On the role of the points at infinity in Iwasawa theory

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§0. Introduction.

For a number field k denote by k^+ its maximal totally real subfield. Assume k is a CM field, i.e. k is totally imaginary and $[k:k^+]=2$. Let $K=k(\mu_{p^\infty})$, it's also a CM field, and we denote by J the non-trivial automorphism of K/K^+. Let A denote the p-part of the ideal class group of K. We have for $p\neq 2$, $A=A^+\oplus A^-$, where J acts on $A^\pm$ by multiplication by ± 1. Greenberg's conjecture is the assertion: $A^+=0$. The purpose of this paper is to generalize the above definition of the plus and minus parts of the ideal class group to arbitrary number field k, and to poke at Greenberg's conjecture from various elementary perspectives, showing that it amounts to the assertion that the points at infinity do not play any role in Iwasawa theory, thus perhaps making it more plausible at least on the philosophical level (there are no continuous non-constant maps from the reals into the p-adic numbers....). In § 1, inspired by Iwasawa's sheaves [3], but adding the points at infinity into the game, we define $\mathrm{Pic}(K)$ and $\mathrm{Pic}(K;\infty)$, the latter being just the ideal class groups of K, and we explore the (Kummer) duality between the torsion parts of these groups and certain Galois groups. We have a surjective map:

$$\mathrm{Pic}(K) \twoheadrightarrow \mathrm{Pic}(K;\infty).$$

In § 2 we specialize the situation to the case of Iwasawa theory, namely when $K=k(\mu_{p^\infty})$, $[k:\mathbb{Q}]<\infty$, and we are interested in the p-torsion parts of Pic's. We have a map:

$$(*) \qquad \mathrm{Pic}(K)_{p\text{-torsion}} \longrightarrow \mathrm{Pic}(K;\infty)_{p\text{-torsion}}.$$

We then consider the case where K is CM and show that Greenberg's conjecture is equivalent to the assertion that $(*)$ is still surjective. In § 3 we show $(*)$ is (almost always) injective and derive some simple corollaries. In § 4 we explain how the "generalized Greenberg conjecture", i.e. the assertion that (*) is an isomorphism, for all k , imply the "cyclicity conjecture" when k is Galois over $\mathbb{Q}$, cf.[1] for the case when k is abelian over $\mathbb{Q}$, and [4] for the case when k is Galois over $\mathbb{Q}$ and is a CM field. Our definitions allow one to generalize Gross's results [2], and also Kužmin's duality pairings, from the case of CM fields to arbitrary ones. In § 5 we show that (*) is surjective if one replaces the naive topology of § 1 by the flat quasi-finite Grothendieck topology(with the small site where points at infinity can easily be added).

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§ 1.

For a finite number field, k, we denote by $U(k)$ the topological space of all places of k, including the infinite places, with the co-finite topology; thus the open sets of $U(k)$ are of the form $U(k;\nu_1\dots\nu_\ell) \overset{\text{def}}{=} \{\nu \in U(k) \mid \nu\neq\nu_1,\dots,\nu\neq\nu_\ell\}$. Given an extension of number fields, k'/k, we have a continuous map $\pi_{k'/k}: U(k') \to U(k)$, given by restriction of places. For a number field, K, of infinite degree over $\mathbb{Q}$, we let $U(K) = \varprojlim U(k)$, the inverse limit taken over all finite number fields $k \subseteq K$, with respect to the maps $\pi_{k'/k}$, and we give $U(K)$ the inverse limit topology; thus the open sets of $U(K)$ are of the form $U(K;\nu_1\dots\nu_\ell) \overset{\text{def}}{=} \pi^{-1}_{K/k}\, U(k;\nu_1\dots\nu_\ell)$, where $k \subseteq K$ is a finite number field, $\nu_1\dots\nu_\ell \in U(k)$, and $\pi_{K/k}: U(K) \longrightarrow U(k)$ is the natural projection. $U(K)$ is quasi-compact; T_1; every closed set $C \subsetneq U(K)$ is compact Hausdorff; $U(K)$ is irreducible, hence every constant sheaf is flasque and has trivial cohomology in positive dimensions. Given a finite number field $k \subseteq K$, $\nu_1\dots\nu_\ell \in U(k)$, let $\underline{U}(K;\nu_1\dots\nu_\ell) = \{U_i\}$, $U_i = U(K;\nu_1\dots\nu_{i-1},\nu_{i+1}\dots\nu_\ell)$; it's a covering of $U(K)$ such that for $i \neq j$, $U_i \cap V_j = U(K;\nu_1\dots\nu_\ell)$ is independent of $\{i,j\}$, and hence for any sheaf $\mathcal{F}$ on $U(K)$, $\check{H}^p(\underline{U}(K;\nu_1\dots\nu_\ell),\mathcal{F})=0$, $p \geq 2$; as these coverings are co-final in the category of all coverings $H^p(U(K),\mathcal{F}) = 0$, $p \geq 2$.

For an open set $U \subseteq U(K)$ define: $\underline{\mathcal{O}^*}(U) = \{f \in K^* \mid |f|_\nu = 1$ for all $\nu \in U\}$, $\underline{\mathcal{M}}(U) = K^*$; $\underline{\mathcal{O}^*}$ and $\underline{\mathcal{M}}$ are sheaves on $U(K)$, $\underline{\mathcal{O}^*} \hookrightarrow \underline{\mathcal{M}}$, we let $\underline{\mathcal{D}} = \underline{\mathcal{M}}/\underline{\mathcal{O}}^*$ denote the sheaf quotient. For $\nu \in U(K)$ let $\underline{\mathcal{O}}^*_\nu = \{f \in K^* \mid |f|_\nu = 1\}$, $\underline{\mathcal{D}}_\nu = K^*/\underline{\mathcal{O}}^*_\nu$, these are the stalks of $\underline{\mathcal{O}^*}$ and $\underline{\mathcal{D}}$. Also, given a finite number field $k \subseteq K$, $\bar{\nu} = \{\nu_1\dots\nu_\ell\}$, $\nu_i \in U(k)$, let $C = U(K)\setminus U(K;\bar{\nu})$, and define: $\underline{\mathcal{D}}_{\bar{\nu}}(K) = H^\circ_C(U(K),\underline{\mathcal{D}})$, $E_{\bar{\nu}}(K) = H^\circ(U(K;\bar{\nu}),\underline{\mathcal{O}}^*)$; by omitting $\bar{\nu}$ we shall designate the empty set of ν_i's, and we omit K from the notation whenever it is clear from the context, thus e.g. E = roots of unity in K, E_∞ = classical units of K, $E_{p\infty}$ = p-units of K, E_p = elements of K that have absolute value 1 away from p including at infinity. Using the exact sequence

$$(1.1)\qquad * \longrightarrow \underline{\mathcal{O}}^* \longrightarrow \underline{\mathcal{M}} \longrightarrow \underline{\mathcal{D}} \longrightarrow *$$

and the fact that $\underline{\mathcal{M}}$ and $\underline{\mathcal{D}}$ are flasque, we get that $H^p(U(K;\bar{v}),\underline{\mathcal{O}}^*) = 0, p \geq 2$, and $\mathrm{Pic}(K;\bar{v}) \overset{\mathrm{def}}{=\!=} H^1(U(K;\bar{v}),\underline{\mathcal{O}}^*)$ is given by

$$(1.2)\qquad * \longrightarrow E_{\bar{v}} \longrightarrow K^* \xrightarrow{\mathrm{div}'_{\bar{v}}} \underline{\mathcal{D}}(U(K;\bar{v})) \longrightarrow \mathrm{Pic}(K;\bar{v}) \longrightarrow *$$

