A COMPARISON OF DEFORMATIONS AND ORBIT CLOSURE

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A Comparison of Deformations and Orbit Closure

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Consider the $2x2$ matrices with entries in a field k :

$$A_t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \ A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} .$$

If k is the field of real numbers $\mathbb{R}$, $\lim\limits_{t\to 0} A_t = A_0$ has the standard definition. For an arbitrary field k, we may consider the family $\{A_t\}_{t \in k}$ as a subset of the variety of $2x2$ matrices over k and say that $\lim\limits_{t\to 0} A_t = A_0$ in the sense that A_0 is in the (Zariski) closure of the set $\{A_t\}_{t \neq 0}$. In studying limits in the algebraic sense, two different viewpoints have arisen: deformations and orbit closure.

In deformation theory, the above example would be written

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$$A_t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and we would say that A_t is a deformation of A_0. As an orbit closure example, we would note that $A_t = S_t A_1 S_t^{-1}$ where

$$S_t = \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}$$

and so A_0 is in the (Zariski) closure of the orbit of A_1 under the conjugation action of $GL_2(k)$ on the variety of 2x2 matrices. For orbit closure, we say A_0 is a degeneration of A_1. (Note the duality in viewpoint between "deformation" and "degeneration".)

The family

$$B_t = \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}$$

is a deformation of B_0, but B_0 is not a degeneration of B_1. It is easy to verify that

$$C_0 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

is a degeneration of

$$C_1 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

but no one-parameter family giving a deformation of C_0

contains C_1 .

Deformation theory was originally developed in the category of analytic structures (see, for instance [12] and [15]). The basic ideas of the deformation theory of analytic structures motivated the deformation theory of algebraic manifolds (see, for instance, [2] and [19]), and of algebras (see, for instance, [9] and [17]).

The orbit closure questions appear only in situations which can be formulated in terms of group actions. Nevertheless, there are many categories in which both deformations and orbit closure may be considered: nxn matrices [8], associative algebras [9],Lie algebras ([17], [21], [6], [10]), representations of a group or of an algebra ([13], [16]), representations of a quiver [14], linear systems of differential equations [20], etc. (This list is far from complete.)

Understanding the differences between deformations and degenerations, we were surprised to find a common formulation for these viewpoints, which we present here. In fact, we establish that if a finite dimensional Lie algebra μ_0 is in the closure of the orbit of a Lie algebra μ_1 , then there is such a deformation family of μ_0 , which contains a Lie algebra isomorphic to μ_1 .

In order to make the exposition readable, we will concentrate on one category, that of n-dimensional Lie algebras, although occasionally we will use examples from the category of nxn matrices. In the following, the reader may substitute the category of his or her choice.

We would like to thank Fritz Grunewald for pointing out that every degeneration of finite dimensional Lie algebras can be realized by a deformation.

1. Deformations and Degenerations

Throughout this paper, we consider an n-dimensional Lie algebra as an element of $\mathrm{Hom}(\Lambda^2 V,V)$, where V is an n-dimensional vector space over an algebraically closed field k . The set of Lie algebras is an algebraic subset L of $\mathrm{Hom}(\Lambda^2 V,V)$, and the general linear group $GL_n(k)$ acts on L by:

$$(g \cdot \mu)(x,y) = g(\mu(g^{-1}x, g^{-1}y)) \ .$$

The orbits under this action are the isomorphism classes, and we say that μ_1 degenerates to μ_0 , or μ_0 is a <u>degeneration</u> of μ_1 , if μ_0 is in $\overline{O(\mu_1)}$, the Zariski closure of the orbit of μ_1 . For example, every Lie algebra degenerates to the abelian Lie algebra via:

$$(t^{-1}I \cdot \mu)(x,y) = t^{-1}\mu(tx,ty) = t\mu(x,y) \ .$$

Then $\lim t^{-1}I \cdot \mu = \mu_0$, where $\mu_0(x,y) = 0$.

The intuitive definition of a <u>deformation</u> of μ_0 is a one-parameter family

$$\mu(t) = \mu_0 + t\varphi_1 + t^2\varphi_2 + \ldots$$

where $\varphi_i \in \mathrm{Hom}(\Lambda^2 V, V)$ and $\mu(t) \in L$ for each $t \in k$. The example above is a deformation; we have

$$\mu(t) = \mu_0 + t\mu \ .$$

Note that this definition of deformation does not require the vector space V to be finite dimensional; in fact, deformations of infinite dimensional Lie algebras have been studied (see, for instance, [5] and [7]) and the infinite dimensional case is of great interest to physicists (see, for instance, [4]). Because the orbit closure formulation requires the Lie algebra structures to lie in a variety (i.e. a finite number of structure constants), deformation theory for infinite dimensional Lie algebras has no reasonable orbit closure analog.

Of course, even in the case of finite dimensional Lie algebras, not every deformation is a degeneration and vice versa, as we demonstrated in the introduction for the case of matrices. One point in the intersection of these two theories is the following.

<u>Proposition 1.1</u> If $\mu(t)$ is a deformation of μ_0 parametrized by t , then $\mu_0 \in \overline{\bigcup_{t \in k} O(\mu(t))}$.

But a more unexpected connection between deformation and orbit closure arises when one considers formal deformations more generally. From the viewpoint of formal deformations, we consider a deformation $\mu(t)$ not as a family of Lie algebra structures, but as a Lie algebra over the field $k((t))$. Then a natural generalization is to allow more parameters, i.e. use $k[[t_1,\dots,t_r]]$ or consider k-algebras other than power series rings. (By "k-algebras" we mean associative, commutative k-algebras with identity.)

<u>Definition</u> Let the parameter ring A be a local finite dimensional algebra over k and let μ_0 be a Lie algebra over k (not necessarily finite dimensional). If μ_A is a Lie algebra in $\mathrm{Hom}(\Lambda^2 V,V)$, where V is a free A-module, then for a morphism $f : A \longrightarrow B$, $\mu_A \otimes_A B$ is a Lie algebra in $\mathrm{Hom}(\Lambda^2(V\otimes B), V\otimes B)$ which is defined in the natural way. A <u>formal</u> <u>deformation</u> of μ_0 parameterized by A is a Lie algebra μ_A over A such that

$$\mu_A \otimes_A k = \mu_0$$

where the tensor product is defined by the residue map $A \longrightarrow A/m_A = k$.

