A Note on Smooth Toral Reductions of Spheres

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In previous work [BGMR1, BGMR2] the authors and E. Rees, using 3-Sasakian toral reductions of the standard $(4n-1)$-dimensional sphere, constructed compact 3-Sasakian 7-manifolds with arbitrary second Betti number. These were the first examples of smooth compact Einstein manifolds of positive scalar curvature with arbitrary second Betti number. This toral reduction procedure actually constructs more general families of 3-Sasakian $(4n-1)$-dimensional orbifolds $\mathcal{S}(\Omega)$. However, to obtain $(4n-1)$-manifolds with $b_2 > 1$ one should answer two separate questions: (1) What are the orbifold Betti numbers of $\mathcal{S}(\Omega)$? and (2) When is $\mathcal{S}(\Omega)$ smooth?

In [BGMR1] sufficient conditions for guaranteeing smoothness for general toral reductions were given, and in Proposition 1.4 below we show that these conditions are essentially also necessary. The Betti number computations in [BGMR1] were carried out only for dimension seven. In this case the we gave a $2k$-parameter family of smooth 3-Sasakian 7-manifolds with second Betti number k. Contemporaneously, R. Bielawski [Bi1] computed Betti numbers of all these toral examples; more precisely, he showed that

$$b_{2i} = \binom{k+i-1}{i}$$

for $i \leq n-k-1$ for any 3-Sasakian orbifold obtained by 3-Sasakian reduction of S^{4n-1} by a k-dimensional torus. Since the odd Betti numbers b_{2i+1} of any 3-Sasakian manifold are known to vanish [GS] for $i \leq n-k-1$ (for orbifolds this follows from results of [BG]), Bielawski's results and Poincaré duality would determine the rational homology of any 3-Sasakian manifold obtained from toral reduction. However, there remains the question as to when these higher dimensional orbifolds can be smooth manifolds. In this note we give mod 2 obstructions to smoothness which show, somewhat surprisingly, that if the reduced 3-Sasakian orbifold $\mathcal{S}$ is a smooth manifold of dimension greater than 7 then $b_2 \leq 4$, and if the dimension of the reduced 3-Sasakian manifold is greater than 15 then $b_2 \leq 1$. That is, we prove

Theorem A: *Let $\mathcal{S}$ be a 3-Sasakian manifold obtained by 3-Sasakian reduction of S^{4n-1} by a torus T^k, and assume that $\mathcal{S}$ is not a sphere nor a smooth quotient of a sphere by a finite group. If $k>1$ then dim $\mathcal{S} = 7, 11, 15$. In addition, if $k>4$ then dim $\mathcal{S} = 7$.*

After proving Theorem A we show that these bounds are sharp in that there exist infinite families of manifold examples in all cases not eliminated by Theorem A. This also shows that any additional mod p obstructions to smoothness for p an odd prime give no further constraints on the Betti numbers.

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§1. Proof of Theorem A

Recall from [BGMR1] that any k-torus action on S^{4n-1} that preserves the 3-Sasakian structure on the sphere is given by a quaternionic representation of the k-torus T^k on a quaternionic vector space $\mathbb{H}^n$. This action can be described by left multiplication on a quaternionic column vector $\mathbf{u}=(u_1,\cdots,u_n)^t$ by a diagonal matrix $h_\Omega(\tau_1,..,\tau_k)\in T^n\subset Sp(n)$

1.1
$$h_\Omega(\tau_1,..,\tau_k)=\begin{pmatrix}\prod_{i=1}^k\tau_i^{a_1^i} & \dots & 0\\ \vdots & \ddots & \vdots\\ 0 & \dots & \prod_{i=1}^k\tau_i^{a_n^i}\end{pmatrix},$$

where $(\tau_1,..,\tau_k)\in S^1\times\cdots\times S^1=T^k$ are the complex coordinates on T^k, and $a_j^i\in\mathbb{Z}$. This representation gives rise to the integral weight matrix