We denote by $\mathrm{div}'_{\bar{v}} : K^* \longrightarrow \underline{\mathcal{D}}(U(K;\bar{v}))$, and $\mathrm{div}_{\bar{v}} : K^* \longrightarrow \underline{\mathcal{D}}_{\bar{v}}$ the obvious maps, thus e.g. $\mathrm{div}'_\infty(f)$ is just the ideal generated by f, and so $\mathrm{Pic}(K;\infty)$ is the ideal class group of K. $\mathrm{Pic}(K;\infty) = \varinjlim \mathrm{Pic}(k;\infty)$, the limit taken over finite $k \subsetneq K$ with the obvious maps, and as a direct limit of finite groups $\mathrm{Pic}(K;\infty)$ is a torsion group. The "minus eigenspace of the ideal class group" alluded to in the introduction is just $\mathrm{Pic}(K)_{\mathrm{torsion}}$.

For two open sets $U_1 = U(K;\bar{v})$, $U_2 = U(K;\bar{v}\bar{u})$, $C = U_1 \setminus U_2$, we get by the relative cohomology sequence for $\underline{\mathcal{O}}^*$, using the observations $H^0_C(U_1,\underline{\mathcal{O}}^*) = 0$, $H^1_C(U_1,\underline{\mathcal{O}}^*) = \mathcal{D}_{\bar{u}}$, $H^2_C(U_1,\underline{\mathcal{O}}^*) = 0$, the following exact sequence:

$$(1.3)\qquad * \longrightarrow E_{\bar{v}} \longrightarrow E_{\bar{v}\bar{u}} \xrightarrow{\mathrm{div}_{\bar{u}}} \underline{\mathcal{D}}_{\bar{u}} \longrightarrow \mathrm{Pic}(K;\bar{v}) \longrightarrow \mathrm{Pic}(K;\bar{v}\bar{u}) \longrightarrow *$$

Thus e.g. with $\bar{v}$ empty and $\bar{u} = \infty$ we get:

$$(1.4)\qquad * \longrightarrow E \longrightarrow E_\infty \xrightarrow{\mathrm{div}_\infty} \underline{\mathcal{D}}_\infty \longrightarrow \mathrm{Pic}(K) \longrightarrow \mathrm{Pic}(K;\infty) \longrightarrow *$$

Hence for any natural number n we get:

$$(1.5)\qquad * \longrightarrow \left(\mathcal{D}_{\bar{u}} \big/ \mathrm{div}_{\bar{u}}(E_{\bar{u}\bar{v}})\right)[n] \longrightarrow \mathrm{Pic}(K;\bar{v})[n] \longrightarrow \mathrm{Pic}(K;\bar{v}\bar{u})[n] \longrightarrow$$

$$\longrightarrow \left(\mathcal{D}_{\bar{u}} \big/ \mathrm{div}_{\bar{u}}(E_{\bar{u}\bar{v}})\ n\underline{\mathcal{D}}_{\bar{u}}\right) \longrightarrow \mathrm{Pic}(K;\bar{v})/_n \longrightarrow \mathrm{Pic}(K;\bar{v}\bar{u})/_n \longrightarrow *$$

Let $\underline{\mathcal{M}}^{(n)} = \underline{\mathcal{M}} \underset{\underline{\mathcal{D}}}{\times} n\cdot\underline{\mathcal{D}}$, $\underline{\mathcal{M}}^{(n)}(U(K;\bar{v})) = \left\{ f \in K^* \mid \mathrm{div}'_{\bar{v}}(f) \in n\cdot\underline{\mathcal{D}}(U(K;\bar{v})) \right\}$.

From

$$\begin{array}{ccccccccc} * & \longrightarrow & n\cdot\underline{\mathcal{O}}^* & \longrightarrow & n\cdot\underline{M} & \longrightarrow & n\cdot\mathcal{D} & \longrightarrow & * \\ & & \downarrow & & \downarrow & & \| & & \\ * & \longrightarrow & \underline{\mathcal{O}}^* & \longrightarrow & \underline{M}^{(n)} & \longrightarrow & n\cdot\mathcal{D} & \longrightarrow & * \end{array}$$

we get $\underline{\mathcal{O}}^*/n\cdot\underline{\mathcal{O}}^* \cong \underline{M}^{(n)}/n\cdot\underline{M}$, and using the long cohomology sequence associated to $* \longrightarrow n\cdot\underline{\mathcal{O}}^* \longrightarrow \underline{\mathcal{O}}^* \longrightarrow \underline{M}^{(n)}/n\cdot M \longrightarrow *$ over $U(K;\bar{\nu})$, we get

$$(1.6)\quad * \longrightarrow E^n_{\bar{\nu}} \longrightarrow E_{\bar{\nu}} \longrightarrow \underline{M}^{(n)}(U(K;\bar{\nu}))/(K^*)^n \longrightarrow \mathrm{Pic}(K;\bar{\nu}) \xrightarrow{\cdot n}$$

$$\longrightarrow \mathrm{Pic}(K;\bar{\nu}) \longrightarrow H^1(U(K;\bar{\nu});\ \underline{M}^{(n)}) \longrightarrow *$$

Assume $\mu_n \subseteq K$. Let $M^{(n)}_{\bar{\nu}} = K(f^{1/n}; f\in\underline{M}^{(n)}(U(K;\bar{\nu})))$,

$$N^{(n)}_{\bar{\nu}} = K(e^{1/n};\ e\in E_{\bar{\nu}}) \ , \quad X^{(n)}_{\bar{\nu}} = \mathrm{Gal}(M^{(n)}_{\bar{\nu}}/N^{(n)}_{\bar{\nu}}) .$$

By Kummer theory we have a perfect duality

$$X^{(n)}_{\bar{\nu}} \times \underline{M}^{(n)}(U(K;\bar{\nu}))/E_{\bar{\nu}}(K^*)^n \xrightarrow{<,>^{(n)}_{\bar{\nu}}} \mu_n$$

Hence from the exact sequence (1.6) we get

<u>Proposition:</u> $X^{(n)}_{\bar{\nu}}$ is dual to $\mathrm{Pic}(K;\bar{\nu})[n]$.

Similarly, for a super-natural number m , letting $\mu_m = \varinjlim_{n/m} \mu_n$, $\mathrm{Pic}(K;\bar{\nu})[m] = \varinjlim_{n/m} \mathrm{Pic}(K;\bar{\nu})[n]$, $M^{(m)}_{\bar{\nu}} = \varinjlim_{n/m} M^{(n)}_{\bar{\nu}}$, $N^{(m)}_{\bar{\nu}} = \varinjlim_{n/m} N^{(n)}_{\bar{\nu}}$, $X^{(m)}_{\bar{\nu}} = \varprojlim_{n/m} X^{(n)}_{\bar{\nu}} = \mathrm{Gal}(M^{(m)}_{\bar{\nu}}/N^{(m)}_{\bar{\nu}})$, we get

<u>Corollary:</u> Assume $\mu_m \subseteq K$. $X^{(m)}_{\bar{\nu}}$ is dual to $\mathrm{Pic}(K;\bar{\nu})[m]$.