More generally, if A is a complete local k-algebra (i.e. $A = \varprojlim A/m_A^n$) such that A/m_A^n is finite dimensional for all n, then a deformation of the Lie algebra μ_0 parametrized by A is a Lie algebra μ_A over A such that $\mu_A = \varprojlim \mu_n$, where μ_n is a deformation of μ_0

parametrized by A/m_A^{n+1} . Two deformations μ_A and μ'_A of μ_0 parametrized by A are equivalent if there is a Lie algebra isomorphism $\mu_A \approx \mu'_A$ which induces the identity map on $\mu_A \otimes k = \mu_0$.

In the case that the parametrization algebra A is $k[[t]]$, this definition coincides with Gerstenhaber's concept of deformation [9].

The analogous viewpoint in the theory of orbit closure is the following characterization of orbit closure, which we present here for the category of n-dimensional Lie algebras, although it holds for many algebraic group actions on varieties.

Theorem 1.2 [10] Let μ_0 and μ_1 be n-dimensional Lie algebras over k . The Lie algebra μ_0 is a degeneration of μ_1 (i.e. $\mu_0 \in \overline{O(\mu_1)}$) if and only if there is a discrete valuation k-algebra A with residue field k , whose quotient field K is finitely generated over k of transcendence degree one, and there is an n-dimensional Lie algebra μ_A over A such that

$$\mu_A \otimes K \approx \mu_1 \otimes K$$

and $\quad \mu_A \otimes k = \mu_0 \quad .$

The orbit closure example given at the beginning of this section would be characterized as follows. Let $A = k[t]_{<t>}$, the polynomial ring localized at the prime ideal $<t>$, and let $\mu_A = t\mu$. Then μ is K-isomorphic to μ_A via the isomorphism $t^{-1}I$ $(K = k(t))$, and $\mu_A \otimes k = \mu_0$.

From Theorem 1.2, and ignoring the conditions on A specified in the definition of formal deformation, we could say that $\mu_0 \in \overline{O(\mu_1)}$ if and only if μ_1 is K-isomorphic to a formal deformation of μ_0 parametrized by A, for some A satisfying the conditions of Theorem 1.2. A crucial difference between the definition of formal deformation and the statement of Theorem 1.2 is that, in the former, the k-algebra A is Artinian, and in the latter it is Noetherian.

On the other hand, if we consider the completion $\hat{A}$ of the discrete valuation k-algebra A from Theorem 1.2, then we see that every degeneration can be realized by a deformation. If μ_A is the Lie algebra over A defining the degeneration (i.e. $\mu_A \otimes k = \mu_0$ and $\mu_A \otimes K \approx \mu_1 \otimes K$), let $\mu_n = \mu_A \otimes A/m_A^{n+1}$ and let $\mu_{\hat{A}} = \varprojlim \mu_n$. Then $\mu_n \otimes k = \mu_0$ for all n. Thus $\mu_{\hat{A}}$ is a formal deformation of μ_0. And so we have:

<u>Proposition 1.3</u> If μ_0 is in the boundary of the orbit of μ_1, then this degeneration defines a non-trivial deformation of μ_0.

(A deformation μ_A of μ_0 is trivial if it is equivalent to $\mu_0 \otimes A$.)

Since every degeneration can be realized by a deformation, then the existence of non-trivial degenerations to μ_0 implies the existence of non-trivial deformations of μ_0 . For a counterexample of the converse in a different category, consider conjugacy classes of nxn matrices. We know from [8] that a matrix with one Jordan block for each eigenvalue is a degeneration of no other non-equivalent matrix, but every matrix has non-trivial deformations.

So far in this comparison of deformation and degeneration, we have considered only the one-parameter case. Even though Theorem 1.2 specifies that the quotient field K of the k-algebra A has transcendence degree one over k (one parameter), the proof of the theorem does not require such a restriction. (The theorem is stated in this way to establish that the degeneration <u>can</u> be realized by such an A , not that it <u>must</u> be.) And so, just as one may generalize Gerstenhaber's concept to include k-algebras like $k[[t_1,\ldots,t_r]]$ one may also realize orbit closure by k-algebras with transcendence degree greater than one.

2. Versal Deformations and Versal Degenerations

An important concept in deformation theory is that of a versal deformation, that is, one deformation which induces all others. Since this deformation is not unique, we call it "versal" rather than "universal".

Definition A deformation μ_R of μ_0 parametrized by a complete local k-algebra R is called formally versal if for any deformation μ_A of μ_0 parametrized by a complete local k-algebra A, there is a morphism $f : R \longrightarrow A$ such that the induced map $m_R/m_R^2 \longrightarrow m_A/m_A^2$ is unique and $\mu_R \otimes_R A$ is equivalent to μ_A.

The following theorem establishes the existence of a versal deformation in the case that the 2-cohomology space with coefficients in the adjoint representation is finite dimensional. Of course, this condition always holds for finite dimensional Lie algebras.

Theorem 2.1 [7] Let μ_0 be a Lie algebra over k (not necessarily finite dimensional). If $H^2(\mu_0,\mu_0)$ is finite dimensional, then there is a formal versal deformation of μ_0.

This theorem was established by applying a theorem of Schlessinger [19, 2.11] to the category of Lie algebras. Schlessinger's construction of a versal deformation is based on the fact that the parameter ring A is Artinian. His construction does not provide a method for computing versal deformations, and, for a given Lie algebra, it remains a difficult problem to compute a versal deformation.

One might ask if such a versal object exists in the case of orbit closure, or even if such an idea makes sense. First we note that the statement analogous to "μ_1 is a deformation of μ_0" is the dual statement "μ_0 is a degeneration of μ_1". The existence of a versal deformation depended on the fact that the parameter rings were Artinian, and the analogous rings in the orbit closure case are Noetherian. Therefore we might expect such a versal object to induce degenerations, the dual of deformations. With this in mind, we state the following definition.