1.2
$$\Omega=\begin{pmatrix}a_1^1 & a_2^1 & \dots & a_k^1 & \dots & a_n^1\\ a_1^2 & a_2^2 & \dots & a_k^2 & \dots & a_n^2\\ \vdots & \vdots & \ddots & \vdots & \dots & \vdots\\ a_1^k & a_2^k & \dots & a_k^k & \dots & a_n^k\end{pmatrix}.$$

The 3-Sasakian quotient space $\mathcal{S}(\Omega)$ is obtained from a moment map $\mu_\Omega: S^{4n-1}\longrightarrow\mathfrak{t}_k\otimes\mathbb{R}^3$ by defining $\mathcal{S}(\Omega)=\mu^{-1}(0)/T^k$. For the explicit form of μ_Ω we refer to [BGMR1]. It is a quadratic form on the quaternionic vector space $\mathbb{H}^n$ and this fact makes its analysis accessible.

Two weight matrices $\Omega,\Omega'\in\mathcal{M}_{k\times n}(\mathbb{Z})$ are equivalent if there are $A\in GL(k,\mathbb{Z})$ and $w\in\mathcal{W}(Sp(n))$ such that $\Omega'=A\Omega w$. Equivalent weight matrices give rise to isomorphic 3-Sasakian quotients $\mu_{\Omega'}^{-1}(0)/T^k(\Omega')\simeq\mu_\Omega^{-1}(0)/T^k(\Omega)$. The question of whether or not the converse is true is more subtle and will be addressed elsewhere. The conditions on Ω that determine the behavior of the quotient are given by the proposition below. We say that $\Omega\in\mathcal{M}_{k\times n}(\mathbb{Z})$ is *non-degenerate* if all its k by k minor determinants are non-vanishing; i.e., if for all sequences $1\le\alpha_1<\cdots<\alpha_k\le n$,

1.3
$$\Delta_{\alpha_1\ldots\alpha_k}=\begin{vmatrix}a_{\alpha_1}^1 & \dots & a_{\alpha_k}^1\\ \vdots & & \vdots\\ a_{\alpha_1}^k & \dots & a_{\alpha_k}^k\end{vmatrix}\neq 0.$$

Otherwise, Ω is called *degenerate*. Since we deal with an effective action of T^k, we can and shall assume that Ω has rank k.

Proposition 1.4: *Let $\mathcal{S}(\Omega) = \mu_\Omega^{-1}(0)/T^k(\Omega)$ be a 3-Sasakian quotient space obtained by 3-Sasakian reduction [BGM1] from the standard sphere S^{4n-1} by a torus T^k. Then*

(i) if Ω is non-degenerate, $\mathcal{S}(\Omega)$ is an orbifold.

(ii) If Ω is degenerate, then either $\mathcal{S}(\Omega)$ is singular or it is a manifold obtained by reduction of a lower dimensional sphere $S^{4n-4r-1}$ by a torus $T^{k-r}(\Omega')$ or a finite quotient of such, where $1 \leq r \leq k$ and Ω' is non-degenerate. (When $r = k$ the quotient is the sphere $S^{4n-4k-1}$).

(iii) Assuming that Ω is non-degenerate $\mathcal{S}(\Omega)$ is a smooth manifold if and only if

$$\gcd(\Delta_{\alpha_2\cdots\alpha_{k+1}}, \ldots, \Delta_{\alpha_1\cdots\hat{\alpha}_s\cdots\alpha_{k+1}}, \ldots, \Delta_{\alpha_1\cdots\alpha_k}) = g$$

for all sequences $1 \leq \alpha_1 < \cdots < \alpha_s < \cdots < \alpha_{k+1} \leq n$, where g denotes the k^{th} determinantal divisor of Ω.

