Values of the Euler phi function not divisible by a prescribed odd prime

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Abstract

Let φ denote Euler's phi function. For a fixed odd prime q we give an asymptotic series expansion in the sense of Poincaré for the number $\mathcal{E}_q(x)$ of $n \le x$ such that $q \nmid \varphi(n)$. Thereby we improve on a recent theorem by B.K. Spearman and K.S. Williams [Ark. Mat. **44** (2006), 166–181]. Furthermore we resolve, under the Generalized Riemann Hypothesis, which of two approximations to $\mathcal{E}_q(x)$ is asymptotically superior using recent results of Y. Ihara on the Euler-Kronecker constant of a number field.

1 Introduction

Let φ denote Euler's phi function. For a fixed odd prime q we set $\mathcal{E}_q = \{n|q \nmid \varphi(n)\}$ and let $\mathcal{E}_q(x)$ denote the associated counting function. (If $\mathcal{A}$ is any set of integers, then by the associated counting function $\mathcal{A}(x)$ we denote the cardinality of the elements a in $\mathcal{A}$ such that $a \le x$.) Spearman and Williams [16] proved that, as x tends to infinity,

$$\mathcal{E}_q(x) = \frac{e(q)x}{\log^{1/(q-1)} x}\Big(1 + O_\epsilon(\frac{1}{\log^{1-\epsilon} x})\Big), \tag{1}$$

with

$$e(q) = \frac{(q+1)(q-1)^{\frac{q-2}{q-1}}\Gamma(\frac{1}{q-1})\sin(\frac{\pi}{q-1})}{2^{\frac{q-3}{2(q-1)}}q^{\frac{3(q-2)}{2(q-1)}}\pi^{\frac{3}{2}}(h(q)R(q)C(q))^{\frac{1}{q-1}}}, \tag{2}$$

where $h(q)$ denotes the class number of the cyclotomic field $K(q) := \mathbb{Q}(\zeta_q)$ and $R(q)$ its regulator. Spearman and Williams gave a rather involved description of $C(q)$, see Section 3, but we will show that actually $C(q) = C(q,1)$, where for $\mathrm{Re}(s) > 1/2$,

$$C(q,s) = \prod_{\substack{p \ne q \\ f_p \ge 2}} \Big(1 - \frac{1}{p^{sf_p}}\Big)^{\frac{q-1}{f_p}}, \tag{3}$$

where the sum is over all primes $p \ne q$ such that f_p, the smallest integer $k \ge 1$ such that $p^k \equiv 1 (\mathrm{mod}\ q)$, satisfies $f_p \ge 2$. One has $C(3) = \prod_{p \equiv 2 (\mathrm{mod}\ 3)}(1 - 1/p^2)$ for example (this is Lemma 3.1 of [16]).

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The goal of this note is to point out that the theory of Frobenian functions allows one to prove an estimate for $\mathcal{E}_q(x)$ which is much more precise than (1), namely (5). Moreover, we will show that making use of the Euler product for the Dedekind zeta function of a cyclotomic number field, cf. (9), leads to a simplification of the arguments of Spearman and Williams. It allows one for example to infer that $C(q) = C(q,1)$ and to give a very short proof of the estimate (32).

The theory of Frobenian functions was initiated by Landau [5] (and, independently, but only heuristically, by Ramanujan [8]), continued by Bernays (of later fame in logic) in his PhD thesis and much later by Serre [14] and brought in its present state by Odoni, see e.g. [10].

For our purposes Theorem 1, more or less implicit in the work of Landau already, will do. Before stating it, we first define what a Frobenian set of primes is. A set of primes $\mathcal{P}$ is called Frobenius of density δ, if there exists a finite Galois extension $K/\mathbb{Q}$ and a subset H of $G := \mathrm{Gal}(K/\mathbb{Q})$ such that

- H is stable under conjugation;
- $|H|/|G| = \delta$;
- for every prime p, with at most finitely many exceptions, one has p in $\mathcal{P}$ if $\sigma_p(K/\mathbb{Q})$ is in H, where $\sigma_p(K/\mathbb{Q})$ denotes the Frobenius map of p in G (defined modulo conjugation in case p does not divide the discriminant of K).

Theorem 1 [14]. *Let E be a set of integers and E' its complement in the set of natural numbers. Suppose that E' is multiplicative, that is if a and b are coprime positive integers, then*

$$ab \in E' \iff \{a \in E' \text{ or } b \in E'\}.$$

Put $h(s) = \sum_{n\in E'} n^{-s}$. Let P be the set of primes that are in E. Suppose that P is Frobenian of density δ, with $0 < \delta < 1$. Then $h(s)/s$ has an expansion around the point $s = 1$ of the form

$$\frac{h(s)}{s} = \frac{1}{(s-1)^{1-\delta}}(c_0 + c_1(s-1) + \cdots + c_k(s-1)^k + \cdots).$$

Furthermore, for every integer $k \geq 2$ we have

$$E'(x) = \frac{x}{\log^\delta x}\Big(e_0 + \frac{e_1}{\log x} + \cdots + \frac{e_k}{\log^k x} + O(\frac{1}{\log^{k+1} x})\Big),$$

with $e_j = c_j/\Gamma(1-j-\delta)$.

In our problem at hand it turns out that P is the set of primes $p \equiv 1 (\mathrm{mod}\ q)$. But this is precisely the set of primes p that split completely in $K(q)$ and thus $\zeta_{K(q)}(s)$, the Dedekind zeta function of $K(q)$, comes into play. We put $\alpha(q) := \mathrm{Res}_{s=1}\zeta_{K(q)}(s)$. The reader unfamiliar with this material is referred to Section 2.

Theorem 2 *Let q be an odd prime. Put*

$$h_q(s) = \frac{(1-q^{-2s})\zeta(s)}{(C(q,s)(1-q^{-s})\zeta_{K(q)}(s))^{\frac{1}{q-1}}}. \tag{4}$$

Then $h_q(s)/s$ has an expansion around the point $s = 1$ of the form

$$\frac{h_q(s)}{s} = \frac{1}{(s-1)^{(q-2)/(q-1)}}\Big(c_0(q) + c_1(q)(s-1) + \cdots + c_k(q)(s-1)^k + \cdots\Big),$$

For any $k \geq 2$ we have

$$\mathcal{E}_q(x) = \frac{x}{\log^{1/(q-1)} x}\Big(e_0(q) + \frac{e_1(q)}{\log x} + \cdots + \frac{e_k(q)}{\log^k x} + O(\frac{1}{\log^{k+1} x})\Big), \tag{5}$$

where $e_j(q) = c_j(q)/\Gamma(\frac{q-2}{q-1} - j)$. In particular,

$$e_0(q) = \frac{(1-\frac{1}{q^2})}{\Gamma(\frac{q-2}{q-1})(C(q)(1-\frac{1}{q})\alpha(q))^{\frac{1}{q-1}}}.$$

On using that

$$\Gamma(\frac{1}{q-1})\Gamma(\frac{q-2}{q-1}) = \frac{\pi}{\sin\frac{\pi}{q-1}}$$

and formula (8), it is seen after some easy computation that $e(q) = e_0(q)$. Thus the estimate (1), that is the theorem of Spearman and Williams, is a weaker form of Theorem 2.

In the second part of the paper we deal with the problem of whether the

$$\frac{e(q)x}{\log^{1/(q-1)} x} \quad \text{– naive –} \quad \text{or } e(q)\int_2^x \frac{dt}{\log^{1/(q-1)} t} \quad \text{– Ramanujan type–}$$

approximation yields -asymptotically- a better approximation to $\mathcal{E}_q(x)$. For every odd prime q this can be decided using Theorem 5. Using recent results of Ihara [3], which assume the Generalized Riemann Hypothesis (GRH) to be true, we will establish the following theorem.

