Zariski pairs of index 19 and the Mordell-Weil groups of extremal elliptic K3 surfaces

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Introduction

Let $\varphi : \mathcal{E} \to \mathbf{P}^1$ be a semi-stable elliptic K3 surface with a section s_0; φ is called "*extremal*" if φ has exactly 6 singular fibers $I_1,...,I_6$. In their paper [MP2], Miranda and Persson give a complete list for the configurations of I_n fibers for semi-stable elliptic K3 surfaces; and show that there are 112 extremal cases. In [MP3], they go on to study the Mordell-Weil group, $MW(\mathcal{E})$, for the 112 cases, and show that $MW(\mathcal{E})$ is determined by the configurations of I_n fibers, namely, a six-tuple $[n_1, ..., n_6]$ for 95 of the 112 cases. There remain 17 cases with potential ambiguity. Miranda and Persson give all possible cases for $MW(\mathcal{E})$, but they do not give any single example of two extremal semi-stable K3 surfaces, $\mathcal{E}_1$ and $\mathcal{E}_2$, such that

(i) they have the same numerical data for the six-tuple $[n_1, ..., n_6]$, and

(ii) $MW(\mathcal{E}_1) \not\cong MW(\mathcal{E}_2)$.

Our first purpose of this note is to give two such examples of $\mathcal{E}_1$ and $\mathcal{E}_2$. We construct $\mathcal{E}_1$ and $\mathcal{E}_2$ in a rather classical way, namely, a method of double sextics. Consider sextics B_1 and B_2 as below, and we shall construct $\mathcal{E}_1$ and $\mathcal{E}_2$ as double coverings of $\mathbf{P}^2$ branched along B_1 and B_2, respectively.

Example 1.

(i) $B_i = C_1^{(i)} + C_2^{(i)}$, $(i, j = 1, 2)$ where $C_j^{(i)}$ are nodal cubics; $C_1^{(i)}$ and $C_2^{(i)}$ meet at only one point p_i, and

(ii) p_1 is an inflection point for both $C_1^{(1)}$ and $C_2^{(1)}$, while p_2 is neither inflection point of $C_1^{(2)}$ nor $C_2^{(2)}$.

Example 2.

(i) $B_i = C_1^{(i)} + C_2^{(i)} + C_3^{(i)}$, $(i = 1, 2,\ j = 1, 2, 3)$ where $C_j^{(i)}$ are curves of degree j ($C_3^{(i)}$ are nodal cubics and $C_2^{(i)}$ are smooth conics). Let p_3^i be the nodal points. Then we have:

[1]Partially supported by CAICYT PB94-0291

- $C_3^{(i)}$ and $C_2^{(i)}$ meet at only one point $p_{23}^i \neq p_3^i$;
- $C_3^{(i)}$ and $C_1^{(i)}$ meet at only one point $p_{13}^i \neq p_3^i$;
- $C_1^{(i)}$ and $C_2^{(i)}$ meet transversally at two points.

It is clear that:
- These points are ordinary double points of B_i,
- p_{23}^i are not inflection points of $C_3^{(i)}$ and
- p_{13}^i are inflection points of $C_3^{(i)}$.

(ii) Let L_i be the tangent line of $C_3^{(i)}$ at p_{23}^i. It is easy to see from the group structure of $C_3^{(i)} \setminus \{p_3^i\}$ that the other intersection point q_3^i of L_i and $C_3^{(i)}$ is also an inflection point. Then B_1 is determined by $q_3^1 = p_{23}^1$ and B_2 is determined by $q_3^2 \neq p_{23}^2$.

We shall show that these two pairs of sextics do exist in §1. Let $f_i' : \mathcal{E}_i' \to \mathbf{P}^2$ $(i = 1, 2)$ be double coverings branched along B_i $(i = 1, 2)$, respectively.

Let $\mu_i : \mathcal{E}_i \to \mathcal{E}_i'$ $(i = 1, 2)$ be the canonical resolutions of $\mathcal{E}_i'$ $(i = 1, 2)$. We have a commutative diagram as follows:

$$\begin{array}{ccc} \mathcal{E}_i' & \stackrel{\mu_i}{\leftarrow} & \mathcal{E}_i \\ f_i' \downarrow & & \downarrow f_i \\ \mathbf{P}^2 & \stackrel{q_i}{\leftarrow} & \Sigma_i, \end{array}$$

where q_i is a succession of blowing-ups such that the induced double covering f_i is finite. Then we have the following:

Theorem 0.1a. *For Example 1, both $\mathcal{E}_1$ and $\mathcal{E}_2$ are K3 surfaces with extremal elliptic fibrations, $\varphi_i : \mathcal{E}_i \to \mathbf{P}^1$ $(i = 1, 2)$ such that*

(i) both φ_1 and φ_2 have singular fibers, I_{18}, I_2, $4I_1$, and

(ii) $MW(\mathcal{E}_1) \cong \mathbf{Z}/3\mathbf{Z}$; $MW(\mathcal{E}_2) \cong \{0\}$.

Theorem 0.1b. *For Example 2, both $\mathcal{E}_1$ and $\mathcal{E}_2$ are K3 surfaces with extremal elliptic fibrations, $\varphi_i : \mathcal{E}_i \to \mathbf{P}^1$ $(i = 1, 2)$ such that*

(i) both φ_1 and φ_2 have singular fibers, I_{12}, I_6, $2I_2$, $2I_1$, and

(ii) $MW(\mathcal{E}_1) \cong \mathbf{Z}/6\mathbf{Z}$; $MW(\mathcal{E}_2) \cong \mathbf{Z}/2\mathbf{Z}$.

Our second purpose is to consider a geometric application of Theorems 0.1a and b. In fact, we shall apply them to construct Galois coverings branched along B_1 and B_2 as follows:

Theorem 0.2. *Let (B_1, B_2) be a pair of sextics in either Example 1 or Example 2. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be elliptic K3 surfaces as above. Then there exists a Galois covering of $\mathbf{P}^2$ branched along B_i with the third symmetric group as its Galois group, if and only if $MW(\mathcal{E}_i)$ has a 3-torsion element.*

An immediate consequence of Theorem 0.2 is

Corollary 0.3.

$$\pi_1(\mathbf{P}^2 \setminus B_1) \not\cong \pi_1(\mathbf{P}^2 \setminus B_2)$$

Corollary 0.3 also follows from computation of the Alexander polynomial of B_i, $(i = 1, 2)$ as follows:

Theorem 0.4. *Let Δ_i be the Alexander polynomial of B_i, $i = 1, 2$. Then, both in Example 1 and in Example 2, $\Delta_1 = t^4 + t^2 + 1$ and $\Delta_2 = 1$.*

Note that in both examples, the pair (B_1, B_2) has the same combinatorics. Hence Theorem 0.3 shows that the pair (B_1, B_2) is a Zariski pair for each case (see [A] for definition of a Zariski pair).