Before giving the proof of this proposition, we recall some terminology from [BGM1]. An $\Omega \in \mathcal{M}_{k\times n}(\mathbb{Z})$ that satisfies the two conditions of Proposition 1.3 was called *admissible* in [BGMR1]. Recall also that if $g > 1$ then the T^k action is not effective, and there is a normal subgroup Γ of order g such that the factor group $T^k/\Gamma \simeq T^k$ acts effectively. Thus, we may assume without loss of generality that $g = 1$; i.e., Ω is in *reduced* form. The subset of reduced admissible matrices in $\mathcal{M}_{k\times n}(\mathbb{Z})$ is denoted by $\mathcal{A}_{k\times n}(\mathbb{Z})$. It is invariant under the action of $GL(k,\mathbb{Z}) \times \mathcal{W}(Sp(n))$. Furthermore, the unordered set $\{|\Delta_s|\}$ of absolute values of all k by k minor determinants is an invariant of $\mathcal{A}_{k\times n}(\mathbb{Z})$.

Proof of Proposition 1.4: (i) and the if part of (iii) are proved in [BGMR1]. To prove the only if part of (iii), we show that if the gcd conditions are not satisfied, the quotient is not smooth. Assuming that Ω in reduced form, suppose there are $k+1$ columns such that the gcd of the proposition is $= r > 1$. Then, putting the $n-k-1$ quaternionic coordinates that do not correspond to the $k+1$ columns equal to zero, one shows as in Lemma 3.3 of [BGM2] that the leaf holonomy group of the foliation on $\mu_\Omega^{-1}(0)$ generated by $\mathfrak{t}_k$ is $\mathbb{Z}_r$. But this is the local uniformizing group of an orbifold which cannot be reduced to the identity. This proves (iii).

To prove (ii) we make use of the hyperkähler cone $C(\mathcal{S})$ over any complete 3-Sasakian orbifold $\mathcal{S}$ [BGM1]. $C(\mathcal{S})$ is a hyperkähler orbifold with its cone metric which admits a non-isometric local homothety. The following lemma is a slight orbifold modification of Lemma 2 Chapter VI, section 3 of [KN].

Lemma 1.5: *Let $\mathcal{S}$ be a complete 3-Sasakian orbifold and $C(\mathcal{S})$ its hyperkähler cone, then if the hyperkähler metric g_C on $C(\mathcal{S})$ extends smoothly to the cone point, it is flat. Moreover, if $C(\mathcal{S})$ (including the cone point) is a smooth manifold, then $\mathcal{S}$ is a sphere.*

Let $\Omega \in \mathcal{M}_{k\times n}(\mathbb{Z})$ be degenerate of rank k. By using an element of $GL(k,\mathbb{Z})$ if necessary we can make a row with the maximal number of zero entries the last row.

We reduce our considerations to the case where all rows have at least two non-vanishing entries. If the last row has only one non-vanishing entry it must be a 1, and using the Weyl group $\mathcal{W}(Sp(n))$ if necessary we can make this the last column. One now easily sees by solving the moment map equations for the last row, that this reduces $\mathcal{S}(\Omega)$ to a reduction of S^{4n-5} by the $k-1$ torus $T^{k-1}(\Omega')$ with Ω' equal to Ω with its last row and column removed. We continue in this way until we remove all rows with only one non-vanishing entry (up to a $GL(k,\mathbb{Z})$ transformation) and the corresponding columns. This gives a $\Omega_1 \in \mathcal{M}_{k-r,n-r}(\mathbb{Z})$ with $0 \leq r \leq k$. Note that if $r=k$ we have reduced S^{4n-1} to $S^{4n-4k-1}$ and we are done. So we can assume that Ω is not equivalent to any matrix with only one non-vanishing entry in any row.

