CRITERIA FOR THE DENSITY PROPERTY OF COMPLEX MANIFOLDS

SHULIM KALIMAN AND FRANK KUTZSCHEBAUCH

1. Introduction

The ground-breaking papers of Andersén and Lempert ([1], [2]) established remarkable properties of the automorphism group of $\mathbb{C}^n$ ($n \geq 2$) which imply, in particular, that any local holomorphic phase flow on a Runge domain Ω in $\mathbb{C}^n$ can be approximated by global holomorphic automorphisms of $\mathbb{C}^n$ (for an exact statement see Theorem 2.1 in [5]).

The next step in the development of the Andersén-Lempert theory was made by Varolin who extended it from Euclidean spaces to a wider class of algebraic complex manifolds. He realized also that following density property is crucial for this theory.

1.1**. Definition.** A complex manifold X has the density property if in the compact-open topology the Lie algebra $\mathrm{Lie}_{\mathrm{hol}}(X)$ generated by globally integrable holomorphic vector fields on X is dense in the Lie algebra $\mathrm{VF}_{\mathrm{hol}}(X)$ of all holomorphic vector fields on X. An affine algebraic manifold has the algebraic density property if the Lie algebra $\mathrm{Lie}_{\mathrm{alg}}(X)$ generated by globally integrable algebraic vector fields on it coincides with the Lie algebra $\mathrm{VF}_{\mathrm{alg}}(X)$ of all algebraic vector fields on it (clearly the algebraic density property implies the density property).

In this terminology the main observation of the Andersén-Lempert theory says that $\mathbb{C}^n$ ($n \geq 2$) has the algebraic density property. Varolin and Toth ([14], [12], [13]) established the density property for some manifolds including semi-simple complex Lie groups and some homogenous spaces of semi-simple Lie groups. Their proof relies heavily on representation theory and does not for example lead to an answer in the case of other linear algebraic groups.

In this paper we suggest new effective criteria for the density property. This enables us to give a trivial proof of the original Andersén-Lempert result and to establish (almost free of charge) the algebraic density property for all linear algebraic groups different from tori or $\mathbb{C}_+$. Actually this approach allows to handle a more delicate algebraic volume-density property, but we omit this fact here since our aim is to give

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a short presentation of this remarkable approach. Instead we tackle another open question (asked among others by F. Forstnerič): the density of algebraic vector fields on Euclidean space vanishing on a codimension 2 subvariety.

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2. The Andersén-Lempert observation.

Our method requires two ingredients. The first one is a homogeneity property reflected in the following.

2.1**. Definition.** Let X be an algebraic manifold and $x_0 \in X$. A finite subset M of the tangent space $T_{x_0}X$ is called a generating set if

(1) the group of algebraic automorphisms G acts transitively on X, and

(2) the image of M under the action of the isotropy group G_{x_0} generate the whole space $T_{x_0}X$.

The manifold X will be called tangentially semi-homogeneous if it admits a generating set consisting of one vector.

Theorem 1. *Let X be an algebraic manifold with algebra of regular functions $\mathbb{C}[X]$, and L be a submodule of the $\mathbb{C}[X]$-module of all vector fields such that $L \subset \mathrm{Lie}_{\mathrm{alg}}(X)$. Suppose that the fiber of L over some $x_0 \in X$ contains a generating set. Then X has the algebraic density property.*

Proof. Treat TX and L as a coherent sheaf and its subsheaf. The action of $\alpha \in \mathrm{Aut}\, X$ maps L onto another coherent subsheaf L_α of TX. The sum of such subsheaves with α running over $\mathrm{Aut}\, X$ is a coherent subsheaf $\mathcal{E}$ of TX. Let $\mathfrak{m}$ be the maximal ideal for x_0. Then condition (2) from Definition 2.1 implies that $\mathcal{E}/\mathfrak{m}\mathcal{E}$ coincides with $T_{x_0}X$, and condition (1) implies that this is true for every point in X. Thus $\mathcal{E} = TX$ ([8], Chapter II, exercise 5.8). Clearly, all global sections of $\mathcal{E}$ are in $\mathrm{Lie}(X)$ which concludes the proof.

□

The proof of the following corollary reflects the second ingredient of our method.

2.2**. Corollary.** (The main observation of the Andersén-Lempert theory) *For $n \geq 2$ the space $\mathbb{C}^n$ has the algebraic density property.*

Proof. Let $x_1, \ldots, x_n$ be a coordinate system on $\mathbb{C}^n$ and $\delta_i = \partial/\partial x_i$ be the partial derivative, i.e. $\mathrm{Ker}\, \delta_i$ is the ring of polynomials independent of x_i. Hence the polynomial ring $\mathbb{C}^{[n]}$ is generated as a vector space by elements of $\mathrm{Ker}\, \delta_1 \cdot \mathrm{Ker}\, \delta_2$. Note also that for $f_i \in \mathrm{Ker}\, \delta_i$ the algebraic vector fields $f_i\delta_i$ and $x_i f_i \delta_i$ are globally integrable. Then the field

$$[f_1\delta_1, x_1 f_2 \delta_2] - [x_1 f_1 \delta_1, f_2\delta_2] = f_1 f_2 \delta_2$$

belongs to $\mathrm{Lie}_{\mathrm{alg}}(X)$ since $x_1 f_2 \in \mathrm{Ker}\, \delta_2$. Thus $\mathrm{Lie}_{\mathrm{alg}}(X)$ contains all algebraic fields proportional to δ_2. Since $\mathbb{C}^n$ is clearly tangentially semi-homogeneous Theorem 1 implies the desired conclusion. □

One can formalize this argument as follows.