2. Let k be a finite number field and let $K=k(\mu_{p^\infty})$. We apply the corollary of §1 with $m=p^\infty$ and with $\bar{\nu}=p\infty$, ∞, p, empty set, respectively. We get (omitting $(p\infty)$ from our notations):

$M_{\infty p}$ = maximal p-abelian unramified-outside-p extension of K.

$N_{\infty p} = K(E_{\infty p}^{1/p^\infty})$ the field obtained by adding all p-power roots of all p-units to K .

$M_\infty = M_{\infty p}$, since $\underline{\mathcal{D}}_p$ is p-divisible.

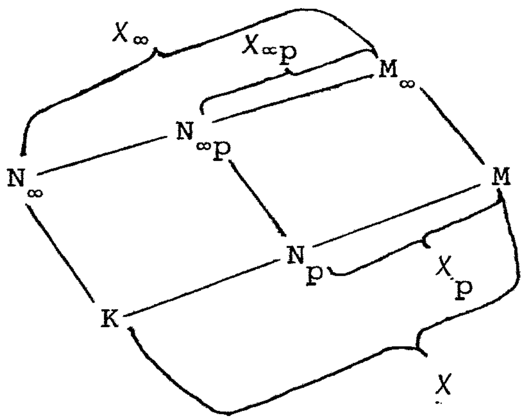
$N_\infty = K(E_\infty^{1/p^\infty})$

$M_p = M = K(f^{1/p^n};\ \mathrm{div}(f) \in p^n \cdot \underline{\mathcal{D}})$

$N_p = K(E_p{}^{1/p^\infty})$

$N = K$

We have a diagram of fields:



The following questions naturally arise:

(Q) Is $M_\infty = N_\infty M$?

$(Q)_p$ Is $M_\infty = N_{\infty p} M$?

(q) Is $N_\infty \cap M = K$?

$(q)_p$ Is $N_{\infty p} \cap M = N_p$?

Assume k is a CM field. Denote by J the non-trivial automorphism of k/k^+. In this case also $K=k(\mu_{p^\infty})$ is a CM field, and we denote again by J the non-trivial automorphism of K/K^+. J acts via inner conjugation on $Gal(M_\infty/K)$. By Hilbert theorem 90 we have $K^*[1+J]= K^{*1-J}$. For a place v/∞ of K, choose an embedding $\sigma_v: K \hookrightarrow \mathbb{C}$, representing v, and let $\overline{\sigma_v}$ denote the conjugate embedding. We have for $f \in K^*$, $||f||_v = f^{\sigma_v} \cdot f^{\overline{\sigma_v}} = (f^{1+J})^{\sigma_v}$, hence $\underline{O}_v^* = K^{*1-J}$, and σ_v induces an isomorphism $K^{*1+J} \cong \underline{D}_v$. From this we see that $\mathrm{div}_\infty(f) \in p^n \cdot \underline{D}_\infty$ if and only if $f^{1+J} \in (K^{*1+J})^{p^n}$, or equivalently, if and only if $f \in K^{*1-J}(K^*)^{p^n}$. Since Kummer duality interchanges the plus and minus eigenspaces for the action of J, i.e. $\langle Jx, f\rangle^{(p^n)}_v = \langle x, -Jf\rangle^{(p^n)}_v$, we see that $X=Gal(M/K)$ is the plus eigenspace of $Gal(M_\infty/K)$. We also have $E_\infty(K)[1-J]=E_\infty(K^+)$, and by Dirichlet unit theorem, $E_\infty(K)^{1-J} \simeq E_\infty(K)/E_\infty(K)[1-J]$ is torsion, hence $E_\infty(K)^{1-J}= E(K)$. Similarly $E_p(K) = E_{p\infty}(K)[1+J]$ are the minus p-units of K. From these observations it follows easily that (q) and $(q)_p$ have an affirmative answer. Also, the restriction map, $X_\infty \longrightarrow X$, is surjective, and its kernel, $Gal(M_\infty/N_\infty M)$, is just the minus eigenspace of X_∞. Dualy, the map coming from the relative cohomology sequence, $Pic(K)[p^\infty] \longrightarrow Pic(K;\infty)[p^\infty]$, is injective, and its cokernel is just the plus eigenspace of the p-part of the ideal class group of K, $Pic(K,\infty)\ [p^\infty]^{1+J}$. Thus, Greenberg's conjecture is equivalent to the assertion that $X_\infty \longrightarrow X$ is an isomorphism; i.e. the answer to (Q) is also affirmative. In [1] Greenberg shows that if all totally real fields k_n^+, the layers of the cyclotomic $\mathbb{Z}_p$-extension K^+/k^+, satisfy the Leopoldt conjecture, then $N_{\infty p}=N_\infty N_p$, and so (Q) follows from $(Q)_p$; that is, the contribution to the plus eigenspace of $Pic(K;\infty)[p^\infty]$ coming from p-ideals in trivial.

In the classical proof of Dirichlet's unit theorem it is proved that the $\mathbb{R}$-linear map, $\lambda_\infty : \mathbb{R}\otimes E_\infty(k) \longrightarrow \mathbb{R}\otimes I_\infty(k)$, is injective, where $I_\infty(k)$ is the free abelian group on k-places $v|\infty$, and for $e\in E_\infty(k)$, $\lambda_\infty(e) = \sum_{v/\infty} \log_\infty(\mathbb{N}_{k_v/\mathbb{R}}(e))\otimes v$, $\log_\infty$ is the usual log on the positive reals and is extended to all $\mathbb{R}$ by $\log_\infty(-r)=\log(r)$ for $r>o$. In [2] Gross conjectures, (for CM fields k), that the p-adic analogue of this holds, namely the injectivity of the $\mathbb{Q}_p$-linear map, $\lambda_p : \mathbb{Q}_p\otimes E_p(k) \longrightarrow \mathbb{Q}_p\otimes I_p(k)$, where $I_p(k)$ is the free abelian group on k-places v/p, and for $e\in E_p(k)$, $\lambda_p(e) = \sum_{v/p} \log_p (\mathbb{N}_{k_v/\mathbb{Q}_p}(e)) \otimes v$, $\log_p$ is Iwasawa's log. It is proved (for CM fields) that this is equivalent to $X(-1)_\Gamma \cong Gal(N_p/K)(-1)_\Gamma$ $(\cong \mathbb{Z}_p^r$, $r=\dim_{\mathbb{Q}_p} \mathbb{Q}_p\otimes E_p(k))$ or equivalently, to $X_p(-1)_\Gamma = 0$, $\Gamma = Gal(K/k)$.