Definition Let μ_1 be an n-dimensional Lie algebra. A versal degeneration of μ_1 is an n-dimensional Lie algebra μ_R over a k-algebra R such that for any n-dimensional Lie algebra μ_A over a discrete valuation k-algebra A which defines a degeneration μ_0 of μ_1 (i.e. $\mu_A \otimes K \approx \mu_1 \otimes K$, where K is the quotient field of A, and $\mu_A \otimes k = \mu_0$), there is a morphism $f : R \longrightarrow A$ such that $(\mu_R \otimes A) \otimes K$ and $\mu_A \otimes K$ are isomorphic over K.

To construct a k-algebra R and a versal degeneration μ_R, we use the algebraic geometry involved. The coordinate ring of the algebraic set of n-dimensional Lie algebras is $k[X_{ijk}]/I$, where the X_{ijk} are the coordinate functions of the structure constants and I is generated by the anti-commutativity and Jacobi conditions. Let R be the coordinate ring of $\overline{O(\mu_1)}$; then $R = k[X_{ijk}]/J$, for some ideal J containing I . Let μ_R be the Lie algebra over R defined by the structure constants $(\bar{X}_{ijk})$ ($\bar{X}_{ijk}$ is the image of X_{ijk} in the quotient ring R); i.e. for $e_i = (0,\ldots,1,\ldots 0)$ in R^n , let

$$\mu_R(e_i,e_j) = \sum_k \bar{X}_{ijk}\, e_k \ .$$

The elements of $\overline{O(\mu_1)}$ (the degenerations of μ_1) are the Lie algebras over k derived from μ_R . An element μ_0 of the algebraic set $\overline{O(\mu_1)}$ defines the evaluation morphism

$$e_0 : R \longrightarrow k \quad \text{given by} \quad \bar{X}_{ijk} \longrightarrow a_{ijk} \ ,$$

where μ_0 has structure constants (a_{ijk}) relative to a fixed basis of k^n . From the definition of e_0 , we have:

$$\mu_R \otimes_{e_0} k = \mu_0 \ .$$

Thus the coordinate ring R of $\overline{O(\mu_1)}$ and the Lie algebra μ_R are natural candidates for a versal degeneration.

<u>Theorem 2.2</u> The Lie algebra μ_R , where R is the coordinate ring of $\overline{O(\mu_1)}$, is a versal degeneration of μ_1 .

<u>Proof</u>: Let μ_A define a degeneration of μ_1 , i.e. $\mu_A \otimes K \approx \mu_1 \otimes K$, where K is the quotient field of A , and $\mu_A \otimes k = \mu_0$. Let $f_1 : R \longrightarrow k$ be given by $f_1(\bar{X}_{ijk}) = c_{ijk}$, where (c_{ijk}) are the structure constants for μ_1 . Let $f = i \circ f_1$ where i is the inclusion of k into A . It follows that

$$(\mu_R \otimes_f A) \otimes K = \mu_1 \otimes K \approx \mu_A \otimes K .$$

<u>Remark</u>: Although the versal degeneration μ_R which we constructed is not defined over a local ring (one of the conditions in Theorem 1.2), for a given degeneration μ_0 of μ_1 we can choose a localization R_M of R such that

$$\mu_{R_M} \otimes k = \mu_0 ,$$

where $\mu_{R_m} = \mu_R \otimes R_M$. Simply let M be the maximal ideal of R corresponding to μ_0 $(M = \ker e_0)$. A natural question is: does μ_{R_M} define a degeneration of μ_1 to μ_0 ? That is, do we have

$$\mu_{R_M} \otimes K \approx \mu_1 \otimes K ,$$

where K is the quotient field of R ?

<u>Remark</u>: An analytic version of deformation and versal deformation which exploits the orbit structure is considered, for instance, in the case of $n \times n$ matrices over $\mathbb{C}$, by Arnold [1]. (In particular, the deformations $A(\lambda)$ in $k[[t_1,\dots,t_r]]$ and parameter changes $\varphi : \mathbb{C}^r \longrightarrow \mathbb{C}^s$ are required to be holomorphic at 0.) He shows that a deformation $A(\lambda)$ is a versal deformation of $A_0 = A(0)$ if and only if A is transversal to the orbit (conjugacy class) of $A(0)$ at 0 (i.e. the tangent space to the manifold of matrices at $A(0)$ is the sum of the tangent space to the orbit at $A(0)$ and the image under A_* of the parameter space $\mathbb{C}^r$). It is natural to consider the same idea from an algebraic viewpoint, and, in fact, an algebraic formulation of this idea appears in [6].

3. Rigidity and Cohomology

For both orbit closure and deformation theory, we may consider rigidity. In the first case, a rigid Lie algebra μ is one whose orbit is open (and so no Lie algebra not isomorphic to η degenerates to μ). In the second case, a (formally) rigid Lie algebra is one which has no non-trivial formal deformations. From Proposition 1.3, we see that if μ_0 is rigid in the sense of deformation theory, then there are no non-trivial degenerations to μ_0 . However, the absence of non-trivial degenerations does not necessarily imply that the orbit is open (for conjugacy classes of matrices, no orbit is open).

In both cases we have the same rigidity theorem: a Lie algebra μ is rigid if the 2-cohomology of μ with coefficients in the adjoint representation $H^2(\mu,\mu)$ vanishes. (For orbit closure see [17]; for deformation theory see [9].) For instance, if a finite dimensional Lie algebra μ is semisimple or if μ is a Borel subalgebra of a finite dimensional semisimple Lie algebra, then $H^2(\mu,\mu) = 0$ and so μ is rigid with respect to orbit closure and with respect to deformation [3, 24.1].

In the category of commutative algebras, rigidity with respect to deformation is equivalent to the vanishing of the symmetric 2-cohomology space $H^2(\mu,\mu)^s$ [11].

In the case the orbit closure, the proof of the rigidity theorem is based on the idea that there is an injection

$$\frac{\text{tangent space of } \mu \text{ to } L}{\text{tangent space of } \mu \text{ to } O(\mu)} \longrightarrow H^2(\mu,\mu)$$

and from this it follows that μ is rigid if $H^2(\mu,\mu) = 0$.

In the case of deformations, the elements of $H^2(\mu,\mu)$ correspond to infinitesimal deformations.

<u>Definition</u> A deformation μ_A of μ_0 parametrized by A is of <u>order</u> r if $m_A^{r+1} = 0$. A deformation of order 1 is called an <u>infinitesimal</u> <u>deformation</u>.