Next consider the cone $C(\mathcal{S}(\Omega))$ where we can assume that every representative in the $GL(k,\mathbb{Z})$ equivalence class $[\Omega]$ has at least two non-vanishing entries in every row. So Ω can be put in Hermite normal form:

$$\Omega = \begin{pmatrix} 0 & \cdots & 0 & a^1_{l_1} & a^1_2 & \cdots & \cdots & a^1_k & a^1_{k+1} & & \cdots & a^1_n \\ 0 & \cdots & 0 & 0 & \cdots & a^2_{l_2} & \cdots & a^2_k & a^2_{k+1} & & \cdots & a^2_n \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & & & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & a^k_{l_k} & \cdots & a^k_n \end{pmatrix}, \tag{1.6}$$

where $a^j_{l_j} \neq 0$, $n - l_k \geq 2$, and if $i<j$ then $l_i < l_j$. We can assume that the row with the most zeroes is the last row and by using permutations of the Weyl group $\mathcal{W}(Sp(n))$ if necessary, that all zero entries are the first $l_k - 1$ entries with $n-2 \geq l_k \geq k+1$; i.e., all $a^k_j \neq 0$ for $l_k \leq j \leq n$. Moreover, since Ω is degenerate $l_k \geq k+1$.

We consider the hyperkähler quotient reduction by the circle defined by the last row. We obtain $\mathbb{H}^{l_k-1} \times C(\mathcal{S}(a^k_{l_k}, \cdots, a^k_n))$, where $l_k - 1 \geq k$. Now the remaining components of the moment map induce a moment map $\nu : \mathbb{H}^{l_k-1} \times C(\mathcal{S}(a^k_{l_k}, \cdots, a^k_n)) \longrightarrow \mathfrak{t}_{k-1} \otimes \mathbb{R}^3$. Let Ω_1 denote Ω with its last row removed, and let $\Omega_2 \in \mathcal{M}_{k-1\times l_k-1}(\mathbb{Z})$ denote the matrix formed from the first $l_k - 1$ columns of Ω_1. Since $l_i < l_j$ it follows from 1.6 that Ω_2 has rank $k-1$. So Ω_2 induces a linear map $L_{\Omega_2} : \mathfrak{t}_{l_k-1} \longrightarrow \mathfrak{t}_{k-1}$ whose kernel K has dimension $l_k - k \geq 1$. Thus, if Σ is the subvariety of $\nu^{-1}(0) \subset \mathbb{H}^{l_k-1} \times C(\mathcal{S}(a^k_{l_k}, \cdots, a^k_n))$ obtained by choosing the cone point of the second factor, we have dim $\Sigma \geq 4$. This subvariety is invariant under the $T^{k-1}(\Omega_1)$ action, so if we choose a point $s \in \Sigma$, the entire orbit $T^{k-1}(\Omega_1) \cdot s$ lies in Σ. So if s is a singular point of the variety $\nu^{-1}(0)$ in the sense that its Zariski tangent space has dimension larger than its dimension, then this inequality passes to the space of orbits. So the quotient space $\mu^{-1}(0)/T^k(\Omega) = \nu^{-1}(0)/T^{k-1}(\Omega_1)$ is singular.

On the other hand if $s \in \Sigma$ is a Zariski smooth point of $\nu^{-1}(0)$, there is a sequence $q_i \in \nu^{-1}(0)$ converging to $s \in \Sigma$ such that for $i \neq j$ q_i and q_j lie on distinct $T^{k-1}(\Omega_1)$ orbits. Furthermore, the $T^{k-1}(\Omega_1)$ action is locally free near s, so locally the projection $\pi : \nu^{-1}(0) \longrightarrow \mu^{-1}(0)/T^k(\Omega)$ is an orbifold Riemannian submersion with respect to a unique hyperkähler metric g on the quotient $\mu^{-1}(0)/T^k(\Omega)$. From the construction the local uniformizing groups are isometries of the induced metric on $\nu^{-1}(0)$. It follows that the metric g on $\mu^{-1}(0)/T^k(\Omega)$ is isometric to the horizontal component g_H of the metric on $\nu^{-1}(0)$. If there are more than two non-zero entries in the last row of Ω, that is if $l_k < n-1$, it follows from [BGM1] that $C(\mathcal{S}(a^k_{l_k},\cdots,a^k_n))$ is not flat. So Lemma 1.5 implies that g_H does not extend smoothly to s. Thus, g is not smooth at $\pi(s)$. On the other hand if $l_k = n-1$, then $\mathcal{S} = S^3/\mathbb{Z}_{a^k_{n-1}+a^k_n}$, so $\mathbb{H}^{l_k-1}\times C(\mathcal{S}(a^k_{n-1},a^k_n))$ is covered by the flat hyperkähler vector space $\mathbb{H}^{n-2}\times\mathbb{H}$. Furthermore, since the deck transformations are hyperkähler isometries, the 3-Sasakian quotient space is just a finite quotient of a reduction by lower dimensional sphere. This completes the proof of Proposition 1.4. ∎

