Singularities in hyperkähler geometry

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This is a survey of some of the work done in 1993-99 on resolution of singularities in the context of hyperkähler geometry. We define a singular hypercomplex variety and its desingularization; similar methods are applied to desingularising coherent sheaves. We relate the singularities of reflexive sheaves over hyperkähler manifolds to quaternionic-Kähler geometry. Finally, we study holomorphic symplectic orbifolds and their resolutions.

Contents

1 Introduction

This introduction is a quick summary of the works presented in this paper. The reader who is not sufficiently acquainted with the hyperkähler geometry is advised to start with Section 2.

Hyperkähler manifold is a Riemannian manifold M with three complex structures I, J, K, $I \circ J = -J \circ I = K$, such that M is Kähler with respect to I, J and K. Clearly, the operators I, J, K define a quaternion action on the tangent space to M. Hyperkähler manifolds are quaternionic analogues of the usual Kähler manifolds.

The hyperkähler manifold is, by definition, smooth. However, there were attempts to introduce singularities in hyperkähler geometry, starting from [De], [S] (Deligne and Simpson; see Definition 3.13). More recently, D.Kaledin (unpublished Ph.D. thesis) and A.Dancer – A.Swann ([DS]) studied singular varieties, appearing as a result of hyperkähler reduction. Unfortunately, as Kaledin noticed, the hyperkähler reduction does not result in the Deligne-Simpson's type singular hyperkähler varieties.

There is an obvious source of examples of singular hyperkähler varieties. Let M be a hyperkähler manifold, I, J, K the standard complex structures on M, and $X \subset M$ a closed subset. The subset X is called **trianalytic** if X is complex analytic with respect to I, J and K. As [V-h], Remark 4.4 implies, trianalytic subsets are singular hyperkähler, in the sense of Deligne and Simpson.

A group of unitarian quaternions is naturally isomorphic to $SU(2)$. This defines an $SU(2)$-action on the tangent space to a hyperkähler manifold M. If M is compact, this action defines a natural action of $SU(2)$ on the cohomology of M.

Another source of examples of singular hyperkähler varieties is given by the theory of hyperholomorphic bundles. A hyperholomorphic bundle over a compact hyperkähler manifold is a stable holomorphic bundle with first and second Chern classes $SU(2)$-invariant. In [V1], it was shown that the moduli space of hyperholomorphic bundles is singular hyperkähler.

The definition of Deligne and Simpson was studied in [V-d], [V-d2] and

[V-h]. It was found that the singularities of the singular hyperkähler varieties are remarkably simple. A canonical desingularization was constructed (Theorem 5.1); the desingularization is a smooth hyperkähler manifold.

This means that hyperkähler varieties (in the sense of Deligne and Simpson) are "almost" non-singular. Indeed, the canonical desingularization is provided by normalization.

It is possible that there is a more relaxed notion of a hyperkähler variety, which allows for more varied singularities. We study the singular structures in hyperkähler geometry, hoping to come across such a notion.

There is a notion of hyperholomorphic connection on a vector bundle B over a hyperkähler manifold M (Definition 4.3). It is a "hyperkähler analogue" of the usual (1,1) connections on holomorphic bundles. When M is compact, such connection exists if and only if B is a direct sum of stable bundles with first and second Chern classes $SU(2)$-invariant. In such a case, the hyperholomorphic connection is unique.

A similar result exists for stable sheaves (Theorem 6.11). If F is a reflexive stable coherent sheaf over a hyperkähler manifold, and the first and second Chern classes of F are $SU(2)$-invariant, then F admits a unique hyperholomorphic connection with admissible type of singularities (Definition 6.5). The study of such sheaves (called **hyperholomorphic sheaves**, Definition 6.9) is done by the same methods as the study of hyperkähler varieties. A version of desingularization theorem holds in this situation as well (Theorem 6.12).

It is easy to see that hyperkähler manifolds admit a holomorphic symplectic form (Subsection 2.1). Conversely, a compact holomorphic symplectic Kähler manifold admits a natural hyperkähler structure (this follows from Calabi conjecture, proven by S.-T. Yau (Theorem 2.8). Therefore, to study compact holomorphic symplectic Kähler manifold we need to learn about holomorphic symplectic geometry.

What is "a singular holomorphic symplectic variety"? This is not clear. However, the most natural generalization of holomorphic symplectic manifold is a holomorphic symplectic orbifold, that is, a variety which is locally isomorphic to a quotient of a holomorphic symplectic manifold by a finite group action.

Let M be a holomorphic symplectic manifold, and G a finite group acting on M preserving the symplectic structure. It is natural to consider the quotient M/G as a holomorphic symplectic orbifold. Suppose we have a resolution of singularities $\widetilde{M} \longrightarrow M/G$ with $\widetilde{M}$ a smooth holomorphically symplectic manifold. Such a situation arises, for instance, when $M = S^n$

is a product of n copies of a holomorphic symplectic surface S, $G = S_n$ the symmetric group and $\widetilde{M}$ a Hilbert scheme of S. Another instance when such a sutuation arises is described in [KV2]. Suppose that $T^{[n]}$ is a so-called "generalized Hilbert scheme" of a torus T, and $X \subset T^{[n]}$ a complex subvariety which survives a generic deformation of $T^{[n]}$ (that is, for any deformation of $T^{[n]}$, there exists a flat deformation of $X \subset T^{[n]}$). From a definition of $T^{[n]}$ (see e.g. [Bea]) it follows that $T^{[n]}$ is equipped by a generically finite map $\pi : T^{[n]} \longrightarrow T^{n+1}$. It was proven in [KV2] that in the above assumptions, $\pi(X)$ is a quotient of a torus by a Coxeter group action on it, and that $\pi : X \longrightarrow \pi(X)$ is a holomorphic symplectic resolution of $\pi(X)$.

It is not clear how the holomorphically symplectic resolutions are related to the hyperkähler geometry. For instance, it is not clear, even in the most simple cases, whether a Hilbert scheme of a non-compact hyperkähler surface is hyperkähler. Still, the desingularizations of hyperkähler orbififolds is one of the most common ways of obtaining hyperkähler manifolds.

The work [KV2] goes some way explaining why the ubiquitous Coxeter groups appear in the study of subvarieties of generalized Kummer varieties. Given a holomorphic symplectic manifold M and a finite group G acting on M by symplectomorphisms, let $\widetilde{M}$ be a holomorphic symplectic resolution of M/G. Then, G is generated by **symplectic reflections**, that is, by automorphisms with fixed set of codimension 2 (see Section 7 for detail).

2 Hyperkähler manifolds

2.1 Hyperkähler manifolds

This subsection contains a compression of the basic and best known results and definitions from hyperkähler geometry, found, for instance, in [Bes] or in [Bea].

Definition 2.1: Let M be a smooth manifold, equipped with an action of quaternion algebra in TM. Then M is called **an almost hypercomplex manifold**.

Let M be an almost hypercomplex manifold and $I, J \neq \pm I$ quaternions satisfying $I^2 = J^2 = -1$. Clearly, I, J define almost complex structures on M.

Proposition 2.2: [K1] In the above situation, assume that the almost complex structures I and J are integrable. Let $K \in \mathbb{H}$ be a quaternion satisfying $K^2 = -1$. Then K defines an integrable complex structure on M. ∎

Definition 2.3: Let M be a smooth manifold, and I, J, K almost complex structures satisfying $I \circ J = -J \circ I = K$. Assume that I and J are integrable. Then M is called a hypercomplex manifold.

Remark 2.4: *A posteriori*, we obtain that every quaternion satisfying $K^2 = -1$ defines an integrable complex structure on a hypercomplex manifold (Proposition 2.2).

Definition 2.5: ([Bes]) A hyperkähler manifold is a hypercomplex manifold equipped with a Riemannian metric $(\cdot,\cdot)$, such that I, J, K are Kähler complex structures with respect to $(\cdot,\cdot)$.

The notion of a hyperkähler manifold was introduced by E. Calabi ([C]).

Clearly, a hyperkähler manifold has a natural action of the quaternion algebra $\mathbb{H}$ in its real tangent bundle TM. Therefore its complex dimension is even. For each quaternion $L \in \mathbb{H}$, $L^2 = -1$, the corresponding automorphism of TM is an almost complex structure. It is easy to check that this almost complex structure is integrable ([Bes]).

Definition 2.6: Let M be a hyperkähler (or hypercomplex) manifold, and L a quaternion satisfying $L^2 = -1$. The corresponding complex structure on M is called **an induced complex structure**. The M, considered as a complex manifold, is denoted by (M, L).

Definition 2.7: Let M be a complex manifold and Θ a closed holomorphic 2-form over M such that $\Theta^n = \Theta \wedge \Theta \wedge ...$, is a nowhere degenerate section of a canonical class of M ($2n = dim_{\mathbb{C}}(M)$). Then M is called **holomorphically symplectic**.

Let M be a hyperkähler manifold; denote the Riemannian form on M by $(\cdot,\cdot)$. Let the form $\omega_I := (I(\cdot),\cdot)$ be the usual Kähler form which is closed and parallel (with respect to the Levi-Civitta connection). Analogously defined forms ω_J and ω_K are also closed and parallel.

A simple linear algebraic consideration ([Bes]) shows that the form $\Theta := \omega_J + \sqrt{-1}\omega_K$ is of type $(2,0)$ and, being closed, this form is also holomorphic. Also, the form Θ is nowhere degenerate, as another linear algebraic argument shows. It is called **the canonical holomorphic symplectic form of a manifold M**. Thus, for each hyperkähler manifold M, and an induced complex structure L, the underlying complex manifold (M, L) is holomorphically symplectic. The converse assertion is also true:

Theorem 2.8: ([Bea], [Bes]) Let M be a compact holomorphically symplectic Kähler manifold with the holomorphic symplectic form Θ, a Kähler class $[\omega] \in H^{1,1}(M)$ and a complex structure I. Let $n = \dim_{\mathbb{C}} M$. Assume that $\int_M \omega^n = \int_M (Re\Theta)^n$. Then there is a unique hyperkähler structure $(I, J, K, (\cdot,\cdot))$ over M such that the cohomology class of the symplectic form $\omega_I = (\cdot, I\cdot)$ is equal to $[\omega]$ and the canonical symplectic form $\omega_J + \sqrt{-1}\,\omega_K$ is equal to Θ.