From Section 1, recall the definition of a formal deformation parametrized by a complete local ring. A deformation of μ_0 parametrized by A is a projective limit $\varprojlim \mu_n$, where μ_n is a deformation of μ_0 parametrized by A/m_A^{n+1} . Then if μ_A is a deformation parametrized by a complete local ring, the Lie algebra μ_r is a deformation of order r .

For instance, if $A = k[[t_1,\dots,t_s]]$, and μ_A is a deformation of μ_0 , then μ_1 can be written

$$\mu_1 = \mu_0 + \sum_{i=1}^{s} t_i \varphi_i .$$

It follows from the Jacobi identity that φ_i is a 2-cocycle for all i . If φ_i is a 2-coboundary for some i , then there is an equivalent deformation where the t_i-term is zero and at least one of the non-zero terms of lowest degree involving t_i has a coefficient which is a 2-cocycle not cohomologous to zero [9]. It follows that if $H^2(\mu_0,\mu_0) = 0$, then every infinitesimal deformation of μ_0 is trivial.

In the case of an arbitrary complete local ring, the rigidity theorem is established by a similar argument.

If $H^2(\mu_0,\mu_0) \neq 0$, then a maximal set of non-trivial pairwise non-equivalent infinitesimal deformations forms a basis of $H^2(\mu_0,\mu_0)$. (For $\varphi \in H^2(\mu_0,\mu_0)$, choose one of the generators c of the parameter ring A ; then $\mu_0 + c\varphi$ is an infinitesimal deformation of μ_0 .)

The 3-cohomology space $H^3(\mu_0,\mu_0)$ can be interpreted as obstructions to extending an infinitesimal deformation to a higher order deformation. These obstructions are closely connected with the Massey operations in the cohomology space. (See [7].)

The 3-cohomology space also appears in the theory of degenerations. In the case of degenerations of Lie algebras over $\mathbb{R}$ or $\mathbb{C}$, there is an analytic map from $H^2(\mu,\mu)$ to $H^3(\mu,\mu)$ whose zeroes parametrize a neighbourhood of μ [18]. In particular, if this map is injective, then the orbit of μ is open.

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ON A POTENTIAL FUNCTION FOR THE WEIL-PETERSSON METRIC ON TEICHMÜLLER SPACE

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§0 Introduction

In 1956 Weil suggested a Riemannian metric on Teichmüller space and in [1] Ahlfors proved it was Kähler, Somewhat later he showed that it had negative Ricci and holomorphic sectional curvature. In [7] the author showed that the sectional curvature is negative. In 1982 we proved the existence of a potential function for this metric. In the ensueing years this result has been used by several authors [5],[8]. Recently [6] it was used in Jost's own computation of the curvature of Teichmüller space, and was rediscovered by Wolf [8] in his 1986 thesis. The growing interest in this result makes it worthwhile to have a proof in the literature.

§1 Preliminaries

Let M be an oriented compact, $\partial M = \phi$* and let $\mathcal{M}_{-1}$ be the Tame Frechét manifold [2] of Riemannian metrics of constant negative curvature on M. The tangent space of $\mathcal{M}_{-1}$ at a metric, $g, T_g\mathcal{M}_{-1}$ consists of those $(0,2)$ tensors h on M satisfying the equation

$$(1.1) \qquad -\Delta(\mathrm{tr}_g h) + \delta_g\delta_g h + \frac{1}{2}(\mathrm{tr}_g h) = 0$$

where $\mathrm{tr}_g h = g^{ij}h_{ij}$ is the trace of h w.r.t. the metric tensor g_{ij}, $\delta_g\delta_g h$ is the double covariant divergence of h w.r.t. g and Δ is the Laplace-Beltrami operator on functions. For example see [2] for details.

Let $\mathcal{D}_0$ be the Tame Frechét Lie group [2] of diffeomorphisms of M which are homotopic to the identity. Then $\mathcal{D}_0$ acts on

* the case with boundary follows analogously

$\mathcal{M}_{-1}$ by pull back, i.e. $f \longrightarrow f^*g$. Teichmüller space is then defined as

$$(1.2) \qquad \mathcal{T}(M) = \mathcal{M}_{-1}/\mathcal{D}_0 \quad .$$

In [2],[5] we show that $\mathcal{T}(M)$ is a C^∞ finite dimensional manifold diffeomorphic to $\mathbb{R}^q$, $q = 6\,(\text{genus } M) - 6$. The L_2-metric on $\mathcal{M}_{-1}$ is given by the inner product

$$(1.3) \qquad \langle\langle h,k\rangle\rangle_g = \frac{1}{2}\int_M \text{trace }(HK)\,d\mu_g$$

where $H = g^{-1}h$, $K = g^{-1}k$ are the $(1;1)$ tensors on M obtained from h and k via the metric g, or "by raising an index", i.e.

$$H^i_j = g^{ik}h_{kj}$$

and similarly for K. Finally μ_g is the volume element induced on M by g and the given orientation.

The inner product (1.3) is $\mathcal{D}_0$ invariant. Thus $\mathcal{D}_0$ acts smoothly on $\mathcal{M}_{-1}$ as a group of isometries with respect to this metric, and consequently we have an induced metric on $\mathcal{T}(M)$ in such a way that the projective map $\pi : \mathcal{M}_{-1} \longrightarrow \mathcal{M}_{-1}|\mathcal{D}_0$ becomes a Riemannian submersion [2]. In [3] it is shown that this induced metric is precisely the metric originally introduced by Weil.

Let $\langle,\rangle$ be the induced metric on $\mathcal{T}(M)$. We can characterize $\langle,\rangle$ as follows. From [2] we can show that given $g \in \mathcal{M}_{-1}$ every $h \in T_g\mathcal{M}_{-1}$ can be uniquely written as

$$(1.4) \qquad h = h^{TT} + L_X g$$

where $L_X g$ is the Lie derivative of g w.r.t. some (unique X) and h^{TT} is a trace free, divergence free, symmetric tensor. Moreover the decomposition (1.4) is L_2-orthogonal. Recall that a conformal coordinate system (where $g_{ij} = \lambda\delta_{ij}$, λ some smooth positive function) is also a complex holomorphic coordinate system. In this system

$$h^{TT} = \text{Re}(\xi(z)dz^2)$$

where Re is "real part" and $\xi(z)dz^2$ is a holomorphic quadratic differential. In fact trace free, divergence free symmetric two tensors are precisely the real parts of holomorphic quadratic differentials.

