Perturbation theory for quasiperiodic solutions of infinite–dimensional Hamiltonian systems

3. Proof of the main theorem

by

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Absender:

Stein mit den Händen zu fassen
Da kam aber eine neue Welle,
riß ihn vom Stein ab, und stürzte
ihn nachdem mit dem Kopf an den
Stein. In dieser Zeit bin ich nicht
weit von ihm zum Ufer gelangt,
und als ich kam zu dem Orte, wo
er war, erblickte ich ihn im Wasser,
bin zu ihm geschwommen, habe ihn
herausgezogen, er aber war schon tot.
Ich habe ihn dort in Batz beerdigt.
Ich schreibe Ihnen nur diese kurze
Karte, Sie wissen doch, daß mein
Herz mir wirklich zerbrochen ist.
Ihr Paul Alexandroff.

Postkarte
mit Antwortkarte

Herrn Professor
Hausdorff
~~Bonn a/Rh.~~
~~Hindenburgstraße~~

[redacted]

Contents

This paper is devoted to a proof of the Theorem 1.1 formulated in [2]. In §1 we reformulate the theorem in more general form (Theorem 1.1 + Statement 1.2) and prove it in § 2. The proof is based on lemmas from §§ 3 – 5.

We use the notations from [1,2] and some new one. A list of them is given at the end of the paper. Sometimes we refer the reader to the formulas from [2]. We write (2.2.3) for the formula (2.3) from [2] and so on. We use the abbreviations r.h.s. (l.h.s) for "right—hand—side" ("left—hand—side") and write ε_0 instead of ε. By " $\varepsilon_0 << 1$" ("K >> 1") we mean "positive ε_0 is small enough" (" K is large enough").

1. Preliminary transformations.

In a symplectic Hilbert scale $\{Z,\{Z_s \mid s \in \mathbb{R}\}, \alpha(a) = \langle \bar{J}^Z(a)\, dz, dz \rangle_Z\}$ (see [1]) we study a Hamiltonian equation with a hamiltonian

$$\mathcal{H}(z;a,\varepsilon_0) = \frac{1}{2} \langle A^Z(a)\, z,z \rangle_Z + \varepsilon_0\, H(z;a,\varepsilon_0)\,;$$

i. e. the equation

$$\dot{z} = J^Z(a)(A^Z(a)z + \varepsilon_0 \nabla H(z;a,\varepsilon_0))\,,\ \ J^Z(a) = -(\bar{J}^Z(a))^{-1}. \tag{1.1}$$

Here $a \in \mathfrak{A} \subset\subset \mathbb{R}^n$ is a n–dimensional parameter, $\varepsilon_0 \in [0,1]$ is a small parameter, H is an analytical function, $J^Z(a)$, $A^Z(a)$ are linear operators and for some Hilbert basis $\{\varphi_j^\pm \mid j \geq 1\}$ of the space Z the following relations take place:

$$J^Z(a)\,\varphi_j^{\pm} = \mp\,\lambda_j^J(a)\,\varphi_j^{\mp} \qquad \forall j\,,\ \forall a\,, \tag{1.2}$$

$$A^Z(a)\,\varphi_j^{\pm} = \lambda_j^A(a)\,\varphi_j^{\pm} \qquad \forall j\,,\ \forall a\,. \tag{1.3}$$

For the exact assumptions on equation (1.1) see [2].

1.1. Change of the symplectic structure.

The numbers $\{\lambda_j^J(a)\}$ are nonzero $\forall j,a$ and are positive for all j large enough (see (2.1.4), (2.1.18)). So after unessential exchange $\varphi_j^{\pm}$ on $\varphi_j^{\mp}$ for some finite number of indexes j we may suppose that $\lambda_j^J(a) > 0$ $\forall j,a$. Let us consider a linear operator L_a which maps $\varphi_j^{\pm}$ into $(\lambda_j^J(a))^{1/2}\,\varphi_j^{\pm}$, $j = 1, 2, \dots$. By assumption (2.1.18) this operator defines an isomorphism of the scale $\{Z_s\}$ of order $d_J/2$, $L_a : Z_s \xrightarrow[\sim]{} Z_{s-d_J/2}$ $\forall s$. It is selfadjoint in Z with the domain of definition $Z_{d_J/2}$. By Corollary 2.3 from [1] the mapping L_a^{-1} transforms solutions of the Hamiltonian equation (1.1) in the symplectic Hilbert scale $\{Z, \{Z_s\}, \alpha(a)\}$ into solutions of a Hamiltonian equation with a hamiltonian $\mathscr{H}_1(z;a,\varepsilon_0) = \frac{1}{2} < A_1(a)\,z,z >_Z + \varepsilon_0\,H_1(z;a,\varepsilon_0)$ in a symplectic Hilbert scale $\{Z, \{Z_s\}, \alpha_1(a) = < \bar{J}_1(a)\,dz, dz >_Z\}$. Here

$$\bar{J}_1(a) = L_a \bar{J}^Z(a)\,L_a,\ \ A_1(a) = L_a A^Z(a)\,L_a,\ \ H_1 = H(L_a\,z;a,\varepsilon_0)\,.$$

By the definition of the operator L_a and by (1.2) one has

$$\bar{J}_1(a)\,\varphi_j^{\pm} = \mp\,\varphi_j^{\mp} \qquad \forall j,\ \forall a\,. \tag{1.4}$$

So operator $\bar{J}_1(a)$ does not depend on the parameter a, $J_1 = -(\bar{J}_1)^{-1} = \bar{J}_1$, and a Hamiltonian equation with the hamiltonian $\mathscr{H}_1$ has the form

$$\dot{z} = J_1(A_1(a)\, z + \varepsilon_0 \nabla H_1(z;a,\varepsilon_0)) \tag{1.5}$$

and

$$\nabla H_1 = L_a \nabla H_1(L_a\, z;a,\varepsilon_0) \tag{1.6}$$

Let us denote by $\mathscr{L}(Z_s;\, Z_{s_1})$ the space of linear continuous operators from Z_s to Z_{s_1} with the operator norm $\|\cdot\|_{s,\, s_1}$, and by $L : \mathfrak{A} \longrightarrow \mathscr{L}(Z_s;\, Z_{s-d_J/2})$ the mapping $a \mapsto L_a$.

Lemma 1.1. For every s

$$\mathrm{Lip}(L : \mathfrak{A} \longrightarrow \mathscr{L}(Z_s;\, Z_{s-d_J/2})) \le C\,. \tag{1.7}$$

For every s and every a

$$\|\, L_a \,\|_{s,s-d_J/2} + \|\, L_a^{-1} \,\|_{s,s+d_J/2} \le C_1\,. \tag{1.8}$$

Proof. An operator $L_{a_1} - L_{a_2}$ is diagonal in the basis $\{\varphi_j^{\pm}\}$ with eigenvalues $\Delta \ell_j^{\pm} = (\lambda_j^J(a_1))^{1/2} - (\lambda_j^J(a_2))^{1/2}$. For the assumptions (2.1.18), (2.2.19)

$$|\Delta \varphi_j^{\pm}| \le \frac{K_1 |a_1 - a_2|\, j^{d_J}}{2 \min (\lambda_j^J(a_1)^{1/2},\ \lambda_j^J(a_2)^{1/2})} \le \frac{K_1^{3/2}}{2} |a_1 - a_2|\, j^{d_J/2}$$

and inequality (1.7) results from (2.1.2). Inequaltiy (1.8) results from (2.1.2) and (2.1.18). ∎

Let us denote $d' = d + \frac{1}{2} d_J$ and $T_a(I) = L_a^{-1} T(I)$, $\mathcal{T}_a = \cup \{T_a(I) \mid I \in \mathcal{J}\}$, $O^c_{d',a} = L_a^{-1} O_d^c$. Then

$$T_a(I) = \{ \sum_{j=1}^{n} \alpha_j^{\pm} \varphi_j^{\pm} | \alpha_j^{+2} + \alpha_j^{-2} = 2 I_j^a, j = 1,\dots,n\}, \qquad I_j^a = \frac{I_j}{\lambda_j^J(a)}$$

and by the assumption (2.1.13) and estimate (1.8)

$$\mathrm{dist}_Z(\mathcal{T}_a ; Z^c_{d'} \setminus O^c_{d',a}) > \gamma > 0 \qquad \forall a \in \mathfrak{A}. \qquad (1.9)$$

By the analyticity assumption (2.1.15), Lemma 1.1 and identity (1.6) one can see that the mappings

$$H_1 : O^c_{d',a} \times \mathfrak{A} \times [0,1] \longrightarrow \mathbb{C}$$

$$\nabla H_1 : O^c_{d',a} \times \mathfrak{A} \times [0,1] \longrightarrow Z^c_{d'-d_H-d_J} \qquad (1.10)$$

are complex–analytical with respect to the first variable and Lipschitz with respect to the second one unifomly with respect to $\varepsilon_0 \in [0,1]$.

The operator $A_1(a)$ is an isomorphism of the scale $\{Z_s\}$ of the order $d_1 = d_A + d_J$ and

$$A_1(a)\,\varphi_j^{\pm} = \lambda_j(a)\,\varphi_j^{\pm} \qquad \forall j, \forall a . \tag{1.11}$$

Equation (1.5) satisfies conditions 2) of the theorem with $d_A' = d_A + d_J$, $d_J' = 0$, $d_H' = d_H + d_J$. So it is sufficient to prove the theorem in a case $d_J = 0$.

1.2. A change of parameter.

The statements of the theorem are local with respect to the parameter a. So one may replace the set $\mathfrak{A}$ of parameters a by arbitrary δ_a–neighbourhood $\mathfrak{A}(a_0,\delta_a)$ of the point a_0 in $\mathfrak{A}$. If positive δ_a is sufficiently small, then for the assumptions (2.1.7), (2.1.8) the mapping

$$\omega : \mathfrak{A}(a_0,\delta_a) \longrightarrow \mathbb{R}^n, \qquad a \mapsto \omega(a) = (\lambda_1(a),\dots,\lambda_n(a))$$

is a C^1–differomorphism on some neighbourhood Ω_0 of the point $\omega_0 = \omega_0(a_0)$ and

$$\text{Lip}\,\omega + \text{Lip}\,\omega^{-1} \le K^1, \tag{1.12}$$

$$\text{diam}\,\Omega_0 \le K^1 \delta_{a,} \tag{1.13}$$

$$K^{-1}\delta_a^n \le \text{mes}\,\Omega_0 \le K\,\delta_a^{\,n} . \tag{1.14}$$

So Lipschitz dependence on the parameter $a \in \mathfrak{A}(a_0,\delta_a)$ is equivalent to Lipschitz dependence on the parameter $\omega \in \Omega_0$.

1.3. A transition to angle variables

In what follows we use the notation $O(Q,\delta,B)$ for the δ–neighbourhood of a subset Q of a metric space B ; for a Banach space Z we write $O(\delta,Z)$ instead of $O(0,\delta,Z)$.

Let us set $Z^0 \subset Z$ be equal to 2n–dimensional linear span of the vectors $\{\varphi_j^{\pm} \mid j \le n\}$ and $Y_s \subset Z_s$, $s \in \mathbb{R}$, be equal to the closure in Z_s of a linear span of the vectors $\{\varphi_j^{\pm} \mid j \ge n+1\}$ and $Y = Y_0$. For a vector from Z^0 let $\{\chi_j^{\pm} \mid 1 \le j \le n\}$ be its coefficients for the basis $\{\varphi_j^{\pm} \mid j \le n\}$. In some small enough neighbourhood of a torus $T_a(I)$ let us change coordinates $\{\chi_j^{\pm}\}$ to (q,ξ), $q \in \mathbb{T}^n$, $\xi \in O(3\delta_0,\mathbb{R}^n)$ $(\delta_0 << 1)$:

$$q_j = \operatorname{Arg}(\chi_j^- + i\chi_j^+), \qquad \xi_j = \tfrac{1}{2}\left[{\chi_j^+}^2 + {\chi_j^-}^2\right] - I_j^a . \tag{1.15}$$

Let us consider toroidal spaces $\mathscr{Y}_s = \mathbb{T}^n \times \mathbb{R}^n \times Y_s$, $s \in \mathbb{R}$, with a natural metric dist_s and tangent spaces $T_u \mathscr{Y}_s \cong \mathbb{R}^n \times \mathbb{R}^n \times Y_s = E_s$, $u \in \mathscr{Y}_s$. Let J be a restriction on Y_s of the operator J_1, i.e. $J\varphi_j^{\pm} = \mp \varphi_j^{\mp}$ $\forall j \ge n+1$ (see (1.4)); let

$$J^T : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \times \mathbb{R}^n, \ (\delta q, \delta\xi) \mapsto (\delta\xi, -\delta q) ,$$

and

$$J^{\mathscr{Y}} = J^T \times J^Y : E_s = (\mathbb{R}^n \times \mathbb{R}^n) \times Y_s \longrightarrow E_s .$$

Let us introduce in $\mathscr{Y}_s$, $s \ge 0$, a symplectic structure with the help of 2–form $\alpha^{\mathscr{Y}} = \langle J^{\mathscr{Y}} d\eta, d\eta \rangle_E$. The triple $\{\mathscr{Y}_0, \{\mathscr{Y}_s\}, \alpha^{\mathscr{Y}}\}$ is a toroidal symplectic Hilbert scale. See [1], § 4, for details.

For the fixed $s \in \mathbb{R}$, $I \in \mathcal{J}$, $\omega \in \Omega_0$ and $\delta_0 << 1$ let us consider a map

$$L : \mathbb{T}^n \times O(3\delta_0, \mathbb{R}^n) \times Y_s \longrightarrow Z_s, \quad (q,\xi,y) \mapsto \sum_{j=1}^{n} \chi_j^{\pm}\varphi_j^{\pm} + y$$

(see (1.15)). It defines a complex–analytical diffeomorphism of a domain

$$Q^c(s) = O(\mathbb{T}^n \times \{0\} \times \{0\}, 3\delta_0, \mathscr{Y}_s^{\,c}) \subset \mathscr{Y}_s^{\,c} = (\mathbb{C}^n / 2\pi\mathbb{Z}^n) \times \mathbb{C}^n \times Y_s^{\,c} \tag{1.16}$$

on a complex neighbourhood of $T_a(I)$ in Z_s. This diffeomorphism is Lipschitz on I and on ω (via the dependence $a = a(\omega)$), i.e.

$$L \in \mathscr{A}^R_{\Omega_0 \times \mathcal{J}}(Q^c(s); Z_s^{\,c}) \tag{1.17}$$

for all s.