By Proposition 1.4 we need only consider non-degenerate Ω. So let $\Omega\in\mathcal{A}_{k\times n}(\mathbb{Z})$. Since Ω is reduced there is a k by k minor determinant that is odd. By permuting columns if necessary this minor can be taken to be the first k columns. Now consider the mod 2 reduction $\mathcal{M}_{k\times n}(\mathbb{Z})\longrightarrow\mathcal{M}_{k\times n}(\mathbb{Z}_2)$. We have the following commutative diagram

1.7
$$\begin{array}{ccc} GL(k,\mathbb{Z})\times\mathcal{M}_{k\times n}(\mathbb{Z}) & \longrightarrow & \mathcal{M}_{k\times n}(\mathbb{Z}) \\ \big\downarrow & & \big\downarrow \\ GL(k,\mathbb{Z}_2)\times\mathcal{M}_{k\times n}(\mathbb{Z}_2) & \longrightarrow & \mathcal{M}_{k\times n}(\mathbb{Z}_2). \end{array}$$

Let $\tilde\Omega\in\mathcal{A}_{k\times n}(\mathbb{Z}_2)$ denote the mod 2 reduction of $\Omega\in\mathcal{A}_{k\times n}(\mathbb{Z})$. Since the first k by k minor determinant of Ω is odd, the mod 2 reduction of this minor in $\tilde\Omega$ is invertible. Thus, we can use the $GL(k,\mathbb{Z}_2)$ action to put $\tilde\Omega$ in the form

1.8
$$\tilde\Omega = \begin{pmatrix} 1 & 0 & \dots & 0 & a^1_{k+1} & \cdots & a^1_n \\ 0 & 1 & \dots & 0 & a^2_{k+1} & \cdots & a^2_n \\ \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \cdots & 1 & a^k_{k+1} & \dots & a^k_n \end{pmatrix}$$

with $a^i_j\in\mathbb{Z}_2$.

Lemma 1.9: *The set $\mathcal{A}_{k\times n}(\mathbb{Z})$ is empty for $n>k+2$ and $k>4$.*

Proof: We analyze the second admissibility condition. Since Ω is reduced the second condition of Proposition 1.4 implies

1.10
$$(\tilde\Delta_{\alpha_2\cdots\alpha_{k+1}},\ldots,\tilde\Delta_{\alpha_1\cdots\hat\alpha_s\cdots\alpha_{k+1}},\ldots,\tilde\Delta_{\alpha_1\cdots\alpha_k})\neq(0,\cdots,0)$$

for all sequences $1\le\alpha_1<\cdots<\alpha_s<\cdots<\alpha_{k+1}\le n$, where $\tilde\Delta$ denotes the mod 2 reduction of Δ. This condition says that every subset of $k+1$ columns of $\tilde\Omega$ spans $(\mathbb{Z}_2)^k$, or equivalently, that every k by $k+1$ submatrix of $\tilde\Omega$ has rank k. By considering $k-1$ of the first k columns and 2 of last $n-k$ columns, this condition implies

1.11
$$(a^j_l,a^j_m)\neq(0,0)$$

for all $j=1,\cdots,k$, and $k+1\le l<m\le n$. Similarly, by considering $k-2$ of the first k columns and 3 of last $n-k$ columns 1.10 implies

1.12
$$\begin{pmatrix} a^i_l & a^i_m & a^i_r \\ a^j_l & a^j_m & a^j_r \end{pmatrix}\neq\begin{pmatrix} 1&1&1\\1&1&1\end{pmatrix},\qquad \begin{pmatrix} a^i_l & a^i_m & a^i_r \\ a^j_l & a^j_m & a^j_r \end{pmatrix}\neq\begin{pmatrix} 0&1&1\\0&1&1\end{pmatrix},$$

where the last inequality is understood to be up to column permutation.