Theorem 3 (GRH). *Let q be an odd prime. For $q \leq 67$ the Ramanujan type approximation is asymptotically better than the naive approximation for $\mathcal{E}_q(x)$, for all remaining primes the naive approximation is asymptotically better.*

2 Preliminaries

For a general number field K we have, for $\mathrm{Re}(s) > 1$,

$$\zeta_K(s) := \sum_{\mathfrak{a}} \frac{1}{N\mathfrak{a}^s} = \prod_{\mathfrak{p}} \frac{1}{1 - N\mathfrak{p}^{-s}}.$$

The letter $\mathfrak{a}$ will be used to denote a non-zero ideal in $\mathcal{O}_K$, the ring of integers of K, and $\mathfrak{p}$ will be used to denote a non-zero prime ideal in $\mathcal{O}_K$. This function is the *Dedekind zeta function* of K. It is known that the sum and product converge for $\mathrm{Re}(s) > 1$, that $\zeta_K(s)$ can be analytically continued to a neighborhood of 1 (in fact, to the whole complex plane), and that at $s = 1$ it has a simple pole.

Let α_K denote the residue of the pole at $s=1$. It is known that

$$\alpha_K = \frac{2^{r_1}(2\pi)^{r_2} h_K R_K}{w_K \sqrt{|d_K|}}, \tag{6}$$

where r_1 is the number of real infinite primes, r_2 is the number of complex infinite primes, $h(K)$ is the class number of K, $R(K)$ is the regulator of K, $w(K)$ is the number of roots of unity in K, and $D(K)$ is the discriminant of K.

Around $s=1$ we have the Laurent expansion

$$\zeta_K(s) = \frac{\alpha_K}{s-1} + \gamma_K + \gamma_1(K)(s-1) + \gamma_2(K)(s-1)^2 + \cdots \tag{7}$$

The constants $\gamma_j(\mathbb{Q})$ are known as the Stieltjes constants. In particular, we have $\gamma_K = \gamma$, with γ the Euler-Mascheroni constant. The constant $\mathcal{EK}_K := \gamma_K/\alpha_K$ is called the *Euler-Kronecker constant* in Ihara [3] and Tsfasman [17], the reason for this being that in the case when K is imaginary quadratic the well-known Kronecker limit formula expresses γ_K in terms of special values of the Dedekind η function.

2.1 Preliminaries on cyclotomic fields

We recall some facts from the theory of cyclotomic fields needed for our proofs. For a nice introduction to cyclotomic fields see [19].

The following result, see e.g. [9, Theorem 4.16], describes the splitting of primes in the ring of integers of a cyclotomic field.

Lemma 1 *(cyclotomic reciprocity law). Let $K = \mathbb{Q}(e^{2\pi i/m})$. If the rational prime p does not divide m and f is the least natural number such that $p^f \equiv 1 (\mathrm{mod}\ m)$, then (p) (considered as an ideal in the ring of integers of K) equals $\mathfrak{p}_1 \cdots \mathfrak{p}_g$ with $g = \varphi(m)/f$, all $\mathfrak{p}_i$'s distinct and of degree f.*

However, if p divides m, $m = p^a m_1$ with $p \nmid m_1$ and f is the least positive integer such that $p^f \equiv 1 (\mathrm{mod}\ m_1)$, then $(p) = (\mathfrak{p}_1 \cdots \mathfrak{p}_g)^e$ with $e = \varphi(p^a)$, $g = \varphi(m_1)/f$, all $\mathfrak{p}_i$'s being distinct and of degree f.

In case $K = K(q)$, we have $R(K) = r(q)$, $h(K) = h(q)$, $r_1 = 0$, $r_2 = q-1$, $w(K(q)) = 2q$ (as $K(q)$ contains exactly $\{\pm 1, \pm\omega_q, \pm\omega_q^2, \ldots, \pm\omega_q^{q-1}\}$ as roots of unity, with $\omega := e^{2\pi i/(q-1)}$) and furthermore $D(K(q)) = (-1)^{q(q-1)/2} q^{q-2}$, and thus we obtain from (6) that

$$\alpha(q) = \mathrm{Res}_{s=1}\zeta_{K(q)}(s) = 2^{\frac{q-3}{2}} q^{-\frac{q}{2}} \pi^{\frac{q-1}{2}} h(q) R(q). \tag{8}$$

For cyclotomic fields $K(q)$ the Euler product for $\zeta_{K(q)}(s)$ can be written down explicitly using the "cyclotomic reciprocity law". We find that

$$\begin{aligned} \zeta_{K(q)}(s) &= (1-\frac{1}{q^s})^{-1} \prod_{p \neq q} \Big(1 - \frac{1}{p^{sf_p}}\Big)^{\frac{1-q}{f_p}} \\ &= (1-\frac{1}{q^s})^{-1} \prod_{p \equiv 1 (\mathrm{mod}\ q)} \Big(1-\frac{1}{p^s}\Big)^{1-q} \prod_{p\neq q,\ p \not\equiv 1 (\mathrm{mod}\ q)} \Big(1 - \frac{1}{p^{sf_p}}\Big)^{\frac{1-q}{f_p}}. \end{aligned} \tag{9}$$

Thus we can write

$$\zeta_{K(q)}(s) = (1-\frac{1}{q^s})^{-1} \frac{g(s)^{1-q}}{C(q,s)}, \tag{10}$$

where

$$g(s) := \prod_{p \equiv 1 (\mathrm{mod}\ q)} \Big(1 - \frac{1}{p^s}\Big).$$

Let k be a natural number and χ a character modulo k. Let χ_0 be the principal character modulo k. The Dirichlet L-series corresponding to χ is given by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

where $s = \sigma + it \in \mathbb{C}$. For $\chi \neq \chi_0$ the latter series converges for $\sigma > 0$ and

$$L(1, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n} = \prod_p \Big(1 - \frac{\chi(p)}{p}\Big)^{-1} \neq 0.$$

Let g be a primitive root modulo q. For any integer not divisible by q we define the index, denoted by $\mathrm{ind}_g(n)$, of n with respect to g modulo $q-1$ by

$$n \equiv g^{\mathrm{ind}_g(n)} (\mathrm{mod}\ q).$$

Associated with g we define a character χ_g modulo q by

$$\chi_g(n) = \begin{cases} \omega^{\mathrm{ind}_g(n)} & \text{if } q \nmid n; \\ 0 & \text{otherwise.} \end{cases}$$

There are exactly $\varphi(q) = q - 1$ characters modulo q. They are $\chi_0,\ \chi_g,\ \chi_g^2, \cdots, \chi_g^{q-2}$, where $\chi_g^{q-1} = \chi_0$, the trivial character.

It is well-known that

$$\zeta_{K(q)}(s) = \zeta(s) L(s, \chi_g) L(s, \chi_g^2) \cdots L(s, \chi_g^{q-2}). \tag{11}$$

3 The constant $C(q)$

In this section, for the convenience of the reader, we repeat the definition of Spearman and Williams of $C(q)$ and, moreover, we will show that $C(q) = C(q, 1)$.

Spearman and Williams put

$$C(q, r, \chi_q) := \prod_{\chi_g(p) = \omega^r} \Big(1 - \frac{1}{p^{(q-1)/(r,q-1)}}\Big),$$

where the product is taken over all primes p such that $\chi_g(p) = \omega^r$ and $(r, q-1)$ denotes the greatest common divisor of r and $q-1$. Then they define

$$C(q) := \prod_{r=1}^{q-2} C(q, r, \chi_g)^{(r,q-1)}.$$

From this definition it is not a priori clear that it does not depend on the choice of the primitive root g. However, Spearman and Williams show that it indeed does not depend on the choice of the primitive root g.