The subspaces $Z^0 \subset Z$, $Y \subset Z$ are skew–ortogonal with respect to the 2–form $\alpha_1 = \langle \bar{J}_1\, dz, dz \rangle_Z$. A restriction of α_1 on Z^0 is of the form $d\chi^- \wedge d\chi^+$ and, so, it is equal to $d\xi \wedge dq$ (see [A]). A restriction of the form $\alpha^{\mathscr{Y}}$ on Z^0 is $d\xi \wedge dq$, too. Hence

$$L^*\alpha_1 = L^*(d\chi^- \wedge d\chi^+ + \langle J^1 dy, dy \rangle_Z) = d\xi \wedge dq + \langle J\, dy, dy \rangle_Y = \alpha^{\mathscr{Y}}$$

and the map L is canonical. So the equation (1.1) in the coordinates (q,ξ,y) is Hamiltonian with the hamiltonian

$$\mathscr{H}_0(q,\xi,y;\omega,I,\varepsilon_0) = \text{const} + \sum_{j=1}^{n} \xi_j\omega_j + \tfrac{1}{2} < A(\omega)\, y,y > + \varepsilon_0 H^0(q,\xi,y;\omega,I,\varepsilon_0) \qquad (1.18)$$

(see [1], Proposition 4.1). Here we use the identity

$$\tfrac{1}{2} < A_1(\omega)\, z_0, z_0 >_Z = \tfrac{1}{2}\sum_{j=1}^{n} \lambda_j(\omega)\left[\chi_j^{+2} + \chi_j^{-2}\right] = \sum \omega_j\xi_j$$

$$\forall z_0 = \sum_{j=1}^{n} \chi_j^{+}\,\varphi_j^{+} + \chi_j^{-}\,\varphi_j^{-} \in Z^0\,,$$

denote by $A(\omega)$ a restriction of the operator $A_1(\omega)$ on the space Y and denote by $< \cdot\,,\cdot > = < \cdot\,,\cdot >_Y$ a scalar product in Y induced from Z. The Hamiltonian equations have a form

$$\dot{q}_j = \omega_j + \varepsilon_0 \frac{\partial}{\partial\xi_j} H^0, \qquad \dot{\xi}_j = -\varepsilon_0 \frac{\partial}{\partial q_j} H^0$$

$$\dot{y} = J(A(\omega)y + \varepsilon_0 \nabla_y H^0)\,. \qquad (1.19)$$

Let us set $\Theta_0 = \Omega_0 \times \mathcal{I}$. A Borel set $\mathcal{I}$ is the same as in (2.1.11), i.e.

$$\mathcal{I} \subset \{I \in \mathbb{R}^n \,|\, K^{-1} \le I_j \le K \qquad \forall j = 1,\dots,n\}\,. \qquad (1.20)$$

It results from (1.9), (1.17) and from the analyticity of the mappings (1.10) that $\forall\varepsilon_0 \in [0,1]$

$$|H^0(\cdot\,;\cdot\,,\varepsilon_0)|^{Q^c(d'),\theta_0} \le K_1', \ \|\nabla_y H^0(\cdot\,;\cdot\,,\varepsilon_0)\|^{Q^c(d'),\theta_0}_{d-d_H-d_J} \le K_1'$$

if in (1.16) $\delta_0 << 1$.

The operator $A(\omega)$ has the double spectrum $\{\lambda_j(\omega)\,|\,j = n+1, n+2, \ldots\}$ and the operator $JA(\omega)$ has the spectrum $\{\pm i\lambda_j(\omega)\,|\,j \ge n+1\}$. Let us shift the numeration:

$$\lambda_j(\omega) := \lambda_{j+n}(\omega), \qquad \varphi_j^{\pm} := \varphi_{j+n}^{\pm}, \qquad \lambda_j^{(s)} := \lambda_{j+n}^{(s)}$$

and redenote

$$d_H := d_H + d_J, \qquad d := d' = d + \tfrac{1}{2} d_J .$$

Then by the condition (2.1.2) the set of vectors $\{\varphi_j^{\pm}\,\lambda_j^{(-s)}\,|\,j \ge 1\}$ is a Hilbert basis of Y_s and for some new K

$$K^{-1} j^s \le \lambda_j^{(s)} \le K j^s, \ \ \lambda_j^{(-s)} = (\lambda_j^{(s)})^{-1} \ \ \forall j \ge 1, \forall s \in \mathbb{R} \tag{1.21}$$

By this condition the scale $\{Y_s\}$ is interpolational. See below appendix A.

For the shifted sequence $\{\lambda_{j0} = \lambda_j(\omega_0)\}$ relation (2.1.16) takes place with the same d_1, some new r , $K_2^{\,1}, \ldots, K_2^{\,r-1}$, $d_{1,1}, \ldots, d_{1,r-1}$ and some new K^1 . For all $j \ge 1$, $\omega \in \Omega_0$

$$A(\omega)\,\varphi_j^{\pm} = \lambda_j(\omega)\,\varphi_j^{\pm} \,,\ J\,\varphi_j^{\pm} = \mp\,\varphi_j^{\mp} \tag{1.22}$$

and

$$\lambda_j(\omega) > 0 \qquad \forall j \geq j_o . \tag{1.22'}$$

Theorem 1.1 from [2] may be reformulated for equations (1.19). Here we formulate some more general result. For to do it , we suppose that the operator $A(\omega)$ depends on ε_0 , $A = A(\omega,\varepsilon_0)$; so $\lambda_j = \lambda_j(\omega,\varepsilon_0)$, and $\lambda_{j0} = \lambda_{j0}(\varepsilon_0)$. We suppose that

$$\varepsilon_0 \, H^0 = \varepsilon_0 H_0(q,\xi,y;\theta,\varepsilon_0) + H^3(q,\xi,y;\theta,\varepsilon_0) ,$$

functions H_0, H^3 may be continued to complex–analytic functions on a domain $Q^c(d)$, $d \geq \frac{1}{2} d_1$. It is supposed that $\forall \varepsilon_0 \in [0,1]$

$$|H_0(\cdot\,;\cdot\,;\varepsilon_0)|^{Q^c(d),\Theta_0} + \|\nabla_y H_0(\cdot\,;\cdot\,;\varepsilon_0)\|^{Q^c(d),\Theta_0}_{d-d_H{}^0} \leq K_1 \tag{1.23}$$

and $\forall \mathfrak{h} = (q,\xi,y) \in Q^c(d)$

$$|H^3(\mathfrak{h}\,;\cdot\,,\varepsilon_0)|^{\Theta_0,\mathrm{Lip}} \leq K_1(|\xi|^2 + |\xi|\,\|y\|_d + \|y\|_d{}^3) ,$$

$$|\nabla_\xi H^3(\mathfrak{h}\,;\cdot\,,\varepsilon_0)|^{\Theta_0,\mathrm{Lip}} \leq K_1(|\xi| + \|y\|_d) , \tag{1.24'}$$

$$\|\nabla_y H^3(\mathfrak{h}\,;\cdot\,,\varepsilon_0)\|^{\Theta_0,\mathrm{Lip}}_{d-d_H} \leq K_1(|\xi| + \|y\|_d{}^2) . \tag{1.24}$$

Here

$$d_H \leq d_1 - 1, \; d_H{}^0 \leq 0 \;, \; d_H^0 < d_1 - 1 . \tag{1.25}$$

In the terms of the decomposition $\varepsilon_0 H^0 = \varepsilon_0 H_0 + H^3$ the results may be formulated in more exact way, important for some applications (for example, to perturbed KdV equation).

Theorem 1.1. Let the conditions (1.20) – (1.25) hold together with

1) $d_1 \geq 1$ and

$$| \lambda_{j0} - K_2 j^{d_1} - K_2{}^1 j^{d_{1,1}} - ... - K_2{}^{r-1} j^{d_{1,r-1}} | \leq K_1 j^{d_{1,r}} \tag{1.26}$$

for some $K_2 = K_2(\varepsilon_0) > 0$, $r \geq 1$, $K_2{}^j = K_2{}^j(\varepsilon_0) \in \mathbb{R}$ $(j = 1, \dots, r-1)$ and for $d_1 > d_{1,1} > \dots > d_{1,r-1} > d_{1,r}$ such that

$$d_1 - 1 > d_{1,r}, \qquad K_3^{-1} \leq K_2(\varepsilon_0) \leq K_3, \qquad | K_2{}^j(\varepsilon_0) | \leq K_3 \qquad \forall j, \forall \varepsilon_0 ;$$

moreover

$$\mathrm{Lip}(\lambda_j : \Omega_0 \longrightarrow \mathbb{R}) \leq K_1 j^{d_{1,r}} \qquad \forall j, \forall \varepsilon_0 ; \tag{1.27}$$

2) if $d_H > 0$ then it is required that for $d_c = d + d_1 - 1 - d_H{}^0$ weak in $\mathcal{Y}_{d_c}$ solutions of (1.19) with initial conditions in an arbitrary set $O(\mathbb{T}^n \times \{0\} \times \{0\}, D, \mathcal{Y}_c)$ exist for some time $T = T(D) > 0$ and stay inside a set $O(\mathbb{T}^n \times \{0\} \times \{0\}, K_4, \mathcal{Y}_c)$

with some $K_4 = K_4(D)$.

Then there exist integer j_1, M_1 such that if a condition

$$3) \quad |s \cdot \omega_0 + \ell_1\lambda_{10} + \ell_2\lambda_{20} + \dots + \ell_{j_1}\lambda_{j_1 0}| \geq K_5 > 0 \tag{1.28}$$

$$\forall s \in \mathbf{Z}^n, \qquad |s| \leq M_1, \quad \forall \ell \in \mathbf{Z}^{j_1}, \qquad 1 \leq |\ell_1| + \dots + |\ell_{j_1}| \leq 2 ,$$

is satisfied, then for sufficiently small $\varepsilon_0 > 0$ there exist $\delta_a > 0$ sufficiently small and independent on ε_0 (see (1.13), (1.14)), a Borel subset $\Theta_{\varepsilon_0} \subset \Theta_0$ and analytic embeddings

$$\sum\nolimits^{\varepsilon_0}_{(\omega,I)} : \mathbf{T}^n \longrightarrow \mathscr{Y}_{d_c}, \quad (\omega,I) \in \Theta_{\varepsilon_0}, \quad d_c = d + d_1 - d_H^0 - 1 ,$$

with the following properties:

$$\text{a)} \quad \text{mes}\, \Theta_{\varepsilon_0}[I] \longrightarrow \text{mes}\, \Omega_0 \quad (\varepsilon_0 \longrightarrow 0) \tag{1.29}$$

uniformly with respect to $I \in \mathscr{J}$;

b) the mapping

$$\sum\nolimits^{\varepsilon_0} : \mathbb{T}^n \times \Theta_{\varepsilon_0} \longrightarrow \mathscr{Y}_{d_c}, \qquad (q,\omega,I) \mapsto \sum\nolimits^{\varepsilon_0}_{(\omega,I)}(q) , \tag{1.30}$$

is Lipschitz and is close to the mapping

$$\sum^{0} : \mathbf{T}^n \times \Theta_{\varepsilon_0} \longrightarrow \mathcal{Y}_{d_c}, \qquad (q,\omega,I) \mapsto (q,0,0) \in \mathcal{Y}_{d_c}.$$

That is

$$\mathrm{dist}_{d_c}\Big[\sum^{0}(q;\omega,I), \sum^{\varepsilon_0}(q;\omega,I)\Big] \le C_\varrho \varepsilon_0^{\varrho} \qquad \forall \varrho < 1/3\,,$$

$$\mathrm{Lip}\Big[\sum^{0} - \sum^{\varepsilon_0} : \mathbf{T}^n \times \Theta_{\varepsilon_0} \longrightarrow \mathcal{Y}_{d_c}\Big] \le C_\varrho \varepsilon_0^{\varrho} \qquad \forall \rho < 1/3\,; \tag{1.31}$$

c) every torus $\sum^{\varepsilon_0}_{(\omega,I)}(\mathbb{T}^n)$, $(\omega,I) \in \Theta_{\varepsilon_0}$, is invariant for the equations (1.19) and is filled with weak in $\mathcal{Y}_d$ solutions of the form $z^{\varepsilon_0}(t) = \sum^{\varepsilon_0}_{(\omega,I)}(q + \omega' t)$, $q \in \mathbb{T}^n$, $\omega' = \omega'(\omega,I,\varepsilon_0) \in \mathbb{R}^n$ and

$$|\,\omega - \omega'\,| \le C\,\varepsilon_0^{1/3}\,; \tag{1.32}$$

d) if $d_H \le 0$ or if $d_H > 0$ and $\forall (q,\xi,y) \in Q^c(d) \cap \mathcal{Y}_d$ we have

$$< J(\nabla_y H^3(q,\xi,y))_* \,|\, _Y \delta y, \delta y >_d \;\le C\, \|\delta y\|_d^2\, \|y\|_d \qquad \forall \delta y \in Y_\infty, \forall \theta,\varepsilon\,,$$

then all Liapunow exponents of the solutions $z^{\varepsilon_0}(t)$ are equal to zero.

Statement 1.2. Under the assumptions of Theorem 1.1 a sharper form of estimates (1.31), (1.32) is true:

$$\operatorname{dist}_{d_c}\left[\sum^{0}(q;\omega,I), \sum^{\varepsilon_0}(q;\omega,I)\right] \le C\,\varepsilon_0\,, \tag{1.33}$$

$$|\,\omega-\omega'\,| \le C_1\,\varepsilon_0\,. \tag{1.34}$$

Remarks.

1) For a rather general theorem applicable to verify the assumption 2) of Theorem 1.1 see [K].

2) For a discussion of assumptions (1.23) – (1.25) see § 7.1 below.

2. Proof of Theorem 1.1

We extend the scalar product $< \cdot , \cdot >$ to a bilinear over $\mathbb{C}$ map $Y^c \times Y^c \longrightarrow \mathbb{C}$ and denote by $\mathscr{L}^s(Y_{s_1}{}^c ; Y_{s_2}{}^c)$ a subspace of operators $L \in \mathscr{L}(Y^c_{s_1} ; Y_{s_2}{}^c)$ symmetric with respect to $< \cdot , \cdot >$, i.e. $< Ly_1, y_2 > = < y_1, Ly_2 > \ \forall y_1, y_2 \in Y^c_\infty$. We denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We shall use the following domains in $\mathbb{C}^n / 2\pi \mathbb{Z}^n$, $\mathscr{Y}_s$ and $\mathscr{Y}_s{}^c$:

$$U(\delta) = \{\xi \in \mathbb{C}^n / 2\pi \mathbb{Z}^n \mid |\mathrm{Im}\, \xi| < \delta\},$$

$$O^c(\xi_0, \xi_1, \xi_2; \mathscr{Y}_s{}^c) = U(\xi_0) \times O(\xi_1, \mathbb{C}^n) \times O(\xi_2, Y_s{}^c) .$$

Let us fix some

$$\gamma_* \in (0,1], \qquad \rho \in (0, \tfrac{1}{3}) \tag{2.1}$$

and set $\gamma_0 = 2(1^{-2} + 2^{-2} + \dots)$,

$$e_m = \begin{cases} 0, & m = 0 \\ (1^{-2} + \dots + m^{-2})\gamma_0{}^{-1}, & m \geq 1 , \end{cases} \tag{2.2}$$

$$\varepsilon_m = \varepsilon_0{}^{(1+\varrho)^m}, \quad \delta_m = \delta_0(1 - e(m)), \qquad U_m = U(\delta_m) , \tag{2.3}$$

$$O_m{}^c = O^c(\delta_m, \varepsilon_m{}^{2/3}, \varepsilon_m{}^{1/3}; \mathscr{Y}_d^c) , \quad O_m = O^c_m \cap \mathscr{Y}_d .$$

We shall need some subdomains of U_m and $O_m{}^c$. For this end let us set

$$\delta_m^j = \frac{6-j}{6}\,\delta_m + \frac{j}{6}\,\delta_{m+1}\,,\ 0 \le j \le 5 \tag{2.4}$$

(so $\delta_m = \delta_m{}^0 > \delta_m{}^1 > \dots > \delta_m{}^5$) , and denote

$$O_m{}^{jc} = O^c(\delta_m{}^j, (2^{-j}\varepsilon_m)^{2/3}, (2^{-j}\varepsilon_m)^{1/3}; \mathscr{Y}_d{}^c),\ U_m{}^j = U(\delta_m{}^j)\,.$$

If $\varepsilon_0 << 1$ then $2^{-j}\varepsilon_m > \varepsilon_{m+1}$, $j = 1, \dots, 5$, and so the domains $O_m{}^{jc}$ are neighbourhoods of O_{m+1}^c, $O_m{}^c \supset O_m{}^{1c} \supset \dots \supset O_m{}^{5c} \supset O_{m+1}^c$.