Hence, it follows that, up to column and row permutations, that any four triples of the last $n-k$ columns of an admissible $\tilde{\Omega}$ must have the form

1.13
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

So we see that we cannot add another row without violating the above conditions. It follows that $k \leq 4$. ∎

We can also use our analysis to prove:

LEMMA 1.14: *Let $\Omega \in \mathcal{A}_{k\times n}(\mathbb{Z})$ and let $k > 1$, then $n-k \leq 4$.*

PROOF: By Lemma 1.9 we need only check this for $k = 2,3,4$. We consider the case $k = 2$ as the cases $k = 3,4$ are similar. As before, let $\tilde{\Omega}$ denote the mod 2 reduction of Ω. Then by 1.12, up to $GL(2,\mathbb{Z})$ transformations and column permutations, any $\Omega \in \mathcal{A}_{2\times 6}(\mathbb{Z})$ will have a reduced matrix of the form

$$\tilde{\Omega} = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

One cannot add another column to this matrix since 1.11 implies that this must be $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ which is forbidden by 1.12. ∎

Lemmas 1.9 and 1.15 now imply Theorem A.

§2. Some Examples in Dimension 11 and 15

In [BGM1] we constructed 3-Sasakian manifolds obtained by toral reduction for $k = 1$ for every n and in [BGMR1] we constructed 3-Sasakian 7-manifolds with arbitrary k. Thus, to show that Theorem A is sharp we need only construct 11 and 15 dimensional manifold examples when $k = 2,3,4$. These last constructions give explicit examples of non-regular 3-Sasakian manifolds for which the Betti number relations given in [GS] for regular 3-Sasakian manifolds fail as Bielawski [Bi1] showed that the relations can hold for a 3-Sasakian toral quotient if and only if $k = 1$. Some of the calculations performed in this section were tedious, especially for dimension 15, and were done with the aid of MAPLE symbolic manipulation program.

We begin by assuming that $k = 4$. We take Ω to be in the special form

2.1
$$\Omega = \begin{pmatrix} \mathbb{I}_4 & A \end{pmatrix},$$

where $\mathbb{I}_4$ denotes the 4 by 4 identity matrix and $A \in \mathcal{M}_{4\times 3}(\mathbb{Z})$ or $A \in \mathcal{M}_{4\times 4}(\mathbb{Z})$. It is straightforward to show

LEMMA 2.2: *Let Ω be as above.*

(i) The first admissibility condition implies that all the entries (a^i_j) of the matrix A are non-vanishing and that all minor determinants are non-vanishing.

(iia) If $A \in \mathcal{M}_{4\times 3}(\mathbb{Z})$ then the second admissibility condition is equivalent to

$$\gcd(a^j_l, a^j_m) = 1$$

and

$$\gcd\left(\begin{vmatrix} a^i_l & a^i_m \\ a^j_l & a^j_m \end{vmatrix}, \begin{vmatrix} a^i_l & a^i_r \\ a^j_l & a^j_r \end{vmatrix}, \begin{vmatrix} a^i_m & a^i_r \\ a^j_m & a^j_r \end{vmatrix}\right) = 1$$

for all $1 \le i < j \le 4$, and $1 \le l < m < r \le 3$.