For a sextic, C, with only simple singularities, we define the index $i(C)$, of C to be the sum of all the subindices of all its singularities x_n ($x_n \in \{a, d, e\}$) (See [P]). It is known that $0 \le i(C) \le 19$; and we call a sextic maximizing if $i(C) = 19$.

In Example 1 both of B_i $(i = 1, 2)$ have the set of singularities $2a_1 + a_{17}$; in Example 2, both of B_i $(i = 1, 2)$ have the set of singularities $3a_1 + a_5 + a_{11}$. Hence each pair of our examples is a Zariski pair of index 19, *i.e.*, a Zariski pair for maximizing sextics. In [A], [D], [T2], [T3] and [Z], we can find some examples for Zariski pairs, but there is no example of a Zariski pair for maximizing sextics. Hence our examples are essentially new. It is, however, still unknown whether a Zariski pair for irreducible maximizing sextics exists or not. It will be interesting to find such a example.

Remark. Corollary 0.3 also follows from direct computations for $\pi_1(\mathbf{P}^2 \setminus B_i)$ $(i = 1, 2)$ in the case of Example 1. In fact, the first author figures out that $\pi_1(\mathbf{P} \setminus B_1)$ is isomorphic to $\mathbf{Z} * \mathbf{Z}/3\mathbf{Z}$ ($*$ means free product) while $\pi_1(\mathbf{P} \setminus B_2)$ is isomorphic to $\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z}$ (Their computations are not easy, though). In this article, however, we pay an attention to give a geometric application of the Mordell-Weil group of an elliptic surface, which has been an arithmetic object so far. This makes our result more interesting.

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Notations and Conventions.

Throughout this article, the ground field will always be the complex number field $\mathbf{C}$.

$\mathbf{C}(X) :=$ the rational function field of X.

Let X be a normal variety, and let Y be a smooth variety. Let $\pi : X \to Y$ be a finite morphism from X to Y. We define the branch locus, $\Delta(X/Y)$, of f as follows:

$$\Delta(X/Y) = \{y \in Y \mid \sharp(\pi^{-1}(y)) < deg\pi\}.$$

An $\mathcal{S}_3$ covering always means a Galois covering having the third symmetric group, $\mathcal{S}_3$, as it Galois group.

Let $\pi : X \to Y$ be an $\mathcal{S}_3$ covering of Y. Morphisms, β_1 and β_2, and the variety $D(X/Y)$ always mean those defined in §3.

Let S be a finite double covering of a smooth projective surface Σ. The "*canonical resolution*" of S always means the resolution given by Horikawa in [H].

For singular fibers of an elliptic surface, we use the notation of Kodaira [K].

For a singular fiber of type I_b, we shall label its irreducible components in the same way as in [K], p. 566. Namely, letting F denote a singular fiber of type I_b over a point, we have

$$F = \Theta_0 + \Theta_1 + \cdots + \Theta_{b-1},$$

where $\Theta_0\Theta_1 = \Theta_1\Theta_2 = \cdots = \Theta_{b-1}\Theta_0 = 1$. In particular, Θ_0 denotes the irreducible component of F meeting s_0, where s_0 is a fixed section of the elliptic surface.

For the configuration of singular fibers, we shall use the same notations as those in [M-P2].

Let D_1, D_2 be divisors.

$D_1 \sim D_2$: linear equivalence of divisors.

$D_1 \approx D_2$: algebraic equivalence of divisors.

$D_1 \approx_{\mathbf{Q}} D_2$: $\mathbf{Q}$-algebraic equivalence of divisors.

For singularities of a plane curve, we shall use the same notation as in [P].

§1 Preliminaries

1. Two sextics B_1 and B_2

We shall give explicit examples for the pairs of B_1 and B_2 in Introduction.

Example 1. We shall start with B_1.

B_1: Consider a pencil of cubics generated by a nodal cubic and a triple line which is tangent at an inflection point of the given nodal cubic. The general element of this pencil is a smooth cubic and two generic elements intersect at a common inflection point p_1 with intersection number 9. We have exactly two nodal cubics $C_1^{(1)}$ and $C_2^{(1)}$ in such a pencil. We give explicit equations as follows:

$$C_1^{(1)} : Z(X^2 + Y^2) + X^3 = 0,$$

and

$$C_2^{(1)} : -\frac{27}{4}\{Z(X^2 + Y^2) + X^3\} + Z^3 = 0.$$

Here $[X : Y : Z]$ denotes a homogeneous coordinate of $\mathbf{P}^2$. Consider sextic curve defined by the equation

$$\{Z(X^2 + Y^2) + X^3\}\left\{-\frac{27}{4}\{Z(X^2 + Y^2) + X^3\} + Z^3\right\} = 0$$

Then we can easily check that this sextic gives an example for B_1.

B_2: Let C_1' and C_2' be cubics defined by the equations as follows:

$$C_1' : Y(XZ - Y^2) - X^3 = 0,$$

and

$$C_2' : Z(XZ - Y^2) - X^2Y = 0.$$

Consider a pencil of cubics $\Lambda = \{\lambda_0 C_1' + \lambda_1 C_2'\}_{[\lambda_0:\lambda_1]\in\mathbf{P}^1}$. The general element is a smooth cubic and two generic elements intersect at one point $p_2 = [0 : 0 : 1]$ with intersection number 9; p_2 is not an inflection point for any cubic in the pencil. Besides C_1', there are exactly three nodal cubics in the pencil, for $[\lambda_0 : \lambda_1] = [1 : 3], [1 : 3\mu], [1 : 3\mu^{-1}]$, where $\mu = exp(2\pi\sqrt{-1}/3)$.