2.3. **Definition.** Let δ_1 and δ_2 be commuting algebraic vector fields on an affine algebraic manifold X such that δ_1 is a locally nilpotent derivation on its algebra of regular functions $\mathbb{C}[X]$, and δ_2 is either also locally nilpotent or semi-simple (i.e δ_i generates an algebraic action of H_i on X where $H_1 \simeq \mathbb{C}_+$ and H_2 is either $\mathbb{C}_+$ or $\mathbb{C}^*$). We say that δ_1 and δ_2 are compatible if

(1) the induced $(H_1 \times H_2)$-action is not degenerate (i.e., its general orbits are two-dimensional), and

(2) the vector space $\operatorname{Span}(\operatorname{Ker}\delta_1 \cdot \operatorname{Ker}\delta_2)$ generated by elements from $\operatorname{Ker}\delta_1 \cdot \operatorname{Ker}\delta_2$ contains a nonzero ideal in $\mathbb{C}[X]$.

Theorem 2. *Let X be a smooth affine algebraic manifold with finitely many pairs of two compatible vector fields $\{\delta_1^k, \delta_2^k\}_{k=1}^m$ such that for some point $x_0 \in X$ vectors $\{\delta_2^k(x_0)\}_{k=1}^m$ form a generating set. Then X has the algebraic density property.*

Proof. Let δ_1 and δ_2 be one of our pairs. Since they commute, δ_1 is a locally nilpotent derivation on the affine domain $\operatorname{Ker}\delta_2$. It is not identically zero since otherwise the $(H_1 \times H_2)$-action is degenerate. Thus one can choose an element a of degree 1 with respect to δ_1 in $\operatorname{Ker}\delta_2$, i.e. $b = \delta_1(a) \in \operatorname{Ker}\delta_1 \setminus 0$. Let $f_i \in \operatorname{Ker}\delta_i$. Then $[af_1\delta_1, f_2\delta_2] - [f_1\delta_1, af_2\delta_2] = -bf_1f_2\delta_2$. The last vector field is from $\operatorname{Lie}_{\rm alg}(X)$ and since δ_1 and δ_2 are compatible, condition (2) from Definition 2.3 implies that sums of such vector fields include every vector field of form $I\delta_2$ where I is a nonzero ideal in $\mathbb{C}[X]$. Applying this argument to all compatible pairs we se that $\operatorname{Lie}_{\rm alg}(X)$ contains all linear combinations of δ_2^k with coefficients in some nonzero ideal $J \subset \mathbb{C}[X]$. Since under a small perturbation of x_0 the set $\{\delta_2^k(x_0)\}_{k=1}^m$ remains a generating set we can suppose that x_0 does not belong to the zero locus of J. Hence by Theorem 1 X has the algebraic density property. □

2.4. **Remark.** If X is tangentially semi-homogenous and furthermore any non-zero tangent vector (at any point) is a generating set, then Theorem 2 implies that for the algebraic density property a single pair of compatible vector fields is enough.

2.5. **Corollary.** *Let X_1 and X_2 be algebraic manifolds such that each X_i admits a finite number of integrable algebraic vector fields $\{\delta_i^k\}_{k=1}^{m_i}$ whose values at some point $x_i \in X_i$ form a generating set and, furthermore, in the case of X_1 these vector fields are locally nilpotent. Then $X_1 \times X_2$ has the algebraic density property.*

Proof. Note that δ_1^k and δ_2^j generate compatible integrable vector fields on $X_1 \times X_2$ which we denote by the same symbols. Applying isotropy groups one can suppose that $\{\delta_i^k(x_i)\}$ is a basis of $T_{x_i}X_i$. In order to show that the set of vectors $M = \{0 \times \delta_2^k(x_2)\}$ form a generating set in $T_{x_1 \times x_2}(X_1 \times X_2)$ we need the following fact that is obvious in a local coordinate system.

Claim. *Let X be a complex manifold and let ν be a vector field on X. Suppose that f is a holomorphic function from* $\operatorname{Ker}\nu$ *and $x_0 \in f^{-1}(0)$. Then phase flow induced by* *the vector field $f\nu$ generates a linear action on the tangent space $T_{x_0}X$ given by the formula $w \to w + df(w)\nu(x_0)$ where df is the differential and $w \in T_{x_0}X$. In particular, the span of the orbit of w under this phase flow contains vector $df(w)\nu(x_0)$.*

Applying this claim for $\nu = \delta_1^j$ we see that the orbit of M under the isotropy group of $x_1 \times x_2$ contains all vectors of form $\delta_1^j(x_1) \times \delta_2^k(x_2)$. Thus M is a generating set and we are done by Theorem 2.

