_{[\tilde\theta,\infty)}) \to H_*^{\mathrm{l.f.}}(\tilde X)$$

and Lemma 8.11. □

Lemma 8.13. *We have*

$$\ker\Big(H_*(\tilde X)\to H_*^{\mathrm{Nov},-}(\tilde X)\oplus H_*^{\mathrm{Nov},+}(\tilde X)\Big)\subseteq \operatorname{img}\big(H_*(\tilde X_{\tilde\theta})\to H_*(\tilde X)\big),$$

where all maps are induced by the tautological inclusions.

Proof. This follows from the following commutative diagram of exact Mayer–Vietoris sequences:

$$\begin{array}{ccccc}
H^{\mathrm{l.f.}}_{*+1}(\tilde X) & \xrightarrow{\partial} & H_*(\tilde X) & \longrightarrow & H_*^{\mathrm{Nov},-}(\tilde X)\oplus H_*^{\mathrm{Nov},+}(\tilde X)\\
\| & & \uparrow & & \uparrow\\
H^{\mathrm{l.f.}}_{*+1}(\tilde X) & \xrightarrow{\partial} & H_*(\tilde X_{\tilde\theta}) & \longrightarrow & H_*^{\mathrm{Nov},-}(\tilde X_{(-\infty,\tilde\theta]})\oplus H_*^{\mathrm{Nov},+}(\tilde X_{[\tilde\theta,\infty)})
\end{array}$$

A similar argument was used in [17, Lemma 2.5]. □

Theorem 8.14. *The inclusion $\iota\colon X_\theta=\tilde X_{\tilde\theta}\to\tilde X$ induces a canonical isomorphism*

$$H_*(X_\theta)_{\mathrm{reg}}=\frac{D}{(K_-+K_+)\cap D}\xrightarrow{\cong}\ker\Big(H_*(\tilde X)\to H_*^{\mathrm{Nov},-}(\tilde X)\oplus H_*^{\mathrm{Nov},+}(\tilde X)\Big),$$

intertwining R_{reg} with the monodromy isomorphism induced by the deck transformation $\tau\colon\tilde X\to\tilde X$ on the right hand side. Moreover, working with coefficients in a field, and assuming that $H_*(X_\theta)$ is finite dimensional, the common kernel on the right hand side above coincides with

$$\ker\big(H_*(\tilde X)\to H_*^{\mathrm{Nov},-}(\tilde X)\big)=\ker\big(H_*(\tilde X)\to H_*^{\mathrm{Nov},+}(\tilde X)\big).$$

Particularly, in this case the latter two kernels are finite dimensional too.

Proof. It follows immediately from Lemma 8.12 and 8.13 that $\iota_*\colon H_*(X_\theta)\to H_*(\tilde X)$ induces an isomorphism

$$\frac{(K_-+D_+)\cap(D_-+K_+)}{K_-+K_+}\xrightarrow{\cong}\ker\Big(H_*(\tilde X)\to H_*^{\mathrm{Nov},-}(\tilde X)\oplus H_*^{\mathrm{Nov},+}(\tilde X)\Big).$$

In view of Lemma 8.9, this isomorphism intertwines the isomorphism induced by R on the left hand side, with the monodromy isomorphism on the right hand side. Combining this with Proposition 8.2 we obtain the first assertion. For the second statement it suffices to show

$$\ker\big(H_*(\tilde X)\to H_*^{\mathrm{Nov},+}(\tilde X)\big)\subseteq\ker\Big(H_*(\tilde X)\to H_*^{\mathrm{Nov},-}(\tilde X)\oplus H_*^{\mathrm{Nov},+}(\tilde X)\Big),\qquad(71)$$

as the converse inclusion is obvious, and the corresponding statement for the downward Novikov homology can be derived analogously. To this end, suppose $a\in\ker\big(H_*(\tilde X)\to H_*^{\mathrm{Nov},+}(\tilde X)\big)$. Then there exists k such that τ_*^ka is contained in the image of $H_*(\tilde X_{(-\infty,\tilde\theta]})\to H_*(\tilde X)$. Using the exact Mayer–Vietoris sequence

$$H_*(\tilde X_{\tilde\theta})\to H_*(\tilde X_{(-\infty,\tilde\theta]})\oplus H_*^{\mathrm{Nov},+}(\tilde X_{[\tilde\theta,\infty)})\to H_*^{\mathrm{Nov},+}(\tilde X)$$