Then, take two elements, $C_1^{(2)}$ and $C_2^{(2)}$, of Λ as follows:

$$C_1^{(2)} : Y(XZ - Y^2) - X^3 + 3\{Z(XZ - Y^2) - X^2Y\} = 0,$$

and

$$C_2^{(2)} : Y(XZ - Y^2) - X^3 + 3\mu\{Z(XZ - Y^2) - X^2Y\} = 0.$$

Then, straightforward calculation shows that (i) $C_1^{(2)}$ (resp. $C_2^{(2)}$) has a node at $[1 : 2 : 5]$ (resp. $[\mu^2 : 2 : 5\mu]$), and neither $C_1^{(2)}$ nor $C_2^{(2)}$ has an inflection point at p_2. Moreover, as Λ has only one base point p_2, p_2 is the only intersection point of $C_1^{(2)}$ and $C_2^{(2)}$. Now consider a sextic defined by the equation

$$\begin{aligned}\Big\{Y(XZ - Y^2) - X^3 \quad &+ \quad 3\{Z(XZ - Y^2) - X^2Y\}\Big\} \\ &\times \quad \Big\{Y(XZ - Y^2) - X^3 + 3\mu\{Z(XZ - Y^2) - X^2Y\}\Big\} = 0\end{aligned}$$

Then, this sextic gives an example for B_2.

Example 2. We can take curves with equations below:

$$C_1^{(1)} = C_1^{(2)} : Z = 0, \qquad C_3^{(1)} = C_3^{(2)} : Y^2Z - X^2(X + Z) = 0,$$

$$C_2^{(1)} : Y^2 + (2X + Z)(X + Z) = 0$$

and

$$C_2^{(2)} : 43X^2 - Y^2 - 64Z^2 + 6\sqrt{-3}XY - 48XZ - 48\sqrt{-3}YZ = 0.$$

Note that p_{23}^2 is $[-4 : -4\sqrt{-3} : 1]$.

2. Elliptic fibrations on $\mathcal{E}_i$ $(i = 1, 2)$

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be K3 surfaces defined in Introduction. We shall show that $\mathcal{E}_i$ $(i = 1, 2)$ have elliptic fibrations described in Theorems 0.1a and 0.1b.

We shall start with Example 1. Let B_1 and B_2 be the sextics in Example 1. Choose a node, q_i, on $C_1^{(i)}$. Then, for each q_i, lines through q_i induce an elliptic fibration on $\mathcal{E}_i$. Following to Persson [P], we call these fibrations the standard fibrations centered at q_i $(i = 1, 2)$, and denote them by $\varphi_i : \mathcal{E}_i \to \mathbf{P}^1$ $(i = 1, 2)$. Note that φ_i has a section, $s_0^{(i)}$, determined by $C_1^{(i)}$.

Lemma 1.1. *Let $l_{p_iq_i}$ be the line joining p_i and q_i. Then $l_{p_iq_i}$ meets B_i at two distinct point other than p_i and q_i.*

Proof. Suppose that $l_{p_iq_i}$ is tangent to B_i at a smooth point or passes through the other node. In the former case, φ_i has singular fibers, I_{19} and I_2; this is impossible by

[S1], Theorem 1.1 and rank$NS(\mathcal{E}_i) \leq 20$. In the latter, φ_i has a singular fiber, I_{20}; this is also impossible by the same reasons as the former.

Corollary 1.2. *For each i, φ_i has singular fibers, I_{18} and I_2.*

Proposition 1.3. *The singular fibers of φ_i ($i = 1, 2$) are I_{18}, I_2, $4I_1$.*

Proof. It is enough to show that φ_i has no singular fiber of type II, but this does not occur by Proposition 3.4 in [MP1].

Remark 1.4. For φ_1, Persson proved Proposition 1.3 in [P].

We shall next consider $\mathcal{E}_i$ ($i = 1, 2$) for Example 2. Choose the node $C_3^{(i)}$, and consider the standard fibration centered at the node.

Lemma 1.5. *Let $l_{p_3^i p_{23}^i}$ be the line joining p_3^i and p_{23}^i. Then $l_{p_3^i p_{23}^i}$ does not pass through $C_1^{(i)} \cap C_2^{(i)}$.*

Proof. Suppose that $l_{p_3^i p_{23}^i}$ passes through $C_1^{(i)} \cap C_2^{(i)}$. Then φ_i has singular fibers I_{14}, I_6, and I_2; this is again impossible by the same reason as in Proof of Lemma 1.1.

Thus the configuration of the singular fibers of φ_i is either I_{12}, I_6, $2I_2$, $2I_1$ or I_{12}, I_6, $2I_2$, II, but the latter case does not occur by Proposition 3.4 in [MP1].

§2 Proof of Theorems 0.1a and 0.1b.

Let $\varphi_i : \mathcal{E}_i \to \mathbf{P}^1$ ($i = 1, 2$) be the elliptic fibration as in §1. We shall denote the Mordell-Weil group for $\varphi_i : \mathcal{E}_i \to \mathbf{P}^1$ by $MW(\mathcal{E}_i)$. We shall first prove Theorem 0.1b.

In [MP3], Miranda and Persson shows that $MW(\mathcal{E}_1) \cong \mathbf{Z}/3\mathbf{Z}$. Hence it is enough to prove the following:

Lemma 2.1. $MW(\mathcal{E}_2) \cong \{0\}$.

Proof. Suppose that $MW(\mathcal{E}_2) \not\cong \{0\}$. Then, by Proposition 1.3 and Lemma 3 in [MP3], $MW(\mathcal{E}_2) \cong \mathbf{Z}/3\mathbf{Z}$. Let s be a 3-torsion section in $MW(\mathcal{E}_2)$. Let $\langle , \rangle$ denote Shioda's pairing in [S2]. Then we have $\langle s, s \rangle = 0$ by [S2]. Hence, by the formula (2.5) in [M3] and Theorem 1.3 [S2], we may assume that s hits Θ_6 at the I_{18} fiber and Θ_0 at the I_2 fiber. Then, by looking into the canonical resolution, we can show that $f_2'\mu_2(s)$

is an inflectional tangent line at p_2 for both $C_1^{(2)}$ and $C_2^{(2)}$. This contradicts to our assumption.