Now $L_X g$ is always tangent to the orbit of $\mathcal{D}_0$ through g. We say that $L_X g$ is the <u>vertical</u> part of h in decomposition 1.4. Similarly we say that h^{TT} represents the <u>horizontal</u> part of H. Given $h,k \in T_{[g]}\mathcal{T}(M)$ there are unique horizontal vectors $\tilde{h},\tilde{k} \in T_g\mathcal{M}_{-1}$ such that $d\pi(g)\tilde{h} = h$ and $d\pi(g)\tilde{k} = k$. Then

$$\langle h,k\rangle_{[g]} = \langle\langle \tilde{h},\tilde{k}\rangle\rangle_g \quad .$$

Suppose now that $g_0 \in \mathcal{M}_{-1}$ is fixed and that $s!(M,g) \longrightarrow (M,g_0)$ is a smooth C^1 map homotopic to the identity and is viewed as a map from M with some arbitrary metric $g \in \mathcal{M}_{-1}$ to M with its g_0 metric.

Define the Dirichlet energy of s by the formula

$$E_g(s) = \frac{1}{2}\int_M |ds|^2\, d\mu_g \tag{1.5}$$

where $|ds|^2$ = trace ds^*ds depends on both g and g_0.

By the embedding theorem of Nash-Moser we may assume that (M,g_0) is isometrically embedded in some Euclidean $\mathbb{R}^K$. Thus we can think of $s : (M,g) \longrightarrow (M,g_0)$ as a map into $\mathbb{R}^K$ and Dirichlet's functional takes the equivalent form

$$E_g(s) = \frac{1}{2}\sum_{i=1}^{k} \int g(x)\, \langle \nabla_g s^i(x), \nabla_g s^i(x)\rangle d\mu_g \quad . \tag{1.6}$$

There is another, equivalent, and useful way to express (1.5) and (1.6) using local conformal cordinate systems $g_{ij} = \sigma\delta_{ij}$ and $(g_0)_{ij} = \rho\delta_{ij}$ on (M,g) and (M,g_0) respectively, namely

$$E_g(s) = \frac{1}{4}\int_M [\rho(s(z))|s_z|^2 + \rho(s(z))|s_{\bar{z}}|^2]dzd\bar{z} \tag{1.7}$$

For fixed g, the critical points of E are there said to be <u>harmonic</u> <u>maps</u>. The follwing result is due to Schoen-Yau [9].

<u>Theorem.</u> Given metrics g and g_0 there exists a unique harmonic map $s(g) : (M,g) \longrightarrow (M,g_0)$ which is homotopic to the identity. Moreover $s(g)$ depends differentially on g in any H^r topology,

$r > 2$, and is a C^{∞} diffeomorphism.

Consider now the function

$$g \longrightarrow E_g(s(g)) \quad .$$

This function on $\mathcal{M}_{-1}$ is $\mathcal{D}$-invariant and thus can be viewed as a function on Teichmüller space.

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This function on $\mathcal{M}_{-1}$ is $\mathcal{D}$-invariant and thus can be viewed as a function on Teichmüller space. To see this one must show that

$$E_{f^*g}(s(f^*(g))) = E_g(s(g)) \quad .$$

Let $c(g)$ be the complex structure associated to g, and induced by a conformal coordinate system for g. For $f \in \mathcal{D}_0$, $f : (M,f^*c(g)) \longrightarrow (M,c(g))$ is holomorphic and consequently since the composition of harmonic maps and holomorphic maps is still harmonic we may conclude, by uniqueness that

$$s(f^*g) = s(g) \circ f \quad .$$

Since Dirichlet's functional is invariant under complex holomorphic changes of coordinates it follows immediately that

$$E_{f^*(g)}(s(g) \circ f) = E_g(s(g)) \quad .$$

Consequently for $[g] \in \mathcal{M}_{-1}|\mathcal{D}_0$ define the C^{∞} smooth function

$$\widetilde{E} : \mathcal{M}_{-1}|\mathcal{D}_0 \longrightarrow R$$

by

$$\tilde{E}[g] = E_g(s(g)) \ .$$

§2 The Main Result

Theorem 2.1 $[g_0]$ is the only critical point of $\tilde{E}$. The Hessian of $\tilde{E}$ at $[g_0]$ is given by

$$d^2\tilde{E}[g_0](h,k) = 2\langle h,k\rangle$$

$h,k \in T_{[g_0]}\mathcal{T}(M)$. That is, the second variation of Dirichlet's energy function is (up to a positive constant) Weil-Petersson metric.

Proof. We begin by computing the first derivative $d\tilde{E}[g_0]$. We again view a map $S : (M,g) \longrightarrow (M,g_0)$ as a map into $\mathbb{R}^k$. Consider the two form

$$\xi(z)dz^2 = \sum_{i=1}^{k} (s_z^i)^2 dz^2 = \sum_{i=1}^{k} (\frac{\partial s^i}{\partial z})^2 dz^2 .$$

We start by proving

Proposition 2.2. If $s : (M,g) \longrightarrow (M,g_0)$ is harmonic the form $\xi(z)dz^2$ is a holomorphic quadratic differential on the complex curve $(M,c(g_0))$, and thus $\mathrm{Re}\ \xi(z)dz^2$ represents a trace free, divergence free symmetric two tensor on (M,g_0). Hence $\mathrm{Re}\ \xi(z)dz^2$ is a horizontal tangent vector to $\mathcal{M}_{-1}$ at g_0. Finally

$$(2.3) \qquad d\tilde{E}[g_0]h = -\ \mathrm{Re}\langle\langle \xi(z)dz^2, \tilde{h}\rangle\rangle_{g_0}$$

where $\tilde{h}$ is the horizontal left of $h \in T_{(g_0)}\mathcal{T}(M)$.