Theorem 2.8 follows from the conjecture of Calabi, proven by Yau ([Y]). ∎

Let M be a hyperkähler manifold. We identify the group $SU(2)$ with the group of unitary quaternions. This gives a canonical action of $SU(2)$ on the tangent bundle, and all its tensor powers. In particular, we obtain a natural action of $SU(2)$ on the bundle of differential forms.

Lemma 2.9: The action of $SU(2)$ on differential forms commutes with the Laplacian.

Proof: This is Proposition 1.1 of [V2]. ∎

Thus, for compact M, we may speak of the natural action of $SU(2)$ in cohomology.

The following lemma is clear from the properties of the Hodge decomposition.

Lemma 2.10: Let ω be a differential form over a hyperkähler manifold M. The form ω is $SU(2)$-invariant if and only if it is of Hodge type (p,p) with respect to all induced complex structures on M.

Proof: This is [V1], Proposition 1.2. ∎

2.2 Trianalytic subvarieties in hyperkähler manifolds.

In this subsection, we give a definition and basic properties of trianalytic subvarieties of hyperkähler manifolds. We follow [V2].

Let M be a compact hyperkähler manifold, $\dim_{\mathbb{R}} M = 2m$.

Definition 2.11: Let $N \subset M$ be a closed subset of M. Then N is called **trianalytic** if N is a complex analytic subset of (M, L) for any induced complex structure L.

Let I be an induced complex structure on M, and $N \subset (M, I)$ be a closed analytic subvariety of (M, I), $dim_{\mathbb{C}} N = n$. Consider the homology class represented by N. Let $[N] \in H^{2m-2n}(M)$ denote the Poincare dual cohomology class, so called **fundamental class** of N. Recall that the hyperkähler structure induces the action of the group $SU(2)$ on the space $H^{2m-2n}(M)$.

Theorem 2.12: Assume that $[N] \in H^{2m-2n}(M)$ is invariant with respect to the action of $SU(2)$ on $H^{2m-2n}(M)$. Then N is trianalytic.

Proof: This is Theorem 4.1 of [V2]. ■

The following assertion is the key to the proof of Theorem 2.12 (see [V2] for details).

Proposition 2.13: (Wirtinger's inequality) Let M be a compact hyperkähler manifold, I an induced complex structure and $X \subset (M, I)$ a closed complex subvariety for complex dimension k. Let J be an induced complex structure, $J \neq \pm I$, and ω_I, ω_J the associated Kähler forms. Consider the numbers

$$\deg_I X := \int_X \omega_I^k, \quad \deg_J X := \int_X \omega_J^k$$

Then $\deg_I X \geqslant |\deg_J X|$, and the inequality is strict unless X is trianalytic. ■

Remark 2.14: Trianalytic subvarieties have an action of quaternion algebra in the tangent bundle. In particular, the real dimension of such subvarieties is divisible by 4.

Definition 2.15: Let M be a complex manifold admitting a hyperkähler structure $\mathcal{H}$. We say that M is **of general type** or **generic** with respect to $\mathcal{H}$ if all elements of the group

$$\bigoplus_p H^{p,p}(M) \cap H^{2p}(M,\mathbb{Z}) \subset H^*(M)$$

are $SU(2)$-invariant.

The following result is an elementary application of representation theory.

Proposition 2.16: Let M be a compact manifold, $\mathcal{H}$ a hyperkähler structure on M and S be the set of induced complex structures over M. Denote by $S_0 \subset S$ the set of $L \in S$ such that (M,L) is generic with respect to $\mathcal{H}$. Then S_0 is dense in S. Moreover, the complement $S \backslash S_0$ is countable.

Proof: This is Proposition 2.2 from [V2] ∎

Theorem 2.12 has the following immediate corollary:

Corollary 2.17: Let M be a compact holomorphically symplectic manifold. Assume that M is of general type with respect to a hyperkähler structure $\mathcal{H}$. Let $S \subset M$ be closed complex analytic subvariety. Then S is trianalytic with respect to $\mathcal{H}$.

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2.3 Twistor spaces

Let M be a hyperkähler manifold. Consider the product manifold $X = M \times S^2$. Embed the sphere $S^2 \subset \mathbb{H}$ into the quaternion algebra $\mathbb{H}$ as the subset of all quaternions J with $J^2 = -1$. For every point $x = m \times J \in X = M \times S^2$ the tangent space T_xX is canonically decomposed $T_xX = T_mM \oplus T_JS^2$. Identify $S^2 = \mathbb{C}P^1$ and let $I_J : T_JS^2 \to T_JS^2$ be the complex structure operator. Let $I_m : T_mM \to T_mM$ be the complex structure on M induced by $J \in S^2 \subset \mathbb{H}$.

The operator $I_x = I_m \oplus I_J : T_xX \to T_xX$ satisfies $I_x \circ I_x = -1$. It depends smoothly on the point x, hence defines an almost complex structure on X. This almost complex structure is known to be integrable (see [Sal]).

Definition 2.18: The complex manifold $\langle X, I_x \rangle$ is called *the twistor space* for the hyperkähler manifold M, denoted by $\operatorname{Tw}(M)$. This manifold is equipped with a real analytic projection $\sigma : \operatorname{Tw}(M) \longrightarrow M$ and a complex analytic projection $\pi : \operatorname{Tw}(M) \longrightarrow \mathbb{C}P^1$.

The twistor space $\operatorname{Tw}(M)$ is not, generally speaking, a Kähler manifold. For M compact, it is easy to show that $\operatorname{Tw}(M)$ does not admit a Kähler metric.

3 Hypercomplex varieties

This section is based on [V-h], Section 4 and 8. In this section, we shall state all results for hypercomplex varieties, instead of hyperkähler ones. However, everything we say can be stated (and proven) for hyperkähler varieties (this approach was chosen in [V-d] and [V-d2]).

3.1 Real analytic varieties and complex structures

In this subsection, we follow [GMT] and [V-h], Section 2.

Let I be an ideal sheaf in the ring of real analytic functions in an open ball B in $\mathbb{R}^n$. The set of common zeroes of I is equipped with a structure of ringed space, with $\mathcal{O}(B)/I$ as the structure sheaf. We denote this ringed space by $Spec(\mathcal{O}(B)/I)$.

Definition 3.1: By a **weak real analytic space** we understand a ringed space which is locally isomorphic to $Spec(\mathcal{O}(B)/I)$, for some ideal $I \subset \mathcal{O}(B)$. A **real analytic space** is a weak real analytic space for which the structure sheaf is coherent (i. e., locally finitely generated and presentable).

For every real analytic variety X, there is a natural sheaf morphism of evaluation, $\mathcal{O}(X) \xrightarrow{ev} C(X)$, where $C(X)$ is the sheaf of real analytic functions on X.

Definition 3.2: A **real analytic variety** is a weak real analytic space for which the natural sheaf morphism $\mathcal{O}(X) \longrightarrow C(X)$ is injective.

Let $(X, \mathcal{O}(X))$ be a real analytic space and $N(X) \subset \mathcal{O}(X)$ be the kernel of the natural sheaf morphism $\mathcal{O}(X) \longrightarrow C(X)$. Clearly, the ringed space $(X, \mathcal{O}(X)/N(X))$ is a real analytic variety. This variety is called **a reduction of** X, denoted X_r. The structure sheaf of X_r is not necessarily coherent, for examples see [GMT], III.2.15.

For an ideal $I \subset \mathcal{O}(B)$ we define the module of real analytic differentials on $\mathcal{O}(B)/I$ by

$$\Omega^1(\mathcal{O}(B)/I) = \Omega^1(\mathcal{O}(B)) \Big/ \Big(I \cdot \Omega^1(\mathcal{O}(B)) + dI \Big),$$

where B is an open ball in $\mathbb{R}^n$, and $\Omega^1(\mathcal{O}(B)) \cong \mathbb{R}^n \otimes \mathcal{O}(B)$ is the module of real analytic differentials on B. Patching this construction, we define the sheaf of real analytic differentials on any real analytic space. Likewise, one defines sheaves of analytic differentials for complex varieties and in other similar situations.

Let X be a complex analytic variety. The **real analytic space underlying** X (denoted by $X_{\mathbb{R}}$) is the following object. By definition, $X_{\mathbb{R}}$ is a ringed space with the same topology as X, but with a different structure sheaf, denoted by $\mathcal{O}_{X_{\mathbb{R}}}$. Let $i: U \hookrightarrow B^n$ be a closed complex analytic embedding of an open subset $U \subset X$ to an open ball $B^n \subset \mathbb{C}^n$, and I be an ideal defining $i(U)$. Then

$$\mathcal{O}_{X_{\mathbb{R}}}\Big|_U := \mathcal{O}_{B^n_{\mathbb{R}}}/Re(I)$$

is a quotient sheaf of the sheaf of real analytic functions on B^n by the ideal $Re(I)$ generated by the real parts of the functions $f \in I$.

Note that the real analytic space underlying X needs not be reduced, though it has no nilpotents in the structure sheaf.