Next we shall prove Theorem 0.1b. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be the elliptic K3 surfaces for the sextics in Example 2. Both of them have singular fibers I_{12} and I_6; and we shall label irreducible components of them $\Theta_i^{(12)}$ and $\Theta_i^{(6)}$, i being the labeling introduced in Notations and Convention.

Lemma 2.2. *$MW(\mathcal{E}_1)$ has a 3-torsion.*

Proof. The tangent line at p_{23}^1 gives rise to two sections s^+ and s^-; s^+ is transformed to s^- by the covering transformation. By our construction, we may assume that s^+ meets $\Theta_4^{(12)}$ at the I_{12} fiber, $\Theta_2^{(6)}$ at the I_6 fiber, and Θ_0 at other singular fibers (we take the opposite orientation of the labeling for the irreducible component if necessary). Then we have $\langle s, s\rangle = 0$ by Theorem 8.6, [S2]. Hence, by Lemma 8.2 and Theorem 8.4 in [S2], this implies

$$\begin{aligned} s^+ \approx_{\mathbf{Q}} \; & s_0^{(1)} + 2F - \frac{2}{3}\sum_{i=1}^{4} i\Theta_i^{(12)} - \frac{1}{3}\sum_{i=5}^{11}(12-i)\Theta_i^{(12)} \\ & -\frac{2}{3}\sum_{i=1}^{3} i\Theta_i^{(6)} + \frac{1}{3}\sum_{i=4}^{5}(6-i)\Theta_i^{(6)} \end{aligned}$$

Hence, by Theorem 1.3 [S2], s^+ is a 3-torsion.

Lemma 2.3. *$MW(\mathcal{E}_2)$ has no 3-torsion.*

Proof. Suppose that $MW(\mathcal{E}_2)$ has a 3-torsion, and let s denote the 3-torsion section. Then, by the formula (2.5) in [M3], the equality $\langle s, s\rangle = 0$, and Theorem 1.3 [S2], we can deduce that s meets $\Theta_4^{(12)}$ at the I_n fiber and $\Theta_2^{(6)}$ at the I_6 fiber. Then $f_2'\mu_2(s)$ is a tangent line at p_{23}^2; and it passes through p_{13}^2. This contradicts to our choice for p_{13}^2. Hence $MW(\mathcal{E}_2)$ has no 3-torsion.

By [MP2] Proposition 4.4, both of $MW(\mathcal{E}_i)$ ($i = 1, 2$) have a 2-torsion. Hence we have Theorem 0.1b.

§3 S_3 coverings

In this section, we shall give a brief summary on $\mathcal{S}_3$ coverings. For details, see [T1]. We shall start with the definition of an $\mathcal{S}_3$ covering.

Definition 3.1. *Let Y be a smooth projective variety. A normal variety, X, with a finite morphism $\pi : X \to Y$ is called an $\mathcal{S}_3$ covering of Y if the rational function field, $\mathbf{C}(X)$, of X is a Galois extension of $\mathbf{C}(Y)$ having the third symmetric group, as its Galois group.*

Let $\mathcal{S}_3 = \langle \sigma, \tau | \sigma^2 = \tau^3 = (\sigma\tau)^2 = 1 \rangle$, Let $\mathbf{C}(X)^\tau$ be the invariant subfield of $\mathbf{C}(X)$ by τ. As $\mathbf{C}(X)^\tau$ is a quadratic extension of $\mathbf{C}(Y)$, the $\mathbf{C}(X)^\tau$-normalization of Y is a double covering. We denote it by $D(X/Y)$ and its covering morphism by β_1. Also, X is a cyclic triple covering of $D(X/Y)$, and β_2 denotes the covering morphism from X to $D(X/Y)$. By their definition, $\pi = \beta_1 \circ \beta_2$. With these notations, we have the following proposition:

Proposition 3.2. *Let $f : Z \to Y$ be a smooth finite double covering of a smooth projective variety Y. Let σ be the involution determined by the covering transformation of f. Let D_1, D_2, and D_3 be effective divisors on Z. Suppose that*

(a) D_1 is reduced and non-empty; D_1 and $\sigma^* D_1$ have no common component, and

(b) $D_1 + 3D_2 \sim \sigma^* D_1 + 3D_3$.

Then there exists an $\mathcal{S}_3$ covering, X, of Y such that (i) $D(X/Y) = Z$, and (ii) $D_1 + \sigma^* D_1$ is the branch locus of β_2.

For a proof, see [T1].

We also have the following proposition saying that the "inverse" of Proposition 3.2 holds.

Proposition 3.3. *Let $\pi : X \to Y$ be an $\mathcal{S}_3$ covering and let σ denote the involution on $D(X/Y)$ coming from the covering transformation of β_1. Suppose that $D(X/Y)$ is smooth. Then there exist three effective divisors D_1, D_2 and D_3 on $D(X/Y)$ such that*

(i) D_1 is reduced; D_1 and $\sigma^* D_1$ have no common component,

(ii) $D_1 + 3D_2 \sim \sigma^* D_1 + 3D_3$, and

(iii) $D_1 + \sigma^* D_1$ is the branch locus of β_2.

For a proof, see [T1].

Corollary 3.4. *Let $\pi : S \to \Sigma$ be an $\mathcal{S}_3$ covering of a smooth projective surface Σ, and let D be an irreducible component of $\beta_1(\Delta(S/D(S/\Sigma)))$. If we denote x by any intersection point of D and $\Delta(D(S/\Sigma)/\Sigma)$. Then the intersection multiplicity at x is ≥ 2.*

Proof. This is immediate from Proposition 3.3.

§4 Proof of Theorem 0.2

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be as before, and $f_i : \mathcal{E}_i \to \Sigma_i$ $(i = 1, 2)$ the induced double covering as in Introduction. Suppose that an $\mathcal{S}_3$ covering $\pi_i : S_i \to \mathbf{P}^2$ branched along B_i exists.

Claim. *$\beta_1^{(i)} : D(S_i/\mathbf{P}^2) \to \mathbf{P}^2$ is branched along B_i.*

Proof of Claim. Since $\deg \beta_1^{(i)} = 2$, $\deg \Delta(D(S_i/\mathbf{P}^2)/\mathbf{P}^2))$ is even. Hence Claim follows in the case of Example 1. In the case of Example 2, $\deg \Delta(D(S_i/\mathbf{P}^2)/\mathbf{P}^2)) = C_2^{(i)}$, $C_1^{(i)} \cup C_3^{(i)}$, or B_i; but the first two cases do not occur by Corollary 3.4.