A nice property that $Pic(k)$ has, versus $Pic(k;\infty)$, is its injectivity up the cyclotomic $\mathbb{Z}_p$-extension K/k. Namely, from the exact sequence

$$* \longrightarrow E(K) \longrightarrow K^* \longrightarrow K^*/E(K) \longrightarrow *$$

together with Hilbert theorem $90: H^1(\Gamma,K^*)=0$, and $H^1(\Gamma,E(K))=0$, we get

$$\begin{array}{ccccccccc} * & \longrightarrow & k^*/E(k) & \longrightarrow & \underline{\mathcal{D}}(U(k)) & \longrightarrow & Pic(k) & \longrightarrow & * \\ & & \| & & \downarrow & & \downarrow & & \\ * & \longrightarrow & (K^*/E(K))^\Gamma & \longrightarrow & \underline{\mathcal{D}}(U(K))^\Gamma & \longrightarrow & Pic(K)^\Gamma & \longrightarrow & * \end{array}$$

Applying the snake lemma we get from this

$$* \longrightarrow Pic(k) \longrightarrow Pic(K)^\Gamma \longrightarrow \underline{\mathcal{D}}(U(K))^\Gamma/\underline{\mathcal{D}}(U(k)) \longrightarrow *$$

§ 3. Fix a finite number field k, and let $i_k = [k:k^+]$. Note that, if k_n denotes the layers of the cyclotomic $\mathbb{Z}_p$-extension of k, then $i_k = i_{k_n}$; and that $i_{k(\mu_p)} = i_k$ or $2 \cdot i_k$ according to $\mu_p \subseteq k$ or $\mu_p \not\subseteq k$ respectively. For k a CM field, the requirement $p \nmid i_k$ is just $p \neq 2$ the familiar requirement from Iwasawa theory for such fields.

<u>Proposition:</u> Assume $\mu_p \subseteq k$, $p \nmid i_k$, then $\left(\mathcal{D}_\infty(k) / \mathrm{div}_\infty(k^*) \right)[p] = 0$

Proof. Let $L \supseteq k$, L/k^+ Galois, and without loss of generality $p \nmid [L:k^+]$; we will prove in fact that $(\mathcal{D}_\infty(L)/\mathrm{div}_\infty(k^*))[p] = 0$ For each place $\nu|\infty$ of L, let $J_\nu = \bar{\sigma}_\nu^{-1} \circ \sigma_\nu \in \mathrm{Gal}(L/k^+)$ the "complex conjugation at ν". The map, $\mathrm{div}_\infty : k^* \longrightarrow \mathcal{D}_\infty(L) = \bigoplus_{\nu|\infty} L^{*1+J_\nu}$, is just $\mathrm{div}_\infty(x) = \bigoplus_{\nu|\infty} x^{1+J_\nu}$. Let $\tilde{y} = (\tilde{y}_\nu) \in \mathcal{D}_\infty(L)$ be such that $p \cdot \tilde{y} = \mathrm{div}_\infty(x)$, $x \in k^*$; i.e. $\tilde{y}_\nu = y_\nu^{1+J_\nu}$ for some $y_\nu \in L^*$, and $\tilde{y}_\nu^p = x^{1+J_\nu}$. We have to find $x_1 \in k^*$ such that $\tilde{y} = \mathrm{div}_\infty(x_1)$; i.e. $\tilde{y}_\nu = x_1^{1+J_\nu}$. Let $F = L(x^{1/p})$. If $F = L$, $x^{1/p} \in L$, then since $p \nmid [L:k]$, $x^{1/p} \in k$, and we can take $x_1 = x^{1/p}$; we have $\tilde{y}_\nu^p = (x_1^{1+J_\nu})^p$, and since both $\tilde{y}_\nu$ and $x_1^{1+J_\nu}$ are real and positive at ν, $\tilde{y}_\nu = x_1^{1+J_\nu}$. So assume $F \neq L$, and let $g \circ \alpha$ denote the natural action of $g \in \mathrm{Gal}(L/k^+)$ on $\alpha \in \mathrm{Gal}(F/L)$ via inner conjugation. Since $x^{J_\nu} \equiv x^{-1} \mod (L^*)^p$, we get $J_\nu \circ \alpha = \alpha$ for all $\nu|\infty$. Since $x \in k^*$ and $\mu_p \subseteq k$, we get $h \circ \alpha = \alpha$ for all $h \in \mathrm{Gal}(L/k)$. Thus $(\mathrm{Gal}\ L/k^+)$ acts trivially on $\mathrm{Gal}(F/L)$, and since these groups have relatively prime cardinalities, the central extension $* \longrightarrow \mathrm{Gal}(F/L) \longrightarrow \mathrm{Gal}(F/k^+) \longrightarrow$ $\longrightarrow \mathrm{Gal}(L/k^+) \longrightarrow *$ splits; i.e. $x = x_\circ \cdot e^p$ with $x_\circ \in k^+$, $e \in L$. Since $e^p \in k$ even $e \in k$. We have now $\tilde{y}_\nu^p = x^{1+J_\nu} = x_\circ^2 \cdot (e^{1+J_\nu})^p$. Setting $w_\nu = \tilde{y}_\nu / e^{1+J_\nu}$, $w_\nu^p = x_\circ^2$, hence $w_\nu \in k$, and $w = w_\nu$ is independent of the place ν up to multiplication by a $p^{\underline{th}}$-root of unity; moreover (since $p \neq 2$) $w^{1/2} \in k$. Let now $x_1 = w^{1/2}.e$, then $x_1^{1+J_\nu} = w_{\bar{\nu}} \cdot e^{1+J_\nu} = \tilde{y}_\nu$.

Corollary: $\left(\mathcal{D}_\infty(k) / \operatorname{div}_\infty(E_\infty(k))\right)[p] = 0$ and $\left(\mathcal{D}_\infty(k) / \operatorname{div}_\infty(E_{p\infty}(k))\right)[p] = 0$

Proof. We have only to verify that

$$\left(\operatorname{div}_\infty(k^*) / \operatorname{div}_\infty(E_\infty(k))\right)[p] = \left(k^* / k^- E_\infty(k)\right)[p] = 0 \quad \text{where}$$

$k^- = \ker\left\{\operatorname{div}_\infty : k^* \longrightarrow \mathcal{D}_\infty\right\}$. This follows since $k^*/E(k) \approx E_\infty(k)/E(k) \oplus k^-/E(k) \oplus k^*/k^-(E_\infty(k))$.

Similarly for the second assertion.

Let $K = k(\mu_{p^\infty})$. By the above corollary, and the remark at the beginning of § 3, we get for $p \nmid i_k$:

$$\left(\mathcal{D}_\infty(K) / \operatorname{div}_\infty(E_\infty(K))\right)[p^\infty] = 0 \;, \quad \left(\mathcal{D}_\infty(K) / \operatorname{div}_\infty(E_{p\infty}(K))\right)[p^\infty] = 0$$

By the sequence (1.5), we get an affirmative answers to (q) and $(q)_p$.

Corollary:

$\operatorname{Pic}(K)[p^\infty] \hookrightarrow \operatorname{Pic}(K;\infty)[p^\infty]$, $\operatorname{Pic}(K;p)[p^\infty] \hookrightarrow \operatorname{Pic}(K;p_\infty)\ [p^\infty]$, are injections, or dualy $X_\infty \twoheadrightarrow X$, $X_{p_\infty} \twoheadrightarrow X_p$, are surjections.