We denote by $C, C_1, C_2, \dots$ different positive constants independent of ε_0 and m ; by $C(m), C_1(m), \dots$ different functions of m of the form $C(m) = C_1 m^{C_2}$; by $C^e(m)$, $C_1{}^e(m), \dots$ different functions of the form $\exp C(m)$. By $C_*, C_{*1}, \dots, C_*(m)$, $C_{*1}(m), \dots$ we denote fixed constants and functions of the form $C(m)$. Let us mention that $\forall\, C(m)$, $\forall\, C^e(m)$ and $\forall\, \sigma > 0$

$$C(m) \le \varepsilon_m^\sigma \quad \forall m\,, \qquad C^e(m) \le \varepsilon_m^\sigma \quad \forall m \qquad \text{if } \varepsilon_0 << 1\,.$$

Let $m \in \mathbb{N}_0$ and Θ_m be a Borel subset of $\Theta_0 = \Omega_0 \times \mathscr{J}$ such that

$$\text{mes}\,\Theta_m[I] \ge K_6(1 - \gamma_*\, e(m)) \qquad \forall I \in \mathscr{J}\,. \tag{2.5}$$

Here $K_6 = \text{mes}\,\Omega_0$ and γ_* as in (2.1).

We shall denote a pair $(\omega, I) \in \Theta_m$ by θ and shall omit dependence of functions and sets on the parameter ε_0 . All estimates will be uniform with respect to $\varepsilon_0 \in [0,1]$.

At domain $O_m^{\,c}$ let us consider a hamiltonian depending on the parameter $\theta \in \Theta_m$;

$$\mathscr{H}_m = H_{0m}(q,y;\theta) + \varepsilon_m H_m(q,\xi,y;\theta) + H^3(q,\xi,y;\theta)\,, \tag{2.6}$$

$$H_{0m} = \xi \cdot \Lambda_m(\theta) + \tfrac{1}{2} < A_m(q;\theta)y,y > . \tag{2.7}$$

Here the function H^3 is the same as in (1.24) and

$$\Lambda_m : \Theta_m \longrightarrow \mathbb{R}^n,\ |\Lambda_m(\omega,I) - \omega|^{\Theta_m,\,\mathrm{Lip}} \le \varepsilon_0{}^{\rho} e(m); \tag{2.8}$$

the operator $A_m(q;\theta)$ is equal to $A(\theta) + A_m^1(q;\theta)$ and

$$A_m^1(q;\theta)\,\varphi_j^{\pm} = \beta_{jm}(q;\theta)\,\varphi_j^{\pm} \qquad\qquad \forall j\,, \tag{2.9}$$

$$\beta_{jm} \in \mathscr{A}^R_{\Theta_m}(U_m;\mathbb{C}), \qquad |\beta_{jm}|^{U_m,\Theta_m} \le \varepsilon_0{}^{\rho} e(m)\, j^{d_H^0}\,. \tag{2.10}$$

We suppose that $H_m \in \mathscr{A}^R_{\Theta_m}(O_m^c;\mathbb{C})$ and

$$|H_m|^{O_m^c;\Theta_m} \le C_*(m) \equiv K_7{}^{m+1}\,, \tag{2.11}$$

$$\|\nabla_y H_m\|^{O_m{}^c,\Theta_m}_{d\ -d_H^0} \le \varepsilon_m{}^{-1/3} C_*(m)\,. \tag{2.12}$$

For $m = 0$ the hamiltonian $\mathscr{H}_0$ in (1.18) has a form (2.6) with $\Lambda_0(\omega,I) \equiv \omega$, $A_0^1 \equiv 0$ and the assumptions (2.11), (2.12) are fulfilled by the theorem's assumptions.

Hamiltonian equations with the hamiltonian $\mathscr{H}_m$ have a form

$$\dot{q} = \Lambda_m(\theta) + \nabla_\xi(\varepsilon_m H_m + H^3)\,(q,\xi,y;\theta)\,, \tag{2.13}$$

$$\dot{\xi} = -\nabla_q(\tfrac{1}{2} < A_m(q;\theta)y,y > + (\varepsilon_m H_m + H^3)\,(q,\xi,y;\theta))\,, \tag{2.14}$$

$$\dot{y} = J(A_m(q;\theta)y + \nabla_y(\varepsilon_m H_m + H^3)\,(q,\xi,y;\theta))\,. \tag{2.15}$$

For $m = 0$ these equations coincide with the equations (1.19).

The theorem will be proved via KAM–procedure. For $m = 0,1,2,\ldots$ we shall construct canonical transformation $S_m : O_{m+1} \longrightarrow O_m$ which is well–defined for $\theta \in \Theta_{m+1}$ and transforms the equations (2.13) – (2.15) into Hamiltonian equations in O_{m+1} with a hamiltonian of form (2.6) with $m := m + 1$. For $\theta \in \Theta_{\varepsilon_0} = \cap\Theta_m$ the limit transformation $\sum^{\varepsilon_0} : S_0 \circ S_1 \circ \ldots$ transforms equations (1.19) into an equation in a set $\cap O_m = \mathbb{T}^n \times \{0\} \times \{0\}$. The last one has solutions $(q + t\Lambda_\infty(\theta),0,0)$, $\Lambda_\infty = \lim \Lambda_m$, $q \in \mathbb{T}^n$. So for $\theta \in \Theta_{\varepsilon_0}$ equation (1.19) has desired quasiperiodic solutions of a form $\sum^{\varepsilon_0}(q_0 + t\Lambda_\infty\,, 0\,, 0)$.

Let us extract from H_m a linear on ξ and quadratic on y part:

$$H_m(q,\xi,y;\theta) = h^q(q;\theta) + \xi \cdot h^{1\xi}(q;\theta) + < y,h^y(q;\theta) > +$$
$$+ < y,h^{yy}(q;\theta)y > + H_{3m}(q,\xi,y;\theta)\,, \tag{2.16}$$

$$H_{3m} = O(|\xi|^2 + \|y\|_d^3 + |\xi|\ \|y\|_d)\,. \tag{2.17}$$

Here $h^q \in \mathbb{C}$, $h^{1\xi} \in \mathbb{C}^n$, $h^y \in Y^c$ and h^{yy} is an operator in the scale $\{Y_s\}$. We may vary H_m on a constant depending on θ and so may suppose that

$$\int h^q(q;\theta)\, dq/(2\pi)^n = 0\,. \tag{2.18}$$

Here and in what follows

$$\int f(q)\, dq/(2\pi)^n = (2\pi)^{-n} \int_{\mathbb{T}^n} f(q)\, dq$$

for an arbitrary vector–valued function integrable on $\mathbb{T}^n$. Let us define a function $h^{0\xi}(\theta) = \int h^{1\xi}(q;\theta) dq/(2\pi)^n$ and set

$$h^\xi(q;\theta) = h^{1\xi} - h^{0\xi}, \qquad \Lambda_{m+1} = \Lambda_m + \varepsilon_m h^{0\xi} \tag{2.19}$$

and rearrange the terms of $\mathscr{H}_m$ in the following way:

$$\mathscr{H}_m = H'_{0m}(q,\xi,y;\theta) + \varepsilon_m(H_{2m} + H_{3m}) + H^3 \tag{2.20}$$

Here

$$H'_{0m} = \xi \cdot \Lambda_{m+1}(\theta) + \tfrac{1}{2} < A_m(q;\theta)y,y > \,,\ H_{3m} = H_m - H_{2m},$$

$$H_{2m} = h^q + \xi \cdot h^\xi + < y,h^y > + < y,h^{yy}y > \,.$$

Lemma 2.1. If $\varepsilon_0 << 1$ then

a)
$$|h^q|^{U_m,\Theta_m} \le C_*(m), \tag{2.21}$$

$$|h^\xi|^{U_m,\Theta_m} \le 2C_*(m)\varepsilon_m^{-2/3},\ |h^{0\xi}|^{\Theta_m,Lip} \le C_*(m)\varepsilon_m^{-2/3}, \tag{2.22}$$

$$\|h^y\|^{U_m,\Theta_m}_{d-d_H^0} \le C_*(m)\,\varepsilon_m^{-1/3}; \tag{2.23}$$

b)
$$h^{yy}(q,\theta) \in \mathscr{L}^s(Y_d, Y_{d-d_H^0}) \qquad \forall q \in \mathbf{T}^n, \forall\theta$$

$$\|h^{yy}\|^{U_m,\Theta_m}_{d\,,\,d-d_H^0} \le C_*(m)\,\varepsilon_m^{-2/3}; \tag{2.24}$$

c) if in (2.11) $K_7 >> 1$ then

$$|H_{3m}|^{O_{m+1},\Theta_m} + \varepsilon_m^{1/3}\|\nabla_y H_{3m}\|^{O^c_{m+1},\Theta_m}_{d-d_H^0} \le \tfrac{1}{8}C_*(m+1)\varepsilon_m^\rho; \tag{2.25}$$

d)
$$H'_{0m}, H_{2m}, H_{3m} \in \mathscr{A}^R_{\Theta_m}(O_m^c; \mathbb{C});$$

e)
$$|\Lambda_{m+1}(\theta) - \omega|^{\Theta_m,Lip} \le \varepsilon_0{}^\rho e(m+1). \tag{2.26}$$

Proof.

a) The estimate (2.21) results from (2.11) because $h^q(q;\theta) = H_m(q,0,0;\theta)$. To estimate the mapping $h^{1\xi}$ let us define a function of an argument $z \in \mathbb{C}$, $|z| < \varepsilon_m{}^{2/3}$:

$$z \longrightarrow H_m(q,z\xi,0;\theta),\ \xi \in \mathbb{C}^n,\ |\xi| \leq 1 .$$

For (2.11) its module is no greater than $C_*(m)$ and for the Cauchy estimate its derivative at zero is no greater than $\varepsilon_m^{-2/3}\,C_*(m)$. So $|\xi \cdot h^{1\xi}(q;\theta)| \leq \varepsilon_m^{-2/3}\,C_*(m)$ $\forall |\xi| \leq 1$ and $|h^{1\xi}| \leq \varepsilon_m^{-2/3}\,C_*(m)$. By considering a function $z \longrightarrow H_m(q,z\xi,0;\theta_1) - H_m(q,z\xi,0;\theta_2)$, one can get an analogous estimate for the Lipschitz constant on θ . So $|h^{1\xi}|^{U_m,\Theta_m} \leq \varepsilon_m^{-2/3}\,C_*(m)$. From this estimates results (2.22).

The estimate (2.23) results from (2.12) with $y = 0$.

b) Let us consider a map

$$\{|z| < \varepsilon_m^{1/3}\} \longrightarrow Y^c_{d-d_H^0}, \qquad z \mapsto \nabla_y H_m(q,0,zy;\theta) , \tag{2.27}$$

($\|y\|_d \leq 1$). Its derivative at zero is equal to $h^{yy}(q;\theta)y$. So by (2.12) and Cauchy estimate

$$\|h^{yy}y\|^{U_m,\Theta_m}_{d-d_H^0} \leq \varepsilon_m^{-2/3}\,C_*(m) \qquad \forall\ \|y\|_d \leq 1 .$$

The last estimate implies (2.24). The inclusion $h^{yy} \in \mathscr{L}^s(Y_d;Y_{d-d_H^0})$ results from the general fact that Hessian of a function is a symmetric linear operator.

c) Let $\mathfrak{h} = (q,\xi,y) \in O^c_{m+1}$ and $\nu = \varepsilon_m^{\rho/3}$. Then $(q,(z/\nu)^2\xi,(z/\nu)y) \in O_m^c$ for $z \in \mathbb{C}$, $|z| \leq 1$. Let us consider a function $z \longrightarrow H_m(q,(z/\nu)^2\xi,\ (z/\nu)y;\theta)$ and its Taylor series at zero:

$$H_m(q,(\tfrac{z}{\nu})^2\xi,\ (\tfrac{z}{\nu})y;\theta) = h_0 + h_1 z + h_2 z^2 + \dots .$$

By (2.11), $|h_k| \leq C_*(m)\ \forall k$. Since $H_{3m}(\mathfrak{h};\theta) = h_3\nu^3 + h_4\nu^4 + \dots$, then

$$|H_{3m}(\mathfrak{h};\theta)| = |h_3\nu^3 + h_4\nu^4 + \dots| \leq \frac{C_*(m)\ \varepsilon_m{}^{\rho}}{1-\varepsilon_m{}^{\rho/3}} \leq \frac{C_*(m+1)\,\varepsilon_m{}^{\rho}}{8}$$

if $K_7 >> 1$. In a similar way one can estimate a Lipschitz constant of H_{3m} .

To estimate $\nabla_y H_{3m}$ let us consider a map

$$z \longrightarrow \nabla_y H_m(q,(\tfrac{z}{\nu})^2\xi,\ (\tfrac{z}{\nu})y;\theta) = h_0{}' + h_1{}' z + \dots \in Y^c_{d-d_H^0} .$$

By (2.12) $\|h_k'\|_{d-d_H^0} \leq \varepsilon_m^{-1/3}\ C_*(m)\ \forall k$. So

$$\|\nabla_y H_{3m}(\mathfrak{h};\theta)\|_{d-d_H^0} = \|h_2'\nu^2 + h_3'\nu^3 + \dots\|_{d-d_H^0} \leq$$
$$\leq \frac{\nu^2}{1-\nu}\varepsilon_m^{-1/3}\ C_*(m) \leq \frac{1}{8}\,\varepsilon_{m+1}^{-1/3}\,\varepsilon_m^{\rho}\ C_*(m+1) .$$

A similar estimate is true for the Lipschitz constant, so (2.25) is proved.

d) The analyticity of the functions is evident. Their real–valuedness for real (q,ξ,y) results from the real–valuedness of $\mathscr{H}_m$.

e) The estimate results from (2.8), (2.19), (2.22). ∎

Let us consider an auxiliary hamiltonian $\varepsilon_m F$,

$$F = f^q(q;\theta) + \xi \cdot f^\xi(q;\theta) + \langle y, f^y(q;\theta)\rangle + \langle y, f^{yy}(q;\theta)y\rangle ,$$

and the corresponding Hamiltonian equations

$$\dot q = \varepsilon_m \nabla_\xi F, \qquad \dot\xi = -\varepsilon_m \nabla_q F, \qquad \dot y = \varepsilon_m J\nabla_y F . \tag{2.28}$$

A flow of these equations consists of canonical transformations $\{S^t\}$ of the phase space (see [1], Theorem 2.4). Let us set $S_m = S^1$ and denote $(q,\xi,y) = \mathfrak{h}$. Then

$$\mathscr{H}_m(S_m(\mathfrak{h};\theta);\theta) = \mathscr{H}_m(\mathfrak{h};\theta) + \varepsilon_m\{F, \mathscr{H}_m\} + O(\varepsilon_m{}^2) .$$

Here $\{\cdot,\cdot\}$ is a Poisson bracket; see [1], Proposition 4.3. So if $\mathfrak{h}, S_m(\mathfrak{h}) \in O_m$, then by (1.24′), (2.20) and (2.25)

$$\mathscr{H}_m(S_m(\mathfrak{h})) = H'_{0m}(\mathfrak{h}) + \varepsilon_m(H_{2m}(\mathfrak{h}) + \{F(\mathfrak{h}), H'_{0m}(\mathfrak{h})\}) +$$

$$+ O(\varepsilon_m{}^{1+\rho}) = H'_{0m}(\mathfrak{h}) + \varepsilon_m(H_{2m}(\mathfrak{h}) - \nabla_q F(\mathfrak{h}) \cdot \nabla_\xi H'_{0m}(\mathfrak{h}) +$$

$$+ \nabla_\xi F \cdot \nabla_q H'_{0m} + \langle J\nabla_y F(\mathfrak{h}), \nabla_y H'_{0m}(\mathfrak{h})\rangle) + O(\varepsilon_m{}^{1+\rho})$$

(we omit the parmeter θ) . As

$$\nabla_\xi H'_{0m} = \Lambda_{m+1}, \quad \nabla_y H'_{0m} = A_m y, \quad \nabla_q H'_{0m} = \tfrac{1}{2}\langle \nabla_q A_m y, y\rangle ,$$

then we may denote

$$\omega' = \Lambda_{m+1}(\omega;\theta), \qquad \frac{\partial}{\partial\omega'} = \sum \omega'_j \frac{\partial}{\partial q_j} \tag{2.29}$$

and rewrite $\mathscr{H}_m \circ S_m$ as follows:

$$\mathscr{H}_m(S_m(\mathfrak{h};\theta);\theta) = H'_{0m} + \varepsilon_m [\tfrac{1}{2} f^{\xi} \cdot < \nabla_q A_m\, y,y > - \partial f^q/\partial\omega' -$$

$$- \xi \cdot \partial f^{\xi}/\partial\omega' - < y, \partial f^y/\partial\omega' > - < y,(\partial f^{yy}/\partial\omega')y > +$$

$$+ < A_m y, Jf^y > + 2 < A_m y, Jf^{yy} y > + h^q + \xi \cdot h^{\xi} + < y,h^y > +$$

$$+ < y,h^{yy} y >] + O(\varepsilon_m^{1+\rho}) . \tag{2.30}$$

We try to find a transformation S_m such that the contents of the square brackets is $O(\varepsilon_m^{\rho})$. For this end we have to find $f^q, f^{\xi}, f^y, f^{yy}$ solving homological equations:

$$\partial f^q/\partial\omega' = h^q(q;\theta) , \qquad \partial f^{\xi}/\partial\omega' = h^{\xi}(q;\theta) , \tag{2.31}$$

$$\partial f^y/\partial\omega' - A_m(\theta) J f^y = h^y(q;\theta) , \tag{2.32}$$

$$\partial f^{yy}/\partial\omega' + f^{yy} J\, A_m - A_m J\, f^{yy} = h^{yy}(q;\theta) -$$

$$- \Delta h^{yy}(q;\theta) + \tfrac{1}{2} f^{\xi} \cdot \nabla_q A_m(q;\theta) . \tag{2.33}$$

Here Δh^{yy} is an admissible disparity.