Proposition 1 *We have $C(q) = C(q,1)$.*

Proof. By the definition of Spearman and Williams we have

$$C(q) = \prod_{r=1}^{q-2} \prod_{\chi_g(p)=\omega^r} \Big(1 - \frac{1}{p^{(q-1)/(r,q-1)}}\Big)^{(r,q-1)}. \tag{12}$$

We claim that if $\chi_g(p) = \omega^r$, then $f_p = (q-1)/(r,q-1)$. We have $1 = \chi_g(p^{f_p}) = \omega^{rf_p}$. It follows that $(q-1)|rf_p$ and thus $q_r := (q-1)/(r,q-1)$ must be a divisor of f_p. On the other hand since $\chi_g(a) = 1$ iff a is the identity, it follows from $\omega^{rq_r} = \chi_g(p^{q_r}) = 1$ and $q_r|f_p$, that $f_p = q_r$. Thus we can rewrite (12) as

$$C(q) = \prod_{r=1}^{q-2} \prod_{\chi_g(p)=\omega^r} \Big(1 - \frac{1}{p^{f_p}}\Big)^{\frac{q-1}{f_p}}. \tag{13}$$

Note that $p \ne q$ and $f_p \ge 2$ iff $\chi_g(p) = \omega^r$ for some $1 \le r \le q-2$. This observation in combination with (13) and the absolute convergence of the double product (13), then shows that $C(q) = C(q,1)$. □

`Remark`. Proposition 1 says that $1/C(q)$ is the contribution at $s = 1$ of the primes $p \ne q$, $p \not\equiv 1 (\text{mod } q)$ to the Euler product (9) of the Dedekind zeta function of $\mathbb{Q}(\zeta_q)$.

4 Proof of Theorem 2

Proof. We apply Theorem 1. We let E be the set of natural numbers n such that $q|\varphi(n)$. Then the couting function we are after is $E'(x)$. The multiplicativity of φ ensures that E' is a multiplicative set. The set of primes P in E consists of all primes p such that $p \equiv 1 (\text{mod } q)$. This set is Frobenian: it consists precisely of the primes p that split completely in $K(q)$. We have $H =$id, $G \cong (\mathbb{Z}/q\mathbb{Z})^*$, $\delta = 1/(q-1)$ and hence $0 < \delta < 1$. Thus all conditions of Theorem 1 are met.

It remains to determine the c_i. For this we have to compute $h(s) = \sum_{n=1}^{\infty} f_q(n)n^{-s}$, where $f_q(n) = 1$ if $q \nmid \varphi(n)$ and $f_q(n) = 0$ otherwise. Note, cf. Proposition 5.1 of [16], that $n \in E'$, that is $q \nmid \varphi(n)$ iff $n = \prod_{p \not\equiv 1 (\text{mod } q)} p^{e_p}$ or $n = q \prod_{p \not\equiv 1 (\text{mod } q)} p^{e_p}$, where the product is taken over all primes $p \ne q$ with $p \not\equiv 1 (\text{mod } q)$ and the e_p are non-negative integers. Thus we have, for $\text{Re}(s) > 1$,

$$\begin{aligned} h(s) &= (1 + \frac{1}{q^s}) \prod_{\substack{p \not\equiv 1 (\text{mod } q) \\ p \ne q}} \frac{1}{1 - p^{-s}} = (1 - \frac{1}{q^{2s}}) \prod_{p \not\equiv 1 (\text{mod } q)} \frac{1}{1 - p^{-s}} \\ &= (1 - q^{-2s})\zeta(s)g(s). \end{aligned} \tag{14}$$

On using (10) to express $g(s)$ in terms of $\zeta_{K(q)}(s)$, we find that $h(s) = h_q(s)$. Now invoke Theorem 1 together with Proposition 1. □

5 On the second order coefficient

5.1 A comparison problem

The prime number states that asymptotically $\pi(x)$, the number of primes $p \le x$, satisfies $\pi(x) \sim x/\log x$. It is well-known that $\text{Li}(x) := \int_2^x dt/\log t$, the logarithmic integral, yields a much better approximation to $\pi(x)$. Likewise one might wonder whether

$$\mathcal{N}_q(x) := \frac{e(q)x}{\log^{1/(q-1)} x} \text{ or } \mathcal{R}_q(x) := e(q)\int_2^x \frac{dt}{\log^{1/(q-1)} t}$$

yields -asymptotically- a better approximation to $\mathcal{E}_q(x)$. The former approximation we will call the '*naive approximation*' and the second the '*Ramanujan type approximation*' to $\mathcal{E}_q(x)$. To be more precise, we say that $\mathcal{N}_q(x)$ is a better approximation to $\mathcal{E}_q(x)$ than $\mathcal{R}_q(x)$ if

$$|\mathcal{N}_q(x) - \mathcal{E}_q(x)| < |\mathcal{R}_q(x) - \mathcal{E}_q(x)|, \tag{15}$$

for all x sufficiently large.

In the history of the theory of Frobenian functions Ramanujan was the first to put forward a problem of this type. If $B(x)$ denotes the counting function of integers $n \le x$ that can be written as sum of two squares, Ramanujan in his first letter (16 Jan. 1913) to Hardy claimed that, for every $r \ge 1$,

$$B(x) = K\int_2^x \frac{dt}{\sqrt{\log t}} + O(\frac{x}{\log^r x}), \tag{16}$$

where K is a certain constant, now called the Landau-Ramanujan constant. Landau [5] had proved in 1908 that $B(x) \sim Kx/\sqrt{\log x}$: a much weaker assertion.

There is some evidence that along with his final letter (12 Jan. 1920) to Hardy, Ramanujan included a manuscript on congruence properties of $\tau(n)$ and $p(n)$, the partition function. In this manuscript, see [1], Ramanujan considers, for various special primes q, the quantity $\sum_{n\le x,\ q\nmid\tau(n)} 1$ and makes claims similar to (16). He defines $t_n = 1$ if $q \nmid \tau(n)$ and $t_n = 0$ otherwise. He then typically writes: "It is easy to prove by quite elementary methods that $\sum_{k=1}^n t_k = o(n)$. It can be shown by transcendental methods that

$$\sum_{k=1}^n t_k = C_q \int_2^n \frac{dx}{(\log x)^{\delta_q}} + O\Big(\frac{n}{(\log n)^r}\Big), \tag{17}$$

where r is any positive number". In each case he gave specific values of C_q and δ_q.

The above claims (16) and (17) were shown to be asymptotically correct by Landau, respectively Rankin. Shanks [15] showed that (16) is false for every $r > 2$. Likewise, the author [6] showed that all claims of the format (17) in the unpublished manuscript to be false for every $r > 2$. The proof involves computing the Euler-Kronecker constant for the generating series $\sum_{k=1}^\infty t_k k^{-s}$ with several decimals of accuracy.

The comparison problem (15) can be studied by computing the second order coefficient $e_1(q)$ in (5) with enough precision. So that is what we set out to do. This will require an excursion in generalized von Mangoldt functions associated to multiplicative functions.

5.2 Generalized von Mangoldt functions

Let f be a nonnegative real-valued multiplicative function. We denote the formal Dirichlet series $F(s) := \sum_{n=1}^{\infty} f(n)n^{-s}$ associated to f by $L_f(s)$. We define $\Lambda_f(n)$ by

$$-\frac{L_f'(s)}{L_f(s)} = \sum_{n=1}^{\infty} \frac{\Lambda_f(n)}{n^s}.$$

The notation suggests that $\Lambda_f(n)$ is an analogue of the von Mangoldt function. Indeed, if $f = \mathbf{1}$, then $L_f(s) = \zeta(s)$ and $\Lambda_f(n) = \Lambda(n)$.