Hence $\beta_1^{(i)} : D(S_i/\mathbf{P}^2) \to \mathbf{P}^2$ coincides with $\mathcal{E}_i' \to \mathbf{P}^2$. Thus we have the following commutative diagram:

$$\begin{array}{ccccc} \Sigma_i & \stackrel{f_i}{\leftarrow} & \mathcal{E}_i & \stackrel{g_i}{\leftarrow} & \tilde{S}_i \\ \downarrow q_i & & \downarrow \mu_i & & \downarrow \\ \mathbf{P}^2 & \stackrel{\beta_1^{(i)}}{\leftarrow} & D(S_i/\mathbf{P}^2) & \stackrel{\beta_2^{(i)}}{\leftarrow} & S_i, \end{array}$$

where $\tilde{S}_i$ denotes the $\mathbf{C}(S_i)$-normalization of $\mathcal{E}_i'$. As β_1 is branched along B_i, and the Galois group is $\mathcal{S}_3$, β_2 is branched at most at $\mathrm{Sing}(D(S_i/\mathbf{P}^2))$. Also, since the local fundamental group of an A_1 singularity is $\mathbf{Z}/2\mathbf{Z}$, $\beta_2^{(i)}$ is not branched at A_1 singularities of $\mathcal{E}_i'$. Hence $\Delta(\tilde{S}_i/\mathcal{E}_i)$ is contained in the exceptional set of the A_{17} singularity for Example 1 and those of the A_{11} and A_5 singularities. Therefore $\Delta(\tilde{S}_i/\Sigma_i)$ consists of irreducible components of the I_{18} fiber not meeting $s_0^{(i)}$ in the case of Example 1, while it consists of those of the I_{12} and I_6 fibers not meeting $s_0^{(i)}$ in the case of Example 2. Thus, by Proposition 3.3, we have the following proposition:

Proposition 4.1. *Suppose that an $\mathcal{S}_3$ covering $\pi_i : S_i \to \mathbf{P}^2$ branched along B_i exists. Then there are three effective divisors D_1, D_2 and D_3 on $\mathcal{E}_i$ enjoying the following properties:*

(i) D_1 is reduced; D_1 and $\sigma_i^* D_1$ have no common component, where σ_i denotes the

covering transformation determined by f_i.

(ii) Every irreducible component of D_1 satisfies the following:

For Example 1, it is an irreducible component of the exceptional set of the A_{17} singularity on $\mathcal{E}'_i$.

For Example 2, it is an irreducible component of the exceptional sets of the A_{11} and A_5 singularities on $\mathcal{E}'_i$.

(iii) $D_1 + 3D_2 \sim \sigma_1^* D_1 + 3D_3$.

(iv) $D_1 + \sigma_1^* D_1$ is the branch locus of g_i.

Conversely, by Proposition 3.1 and the observation as above, we have the following:

Proposition 4.2. *Let D_1, D_2 and D_3 be three effective divisors on $\mathcal{E}_i$ as follows:*

(i) D_1 is reduced; D_1 and $\sigma_i^* D_1$ have no common component, where σ_i denotes the covering transformation determined by f_i.

(ii) Every irreducible component of D_1 satisfies the following:

For Example 1, it is an irreducible component of the exceptional set of the A_{17} singularity on $\mathcal{E}'_i$.

For Example 2, it is an irreducible component of the exceptional sets of the A_{11} and A_5 singularities on $\mathcal{E}'_i$.

(iii) $D_1 + 3D_2 \sim \sigma_1^* D_1 + 3D_3$.

Then there exists an S_3 covering of $\mathbf{P}^2$ branched along B_i.

Now we shall go on to prove Theorem 0.2.

Proposition 4.3. *If there exists an S_3 covering of $\mathbf{P}^2$ branched along B_i, then $MW(\mathcal{E}_i)$ has a 3-torsion.*

Proof. Suppose that such a covering $\pi : S \to \mathbf{P}^2$ exists. Then, by Claim, $D(S/\mathbf{P}^2) = \mathcal{E}'_i$. Hence there exist three divisors, D_1, D_2 and D_3, satisfying the four conditions in Proposition 4.2. By the same argument in §4 Claim in [T1], we can show that $MW(\mathcal{E}_i)$ has a 3-torsion element.

Now we shall next prove the converse of Proposition 4.3, by which we have Theorem 0.2.

Proposition 4.4. *If $MW(\mathcal{E}_i)$ has a 3-torsion, there exists an S_3 covering of $\mathbf{P}^2$ branched along B_i.*

Proof. By Theorems 0.1a and b, there is no 3-torsion in $MW(\mathcal{E}_2)$. Hence we shall prove Proposition 4.3 when $i = 1$. By Proposition 4.2, it is enough to show that the three effective divisors D_1, D_2 and D_3 on $\mathcal{E}_1$ satisfying the conditions (i), (ii) and (iii) exist. We shall first consider the case of Example 1. Let s be the corresponding section of $\varphi_1 : \mathcal{E}_1 \to \mathbf{P}^1$. Then, by [S2] and [M3], we have

$$s \approx_{\mathbf{Q}} s_0^{(1)} + 2F - \sum_{i=1}^{17} \alpha_i \Theta_i,$$

where

$$\alpha_i = \begin{cases} \frac{2}{3}i & (1 \leq i \leq 6) \\ \frac{18-i}{3} & (7 \leq i \leq 17) \end{cases};$$

$s_0^{(1)}$ is the section as in §1, and F is a class of a fiber. As $\Theta_{18-i} = \sigma_1^* \Theta_i$ and $\mathcal{E}_1$ is simply connected, we can rewrite this relation as follows:

$$\begin{aligned} 3s \sim\ & 3s_0^{(1)} + 6F + (\Theta_1 + \sigma_1^*\Theta_2 + \Theta_4 + \sigma_1^*\Theta_5 + \Theta_7 + \sigma_1^*\Theta_8) \\ & -\sigma_1^*(\Theta_1 + \sigma_1^*\Theta_2 + \Theta_4 + \sigma_1^*\Theta_5 + \Theta_7 + \sigma_1^*\Theta_8) \\ & -3(\sigma_1^*\Theta_2 + 3\Theta_4 + 2\sigma_1^*\Theta_5 + 3\sigma_1^*\Theta_8 \\ & \Theta_1 + \Theta_2 + 2\Theta_3 + \sigma_1^*\Theta_3 + \sigma_1^*\Theta_4 \\ & +3\Theta_5 + 4\Theta_6 + 2\sigma_1^*\Theta_6 + 2\sigma_1^*\Theta_7 \\ & 4\Theta_7 + 3\Theta_8 + 3\Theta_9). \end{aligned}$$