Lemma 2.2. If $\varepsilon_0 << 1$ then there exists a Borel subset $\Theta_{m+1} \subset \Theta_m$ such that

$$\text{mes}\,(\Theta_m \setminus \Theta_{m+1})[I] \le \gamma_* K_6 (m+1)^{-2} / \gamma_0 \qquad \forall I \tag{2.34}$$

and for all $\theta \in \Theta_{m+1}$

a) equations (2.31) have solutions $f^q \in \mathscr{A}^R_{\Theta_{m+1}}(U_m^1;\mathbb{C})$, $f^\xi \in \mathscr{A}^R_{\Theta_{m+1}}(U_m^1;\mathbb{C}^n)$ and

$$|f^q|^{U_m^1,\Theta_{m+1}} \le C(m), \quad |f^\xi|^{U_m^1,\Theta_{m+1}} \le \varepsilon_m^{-2/3} C(m)\,; \tag{2.35}$$

b) equation (2.32) has an analytical solution $f^y \in \mathscr{A}^R_{\Theta_{m+1}}(U_m^2; Y^c_{d-d_H^0} + d_1)$ and

$$\|f^y\|^{U_m^2,\Theta_{m+1}}_{d-d_H^0 + d_1} \le C^e(m)\,\varepsilon_m^{-1/3}\,; \tag{2.36}$$

c) there exist $\Delta h^{yy} \in \mathscr{A}^R_{\Theta_{m+1}}(U_m^2; \mathscr{L}^s(Y_d, Y_{d-d_H^0}))$ such that

$$\Delta h^{yy} \varphi_j^{\pm} = b_j(q;\theta)\varphi_j^{\pm} \qquad \forall j, \forall q\,, \tag{2.37}$$

$$|b_j|^{U_m^2,\Theta_{m+1}} \le C(m)\varepsilon_m^{-2/3} \qquad \forall j\,, \tag{2.38}$$

equation (2.33) has a solution f^{yy} belonging to the same class as Δh^{yy},

$$\|f^{yy}\|^{U_m^2,\Theta_{m+1}}_{a\,,\,a+\,\Delta d} \le C^e(m)\,\varepsilon_m^{-2/3} \qquad \forall a \in [-d-\Delta d, d]\ , \tag{2.39}$$

(here $\Delta d = d_1 - d_H^{\,0} - 1$) and

$$\|A_m J f^{yy} - f^{yy} J\, A_m\|^{U_m^2,\Theta_{m+1}}_{d\,,\,d-d_H^0} \le C^e(m)\,\varepsilon_m^{-2/3}\,. \tag{2.40}$$

A proof of the lemma is given below in § 3.

Let us denote

$$\Pi_{\mathscr{Y}} : \mathscr{Y}^c \times \Theta_0 \longrightarrow \mathscr{Y}^c, \qquad (\mathfrak{h},\theta) \mapsto \mathfrak{h}\ , \tag{2.41}$$

$$\Pi_\Theta : \mathscr{Y}^c \times \Theta_0 \longrightarrow \Theta_0, \qquad (\mathfrak{h},\theta) \mapsto \theta\ , \tag{2.42}$$

let Π_q, Π_ξ, Π_y be projectors of $\mathscr{Y}^c = (\mathbb{C}^n/\,2\pi\,\mathbb{Z}^n) \times \mathbb{C}^n \times Y^c$ on the first, second and third term respectively and let S_m be a time one shift along the trajectories of the system (2.28).

Let $d_c = d + d_1 - d_H^0 - 1$ and $O^{\,c}_{m,d_c} = O_m^{\,c} \cap \mathscr{Y}^c_{d_c}$ with the norm dist d_c. O^c_{m,d_c} is dense in O^c_m and is unbounded in $\mathscr{Y}^c_{d_c}$.

We may identify the torus $\mathbb{T}^n$ with a measurable subset $T^{(n)} \subset \mathbb{R}^n$,

$$\{q \in \mathbb{R}^n \mid\ |q_j| < \pi\ \ \forall j\} \subset T^{(n)} \subset \{q \in \mathbb{R}^n \mid\ |q_j| \le \pi\ \ \forall j\}$$

(the map $\mathbf{T}^n \longrightarrow T^{(n)}$ is one–to–one, measurable and discontinuous), and may identify $\mathscr{Y}$ with a subset $T^{(n)} \times \mathbb{R}^n \times Y$ of $E = \mathbb{R}^{2n} \times Y$. The identifications depend on a choice of $T^{(n)}$, but if $\mathrm{dist}_{\mathscr{Y}}(\mathfrak{h}_1, \mathfrak{h}_2) < \pi$ then the point in E corresponding to $\mathfrak{h}_1 - \mathfrak{h}_2$ does not depend on $T^{(n)}$. We shall use these identifications and treat a difference of two close $\mathscr{Y}$–valued (or $\mathbb{T}^n$–valued) maps as a E–valued ($\mathbb{R}^n$–valued) map.

Lemma 2.3. If $\varepsilon_0 << 1$ then

a)
$$S_m \in \mathscr{A}^R_{\Theta_{m+1}}(O_m^{4c}; O_m{}^c) \tag{2.43}$$

and

$$| S_m - \Pi_{\mathscr{Y}} |_{E_{d_c}}^{O_m^{5c} \times \Theta_{m+1}, \mathrm{Lip}} \le \varepsilon_m^{\rho} . \tag{2.44}$$

More precisely,

$$| \Pi_q \circ (S_m - \Pi_{\mathscr{Y}}) |^{O_m^{5c} \times \Theta_{m+1}, \mathrm{Lip}} \le C(m)\, \varepsilon_m^{1/3} , \tag{2.45}$$

$$| \Pi_\xi \circ (S_m - \Pi_{\mathscr{Y}}) |^{O_m^{5c} \times \Theta_{m+1}, \mathrm{Lip}} \le C^e(m)\, \varepsilon_m , \tag{2.46}$$

$$\| \Pi_y \circ (S_m - \Pi_{\mathscr{Y}}) \|_{d_c}^{O_m^{5c} \times \Theta_{m+1}, \mathrm{Lip}} \le C^e(m)\, \varepsilon_m^{2/3} . \tag{2.47}$$

b) A restriction of S_m on O_{m+1} is a canonical transformation which transforms equations (2.13) – (2.15) on the domain O_m into Hamiltonian equations with a hamiltonian $\mathscr{H}_{m+1}$ of the form (2.6) with $m := m+1$ on the domain O_{m+1}.

The lemma is proved in § 5.

Let us set $\Theta_{\varepsilon_0} = \cap \Theta_m$. Then Θ_{ε_0} is a Borel set. For the definition of γ_0 and e_m (see (2.2)) and for (2.5)

$$\text{mes}\, \Theta_{\varepsilon_0}[I] \geq (1 - \tfrac{1}{2}\gamma_*)\, \text{mes}\, \Omega_0 \qquad \forall I \in \mathcal{J}. \tag{2.48}$$

For $\theta \in \Theta_{\varepsilon_0}$ and $r, N \in \mathbb{N}_0$ let us set

$$\sum\nolimits^{r}_{r+N+1}(\cdot\,;\theta) = S_r(\cdot\,;\theta) \circ \ldots \circ S_{r+N}(\cdot\,;\theta) : O^c_{r+N+1} \longrightarrow O_r^{\,c} \tag{2.49}$$

and let us set $\sum^{r}_{r}$ be equal to the identical map of $O_r^{\,c}$.

<u>Lemma 2.4</u>. For all $r, m \geq 0$

$$|\sum\nolimits^{r}_{r+m} - \Pi_{\mathscr{Y}}|^{O^c_{r+m,d_c} \times \Theta_{m+1},\mathrm{Lip}}_{E_{d_c}} \leq 3\,\varepsilon_r^{\rho}, \tag{2.50}$$

<u>Proof</u>. Let us denote the l.h.s. in (2.50) by D^{r}_{r+m}. One may rewrite the identity $\sum^{r}_{r+m}(\mathfrak{h};\theta) = S_r(\sum^{r+1}_{r+m}(\mathfrak{h};\theta);\theta)$ in a form

$$\sum\nolimits_{r+m}^{r} - \Pi_{\mathcal{Y}} = (S_r - \Pi_{\mathcal{Y}}) \circ (\sum\nolimits_{r+m}^{r+1} \times \Pi_{\Theta}) + \sum\nolimits_{r+m}^{r+1} - \Pi_{\mathcal{Y}} .$$

So by (2.44) we get an estimate

$$D_{r+m}^{r} \leq \varepsilon_r^{\rho} (D_{r+m}^{r+1} + 2) + D_{r+m}^{r+1}. \tag{2.51}$$

As $D_{r+m}^{r+m} = 0$, then the lemma's assertion results by the induction. ■

Let us denote $T_0^{\,n} = \mathbf{T}^n \times \{0\} \times \{0\}$ and $O_\infty^c = U(\delta_0/2) \times \{0\} \times \{0\} \subset \mathcal{Y}_\infty^c$. Then $T_0^{\,n} \subset O_\infty^c$ and O_∞^c lies in O_m^c for every $m \geq 1$ as $\delta_m > \frac{1}{2}\delta_0$ $\forall m$.

<u>Lemma 2.5</u>. If $\varepsilon_0 << 1$ then $\forall m \in N_0$ the maps $\sum\nolimits_{m+N}^{m} : O_\infty^c \times \Theta_{\varepsilon_0} \longrightarrow \mathcal{Y}_{d_c}^c$ $(N \longrightarrow \infty)$ converge to a map $\sum\nolimits_{\infty}^{m} : O_\infty^c \times \Theta_{\varepsilon_0} \longrightarrow \mathcal{Y}_{d_c}^c$ such that

a) for every θ the map $\sum\nolimits_{\infty}^{m} (\cdot\,;\theta) : O_\infty^c \longrightarrow \mathcal{Y}_{d_c}^c$ is complex–analytical;

b)

$$\sum\nolimits_{p}^{m}(\cdot\,;\theta) \circ \sum\nolimits_{\infty}^{p}(\cdot\,;\theta) = \sum\nolimits_{\infty}^{m}(\cdot\,;\theta)\,;\ \ \forall 0 \leq m \leq p,\ \ \forall \theta \in \Theta_{\varepsilon_0} \tag{2.52}$$

c) $$|\sum\nolimits_{\infty}^{m} - \Pi_{\mathcal{Y}}|_{E_{d_c}}^{O_\infty^c \times \Theta_{\varepsilon_0}, Lip} \leq 3\,\varepsilon_m^{\rho}\,; \tag{2.53}$$

d) $$\|\Pi_y \circ \sum\nolimits_{\infty}^{m}(\mathfrak{h};\theta)\|_{d_c} \leq \varepsilon_m^{\;1/3+\rho} \qquad \forall(\mathfrak{h},\theta) \in T_0^n \times \Theta_{\varepsilon_0}\,, \tag{2.54}$$

$$|\Pi_\xi \circ \sum_{\infty}^{m}(\mathfrak{h};\theta)| \le \tfrac{1}{4}\varepsilon_m^{\,2/3} \qquad \forall(\mathfrak{h},\theta) \in T_0^n \times \Theta_{\varepsilon_0}\,. \tag{2.55}$$

<u>Proof</u>. Let $\mathfrak{h}_0 \in O_\infty^c$ and for $j \ge 1$ let $\mathfrak{h}_j = \sum_{m+j}^{m}(\mathfrak{h}_0;\theta)$. Then by (2.44), (2.50)

$$\operatorname{dist}_{d_c}(\mathfrak{h}_{N+1}, \mathfrak{h}_N) = \operatorname{dist}_{d_c}\Big(\sum_{m+N}^{m}(S_{m+N}(\mathfrak{h}_0;\theta)), \sum_{m+N}^{m}(\mathfrak{h}_0;\theta)\Big) \le$$

$$\le (1 + 3\,\varepsilon_m^\rho)\,\varepsilon_{m+N}^\rho \le 2\,\varepsilon_{m+N}^\rho\,.$$

So the sequence $\{\mathfrak{h}_j\}$ is fundamental and converges to a point $\mathfrak{h}_\infty \in \mathscr{Y}_{d_c}^c$. The r.h.s. of the last estimate does not depend on $\mathfrak{h}_0$. So the sequence $\{\sum_{m+N}^{m}(\cdot\,;\theta)\}$ converges uniformly in O_∞^c to an analytical map $\sum_\infty^m(\cdot\,;\theta) : O_\infty^c \longrightarrow \mathscr{Y}_{d_c}^c$, $\sum_\infty^m(\mathfrak{h}_0;\theta) = \mathfrak{h}_\infty$. The relations (2.52) take place and the items a), b) are proved.

The estimate (2.53) results from (2.50) by going to a limit.

To prove (2.54), (2.55) let us take $\mathfrak{h} \in O_\infty^c$ and set $\mathfrak{h}^{m+N+1} = \mathfrak{h}$,

$$\mathfrak{h}^j = \sum_{m+N+1}^{j}(\mathfrak{h}^{m+N+1};\theta) \in O_j^c \;\forall j \in [m, m+N] \cap \mathbb{N}_0\,.$$

Then $\mathfrak{h}^j = S_j(\mathfrak{h}^{j+1};\theta)$ and by (2.47)

$$\|\Pi_y \mathfrak{h}^j\|_{d_c} \le \|\Pi_y \mathfrak{h}^{j+1}\|_{d_c} + \tfrac{1}{2}\varepsilon_j^{\rho+1/3}, \qquad m \le j \le m+N\,.$$

As $\Pi_y \mathfrak{h}^{m+N+1} = 0$, then $\|\Pi_y \mathfrak{h}^m\|_{d_c} \le \varepsilon_m^{\rho+1/3}$. So $\|\Pi_y \circ \sum_{m+N+1}^{m} (\mathfrak{h};\theta)\|_{d_c} \le \varepsilon_m^{\rho+1/3}$ and by going to limit when $m \longrightarrow \infty$ one gets (2.54).