Consider the sheaf $\mathcal{O}_X$ of holomorphic functions on X as a subsheaf of the sheaf $C(X, \mathbb{C})$ of continuous $\mathbb{C}$-valued functions on X. The sheaf $C(X, \mathbb{C})$ has a natural automorphism $f \longrightarrow \overline{f}$, where $\overline{f}$ is complex conjugation. By definition, the section f of $C(X, \mathbb{C})$ is called **antiholomorphic** if $\overline{f}$ is holomorphic. Let $\mathcal{O}_X$ be the sheaf of holomorphic functions, and $\overline{\mathcal{O}}_X$ be the sheaf of antiholomorphic functions on X. Let $\mathcal{O}_X \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X \xrightarrow{i} \mathcal{O}_{X_{\mathbb{R}}} \otimes \mathbb{C}$ be the natural multiplication map.

Claim 3.3: Let X be a complex variety, $X_{\mathbb{R}}$ the underlying real analytic space. Then the natural sheaf homomorphism $i: \mathcal{O}_X \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X \longrightarrow \mathcal{O}_{X_{\mathbb{R}}} \otimes \mathbb{C}$ is injective. For each point $x \in X$, i induces an isomorphism on x-completions of $\mathcal{O}_X \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X$ and $\mathcal{O}_{X_{\mathbb{R}}} \otimes \mathbb{C}$.

Proof: Clear from the definition. ∎

In the assumptions of Claim 3.3, let

$$\Omega^1(\mathcal{O}_{X_{\mathbb{R}}}), \quad \Omega^1(\mathcal{O}_X \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X), \quad \Omega^1(\mathcal{O}_{X_{\mathbb{R}}} \otimes \mathbb{C})$$

be the sheaves of real analytic differentials associated with the corresponding sheaves of rings. There is a natural sheaf map

$$\Omega^1(\mathcal{O}_{X_\mathbb{R}}) \otimes \mathbb{C} = \Omega^1(\mathcal{O}_{X_\mathbb{R}} \otimes \mathbb{C}) \longrightarrow \Omega^1(\mathcal{O}_X \otimes_\mathbb{C} \overline{\mathcal{O}}_X), \tag{3.1}$$

correspoding to the monomorphism

$$\mathcal{O}_X \otimes_\mathbb{C} \overline{\mathcal{O}}_X \hookrightarrow \mathcal{O}_{X_\mathbb{R}} \otimes \mathbb{C}.$$

Claim 3.4: Tensoring both sides of (3.1) by $\mathcal{O}_{X_\mathbb{R}} \otimes \mathbb{C}$ produces an isomorphism

$$\Omega^1(\mathcal{O}_X \otimes_\mathbb{C} \overline{\mathcal{O}}_X) \bigotimes_{\mathcal{O}_X \otimes_\mathbb{C} \overline{\mathcal{O}}_X} \left(\mathcal{O}_{X_\mathbb{R}} \otimes \mathbb{C}\right) = \Omega^1(\mathcal{O}_{X_\mathbb{R}} \otimes \mathbb{C}).$$

Proof: Clear. ■

According to the general results about differentials (see, for example, [H], Chapter II, Ex. 8.3), the sheaf $\Omega^1(\mathcal{O}_X \otimes_\mathbb{C} \overline{\mathcal{O}}_X)$ admits a canonical decomposition:

$$\Omega^1(\mathcal{O}_X \otimes_\mathbb{C} \overline{\mathcal{O}}_X) = \Omega^1(\mathcal{O}_X) \otimes_\mathbb{C} \overline{\mathcal{O}}_X \oplus \mathcal{O}_X \otimes_\mathbb{C} \Omega^1(\overline{\mathcal{O}}_X).$$

Let $\widetilde{I}$ be an endomorphism of $\Omega^1(\mathcal{O}_X \otimes_\mathbb{C} \overline{\mathcal{O}}_X)$ which acts as a multiplication by $\sqrt{-1}$ on

$$\Omega^1(\mathcal{O}_X) \otimes_\mathbb{C} \overline{\mathcal{O}}_X \subset \Omega^1(\mathcal{O}_X \otimes_\mathbb{C} \overline{\mathcal{O}}_X)$$

and as a multiplication by $-\sqrt{-1}$ on

$$\mathcal{O}_X \otimes_\mathbb{C} \Omega^1(\overline{\mathcal{O}}_X) \subset \Omega^1(\mathcal{O}_X \otimes_\mathbb{C} \overline{\mathcal{O}}_X).$$

Let $\underline{I}$ be the corresponding $\mathcal{O}_{X_\mathbb{R}} \otimes \mathbb{C}$-linear endomorphism of

$$\Omega^1(\mathcal{O}_{X_\mathbb{R}}) \otimes \mathbb{C} = \Omega^1(\mathcal{O}_X \otimes_\mathbb{C} \overline{\mathcal{O}}_X) \otimes_{\mathcal{O}_X \otimes_\mathbb{C} \overline{\mathcal{O}}_X} \left(\mathcal{O}_{X_\mathbb{R}} \otimes \mathbb{C}\right).$$

A quick check shows that $\underline{I}$ is *real*, that is, comes from the $\mathcal{O}_{X_{\mathbb{R}}}$-linear endomorphism of $\Omega^1(\mathcal{O}_{X_{\mathbb{R}}})$. Denote this $\mathcal{O}_{X_{\mathbb{R}}}$-linear endomorphism by

$$I: \Omega^1(\mathcal{O}_{X_{\mathbb{R}}}) \longrightarrow \Omega^1(\mathcal{O}_{X_{\mathbb{R}}}),$$

$I^2 = -1$. The endomorphism I is called **the complex structure operator on the underlying real analytic space**. In the case when X is smooth, I coinsides with the usual complex structure operator on the cotangent bundle.

Definition 3.5: Let M be a weak real analytic space, and

$$I: \Omega^1(\mathcal{O}_M) \longrightarrow \Omega^1(\mathcal{O}_M)$$

be an endomorphism satisfying $I^2 = -1$. Then I is called **an almost complex structure on M**.

3.2 Almost complex structures on real analytic varieties and integrability

In this Subsection, we follow [V-h], Section 2.

From the definition (see [V-h], Lemma 2.6), it follows that a real analytic variety underlying a given complex variety is equipped with a natural almost complex structure. The corresponding operator is called **the complex structure operator in the underlying real analytic variety.**

The following theorem is quite easy to prove.

Theorem 3.6: Let X, Y be complex analytic varieties, and

$$f_{\mathbb{R}}: X_{\mathbb{R}} \longrightarrow Y_{\mathbb{R}}$$

be a morphism of underlying real analytic varieties which commutes with the complex structure. Then there exist a morphism $f: X \longrightarrow Y$ of complex analytic varieties, such that $f_{\mathbb{R}}$ is its underlying morphism.

Proof: This is [V-h], Theorem 2.10. ■

From Theorem 3.6, it follows that the complex structure on X is uniquely determined by the complex structure on the underlying real analytic variety.

Definition 3.7: Let M be a real analytic variety, and

$$I: \Omega^1(\mathcal{O}_M) \longrightarrow \Omega^1(\mathcal{O}_M)$$

be an endomorphism satisfying $I^2 = -1$. Then I is called **an almost complex structure on** M. If there exist a structure $\mathfrak{C}$ of complex variety on M such that I appears as the complex structure operator associated with $\mathfrak{C}$, we say that I is **integrable**. Theorem 3.6 implies that this complex structure is unique if it exists.

3.3 Hypercomplex varieties: the definition

Definition 3.8: Let M be a real analytic variety equipped with almost complex structures I, J and K, such that $I \circ J = -J \circ I = K$. Then M is called **an almost hypercomplex variety.**

An almost hypercomplex variety is equipped with an action of quaternion algebra in its differential sheaf. Each quaternion $L \in \mathbb{H}$, $L^2 = -1$ defines an almost complex structure on M. Such an almost complex structure is called **induced by the hypercomplex structure**.

Definition 3.9: Let M be an almost hypercomplex variety. We say that M is **hypercomplex** if there exist a pair of induced complex structures $I_1, I_2 \in \mathbb{H}$, $I_1 \neq \pm I_2$, such that I_1 and I_2 are integrable.

Caution: Not everything which looks hypercomplex satisfies the conditions of Definition 3.9. Take a quotient M/G of a hypercomplex manifold by an action of a finite group G, acting compatible with the hypercomplex structure. Then M/G is *not* hypercomplex, unless G acts freely.

Claim 3.10: Let M be a hypercomplex manifold. Then M is a hypercomplex variety in the sense of Definition 3.9.

Proof: Let I, J be induced complex structures. We need to identify $(M, I)_{\mathbb{R}}$ and $(M, J)_{\mathbb{R}}$ in a natural way. These varieties are canonically identified as C^∞-manifolds; we need only to show that this identification is real analytic. This is [V3], Proposition 6.5. ∎

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Remark 3.11: Trianalytic subvarieties of hyperkähler manifolds are obviously hypercomplex. Define trianalytic subvarieties of hypercomplex varieties as subvarieties which are complex analytic with respect to all induced complex structures. Clearly, trianalytic subvarieties of hypercomplex varieties are equipped with a natural hypercomplex structure. Another example of a hypercomplex variety is given in Corollary 4.11. For additional examples, see [V3].

3.4 Hypercomplex varieties and twistor spaces

For a hypercomplex variety, it is clear how to define the twistor space, which is a complex variety (see [V-h], Section 7 for details). This definition coinsides with the usual one for the hypercomplex manifolds in the smooth case.

Following [HKLR], Deligne and Simpson defined hypercomplex varieties in terms of their twistor spaces. This is done as follows.

Let M be a hypercomplex variety and Tw its twistor space. Consider the unique anticomplex involution $\iota_0 : \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ with no fixed points. This involution is obtained by central symmetry with center in 0 if we identify $\mathbb{C}P^1$ with a unit sphere in $\mathbb{R}^3$. Let $\iota : \mathrm{Tw} \longrightarrow \mathrm{Tw}$ be an involution of the twistor space mapping $(s,m) \in S^2 \times M = \mathrm{Tw}$ to $(\iota_0(s), m)$. Clearly, ι is anticomplex.