□

2.6. **Remark.** The reason why we use the locally nilpotent δ_1^j in the above proof as ν and not (the possibly semi-simple) δ_2^j is the following: The vector field $f\delta_2^j$ with $f \in \operatorname{Ker} \delta_2^j$ may not generate an algebraic action while $f\delta_1^j$ with $f \in \operatorname{Ker} \delta_1^j$ always generates an algebraic action. It is worth mentioning if one wants to prove density property instead of algebraic density property the use of δ_2^j is permissible.

2.7. **Example.** (1) Let $X = \mathbb{C}^k \times (\mathbb{C}^*)^l$ with $k \geq 1$ and $k+l \geq 2$. Then X has algebraic density property by Corollary 2.5.

(2) If G is a semi-simple group then it is tangentially semi-homogeneous since the adjoint action of G generates an irreducible representation on the tangent space $\mathfrak{g}$ at the identity e (in particular, any nonzero vector in T_eG is a generating set). Let X be $SL_n(\mathbb{C})$ with $n \geq 3$, i.e. X is tangentially semi-homogeneous. Then every $x \in X$ is a matrix (c_{kj}) with determinant 1. Set δ_i, $i = 1, 2$ by formulas $\delta_i(c_{ij}) = c_{nj}$ and $\delta_i(c_{kj}) = 0$ for $k \neq i$. These associated $\mathbb{C}_+$-actions are commutative and free. Note that constants and functions depending on $c_{kj}, k \neq i$ only are in $\operatorname{Ker} \delta_i$. Therefore, condition (2) of Definition 2.3 holds and, in fact δ_1 and δ_2 are compatible. Thus $SL_n(\mathbb{C})$ has the algebraic density property when $n \geq 3$.

3. Density of Affine Algebraic Groups Different from Tori or $\mathbb{C}_+$.

Notation. In this section a group H_1 is isomorphic to $\mathbb{C}_+$ and H_2 is isomorphic either to $\mathbb{C}_+$ or $\mathbb{C}^*$. Suppose that there is a non-degenerate algebraic action of $H_1 \times H_2$ on an affine algebraic manifold X. That is, H_i generate an algebraic vector field δ_i on X. Recall, that the algebraic quotient $X_i = X//H_i$ is a normal quasi-affine variety [Wi03]. Let $\rho_i : X \to X_i$ be the quotient morphism. Set $\rho = (\rho_1, \rho_2) : X \to Y := X_1 \times X_2$ and Z equal to the closure of $\rho(X)$ in Y. For any quasi-affine algebraic variety T we denote by $\mathbb{C}[T]$ its algebra of regular functions and G below will be a linear algebraic group whose action on algebraic varieties will be always algebraic.

We start with a geometric reformulation of condition (2) in Definition 2.3.

3.1. **Lemma.** *In the notation as before δ_1 and δ_2 are compatible iff ρ is a birational morphism into Z and $W := Z \setminus \rho(X)$ is of a codimension at least 2 in Z.*

Proof. Every nonzero element of $\operatorname{Span}(\operatorname{Ker} \delta_1 \cdot \operatorname{Ker} \delta_2)$ is of the form $g \circ \rho$ where $g \in \mathbb{C}[Z] = \mathbb{C}[Y]|_Z$. Thus $\operatorname{Span}(\operatorname{Ker} \delta_1 \cdot \operatorname{Ker} \delta_2)$ coincides with the subalgebra $\rho^*(\mathbb{C}[Z]) \subset \mathbb{C}[X]$ which does not separate points of $\rho^{-1}(z)$ for any $z \in Z$. Hence if $\rho : X \to Z$ is not birational $\rho^*(\mathbb{C}[Z])$ cannot contain a nonzero ideal of $\mathbb{C}[X]$, i.e. δ_1 and δ_2 are not compatible.

Assume now that W contains a divisor D in Z. There is a rational function f on Z so that it has poles on D and nowhere else. In particular, $f \circ \rho$ is regular on X. On the other hand for n sufficiently large and g as before gf^n has poles on D and cannot be a regular function on Z. Thus $(gf^n) \circ \rho \notin \mathrm{Span}(\mathrm{Ker}\, \delta_1 \cdot \mathrm{Ker}\, \delta_2)$ and the last vector space cannot contain a nonzero ideal in this case.

Suppose now that $\rho : X \to Z$ is birational and W is of codimension at least 2 in Z. If Z is normal then any regular function on $\rho(X)$ extends to Z by the Hartogs theorem. This implies that the image of the defining ideal of the closure $\bar{W}$ of W in Z under the homomorphism ρ^* is an ideal in $\mathbb{C}[X]$. In the general case any regular function on $\rho(X)$ extends as a regular function to a normalization Z_0 of Z, i.e. we have a homomorphism $\rho_0^* : \mathbb{C}[Z_0] \to \mathbb{C}[X]$. Choose a divisor $E_0 \subset Z_0$ for which the restriction of the normalization morphism $\nu : Z_0 \to Z$ to $Z_0 \setminus E_0$ is an embedding and $\nu^{-1}(\bar{W}) \subset E_0$. Then ρ_0^* sends any nonzero ideal $J_0 \subset \mathbb{C}[Z_0]$ that is contained in the defining ideal of E_0, into a nonzero ideal of $\mathbb{C}[X]$. Treat $\mathbb{C}[Z] \simeq \nu^*(\mathbb{C}[Z])$ as a subalgebra of $\mathbb{C}[Z_0]$. Since ν is finite $\mathbb{C}[Z_0]$ is generated over $\mathbb{C}[Z]$ by a finite number of functions of form f_i/g_i, $i = 1, 2, \ldots, n$ where f_i and g_i are regular on Z. Consider the intersection $J \subset \mathbb{C}[Z]$ of the defining ideal of $E = \nu(E_0)$ and the principal ideal generated by $\prod_{i=1}^n g_i$. By construction $J_0 = J\mathbb{C}[Z_0] \subset \mathbb{C}[Z]$. Hence $\rho^*(\mathbb{C}[Z])$ contains a nonzero ideal of $\mathbb{C}[X]$ and, therefore, δ_1 and δ_2 are compatible.