Estimate (2.55) results from (2.46) because for the last $|\Pi_\xi \mathfrak{h}^j| \le |\Pi_\xi \mathfrak{h}^{j+1}| + C^e(j)\, \varepsilon_j$. ■

As $\Lambda_0(\omega,I) \equiv \omega$ then by (2.19) with $m = 0, 1, \dots, r-1$

$$\Lambda_r(\omega,I) = \omega + \varepsilon_0 h_0^{0\xi} + \varepsilon_1 h_1^{0\xi} + \dots + \varepsilon_{r-1} h_{r-1}^{0\xi}. \tag{2.56}$$

Here the vector–function $h_j^{0\xi}$ corresponds to the hamiltonian $\mathscr{H}_m$ with $m = j$. So

$$|\varepsilon_j h_j^{0\xi}|^{\Theta_{\varepsilon_0},\mathrm{Lip}} \le C(j)\, \varepsilon_j^{1/3}, \tag{2.57}$$

the maps $\Lambda_r : \Theta_{\varepsilon_0} \longrightarrow \mathbb{R}^n$ $(r \longrightarrow \infty)$ converge to a Lipschitz one

$$\Lambda_\infty : \Theta_{\varepsilon_0} \longrightarrow \mathbb{R}^n,\ \Lambda_\infty = \omega + \varepsilon_0 h_0^{0\xi} + \varepsilon_1 h_1^{0\xi} + \dots \tag{2.58}$$

and by (2.57)

$$|\Lambda_\infty(\omega,I) - \omega|^{\Theta_{\varepsilon_0},\mathrm{Lip}} \le C\, \varepsilon_0^{1/3}. \tag{2.59}$$

Let us fix $\theta_0 \in \Theta_{\varepsilon_0}$ and denote $\omega_m = \Lambda_m(\theta_0)$, $m \le \infty$. Then by (2.56), (2.57)

$$|\omega_m - \omega_{m+p}| \le C(m)\,\varepsilon_m^{1/3} \qquad \forall m, \forall p \ge 1\,. \tag{2.60}$$

Let us consider a curve $t \mapsto \mathfrak{h}_\infty(t) = (q_0 + t\omega_\infty',0,0)$, $0 \le t \le 1$, on the torus $T_0^{\,n} = \mathbb{T}^n \times \{0\} \times \{0\}$. The map $\sum_\infty^m(\cdot\,;\theta_0)$, $m \ge 0$, transforms it into a curve $\mathfrak{h}_m(t) = (q_m(t),\xi_m(t),y_m(t)) \in O_m$. By the estimates (2.53) – (2.55)

$$\operatorname{dist}\,(q_m(t),\, q_m(0) + t\omega_\infty') \le C\,\varepsilon_m^{\rho}\,, \tag{2.61}$$

$$\|y_m(t)\|_{d_c} \le \varepsilon_m^{1/3+\rho}\,, \tag{2.62}$$

$$|\xi_m(t)| \le \tfrac{1}{4}\,\varepsilon_m^{2/3}\,. \tag{2.63}$$

The cases $d_H > 0$ and $d_H \le 0$ must be considered separately.

A) $d_H > 0$. For the assumption 2) of Theorem 1.1 system (2.13) – (2.15) with hamiltonian $\mathscr{H}_0$ and initial condition $\mathfrak{h}_0(0) \in \mathscr{Y}_{d_c}$ has a weak in $\mathscr{Y}_{d_c}$ solution $\mathfrak{h}^0(t)$, $0 \le t \le T$, and $|\xi^0(t)| + \|y^0(t)\|_{d_c} \le C_1$. Let $\{\mathfrak{h}_N^{\,0} = (q_N^0(t),\ \xi_N^0(t),\ y_N^0(t))\,|\,N = 1,2,\ldots\}$ be a sequence of strong in $\mathscr{Y}_{d_c}$ solutions which converge to $\mathfrak{h}^0$ in $\mathscr{Y}_{d_c}$. Then

$$|\xi_N^0(t)| + \|y_N^0(t)\|_{d_c} \le C_2 \tag{2.64}$$

for some C_2 and for all $0 \le t \le T_0 = \min\,\{T,1\}$.

The canonical transformation $(\sum_m^0)^{-1}$ transforms $\mathfrak{h}_N^0$ into a strong in $\mathscr{Y}_d$ solution $\mathfrak{h}_N^m = (q_N^m(t), \xi_N^m(t), y_N^m(t))$ of a system with Hamiltonian $\mathscr{H}_m$, N = 1 , 2 ... .

The solution $\mathfrak{h}_N^m(t)$ is well–defined while $\mathfrak{h}_N^0(t)$ stays inside domain $\sum_m^0(O_m)$, i.e. for $t \in [0,T_{mN}]$ with some $T_{mN} \leq T_0$. When $N \longrightarrow \infty$ the solutions converge to a weak solution $\mathfrak{h}^m(t)$. As $d + d_H \leq d_c$, then by (2.64) and (2.50) with $r = 0$

$$\|y_N^m\|_{d+d_H} \leq \|y_N^m\|_{d_c} \leq C_2 + 1 \qquad \forall m, N . \tag{2.65}$$

As $\mathfrak{h}^m(0) = \mathfrak{h}_m(0)$, then by (2.61) – (2.63) with $t = 0$

$$\|y_N^m(0)\|_d \leq 3\, \varepsilon_m{}^{1/3+\rho},\ |\xi_N^m(0)| \leq \tfrac{1}{3}\, \varepsilon_m{}^{2/3}$$
$$\operatorname{dist}(q_0, q_N^m(0)) \leq 3\, C\, \varepsilon_m{}^{\rho} \tag{2.66}$$

for $N \geq N_m >> 1$. For $0 \leq t \leq T_{mN}$ one gets from the equation (2.15) an identity

$$\frac{1}{2}\frac{d}{dt}\|y_N^m(t)\|_d^2 = < J\, A_m y_N{}^m, y_N^m >_d +$$
$$+ < J\, \nabla_y\, H^3, y_N^m >_d + \varepsilon_m < J\, \nabla_y H_m, y_N^m >_d . \tag{2.67}$$

As the operator $J\, A_m(q)$ is antiselfadjoint in Y_d then $< J\, A_m\, y_N^m, y_N^m >_d = 0$. So by (2.12), (1.24) and (2.65) we have for $y_N^m(t) \in O_m$

$$\frac{1}{2}\frac{d}{dt}\|y_N^m(t)\|_d^2 \leq \|\nabla_y H^3\|_{d-d_H}\|y_N^m\|_{d+d_H} + \varepsilon_m\|y_N^m\|_d\,\|\nabla_y H_m\|_d \leq$$

$$\leq (C_2+1)\,K_1(\|y_N^m\|_d^2 + |\xi_N^m|) + C(m)\,\varepsilon_m{}^{2/3}\,\|y_N^m\|_d\,.$$

So $\frac{d}{dt}\|y_N^m(t)\|_d^2 \leq C'\varepsilon_m^{2/3}$ and by (2.66)

$$\|y_N^m(t)\|_d^2 \leq \tfrac{1}{9}(1+Ct)\,\varepsilon_m{}^{2/3} \qquad \forall 0 \leq t \leq T_{mN}\,. \tag{2.68}$$

In a similar way by (2.13), (2.14), (2.14′) and (2.66) we get estimates on $\xi(t)$, $q(t)$:

$$\begin{aligned} &\mathrm{dist}(q_N^m(t),\, q_0 + t\omega_m') \leq C(1+t)\,\varepsilon_m^{\rho}\,, \\ &|\xi_N^m(t)| \leq \tfrac{1}{3}(1+Ct)\,\varepsilon_m{}^{2/3} \qquad \forall 0 \leq t \leq T_{mN}\,. \end{aligned} \tag{2.69}$$

So the solution $\mathfrak{h}_N^m$ stays inside $O_m \cap O_m^{1c}$ for $0 \leq t \leq C_1^{-1}$ (i.e. one may take $T_{mN} = C_1^{-1}$) and for such a "t" estimates (2.68), (2.69) are valid for $\mathfrak{h}^m(t)$, too. For the inequalities (2.61) – (2.63), (2.68), (2.69) and (2.60) $\mathrm{dist}_d\,(\mathfrak{h}^m(t),\, \mathfrak{h}_m(t)) \leq C\,\varepsilon_m^{\rho}$ $\forall 0 \leq t \leq C_1^{-1}$. The mapping $\sum_m^0(\cdot\,;\theta) : \mathscr{Y}_d \longrightarrow \mathscr{Y}_d$ is Lipschitz by Lemma 2.4. So $\mathrm{dist}_d\,(\mathfrak{h}^0(t),\, \mathfrak{h}_0(t)) \leq C'\varepsilon_m^{\rho}$ $\forall t \in [0, C_1^{-1}]$ for arbitrary m . Hence $\mathfrak{h}^0 = \mathfrak{h}_0$ and $\sum_\infty^0(\mathfrak{h}_\infty(t);\theta)$ is a weak in $\mathscr{Y}_d$ solution of the initial Hamiltonian system.

B) $d_H \leq 0$. Let $\mathfrak{h}(t)$ be some strong solution of the system with hamiltonian $\mathscr{H}_m$, staying inside $O_m \cap O_m{}^{1c}$ for $0 \leq t \leq T$. Taking the inner product in Y_d of equation (2.15) by $y(t)$ we obtain

$$\frac{1}{2}\frac{d}{dt}\|y(t)\|_d^2 = \varepsilon_m < J\,\nabla_y H_m, y >_d + < J\,\nabla_y H^3, y >_d \le$$

$$\le \varepsilon\|y\|_d\|\nabla_y H_m\|_d + \|y\|_d\, K_1\,(\|y\|_d^2 + |\xi|)\,,$$

and $\|y(t)\|_d \le \|y(0)\|_d + \frac{1}{3}\,t\,\varepsilon_m{}^{1/3}$ for $0 \le t \le T$. By equations (2.13), (2.14) we have

$$|\xi(t)| \le |\xi(0)| + \frac{1}{3}\,t\,\varepsilon_m{}^{2/3}\,,\ \text{dist}\,(q(t), q(0) + t\omega_m) \le Ct\,\varepsilon_m^{\delta} \quad (0 \le t \le 1)\,.$$

So if

$$\|y(0)\|_d \le \frac{1}{3}\,\varepsilon_m{}^{1/3},\ |\xi(0)| \le \frac{1}{3}\,\varepsilon_m{}^{2/3},\ q(0) \in \mathbb{T}^n\,, \tag{2.70}$$

then the solution $\mathfrak{h}(t)$ stays inside $O_m \cap O_m{}^{1c}$ for $0 \le t \le 1$. If $\mathfrak{h}^m(t)$ is a weak solution of equations with hamiltonian $\mathscr{H}_m$ and $\mathfrak{h}^m(0) = \mathfrak{h}_m(0)$, then (2.70) is true by (2.61) – (2.63) with $t = 0$. So by Theorem 3.1 from [1] solution $\mathfrak{h}^m(t)$ exists for $0 \le t \le 1$ and for this solution estimates (2.68), (2.69) take place. So as in the case A) we see that $\mathfrak{h}_0(t) \equiv \mathfrak{h}^0(t)$, i.e. $\sum_\infty^0(\mathfrak{h}_\infty(t);\theta)$ is a weak in $\mathscr{Y}_d$ solution.

Now the assertions b) – c) of the theorem are proved by setting
$\sum^{\varepsilon_0}(q;\theta) = \sum_\infty^0(q,0,0;\theta)$, because estimate (1.31) results from (2.53) and (1.32) results from (2.59).

In order to prove the assertion a), we set in (2.1) $\gamma_* = \gamma_*(M) \searrow 0$, where M is a natural parameter tending to infinity. Assertions b) – c) are valid for $\varepsilon_0 = \varepsilon_0(M) > 0$, and we may assume that $\varepsilon_0(M) \searrow 0$. Then by (2.48) for $\varepsilon_0 \in (\varepsilon_0(M+1),\ \varepsilon_0(M)]$

mes Ω_0 – mes $\Theta_{\varepsilon_0}[I] \le \gamma_*(M) \searrow 0$ and the assertion is proved.

For to prove assertion d) let us mention that Liapunov exponents are stable under a change of phase variable. So exponents of a solution $\mathfrak{h}_0(t)$ of equations (2.13) – (2.15) with $m = 0$ are equal to ones of the solution $\mathfrak{h}_m(t) = (\sum_m^0)^{-1} \mathfrak{h}_0(t)$ of the equations with $m = m$. Let $\delta\mathfrak{h} = (\delta q, \delta\xi, \delta y)(t)$ be a strong solution of the variational equations for (2.13) – (2.15) along $\mathfrak{h}_m(t)$:

$$\delta\dot{q} = \nabla_\xi(\varepsilon_m H_m + H^3)(\mathfrak{h}_m(t))_*(\delta q, \delta\xi, \delta y),$$

$$\delta\dot{\xi} = -\nabla_q(H_{0m} + \varepsilon_m H_m + H^3)(\mathfrak{h}_m(t))_*(\delta q, \delta\xi, \delta y), \tag{2.71}$$

$$\delta\dot{y} = J[A_m(q_m(t))\delta y + (\delta q \cdot \nabla q\, A_m(q_m(t)))y + \nabla_y(\varepsilon_m H_m + H^3)(\mathfrak{h}_m(t))_*(\delta\mathfrak{h})].$$

Taking the inner product in E_d of these equations with $\delta\mathfrak{h}(t)$ we get an inequality:

$$\frac{d}{dt}\|\mathfrak{h}(t)\|_d \le \varepsilon_m{}^\rho \|\mathfrak{h}(t)\|_d .$$

The same is true after the change $t \longrightarrow -t$. So modules of the exponents of variational equations do not exceed $\varepsilon_m{}^\rho$. As m is arbitrary, they are equal to zero.

3. Proof of Lemma 2.2 (solving of homological equations)

In § 3 – 5 we write ε, δ instead of ε_m, δ_m and sometimes we omit the argument θ for functions and maps. In the deductions of estimates, we use systematically the conditions $\varepsilon_0 << 1$, $\delta_a << 1$. We denote $\mathbb{Z}_0^s = \mathbb{Z}^s \setminus \{0\}$, $\mathbb{Z}_0 = \mathbb{Z} \setminus \{0\}$.