Definition 3.12: Let $s : \mathbb{C}P^1 \longrightarrow \mathrm{Tw}$ be a section of the natural holomorphic projection $\pi : \mathrm{Tw} \longrightarrow \mathbb{C}P^1$, $s \circ \pi = Id\Big|_{\mathbb{C}P^1}$. Then s is called **the twistor line**. The space Sec of twistor lines is finite-dimensional and equipped with a natural complex structure, as follows from deformation theory ([Do]).

Let Sec^ι be the space of all lines $s \in \mathrm{Sec}$ which are fixed by ι. The space Sec^ι is equipped with a structure of a real analytic space. We have a natural map $\tau : M_{\mathbb{R}} \longrightarrow \mathrm{Sec}^\iota$ associating to $m \in M$ the line $s : \mathbb{C}P^1 \longrightarrow \mathrm{Tw}$, $s(x) = (x,m) \in S^2 \times M = \mathrm{Tw}$. Such twistor lines are called **horizontal twistor lines.** Denote the set of horizontal twistor lines by $\mathrm{Hor} \subset \mathrm{Sec}$.

The linear algebra of quaternions implies that the normal bundle of a horizontal twistor line $s \cong \mathbb{C}P^1$ is a direct sum of several copies of $O(1)$. A section of $O(1)$ is uniquely determined by its values in two distinct points. Therefore, (at least if M is smooth), through every two generic points in a neighbourhood of s passes a unique deformation of s (if this statement needs a justification, see [V-h], (7.2)).

This motivates the following definition, proposed by Delidne and Simpson ([De], [S]).

Definition 3.13: (Hypercomplex spaces) Let Tw be a complex analytic space, $\pi : \operatorname{Tw} \longrightarrow \mathbb{C}P^1$ a holomorphic map, and $\iota : \operatorname{Tw} \longrightarrow \operatorname{Tw}$ an anti-complex automorphism, such that $\iota \circ \pi = \pi \circ \iota_0$. Let Sec be the space of sections of π equipped with a structure of a complex analytic space, and Sec^ι be the real analytic space of sections s of π satisfying $s \circ \iota_0 = \iota \circ s$. Let Hor be a connected component of Sec^ι. Then $(\operatorname{Tw}, \pi, \iota, \operatorname{Hor})$ is called **a hypercomplex space** if

(i) For each point $x \in \operatorname{Tw}^r$, there exists a unique line $s \in \operatorname{Hor}^r$ passing through x, where Tw^r, Hor^r is a reduction of Tw, Hor.

(ii) Let $s \in \operatorname{Hor}$, and $U \subset \operatorname{Tw}$ be a neighbourhood of s such that an irreducible decomposition of U coinsides with the irreducible decomposition of Tw in a neighbourhood of $s \subset \operatorname{Tw}^r$. Let

$$\overline{X} := \pi^{-1}(I) \times \pi^{-1}(J) \cap U \times U,$$

where I, J distinct points of $\mathbb{C}P^1$. Let $p_{IJ} : U \longrightarrow \overline{X} \subset \pi^{-1}(I) \times \pi^{-1}(J)$ be the evaluation map, $s \longrightarrow (s(I), s(J))$. Then there exist a closed subspace $X \subset \overline{X}$, obtained as a union of some of irreductible components of $\overline{X}$, and an open neighbourhood $V \subset \operatorname{Sec}$ of $s \in \operatorname{Sec}$, such that p_{IJ} is an open embedding of V to X.

For varieties, this definition is equivalent to Definition 3.9 ([V-h], Theorem 8.1).

4 Hyperholomorphic bundles

4.1 Hyperholomorphic bundles: the definition

This subsection contains several versions of a definition of hyperholomorphic connection in a complex vector bundle over a hyperkähler manifold. We follow [V1].

Let B be a holomorphic vector bundle over a complex manifold M, ∇ a connection in B and $\Theta \in \Lambda^2 \otimes End(B)$ be its curvature. This connection is called **compatible with a holomorphic structure** if $\nabla_X(\zeta) = 0$ for any holomorphic section ζ and any antiholomorphic tangent vector field $X \in T^{0,1}(M)$. If there exists a holomorphic structure compatible with the given Hermitian connection then this connection is called **integrable**.

One can define a **Hodge decomposition** in the space of differential forms with coefficients in any complex bundle, in particular, $End(B)$.

Theorem 4.1: Let ∇ be a Hermitian connection in a complex vector bundle B over a complex manifold. Then ∇ is integrable if and only if $\Theta \in \Lambda^{1,1}(M, \operatorname{End}(B))$, where $\Lambda^{1,1}(M, \operatorname{End}(B))$ denotes the forms of Hodge type (1,1). Also, the holomorphic structure compatible with ∇ is unique.

Proof: This is Proposition 4.17 of [Ko], Chapter I. ■

This result has the following more general version:

Proposition 4.2: Let ∇ be an arbitrary (not necessarily Hermitian) connection in a complex vector bundle B. Then ∇ is integrable if and only its $(0,1)$-part has square zero.

■

This proposition is a version of Newlander-Nirenberg theorem. For vector bundles, it was proven by Atiyah and Bott.

Definition 4.3: Let B be a Hermitian vector bundle with a connection ∇ over a hyperkähler manifold M. Then ∇ is called **hyperholomorphic** if ∇ is integrable with respect to each of the complex structures induced by the hyperkähler structure.

As follows from Theorem 4.1, ∇ is hyperholomorphic if and only if its curvature Θ is of Hodge type (1,1) with respect to any of complex structures induced by a hyperkähler structure.

As follows from Lemma 2.10, ∇ is hyperholomorphic if and only if Θ is a $SU(2)$-invariant differential form.

Example 4.4: (Examples of hyperholomorphic bundles)

(i) Let M be a hyperkähler manifold, and TM be its tangent bundle equipped with the Levi–Civita connection ∇. Consider a complex structure on TM induced from the quaternion action. Then ∇ is a Hermitian connection which is integrable with respect to each induced complex structure, and hence, is Yang–Mills.

(ii) For B a hyperholomorphic bundle, all its tensor powers are also hyperholomorphic.

(iii) Thus, the bundles of differential forms on a hyperkähler manifold are also hyperholomorphic.

4.2 Stable bundles and Yang–Mills connections.

This subsection is a compendium of the most basic results and definitions from the Yang–Mills theory over Kähler manifolds, concluding in the fundamental theorem of Uhlenbeck–Yau [UY].

Definition 4.5: Let F be a coherent sheaf over an n-dimensional compact Kähler manifold M. We define $\deg(F)$ as

$$\deg(F) = \int_M \frac{c_1(F) \wedge \omega^{n-1}}{vol(M)}$$

and $\operatorname{slope}(F)$ as

$$\operatorname{slope}(F) = \frac{1}{\operatorname{rank}(F)} \cdot \deg(F).$$

The number $\operatorname{slope}(F)$ depends only on a cohomology class of $c_1(F)$.

Let F be a coherent sheaf on M and $F' \subset F$ its proper subsheaf. Then F' is called **destabilizing subsheaf** if $\operatorname{slope}(F') \geqslant \operatorname{slope}(F)$

A coherent sheaf F is called **stable** [1] if it has no destabilizing subsheaves. A coherent sheaf F is called **semistable** if for all destabilizing subsheaves $F' \subset F$, we have $\operatorname{slope}(F') = \operatorname{slope}(F)$.

Later on, we usually consider the bundles B with $deg(B) = 0$.

Let M be a Kähler manifold with a Kähler form ω. For differential forms with coefficients in any vector bundle there is a Hodge operator $L : \eta \longrightarrow \omega \wedge \eta$. There is also a fiberwise-adjoint Hodge operator Λ (see [GH]).

Definition 4.6: Let B be a holomorphic bundle over a Kähler manifold M with a holomorphic Hermitian connection ∇ and a curvature $\Theta \in \Lambda^{1,1} \otimes End(B)$. The Hermitian metric on B and the connection ∇ defined by this metric are called **Yang-Mills** if

$$\Lambda(\Theta) = constant \cdot \operatorname{Id}\Big|_B,$$

[1] In the sense of Mumford-Takemoto

where Λ is a Hodge operator and $\operatorname{Id}\Big|_B$ is the identity endomorphism which is a section of $End(B)$.

Further on, we consider only these Yang–Mills connections for which this constant is zero.

A holomorphic bundle is called **indecomposable** if it cannot be decomposed onto a direct sum of two or more holomorphic bundles.

The following fundamental theorem provides examples of Yang--Mills bundles.

Theorem 4.7: (Uhlenbeck-Yau) Let B be an indecomposable holomorphic bundle over a compact Kähler manifold. Then B admits a Hermitian Yang-Mills connection if and only if it is stable, and this connection is unique.

Proof: [UY]. ∎

4.3 Hyperholomorphic connections and Yang-Mills theory

In this subsection, we apply Yang-Mills theory to hyperholomorphic connections. We follow [V1].

Proposition 4.8: Let M be a hyperkähler manifold, L an induced complex structure and B be a complex vector bundle over (M,L). Then every hyperholomorphic connection ∇ in B is Yang-Mills and satisfies $\Lambda(\Theta) = 0$ where Θ is a curvature of ∇.

Proof: We use the definition of a hyperholomorphic connection as one with $SU(2)$-invariant curvature. Then Proposition 4.8 follows from the

Lemma 4.9: Let $\Theta \in \Lambda^2(M)$ be a $SU(2)$-invariant differential 2-form on M. Then $\Lambda_L(\Theta) = 0$ for each induced complex structure L.[2]

Proof: This is Lemma 2.1 of [V1]. ∎

Let M be a compact hyperkähler manifold, I an induced complex structure. For any stable holomorphic bundle on (M,I) there exists a unique Hermitian Yang-Mills connection which, for some bundles, turns out to be hyperholomorphic. It is possible to tell when this happens.