□

3.2. **Lemma.** *Let X, H_i, X_i, and ρ_i be as in the beginning of this section and Γ be a finite group acting on X so that this action preserves the subrings $\mathbb{C}[X_1]$ and $\mathbb{C}[X_2]$, i.e. Γ acts on X_1 and on X_2. Suppose that $X' = X//\Gamma$ is normal, $X_i' = X_i//\Gamma$, and that $\rho_i' : X' \to X_i'$ is the dominant morphism induced by ρ_i. In particular, we treat $\mathbb{C}[X_i']$ as a subalgebra of $\mathbb{C}[X']$. Let $\mathrm{Span}(\mathbb{C}[X_1] \cdot \mathbb{C}[X_2])$ contain a nonzero ideal of $\mathbb{C}[X]$. Then $\mathrm{Span}(\mathbb{C}[X_i'] \cdot \mathbb{C}[X_2'])$ contains a nonzero ideal of $\mathbb{C}[X']$.*

Proof. Set $\rho' = (\rho_1', \rho_2') : X' \to Y' := X_1' \times X_2'$ and Z' equal to the closure of $\rho'(X')$ in Y'. There is a natural action of the group $\Gamma \times \Gamma$ on $Y = X \times X$ such that $Y' = Y//(\Gamma \times \Gamma)$. In particular, $q \circ \rho = \rho' \circ p$ where $p : X \to X'$ and $q : Y \to Y'$ are the quotient morphisms of the actions of Γ and $\Gamma \times \Gamma$ respectively. Thus $Z' = q(Z)$ and $\rho'(X') = q(\rho(X))$. By Lemma 3.1 $\rho : X \to Z$ is birational and $Z \setminus \rho(X)$ is of codimension at least 2 in Z. Since q is finite $Z' \setminus \rho'(X')$ is of codimension at least 2 in Z'. Thus it suffices to prove that $\rho' : X' \to Z'$ is birational.

For points $x_1, x_2 \in X$ we set $x_i' = p(x_i)$, $y_i = \rho(x_i) = (\rho_1(x_i), \rho_2(x_i))$, and $y_i' = q(y_i) = \rho'(x_i')$. Assume that ρ' is not birational, i.e. there are a general point x_1 and and a point $x_2 \neq x_1$ such that $y_1' = y_2'$ but $x_1' \neq x_2'$ (i.e. x_1 and x_2 do not belong to the same orbit of Γ on X). The last equality implies that y_1 and y_2 are in the same orbit of the $(\Gamma \times \Gamma)$-action on Y, i.e. there $\alpha_1, \alpha_2 \in \Gamma$ such that $\alpha_1(\rho_1(x_1)) = \rho_1(x_2)$ and $\alpha_2(\rho_2(x_1)) = \rho_2(x_2)$. Note that $\alpha_k(\rho_j(x_i)) = \rho_j(\alpha_k(x_i))$ since by construction the Γ-actions commute with the morphisms ρ_j. Thus replacing x_1 by $\alpha_2(x_1)$ and α_1 by

$\alpha_1 \circ \alpha_2^{-1}$ we have equalities

$$\rho_1(\alpha(x_1)) = \rho_1(x_2) \text{ and } \rho_2(x_1) = \rho_2(x_2).$$

Since $H_1 \simeq \mathbb{C}_+$ acts on X_2 and $\rho_2(x_1) \in X_2$ is a general point we see that H_1-orbit O of the fiber $F = \rho_2^{-1}(\rho_2(x_1))$ is isomorphic to $F \times H_1$ and x_1, x_2, and $\alpha(x_1)$ are in O. Any element of Γ sends O into a similar orbit because Γ-actions commute with ρ_i. Hence $\alpha_1(O) = O$ and we have an action of the finite cyclic group generated by α on O. It commutes with the morphism $\rho_2|_O : O \to H_1$. Thus this cyclic group acts on $H_1 \simeq \mathbb{C}_+$. Since the only finite group on $\mathbb{C}$ that commutes with translations is a trivial one, we see that the orbit of x_1 under this action is contained in the fiber of ρ_2. That is, $\rho_2(\alpha(x_1)) = \rho_2(x_1) = \rho_2(x_2)$. Since x_1 is general we can suppose that x_1 and x_2 are in a dense open subset of X on which ρ is an embedding. Therefore, $\alpha(x_1) = x_2$ contrary to the assumption that x_1 and x_2 are not in the same orbit of Γ. Thus ρ' is birational.