Proof (of 2.2)

We have Dirichlet's functional

$$E(g,s) = \frac{1}{2} \sum_{i=1}^{k} \int_M g(x)\ (\nabla_g s^i, \nabla_g s^i) d\mu_g \ .$$

Suppose s is harmonic. Let Ω denote the second fundamental form of $(M,g_0) \subset \mathbb{R}^k$. Thus for each $p \in M$, $\Omega(p) : T_pM \times T_pM \longrightarrow T_pM^{\perp}$. Let Δ denote the (non-linear) Laplacian of maps from (M,g) to (M,g_0) and Δ_β denote the Laplace-Betrami operator on functions. Then if s is harmonic we have

(2.4) $$0 = \Delta s = \Delta_\beta s + \sum_{j=1}^{2} \Omega(s)(ds(e_j), ds(e_j))$$

$e_1(p)$, $e_2(p)$ on orthonormal basis for T_pM with respect to g. $\xi(z)dz^2$ will be holomorphic of

$$\frac{\partial}{\partial \bar{z}}\left(\sum_{i=1}^{k} \frac{\partial s^i}{\partial z} \cdot \frac{\partial s^i}{\partial z}\right) = 0 \quad .$$

But this is equal to

$$\frac{2}{\sigma} \cdot \sum_{i=1}^{k} \Delta_\beta s^i \cdot \frac{\partial s^i}{\partial z}$$

where in conformal coordinates $g_{ij} = \sigma\delta_{ij}$. By (2.4) we see that this, in time, is equal to

$$-\frac{2}{\sigma} \sum_{i=1}^{k} \sum_{j=1}^{2} \Omega^i(s)(ds(e_j), ds(e_j)) \cdot \frac{\partial s^i}{\partial z}$$

$$= -\frac{2}{\sigma} \sum_{j=1}^{2} \left\{ \sum \Omega(s)(ds(e_j), ds(e_j)) \cdot \frac{\partial s}{\partial x} + i\Omega(s)(ds(e_j), ds(e_i)) \cdot \frac{\partial s}{\partial y}\right\} \quad .$$

Since $\Omega(p)$ takes values in $T_pM^\perp$ it follows that both the real and imaginary parts of the expression vanish. Thus $\xi(z)dz^2$ is holomorphic.

Recall that s is harmonic iff $\frac{\partial E}{\partial s}(g,s) = 0$. We now compute $\frac{\partial E}{\partial g}$. If we have local coordinates represented by $(x,y) \in W$, then in this coordinate system

$$E(g,s) = \frac{1}{2} \sum_{\ell=1}^{k} \int_M g(x) \langle G^{-1}\nabla s^\ell, \nabla s^\ell\rangle_{\mathbb{R}^2} \sqrt{\det G}\, dxdy$$

where ∇s^ℓ is the vector $\left(\frac{\partial s^\ell}{\partial x}, \frac{\partial s^\ell}{\partial y}\right)$, G is the matrix $\{g_{ij}\}$ of g and $\langle,\rangle_{\mathbb{R}^2}$ denotes the ordinary $\mathbb{R}^2$ inner product and $\sqrt{\det G}$ dxdy is the local representation of $d\mu_g$. In the following computation we adopt the convention, that summations over the index ℓ will be understood.

(2.5) $$\frac{\partial E}{\partial g}(g_0,s)\tilde{h} = -\int \langle G_0^{-1} H G_0 \nabla s^\ell, \nabla s^\ell\rangle \sqrt{\det G_0}\, dxdy$$

$$+ \frac{1}{2} \int \langle G_0^{-1} \nabla s^\ell, \nabla s^\ell\rangle \frac{\operatorname{trace} H}{\sqrt{\det G_0}}\, dxdy$$

where $H = \{\tilde{h}_{ij}\}$ is the matrix of the symmetries tensor h in these coordinates. Here we use the fact that the derivative of $G \longrightarrow G^{-1}$ is $H \longrightarrow G^{-1}HG^{-1}$. Suppose we look at this first derivative in conformal coordinates $(g_0)_{ij} = \lambda\delta_{ij}$. Then if $\tilde{h}$ is horizontal the second term in (2.5) vanishes (h is trace free) and

$$\frac{\partial E}{\partial g}(g_0,s)\tilde{h} = -\int \frac{1}{\lambda} \langle \nabla s^\ell, \nabla s^\ell \rangle_{\mathbb{R}^2} \, dxdy$$

$$= -\int \frac{1}{\lambda} \left\{\tilde{h}_{11}\left(\frac{\partial s^\ell}{\partial x}\right)^2 + 2\tilde{h}_{12}\left(\frac{\partial s^\ell}{\partial x}\right)\left(\frac{\partial s^\ell}{\partial y}\right) + \tilde{h}_{22}\left(\frac{\partial s^\ell}{\partial y}\right)^2\right\}dxdy \quad .$$

Since $h_{11} = -h_{22}$ this is equal to

$$(2.6) \qquad -\int \frac{1}{\lambda}\left\{\tilde{h}_{11}\left[\left(\frac{\partial s^\ell}{\partial x}\right)^2 - \left(\frac{\partial s^\ell}{\partial x}\right)^2\right. + 2\tilde{h}_{12}\left(\frac{\partial s^\ell}{\partial x}\right)\left(\frac{\partial s^\ell}{\partial y}\right)\right\} dxdy \quad .$$

Now

$$\left(\frac{\partial s^\ell}{\partial x} - i\frac{\partial s^\ell}{\partial y}\right)(dx + dy)^2 = \xi(z)dz^2$$

is a quadratic differential. But

$$\mathrm{Re}(\xi(z)dz^2) = \left[\left(\frac{\partial s^\ell}{\partial x}\right)^2 - \left(\frac{\partial s^\ell}{\partial y}\right)^2\right]dx^2 + \left[\left(\frac{\partial s^\ell}{\partial y}\right)^2 - \left(\frac{\partial s^\ell}{\partial x}\right)^2\right]dy^2 + 4\left(\frac{\partial s^\ell}{\partial x}\right)\left(\frac{\partial s^\ell}{\partial y}\right)dxdy .$$

If s is harmonic $\mathrm{Re}(s(z)dz^2)$ is a trace free divergence free tensor. Let us compute

$$\langle\langle \mathrm{Re}\ \xi(z)dz^2, \tilde{h}\rangle\rangle_{g_0} \quad .$$