[2]By Λ_L we understand the Hodge operator Λ associated with the Kähler complex structure L.

Theorem 4.10: Let B be a stable holomorphic bundle over (M,I), where M is a hyperkähler manifold and I is an induced complex structure over M. Then B admits a compatible hyperholomorphic connection if and only if the first two Chern classes $c_1(B)$ and $c_2(B)$ are $SU(2)$-invariant.[3]

Proof: This is Theorem 2.5 of [V1]. ∎

From Theorem 4.10 it follows that hyperholomorphic bundles can be described in two ways: either as holomorphic objects over (M,I), or as certain types of connections. The first definition implies that there exists a complex structure on the moduli of hypercolomorphic connections. The second implies that if we replace I by another induced complex structure, the C^∞-structure of the moduli of hyperholomorphic connections remains the same. In other words, the real analytic variety underlying underlying the moduli of hyperholomorphic connections admits a set of complex structures parametrized by $\mathbb{C}P^1$. Using Kodaira relations, it is easy to check that these complex structures satisfy quaternionic relations. We obtain the following

Corollary 4.11: ([V-h], Subsection 10.2) The moduli space of hyperholomorphic bundles is singular hyperkähler.

5 Hypercomplex varieties: the desingularization

The Desingularization Theorem is stated as follows.

Theorem 5.1: (Desingularization theorem) Let M be a hypercomplex variety I an integrable induced complex structure. Let

$$\widetilde{(M,I)} \xrightarrow{n} (M,I)$$

be the normalization of (M,I). Then $\widetilde{(M,I)}$ is smooth and has a natural hypercomplex structure $\mathcal{H}$, such that the associated map $n: \widetilde{(M,I)} \longrightarrow (M,I)$ agrees with $\mathcal{H}$. Moreover, the hypercomplex manifold $\widetilde{M} := \widetilde{(M,I)}$ is independent from the choice of induced complex structure I.

Proof: This is [V-h], Theorem 6.2 ∎

[3] We use Lemma 2.9 to speak of action of $SU(2)$ in cohomology of M.

In this Section, we give a sketch of a proof of Theorem 5.1. This sketch assumes some background in commutative algebra. Some readers might prefer to read [V-h], where we don't skip these details.

The idea of the proof is following. First of all, we prove Theorem 5.1 under an addition assumption, called LHS (locally homogeneous singularities). Then, we prove that LHS always holds for hypercomplex varieties. LHS is the following beast.

Definition 5.2: (local rings with LHS) Let A be a local ring. Denote by $\mathfrak{m}$ its maximal ideal. Let A_{gr} be the corresponding associated graded ring for the $\mathfrak{m}$-adic filtration. Let $\hat{A}$, $\widehat{A_{gr}}$ be the $\mathfrak{m}$-adic completion of A, A_{gr}. Let $(\hat{A})_{gr}$, $(\widehat{A_{gr}})_{gr}$ be the associated graded rings, which are naturally isomorphic to A_{gr}. We say that A **has locally homogeneous singularities** (LHS) if there exists an isomorphism $\rho: \ \hat{A} \longrightarrow \widehat{A_{gr}}$ which induces the standard isomorphism $i: \ (\hat{A})_{gr} \longrightarrow (\widehat{A_{gr}})_{gr}$ on associated graded rings.

Definition 5.3: (SLHS) Let X be a complex or real analytic space. Then X is called **a space with locally homogeneous singularities** (SLHS) if for each $x \in X$, the local ring $\mathcal{O}_x X$ has locally homogeneous singularities.

To say that a ring is graded is the same as to say that it is equipped with an action of $\mathbb{C}^*$. Therefore, a local ring is LHS if and only if its completion is equipped with an action ρ of $\mathbb{C}^*$, and ρ acts on its tangent space by dilatations. This is why we use the word the word "homogeneous".

The following proposition was the main result of [V-d].

Proposition 5.4: Let M be a hypercomplex variety, and I an induced complex structure. Assume that the complex variety (M, I) has locally homogeneous singularities (LHS). Then the normalization of (M, I) is smooth.

Proof: Let O_x be the adic completion of the localization of the structure sheaf $\mathcal{O}_{(M,I)}$ in $x \in M$. The normalization is compatible with the adic completions ([M], Chapter 9, Proposition 24.E). Therefore to prove that the normalization of (M, I) is smooth we need only to show that the normalization of O_x is regular. Since (M, I) is LHS, the ring O_x is isomorphic to the completion of the coordinate ring $\mathcal{O}(Z_x)$ of the Zariski tangent cone Z_x of (M, I). Therefore, it suffices to show that the normalization of Z_x is smooth. On the other hand, by [V-h], Theorem 4.5, the Zariski tangent cone Z_x of

(M, I) is hypercomplex (this is easy to see from the differential-geometric definition of the Zariski tangent cone). Moreover, the natural embedding of the Zariski tangent cone to the Zariski tangent space $T_x M$ is compatible with the hypercomplex structure (the space $T_x M$ is quaternionic, which follows immediately from the definition of a hypercomplex structure). The manifold $T_x M$ is hyperkähler. It is well known (see, for instance, [V3]) that trianalytic subvarieties of hyperkähler manifolds are completely geodesic. Since the manifold $T_x M$ is flat, a completely geodesic subvariety must be a union of planes. But the normalization of a union of planes is smooth. This finishes the proof of Proposition 5.4; for more details, please read [V-h] and [V-d2]. ∎

To finish the sketch of the proof of Theorem 5.1, it remains to prove the following proposition, which is the main result of [V-d2].

Proposition 5.5: Let M be a hypercomplex variety. Then M is a space with locally homogeneous singularities (SLHS).

Proof: Let I be an induced complex structure, $x \in M$ a point and O_x the adic completion of the localization $\mathcal{O}_x(M, I)$ of the structure ring of (M, I). To produce a grading on O_x, we need to construct an action ρ of $\mathbb{C}^*$ on O_x, such that ρ acts by dilatations on the tangent space $T_x(M, I)$. This is done geometrically as follows.

Let $x \in M$ be a point, $\pi : \operatorname{Tw} \longrightarrow \mathbb{C}P^1$ a twistor space of M, and $s_x : \mathbb{C}P^1 \longrightarrow \operatorname{Tw}$ the line corresponding to the set (i, x) where i runs through $\mathbb{C}P^1$ (such lines are called **horizontal twistor lines**, see Definition 3.12). As we have mentioned before, for "generic" pair of points (α, β) sufficiently close to s_x, there exists a unique twistor line s passing through α and β. To be more precise, let $I, I' \in \mathbb{C}P^1$ be distinct induced complex structures. Then s_x has a neighbourhood U such that for all $\alpha \in \pi^{-1}(I) \cap U$, $\beta \in \pi^{-1}(I') \cap U$, there exists a unique twistor line $s_{\alpha,\beta} : \mathbb{C}P^1 \longrightarrow \operatorname{Tw}$ passing through α, β. Fix a point $I'' \in \mathbb{C}P^1$ which is distinct from I and I'. Let $\delta = (I'', x) \in (\mathbb{C}P^1, M) \cong \operatorname{Tw}$ be the corresponding point of s_x. For each $\alpha \in \pi^{-1}(I) \cap U$ there exists a unique twistor line $s_{\alpha,\delta}$ passing through α and δ. Evaluating this map at I', we obtain a point β in $\pi^{-1}(I') \cap U$. Consider this as an operation producing β from α. Clearly, this way we obtain an isomorphism

$$\hat{\mathcal{O}}_x(M, I') \longrightarrow \hat{\mathcal{O}}_x(M, I)$$

where $\hat{\mathcal{O}}_x(M, I')$, $\hat{\mathcal{O}}_x(M, I)$ is an adic completion of a localization of a ring of regular functions on (M,I'), (M,I). This isomorphism depends from the parameter I''. Varying I'', we obtain different isomorphisms between $\hat{\mathcal{O}}_x(M,I')$ and $\hat{\mathcal{O}}_x(M,I)$. A composition of two such isomorphisms is an automorphism of $\hat{\mathcal{O}}_x(M,I)$. A simple linear-algebraic argument shows that this automorphism acts as a dilatation on the tangent space T_xM (Lemma 5.13, [V-h]). This proves Proposition 5.5. ∎

6 Hyperholomorphic sheaves and their singularities

6.1 Stable sheaves and Yang-Mills connections

In [BS], S. Bando and Y.-T. Siu developed the machinery allowing one to apply the methods of Yang-Mills theory to torsion-free coherent sheaves. In the course of this paper, we apply their work to generalise the results of [V1]. In this Subsection, we give a short exposition of their results.

Definition 6.1: Let X be a complex manifold, and F a coherent sheaf on X. Consider the sheaf $F^* := \mathcal{H}om_{\mathcal{O}_X}(F, \mathcal{O}_X)$. There is a natural functorial map $\rho_F : F \longrightarrow F^{**}$. The sheaf F^{**} is called **a reflexive hull**, or **reflexization** of F. The sheaf F is called **reflexive** if the map $\rho_F : F \longrightarrow F^{**}$ is an isomorphism.

Remark 6.2: For all coherent sheaves F, the map $\rho_{F^*} : F^* \longrightarrow F^{***}$ is an isomorphism ([OSS], Ch. II, the proof of Lemma 1.1.12). Therefore, a reflexive hull of a sheaf is always reflexive.

Claim 6.3: Let X be a Kähler manifold, and F a torsion-free coherent sheaf over X. Then F (semi)stable if and only if F^{**} is (semi)stable.