□

3.3. **Proposition.** *Let G act on an affine algebraic manifold X so that $Z := X//G$ is affine (which is always true when G is reductive) and the quotient morphism $\rho : X \to Z$ is a principal algebraic G-bundle. Let $H_1 \times H_2$ be an algebraic subgroup of G and the action of H_i on G generated by left multiplication correspond to a derivation δ_i^0 on $B = \mathbb{C}[G]$ such that δ_1^0 and δ_2^0 are compatible. Suppose that the induced H_i-actions on X correspond to derivations δ_i on $\mathbb{C}[X]$. Then δ_1 and δ_2 are compatible.*

Proof. It suffices to check condition (2) in Definition 2.3 since condition (1) is obvious. Note that the kernels of δ_1 and δ_2 contain $\mathbb{C}[Z]$. Let $I_0 \subset B$ be the largest ideal contained in $\operatorname{Span}(\operatorname{Ker}\delta_1^0 \cdot \operatorname{Ker}\delta_2^0)$ and $F \subset G$ be its zero locus. In particular $I^k \subset \operatorname{Span}(\operatorname{Ker}\delta_1^0 \cdot \operatorname{Ker}\delta_2^0)$ for some $k \geq 1$ where $I \subset B$ is the defining ideal of F. The analogue of F in each fiber $\rho^{-1}(z) \simeq G$ is determined independently of the trivialization since I_0 is the largest ideal contained in $\operatorname{Span}(\operatorname{Ker}\delta_1^0 \cdot \operatorname{Ker}\delta_2^0)$. Thus there is a subvariety $\mathcal{F}$ of X such that $\mathcal{F} \cap \rho^{-1}(z)$ plays the role of this F in $\rho^{-1}(z) \simeq G$ for every $z \in Z$ (i.e. $\rho|_{\mathcal{F}} : \mathcal{F} \to Z$ is a locally trivial F-fibration). Let $J \subset A$ be the defining ideal of $\mathcal{F}$. Show that $J^k \subset \operatorname{Span}(\operatorname{Ker}\delta_1 \cdot \operatorname{Ker}\delta_2)$.

Choose a finite covering $\{U_i\}$ of Z such that for $V_i = \rho^{-1}(U_i)$ and $\mathcal{F}_i = \mathcal{F} \cap V_i$ pair $(V_, \mathcal{F}_i)$ is isomorphic to $(U_i \times G, U_i \times F)$ over U_i. Hence one can see that the localization of $\operatorname{Span}(\operatorname{Ker}\delta_1 \cdot \operatorname{Ker}\delta_2)$ to V_i contains the k-th power $L \simeq \mathbb{C}[U_i] \otimes I^k$ of the defining ideal of $\mathcal{F}_i$ in $C_i = \mathbb{C}[V_i] \simeq \mathbb{C}[U_i] \otimes B$. We can suppose that $U_i = Z \setminus f_i^{-1}(0)$ where f_i is a regular function on Z. Thus for every $a \in J^k$ there exists $k_i > 0$ such that $af_i^{k_i}$ is in $\operatorname{Span}(\operatorname{Ker}\delta_1 \cdot \operatorname{Ker}\delta_2)$. By Hilbert's Nullstellensatz there are regular functions g_i on Z such that $\sum_i f_i^{k_i} g_i \equiv 1$. Since g_i is in the kernel of δ_1 we see that $a \in \operatorname{Span}(\operatorname{Ker}\delta_1 \cdot \operatorname{Ker}\delta_2)$ which concludes the proof.

□

Theorem 3. *Let G be a linear algebraic group whose connected component is different from a torus or $\mathbb{C}_+$. Then G has the algebraic density property.*

Proof. Since all components of G are isomorphic as varieties we can suppose that G is connected. Recall that the unipotent radical R of G is an algebraic subgroup of G ([4], p. 183). By Mostow's theorem [11] (see also [4], p. 181) G contains a (Levi) maximal closed reductive algebraic subgroup L (which is, in particular, affine) such that G is the semi-direct product of L and R, i.e. G is isomorphic as affine variety to the product $R \times L$. In case L is trivial $G = R \simeq \mathbb{C}^n, n \geq 2$ and we are done by Corollary 2.2. In the case of both R and L being nontrivial we are done by Corollary 2.5 with R playing the role of X_1 and L of X_2.

Thus it remains to cope with reductive groups G. Let $Z \simeq (\mathbb{C}^*)^n$ denote the center of G and S its semisimple part. First we suppose that Z is nontrivial. The case when G is isomorphic as group to the direct product $S \times Z$ can be handled as above by Corollary 2.5 with S playing the role of X_1 and Z of X_2. In particular we have a finite set of pairs of compatible vector fields $\{\delta_1^k, \delta_2^k\}$ as in Theorem 2. Furthermore one can suppose that the fields δ_1^k correspond to one parameter subgroups of S isomorphic to $\mathbb{C}_+$ and δ_2^k to one parameter subgroups of Z isomorphic to $\mathbb{C}^*$. In the general case G is the factor group of $S \times Z$ by a finite (central) normal subgroup Γ. Since Γ is central the fields δ_1^k, δ_2^k induce integrable vector fields $\tilde{\delta}_1^k$, $\tilde{\delta}_2^k$ on G while $\tilde{\delta}_2^k(x_0)$ is a generating set for some $x_0 \in G$. By Lemma 3.2 the pairs $\{\tilde{\delta}_1^k, \tilde{\delta}_2^k\}$ are compatible and the density property for G follows again from Theorem 2.