Theorem 4 *In case f is a multiplicative function satisfying $0 \le f(p^r) \le c_1 c_2^r$, $c_1 \ge 1$, $1 \le c_2 < 2$ and $\sum_{p\le x} f(p) = \tau \mathrm{Li}(x) + O(x \log^{-2-\rho} x)$, where τ and ρ are positive real fixed numbers, then there exists a constant B_F such that*

$$\sum_{n\le x} \frac{\Lambda_f(n)}{n} = \tau \log x + B_f + O(\log^{-\rho} x).$$

Moreover, we have

$$\sum_{n\le x} f(n) = \lambda_1(f) x \log^{\tau-1} x \Big(1 + (1+o(1))\frac{\lambda_2(f)}{\log x}\Big),$$

where $\lambda_2(f) = (1-\tau)(1+B_f)$. Alternatively we have

$$B_f = -\lim_{s\to 1+0} \Big(\frac{L_f'(s)}{L_f(s)} + \frac{\tau}{s-1}\Big). \tag{18}$$

Proof. Identity (18) is Lemma 1 of [6]. The remainder is Theorem 4 of [7]. □

Recall that $f_q(n) = 1$ if $q \nmid \varphi(n)$ and $f_q(n) = 0$ otherwise. Note that $L_{f_q}(s) = h_q(s)$. Using equation (14) we find

$$-\frac{L_{f_q}'(s)}{L_{f_q}(s)} = \frac{\log q}{q^s+1} + \sum_{p\ne q,\ p\not\equiv 1 (\mathrm{mod}\ q)} \frac{\log p}{p^s-1} = -\frac{2\log q}{q^{2s}-1} + \sum_{p\not\equiv 1(\mathrm{mod}\ q)} \frac{\log p}{p^s-1},$$

from which we infer that

$$\Lambda_{f_q}(n) = \begin{cases} (-1)^{r+1}\log q & \text{if } n = q^r,\ r\ge 1; \\ \log p & \text{if } n = p^r,\ p \ne q,\ p \not\equiv 1(\mathrm{mod}\ q),\ r \ge 1; \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

We have $h_q(s) = \sum_{n=1}^{\infty} f_q(n) n^{-s}$. By the prime number theorem for arithmetic progressions one has $\sum_{p\le x} f_q(p) = \tau_q \mathrm{Li}(x) + O_q(x\log^{-2-\rho} x)$, with $\tau_q = (q-2)/(q-1)$ and $\rho > 0$ arbitrary. We thus infer by Theorem 4 that

$$\sum_{n\le x} \frac{\Lambda_{f_q}(n)}{n} = \tau_q \log x + B_{f_q} + O_{\rho,q}(\log^{-\rho} x). \tag{20}$$

From Theorem 4, (5) and (20), we infer the following result.

Lemma 2 *Let q be an odd prime. Then*

$$\frac{e_1(q)}{e_0(q)} = \frac{1}{q-1}\Big(1 + B_{f_q}\Big),$$

where

$$B_{f_q} = \lim_{x\to\infty}\Big(\sum_{n\le x}\frac{\Lambda_{f_q}(n)}{n} - (\frac{q-2}{q-1})\log x\Big). \tag{21}$$

The following lemma now shows that our comparison problem can be reduced to a comparison problem for B_{f_q}:

Lemma 3 *The naive approximation gives an asymptotically better approximation to $\mathcal{E}_q(x)$ than the Ramanujan type approximation if $B_{f_q} < -1/2$ (that is in this case inequality (15) holds for all x sufficiently large). If $B_{f_q} > -1/2$ it is the other way around.*

Proof. The result easily follows on noting that, as $x \to \infty$,

$$\mathcal{E}_q(x) = \frac{e(q)x}{\log^{1/(q-1)} x}\Big(1 + \frac{1}{q-1}\frac{(1+B_{f_q})}{\log x} + O_q\Big(\frac{1}{\log^2 x}\Big)\Big),$$

and

$$e(q)\int_2^x \frac{dt}{\log^{1/(q-1)} t} = \frac{e(q)x}{\log^{1/(q-1)} x}\Big(1 + \frac{1}{(q-1)\log x} + O_q\Big(\frac{1}{\log^2 x}\Big)\Big),$$

where the first estimate is a consequence of (5) and Lemma 2 and the latter follows by partial integration. □

We will work out the limit result (21) more explicitly and then use it to approximate B_{f_q}.

Lemma 4 *Put*

$$H_{f_q}(x) = -\frac{2\log q}{q^2-1} + \sum_{\substack{p\le x\\ p\not\equiv 1 (\mathrm{mod}\ q)}}\frac{\log p}{p-1} - (\frac{q-2}{q-1})\log x,$$

and

$$J_{f_q}(x) = -\gamma - \frac{2\log q}{q^2-1} - \sum_{\substack{p\le x\\ p\equiv 1 (\mathrm{mod}\ q)}}\frac{\log p}{p-1} + \frac{\log x}{q-1}.$$

Then $\lim_{q\to\infty} H_{f_q}(x) = \lim_{x\to\infty} J_{f_q}(x) = B_{f_q}$.

In Table 2 one finds computations of $J_{f_q}(10^5)$, $J_{f_q}(10^6)$ and $J_{f_q}(10^7)$. These computations are very easily implemented in MAPLE, say, and give an idea of the true value of B_{f_q}, but unfortunately cannot be used to approximate B_{f_q} with a prescribed degree of accuracy. To that end we will use (18) instead of (21) in Section 5.3.

The proof of Lemma 4 makes use of the following result.

Lemma 5 *For every $\rho > 0$ we have We have*

$$\sum_{p\le x}\frac{\log p}{p-1} = \log x - \gamma + O_\rho(\log^{-\rho} x).$$

Proof. We have

$$\sum_{p\le x}\frac{\log p}{p-1}=\sum_{n\le x}\frac{\Lambda(n)}{n}+\sum_{\substack{p\le x\\ p^r>x}}\frac{\log p}{p^r}, \tag{22}$$

where the sum is over all pairs (p,r) with $p^r>x$. Now note that

$$\sum_{\substack{p\le x\\ p^r>x}}\frac{\log p}{p^r}=O\Big(\frac{1}{x}\sum_{p\le x^{2/3}}\log p\Big)+O(\frac{1}{x^{4/3}}\sum_{x^{2/3}<p\le x}\log p\Big)=O(\frac{1}{x^{1/3}}), \tag{23}$$

where we used the estimate $\sum_{p\le x}\log p=O(x)$. We apply Theorem 4 with $f=\mathbf{1}$ (and so $L_f(s)=\zeta(s)$). The constant B_f was first identified by de la Vallée-Poussin [18], who proved that

$$\gamma=-\lim_{x\to\infty}\Big(\sum_{n\le x}\frac{\Lambda(n)}{n}-\log x\Big),$$

which shows that $B_f=-\gamma$. (Alternatively, we find from (7) and $\gamma(\mathbb{Q})=\gamma$ by logarithmic derivation that

$$\frac{\zeta'(s)}{\zeta(s)}=-\frac{1}{s-1}+\gamma+O(s-1). \tag{24}$$

This in combination with (18) also shows that $B_f=-\gamma$.) It then follows by Theorem 4 and the prime number theorem that

$$\sum_{n\le x}\frac{\Lambda(n)}{n}=\log x-\gamma+O_\rho(\log^{-\rho}x).$$

The result now follows on combining the latter estimate with (22) and (23). □

Proof of Lemma 4. Follows from (20) and (19) on invoking (23) and Lemma 5. □

5.3 Determining B_{f_q} using logarithmic derivation of $f_q(s)$

In this section we use (18) to calculate B_{f_q}. Recall that $L_{f_q}(s)=h_q(s)$. The problem of studying the numerical behaviour of $\mathcal{EK}_{K(q)}$ and the prime sum with enough accuracy will be considered in subsequent sections.