The assertions of the lemma will be proved for $\Theta_{m+1} = \Theta_m \setminus (\Theta^1 \cup \Theta^2 \cup \Theta^3)$, where Θ^p are Borel sets, and for $p = 1, 2, 3$

$$\text{mes}\, \Theta^p[I] \leq \gamma_* K_6 (m+1)^{-2} / (3\gamma_0) \qquad \forall I. \tag{3.1}$$

By (2.8) the map

$$\Theta_m[I] \ni \omega \mapsto \omega' = \Lambda_{m+1}(\omega, I) \tag{3.2}$$

for all I is a Lipschitz homeomorphism, changing the Lebesgue measure by a factor no greater than two. I.e. for every Borel subset $\Omega \subset \Theta_m[I]$

$$\tfrac{1}{2}\, \text{mes}\, \Omega \leq \text{mes}\, \Lambda_{m+1}(\Omega, I) \leq 2\, \text{mes}\, \Omega \tag{3.3'}$$

(see Appendix C, Treorem C 1). Besides,

$$|\Lambda_{m+1} - \omega|^{\Theta_m[I], \text{Lip}} \leq C\, \varepsilon_0^{\rho}, \; |\Lambda_{m+1}(\omega, I) - \omega_0| \leq C(\varepsilon_0^{\rho} + \delta_a) \qquad \forall \omega \in \Theta_m[I]. \tag{3.3}$$

Therefore, if

$$\Theta^1 = \cup \{\Theta_s{}^1 \mid s \in \mathbb{Z}_0{}^n\} , \Theta_s^1 = \{\theta \in \Theta_m \mid |\omega'(\theta) \cdot s| \leq$$

$$\leq [(m+1)^2 |s|^n C]^{-1}\} , \tag{3.4}$$

then

$$\text{mes}\, \Theta^1 [I] \leq \sum_{s \in \mathbb{Z}_0^n} \text{mes}\ \Theta_s^1\ [I] \leq$$

$$\leq 2 \sum_{s \in \mathbb{Z}_0^n} \text{mes}\ \{\omega' \mid |\omega' \cdot s| \leq (m+1)^{-2} |s|^{-n} C^{-1}\} \leq$$

$$\leq \frac{2}{C(m+1)^2} \sum_{s \in \mathbb{Z}_0^n} C_1 |s|^{-n-1} \leq 2\, C_2 C^{-1} (m+1)^{-2}$$

and condition (3.1) is satisfied if $C >> 1$. For $\theta \in \Theta_m \setminus \Theta^1$, $q \in U_m{}^1$, the solutions of equations (2.31) are given by convergent trigonometric series and satisfy the estimates (2.35) (see [A, Sec. 4.2] and Lemmas B1, B2 in Appendix B below).

We turn to the equation (2.33) (a proof of the assertion b) on the equation (2.32) is much simpler, a sketch of it is given at the end of the section). For $j \in \mathbb{Z}_0$ we set

$$w_j = (\varphi_{|j|}^{+} + (\text{sgn } j) i\, \varphi_{|j|}^{-}) / \sqrt{2} .$$

Then $\{w_j \lambda_{|j|}^{(-s)} \mid j \in \mathbb{Z}_0\}$ is a Hilbert basis of a space $Y_s{}^c$, $s \in \mathbb{R}$. For complex numbers χ_j, $j \in \mathbb{Z}_0$, we denote by diag (χ_j) an operator in Y^c which maps w_j to

$\chi_j w_j \quad \forall j \in \mathbb{Z}_0$. In particular by (2.9)

$$J\,A_m(q;\theta) = \mathrm{diag}\,(i\lambda_j^1(q;\theta)) \tag{3.5}$$

with $\lambda_j^1(q;\theta) = \lambda_j(\omega) + \beta_{jm}(q;\theta) \quad \forall j \in \mathbb{Z}_0$. Here for $j \in \mathbb{N}$ $\lambda_{-j}(\omega) = -\lambda_j(\omega)$, $\beta_{-jm}(q;\theta) = -\beta_{jm}(q;\theta)$. By (2.10) and (1.27) $\forall j \in \mathbb{Z}_0$

$$|\lambda_j^1(\cdot\,;\cdot) - \lambda_j(\cdot)|^{U_m;\Theta_m} \le \varepsilon_0{}^{\rho}|j|^{d_H^0},$$

$$|\lambda_j^1(q;\theta) - \lambda_{j0}| \le C(\delta_a|j|^{d_{1,r}} + \varepsilon_0{}^{\rho}|j|^{d_H^0}) \quad \forall q,\theta \tag{3.6}$$

(here $\lambda_{j0} = \lambda_j(\omega_0)$) . Let us choose functions $b_j(q;\theta)$, $j \in \mathbb{N}$ (see (2.37), (2.38)), as follows:

$$b_j(q;\theta) = \frac{1}{2}\sum_{\sigma=\pm} < (\frac{1}{2} f^{\xi} \cdot \nabla_q A_{m+1} + h^{yy})\,\varphi_j^{\sigma} , \varphi_j^{\sigma} >$$

and define an operator Δh^{yy} , $\Delta h^{yy}\,\varphi_j^{\pm} = b_j(q)\,\varphi_j^{\pm} \quad \forall j \in \mathbb{N}$. By (2.24) and (2.10), (2.35)

$$|b_j|^{U_m^1;\Theta_{m+1}} \le C(m)\,\varepsilon_m{}^{-2/3}\, j^{d_H^0} \qquad \forall j \in \mathbb{N}\,. \tag{3.7}$$

So operator Δh^{yy} satisfies (2.37), (2.38).

Let us denote $h^{1yy}(q;\theta) = h^{yy} + \frac{1}{2} f^{\xi} \cdot \nabla_q A_m - \Delta h^{yy}$. Then by (2.24), (2.35) and (3.7)

$$\|h^{1yy}\|^{U_m^1, \Theta_{m+1}}_{d\,,\,d-d_H^0} \leq C(m)\,\varepsilon_m^{-2/3} \tag{3.8}$$

As operators J and A_m commute we may write equation (2.33) as follows:

$$\frac{\partial}{\partial\omega} f^{yy} + [f^{yy}, J\,A_m] = h^{1yy} \tag{3.9}$$

Let us fix for a moment some functions $W_j(q;\theta)$, $j \in \mathbb{Z}_0$, such that $W_j = -W_{-j}$, and

$$W_j \in \mathscr{A}^R_{\Theta_{m+1}}(U_m{}^1\,;\,\mathbb{C})\,,\ |W_j|^{U_m{}^1, \Theta_{m+1}} \leq C(m) \qquad \forall j \tag{3.10}$$

(they will be chosen later) and denote $W(q;\theta) = \mathrm{diag}(\exp i\, W_j(q;\theta))$. Then

$$\frac{\partial}{\partial\omega} W^{\pm 1}(q;\theta) = \pm\, \mathrm{diag}\,(i \frac{\partial}{\partial\omega} W_j(q,\theta))\; W^{\pm 1}(q;\theta)\,.$$

So if we substitute into (3.9)

$$f^{yy} = W F^{yy}\, W^{-1}\,,\ \ h^{1yy} = W H^{yy}\, W^{-1}\,, \tag{3.11}$$

then by (3.5) we get for F^{yy} an equation

$$\frac{\partial}{\partial\omega} F^{yy} + [F^{yy}\,, \mathrm{diag}\,(i\,(\lambda_j^1 - \frac{\partial}{\partial\omega} W_j))] = H^{yy}\,. \tag{3.12}$$

Let us take functions W_k be solutions of equations

$$\frac{\partial}{\partial \omega} W_k(q;\theta) = \lambda_k^1(q;\theta) - \lambda_k'(\theta) \ , \ \lambda_k' = \int \lambda_k^1(q;\theta)\, dq/(2\pi)^n \ . \tag{3.13}$$

If $\theta \in \Theta_m \setminus \Theta^1$ then the equations (3.13) may be solved just as equations (2.31) and by (3.6) estimates (3.10) take place for the solutions W_j, $j \in \mathbb{Z}_0$. By (3.10), (3.11) $\| \cdot \|_{a\ ,\ b}^{U_m^1, \Theta_{m+1}}$–norms of operators h^{1yy} and H^{yy}, f^{yy} and F^{yy} differ by a factor no greater than $C^e(m)$. Thus to get estimate (2.39) for a solution of (2.33) is equivalent to get it for one of (3.12).

Let us mention that a matrix $\{F_{jk}\}$ of operator F^{yy} in the basis $\{W_j \,|\, j \in \mathbb{Z}_0\}$ is equal to $F_{jk} = \langle F^{yy} W_k , W_{-j} \rangle$ and the same is true for a matrix $\{H_{jk}\}$ of operator H^{yy}. So we may apply quadratic forms corresponding to the operators in l.h.s. and r.h.s. of (3.12) to vectors W_k, W_{-j} and get equations on the matrix elements $F_{jk}(q;\theta)$:

$$\frac{\partial}{\partial \omega} F_{jk}(q;\theta) + (\lambda_k'(\theta) - \lambda_j'(\theta))\, F_{jk} = H_{jk}(q;\theta) \ . \tag{3.14}$$

For a vector–function $f(q)$, $q \in \mathbf{T}^n$, we denote by $\widehat{f}(s)$, $s \in \mathbb{Z}^n$, its Fourier coefficients: $f(q) = \sum \widehat{f}(s)\, e^{iq \cdot s}$. By (3.8) and Lemma B1

$$\| \widehat{H}^{yy}(s) \|_{d\ ,\ d-d_H^0}^{\Theta_{m+1}, \mathrm{Lip}} \le C^e(m)\, \varepsilon_m^{-2/3} e^{-5/6\ \delta |s|} \ . \tag{3.15}$$

For the diagonal elements $\{h_{jj}^1\}$ of the matrix of operator h^{1yy} we have:

$$h_{jj}^1(q) = \tfrac{1}{2} \langle h^{1yy}(q)(\varphi_{|j|}^+ + i(\mathrm{sgn}\ j)\, \varphi_{|j|}^-) , \varphi_{|j|}^+ - i(\mathrm{sgn}\ j)\, \varphi_{|j|}^- \rangle$$

$$= \frac{1}{2} \left(< h^{1yy}(q)\, \varphi^{+}_{|j|} , \varphi^{+}_{|j|} > + < h^{1yy}(q)\, \varphi^{-}_{|j|} , \varphi^{-}_{|j|} > \right) .$$

So by the definitions of the functions b_j and operator h^{1yy}, $h^{1}_{jj}(q) \equiv 0 \quad \forall j$ and the same is true for the operator H^{yy}:

$$H_{jj}(q;\theta) \equiv 0 \qquad \forall j \tag{3.17}$$

By (3.17) equations (3.14) are equivalent to the following relations on Fourier coefficients:

$$i(\omega' \cdot s - \lambda'_j + \lambda'_k)\, \widehat{F}_{kj}(s) = \begin{cases} 0 & \text{if } k = j , \\ \widehat{H}_{kj} & \text{if } k \neq j . \end{cases}$$

Let us choose $\widehat{F}_{kk}(s) \equiv 0 \quad \forall k \in \mathbb{Z}_0$ and denote

$$D(k,j,s;\theta) = \begin{cases} i(\omega' \cdot s - \lambda'_j + \lambda'_k) , & j \neq k , \\ i , & j = k . \end{cases}$$

Then

$$\widehat{F}_{kj}(s) = \widehat{H}_{kj}(s;\theta)\, D^{-1}(k,j,s;\theta) . \tag{3.18}$$

<u>Lemma 3.1</u>. There exists a Borel subset $\Theta^2 \subset \Theta_m$ with the property (3.1) and a constant $c > 0$ such that if $\varepsilon_0 << 1$ and $\delta_a << 1$ then for all $\theta \in \Theta_m \setminus (\Theta^1 \cup \Theta^2)$ and for all $j,k \in \mathbb{Z}_0$, $j \neq k$, $s \in \mathbb{Z}^n$ the following estimate takes place:

$$|D^{-1}(k,j,s;\cdot)|^{\Theta_m \setminus \Theta^2,\mathrm{Lip}} \le$$

$$\le C(m)(1+|s|)^{2c+1}(1+|\lambda_{k0}-\lambda_{j0}|)^{-1}. \qquad (3.19)$$

The proof is given in § 4 below.

For a map $g(k,p)$, $g: \mathbb{Z}_0 \times P \longrightarrow \mathbb{C}$, where P is an abstract set, we denote

$$|g(k,p)|_{\ell^r(k)} = \Big(\sum_{k\in\mathbb{Z}_0} |g(k,p)|^r\Big)^{1/r}$$

and treat g as a map from P to $\ell^r(\mathbb{Z}_0)$.

We have to estimate the norm of operator $F^{yy}(q;\theta)$. By Lemmas B 1, B 2 this is equivalent to estimate the operator norms of Fourier coefficients $\widehat{F}^{yy}(s)$. For this end we :

(1) estimate matrix coefficients $\widehat{H}_{kj}(s)$ of operator $\widehat{H}^{yy}(s)$,
(2) estimate coefficients $\widehat{F}_{kj}(s)$ via the relation (3.18);
(3) estimate the norm of a matrix $\widehat{F}^{yy}(s)$ via coefficients $\widehat{F}_{kj}(s)$.

Step (1) is rather simple. Indeed, the matrix of operator $\widehat{H}^{yy}(s) : Y^c_d \longrightarrow Y^c_{d-d^0_H}$ with respect to basises $\{\lambda_k^{(-d)} w_k \,|\, k \in \mathbb{Z}_0\} \subset Y^c_d$, $\{\lambda_k^{(-d+d^0_H)} w_k \,|\, k \in \mathbb{Z}_0\} \subset Y^c_{d-d^0_H}$ is equal to $\{\lambda_k^{(-d+d^0_H)} \widehat{H}_{kj}(s)\, \lambda_j^{(-d)}\}$. So by Lemma B1 $\forall j \in \mathbb{Z}_0$

$$\left|\, |k|^{d-d_H^0}\, \widehat{H}_{kj}(s)\, |j|^{-d} \right|_{\ell^2(k)}^{\Theta_m, \mathrm{Lip}} \le C \, \| \widehat{H}^{yy}(s) \|_{d,\, d-d_H^0}^{\Theta_m, \mathrm{Lip}} \le$$

$$\le C(m)\, \varepsilon^{-2/3}\, e^{-5/6\, \delta |s|} \,. \qquad (3.20)$$

Step (2) results from (3.18) and Lemma 3.1:

$$| \widehat{F}_{kj}(s) |^{\Theta_m \setminus (\Theta^1 \cup \Theta^2), \mathrm{Lip}} \le \frac{| \widehat{H}_{kj}(s) |^{\Theta_m, \mathrm{Lip}}\, T_1}{1 + |\lambda_{k0} - \lambda_{j0}|},$$

$$T_1 = C_1(m)(1 + |s|)^{2c+1} \,. \qquad (3.21)$$

For Step (3) we have to glue the estimates (3.20), (3.21) in order to obtain estimates on $\widehat{F}^{yy}(s)$ and $F^{yy}(q)$. The operator $\widehat{F}^{yy}(s)$ from the space Y_d^c with the basis $\{\lambda_j^{(-d)} w_j\}$ into the space $Y_{d_c}^c$, $d_c = d - d_H^0 + d_1 - 1$, with the basis $\{\lambda_j^{(-d_c)} w_j\}$ has a matrix

$$\{\lambda_k^{(d_c)}\, \widehat{F}_{kj}\, \lambda_j^{(-d)}\} \,. \qquad (3.22)$$

Let us denote by $\pi_{k,j}$ $(j,k \in \mathbb{Z}_0)$ a function $\pi_{k,j} = 1 - \delta_{k,j}$. Then for (3.21) we have a trivial estimate for ℓ^1–norm of the column number j and its Lipschitz coefficient:

$$| \lambda_k^{(d_0)}\, \widehat{F}_{kj}(s)\, \lambda_j^{(-d)} |_{\ell^1(k)}^{\Theta_{m+1}, \mathrm{Lip}} \le \qquad (3.23)$$

$$\leq T_1 \| \, |k|^{d-d_H^0} \widehat{H}_{kj}(s) \, |j|^{-d} \|_{\ell^2(k)}^{\Theta_m,Lip} \, \| \frac{\pi_{k,j} \, |k|^{d_1-1}}{1 + |\lambda_{k0} - \lambda_{j0}|} \|_{\ell^2(k)}$$

For to estimate the r.h.s. we need the following statement:

Lemma 3.2. If j_1 in (1.28) is large enough then $\forall j, k \in \mathbb{N}$

$$C_1 \, |j^{d_1} - k^{d_1}| \geq |\lambda_{j0} - \lambda_{k0}| \geq C_1^{-1} \, |j^{d_1} - k^{d_1}| \, . \tag{3.24}$$

If $j > k > 0$ then

$$|\lambda_{j0} - \lambda_{k0}| \geq C_2 j^{d_1-1} \, . \tag{3.25}$$

Proof. For $j = k$ the inequalities (3.24) are evident. So we may suppose that $j > k$. Then for the assumption (1.26)

$$\lambda_{j0} - \lambda_{k0} = K_2(j^{d_1} - k^{d_1}) + \Delta(j, k) \, ,$$

$$|\Delta(j, k)| \leq C \sum_{l=1}^{r-1} (j^{d_{1,l}} - k^{d_{1,l}}) + K_1 \, j^{d_{1,r}} + K_1 \, k^{d_{1,r}} \leq$$

$$\leq C_1(j)(j^{d_1} - k^{d_1}) + K_1 \, j^{d_{1,r}} + K_1 \, k^{d_{1,r}} \leq C_2(j)(j^{d_1} - k^{d_1})$$

and $C_1(j)$, $C_2(j) \longrightarrow 0$ as $j \longrightarrow \infty$ (one has to mention that $j^{d_1} - k^{d_1} \geq j^{d_1} - (j-1)^{d_1} \geq C \, j^{d_1-1}$ and so $(j^{d_1} - k^{d_1}) \, j^{-d_{1,r}} \longrightarrow \infty$ $(j \longrightarrow \infty)$ because $d_{1,r} < d_1 - 1)$. Now the estimate (3.24) is proved for j greater than some

C_* . For $j, k \le C_*$ it is true with some $C_1 >> 1$ because $\inf\{ |\lambda_{j0} - \lambda_{k0}| \; | 1 \le k < j \le C_* \} > 0$ for the assumption (1.28) with $s = 0$ and $-\ell_k = \ell_j = 1$ (one has to take $j_1 \ge C_*$).