Since X_∞ is a finitely generated torsion $\Lambda = \mathbb{Z}_p[[\Gamma]]$ module, $\Gamma = \operatorname{Gal}(K/k)$, we get

Corollary: X is a finitely generated torsion Λ-module.

Greenberg's conjecture (Q), i.e. the surjectivity of $Pic(K)[p^\infty] \longrightarrow Pic(K;\infty)[p^\infty]$, is now equivalent to the surjectivity of $Pic(K)[p^n] \longrightarrow Pic(K;\infty)[p^n]$ for all n, and this, via the sequence (1.5), is equivalent to the injectivity of

$$\mathcal{D}_\infty(K) \big/ div_\infty(E_\infty(K))\cdot p^n \mathcal{D}_\infty(K) \longrightarrow Pic(K) \big/ p^n(Pic(K)) .$$

Writing this explicitly and applying once more the above proposition, we get that (Q) is equivalent to the "down-to-earth" assertion:

(Q)' given $f\in K^*$, such that its associated ideal, $div'_\infty(f)$, is a p^n th power there exists $g\in K^*$, and a unit $e\in E_\infty(K)$, such that $div_\infty(f) = div_\infty(g^{p^n}.e)$.

§ 4. Fix a finite number field k such that $\mu_p\subseteq k$ and k is Galois over $\mathbb{Q}$. For simplicity we assume also $p\nmid i_k$. Let $\Delta=Gal(k/\mathbb{Q})$, $\Delta^-=Gal(k^+/\mathbb{Q})$, $K=(\mu_{p^\infty})$, $\Gamma=Gal(K/k)$, $G=Gal(K/\mathbb{Q})$, $G^-=Gal(K/k+)$; note that $G^-=\Gamma\times\Delta^-$, but in general the central extension

$$* \longrightarrow \Gamma \longrightarrow G \longrightarrow \Delta \longrightarrow *$$

does not split. Let $\Lambda=\mathbb{Z}_p[[\Gamma]]$, $\Omega=\mathbb{Z}_p[[G]]$, $\Omega^-=\mathbb{Z}_p[[G]] = \Lambda[\Delta^-]$ For any $\mathbb{Z}_p$-module M, set $\tilde{M}=M\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$. We have $\tilde{\Lambda}\subseteq\tilde{\Omega}^-\subseteq\tilde{\Omega}$; $\tilde{\Lambda}$ is a principal ideal domain; and $\tilde{\Lambda}$ is a central subalgebra of $\tilde{\Omega}$, $\tilde{\Omega}$ is a free $\tilde{\Lambda}$-module of rank $[k:\mathbb{Q}]$. Let $\omega: \Delta \twoheadrightarrow Gal(\mathbb{Q}(\mu_p)/\mathbb{Q}) \xrightarrow{\cong} \mu_{p-1}$, $\xi^\delta = \xi^{\omega(\delta)}$ for $\xi\in\mu_p$, $\delta\in\Delta$. Denote by, $sgn: \Delta^- \longrightarrow \{\pm 1\}$, the restriction of ω to Δ^-.

Let $\Delta^\circ=\ker\left\{\mathrm{sgn}\colon \Delta^- \longrightarrow \{\pm 1\}\right\}=\mathrm{Gal}(k/k^\circ)$, where k° is the maximal CM subfield of k . Let

$$S^- = \frac{1}{\#\Delta^-} \sum_{\delta\in\Delta^-} \mathrm{sgn}(\delta)\cdot\delta \in \mathbb{Z}_p[\Delta^-]$$

denote the associated idempotent; it can be viewed as an element of Ω^-, or as an element of the center of Ω . Let $\mathcal{Y}=\mathrm{Gal}(M_\infty/K)$, M_∞ the maximal p-abelian p-ramified extension of K ; $\mathcal{Y}$ is a $\mathbb{Z}_p$-module on which G acts continuously via inner conjugation, hence it is an Ω-module. Let $\mathcal{Y}^-=S^-\mathcal{Y}$, again an Ω-module,and denote by M_∞^- the subfield of M_∞ such that $\mathcal{Y}^-=\mathrm{Gal}(M_\infty^-/K)$. Let $\mathcal{Y}_t^-$ denote the Λ-torsion submodule of $\mathcal{Y}^-$, and set $\mathcal{Z} = \mathcal{Y}^-/\mathcal{Y}_t^-$.

4.1) <u>Theorem:</u> For every non-zero prime $\mathfrak{p}$ of Λ .

$$\tilde{\mathcal{Z}}/\mathfrak{p}\tilde{\mathcal{Z}} \cong S^-\tilde{\Omega}/\mathfrak{p}S^-\tilde{\Omega} \qquad \text{as } \tilde{\Omega}\text{-modules.}$$

4.2) <u>Corallary</u>: For every non-zero $f\in\Lambda$, $\tilde{\mathcal{Z}}/f\cdot\tilde{\mathcal{Z}}$ is a cyclic $\tilde{\Omega}$-module.

Proof of theorem: Let $\mathfrak{U}=\prod_{\mathfrak{p}|p} \mathcal{O}^*_{\mathfrak{p}}$, $\mathcal{O}^*_{\mathfrak{p}}$ the units of the completion of k^* at $\mathfrak{p}$. It is easily verified that $\tilde{\mathfrak{U}}\approx \mathbb{Q}_p[\Delta]$ as $\mathbb{Q}_p[\Delta]$-module. Class field theory gives us a $\mathbb{Q}_p[\Delta]$-surjection $\tilde{\mathfrak{U}}\twoheadrightarrow \mathrm{Gal}\,(M_0^-/k)^\sim$, where M_0^- is the maximal subextension of M_∞^- which is abelian over k . Note that we have $S^-\mathrm{Gal}(M_0^-/k)^\sim = S^-\mathrm{Gal}(M_0^-/K)^\sim = \mathrm{Gal}(M_0^-/K)^\sim = \tilde{\mathcal{Z}}_\Gamma$. Thus we have a $\mathbb{Q}_p[\Delta]$-surjection:

4.3)
$$S^-\tilde{\mathfrak{U}} \longrightarrow \tilde{\mathcal{Z}}_\Gamma$$

Let $K^0=k^0(\mu_{p^\infty})$, and let $\mathcal{Z}^0$ be associated to K^0 just as $\mathcal{Z}$ was associated to K , then by [1], $\tilde{\mathcal{Z}}^0$ is a free $\tilde{\Lambda}$-module of rank $[k^+:\mathbb{Q}]$, from which we deduce:

$$\dim_{\mathbb{Q}_p} \tilde{Z}_\Gamma \geq [k^+:\mathbb{Q}]$$

On the other hand, $\dim_{\mathbb{Q}_p} S^-\tilde{\mathfrak{U}} = \dim_{\mathbb{Q}_p} S^-\mathbb{Q}_p[\Delta] = [k^+:\mathbb{Q}]$,

hence (4.3) is an isomorphism, and the proof of the theorem is finished via the following,