Theorem 5 *We have*

$$B_{f_q}=\frac{(3-q)\log q}{(q-1)(q^2-1)}-\gamma+\frac{\mathcal{EK}_{K(q)}}{q-1}+\sum_{\substack{p\ne q\\ p\not\equiv 1(\mathrm{mod}\ q)}}\frac{\log p}{p^{f_p}-1}, \tag{25}$$

or alternatively,

$$B_{f_q}=\frac{(3-q)\log q}{(q-1)(q^2-1)}-(\frac{q-2}{q-1})\gamma+\frac{1}{q-1}\sum_{k=1}^{q-2}\frac{L'(1,\chi_g^k)}{L(1,\chi_g^k)}+\sum_{\substack{p\ne q\\ p\not\equiv 1(\mathrm{mod}\ q)}}\frac{\log p}{p^{f_p}-1}. \tag{26}$$

Proof. On noting that

$$\frac{1}{q-1}\frac{C'(q,s)}{C(q,s)} = \sum_{\substack{p\neq q\\ p\not\equiv 1 (\mathrm{mod}\ q)}} \frac{\log p}{p^{sf_p}-1},$$

we find by logarithmic differentation of (4) that

$$-\frac{h_q'(s)}{h_q(s)} = -\frac{2\log q}{q^{2s}-1} - \frac{\zeta'(s)}{\zeta(s)} + \frac{1}{q-1}\Big(\frac{\zeta_K'(s)}{\zeta_K(s)} + \frac{\log q}{q^s-1}\Big) + \sum_{\substack{p\neq q\\ p\not\equiv 1 (\mathrm{mod}\ q)}} \frac{\log p}{p^{sf_p}-1}. \qquad (27)$$

For notational convenience we put

$$v_q(s) := \sum_{\substack{p\neq q\\ p\not\equiv 1 (\mathrm{mod}\ q)}} \frac{\log p}{p^{sf_p}-1}.$$

By logarithmic differentiation of the Laurent series (7) we find that

$$\frac{\zeta_{K(q)}'(s)}{\zeta_{K(q)}(s)} = -\frac{1}{s-1} + \mathcal{EK}_{K(q)} + O_q(s-1). \qquad (28)$$

On inserting the latter estimate for $\zeta_{K(q)}'(s)/\zeta_{K(q)}(s)$ and the estimate (24) for $\zeta'(s)/\zeta(s)$ in (27) and setting $\tau_q := (q-2)/(q-1)$, we find that

$$-\frac{h_q'(s)}{h_q(s)} - \frac{\tau_q}{s-1} = -\frac{2\log q}{q^{2s}-1} + \frac{\log q}{(q-1)(q^s-1)} + v_q(s) - \gamma + \frac{\mathcal{EK}_{K(q)}}{q-1} + O_q(s-1).$$

Using (18) we conclude that (25) holds true. By using (11) and (24) we find similarly that (26) holds. (Alternatively one merely combines (25) with (29) to arrive at the same conclusion.) This completes the proof. $\square$

`Remark.` Using (31) one can also express B_{f_q} in terms of the non-trivial zeros of $\zeta_{K(q)}(s)$.

5.4 On the Euler-Kronecker constant for $\mathbb{Q}(\zeta_q)$

Put

$$\mathrm{Cyc}_q(x) = \log x - (q-1)\sum_{\substack{p\le x\\ p\neq q}} \frac{\log p}{p^{f_p}-1} - \frac{\log q}{q-1}.$$

We leave it as an exercise to the reader to show that for any $\rho > 0$ we have

$$\mathrm{Cyc}_q(x) = \mathcal{EK}_{K(q)} + O_{q,\rho}(\log^{-\rho} x).$$

By logarithmic differentation from (11) we find that

$$\mathcal{EK}_{K(q)} = \gamma + \sum_{k=1}^{q-2} \frac{L'(1,\chi_g^k)}{L(1,\chi_g^k)}. \qquad (29)$$

For example, cf. [7],

$$\mathcal{EK}_{K(3)} = \gamma + \frac{L'(1,\chi_{-3})}{L(1,\chi_{-3})} = 0.9454972808716807032397499941581890 73\cdots$$

Ihara [4] conjectures that for any $\epsilon > 0$ we have

$$(\frac{1}{2} - \epsilon)\log q < \mathcal{EK}_{K(q)} < (\frac{3}{2} + \epsilon)\log q,$$

for all q sufficiently large. In [3] he remarks that it seems very likely that always $\mathcal{EK}_{K(q)} > 0$. (This was checked numerically for $q \le 8000$ by Mahora Shimura, assuming GRH.) How large Euler-Kronecker constants can get seems to be a much easier problem then how small they can get. Ihara observed that $\mathcal{EK}_K$ can be conspicuously negative and that this occurs when K has many primes having small norm. However, in the case of $\mathbb{Q}(\zeta_q)$ the smallest norm is q and therefore rather large as q increases.

Ihara has given bounds for $\mathcal{EK}_K$ that are valid under GRH. Specalising his result to the case where $K = K(q)$ one obtains, on invoking the cyclotomic reciprocity law, the following proposition.

Proposition 2 *(Ihara* [3, Proposition 2]*.) Assume that GRH holds. Then we have* $\mathrm{low}_q(x) \le \mathcal{EK}_{K(q)} \le \mathrm{upp}_q(x)$, *with*

$$\mathrm{upp}_q(x) = \frac{\sqrt{x}+1}{\sqrt{x}-1}(\log x - \Phi_{K(q)}(x) + l_q(x)) + \frac{2\kappa_q}{\sqrt{x}-1} - 1,$$

$$\mathrm{low}_q(x) = \frac{\sqrt{x}-1}{\sqrt{x}+1}(\log x - \Phi_{K(q)}(x) + l_q(x)) - \frac{2\kappa_q}{\sqrt{x}+1} - 1,$$

where

$$(x-1)\Phi_{K(q)}(x) = (q-1)\sum_{\substack{p^{f_p k}\le x \\ p\ne q}} \Big(\frac{x}{p^{f_p k}} - 1\Big)\log p + \sum_{q^k \le x}\Big(\frac{x}{q^k} - 1\Big)\log q,\ \ (x > 1),$$

where the first sum is over all prime powers $p^{f_p k}$, $k \ge 1$, *such that* $p^{f_p k} \le x$ *and* $p \ne q$,

$$l_q(x) = \frac{(q-1)}{2}\Big(\log\frac{x}{x-1} + \frac{\log x}{x-1}\Big),$$

and $2\kappa_q = (q-2)\log q - (q-1)(\gamma + \log 2\pi)$.

Moreover, both $upp_q(x)$ *and* $low_q(x)$ *tend to* $\mathcal{EK}_{K(q)}$ *as* x *gets large and thus these bounds allow one to calculate* $\mathcal{EK}_{K(q)}$ *with arbitrary precision.*

Theorem 1 of Ihara [3] implies that, under GRH,

$$\mathcal{EK}_{K(q)} \le \Big(\frac{z_q+1}{z_q-1}\Big)(2\log z_q + 1),$$

with $z_q = \frac{(q-2)}{2}\log q$. A simple analysis shows that this implies that, for $q \ge 23$,

$$\mathcal{EK}_{K(q)} \le 2\log(q\log q). \tag{30}$$

As concerns the zeros ρ of $\zeta_{K(q)}(s)$ in the critical strip, the so called non-trivial zeros, we have by [2],

$$\sum_{\zeta_{K(q)}(\rho)=0} \frac{1}{\rho} = \mathcal{EK}_{K(q)} - (q-1)(\log 2+\gamma) + \frac{1}{2}(q-2)\log q - \frac{(q-1)}{2}\log \pi, \qquad (31)$$

where the zeros are counted with possible multiplicity. Since -at least conjecturally - $\mathcal{EK}_{K(q)}$ is small in comparison with $q\log q$ it seems to 'measure' a subtle effect in the distribution of the zeros.

In Table 1 some data concerning $\mathcal{EK}_{K(q)}$ are gathered. The second and third column give $\mathrm{Cyc}_q(10^5)$, respectively $\mathrm{Cyc}_q(10^6)$. The next two columns give $\mathrm{low}_q(x)$ and $\mathrm{upp}_q(x)$ for the value of x recorded in the sixth column. The final value gives the true value of $\mathcal{EK}_{K(q)}$ as computed with MAGMA (computations of L and L' were implemented in MAGMA by Tim Dokchitser).

5.5 Estimating the prime sum in B_{f_q}

Put

$$v(q) := \sum_{\substack{p\neq q \\ p\not\equiv 1 (\mathrm{mod}\ q)}} \frac{\log p}{p^{f_p}-1}.$$

Note that $v(q)$ is the prime sum arising in our expressions for B_{f_q} in Theorem 5. We can estimate this quantity as follows.