Put

$$\begin{aligned} D_1 &= \Theta_1 + \sigma_1^*\Theta_2 + \Theta_4 + \sigma_1^*\Theta_5 + \Theta_7 + \sigma_1^*\Theta_8 \\ D_2 &= s_0^{(1)} + 2F \\ D_3 &= s + \sigma_1^*\Theta_2 + 3\Theta_4 + 2\sigma_1^*\Theta_5 + 3\sigma_1^*\Theta_8 \\ & \quad \Theta_1 + \Theta_2 + 2\Theta_3 + \sigma_1^*\Theta_3 + \sigma_1^*\Theta_4 \\ & \quad +3\Theta_5 + 4\Theta_6 + 2\sigma_1^*\Theta_6 + 2\sigma_1^*\Theta_7 \\ & \quad +4\Theta_7 + 3\Theta_8 + 3\Theta_9. \end{aligned}$$

Then these three divisors satisfy the desired three conditions.

In the case of Example 2, we can similarly show that there exist three divisors with the desired properties. In fact, they are as follows:

$$\begin{aligned}
D_1 &= \Theta_1^{(12)} + \sigma^*\Theta_2^{(12)} + \Theta_4^{(12)} + \sigma^*\Theta_5^{(12)} + \Theta_1^{(6)} + \sigma^*\Theta_2^{(6)} \\
D_2 &= s_0^{(1)} + 2F \\
D_3 &= s + \Theta_1^{(12)} + \Theta_2^{(12)} + 2\Theta_3^{(12)} + 3\Theta_4^{(12)} \\
&\quad + 2\Theta_5^{(12)} + \sigma^*\Theta_2^{(12)} + \sigma^*\Theta_3^{(12)} + \sigma^*\Theta_4^{(12)} + 2\sigma^*\Theta_5^{(12)} \\
&\quad + \Theta_1^{(6)} + \Theta_2^{(6)} + \Theta_3^{(6)} + \sigma^*\Theta_2^{(6)}.
\end{aligned}$$

Note that $\Theta_{12-i}^{(12)} = \sigma^*\Theta_i^{(12)}$ and $\Theta_{6-i}^{(6)} = \sigma^*\Theta_i^{(6)}$; and $\Theta_i^{(12)}$ $(i \neq 6)$ and Θ_i $(i \neq 3)$ are irreducible components of the exceptional sets of the A_{11} and A_5 singularities, respectively.

§5 Proof of Theorem 0.4

We refer to Libgober [L] and Degtyarev [D] for the original definition and properties of the Alexander polynomial of a plane projective curve. As it is shown in these papers, the Alexander polynomial of a curve C of degree d can be computed as follows:

Let $f(x,y,z) = 0$ be the equation of C. Let $\tilde{S}$ be the hypersurface in $\mathbf{P}^3$ defined by the equation $f(x,y,z) = t^d$ and let $p : \tilde{S} \to \mathbf{P}^2$ be the restriction to $\tilde{S}$ of the projection in x, y, z. This map is a d-fold cyclic covering outside C; the monodromy of this covering is generated by the map $\tilde{\tau} : \tilde{S} \to \tilde{S}$ defined by multiplying the coordinate t by $e^{2i\pi/d}$.

Let $\sigma : S \to \tilde{S}$ be a resolution of $\tilde{S}$. Then we have in a natural way a smooth cyclic covering $\tau : S \to \mathbf{P}^2$ with a monodromy transformation $\tau : S \to S$ (S is well-defined up to birational transformations).

Definition 5.1. *The Alexander polynomial Δ_C of C is the characteristic polynomial of the linear automorphism $\tau^* : H^1(S;\mathbf{C}) \to H^1(S;\mathbf{C})$.*

We find in [A] an explicit method to compute the eigenspaces of τ^* in terms of the position of the singularities of C with respect to curves of given degree. We recall it in order to compute the Alexander polynomials of the examples with the notations introduced above. It is clear that all eigenvalues of τ^* are d^{th}-roots of unity. Let us fix $k \in \{0, 1, \ldots, d-1\}$. Let us denote H_k the τ^*-eigenspace of $H^1(S;\mathbf{C})$ for $e^{2ik\pi/d}$. Then,

$$\dim H_k = \dim \operatorname{Coker}(\alpha_k) + \dim \operatorname{Coker}(\alpha_{d-k})$$

where

$$\alpha_k : H^0(\mathbf{P}^2, \mathcal{O}(k-3)) \to \sum_{p \in \operatorname{Sing}(C)} \mathcal{O}_{\mathbf{P},p}/\mathcal{J}_{p,k,d}$$

and α_k is defined as follows:

Choose a line which does not intersect Sing(C) and suppose that it is $Z = 0$. Then $H^0(\mathbf{P}^2; \mathcal{O}(k-3))$ is identified with the polynomials in X, Y of degree $\leq d$. For such a polynomial take the classes modulo the given ideals of their germs at p, $p \in$ Sing(C). We recall the definition of $\mathcal{J}_{p,k,d}$:

Let $\sigma : M \to \mathbf{P}^2$ be an embedded resolution of C at p. We have

$$\sigma^*(C) = \tilde{C} + \sum_{i=1}^{r} m_i E_i,$$

where $\tilde{C}$ is the strict transform of C and $E_1, \ldots, E_r$ are the exceptional components of σ. Let w be a 2-meromorphic form on $\mathbf{P}_2$ which is holomorphic and non-vanishing near p. Let κ_i be the multiplicity of E_i in $\sigma^*\omega$. It means that

$$\sigma^*\omega = K + \sum_{i=1}^{r} \kappa_i E_i,$$

where K is a divisor in M whose support is disjoint from $\sigma^{-1}(p)$. Then $\mathcal{J}_{p,k,d}$ is the ideal of germs $h \in \mathcal{O}_{\mathbf{P},p}$ such that for each $i = 1, \ldots, r$ the multiplicity of E_i in the (local) divisor $\sigma^*(h)$ is greater or equal than

$$-\kappa_i + \left\lceil \frac{km_i}{d} \right\rceil.$$

Examples of $\mathcal{J}_{p,k,6}$. We are going how to compute these ideals for some double points with given local analytic coordinates. We fix $d = 6$. Let $p \in C$ an a_{2r-1}-singular point. Let us take analytic coordinates such that the equation of C near p is $f(x, y) = x^2 - y^{2r} = 0$.