Proof: This is [OSS], Ch. II, Lemma 1.2.4. ∎

Definition 6.4: Let X be a Kähler manifold, and F a coherent sheaf over X. The sheaf F is called **polystable** if F is a direct sum of stable sheaves.

The admissible Hermitian metrics, introduced by Bando and Siu in [BS], play the role of the ordinary Hermitian metrics for vector bundles.

Let X be a Kähler manifold. In Hodge theory, one considers the operator $\Lambda : \Lambda^{p,q}(X) \longrightarrow \Lambda^{p-1,q-1}(X)$ acting on differential forms on X, which is adjoint to the multiplication by the Kähler form. This operator is defined on differential forms with coefficient in every bundle. Considering a curvature Θ of a bundle B as a 2-form with coefficients in $\operatorname{End}(B)$, we define the expression $\Lambda\Theta$ which is a section of $\operatorname{End}(B)$.

Definition 6.5: Let X be a Kähler manifold, and F a reflexive coherent sheaf over X. Let $U \subset X$ be the set of all points at which F is locally trivial. By definition, the restriction $F\Big|_U$ of F to U is a bundle. An **admissible metric** on F is a Hermitian metric h on the bundle $F\Big|_U$ which satisfies the following assumptions

(i) the curvature Θ of (F, h) is square integrable, and

(ii) the corresponding section $\Lambda\Theta \in \operatorname{End}(F\Big|_U)$ is uniformly bounded.

Definition 6.6: Let X be a Kähler manifold, F a reflexive coherent sheaf over X, and h an admissible metric on F. Consider the corresponding Hermitian connection ∇ on $F\Big|_U$. The metric h and the connection ∇ are called **Yang-Mills** if its curvature satisfies

$$\Lambda\Theta \in \operatorname{End}(F\Big|_U) = c \cdot \operatorname{Id}$$

where c is a constant and Id the unit section $\operatorname{Id} \in \operatorname{End}(F\Big|_U)$.

Further in this paper, we shall only consider Yang-Mills connections with $\Lambda\Theta = 0$.

Remark 6.7: By Gauss-Bonnet formule, the constant c is equal to $\deg(F)$, where $\deg(F)$ is the degree of F (Definition 4.5).

One of the main results of [BS] is the following analogue of Uhlenbeck–Yau theorem (Theorem 4.7).

Theorem 6.8: Let M be a compact Kähler manifold, and F a coherent sheaf without torsion. Then F admits an admissible Yang–Mills metric is and only if F is polystable. Moreover, if F is stable, then this metric is unique, up to a constant multiplier.

Proof: In [BS], Theorem 6.8 is proved for Kähler M ([BS], Theorem 3). ∎

6.2 Stable sheaves over hyperkähler manifolds

Let M be a compact hyperkähler manifold, I an induced complex structure, F a torsion-free coherent sheaf over (M, I) and F^{**} its reflexization. Recall that the cohomology of M are equipped with a natural $SU(2)$-action (Lemma 2.9). The motivation for the following definition is Theorem 4.10 and Theorem 6.8.

Definition 6.9: Assume that the first two Chern classes of the sheaves F, F^{**} are $SU(2)$-invariant. Then F is called **stable hyperholomorphic** if F is stable, and **semistable hyperholomorphic** if F can be obtained as a successive extension of stable hyperholomorphic sheaves.

Consider the natural $SU(2)$-action in the bundle $\Lambda^i(M, B)$ of the differential i-forms with coefficients in a vector bundle B. Let $\Lambda^i_{inv}(M, B) \subset \Lambda^i(M, B)$ be the bundle of $SU(2)$-invariant i-forms.

Definition 6.10: Let $X \subset (M, I)$ be a complex subvariety of codimension at least 2, such that $F\Big|_{M\backslash X}$ is a bundle, h be an admissible metric on $F\Big|_{M\backslash X}$ and ∇ the associated connection. Then ∇ is called **hyperholomorphic** if its curvature

$$\Theta_\nabla = \nabla^2 \in \Lambda^2\left(M, \operatorname{End}\left(F\Big|_{M\backslash X}\right)\right)$$

is $SU(2)$-invariant, i. e. belongs to $\Lambda^2_{inv}\left(M, \operatorname{End}\left(F\Big|_{M\backslash X}\right)\right)$.

Theorem 6.11: Let M be a compact hyperkähler manifold, I an induced complex structure and F a reflexive sheaf on (M, I). Then F admits a hyperholomorphic connection if and only if F is polystable hyperholomorphic in the sense of Definition 6.9. Moreover, such a connection is unique.

Proof: This is [V-c], Theorem 3.19. ∎

The proof of Theorem 6.11 is based on an elementary linear algebra argument (see Lemma 2.10).

6.3 Desingularization of hyperholomorphic sheaves

Hyperholomorphic sheaves (at least ones with isolated singularities) can be desingularized in the same fashion as the hyperkähler varieties; in fact, almost the same argument applies to both cases.

Theorem 6.12: Let M be a hyperkähler manifold, not necessarily compact, I an induced complex structure, and F a reflexive coherent sheaf over (M, I) equipped with a hyperholomorphic connection (Definition 6.10). Assume that F has isolated singularities. Let $\widetilde{M} \xrightarrow{\sigma} M$ be a blow-up of (M, I) in the singular set of F, and σ^*F the pullback of F. Then σ^*F is a locally trivial sheaf, that is, a holomorphic vector bundle.

Proof: This is [V-c], Theorem 6.1. ■

The idea of the proof is the following. We apply to F the methods used in the proof of Desingularization Theorem (Theorem 5.1). The main ingredient in the proof of Desingularization Theorem is the existence of a natural $\mathbb{C}^*$-action on the completion $\hat{\mathcal{O}}_x(M, I)$ of the local ring $\mathcal{O}_x(M, I)$, for all $x \in M$. This $\mathbb{C}^*$-action identifies $\hat{\mathcal{O}}_x(M, I)$ with a completion of a graded ring. Here we show that a sheaf F is $\mathbb{C}^*$-equivariant. Therefore, a germ of F at x has a grading, which is compatible with the natural $\mathbb{C}^*$-action on $\hat{\mathcal{O}}_x(M, I)$. Singularities of such reflexive sheaves can be resolved by a single blow-up.

6.4 Quaternionic-Kähler geometry

In this Subsection, we follow [V-c] (Section 7). We give here a number of preliminary constructions, which are used later on to describe the singularities of hyperholomorphic sheaves. These constructions are mostly due to A.Swann and T. Nitta ([Sw], [N1], [N2]).

Definition 6.13: ([Sal], [Bes]) Let M be a Riemannian manifold. Consider a bundle of algebras $\operatorname{End}(TM)$, where TM is the tangent bundle to M. Assume that $\operatorname{End}(TM)$ contains a 4-dimensional bundle of subalgebras $W \subset \operatorname{End}(TM)$, with fibers isomorphic to a quaternion algebra $\mathbb{H}$. Assume, moreover, that W is closed under the transposition map $\perp : \operatorname{End}(TM) \longrightarrow \operatorname{End}(TM)$ and is preserved by the Levi-Civita connection. Then M is called **quaternionic-Kähler**.

Example 6.14: Consider the quaternionic projective space

$$\mathbb{H}P^n = (\mathbb{H}^n \backslash 0)/\mathbb{H}^*.$$

It is easy to see that $\mathbb{H}P^n$ is a quaternionic-Kähler manifold. For more examples of quaternionic-Kähler manifolds, see [Bes].

A quaternionic-Kähler manifold is Einstein ([Bes]), i. e. its Ricci tensor is proportional to the metric: $Ric(M) = c \cdot g$, with $c \in \mathbb{R}$. When $c = 0$, the manifold M is hyperkähler, and its restricted holonomy group is $Sp(n)$; otherwise, the restricted holonomy is $Sp(n) \cdot Sp(1)$. The number c is called **the scalar curvature** of M. Further on, we shall use the term *quaternionic-Kähler manifold* for manifolds with non-zero scalar curvature.

The quaternionic projective space $\mathbb{H}P^n$ has positive scalar curvature.

Let M be a quaternionic-Kähler manifold, and $W \subset \operatorname{End}(TM)$ the corresponding 4-dimensional bundle. For $x \in M$, consider the set $\mathcal{R}_x \subset W\Big|_x$, consisting of all $I \in W\Big|_x$ satisfying $I^2 = -1$. Consider $\mathcal{R}_x$ as a Riemannian submanifold of the total space of $W\Big|_x$. Clearly, $\mathcal{R}_x$ is isomorphic to a 2-dimensional sphere. Let $\mathcal{R} = \cup_x \mathcal{R}_x$ be the corresponding spherical fibration over M, and $\operatorname{Tw}(M)$ its total space. The manifold $\operatorname{Tw}(M)$ is equipped with an almost complex structure, which is defined in the same way as the almost complex structure for the twistor space of a hyperkähler manifold. This almost complex structure is known to be integrable (see [Sal]).

A role of $SU(2)$-invariant 2-forms is played by so-called B_2-forms.

Definition 6.15: Let $SO(TM) \subset \operatorname{End}(TM)$ be a group bundle of all orthogonal automorphisms of TM, and $G_M := W \cap SO(TM)$. Clearly, the fibers of G_M are isomorphic to $SU(2)$. Consider the action of G_M on the bundle of 2-forms $\Lambda^2(M)$. Let $\Lambda^2_{inv}(M) \subset \Lambda^2(M)$ be the bundle of G_M-invariants. The bundle $\Lambda^2_{inv}(M)$ is called **the bundle of B_2-forms**. In a similar fashion we define B_2-forms with coefficients in a bundle.

Definition 6.16: In the above assumptions, let (B, ∇) be a bundle with connection over M. The bundle B is called **a B_2-bundle**, and ∇ is called **a B_2-connection**, if its curvature is a B_2-form.