<u>Lemma</u> [4]: Let Z_1, Z_2 be two $\tilde{\Omega}$-modules, free of finite rank as $\tilde{\Lambda}$-modules, and such that $(Z_1)_\Gamma \cong (Z_2)_\Gamma$, as $\mathbb{Q}_p[\Delta]$-modules. Then, for every non-zero prime p of $\tilde{\Lambda}$,

$$Z_1/pZ_1 \cong Z_2/pZ_2 \quad \text{as } \tilde{\Omega}\text{-modules} .$$

Let T denote the Tate module of Pic(K), $T=\mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, \mathrm{Pic}(K))$. Then, as $\tilde{\Omega}$-module, $\tilde{T} \cong \mathrm{Gal}(L^-/K)^\sim$, where L^-=maximal subextension of M_∞^- unramified over K . Assuming K^+ satisfies Greenberg's conjecture, and that all the fields $k(\mu_{p^n})^+$ satisfy Leopoldt's conjecture, it follows that the image of Y_t^- under $Y^- \twoheadrightarrow \mathrm{Gal}(L^-/K)$ has bounded exponent [1; Proof of Theorems 5], hence that we have an $\tilde{\Omega}$-surjection $\tilde{Z} \twoheadrightarrow \tilde{T}$. As is well known, T is a finitely generated torsion Λ-module, hence $f\cdot\tilde{T} = 0$ for some $f \in \Lambda$, and we get an $\tilde{\Omega}$-surjection $\tilde{Z}/f\cdot\tilde{Z} \twoheadrightarrow \tilde{T}$. By corollary (4.2) we see that $\tilde{T}$ is a cyclic $\tilde{\Omega}$-module. If K satisfies (Q), $T = \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, \mathrm{Pic}(K;\infty))$ is the Tate module of the ideal class group of K, and it also follows that K^+ satisfies Greenberg's conjecture, thus we get

4.4) <u>Theorem</u>: If (Q) has an affirmative answer, and if all the fields $k(\mu_{p^n})^+$ satisfy Leopoldt's conjecture, then

$$\mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, \mathrm{Pic}(K;\infty)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is a cyclic $\tilde{\Omega}$-module, $\tilde{\Omega}=\mathbb{Z}_p[[\mathrm{Gal}(K/\mathbb{Q})]] \otimes \mathbb{Q}_p$.

§ 5. The moral of § 3 is that we can glue the places at infinity together without much loss. So let $U^{\vee}(K)$ denote the topological space obtained from $U(K)$ by glueing together all the points at infinity into one point which we shall denote by ∞. Let $\pi^{\vee}: U(K) \longrightarrow U^{\vee}(K)$ denote the glueing map, and let $\underline{O}^*$ denote the sheaf $\pi^{\vee}_* \underline{O}^*$ on $U^{\vee}(K)$; away from infinity nothing changes: $\underline{O}^*(U^{\vee}(K;\infty^-)) = \underline{O}^*(U(K;\infty\bar{v}))$; at infinity we have stalk $\underline{O}^*_\infty = \{f \in K^* | \mathrm{div}_\infty f = o\}$. Let $\underline{M}$ denote the constant sheaf K^* as befor, and define $\underline{D}^{\vee}$ as the sheaf cokernel $* \longrightarrow \underline{O}^* \longrightarrow \underline{M} \longrightarrow \underline{D}^{\vee} \longrightarrow *$. The stalk of $\underline{D}^{\vee}$ at ∞ is just $\underline{D}^{\vee}_\infty = K^*/\underline{O}_\infty = \mathrm{div}_\infty(K^*) \cong \underline{D}_\infty$. We have $\mathrm{Pic}^{\vee}(K;\infty) \overset{\mathrm{def}}{=} H^1(U^{\vee}(K;\infty), \underline{O}^*) \cong \mathrm{Pic}(K;\infty)$, and from the diagram

$$\begin{array}{ccccccccc} * & \longrightarrow & \mathcal{D}^{\vee}_\infty/\mathrm{div}_\infty(E\ ) & \longrightarrow & H^1(\mathcal{U}^{\vee}(K), \underline{O}^*) & \longrightarrow & H^1(U^{\vee}(K;\infty), \underline{O}^*) & \longrightarrow & * \\ & & \downarrow & & \downarrow & & \downarrow & & \\ * & \longrightarrow & \mathcal{D}_\infty/\mathrm{div}_\infty(E_\infty) & \longrightarrow & H^1(\mathcal{U}\ (K), \underline{O}^*) & \longrightarrow & H^1(\mathcal{U}\ (K;\infty), \underline{O}^*) & \longrightarrow & * \end{array}$$

where the top row is the analouge of the relative cohomology sequence (1.4) for $\mathcal{U}^{\vee}$ (or equivalently, from the Leray spectral sequence for $\pi^{\vee}$), we get for $\mathrm{Pic}^{\vee}(K) \overset{\mathrm{def}}{=} H^1(U^{\vee}(K), \underline{O}^*)$

(5.1) $$* \longrightarrow \mathrm{Pic}^{\vee}(K) \longrightarrow \mathrm{Pic}(K) \longrightarrow \mathcal{D}_\infty/\mathcal{D}_\infty^{\vee} \longrightarrow *.$$

The point is that by the proposition of § 3, for $p \nmid i_K$, $\mathcal{D}_\infty/\mathcal{D}^{\vee}_\infty$ has no p-torsion, hence

(5.2) $$\mathrm{Pic}^{\vee}(K)[p^\infty] \cong \mathrm{Pic}(K)\ [p^\infty];\ * \longrightarrow \mathrm{Pic}^{\vee}(K) \otimes \mathbb{Q}_p/\mathbb{Z} \longrightarrow \mathrm{Pic}(K) \otimes \mathbb{Q}_p/\mathbb{Z} \longrightarrow$$

$$\longrightarrow \mathcal{D}_\infty/\mathcal{D}^{\vee}_\infty \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow *$$

These remark will motivate the definition below, where we need infinity to be glued for some technical reasons (see the proof of the Descent Lemma below).

Fix a finite number field k. Let $\tilde{U}(k)$ denote the Grothendieck topology with objects $Y=(\mathrm{spec}\ B_Y, S_Y)$, where B_Y is a flat, quasi-finite, seperated, locally of finite type, $\mathcal{O}_k$-algebra; hence $B_Y \otimes \mathbb{Q} = \prod_{i\in I} L_i$, where L_i/k is a finite extension of number fields, and $I=I_Y$ is the finite set of components of Y; $S_Y \subseteq I_Y$ is just a subset to be thought of as added points at infinity. Maps in $\tilde{U}(k)$ are the usual maps of spec's. Coverings of $\tilde{U}(k)$ are finite "surjective" families where by surjective we mean the usual faithfully flatness on the finite part and the surjectivity of the induced map on the points at infinity.

Remark: If one is interested only in the p-torsion part of $\tilde{\mathrm{Pic}}(K)=H^1(\tilde{U}(K),\underline{\mathcal{O}}^*)$, one can also impose the condition on B_Y that it will be êtale outside p.