we conclude, that τ_*^ka is contained in the image of $H_*(\tilde X_{\tilde\theta})\to H_*(\tilde X)$. Thus τ_*^ka is contained in $\iota_*(D_+)$, see Lemma 8.12. Since $H_*(X_\theta)$ is assumed to be a finite dimensional vector space, we have $\iota_*(D_-)=\iota_*(D)=\iota_*(D_+)$, see (68). Using Lemma 8.12 we thus conclude τ_*^ka is contained in the kernel on the right hand side of (71). Since this common kernel is invariant under the isomorphism $\tau_*\colon H_*(\tilde X)\to H_*(\tilde X)$, we conclude that a has to be contained in the common kernel too, whence the theorem. □

Clearly, Theorem 8.14 and Proposition 8.3 imply Theorem 1.4.

9. Appendix (an example)

Consider the space X is obtained from Y indicated in picture below by identifying its right end Y_1 (a union of three circles) to the left end Y_0 (a union of three circles) following the map $\phi\colon Y_1 \to Y_0$ given by the matrix

$$\begin{pmatrix} 1 & 1 & 2 \\ -3 & 4 & 2 \\ -2 & 1 & 2 \end{pmatrix}.$$

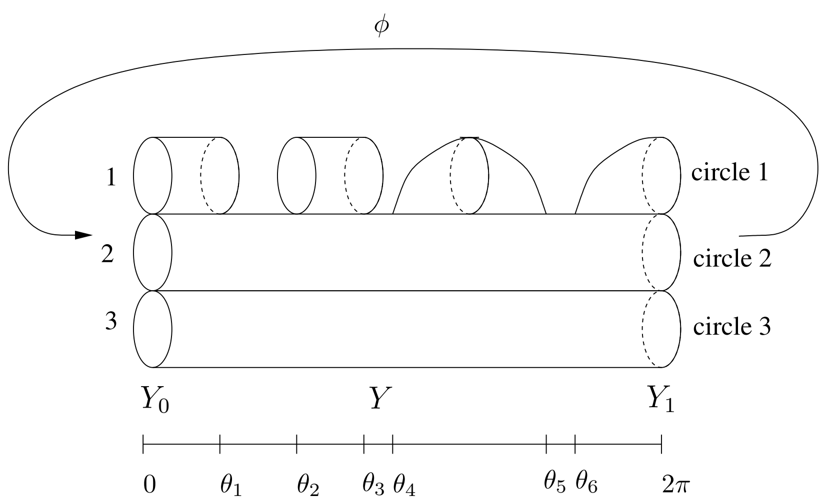


Figure 2. Example of r-invariants for a circle valued map

The meaning of this matrix is that the first circle is divided in 6 equal parts ; the first part go around the first circle clockwise the next 3 over the second counterclockwise to cover this circle three times and the last two also counterclockwise to cover the third circle twice. Similarly with he other two circles. The map $f\colon X \to S^1$ is induced by the projection of Y on the interval $[0, 2\pi]$.

The bar codes and the Jordan blocks are collected in the following table. Their calculation was done in [1] as an illustration of the algorithm proposed in that paper.

map ϕ	r-invariants		
circle 1: 1 time around circle 1	dimension	bar codes	Jordan cells
-3 times around 2, - 2 times around 3	0		$(1,1)$
circle 2: 1 time around circle 1	1	$(\theta_6, \theta_1 + 2\pi]$	$(3,2)$
, 4 times around 2, 1 time around 3		$[\theta_2, \theta_3]$	
circle 3: 2 time around 1,		(θ_4, θ_5)	
2 times around 2, 2 times around 3			

Simply by looking at the picture the reader can notice the contribution the closed $1-$closed bar code $[\theta_2, \theta_3]$ with one unit to the Betti number $\beta_1(X)$ the contribution of the $1-$open bar code (θ_4, θ_5) with one unit to the Betti number

$\beta_2(X)$ and the lack of contribution to homology of the open closed bar code $(\theta_6, \theta_1 + 2\pi]$.

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Dept. of Mathematics, The Ohio State University, 231 West 18th Avenue, Columbus, OH 43210, USA.
E-mail address: burghele@mps.ohio-state.edu

Department of Mathematics, University of Vienna, Nordbergstrasse 15, A-1090 Vienna, Austria.
E-mail address: stefan.haller@univie.ac.at