Thus we are left with the case when G is semisimple. If G is not simple it is isomorphic to the factor group of a product of two semisimple groups $S_1 \times S_2$ by a finite (central) subgroup Γ and this case can be handled exactly as before.

Now it remains to consider simple G. In the case of $G \simeq SL_2(\mathbb{C})$ or $G \simeq PSL_2(\mathbb{C})$ we just refer to [12] or [10].

If the Dynkin diagram of G contains at least three nodes then two of them are not connected by an edge which implies that the $\mathfrak{sl}_2$-subalgebras of the Lie algebra of G corresponding to these roots commute (e.g., see Serre's relations on p. 337 in [6]). Thus G contains two $SL_2(\mathbb{C})$ that commute and have only the identical element in common. The existence of two compatible vector fields δ_1 and δ_2 on $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$ implies by Proposition 3.3 their existence on G. Since a semi-simple Lie group is tangentially semi-homogenous (see Example 2.7) the algebraic density property for G follows again from Theorem 2.

Since we already excluded the case of $G \simeq SL_2(\mathbb{C})$ or $G \simeq PSL_2(\mathbb{C})$ it remains to consider the Dynkin diagrams with two nodes, i.e. it is either (A2), or (B2)$\simeq$(C2), or (G2). In the case of (A2) $S \simeq SL_3(\mathbb{C})$ and we saw already that this groups has a pair of compatible free $\mathbb{C}_+$-actions (see Example 2.7). In the case of (B2) or (G2) the consideration of the root systems (e.g., [6], p. 332) shows that S has again two $SL_2(\mathbb{C})$-subgroups that commute and have only the identical element in common. We are done. □

4. Codimension 2 Case.

Notation. In this section X will be a closed affine algebraic subvariety of $\mathbb{C}^n$ whose codimension $n-k$ is at least 2, and $Z = \mathbb{C}^n \setminus X$. By I we denote the defining ideal of X in $\mathbb{C}^{[n]}$. For any affine algebraic variety Y its algebra of regular functions will be denoted by $\mathbb{C}[Y]$ and $\mathrm{AVF}_L(\mathbb{C}^n)$ will be the Lie algebra of algebraic vector fields on $\mathbb{C}^n$ whose coordinate functions are contained in an ideal $L \subset \mathbb{C}^{[n]}$.

4.1**. Lemma.** *In this notation Z is tangentially semi-homogeneous.*

Proof. By a theorem of Gromov [7] and Winkelmann [15] Z is homogenous. More precisely, consider a general linear projection $p : \mathbb{C}^n \to \mathcal{H} \simeq \mathbb{C}^{n-1}$ and a nonzero constant vector field ν such that $p_*(\nu) = 0$. Then $p(X)$ is a subvariety of codimension at least 1 in $\mathcal{H}$. For every regular function h on $\mathcal{H}$ that vanishes on $p(X)$ the vector field $h\nu$ generates a $\mathbb{C}_+$-action on Z. Changing $\mathcal{H}$ we get a transitive action.

Consider a general point $z \in Z$ whose projection $z_0 \in \mathcal{H}$ is not in $p(X)$. Suppose that h has a simple zero at z_0. By the Claim in the proof of Corollary 2.5 the $\mathbb{C}_+$-action generated by $h\nu$ acts on $T_z Z$ by the formula $w \to w + dh(w)\nu(w)$ where dh is the differential of h and $w \in T_z Z$. Since ν may be chosen as a general constant vector field on $\mathbb{C}^n$ we see that G induces an irreducible representation on $T_z Z$ which implies tangential semi-homogeneity.

□

Theorem 4. *There is an ideal $L \subset \mathbb{C}^{[n]}$ whose radical is I such that $\mathrm{Lie}_{\mathrm{alg}}(Z)$ contains $\mathrm{AVF}_L(\mathbb{C}^n)$.*

Proof. Suppose that $x_1, \ldots, x_n$ is a coordinate system, $p_i : \mathbb{C}^n \to \mathbb{C}^{n-1}$ is a projection to the coordinate hyperplane $\mathcal{H}_i = \{x_i = 0\}$, and h_i is a nonzero function on $\mathcal{H}_i$ that vanishes on $p_i(X)$. Set $\delta_i = \partial/\partial x_i$ and choose $f_i \in \operatorname{Ker} \delta_i$. Then $f_i h_i \delta_i$ is a globally integrable algebraic vector field on $\mathbb{C}^n$ that vanishes on X, i.e. it generates a $\mathbb{C}_+$-action on Z. Then