Note that $\tilde{U}(k)$ has fibre products hence finite inverse limit exist in $\tilde{U}(k)$. Namely, given $Y_i \longrightarrow Y$, $i=1,2$, we have $B_{Y_1 \times_Y Y_2} = B_{Y_1} \otimes_{B_Y} B_{Y_2}$, and there are maps $\varphi_i : I_{Y_1 \times_Y Y_2} \longrightarrow I_{Y_i}$, and $S_{Y_1 \times_Y Y_2} = \varphi_1^{-1}(S_{Y_1}) \cap \varphi_2^{-1}(S_{Y_2})$.

$\tilde{U}(k)$ has a final object $(\mathrm{spec}\ \mathcal{O}_k, \infty)$ which we shall denote by $U^{\vee}(k)$.

If k'/k is a finite extension of finite fields, then $\tilde{U}(k') = \tilde{U}(k)/U^{\vee}(k') \hookrightarrow \tilde{U}(k)$, and we have a continuous map of topologies $\pi_{k'/k} : \tilde{U}(k') \longrightarrow \tilde{U}(k)$, $\pi^{-1}_{k'/k}(Y) = Y \times_{U^{\vee}(k)} U^{\vee}(k')$, $Y \in \mathrm{Obj}\ \tilde{U}(k)$.

If K is a number field of infinite degree over $\mathbb{Q}$, let $\tilde{U}(K) = \varinjlim \tilde{U}(k)$, the limit taken over finite $k \subseteq K$ with respect to the maps $\pi_{k'/k}$, and we give $\tilde{U}(K)$ the (inverse) limit Grothendieck topology. Thus for a sheaf $F \in S(\tilde{U}(k))$, letting $F_{k'} \in S(\tilde{U}(k'))$, $F_K \in S(\tilde{U}(K))$, denote the induced sheaves,we have $H^*(\tilde{U}(K), F_K) = \varinjlim_{k'} H^*(\tilde{U}(k', F_{k'})$. The objects of $\tilde{U}(K)$ can be thought of as $Y = (\mathrm{spec}\ B_Y, S_Y)$ where B_Y is an algebra as above but over $\mathcal{O}_K$, and S_Y a set of "infinite places"; for such a Y, define $\underline{\mathcal{O}}^*(Y) = \{f \in B_Y^*$ such that $\mathrm{div}_\infty f \equiv o$ for all $\infty \in S_Y\}$. $\underline{\mathcal{O}}^*$ is a sheaf on $\tilde{U}(K)$, and for $Y \in \mathrm{Obj}\ \tilde{U}(K)$ let $\tilde{\mathrm{Pic}}(Y) = H^1(Y, \underline{\mathcal{O}}^*)$; in particular we have $\tilde{\mathrm{Pic}}(K) = \tilde{\mathrm{Pic}}(U^{\vee}(K))$ and $\tilde{\mathrm{Pic}}(K;\infty) = \tilde{\mathrm{Pic}}(U^{\vee}(K;\infty))$.

We say $Y' \longrightarrow Y$ is an open immersion if it is so on the finite part, i.e. $\mathrm{spec}\ B_{Y'} \longrightarrow \mathrm{spec}\ B_Y$ is an open immersion. These open immersions induce a topology on Y which we shall denote simply by Y . On the other hand one has the induced topology $\tilde{Y} = \tilde{U}(K)/Y$. There is a continuous map of topologies $j : \tilde{Y} \longrightarrow Y$, $j^{-1}(Y') = Y'$ viewed as element of $\tilde{Y}$. Since finite inverse limits exists in Y, j^p (pull back of pre-sheaves) is exact, hence j^* (pull back of sheaves) is exact, and so its adjoint j_* takes injectives to injectives and we get

Leray spectral sequenc:

$$H^p(Y, R^q j_* F) \Longrightarrow H^{p+q}(\tilde{Y}, F) \qquad F \in S(\tilde{Y}),\ Y \in \mathrm{Obj}\ \tilde{U}(K) .$$

To apply this to $\underline{\mathcal{O}}^*$ we need,

Descent Lemma $\quad R^1 j_* \underline{\mathcal{O}}^* \Big|_v = \begin{cases} o & v \nmid \infty \\ \underline{\mathcal{D}}_v^{\vee} \otimes \mathbb{Q}/\mathbb{Z} & v \mid \infty \end{cases}$

Proof. Descent theory shows $R^1 j_* \underline{\mathcal{O}}^*$ has support at the infinite places of Y; fix such $v \in S_Y$. Let $\beta \in R^1 j_* \underline{\mathcal{O}}^* |_v$ be represented by $\tilde{\beta} \in H^1(\bar{Y}/Y', \underline{\mathcal{O}}^*)$, $Y' \subseteq Y$ open, $v \in S_{Y'}$. Without loss of generality Y' and $\bar{Y}$ are connected,

$B_{Y'} \otimes \mathbb{Q} = L$, $B_{\bar{Y}} \otimes \mathbb{Q} = F$, F/L .Galois, $G = \mathrm{Gal}(F/L)$. Taking the limit over smaller Y', we see that we have to compute the 1^{st} cohomology group of the complex of elements having $\mathrm{div}_\infty \equiv o$ inside the exact complex:

$$* \longrightarrow F^* \xrightarrow{\partial^o} (F \underset{L}{\otimes} F)^* \xrightarrow{\partial^1} (F \underset{L}{\otimes} F \underset{L}{\otimes} F)^* .$$

This latter complex is isomorphic to the standard complex of G-chains with values in F^*:

$$* \longrightarrow F^* \xrightarrow{\partial^o} \prod_{\sigma \in G} F^* \xrightarrow{\partial^1} \prod_{\sigma, \tau \in G} F^*$$

Now β is represented by $(f_\sigma) \in \prod_{\sigma \in G} F^*$, with $\mathrm{div}_\infty(f_\sigma) = o$ for all $\sigma \in G$, and since $\partial^1(f_\sigma) = o$ and the above complex is exact (Hilbert Theorem 90) we can assume $f_\sigma = g^{1-\sigma}$ for some $g \in F^*$. Since $\prod_{\sigma \in G} g^{1-\sigma} = g^{[F:L]} \cdot \mathbb{N}_{F/L} g^{-1}$,

$[F:L]\ \mathrm{div}_\infty(g) = -\mathrm{div}_\infty(\mathbb{N}_{F/L} g)$, we can define

$\mathrm{Div}_\infty(g) = \mathrm{div}_\infty(N_{F/L}g) \otimes \frac{-1}{[F:L]} \in \mathcal{D}_\infty^{\vee}(L)\otimes \mathbb{Q}$. Changing g by some $g_o \in F^*$ with $\mathrm{div}_\infty(g_o) = o$ will not change $\mathrm{Div}_\infty(g)$, and if $\mathrm{Div}_\infty(g) = \mathrm{div}_\infty(x)$ for some $x\in L^*$ then $f_\sigma = (g/x)^{1-\sigma}$ or $(f_\sigma) = \partial^\circ(g/x)$ with $\mathrm{div}_\infty(g/x) = o$, i.e. f_σ is a co-boundery. Therefor, the map

$R^1 j_*\underline{\mathcal{O}}^*|_v \longrightarrow \underline{\mathcal{D}}_\infty^{\vee}(L)\otimes \mathbb{Q}/\mathbb{Z}$, $\beta \longmapsto \mathrm{Div}_\infty(g) \pmod{\mathbb{Z}}$, is well defined and injective, and by adding n^{th} roots to L it is easily seen to be surjective.