The B_2-bundles were introduced and studied by T. Nitta in a serie of papers ([N1], [N2] etc.)

Consider the natural projection $\sigma : \operatorname{Tw}(M) \longrightarrow M$. The following observation is clear (Claim 6.17 (ii) is, in fact, an immediate consequence of Claim 6.17 (i), which is proven by linear algebra).

Claim 6.17: ([V-c], Claim 7.13)

(i) Let ω be a 2-form on M. The pullback $\sigma^*\omega$ is of type $(1,1)$ on $\operatorname{Tw}(M)$ if and only if ω is a B_2-form on M.

(ii) Let B be a complex vector bundle on M equipped with a connection ∇, not necessarily Hermitian. The pullback $\sigma^* B$ of B to $\operatorname{Tw}(M)$ is equipped with a pullback connection $\sigma^*\nabla$. Then, ∇ is a B_2-connection if and only if $\sigma^*\nabla$ has curvature of Hodge type $(1,1)$.

∎

Definition 6.18: Let $\operatorname{Tw}(M)$ be the twistor space of a quaternionic-Kähler manifold M. A B_2-bundle F on M gives a holomorphic bundle F' on $\operatorname{Tw}(M)$. We say that F' is a **twistor transform**, or **direct twistor transform** of F.

The B_2-bundle F can be recovered from F' ([V-c], Corollary 7.15). This procedure is called **the inverse twistor transform**.

In [Sw], A.Swann discovered a construction which relies a hyperkähler manifold with a special $\mathbb{H}^*$-action to every quaternionic-Kähler manifold of positive scalar curvature. This is done as followis.

Let $\mathbb{H}^*$ be the group of non-zero quaternions. Consider an embedding $SU(2) \hookrightarrow \mathbb{H}^*$. Clearly, every quaternion $h \in \mathbb{H}^*$ can be uniquely represented as $h = |h| \cdot g_h$, where $g_h \in SU(2) \subset \mathbb{H}^*$. This gives a natural decomposition $\mathbb{H}^* = SU(2) \times \mathbb{R}^{>0}$. Recall that $SU(2)$ acts naturally on the set of induced complex structures on a hyperkähler manifold.

Definition 6.19: Let M be a hyperkähler manifold equipped with a free smooth action ρ of the group $\mathbb{H}^* = SU(2) \times \mathbb{R}^{>0}$. The action ρ is called **special** if the following conditions hold.

(i) The subgroup $SU(2) \subset \mathbb{H}^*$ acts on M by isometries.

(ii) For $\lambda \in \mathbb{R}^{>0}$, the corresponding action $\rho(\lambda): M \longrightarrow M$ is compatible with the hyperholomorphic structure (which is a fancy way of saying that $\rho(\lambda)$ is holomorphic with respect to any of induced complex structures).

(iii) Consider the smooth $\mathbb{H}^*$-action $\rho_e: \mathbb{H}^* \times \operatorname{End}(TM) \longrightarrow \operatorname{End}(TM)$ induced on $\operatorname{End}(TM)$ by ρ. For any $x \in M$ and any induced complex structure I, the restriction $I\Big|_x$ can be considered as a point in the total space of $\operatorname{End}(TM)$. Then, for all induced complex structures I, all $g \in SU(2) \subset \mathbb{H}^*$, and all $x \in M$, the map $\rho_e(g)$ maps $I\Big|_x$ to $g(I)\Big|_{\rho_e(g)(x)}$.

Speaking informally, this can be stated as "$\mathbb{H}^*$-action interchanges the induced complex structures".

(iv) Consider the automorphism of S^2T^*M induced by $\rho(\lambda)$, where $\lambda \in \mathbb{R}^{>0}$. Then $\rho(\lambda)$ maps the Riemannian metric tensor $s \in S^2T^*M$ to $\lambda^2 s$.

Example 6.20: Consider the flat hyperkähler manifold $M_{fl} = \mathbb{H}^n \backslash 0$. There is a natural action of $\mathbb{H}^*$ on $\mathbb{H}^n \backslash 0$. This gives a special action of $\mathbb{H}^*$ on M_{fl}.

The case of a flat manifold $M_{fl} = \mathbb{H}^n \backslash 0$ is the only case where we apply the results of this section. However, the general statements are just as difficult to prove, and much easier to comprehend.

Definition 6.21: Let M be a hyperkähler manifold with a special action ρ of $\mathbb{H}^*$. Assume that $\rho(-1)$ acts non-trivially on M. Then $M/\rho(\pm 1)$ is also a hyperkähler manifold with a special action of $\mathbb{H}^*$. We say that the manifolds (M, ρ) and $(M/\rho(\pm 1), \rho)$ are **hyperkähler manifolds with special action of $\mathbb{H}^*$ which are special equivalent**. Denote by H_{sp} the category of hyperkähler manifolds with a special action of $\mathbb{H}^*$ defined up to special equivalence.

A. Swann ([Sw]) developed an equivalence between the category of quaternionic-Kähler manifolds of positive scalar curvature and the category H_{sp}.

Let Q be a quaternionic-Kähler manifold. The restricted holonomy group of Q is $Sp(n) \cdot Sp(1)$, that is, $(Sp(n) \times Sp(1))/\{\pm 1\}$. Consider the principal bundle $\mathcal{G}$ with the fiber $Sp(1)/\{\pm 1\}$, corresponding to the subgroup

$$Sp(1)/\{\pm 1\} \subset (Sp(n) \times Sp(1))/\{\pm 1\}.$$

of the holonomy. There is a natural $Sp(1)/\{\pm 1\}$-action on the space $\mathbb{H}^*/\{\pm 1\}$. Let

$$\mathcal{U}(Q) := \mathcal{G} \times_{Sp(1)/\{\pm 1\}} \mathbb{H}^*/\{\pm 1\}.$$

Clearly, $\mathcal{U}(Q)$ is fibered over Q, with fibers which are isomorphic to $\mathbb{H}^*/\{\pm 1\}$. We are going to show that the manifold $\mathcal{U}(Q)$ is equipped with a natural hypercomplex structure.

There is a natural smooth decomposition $\mathcal{U}(Q) \cong \mathcal{G} \times \mathbb{R}^{>0}$ which comes from the isomorphism $\mathbb{H}^* \cong Sp(1) \times \mathbb{R}^{>0}$.

Consider the standard 4-dimensional bundle W on Q. Let $x \in Q$ be a point. The fiber $W\Big|_q$ is isomorphic to $\mathbb{H}$, in a non-canonical way. The choices of isomorphism $W\Big|_q \cong \mathbb{H}$ are called **quaternion frames in** q. The set of quaternion frames gives a fibration over Q, with a fiber $\operatorname{Aut}(\mathbb{H}) \cong Sp(1)/\{\pm 1\}$. Clearly, this fibration coincides with the principal bundle $\mathcal{G}$ constructed above. Since $\mathcal{U}(Q) \cong \mathcal{G} \times \mathbb{R}^{>0}$, a choice of $u \in \mathcal{U}(Q)\Big|_q$ determines an isomorphism $W\Big|_q \cong \mathbb{H}$.

Let (q, u) be the point of $\mathcal{U}(Q)$, with $q \in Q$, $u \in \mathcal{U}(Q)\Big|_q$. The natural connection in $\mathcal{U}(Q)$ gives a decomposition

$$T_{(q,u)} U(Q) = T_u \left(\mathcal{U}(Q)\Big|_q \right) \oplus T_q Q.$$

The space $\mathcal{U}(Q)\Big|_q \cong \mathbb{H}^*/\{\pm 1\}$ is equipped with a natural hypercomplex structure. This gives a quaternion action on $T_u \left(\mathcal{U}(Q)\Big|_q \right)$ The choice of $u \in \mathcal{U}(Q)\Big|_q$ determines a quaternion action on $T_q Q$, as we have seen above. We obtain that the total space of $\mathcal{U}(Q)$ is an almost hypercomplex manifold.

Proposition 6.22: (A. Swann) Let Q be a quaternionic-Kähler manifold. Consider the manifold $\mathcal{U}(Q)$ constructed as above, and equipped with a quaternion algebra action in its tangent space. Then $\mathcal{U}(Q)$ is a hypercomplex manifold.

Proof: This is [V-c], Proposition 7.22. ∎

Consider the action of $\mathbb{H}^*$ on $\mathcal{U}(M)$ defined in the proof of Proposition 6.22. This action satisfies the conditions (ii) and (iii) of Definition 6.19. The conditions (i) and (iv) of Definition 6.19 are easy to check (see [Sw] for details). This gives a functor from the category $\mathcal{C}$ of quaternionic-Kähler manifolds of positive scalar curvature to the category H_{sp} of Definition 6.21. This is an equivalence of categories, constructed by A.Swann ([Sw]).

The inverse functor from H_{sp} to C is constructed by taking a quotient of M by the action of $\mathbb{H}^*$. Using the technique of quaternionic-Kähler reduction anf hyperkähler potentials ([Sw]), one can equip the quotient $M/\mathbb{H}^*$ with a natural quaternionic-Kähler structure. We call this equivalence **Swann's formalism for quaternionic-Kähler manifolds**.

6.5 Swann's formalism for vector bundles

Here we use the correspondence constructed by A.Swann to construct a correspondence between B_2-bundles on a quaternionic-Kähler manifold and $\mathbb{H}^*$-invariant hyperholomorphic bundles on the corresponding $\mathbb{H}^*$-invariant hyperkähler manifold. We follow [V-c], Section 8.

For the duration of this Subsection, we fix a hyperkähler manifold M, equipped with a special $\mathbb{H}^*$-action ρ, and the corresponding quaternionic-Kähler manifold $Q = M/\mathbb{H}^*$. Denote the standard quotient map by $\varphi : M \longrightarrow Q$.