This inner product is given locally by the expression

$$(2.7) \qquad \frac{1}{2}\int g_0^{ab} g_0^{cd} k_{ac} \tilde{h}_{bd} d\mu_g$$

where k_{ac} is the coordinate representative of the two tensor $\xi(z)dz^2$. Therefore

$$k_{11} = \left\{\left(\frac{\partial s^\ell}{\partial x}\right)^2 - \left(\frac{\partial s^\ell}{\partial y}\right)^2\right\} , \quad k_{12} = k_{21} = 2\left(\frac{\partial s^\ell}{\partial x}\right)\left(\frac{\partial s^\ell}{\partial y}\right) \quad .$$

Thus in conformal coordinates (2.7) is equal to

$$\int \frac{1}{2\lambda}\{k_{ac}\tilde{h}_{ac}\}dxdy$$

$$= \int \frac{1}{2\lambda}\{k_{11}\tilde{h}_{11} + 2k_{11}\tilde{h}_{12} + k_{22}\tilde{h}_{22}\}dxdy \ .$$

Since $k_{11} = -k_{22}$, $\tilde{h}_{11} = -\tilde{h}_{22}$ this equals

$$\int \frac{1}{\lambda}\{k_{11}\tilde{h}_{11} + k_{12}\tilde{h}_{12}\}dxdy$$

$$= \int \frac{1}{\lambda}\{[(\frac{\partial s^{\ell}}{\partial x})^2 - (\frac{\partial s^{\ell}}{\partial y})^2]\tilde{h}_{11} + 2(\frac{\partial s^{\ell}}{\partial x})(\frac{\partial s^{\ell}}{\partial y})\tilde{h}_{12}\}dxdy \ .$$

Comparing this with expression (2.6) establishes the formula

$$\frac{\partial E}{\partial g}(g_0,s)\tilde{h} = -\langle\langle \mathrm{Re}\ \xi(z)dz^2,\tilde{h}\rangle\rangle_{g_0} \ .$$

However $\tilde{E}[g] = E(g,s(g))$. Since $s(g)$ is harmonic $\frac{\partial E}{\partial s}(g_0,s(g_0)) = 0$. This immediately implies that

$$\frac{\partial \tilde{E}}{\partial g}[g_0]h = -\langle\langle \mathrm{Re}\ \xi(z)dz^2,\tilde{h}\rangle\rangle_{g_0}$$

which establishes 2.2. We should remark that this formula tells us that the gradient of Dirichlet's function on Teichmüller space is represented as a holomorphic quadratic differential.

To complete theorem 2.1 we need to compute a second derivative. Again working locally and thinking of the map s as now being fixed we see that for $\tilde{h},\tilde{k}$ <u>horizontal</u>

$$\frac{\partial^2 E}{\partial g^2}(g_0,s)(\tilde{h},\tilde{k}) = \int \langle G_0^{-1}KG_0^{-1}HG_0^{-1}\nabla s^{\ell},\nabla s^{\ell}\rangle_{\mathbb{R}^2}\sqrt{\det G_0}\ dxdy$$

$$+ \int \langle G_0^{-1}HG_0^{-1}KG_0^{-1}\nabla s^{\ell},\nabla s^{\ell}\rangle_{\mathbb{R}^2}\sqrt{\det G_0}\ dxdy$$

and in conformal coordinates this is equal to

$$\int \frac{1}{\lambda^2}\langle KH\nabla s^{\ell},\nabla s^{\ell}\rangle_{\mathbb{R}^2}\ dxdy + \int \frac{1}{\lambda^2}\langle HK\nabla s^{\ell},\nabla s^{\ell}\rangle\ dxdy$$

$$\int \frac{2}{\lambda^2}\{\tilde{h}_{11}\tilde{k}_{11} + \tilde{h}_{12}\tilde{k}_{12})\ (\frac{\partial s^{\ell}}{\partial x})^2 + (\frac{\partial s^{\ell}}{\partial y})^2]\}dxdy$$

Now at the point g_0, the unique harmonic map s is the identity map of (M,g_0) to itself. Since (M,g_0) is isommetrically immersed in $\mathbb{R}^K$, $s(g_0)^*G_{\mathbb{R}^K} = g_0$; where $G_{\mathbb{R}^K}$ is the Euclidean metric on $\mathbb{R}^K$. But if g_0 is expressed in local conformal coordinates this says exactly that

$$\{(\frac{\partial s^\ell}{\partial x})^2 + (\frac{\partial s^\ell}{\partial y})^2\} = \lambda.$$

Thus at the point g_0, we see that

$$\frac{\partial^2 E}{\partial g^2}(g_0,\mathrm{id})(\tilde{h},\tilde{k}) = \int \frac{2}{\lambda}(\tilde{h}_{11}\tilde{k}_{11} + \tilde{h}_{12}\tilde{k}_{12})\,dxdy$$

Since $\tilde{k}_{11} = -\tilde{k}_{22}$, $\tilde{h}_{11} = -\tilde{h}_{22}$, applying formula (2.7) for the Weil-Petersson metric we see that

$$\frac{\partial^2 E}{\partial g^2}(g_0,\mathrm{id})(\tilde{h},\tilde{k}) = 2\langle\langle\tilde{h},\tilde{k}\rangle\rangle \quad . \tag{2.8}$$

However we are interested in the map

$$\tilde{E}[g] = E(g,s(g)).$$

Clearly

$$\frac{\partial\tilde{E}}{\partial g}[g]h = \frac{\partial E}{\partial g}(g,s(g))\tilde{h} + \frac{\partial E}{\partial s}(g,s(g))\cdot Ds(g)\tilde{h}$$

where $Ds(g)$ represents the derivative of s with respect to g. However the second term is identically zero since $s(g)$ is harmonic implies $\frac{\partial E}{\partial s}(g,s(g)) \equiv 0$. Therefore

$$\frac{\partial^2\tilde{E}}{\partial g^2}[g_0](h,k) = \frac{\partial^2 E}{\partial g^2}(g_0,\mathrm{id})(\tilde{h},\tilde{k}) + \frac{\partial^2 E}{\partial g\partial s}(g_0,\mathrm{id})(\tilde{h},Ds(g_0)\tilde{k})$$

and by 2.8

$$= 2\langle\langle\tilde{h},\tilde{k}\rangle\rangle + \frac{\partial^2 E}{\partial g\partial s}(g_0,\mathrm{id})(\tilde{h},Ds(g_0)\tilde{k}).$$

Theorem 2.1 will now follow immediately from the following.