$$[f_1 h_1 \delta_1, x_1 f_2 h_2 \delta_2] - [x_1 f_1 h_1 \delta_1, f_2 h_2 \delta_2] = f_1 f_2 h_1 h_2 \delta_2$$

belongs to $\mathrm{Lie}_{\mathrm{alg}}(Z)$. Since $\operatorname{Ker} \delta_1 \cdot \operatorname{Ker} \delta_2$ generates the ring of polynomials $\mathbb{C}^{[n]}$ as a vector space we see that $\mathrm{Lie}_{\mathrm{alg}}(Z)$ contains all algebraic fields proportional to δ_2 with coordinate functions in the principal ideal generated by $h_1 h_2$. Since one can perturb x_2 (as a linear function) $\mathrm{Lie}_{\mathrm{alg}}(Z)$ contains all algebraic vector fields whose coordinates are in some (non-zero) ideal L. Since Z is homogenous arguing as in the proof of Theorem 1 one can suppose that the radical of L is I.

□

Though Theorem 4 does not give the algebraic density of the Lie algebra of algebraic vector fields vanishing on X it implies already a strong approximation result generalizing the Andersén-Lempert theorem. We omit its proof since it repeats the arguments in [5] with minor modifications.

Theorem 5. *Let X be an algebraic subvariety of $\mathbb{C}^n$ of codimension at least 2 and Ω be an open set in $\mathbb{C}^n$ ($n \geq 2$). Let $\Phi : [0,1] \times \Omega$ be a map of class $\mathcal{C}^2$ such that for every $t \in [0,1]$ the map $\Phi_t : \Omega \to \mathbb{C}^n$ is injective and holomorphic. Assume that each domain $\Phi_t(\Omega)$ is Runge in $\mathbb{C}^n$ and does not intersect X. If Φ_0 can be approximated on Ω by holomorphic automorphisms of $\mathbb{C}^n$ fixing X, then for every $t \in [0,1]$ the map Φ_t can be approximated on Ω by holomorphic automorphisms of $\mathbb{C}^n$ fixing X.*

The following is a stronger version of a result of Buzzard and Hubbard [3] answering a question by Siu.

4.2. **Corollary.** *Any point z in the complement of an algebraic subset X of $\mathbb{C}^n$ of codimension at least 2 has a neighborhood U in $\mathbb{C}^n \setminus X$ which is biholomorphic to $\mathbb{C}^n$ (such U is called a Fatou-Bieberbach domain).*

Proof. Choose Ω to be a ball around z not intersecting X. Let Φ_t contract this ball radially towards z. The resulting automorphism approximating Φ_1 from Theorem 5 will have an attracting fixed point near z and z will be contained in the basin of attraction. This basin is a Fatou-Bieberbach domain and does not intersect X since the automorphism fixes X. □

However, let us be accurate and establish the algebraic density for algebraic vector fields vanishing on X under an additional assumption.

4.3. **Convention.** *We suppose further in this section that the dimension of the Zariski tangent space T_xX is at most $n-1$ for every point $x \in X$.*

4.4. **Lemma.** *Lie algebra* $\mathrm{Lie}_{\mathrm{alg}}(Z)$ *contains* $\mathrm{AVF}_{I^2}(\mathbb{C}^n)$.

Proof. It suffices to show that for every point $o \in \mathbb{C}^n$ there exists a Zariski neighborhood V and a submodule M_V from $\mathrm{Lie}_{\mathrm{alg}}(\mathbb{C}^n) \cap \mathrm{AVF}_{I^2}(\mathbb{C}^n)$ such that its localization to V coincides with the localization of $\mathrm{AVF}_{I^2}(\mathbb{C}^n)$ to V. Indeed, because of quasi-compactness we can find a finite number of such open sets V_i that covers $\mathbb{C}^n$. Hence the coherent sheaves generated by $\mathrm{AVF}_{I^2}(\mathbb{C}^n)$ and $\sum_i M_{V_i}$ coincide locally which implies that they have the same global sections over affine varieties by Serre's theorem B. In fact, it suffices to show that the localization of M_V to V contains all fields from the localization of $\mathrm{AVF}_{I^2}(\mathbb{C}^n)$ to V that are proportional to some general constant vector field δ which is *our aim now.* By Theorem 4 it is also enough to consider $o \in X$ only and we are going to construct the desired neighborhood V of o as follows.

Claim. For any point $o \in X$, $l \geq \max(k+1, \dim T_oX)$, and a general linear projection $p : \mathbb{C}^n \to \mathcal{H} \simeq \mathbb{C}^l$ one can choose a projection $p_0 : \mathbb{C}^n \to \mathcal{H}_0 \simeq \mathbb{C}^{l-1}$ for which

(i) $p_0 = \varrho \circ p$ where $\varrho : \mathcal{H} \to \mathcal{H}_0$ is a general linear projection, and

(ii) there exists $h \in \mathbb{C}[\mathcal{H}_0] \simeq \mathbb{C}^{[l-1]} \simeq \varrho^*(\mathbb{C}^{[l-1]}) \subset \mathbb{C}^{[l]}$ such that h does not vanish at $p_0(o)$ and $p|_{X \setminus (h\circ p)^{-1}(0)} : X \setminus (h \circ p)^{-1}(0) \to p(X) \setminus h^{-1}(0)$ is an isomorphism.