Applying the Leray spectral sequence, noting that $j_*\underline{\mathcal{O}}^*$ is just $\underline{\mathcal{O}}^*$ on Y we get

$$* \longrightarrow H^1(Y,\underline{\mathcal{O}}^*) \longrightarrow H^1(\tilde{Y},\underline{\mathcal{O}}^*) \longrightarrow H^\circ(Y,R^1 j_*\underline{\mathcal{O}}^*) \longrightarrow *$$

and by the Descent lemma the last term is just $\coprod_{v\in S_Y} \underline{\mathcal{D}}_v^{\vee}\otimes\mathbb{Q}/\mathbb{Z}$.

In particular, for $Y=U^{\vee}(K;\infty)$ we get the classical $\mathrm{Pic}^{\vee}(K;\infty) = \widetilde{\mathrm{Pic}}(K;\infty)$, and for $Y=U^{\vee}(K)$

we get:

(5.3) $$* \longrightarrow \mathrm{Pic}^{\vee}(K) \longrightarrow \widetilde{\mathrm{Pic}}(K) \longrightarrow \underline{\mathcal{D}}_\infty^{\vee}\otimes\mathbb{Q}/\mathbb{Z} \longrightarrow *$$

From this we have

(5.4) $$* \longrightarrow \mathrm{Pic}^{\vee}(K)[p^\infty] \longrightarrow \widetilde{\mathrm{Pic}}(K)[p^\infty] \longrightarrow \underline{\mathcal{D}}_\infty^{\vee}\otimes\mathbb{Q}_p/\mathbb{Z}_p \longrightarrow$$

$$\xrightarrow{\partial} \mathrm{Pic}^{\vee}(K)\otimes\mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \widetilde{\mathrm{Pic}}(K)\otimes\mathbb{Q}_p/\mathbb{Z}_p \longrightarrow *$$

And if we want to compare the p-part of $\widetilde{\mathrm{Pic}}(K)$ with our original Pic(K) we get from (5.4) and (5.2):

$$(5.5)\quad * \longrightarrow \mathrm{Pic}(K)[p^{\infty}] \longrightarrow \widetilde{\mathrm{Pic}}(K)[p^{\infty}] \longrightarrow \underline{\mathcal{D}}_{\infty} \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow$$

$$\longrightarrow \mathrm{Pic}(K)\otimes\mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \widetilde{\mathrm{Pic}}(K)\otimes\mathbb{Q}_p/\mathbb{Z}_p \longrightarrow *$$

On the other hand, since $\tilde{U}$ (or rather $\mathcal{S}(\tilde{U})$) is "nothing more" than the mapping cone category of $\tilde{U}(\infty)$ and the category of infinite places, we have a relative cohomolgy sequence for $\tilde{U}$, that can be compared with the relative cohomology sequence for $U^{\vee}$. We get:

$$\begin{array}{ccccccccccccc}
* & \longrightarrow & E & \longrightarrow & E_{\infty} & \xrightarrow{\mathrm{div}_{\infty}} & \underline{\mathcal{D}}_{\infty}^{\vee} & \longrightarrow & \mathrm{Pic}^{\vee}(K) & \longrightarrow & \mathrm{Pic}^{\vee}(K;\infty) & \longrightarrow & * \\
 & & \Vert & & \Vert & & \downarrow & & \downarrow & & \downarrow & & \\
* & \longrightarrow & E & \longrightarrow & E_{\infty} & \longrightarrow & H^1_{\infty}(\tilde{U},\underline{\mathcal{O}}^*) & \longrightarrow & \widetilde{\mathrm{Pic}}\ (K) & \longrightarrow & \widetilde{\mathrm{Pic}}\ (K;\infty) & \longrightarrow & \cdots
\end{array}$$

From this we can read the bottom row, using (5.3) we get:

$$(5.6)\quad * \longrightarrow \underline{\mathcal{D}}_{\infty}^{\vee}\otimes\mathbb{Q}/\mathrm{div}_{\infty}(E_{\infty}) \longrightarrow \widetilde{\mathrm{Pic}}(K) \longrightarrow \widetilde{\mathrm{Pic}}(K;\infty) \longrightarrow *$$

Since the first term of (5.6) is divisible we get:

$$(5.7)\quad * \longrightarrow \left(\underline{\mathcal{D}}_{\infty}^{\vee}\otimes\mathbb{Q}/\mathrm{div}_{\infty}(E_{\infty})\right)[p^{\infty}] \longrightarrow \widetilde{\mathrm{Pic}}(K)[p^{\infty}] \longrightarrow \widetilde{\mathrm{Pic}}(K;\infty)[p^{\infty}] \longrightarrow *$$

$$(5.8)\qquad \widetilde{\mathrm{Pic}}(K)\otimes\mathbb{Q}_p/\mathbb{Z}_p = \widetilde{\mathrm{Pic}}(K;\infty)\otimes\mathbb{Q}_p/\mathbb{Z}_p$$

To "understand" why we have no difficulty with surjectivity on the p-torsion part for $\widetilde{\mathrm{Pic}}$'s, as opposed to the situation with Pic, note that the first term of (5.7) is nothing but $\mathrm{div}_\infty(E_\infty)\otimes\mathbb{Q}_p/\mathbb{Z}_p = E_\infty\otimes\mathbb{Q}_p/\mathbb{Z}_p$, and so (5.7) can be rewritten as

$$(5.9)\quad * \longrightarrow E_\infty\otimes\mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \widetilde{\mathrm{Pic}}(K)[p^\infty] \longrightarrow \mathrm{Pic}(K;\infty)[p^\infty] \longrightarrow *$$

By Kummer duality the first term of this sequence is dual to $G(N_\infty/K)$, the last is dual to $X_\infty = G(M_\infty/N_\infty)$, at least when $\mu_{p^\infty} \subseteq K$. It is possible to check that (5.9) is just the sequence dual to the extension of Galois groups

$$* \longleftarrow G(N_\infty/K) \longleftarrow G(M_\infty/K) \longleftarrow G(M_\infty/N_\infty) \longleftarrow *$$

i.e. that $\widetilde{\mathrm{Pic}}(K)[p^\infty]$ is dual to the "big" $G(M_\infty/K)$ when $\mu_{p^\infty} \subseteq K$.

In order to give a more standard definition of $\mathrm{Pic}(K)[p^\infty]$, one can use the exact sequence $* \longrightarrow \mu_{p^n} \longrightarrow \mathcal{O}^* \xrightarrow{p^n} \mathcal{O}^* \longrightarrow *$ to get $\widetilde{\mathrm{Pic}}(K)[p^n] = H^1(\tilde{U}(K), \mu_{p^n})$; and than via the relative cohomology sequence for μ_{p^n} with respect to $\tilde{U}(K)$ and $\tilde{U}(K;\infty)$, one gets $H^1(\tilde{U}(K), \mu_{p^n}) = H^1(\tilde{U}(K;\infty), \mu_{p^n})$; thus $\mathrm{Pic}(K)[p^\infty] = H^1(\tilde{U}(K;\infty), \mu_{p^\infty})$.

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