Let us take σ as above. Then

$$\sigma^*(C) = \tilde{C} + 2\sum_{i=1}^{r} iE_i, \quad \kappa_i = i, \quad i = 1, \ldots, r.$$

One can choose local coordinates (x_i, y_i) near a smooth point of E_i such that $y_i = 0$ is the local equation of E_i and $\sigma(x_i, y_i) = (x_i y_i^i, y_i)$. These local equations allow us to compute the multiplicity of E_i in the divisor $\sigma^*(h)$, $h \in \mathcal{O}_{\mathbf{P},p}$. Let $h \in \mathbf{C}\{x, y\}$ and

$k = 0, 1, \ldots, 5$; then $h \in \mathcal{J}_{p,k,6}$ if and only if for all $i = 1, \ldots, r$ the germ $h(x_i y_i^i, y_i)$ is in the ideal of $\mathbf{C}\{x_i, y_i\}$ generated by $y_i^{n_{i,k}}$, where:

$$n_{i,k} := \left\lceil \frac{2ik}{6} \right\rceil - i = \left\lceil \frac{ik}{3} \right\rceil - i.$$

The only cases where the cokernels may be non trivial are $k = 4, 5$. We will drop the other cases.

Case 1. p is an ordinary double point, i.e., $r = 1$.

The powers $n_{1,k}$ obtained are equal to 0. It follows that $\mathcal{O}_{\mathbf{P},p} = \mathcal{J}_{p,k,6}$ and the quotient is trivial, for $k = 4, 5$.

Case 2. p is of type a_5, i.e., $r = 3$ (this case has been made in [A]).

For $k = 4$, $\mathcal{J}_{p,4,6}$ is the maximal ideal $\mathcal{M}$ of $\mathcal{O}_{\mathbf{P},p}$. For $k = 5$, $\mathcal{J}_{p,5,6}$ is the ideal generated by $(x, y^2) = (x) + \mathcal{M}^2$.

Case 3. p is of type a_{11}, i.e., $r = 6$.

For $k = 4$, $\mathcal{J}_{p,4,6}$ is the ideal generated by $(x, y^2) = (x) + \mathcal{M}^2$. For $k = 5$, $\mathcal{J}_{p,5,6}$ is the ideal generated by $(x, y^4) = (x) + \mathcal{M}^4$.

Case 4. p is of type a_{17}, i.e., $r = 9$.

This case has been made in [A]. For $k = 4$, $\mathcal{J}_{p,4,6}$ is the ideal generated by $(x, y^3) = (x) + \mathcal{M}^3$. For $k = 5$, $\mathcal{J}_{p,5,6}$ is the ideal generated by $(x, y^6) = (x) + \mathcal{M}^6$.

We must compute the cokernel of α_k, for $k = 4, 5$. It is clear that it is enough to compute the kernel and apply linear algebra. For each $p \in \mathrm{Sing}(C)$, let us denote

$$\alpha_{p,k} : H^0(\mathbf{P}^2, \mathcal{O}(k-3)) \to \mathcal{O}_{\mathbf{P},p}/\mathcal{J}_{p,k,d}$$

the p-coordinate of α_k. It is easily seen that

$$\mathrm{Ker}\alpha_k = \cap_{p \in \mathrm{Sing}(C)} \mathrm{Ker}\alpha_{p,k}.$$

Let $\mathbf{P}_{p,k}$ the projective space of $\mathrm{Ker}\alpha_{p,k}$. It is in a natural way the space of curves of degree $k-3$ which verify the conditions imposed by the ideal $\mathcal{J}_{p,k,6}$. Then

$$\mathbf{P}_k := \cap_{p \in \mathrm{Sing}(C)} \mathbf{P}_{p,k}.$$

is the projective space of $\mathrm{Ker}\alpha_k$.

Proposition 5.2. *Let C be a reduced curve of degree 6, let $p \in \mathrm{Sing}(C)$ a double point of type a_{2r-1}. If $r > 1$, denote by L (resp. Q) the tangent line to C at p (resp. the conic having maximal contact). Then*

(i) *If $r = 1$, $\mathbf{P}_{p,k}$ is the space of curves of degree $k-3$.*

(ii) *If $r = 3$ and the intersection number of C and L at p is greater than 4, then $Q = 2L$, $\mathbf{P}_{p,4}$ is the space of lines passing through p and $\mathbf{P}_{p,5}$ is the space of conics tangent to C at p.*

(iii) *If $r = 6$ and the intersection number of C and L at p is equal to 4, then $Q \neq 2L$, $\mathbf{P}_{p,4} = \{L\}$ and $\mathbf{P}_{p,5}$ is the pencil of conics generated by Q and $2L$.*

(iv) *If $r = 9$ and the intersection number of C and L at p is equal to 6, then $Q = 2L$, $\mathbf{P}_{p,4} = \{L\}$ and $\mathbf{P}_{p,5} = \{2L\}$.*

(v) *If $r = 9$ and the intersection number of C and L at p is equal to 4, then $Q \neq 2L$, $\mathbf{P}_{p,4}$ and $\mathbf{P}_{p,5}$ are empty sets.*

Proof. The result in (i) is trivial.

Consider now (ii). We begin with $k = 4$. We have seen that $\mathcal{J}_{p,5,6}$ is the maximal ideal in the local ring of p in $\mathbf{P}^2$. Then the quotient is $\mathbf{C}$ and $\alpha_{p,4}$ is the evaluation of the polynomials at p. Then, $\mathbf{P}_{p,4}$ is the space of lines passing by p.

For $k = 5$, it follows from the next observation: Let x', y' be another analytical system of coordinates, such that x' is also tangent to C at p. Then $\mathcal{J}_{p,5,6} = (x) + \mathcal{M}^2$. Let us suppose that $p = [0:0:1]$ and L is $X = 0$. We apply this property to X, Y and we get $\mathcal{J}_{p,5,6} = (X, Y^2)$.