Lemma 6 *For $q \ge 67$ we have*

$$v(q) \le \frac{2[\log(9\log q)][\log q]}{3q}.$$

Our proof of this makes use of the following estimate.

Lemma 7 *For $x \ge 3$ one has*

$$\sum_{p>x} \frac{\log p}{p^2-1} \le \frac{1.055}{x}.$$

Proof. For $x \ge 7481$ one has $0.98x \le \sum_{p\le x}\log p \le 1.017x$, as was shown by Rosser and Schoenfeld [12]. From this one easily infers that for $x \ge 7481$

$$\sum_{p>x} \frac{\log p}{p^k-1} \le \frac{x}{x^k-1}\Big(-0.98 + 1.017\frac{k}{k-1}\Big).$$

A simple further numerical analysis using the latter estimate with $k=2$ then gives the result. □

Proof of Lemma 6. Write

$$v(q) = \sum_{\substack{p<q \\ p\not\equiv 1 (\mathrm{mod}\ q)}} \frac{\log p}{p^{f_p}-1} + \sum_{\substack{p>q \\ p\not\equiv 1 (\mathrm{mod}\ q)}} \frac{\log p}{p^{f_p}-1} = v_1(q) + v_2(q),$$

say. Note that $f_p \ge 3$ in the former sum. If p is odd, then $p^{f_p} = 1 + 2kq$ for some integer k. This together with the observation that $\log t/(t^r - 1)$ is nonincreasing for $t \ge 2$ and $r \ge 2$ fixed and for $r \ge 2$ and $t \ge 2$ fixed, shows that

$$v_1(q) \le \frac{\log 2}{q} + \sum_{k=1}^{m} \frac{1}{3} \frac{\log(1+2kq)}{2kq},$$

where m is the number of odd primes not exceeding q (and thus $m = \pi(q) - 2$).

Now let us refine this estimate for $v_1(q)$ further. Let g be the smallest integer such that $2^g \ge q^2$. We consider the primes $p < q$ for which $f_p \le g$ first. Note that for fixed f there are at most $f - 1$ primes $p < q$ with $p \not\equiv 1 (\mathrm{mod}\ q)$ such that $p^f \equiv 1 (\mathrm{mod}\ q)$. Note that $\sum_{f=2}^{g} (f-1) \le g^2/2$. It follows that the primes p with $f_p \le g$ contribute at most

$$\frac{\log 2}{q} + \sum_{k=1}^{m_1} \frac{1}{3} \frac{\log(1+2kq)}{2kq}$$

to $v_1(q)$, where $m_1 = \min\{m, [g^2/2]\}$. Since trivially $\pi(q) \le (q+1)/2$, we have for $q \ge 3$ that

$$1 + 2kq \le 1 + 2m_1 q \le 1 + 2(\pi(q) - 2)q < q^2$$

and hence, on using that $\sum_{k \le n} 1/k \le \log n + 1$,

$$\sum_{k=1}^{m_1} \frac{1}{3} \frac{\log(1+2kq)}{2kq} \le \frac{\log q}{3q} \sum_{k=1}^{[g^2/2]} \frac{1}{k} \le \frac{\log q}{3q} (\log(\frac{g^2}{2}) + 1) \le \frac{\log q}{3q} (\log(\frac{2\log^2 q}{\log^2 2}) + 1).$$

The primes $2 < p < q$ with $f_p > g$ contribute at most

$$\frac{1}{q^2} \sum_{p<q} \log p \le \frac{1.0012}{q},$$

to $v_1(q)$, where we used the estimate $\sum_{p \le x} \log p < 1.0012x$ valid for $x > 0$ [13, Theorem 6]. We thus find that for $q \ge 3$

$$v_1(q) \le \frac{\log 2}{q} + \frac{\log q}{3q} (\log(\frac{2\log^2 q}{\log^2 2}) + 1) + \frac{1.0012}{q}.$$

On invoking Lemma 7 to estimate $v_2(q)$ we then find that

$$v(q) \le \frac{\log 2}{q} + \frac{\log q}{3q} (\log(\frac{2\log^2 q}{\log^2 2}) + 1) + \frac{1.0012}{q} + \frac{1.055}{q}.$$

On some further analysis the result is easily obtained. □

`Remark`. Note that in case q is a Mersenne prime we have

$$v(q) \ge \frac{\log 2}{2^{f_2} - 1} = \frac{\log 2}{q}.$$

Actually, the only q I have been able to find for which $v(q) > (\log 2)/q$ are the Mersenne primes. It thus is conceivable that if q is not a Mersenne prime, then always $v(q) < (\log 2)/q$. For a given $\epsilon > 0$ it also seems that the primes q for which $v(q) > \epsilon/q$ have density zero. In general $v(q)$ is relatively large if q almost equals a number of the form $p^r - 1$ with p small. For example, if $2q = 3^r - 1$ for some r (e.g. when $r = 3, 7, 13, 71$), then $v(g) > (\log 3)/(2q)$.

5.6 Some numerical data regarding B_{f_q}

It is not difficult to relate B_{f_3} to the constant $B_{g_{3,2}}$ computed with high decimal accuracy in Moree [7, p. 437]. One finds that

$$B_{f_3} = B_{g_{3,2}} + \frac{\log 3}{4} = -\frac{\gamma}{2} + \frac{L'(1,\chi_{-3})}{2L(1,\chi_{-3})} + v(3) = 0.24718078879811624702914196\cdots$$

In Table 2 one finds further values of B_{f_q} with 5 decimal precision. They were computed from (25) using a precise enough approximation of $\mathcal{EK}_{K(q)}$ and $v(q)$. The latter was obtained using that, for $y \geq 3$,

$$\sum_{\substack{p\leq y,\ p\neq q\\ p\not\equiv 1 (\mathrm{mod}\ q)}} \frac{\log p}{p^{f_p}-1} < v(q) \leq \sum_{\substack{p\leq y,\ p\neq q\\ p\not\equiv 1 (\mathrm{mod}\ q)}} \frac{\log p}{p^{f_p}-1} + \frac{1.055}{y},$$

and taking y large enough.

5.7 The proof of Theorem 3

With the following lemma at our disposal we are finally in the position to establish Theorem 3.

Lemma 8 (GRH). *Let $y \geq 3$. We have* $\mathrm{Low}_q(x,y) \leq B_{f_q} \leq \mathrm{Upp}_q(x,y)$ *with*

$$\mathrm{Low}_q(x,y) = \frac{(3-q)\log q}{(q-1)(q^2-1)} - \gamma + \frac{\mathrm{low}_q(x)}{q-1} + \sum_{\substack{p\leq y,\ p\neq q\\ p\not\equiv 1 (\mathrm{mod}\ q)}} \frac{\log p}{p^{f_p}-1},$$

and

$$\mathrm{Upp}_q(x,y) = \frac{(3-q)\log q}{(q-1)(q^2-1)} - \gamma + \frac{\mathrm{upp}_q(x)}{q-1} + \sum_{\substack{p\leq y,\ p\neq q\\ p\not\equiv 1 (\mathrm{mod}\ q)}} \frac{\log p}{p^{f_p}-1} + \frac{1.055}{y}.$$

These estimates allow one to estimate B_{f_q} with arbitrary precision.

Proof. Follows from equality (25) on invoking Proposition 2 and Lemma 7. □

Proof of Theorem 3. Assume GRH. On invoking (25) together with the estimates (30), respectively Lemma 6 for $\mathcal{EK}_{K(q)}$ and $v(q)$, we find that

$$B_{f_q} \leq -\gamma + \frac{2\log(q\log q)}{q} + \frac{2[\log(9\log q)\log q]}{3q},$$

when $q \geq 67$. It is easily proved that the right hand side is monotonically decreasing for $q \geq 67$. On taking $q = 419$ the right hand side equals $-0.50143\ldots$ and so we obtain that $B_{f_q} < -1/2$ for all $q \geq 419$. For $y \geq 3$ we can similarly estimate B_{f_q} as follows:

$$B_{f_q} \leq -\gamma + \frac{2\log(q\log q)}{q} + \sum_{\substack{p< y,\ p\neq q\\ p\not\equiv 1 (\mathrm{mod}\ q)}} \frac{\log p}{p^{f_p}-1} + \frac{1.055}{y}.$$

Using this estimate with y chosen large enough ($y = 1373$ will do) one concludes that also $B_{f_q} < -1/2$ in the range $179 \le q < 419$.