(iib) If $A \in \mathcal{M}_{4\times 4}(\mathbb{Z})$ then the second admissibility condition is equivalent to the conditions in (iia) for all $1 \le i < j \le 4$, and $1 \le l < m < r \le 4$, together with

$$\gcd(M_{i1}, \cdots, M_{i4}, \det A) = 1$$

for $i = 1 \cdots 4$, where M_{ij} denotes ij minor determinant of A.

These equations are still somewhat complicated in their full generality, so we shall present two three parameter families of solutions, namely

2.3
$$A_1 = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1+2^l \\ 1 & 16 & 1+2^m \\ -1 & 3 & 2c \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & 1 & 1+2^{l'} & -1 \\ 1 & 16 & 1+2^{2n} & 3 \\ -1 & 3 & 2c' & -1 \end{pmatrix},$$

where $l, l', m, n \in \mathbb{Z}^+$, and $c, c' \in \mathbb{Z}$.

PROPOSITION 2.4: *Let $\Delta = 2(31c + 6 + 19 \cdot 2^{l-1} - 7 \cdot 2^{m-1})$.*

(i) $\Omega_1 = (\,\mathbb{I}_4 \quad A_1\,)$ is admissible if and only if $c \neq 0$ and is not divisible by 3, and $\Delta \neq 0$ and is not divisible by $7, 19$ nor 31.

(ii) $\Omega_2 = (\,\mathbb{I}_4 \quad A_2\,)$ is admissible if and only if c and all minor determinants of A_2 are non-vanishing, and $c' \not\equiv 0 \pmod 3$, $l' \not\equiv 0 \pmod 4$, $c' \not\equiv 5 \pmod 7$, and $11, 19, 37, 71$ do not divide $\det A = 19 \cdot 2^{2n} - 63 - 148c' - 11 \cdot 2^{l'}$, and the following conditions hold:

$$\begin{aligned} \gcd(3, 4c' + 2^{l'} + 1, 2c' - 2^{l'} - 1) &= 1 \\ \gcd(7, 2^{2n+1} - 2^{l'} + 1, 3 \cdot 2^{l'} + 2^{2n} + 4) &= 1 \\ \gcd(19, 2^{2n} - 2^{l'+4} - 15, 3 \cdot 2^{l'} + 2^{2n} + 4) &= 1 \\ \gcd(25, 32c' - 3 \cdot 2^{2n} - 3, 6c' + 2^{2n} + 1) &= 1. \end{aligned}$$

The conditions in this proposition guarantee that the quotient spaces denoted by $\mathcal{S}(c,l,m)$ and $\mathcal{S}(c',l',n)$ are smooth manifolds of dimension 11 and 15, respectively, which by [Bi1] have $b_2=4$. One can then obtain smooth manifolds with $b_2=3$ or 2 by simply removing one or two rows from the matrices A_1 or A_2 (together with the corresponding columns in $\mathbb{I}_4$ in Ω_1 or Ω_2). It is routine to verify that the three parameter infinite family given by

2.5
$$\begin{aligned} c &\equiv 14 \pmod{21}, \\ l &\not\equiv 1 \pmod{5}, \\ m &\not\equiv \alpha(c) \pmod{18}, \end{aligned}$$

where $2^{\alpha(c)} = 22(31c+6) \pmod{18}$ satisfies the conditions in (i) of Proposition 2.4. This gives examples in dimension 11. (Notice that as 2 is a primitive root of 19 the equation defining $\alpha(c)$ has a unique solution $\pmod{18}$ for each value of c.) Similarly, it is straightforward to verify that the infinite family given by

2.6
$$\begin{aligned} c' &= 2, \\ l' &= 1, \\ n &\equiv 21 \pmod{90}, \end{aligned}$$

satisfies the conditions (ii) of Proposition 2.4. We have arrived at:

THEOREM 2.7: *There exist smooth 3-Sasakian manifolds in dimensions 11 and 15 with second Betti number b_2 equal to 2, 3 and 4.*