Inequality (3.25) results from (3.24) . ∎

For this lemma

$$\left| \frac{\pi_{k,j} \; |k|^{d_1 - 1}}{1 + |\lambda_{k0} - \lambda_{j0}|} \right|^2_{\ell^2(k)} \le C \left[\sum_{k=-\infty}^{|j|-1} + \sum_{k=|j|+1}^{\infty} \right] \frac{|k|^{2(d_1-1)}}{\left[|k|^{d_1} - |j|^{d_1} \right]^2} .$$

After a substitution $k = |j| y$ one can estimate the sums in the r.h.s. via integrals. So

$$\left| \frac{\pi_{k,j} \; |k|^{d_1 - 1}}{1 + |\lambda_{k0} - \lambda_{j0}|} \right|^2_{\ell^2(k)} \le \frac{C_1}{|j|} \left[\int_{-\infty}^{1-|j|^{-1}} + \int_{1+|j|^{-1}}^{\infty} \right] \frac{|y|^{2(d_1-1)} \, dy}{\left[|y|^{d_1} - 1 \right]^2} \le C_2 .$$

By (3.20), (3.23) and for the last estimate , ℓ^1–norm of the column number j of the matrix (3.22) and its Lipschitz constant are no greater than

$$L_1 = C_1(m) \, T_1 \varepsilon^{-2/3} e^{-5/6 \; \delta |s|} .$$

For ℓ^1–norm of the row number k of the matrix (3.22) and its Lipschitz constant we have an estimate:

$$| \lambda_k^{(d_c)} \widehat{F}_{kj}(s) \, \lambda_j^{(-d)} |^{\theta_{m+1}, Lip}_{\ell^1(j)} \le$$

$$\leq C\, T_1 \,\|\, |k|^{d-d_H^0-1}\, \widehat{H}_{kj}(s)\, |j|^{1-d} \|^{\Theta_m,\mathrm{Lip}}_{\ell^2(j)} \left\| \frac{\pi_{k,j}\, |k|^{d_1}}{|j|\,(1+|\lambda_{k0}-\lambda_{j0}|)} \right\|_{\ell^2(j)} \tag{3.26}$$

As $H^{yy}(q) \in \mathscr{L}^s(Y^c_d\,;\, Y^c_{d-d^0_H})$ then by the interpolation theorem (Corollary A2)

$$\|H^{yy}\|^{U^1_m,\Theta_m}_{1-d+d^0_H,1-d} \leq 2\, \|H^{yy}\|^{U^1_m,\Theta_m}_{d\,,\,d-d^0_H} \leq C\, C_*(m)\, \varepsilon^{-2/3}$$

and for the conjugate operator $(H^{yy})^*$ one has an estimate

$$\|(H^{yy})^*\|^{U^1_m,\Theta_m}_{d-1\,,\,d-d^0_H-1} \leq C\, C_*(m)\, \varepsilon^{-2/3}\,.$$

Thus $\forall s,k$

$$\|\, |k|^{d-d_H^0-1}\, \widehat{H}_{kj}(s)\, |j|^{1-d} \|^{\Theta_m,\mathrm{Lip}}_{\ell^2(j)} \leq C(m)\, \varepsilon^{-2/3} e^{-5/6\,\delta|s|} \tag{3.27}$$

and the first factor in the r.h.s. in (3.26) is estimated. For the second one the following estimate is true:

$$\left\| \frac{\pi_{k,j}\, |k|^{d_1}}{|j|\,(1+|\lambda_{k0}-\lambda_{j0}|)} \right\|^2_{\ell^2(j)} \leq$$

$$\le \frac{C}{|k|}\left[\int_{-\infty}^{-|k|^{-1}} + \int_{|k|^{-1}}^{1-|k|^{-1}} + \int_{1+|k|^{-1}}^{\infty}\right] \frac{dy}{y^2(1-\mathrm{sgn}\, y\ |y|^{d_1})^2} \le C_1 \,.$$

For it and for (3.26), (3.27) ℓ^1–norm of the row number k is bounded above by the constant $L_2 = C_2(m)\, T_1\, \varepsilon^{-2/3} \exp -\frac{5}{6}\,\delta|s|$.

So the matrix (3.22) of the operator $\widehat{F^{yy}}(s) : Y^c_d \longrightarrow Y^c_{d_c}$ has columns and rows bounded in ℓ^1–norm together with their Lipschitz constants by $\max(L_1,L_2)$. Hence the norm of the operator is bounded by the same constant; for this classical result see [HLP, Chap. 8] or [HS]. We have got an estimate $\|\widehat{F^{yy}}(s)\|^{\Theta_{m+1},\mathrm{Lip}}_{d\,,\,d_c} \le C(m)\, T_1\, \varepsilon^{-2/3} e^{-5/6\ \delta|s|}$. By it and Lemma B2

$$\|f^{yy}\|^{U^2_m,\Theta_{m+1}}_{d\,,\,d_c} + \sum_{j=1}^{n} \|\frac{\partial}{\partial q_j} f^{yy}\|^{U^2_m,\Theta_{m+1}}_{d\,,\,d_c} \le C^e_1(m)\, \varepsilon^{-2/3} \qquad (3.28)$$

because the norm of f^{yy} is equivalent to the norm of F^{yy} up to a factor $C^e(m)$. So (2.39) is proved for $a = d$. The estimate (2.40) results from the equality (2.33) and from estimate (3.28).

The symmetry of the operators F^{yy} and f^{yy} results from the one of the Fourier coefficients $\widehat{F^{yy}}(s)$ (formula (3.18)). For $q \in \mathbb{T}^n$ the operator $f^{yy}(q)$ is real, i.e. it maps Y_d into $Y_{d-d^0_H}$ because the operators $h^{yy}(q)$, $q \in \mathbb{T}^n$, are real. So $f^{yy}(q) \in \mathcal{L}^s(Y^c_d\,;\,Y^c_{d_c})$. Now the validity of the estimate (2.39) $\forall a \in [-d_c,d]$ results from the estimate for $a = d$, from the symmetry of operator f^{yy} and interpolation theorem (Corollary A2) . The assertion c) is proved.

We give now a sketch of a proof of the assertion b). Let us substitute into (2.32) $f^y = W F^y$, $h^y = WH^y$. Then

$$\frac{\partial}{\partial \omega'} F^y - \left[J A_m - \operatorname{diag}\left[i \frac{\partial}{\partial \omega'} W_j \right]\right] F^y = H^y$$

or

$$(i\omega' \cdot s)\ \widehat{F^y} - \operatorname{diag}(i\ \lambda_j'(\theta))\ \widehat{F^y} = \widehat{H^y} . \tag{3.29}$$

Let

$$\widehat{F^y}(s) = \sum_{j \in \mathbb{Z}_0} \widehat{F}_j(s) w_j\ ,\ \widehat{H^y}(s) = \sum_{j \in \mathbb{Z}_0} \widehat{H}_j(s) w_j$$

Then by (3.29)

$$\widehat{F}_j(s) = D_1^{-1}(j,s;\theta)\ \widehat{H}_j(s),\ D_1(j,s;\theta) = i(s \cdot \omega' - \lambda_j') \tag{3.30}$$

By (2.30), (3.10) and Lemma B1

$$\| \widehat{H^y}(s) \|_{d - d_H^0}^{\Theta_{m+1}, \mathrm{Lip}} \le C^e(m)\ \varepsilon^{-1/3} e^{-5/6\ \delta |s|} . \tag{3.31}$$

For to estimate D_1^{-1} we use an analog of Lemma 3.1 (it will be proved in § 4):

Lemma 3.3. There exists a Borel subset $\Theta^3 \subset \Theta_m$ with the property (3.1) and such that

$$|D_1^{-1}|^{\Theta_m \setminus \Theta^3, Lip} \leq C\, C_{**}(m)\ |s|^{2n+3}. \qquad (3.32)$$

By equality (3.30) and estimates (3.31), (3.32)

$$\|\widehat{F}^y(s)\|_{d-d_H^0}^{\Theta_{m+1}, Lip} \leq C_1^e(m)\ |s|^{2n+5} e^{-5/6\ \delta |s|} \varepsilon^{-1/3}.$$

So by Lemma B2

$$\|f^y\|_{d-d_H^0}^{U_m^2, \Theta_{m+1}} + \|\nabla_q f^y\|_{d-d_H^0}^{U_m^2, \Theta_{m+1}} \leq C^e(m)\, \varepsilon^{-1/3}$$

and the estimate (2.37) results from the equality (2.32). ∎

4. Proof of Lemmas 3.1, 3.3 (estimation of small divisors)

The estimate (3.19) results easily if we prove the following one:

$$|D(k,j,s;\theta)| \geq \frac{|\lambda_{k0} - \lambda_{j0}|}{C_1(m)(1+|s|)^c}$$

$$\forall k \neq j \in \mathbb{Z}_0 \,, \forall s \in \mathbb{Z}^n \,, \forall \theta \in \Theta_m \setminus (\Theta^1 \cup \Theta^2) \,. \qquad (4.1)$$

Indeed, $\mathrm{Lip}\, D^{-1} \leq (\mathrm{Lip}\, D)(\inf|D|)^{-2}$ and by the estimates (3.6), (1.27) $\mathrm{Lip}\, D(k,j,s;\cdot) \leq C(|s| + 1 + \max\{|j|\,,\,|k|\}^{d_1-1})$. So (4.1) and (3.25) imply (3.19).

We may suppose that $|j| \geq |k|$ and $j > 0$ because $|D(k,j,s)| = |D(j,k,s)| = |D(-k,-j,-s)|$. So in what follows

$$j > 0\,,\ |k| \leq j\,,\ k \neq j\,. \qquad (4.2)$$

By estimates (3.6)

$$|\lambda_j' - \lambda_{j0}| \leq C(\delta_a\, |j|^{d_{1,r}} + \varepsilon_0^{\rho}\, |j|^{d_H^0}) \qquad \forall j\,, \forall \theta \qquad (4.3)$$

By this estimate and (4.2), (3.25) we have for $\delta_a, \varepsilon_0 << 1$ inequalities

$$|\lambda_k' - \lambda_j'| \geq |\lambda_{k0} - \lambda_{j0}| - |\lambda_k' - \lambda_{k0}| - |\lambda_j' - \lambda_{j0}| \geq \tfrac{1}{2}|\lambda_{k0} - \lambda_{j0}| +$$

$$+\tfrac{1}{2}C_2|j|^{d_1-1} - C_1(\delta_a + \varepsilon_0^{\delta})|j|^{d_1-1} \geq \tfrac{1}{2}|\lambda_{k0} - \lambda_{j0}| \,. \tag{4.4}$$

If $2|\omega' \cdot s| \leq |\lambda_k' - \lambda_j'|$ then by (4.4) $|D| \geq \frac{1}{2}|\lambda_k' - \lambda_j'| \geq \frac{1}{4}|\lambda_{k0} - \lambda_{j0}|$ and the estimate (4.1) is obtained. So we may suppose below that

$$2|\omega' \cdot s| \geq |\lambda_k' - \lambda_j'| \tag{4.5}$$

In particular, $s \neq 0$. By (4.4), (4.5) and (3.24), (3.25)

$$j^{d_1-1} \leq C|\omega' \cdot s| \,, \tag{4.6}$$

$$|\lambda_{k0} - \lambda_{j0}| \leq C|s| \,, \tag{4.7}$$

$$\left|j^{d_1} - |k|^{d_1}\right| \leq C_1|s| \,. \tag{4.8}$$

Situations $d_1 = 1$ and $d_1 > 1$ have to be considered separately. We start with more difficult one.

A) $d_1 = 1$. Then in (1.25) $d_H^0 < -\chi$ and in (1.26) $d_{1,r} \leq -\chi$ and $d_{1,j} \leq 1 - \chi$ $\forall j$ for some $0 < \chi < 1$. Depending on the relation between k and s, we consider three cases.

A1) $|s| \leq 9\,K_1|k|^{-\chi} + \frac{1}{2}$. Then

$$|k| \leq (18\,K_1)^{1/\chi}, \; |s| \leq 9\,K_1 + \tfrac{1}{2} \tag{4.9}$$

because $|s| \geq 1$, $|k| \geq 1$. By (4.8)

$$j \leq C_1 |s| + |k| \leq C_1(9\,K_1 + \tfrac{1}{2}) + (18\,K_1)^{1/\chi} = C_{1*}$$

Let us take in the assumption 3) of the theorem $j_1 \geq C_{1*}$ and $M_1 \geq 9\,K_1 + \frac{1}{2}$. Then by (3.3), (4.3) and (1.28) with $\ell_j = 1$, $\ell_{|k|} = -\operatorname{sgn} k$ (or $\ell_j = 2$ if $k = -j$)

$$|D| \geq |\omega_0 \cdot s + \lambda_{k0} - \lambda_{j0}| - (9\,K_1 + \tfrac{1}{2})\,|\omega' - \omega_0| -$$

$$- |\lambda_k' - \lambda_{k0}| - |\lambda_j' - \lambda_{j0}| \geq K_5 - C|\varepsilon_0^\rho + \delta_a|\,.$$

Now the estimate (4.1) results from the last one because $0 \leq j \leq C_{1*}$, $|k| \leq (18\,K_1)^{1/\chi}$.