Since p is general the condition on l implies that p is a local isomorphism in a neighborhood of o and, furthermore, since ϱ is also general then by Bertini's theorem

$p_0^{-1}(p_0(o))$ contains only smooth points of X except, may be, for o, i.e. p is a local isomorphism in a neighborhood of each of these points which implies the Claim.

From now on let $l = n-1$. Choose a general coordinate system $\bar{x} = (x_1, \ldots, x_n)$ on $\mathbb{C}^n$ such that $p(\bar{x}) = (x_2, \ldots, x_n)$ and $p_0(\bar{x}) = (x_3, \ldots, x_n)$, i.e. $h = h(x_3, \ldots, x_n)$. Set $V = \mathbb{C}^n \setminus h^{-1}(0)$.

Since $p(X) \cap V \simeq X \cap V$ we have $x_1 = r/h^s$ where r is a polynomial in $x_2, \ldots, x_n$ and $s \geq 0$. Set $\nu_i = \partial/\partial x_i$ for $i \neq 2$, and $\nu_2 = h^s \partial/\partial x_2 + (\partial r/\partial x_2)\partial/\partial x_1$. Then each ν_i is a locally nilpotent derivation and $\operatorname{Ker} \nu_1$ contains the defining ideal I_p of $p(X)$ in $\mathbb{C}[\mathcal{H}] \simeq \mathbb{C}^{[n-1]} \simeq p^*(\mathbb{C}^{[n-1]}) \subset \mathbb{C}^{[n]}$. Furthermore, for $\xi = h^s x_1 - r$ we have $\xi \in \operatorname{Ker} \nu_2$, and ξ (resp. x_2) is of degree 1 with respect to ν_1 (resp. ν_2). This implies that for $f, g \in I_p$ the vector fields that appear in the Lie brackets below are globally integrable and vanish on X:

$$[f\nu_1, \xi g\nu_1] = h^s fg\nu_1, \ [\xi\nu_2, x_2\xi\nu_1] - [x_2\xi\nu_2, \xi\nu_1] = h^s\xi^2\nu_1,$$
$$\text{and } [\xi\nu_2, x_2 f\nu_1] - [x_2\xi\nu_2, f\nu_1] = h^s\xi f\nu_1.$$

The defining ideal of $X \cap V$ is generated by ξ and elements of I_p. Since h is invertible on V from the formulas before we see that the localization of $\operatorname{Lie}_{\text{alg}}(Z)$ to V contains the localization of $\operatorname{AVF}_{I^2}(\mathbb{C}^n)\nu_1$. Since ν_1 is a general constant field we have the desired conclusion.

□

Theorem 6. *Let X be a closed algebraic subset of $\mathbb{C}^n$ of codimension at least 2 such that the Zariski tangent space T_xX has dimension at most $n-1$ for any point $x \in X$. Then $\mathbb{C}^n \setminus X$ has the algebraic density property.*

Proof. Similarly to the proof of Lemma 4.4, it suffices to show that for every point $o \in \mathbb{C}^n$ there exists a Zariski neighborhood V and a submodule from $\operatorname{Lie}_{\text{alg}}(Z) = \operatorname{Lie}_{\text{alg}}(\mathbb{C}^n) \cap \operatorname{AVF}_I(\mathbb{C}^n)$ such that its localization M to V coincides with localization of $\operatorname{AVF}_I(\mathbb{C}^n)$ to V. By Theorem 4 it is enough to consider $o \in X$ and, furthermore, it suffices to show that this localization M contains all elements of $\operatorname{AVF}_I(\mathbb{C}^n)$ proportional to some general constant vector field.

Let ν_i, p, I_p, and ξ have the same meaning as in the proof of Lemma 4.4. Choose ν_1 as this constant vector field. Since I is generated by ξ and I_p one needs to show that all fields of the form $\mu = (\xi g_0 + \sum g_i f_i)\nu_1$ are contained in M where g_0, g_i are regular on V and $f_i \in I_p$. Since p yields an isomorphism between $p(X) \cap V$ and $X \cap V$ there are functions e_0, e_i that do not depend on x_1 and such that $e_0|_X = g_0|_X$ and $e_i|_X = g_i|_X$. Then $\mu = (\xi e_0 + \sum e_i f_i)\nu_1 + a\nu_1$ where a belongs to the localization of I^2 to V (e.g. $a = \xi(g_0 - e_0) + \sum(g_i - e_i)f_i$). Since the first summand in the last formula for μ is globally integrable we have the desired conclusion from Lemma 4.4.

□

4.5. **Remark.** (1) The authors believe that the condition $\dim T_x X \leq n-1$ in Theorem 6 is essential. As a potential counterexample one may try to take X equal to polynomial curve in $\mathbb{C}^3$ with one singular point whose Zariski tangent space is 3-dimensional.

(2) In view of Theorem 6 the assumptions of Theorem 5 can be weakened in case of $\dim T_x X \leq n-1$ to the following extend: The assumption $\Phi_t(\Omega) \cap X = \varnothing$ can be replaced by Φ_t fixes $\Phi_t(\Omega) \cap X$ for all t.

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