For every prime q in the range $71 \le q \le 173$ one searches for appropriate x and y such that $\mathrm{Upp}_q(x, y) < -1/2$. By Lemma 8 it then follows that $B_{f_q} \le \mathrm{Upp}_q(x, y) < -1/2$ for every prime q in this range.

Finally, for every prime $q \le 67$ one searches for appropriate x and y such that $\mathrm{Low}_q(x, y) > -1/2$. By Lemma 8 it then follows that $B_{f_q} \ge \mathrm{Low}_q(x, y) > -1/2$ for every prime q in this range. □

6 On Mertens' theorem for arithmetic progressions

A crucial ingredient in the paper of Spearman and Williams is an asymptotic estimate for $\prod_{\substack{p\le x \\ p\equiv 1 (\mathrm{mod}\ q)}}(1-1/p)$ [16, Proposition 6.3]. They prove that

$$\prod_{\substack{p\le x \\ p\equiv 1 (\mathrm{mod}\ q)}}(1-\frac{1}{p}) = \Big(\frac{qe^{-\gamma}}{(q-1)\alpha(q)C(q)\log x}\Big)^{\frac{1}{q-1}}\Big(1+O_q(\frac{1}{\log x})\Big). \tag{32}$$

An alternative, much shorter proof of the estimate (32) can be obtained on invoking Mertens' theorem for algebraic number fields.

Lemma 9 *Let α_K denote the residue of $\zeta_K(s)$ at $s = 1$. Then,*

$$\prod_{N\mathfrak{p}\le x}\Big(1-\frac{1}{N\mathfrak{p}}\Big) = \frac{e^{-\gamma}}{\alpha_K \log x}\Big(1+O_K(\frac{1}{\log x})\Big),$$

where the product is over all prime ideals $\mathfrak{p}$ in the ring of integers of K whose norm is less than or equal to x.

Proof. Similar to that of the usual Mertens' theorem, see e.g. Rosen [11]. □

We invoke the latter result with $K = K(q)$ and work out the product over the prime ideals more explicitly using the cyclotomic reciprocity law, Lemma 1. One finds, for $x \ge q$, that it equals

$$(1-\frac{1}{q})\prod_{\substack{p\le x \\ p\equiv 1 (\mathrm{mod}\ q)}}\Big(1-\frac{1}{p}\Big)^{q-1}\prod_{\substack{p\le x,\ p\ne q \\ p\not\equiv 1 (\mathrm{mod}\ q)}}\Big(1-\frac{1}{p^{f_p}}\Big)^{\frac{q-1}{f_p}} =$$

$$(1+O_q(\frac{1}{x}))(1-\frac{1}{q})C(q)\prod_{\substack{p\le x \\ p\equiv 1 (\mathrm{mod}\ q)}}\Big(1-\frac{1}{p}\Big)^{q-1},$$

where we used that for $k \ge 2$,

$$\sum_{p>x}\Big(1-\frac{1}{p^k}\Big)^{-1} = 1+O\Big(\sum_{n>x} n^{-k}\Big) = 1+O(x^{1-k}).$$

Thus, on invoking Lemma 9 we find

$$(1-\frac{1}{q})C(q)\prod_{\substack{p\le x \\ p\not\equiv 1 (\mathrm{mod}\ q)}}\Big(1-\frac{1}{p}\Big)^{q-1} = \frac{e^{-\gamma}}{\alpha(q)\log x}\Big(1+O_q(\frac{1}{\log x})\Big),$$

from which (32) is easily deduced.

Table 1: Approximate numerical values of $\mathcal{EK}_{K(q)}$ with q an odd prime

q	$\mathrm{Cyc}_q(10^5)$	$\mathrm{Cyc}_q(10^6)$	$\mathrm{low}_q(x)$	$\mathrm{upp}_q(x)$	x	true
3	$0.9372\cdots$	$0.9431\cdots$	≥ 0.945	≤ 0.946	$3\cdot 10^5$	$0.94549\cdots$
5	$1.7148\cdots$	$1.7181\cdots$	≥ 1.719	≤ 1.722	$3\cdot 10^5$	$1.72062\cdots$
7	$2.0799\cdots$	$2.0865\cdots$	≥ 2.086	≤ 2.090	10^6	$2.08759\cdots$
11	$2.4216\cdots$	$2.4116\cdots$	≥ 2.411	≤ 2.420	10^6	$2.41542\cdots$
13	$2.6022\cdots$	$2.6050\cdots$	≥ 2.601	≤ 2.615	10^6	$2.61075\cdots$
17	$3.5662\cdots$	$3.5832\cdots$	≥ 3.565	≤ 3.592	10^6	$3.58197\cdots$
19	$4.7659\cdots$	$4.7876\cdots$	≥ 4.765	≤ 4.802	10^6	$4.79040\cdots$
23	$2.6185\cdots$	$2.6090\cdots$	≥ 2.594	≤ 2.635	10^6	$2.61128\cdots$
29	$3.0870\cdots$	$3.0932\cdots$	≥ 3.068	≤ 3.132	10^6	$3.09373\cdots$
31	$4.2759\cdots$	$4.3078\cdots$	≥ 4.264	≤ 4.340	10^6	$4.31444\cdots$
37	$4.3149\cdots$	$4.3155\cdots$	≥ 4.262	≤ 4.363	10^6	$4.30493\cdots$
41	$3.9661\cdots$	$3.9649\cdots$	≥ 3.902	≤ 4.020	10^6	$3.97152\cdots$
43	$4.3408\cdots$	$4.3802\cdots$	≥ 4.318	≤ 4.446	10^6	$4.37862\cdots$
47	$4.8142\cdots$	$4.7925\cdots$	≥ 4.717	≤ 4.865	10^6	$4.79939\cdots$
53	$4.3029\cdots$	$4.3370\cdots$	≥ 4.267	≤ 4.392	$2\cdot 10^6$	$4.33773\cdots$
59	$5.4275\cdots$	$5.4285\cdots$	≥ 5.351	≤ 5.501	$2\cdot 10^6$	$5.43351\cdots$
61	$5.0024\cdots$	$5.0618\cdots$	≥ 4.971	≤ 5.127	$2\cdot 10^6$	$5.07108\cdots$
67	$5.3340\cdots$	$5.2876\cdots$	≥ 5.204	≤ 5.384	$2\cdot 10^6$	$5.29213\cdots$
71	$5.2392\cdots$	$5.2336\cdots$	≥ 5.148	≤ 5.343	$2\cdot 10^6$	$5.25525\cdots$
73	$3.9935\cdots$	$4.0650\cdots$	≥ 3.957	≤ 4.157	$2\cdot 10^6$	$4.06694\cdots$
79	$5.0581\cdots$	$5.0004\cdots$	≥ 4.905	≤ 5.132	$2\cdot 10^6$	$4.99827\cdots$
83	$2.9654\cdots$	$3.0295\cdots$	≥ 2.900	≤ 3.139	$2\cdot 10^6$	$3.03313\cdots$
89	$4.1811\cdots$	$4.1574\cdots$	≥ 3.963	≤ 4.341	10^6	$4.16409\cdots$
97	$4.8455\cdots$	$4.8793\cdots$	≥ 4.660	≤ 5.090	10^6	$4.89124\cdots$
101	$5.2782\cdots$	$5.2883\cdots$	≥ 5.073	≤ 5.530	10^6	$5.29701\cdots$
103	$5.1005\cdots$	$5.1326\cdots$	≥ 4.899	≤ 5.368	10^6	$5.14433\cdots$
107	$5.4382\cdots$	$5.5044\cdots$	≥ 5.232	≤ 5.728	10^6	$5.45827\cdots$
109	$6.9373\cdots$	$6.9267\cdots$	≥ 6.664	≤ 7.179	10^6	$6.90663\cdots$
113	$3.9793\cdots$	$4.0425\cdots$	≥ 3.759	≤ 4.288	10^6	$4.02173\cdots$
127	$5.0040\cdots$	$5.0705\cdots$	≥ 4.763	≤ 5.390	10^6	$5.08859\cdots$
131	$2.8372\cdots$	$2.8495\cdots$	≥ 2.550	≤ 3.917	10^6	$2.83682\cdots$
137	$4.9312\cdots$	$4.9205\cdots$	≥ 4.607	≤ 5.303	10^6	$4.93700\cdots$
139	$5.8719\cdots$	$5.8953\cdots$	≥ 5.546	≤ 6.260	10^6	$5.88916\cdots$
149	$6.0227\cdots$	$5.9895\cdots$	≥ 5.611	≤ 6.396	10^6	$5.98342\cdots$
151	$5.1040\cdots$	$5.0604\cdots$	≥ 4.679	≤ 5.474	10^6	$5.04201\cdots$
157	$7.4201\cdots$	$7.4053\cdots$	≥ 7.007	≤ 7.855	10^6	$7.40802\cdots$
163	$5.9314\cdots$	$5.9475\cdots$	≥ 5.522	≤ 6.409	10^6	$5.92966\cdots$
167	$8.1704\cdots$	$8.0129\cdots$	≥ 7.596	≤ 8.520	10^6	$8.03300\cdots$
173	$3.4172\cdots$	$3.3853\cdots$	≥ 2.924	≤ 3.874	10^6	$3.38434\cdots$