An important corollary of this theorem is that the linear Betti number relations obtained by Galicki and Salamon [GS] for regular 3-Sasakian manifolds fail in the non-regular case. Bielawski [Bi1] showed that these relations can hold for a 3-Sasakian toral quotient if and only if $k=1$. Since the Betti number relations in [GS] hold only for dimensions greater than or equal to 11, Theorem 2.7 exhibits the first examples of non-regular 3-Sasakian manifolds where these relations fail. The examples occur in dimensions 11 and 15 where the Betti number relations [GS] are particularly simple, namely $b_2(\mathcal{S}) = b_4(\mathcal{S})$ for dim $\mathcal{S}=11$, and $b_2(\mathcal{S}) = b_6(\mathcal{S})$ for dim $\mathcal{S}=15$.

An important problem not considered in this note is the equivalence problem, that is, given two 3-Sasakian manifolds $(\mathcal{S}(\Omega), g, \xi^a)$ and $(\mathcal{S}(\Omega'), g', \xi'^a)$ when is there a diffeomorphism $f: \mathcal{S}(\Omega) \longrightarrow \mathcal{S}(\Omega')$ such that $f^*g' = g$ and $f_*\xi^a = \xi'^a$. The best approach to this problem is through combinatorial geometry. This is currently under study. In this regard, Bielawski and Dancer [BD] have given a combinatorial approach to hyperkähler reductions by a torus, and Bielawski [Bi2] has recently proven a Delzant-type theorem [Gui] for both hyperkähler and 3-Sasakian toral reductions. Explicitly, Bielawski proves that any compact 3-Sasakian manifold $\mathcal{S}$ of dimension $4n-1$ having a locally smooth effective action of an n-torus T^n preserving the 3-Sasakian structure is T^n equivariantly homeomorphic to a 3-Sasakian quotient of the sphere S^{4d-1} by a subgroup of T^d. If this result could be strengthened to a diffeomorphism, then it would follow from this and Theorem A that there would be no compact 3-Sasakian manifolds of dimension $4n-1$ with $n>4$ and $b_2>1$ whose automorphism group of 3-Sasakian isometries contain an n-torus T^n.

Bibliography

[BG] C.P. Boyer and K. Galicki, *The Twistor Space of a 3-Sasakian Manifold*, Int. J. of Math. 8(1) (1997), 31-60.

[BGM1] C.P. Boyer, K. Galicki, and B.M. Mann, *The Geometry and Topology of 3-Sasakian Manifolds*, J. reine angew. Math. 455 (1994), 183-220.

[BGM2] C.P. Boyer, K. Galicki, and B.M. Mann, *Hypercomplex Structures on Stiefel Manifolds*, Ann. Global Anal. and Geom. 14 (1996), 81-105.

[BGMR1] C.P. Boyer, K. Galicki, B.M. Mann, and E.G. Rees, *Compact 3-Sasakian 7-Manifolds with Arbitrary second Betti Number*, Inventiones Math., to appear.

[BGMR2] C.P. Boyer, K. Galicki, B.M. Mann, and E.G. Rees, *Positive Einstein Manifolds with Arbitrary Second Betti Number*, Balkan Jour. of Geom. and Its Appl. 1(2) (1996), 1-8.

[Bi1] R. Bielawski, *Betti Numbers of 3-Sasakian Quotients of Spheres by Tori*, Bull. London Math. Soc., to appear.

[Bi2] R. Bielawski, *Reconstruction of Hamiltonian T^n Spaces in Quaternionic Riemannian Geometry*, MPI preprint (1997).

[BD] R. Bielawski and A.S. Dancer, *The Geometry and Topology of Toric Hyperkähler Manifolds*, McMaster Univ. preprint (1996).

[GS] K. Galicki and S. Salamon, *On Betti Numbers of 3-Sasakian Manifolds*, Geom. Ded. 63 (1996), 45-68.

[Gui] V. Guillemin, *Moment Maps and Combinatorial Invariants of Hamiltonian T^n-spaces*, Birkhauser, Boston (1994).

[KN] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vol. 1, Interscience Publications, New York, 1963.

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