Lemma 6.23: Let ω be a 2-form over Q, and $\varphi^*\omega$ its pullback to M. Then the following conditions are equivalent

(i) ω is a B_2-form

(ii) $\varphi^*\omega$ is of Hodge type $(1,1)$ with respect to some induced complex structure I on M

(iii) $\varphi^*\omega$ is $SU(2)$-invariant.

Proof: The proof is elementary linear algebra ([V-c], Lemma 8.1). ■

The following proposition is an immediate corollary of Lemma 6.23

Proposition 6.24: ([V-c], Proposition 8.2) Let (B, ∇) be a Hermitian vector bundle with connection over Q, and $(\varphi^* B, \varphi^* \nabla)$ its pullback to M. Then the following conditions are equivalent

(i) (B, ∇) is a B_2-bundle

(ii) The curvature of $(\varphi^* B, \varphi^* \nabla)$ is of Hodge type (1,1) with respect to some induced complex structure I on M

(iii) The bundle $(\varphi^* B, \varphi^* \nabla)$ is hypercomplex.

Proof: Follows from Lemma 6.23 applied to $\omega = \nabla^2$. ■

Let N be a hyperkähler manifold $\sigma : \operatorname{Tw}(N) \longrightarrow N$ its twistor space and B a hyperholomorphic bundle. It is easy to check that the lift $\sigma^* B$ is a holomorphic bundle on $\operatorname{Tw}(N)$. The holomorphic structure $\sigma^* B$ defines the connection on B in a unique way. This is called **direct and inverse twistor transform for hyperholomorphic bundles** ([KV1]). A holomorphic bundle on $\operatorname{Tw}(N)$ is called **compatible with the twistor transform** if it is obtained from some hyperholomorphic bundle on NN.

Proposition 6.24 has the following fundamental corollary

Theorem 6.25: In the above assumptions, let $\mathcal{C}_{B_2}$ be the category of of B_2-bundles on Q, and $\mathcal{C}_{\operatorname{Tw}, \mathbb{C}^*}$ the category of $\mathbb{C}^*$-equivariant holomorphic bundles on $\operatorname{Tw}(M)$ which are compatible with the twistor transform. Consider the functor

$$(\sigma^* \varphi^*)^{0,1} : \mathcal{C}_{B_2} \longrightarrow \mathcal{C}_{\operatorname{Tw}, \mathbb{C}^*},$$

$(B, \nabla) \longrightarrow (\sigma^* \varphi^* B, (\sigma^* \varphi^* \nabla)^{0,1})$, constructed above. Then $(\sigma^* \varphi^*)^{0,1}$ establishes an equivalence of categories.

Proof: This is [V-c], Theorem 8.5. ■

Now, let $x \in R$ be an isolated singularity of a reflexive hyperholomorphic sheaf F over a hyperkähler manifold R. Consider the twistor space $\operatorname{Tw}(N)$ and the horizontal twistor line $s_x : \mathbb{C}P^1 \longrightarrow \operatorname{Tw}(N)$ corresponding to x. Let I be an ideal sheaf of s_x. Denote the associated graded sheaf by $\mathcal{O}(\mathrm{Tw}(N))_{gr}$. Then $Spec(\mathcal{O}(\mathrm{Tw}(N))_{gr})$ is isomorphic to the twistor space of T_xM. Taking an associate graded sheaf of F, we obtain a sheaf F_{gr} over $Spec(\mathcal{O}(\mathrm{Tw}(N))_{gr}) \cong \mathrm{Tw}(T_xM)$. We proved the Desingularization Theorem by establishing the natural $\mathbb{C}^*$-action on the fibers (N, I) of the twistor projection $\pi : \mathrm{Tw}(N) \longrightarrow \mathbb{C}P^1$. As we have shown, the sheaf F is $\mathbb{C}^*$-equivariant with respect to this $\mathbb{C}^*$-action. Therefore, the associated graded sheaf F_{gr} has the same singularities as F.

This reasoning leads to the following theorem.

Theorem 6.26: ([V-c], Theorem 8.15) Let N be a hyperkähler manifold, I an induced complex structure and F a reflexive sheaf on (N, I) admitting a hyperholomorphic connection. Assume that F has an isolated singularity in $x \in N$, and is locally trivial outside of x. Let $\pi : \widetilde{N} \longrightarrow (N, I)$ be the blow-up of (N, I) in x. Consider the holomorphic vector bundle π^*F on $\widetilde{M}$ (Theorem 6.12). Let $C \subset (N, I)$ be the blow-up divisor, $C = \mathbb{C}PT_xM$. Clearly, the manifold C is canonically isomorphic to the twistor space of the quaternionic-Kähler manifold $\mathbb{H}P(T_xN)$. Then π^*F is the twistor transform of a B_2-bundle on $\mathbb{H}P(T_xN)$.

■

T.Nitta ([N2]) has shown that a bundle, obtained by twistor transform, is Yang-Mills (hence, direct sum of stable bundles of the same degree). This leads to the following corollary.

Corollary 6.27: In the assumptions of Theorem 6.26, consider the natural connection ∇ on the bundle $\pi^*F\Big|_C$ obtained from the twistor transform. Then ∇ is Yang-Mills, with respect to the Fubini-Study metric on $C = \mathbb{P}T_xN$, the degree $\deg c_1\left(\pi^*F\Big|_C\right)$ vanishes, and the holomorphic vector bundle $\pi^*F\Big|_C$ is a direct sum of stable bundles of the same degree.

■

7 Crepant resolutions and holomorphic symplectic geometry

Let V be a complex symplectic vector space, and G a finite group acting on V by linear transformations preserving symplectic structure. The variety

$X = V/G$ is usually singular. In this section, we are working with the resolutions of these singularities.

The **crepant** resolutions of X are resolutions $\pi: \widetilde{X} \longrightarrow X$ such that the canonical class of $\widetilde{X}$ is obtained as a pullback of a canonical class of X.

Further on, we shall assume that $\pi: \widetilde{X} \longrightarrow X = V/G$ is a crepant resolution. In such a case, the manifold $\widetilde{X}$ is also holomorphically symplectic, which is quite easy to see ([V-r], Theorem 2.5).

Here is an example of such situation.

Example 7.1: The Hilbert scheme of n points on $\mathbb{C}^2$ provides a crepant resolution of the quotient $(\mathbb{C}^2)^n/S_n$ of $(\mathbb{C}^2)^n$ by the natural action of the symmetric group S_n (this is well known; see e. g. [N]).

A reason to study the symplectic desingularization comes from the hyperkähler geometry. Consider a compact complex torus T, $\dim_{\mathbb{C}} T = 2$, and its n-th Hilbert scheme of points $T^{[n]}$. Let $Alb: T^{[n]} \longrightarrow T$ be the Albanese map. A **generalized Kummer variety** $K^{[n-1]}$ is defined as

$$K^{[n-1]} \subset T^{[n]}, \quad K^{[n-1]} := Alb^{-1}(0).$$

The variety $K^{[n-1]}$ is smooth and holomorphically symplectic ([Bea]). By Calabi-Yau theorem ([Y], [Bea]), the variety $K^{[n]}$ is equipped with a set of hyperkähler structures, parametrized by the Kähler cone.

In [KV2] (Section 4), it was shown that all trianalytic subvarieties of generalized Kummer varieties (at least, for generic hyperkähler structures) are isomorphic to symplectic desingularizations of a quotient of a compact torus by an action of a Coxeter group. This establishes a very interesting relation between the Dynkin diagrams and hyperkähler geometry, and motivates the study of symplectic desingularization of quotient singularities.

In [V-r], this argument was carried a step further, to obtain information about the structure of finite groups $G \subset Sp(V)$ such that V/G admits a symplectic desingularization. This is done as follows. Let $g \in \operatorname{End}(V)$ be a symplectomorphism of finite order. We say that g is a **symplectic reflection** if

$$\operatorname{codim}_V \Big(\{x \in V \ \mid \ g(x) = x\} \Big) = 2,$$

that is, the dimension of the fixed set of g is maximal possible for non-trivial g. This definition parallels that of complex reflections – a complex reflection is an endomorphism of finite order with fixed point set of codimension 1. The main result of [V-r] is the following theorem.

Theorem 7.2: ([V-r], Theorem 3.2) Let V be a symplectic vector space over $\mathbb{C}$, and $G \subset Sp(V)$ a finite group of symplecic transformations. Assume that V/G admits a symplectic resolution. Then G is generated by symplectic reflections.

The proof of Theorem 7.2 is modeled on the proof of well-known theorem about groups generated by complex reflections (that is, endomorphisms which fix a subspace of codimension 1).

Proposition 7.3: ([Bou], Ch. V, §5 Theorem 4) Let V be a complex vector space, and $G \subset GL(V)$ a finite group acting on V. Assume that $X := V/G$ is smooth. Then G is generated by complex reflections. Conversely, if G generated by complex reflections, the quotient V/G is smooth.

Another important ingredient is the following theorem, which is based on elementary arguments from linear algebra.

Definition 7.4: Let $\pi : \widetilde{X} \longrightarrow X$ be a resolution of singularities. The map π is called **semismall** if X admits a stratification $\mathfrak{S}$ with open strata U_i, such that

$$\forall x \in U_i \ \mid \ \dim \pi^{-1}(x) \leqslant \frac{1}{2} \operatorname{codim} U_i$$

Theorem 7.5: Let $\pi : \widetilde{X} \longrightarrow X$ be a crepant resolution of a quotient singularity $X = V/G$, $G \in Sp(V)$. Then π is semismall.

Proof: This statement easily follows from Proposition 4.16 and Proposition 4.5 of [V4] (see also [K1], Proposition 4.4). ∎

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