Table 2: Numerical data related to the evaluation of B_{f_q}

q	$J_{f_q}(10^5)$	$J_{f_q}(10^6)$	$J_{f_q}(10^7)$	true	$v(q)$
3	$+0.2430\cdots$	$+0.2460\cdots$	$+0.2469\cdots$	$+0.24718\cdots$	$0.35164\cdots$
5	$-0.1042\cdots$	$-0.1034\cdots$	$-0.1029\cdots$	$-0.10281\cdots$	$0.07777\cdots$
7	$-0.1347\cdots$	$-0.1336\cdots$	$-0.1334\cdots$	$-0.13348\cdots$	$0.12282\cdots$
11	$-0.3419\cdots$	$-0.3429\cdots$	$-0.3425\cdots$	$-0.34255\cdots$	$0.00910\cdots$
13	$-0.3268\cdots$	$-0.3266\cdots$	$-0.3262\cdots$	$-0.32617\cdots$	$0.04620\cdots$
17	$-0.3584\cdots$	$-0.3574\cdots$	$-0.3576\cdots$	$-0.35751\cdots$	$0.00443\cdots$
19	$-0.3087\cdots$	$-0.3074\cdots$	$-0.3074\cdots$	$-0.30734\cdots$	$0.01100\cdots$
23	$-0.4627\cdots$	$-0.4631\cdots$	$-0.4630\cdots$	$-0.46308\cdots$	$0.00082\cdots$
29	$-0.4703\cdots$	$-0.4701\cdots$	$-0.4701\cdots$	$-0.47009\cdots$	$0.00034\cdots$
31	$-0.4014\cdots$	$-0.4003\cdots$	$-0.4002\cdots$	$-0.40015\cdots$	$0.03658\cdots$
37	$-0.4589\cdots$	$-0.4589\cdots$	$-0.4591\cdots$	$-0.45919\cdots$	$0.00092\cdots$
41	$-0.4797\cdots$	$-0.4797\cdots$	$-0.4794\cdots$	$-0.47957\cdots$	$0.00044\cdots$
43	$-0.4755\cdots$	$-0.4746\cdots$	$-0.4747\cdots$	$-0.47468\cdots$	$0.00021\cdots$
47	$-0.4740\cdots$	$-0.4745\cdots$	$-0.4744\cdots$	$-0.47441\cdots$	$0.00012\cdots$
53	$-0.4956\cdots$	$-0.4949\cdots$	$-0.4949\cdots$	$-0.49494\cdots$	$0.00021\cdots$
59	$-0.4847\cdots$	$-0.4846\cdots$	$-0.4845\cdots$	$-0.48460\cdots$	$0.00006\cdots$
61	$-0.4934\cdots$	$-0.4924\cdots$	$-0.4922\cdots$	$-0.49232\cdots$	$0.00143\cdots$
67	$-0.4970\cdots$	$-0.4977\cdots$	$-0.4976\cdots$	$-0.49767\cdots$	$0.00026\cdots$
71	$-0.5025\cdots$	$-0.5026\cdots$	$-0.5023\cdots$	$-0.50234\cdots$	$0.00061\cdots$
73	$-0.5211\cdots$	$-0.5201\cdots$	$-0.5200\cdots$	$-0.52013\cdots$	$0.00137\cdots$
79	$-0.5125\cdots$	$-0.5132\cdots$	$-0.5132\cdots$	$-0.51332\cdots$	$0.00049\cdots$
83	$-0.5416\cdots$	$-0.5408\cdots$	$-0.5408\cdots$	$-0.54077\cdots$	$0.00007\cdots$
89	$-0.5299\cdots$	$-0.5301\cdots$	$-0.5301\cdots$	$-0.53010\cdots$	$0.00034\cdots$
97	$-0.5270\cdots$	$-0.5266\cdots$	$-0.5265\cdots$	$-0.52657\cdots$	$0.00017\cdots$
101	$-0.5248\cdots$	$-0.5247\cdots$	$-0.5247\cdots$	$-0.52467\cdots$	$0.00001\cdots$
103	$-0.5276\cdots$	$-0.5272\cdots$	$-0.5272\cdots$	$-0.52717\cdots$	$0.00003\cdots$
107	$-0.5262\cdots$	$-0.5256\cdots$	$-0.5260\cdots$	$-0.52609\cdots$	$0.00003\cdots$
109	$-0.5133\cdots$	$-0.5134\cdots$	$-0.5135\cdots$	$-0.51362\cdots$	$0.00002\cdots$
113	$-0.5420\cdots$	$-0.5414\cdots$	$-0.5416\cdots$	$-0.54164\cdots$	$0.00002\cdots$
127	$-0.5318\cdots$	$-0.5313\cdots$	$-0.5311\cdots$	$-0.53121\cdots$	$0.00591\cdots$
131	$-0.5556\cdots$	$-0.5555\cdots$	$-0.5556\cdots$	$-0.55564\cdots$	$0.00002\cdots$
137	$-0.5411\cdots$	$-0.5412\cdots$	$-0.5411\cdots$	$-0.54113\cdots$	$0.00003\cdots$
139	$-0.5348\cdots$	$-0.5346\cdots$	$-0.5346\cdots$	$-0.53471\cdots$	$0.00079\cdots$
149	$-0.5367\cdots$	$-0.5369\cdots$	$-0.5370\cdots$	$-0.53700\cdots$	$0.00000\cdots$
151	$-0.5433\cdots$	$-0.5436\cdots$	$-0.5438\cdots$	$-0.54378\cdots$	$0.00002\cdots$
157	$-0.5297\cdots$	$-0.5298\cdots$	$-0.5298\cdots$	$-0.52986\cdots$	$0.00006\cdots$
163	$-0.5407\cdots$	$-0.5406\cdots$	$-0.5407\cdots$	$-0.54078\cdots$	$0.00001\cdots$
167	$-0.5281\cdots$	$-0.5291\cdots$	$-0.5289\cdots$	$-0.52899\cdots$	$0.00000\cdots$
173	$-0.5575\cdots$	$-0.5576\cdots$	$-0.5576\cdots$	$-0.55769\cdots$	$0.00001\cdots$

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