<u>Proposition 2.9.</u> $Ds(g_0)\tilde{h} = 0$, if $\tilde{h}$ is trace free divergence free.

Proof. In order to compute this derivative we write down the general equation of a harmonic map from a Riemanian manifold (M,g) to a Riemannian manifold $(N,\mathcal{g})$. Namely $f : (M,g) \to (N,\mathcal{g})$ is harmonic if in local coordinates, $f = (f',\dots,f^n)$, $n = \dim N$

$$(2.10) \qquad \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_j} g^{ij}\sqrt{g} \frac{\partial}{\partial x_i} f^{\alpha}\} + \Gamma^{\alpha}_{\gamma\beta} \frac{\partial f^{\gamma}}{\partial x_i} \frac{\partial f^{\beta}}{\partial x_j} g^{ij} = 0$$

where $\Gamma^{\alpha}_{\gamma\beta}$ are the Christofel symbol of the metric $\mathcal{g}$.

If $\dim N = 2 = \dim M$ and we express (2.10) in local conformal coordinates $g_{ij} = \lambda\delta_{ij}$ and $\mathcal{g}_{ij} = \rho\delta_{ij}$ we see that (2.10) is equivalent to

$$(2.11) \qquad f_{z\bar{z}} + (\log \rho)_f f_z f_{\bar{z}} = 0$$

where $(\log \rho)_f = \frac{\rho(f)}{\rho'(f)}$.

In the case under consideration $\mathcal{g}$ is the fixed metric g_0 on M. We now think of f^{α} as depending on g, and let $w^{\alpha} = Df^{\alpha}(\tilde{h})$ be the linearization of f^{α} in the direction $\tilde{h}$. We now differentiate equation (2.10) w.r.t. g in the direction $\tilde{h}$. We first make three important observations. The Christofel symbol Γ^{α}_{β} are fixed and do not depend on g. Second the derivative of $\sqrt{g}$ in a direction $\tilde{h}$ is given by $\tilde{h} \longrightarrow tr_g h/\sqrt{g}$

If $\tilde{h}$ is trace free this derivative vanishes. Thirdly, the derivative of $g^{ij}\sqrt{g}$ in the direction $\tilde{h}$ is $\tilde{h} \longrightarrow -\tilde{h}^{ij}$.

Taking the derivative of (2.10) w.r.t. g in the direction $\tilde{h}$, evaluating it in conformal coordinates $(g_0)_{ij} = \lambda\delta_{ij}$ at $f = id$, and using formula 2.12 for the complex form of $w = w + iw_2$ we see that

$$(2.12) \qquad w_{z\bar{z}} + (\log \lambda)_z w_{\bar{z}} = + \frac{1}{\lambda} \frac{\partial}{\partial x_j} \{\tilde{h}^{\alpha j}\} + \frac{\Gamma^{\alpha}_{ij}\tilde{h}_{ij}}{\lambda^2} \quad .$$

Lemma 2.13 If $\tilde{h}$ is trace free and divergence free, the expression

$$(2.14) \qquad \frac{1}{\lambda} \frac{\partial}{\partial x_j} \{\tilde{h}^{\alpha j}\} + \frac{1}{\lambda^2} \Gamma^{\alpha}_{ij} \tilde{h}_{ij} = 0.$$

Before proving 2.13 let us see how it implies proposition 2.9.

Using 2.12 we see that the linearization $w = Ds(g_0)\tilde{h}$ satisfies

$$w_{z\bar{z}} + (\log \lambda)_z w_{\bar{z}} = 0$$

or

$$\frac{\partial}{\partial z}(\lambda w_{\bar{z}}) = 0 \ .$$

Now this implies that

$$\int \frac{\partial}{\partial z}(\lambda w_{\bar{z}})\overline{w}\, dz \wedge d\bar{z} = 0 \ .$$

Integrating by parts we further see that

$$\int \lambda |w_{\bar{z}}|^2 dz \wedge d\bar{z} = 0 \ .$$

Therefore $w_{\bar{z}} = 0$ and consequently w is a holomorphic vector field on $(M, c(g_0))$. Since (genus M) > 1 this clearly implies that $w \equiv 0$ concluding 2.9.

To prove lemma 2.13 we note that

$$\Gamma^{\alpha}_{ij} = \frac{1}{2\lambda}\{\frac{\partial\lambda}{\partial x_j}\delta_{i\alpha} + \frac{\partial\lambda}{\partial x_i}\delta_{j\alpha} - \frac{\partial\lambda}{\partial x_\alpha}\delta_{ij}\}$$

and that $\tilde{h}^{\alpha j} = \frac{1}{\lambda}\tilde{h}_{\alpha j}$. Since $\tilde{h}$ is divergence free $\frac{\partial}{\partial x_j}\tilde{h}_{\alpha j} = 0$ and so

$$\frac{1}{\lambda}\frac{\partial}{\partial x_j}(\tilde{h}^{\alpha j}) = -\frac{1}{\lambda^3}\tilde{h}_{\alpha j}\frac{\partial\lambda}{\partial x_j} \ .$$

Therefore expression 2.14 equals

$$-\frac{1}{\lambda^3}\frac{\partial\lambda}{\partial x_j}\tilde{h}_{\alpha j} + \frac{1}{2\lambda^3}\{\frac{\partial\lambda}{\partial x_j}\delta_{i\alpha} + \frac{\partial\lambda}{\partial x_i}\delta_{j\alpha} - \frac{\partial\lambda}{\partial x_\alpha}\delta_{ij}\}\tilde{h}_{ij}$$

$$= -\frac{1}{\lambda^3}\frac{\partial\lambda}{\partial x_j}\tilde{h}_{\alpha j} + \frac{1}{2\lambda^3}\frac{\partial\lambda}{\partial x_j}\tilde{h}_{\alpha j} + \frac{1}{2\lambda^3}\frac{\partial\lambda}{\partial x_i}\tilde{h}_{i\alpha} - \frac{1}{2\lambda^3}\frac{\partial\lambda}{\partial x_\alpha}\tilde{h}_{ii} \ .$$

Clearly the sum of the first three terms is zero and since $\tilde{h}$ is trace free the fourth also vanishes. This completes lemma 2.13 and this paper.

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