Let us consider $\alpha_{p,5}$. We can choose $1, X, Y, X^2, XY, Y^2$ as a base in the source and the classes of 1 and Y as a base in the target. It follows that X, X^2, XY generate the kernel, and we get the result.

Consider now (iii). We proceed for $k = 4$ as in the case $k = 5$ for (ii).

Let us consider now $k = 5$. We fix p and L as above and let us suppose that the maximal contact conic Q is $X - Y^2 = 0$ (in affine coordinates X, Y); it is possible because it cannot be $2L$. We recall that $(Q \cdot C)_p \geq 10$ because maximal contact implies that Q passes through (at least) five infinitely near points of C at p, which are double points, and we apply Noether's formula for the intersection number. Let us take also analytical coordinates x, y such that the local equation of C at p is $x^2 - y^{12} = 0$. We deduce that:

$$x = (X - Y^2)u_1(X, Y) + \alpha Y^5 v_1(Y), \quad u_1(0,0) \neq 0, \quad v_1(0) \neq 0 \quad \alpha \in \mathbf{C}$$

and

$$y = Yu_2(X, Y) + \beta X^b v_2(X), \quad u_2(0,0) \neq 0, \quad v_2(0) \neq 0, \quad \beta \in \mathbf{C}, \quad b \geq 1.$$

It is easily seen that such a coordinate change exists if we replace the term $\alpha Y^5 \ldots$ by $\alpha Y^a \ldots$, $a \geq 1$. The fact that L is tangent to C implies that $a \geq 2$ (replace $X = 0$ in the series $x^2 - y^{12} = 0$; the order should be equal to four). The fact about Q implies that $a \geq 10$ (replace $X = Y^2$ in the series $x^2 - y^{12} = 0$; the order should ≥ 10).

Then, $(x, y^4) = (x) + \mathcal{M}^4 = (X - Y^2) + \mathcal{M}^4 = (X - Y^2, Y^4)$. Let us take now $\alpha_{p,5}$. It is easily seen that the kernel is generated by $X - Y^2$ and X^2 and we get the result.

We prove in the same way the results of (iv) and (v) for a_{17} (see [A] for details). Q.E.D.

Proof of Theorem 0.4, Example 1. The singular points of B_1 and B_2 are $a_{17}+2a_1$. Let us denote p_i the a_{17}-singular point. It is clear that $\mathbf{P}_{p_i,k} = \mathbf{P}_k(B_i)$, $k = 4, 5$. Applying Proposition 5.2(iv), we get that $\mathbf{P}_4(B_1)$ and $\mathbf{P}_5(B_1)$ have exactly one point each one. Then,

$$\dim \operatorname{Ker}\alpha_4(B_1) = \dim \operatorname{Ker}\alpha_5(B_1) = 1.$$

By Proposition 5.2(v), we get that $\mathbf{P}_4(B_2) = \mathbf{P}_5(B_1) = \emptyset$. Then,

$$\dim \operatorname{Ker}\alpha_4(B_2) = \dim \operatorname{Ker}\alpha_5(B_2) = 0.$$

In both cases, α_4 (resp. α_5) is a linear map between spaces of dimension 3 (resp. 6). Then the dimension of the cokernel equals the dimension of the kernel.

We get $\Delta_1 = t^4 + t^2 + 1$ and $\Delta_2 = 1$. Q.E.D.

Proof of Theorem 0.4, Example 2.

The singular points of B_1 and B_2 are $a_{11} + a_5 + 3a_1$. If we take again the notations in the introduction p_{13}^i is the a_5-point of B_i and p_{23}^i is the a_{11}-point. It is clear that

$$\mathbf{P}_k(B_i) = \mathbf{P}_{p_{13}^i,k} \cap \mathbf{P}_{p_{23}^i,k}, \quad k = 4, 5.$$

We begin with $k = 4$; by Proposition 5.2(ii) and (iii), $\mathbf{P}_4(B_i)$ is the space of lines tangent to B_i at p_{23}^i and passing through p_{13}^i. By the definition of the pair (B_1, B_2), we get that $\mathbf{P}_4(B_1)$ has exactly one point and $\mathbf{P}_4(B_2) = \emptyset$. Then,

$$\dim \operatorname{Ker}\alpha_4(B_1) = 1 \quad \dim \operatorname{Ker}\alpha_4(B_2) = 0.$$

We apply again Proposition 5.2(ii) and (iii) for $k = 5$. Take the equations for B_1 and B_2 defined in §1. Let $L^{(i)}$ be the tangent line to B_i at p_{23}^i; their equations are:

$$L^{(1)} : X + Z = 0 \qquad L^{(2)} : 5X + \sqrt{-3}Y + 8Z = 0.$$

It is clear that the conic with maximal contact is $C_2^{(i)}$. The tangent line to B_i at $p_{13}^{(i)}$ is $C_1^{(i)}$, whose equation is $Z = 0$ in both cases.

Then $\mathbf{P}_5(B_i)$ is the set of conics in the pencil generated by $C_2^{(i)}$ and $2L^{(i)}$ which are tangent to $C_1^{(i)}$ at p_{13}^i. For $i = 1$, the conic in the pencil passing through p_{13}^1 is exactly $2L^{(1)}$ which is virtually tangent to $C_1^{(i)}$. Then $\mathbf{P}_5(B_1)$ has exactly one point.

For $i = 2$, the conic in the pencil passing through p_{13}^2 has equation

$$13X^2 + \sqrt{-3}XY - 28XZ - 20\sqrt{-3}YZ - 32Z^2 = 0.$$

This conic is not tangent to $C_1^{(2)}$. We find $\mathbf{P}_5(B_1) = \emptyset$. Then,

$$\dim \operatorname{Ker}\alpha_5(B_1) = 1, \quad \dim \operatorname{Ker}\alpha_5(B_2) = 0.$$

As in Example 1, α_4 (resp. α_5) is a linear map between spaces of dimension 3 (resp. 6). Then the dimension of the cokernel equals the dimension of the kernel.

We get $\Delta_1 = t^4 + t^2 + 1$ and $\Delta_2 = 1$. Q.E.D.

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