A2) $|s| > 9\,K_1 k^{-\chi} + \frac{1}{2}$, $|k| \leq C_{*3}(m)\,|s|^{m_0}$. Here $m_0 \geq \chi^{-1}(n+3)$ and a function $C_{*3}(m)$ will be chosen later. By (3.6) and (1.27) Lip $(\lambda_k' - \lambda_j') \leq 3\,K_1 |k|^{-\chi}$ if $\varepsilon_0 << 1$. So

$$|s| \geq 3\,\mathrm{Lip}(\lambda_k' - \lambda_j') + \tfrac{1}{2}. \qquad (4.10)$$

Let

$$T = T(k,j,s) = C_{*4}^{-1}(m)\,|s|^{-m_1}\,|\lambda_{j0} - \lambda_{k0}|, \qquad m_1 = m_0 + n + 2\,,$$

and

$$\Theta'(k,j,s) = \{\theta \in \Theta_m \mid |D(k,j,s;\theta)| \le T\} ,$$

$$\Theta^{2,1} = \cup \{\Theta'(k,j,s) \mid (9 K_1 |k|^{-\chi} + \tfrac{1}{2}) < |s| , |k| \le j ,$$

$$|k| \le C_{*3}(m) |s|^{m_0} , |\lambda_{k0} - \lambda_{j0}| \le C|s|\} .$$

We shall construct a set Θ^2 as $\Theta^2 = \Theta^{2,1} \cup \Theta^{2,2}$ (a set $\Theta^{2,2}$ will be defined later). Therefore, if $\theta \notin \Theta^2$ then $\theta \notin \Theta^{2,1}$ and $|D| \ge T$. So (4.1) is true.

We have to estimate $\text{mes}\, \Theta^{2,1}$ [I] . For this end we estimate $\text{mes}\, \Theta'(k,j,s)$ [I] . By the estimate (3.3$'$) $\text{mes}\, \Theta'(k,j,s)\, [I] \le 2\, \text{mes}\, \Omega'(k,j,s)\, [I]$. Here $\Omega'(k,j,s)$ [I] is the image of the set $\Theta'(k,j,s)$ [I] under the map (3.2). For to estimate $\text{mes}\, \Omega'$ [I] it is enough to estimate one–dimensional Lebesque measure of the intersection of Ω' [I] with an arbitrary line of a form $\{\omega' = \omega'(t) = \eta + t\, s|s|^{-1} | t \in \mathbb{R}\}$, $\eta \in \mathbb{R}^n$. The set of "t" corresponding to this intersection is contained in a set

$$\{t \mid -T \le \Gamma(t) \le T\} , \quad \Gamma(t) = \eta \cdot s + t|s| + (\lambda_k' - \lambda_j')(\omega'(t)) \qquad (4.11)$$

By (3.3) $\text{Lip}\, (\omega : \omega' \mapsto \omega) \le \frac{3}{2}$ if $\varepsilon_0 << 1$ $\forall I \in \mathcal{J}$. So by (4.10) $\text{Lip}\, (t \mapsto (\lambda_k' - \lambda_j')(\omega'(t)) \le \frac{1}{2}|s| - \frac{1}{4}$. Hence for $t_1 > t_2$

$$\Gamma(t_1) - \Gamma(t_2) \ge |s|(t_1 - t_2) - |(\lambda_k' - \lambda_j')(\omega'(t_1)) -$$

$$- (\lambda_k' - \lambda_j')(\omega'(t_2))| \ge |t_1 - t_2|\, (\tfrac{1}{2}|s| + \tfrac{1}{4})$$

and the measure of the set (4.11) is not greater than $2T(\frac{1}{2}|s| + \frac{1}{4})^{-1}$. Since the set $\Omega'[I]$ is bounded and the vector η may be chosen orbitrarily, we have by Fatou lemma: mes $\Omega'(k,j,s)$ $[I] \le CT(|s| + 1)^{-1}$. So

$$\text{mes}\ \Theta^{2,1}[I] \le \sum_{k,j,s} \text{mes}\ \Theta'(k,j,s)\ [I] \le 2\sum_{s\neq 0}\sum_{j,k} \text{mes}\ \Omega'(k,j,s)\ [I]\ .$$

As $|k| \le C_{*3}(m)\,|s|^{m_0}$ and $|j-k| \le C_1|s|$, then we have no more than $C\,C_{*3}(m)\,|s|^{m_0+1}$ admissible pairs (j,k). As $|\lambda_{k0} - \lambda_{j0}| \le C|s|$ then $T \le C|s|^{1-m_1}\,C_{*4}^{-1}(m)$ and

$$\text{mes}\ \Theta^{2,1}[I] \le C\sum_{s\neq 0} \frac{C_{*3}(m)\ |s|^{m_0+1}}{C_{*4}(m)\ |s|^{m_1}} \le \frac{C_1\ C_{*3}(m)}{C_{*4}(m)}\,.$$

Therefore, under a suitable choice of the function $C_{*4}(m)$, depending on a choice of $C_{*3}(m)$, mes $\Theta^{2,1}[I]$ is no greater than one–half of the r.h.s. of (3.1).

A_3) $|k| \ge C_{*3}(m)\,|s|^{m_0}$, $s \neq 0$. Then by (4.2) and (4.7) $j > k > 0$. By (1.26) (with $d_1 = 1$, $d_{1,r} \le -\chi$, $d_{1,j} \le 1-\chi$ $\forall j = 1, \dots, r-1$), (4.3) (with $d_{1,r} < -\chi$, $d_H^0 < -\chi$) and (4.7) we have

$$|\lambda_j' - \lambda_k' - K_2(j-k)| \le C\,k^{-\chi}(|j-k| + 1) \le \tag{4.12}$$

$$\leq C\, C_{*3}^{-\chi}(m)\ |s|^{1-\chi m_0}.$$

Let us set

$$\Omega''(s,N) = \left\{ \omega' \,\middle|\, |\omega' - \omega_0| \leq 1\,,\ |\omega'\cdot s - NK_2| \leq \frac{|\lambda_{j0} - \lambda_{k0}|}{C_{*2}(m)\ |s|^{n+2}} \right\}$$

(a function $C_{*2}(m)$ will be chosen later) and

$$\Theta^{2,2} = \cup\ \{\theta \in \Theta_m \,|\, \omega'(\theta) \in \Omega''(s,N)\}\ .$$

Here we take the union over all $s \in \mathbb{Z}_0^n$ and $N \in \mathbb{Z}$. The set $\Omega''(s,N)$ is empty if $|N| \geq C|s|\ K_2^{-1}$; by (4.7) mes $\Omega''(s,N) \leq C(C_{*2}(m)\ |s|^{n+2})^{-1}$. So by (3.3′)

$$\text{mes}\ \Theta^{2,2}[I] \leq \sum_{s\neq 0}\ \sum_{|N|\leq C|s|/K_2} \frac{C_1}{C_{*2}(m)\ \ |s|^{n+2}} \leq \frac{C_2}{C_{*2}(m)}$$

and mes $\Theta^{2,1}[I]$ is no greater than one–half of r.h.s. of (3.1) if the function $C_{*2}(m)$ is large enough.

If $\theta \notin \Theta^{2,2}$ then by (3.25) and (4.12), by the definition of $\Omega''(s,N)$ and by the inequality $m_0 \geq (n+3)/\chi$

$$|D| = |(\lambda'_j - \lambda'_k - K_2(j-k)) + (K_2(j-k) - \omega'\cdot s)| \geq$$

$$\geq |\lambda_{j0} - \lambda_{k0}|\ C_{*2}^{-1}(m)\ |s|^{-n-2} - C\, C_{*3}^{-\chi}(m)\ |s|^{1-\chi m_0} \geq$$

$$\geq \frac{1}{2} |\lambda_{j0} - \lambda_{k0}| \ C_{*2}^{-1}(m) \ |s|^{-n-2},$$

if $C_{*3}(m)$ is large enough. The inequality (4.1) for $\theta \in \Theta_m \setminus \Theta^{2,2}$ results from the last one.

Now the lemma is proved for $d = 1$ with $\Theta^2 = \Theta^{2,1} \cup \Theta^{2,2}$.

B) $d_1 > 1$. Let us find $\chi \in (0,1)$ such that $d_1 - 1 > \chi$ and $d_{1,r} \leq d_1 - 1 - \chi$.

By the inequality (4.6)

$$|s| \geq C_* \, j^{d_1 - 1}. \tag{4.13}$$

Let us denote $j_* = (12 \ K_1 \ C_*^{-1})^{1/\chi}$, $j_{**} = 3 \, j_*^{d_1 - 1} (K_1 \, j_*^{-\chi} + 1) + 1$ and consider two cases.

B1) $j \leq j_*$, $|s| \leq j_{**}$. In this case the estimate (4.1) results from (4.3) and assumption 3) of the theorem if $j_1 \geq j_*$, $M_1 \geq j_{**}$ and $\varepsilon_0 << 1$, $\delta_a << 1$.

B2) $j > j_*$ or $|s| > j_{**}$. Let the sets $\Theta'(k,j,s)$ and $\Omega'(k,j,s)[I]$ be the same as in the item A2) and

$$\Theta^2 = \cup \{\Theta'(k,j,s) \,|\, j > |k| \, , \, |\lambda_{k0} - \lambda_{j0}| \leq C \, |s| \, , \, |s| \geq C_* j^{\chi}\}.$$

Then for $\theta \in \Theta_m \setminus \Theta^2$ the estimate (4.1) is true. So we have to estimate mes $\Theta^2[I]$.

By (1.27) and (3.6)

$$\operatorname{Lip}(\omega' \mapsto \lambda_r'(\omega')) \le \tfrac{3}{2} j^{d_1-1}(K_1 j^{-\chi} + \varepsilon_0^{\rho}),\ r = k,j .$$

By this estimate and (3.3) we have for the function $\Gamma(t)$ (see (4.11)):

$$\Gamma(t_1) - \Gamma(t_2) \ge |s|\,(t_1 - t_2) - 3\, j^{d_1-1}(K_1 j^{-\chi} + \varepsilon_0^{\rho})\,(t_1 - t_2) .$$

If $j > j_*$ then by (4.13) for $t_1 > t_2$

$$\Gamma(t_1) - \Gamma(t_2) \ge (t_1 - t_2) j^{d_1-1}\,(C_* - 3\,(K_1\, j^{-\chi} + \varepsilon_0^{\rho})) \ge \tfrac{1}{2}\, C_*(t_1 - t_2) ,$$

if $j \le j_*$ then $|s| > j_{**}$ and

$$\Gamma(t_1) - \Gamma(t_2) \ge (t_1 - t_2)(|s| - 3\, j_*^{d_1-1}\,(K_1\, j_*^{-\chi} + 1) \ge t_1 - t_2 .$$

So $\operatorname{mes} \Omega'(k,j,s)[I] \le C_1 T$ and $\operatorname{mes} \Theta^2[I] \le C \sum_{s\ne 0} \sum_{k,j} T(k,j,s)$.

By (4.13) there are no more than $C|s|^{2\chi}$ admissible pairs (j,k) ; by (4.7) $|\lambda_{k0} - \lambda_{j0}| \le C|s|$. So

$$\operatorname{mes} \Theta^2[I] \le C \sum_{s\ne 0} C_{*4}^{-1}(m)\,|s|^{1+2\chi-m_1} \le \frac{C_1}{C_{*4}(m)} ,$$

if $m_1 \ge n + 2 + 2\chi$, and the estimate (3.1) is fulfilled if $C_{*4}(m)$ is large enough.

The lemma is proved. ∎

Proof of the Lemma 3.3. Let us define a set Θ^3 as follows:

$$\Theta^3 = \cup \{\Theta'(j,s) \mid s \in \mathbb{Z}^n, \ j \in \mathbb{Z}_0\},$$

$$\Theta'(j,s) = \{\theta \in \Theta_m \mid \ |s \cdot \omega'(\theta) - \lambda_j'(\theta)| \le C_{**}^{-1}(m)(1 + |s|)^{-n-1}\}.$$

By the assumption (1.26) the set Θ' is empty if $|j| \ge C|s|^{1/d_1}$. By (1.26) and (1.28) with $s = 0$, $|\ell_1| + \ldots + |\ell_{j_1}| = 1$ and M_1 large enough, $|\lambda_j(\theta)| \ge C^{-1}$ $\forall j, \theta$. So by (4.3) this set is empty if $s = 0$ provided that $\varepsilon_0 << 1$, $\delta_a << 1$ and $C_*(m) >> 1$. Thus we may suppose that

$$|j| \le C|s|^{1/d_1}, \ s \ne 0. \tag{4.14}$$

As in the proof of Lemma 3.1 we get that $\operatorname{mes} \Theta'(j,s)[I] \le C\, C_{**}^{-1}(m)\ |s|^{-n-2}$. So by (4.14)

$$\operatorname{mes} \Theta^3[I] \le \frac{C}{C_{**}(m)} \sum_{s \ne 0} \sum_{j \le C|s|} (1 + |s|)^{-n-2} \le \frac{C_1}{C_{**}(m)}$$

and (3.1) is true if $C_{**}(m)$ is large enough. If $\theta \notin \Theta^3$ then $|D| \ge C_{**}(m)^{-1}(1 + |s|)^{-n-1}$ and (3.32) is proved.

5. Proof of Lemma 2.3 (estimation of the change of variables)

Let us denote by $E^{c\,\sigma}_{s,\varepsilon}$, $s \in \mathbb{R}$, $\sigma = \pm$, the space $E^c_s = \mathbb{C}^{2n} \times Y^c_s$ endowed with a norm $\|\cdot\|_{(\sigma,s,\varepsilon)}$,

$$\|(p,\xi,y)\|^2_{(\pm,s,\varepsilon)} = |p|^2 + \varepsilon^{\pm\frac{4}{3}} |\xi|^2 + \varepsilon^{\pm\frac{2}{3}} \|y\|^2_{\mp s} .$$

The following assertion results from the definition.

Lemma 5.1. For all $s \in \mathbb{R}$ the spaces $E^{c\,\pm}_{s,\varepsilon}$ are dual with respect to the bilinear pairing $<\cdot,\cdot>_E : E^c \times E^c \longrightarrow \mathbb{C}$,

$$\|\mathfrak{h}\|_{(\pm,s,\varepsilon)} = \sup_{\|\mathfrak{h}^*\|_{(\mp,s,\varepsilon)} \leq 1} |<\mathfrak{h},\mathfrak{h}^*>_E| .$$

We denote by $\mathrm{dist}_{(s,\varepsilon)}$ a metric in $\mathscr{Y}^c_s$ induced by $\|\cdot\|_{(-,s,\varepsilon)}$.

Let us write down the system (2.28) in a form:

$$\dot{\mathfrak{h}} = \varepsilon\, \mathscr{F}(\mathfrak{h}) \,,\ \mathfrak{h} = \mathfrak{h}(t) = (q(t), \xi(t), y(t)) \,,$$
$$\mathscr{F} = (\mathscr{F}^q, \mathscr{F}^\xi, \mathscr{F}^y),\ \mathscr{F}^q = \nabla_\xi F,\ \mathscr{F}^\xi = -\nabla_q F,\ \mathscr{F}^y = J\nabla_y F . \tag{5.1}$$

If $\varepsilon_0 << 1$, then for $j = 1 , \dots , 5$

$$\mathrm{dist}_{(s,\varepsilon)}\,(O_m{}^{j+1,c}\,,\,O_m{}^{c}\setminus O_m{}^{jc}) \geq C^{-1}(m)\,. \tag{5.2}$$

By Lemma 2.2 and the Cauchy estimate

$$\varepsilon\|\mathcal{F}\|^{O_m{}^{3c}\times\Theta_{m+1},\mathrm{Lip}}_{(-,d\ +\ \Delta d,\varepsilon)} \leq C^e(m)\varepsilon^{1/3}\ ,\Delta d = d_1 - d_H^0 - 1\,. \tag{5.3}$$

By (5.2) and (5.3) for $0 \leq t \leq 1$ and $\varepsilon_0 << 1$ the solution of (5.1) depends analytically on $\mathfrak{h}(0) \in O_m{}^{4c}$ and stays inside $O_m{}^{3c}$. So (2.43) is proved.

For every $\mathfrak{h} \in O_m^{3c}$ the following estimate on a tangent map $\mathcal{F}_*$ results by Lemma 2.2:

$$\|\varepsilon\ \mathcal{F}_*(\mathfrak{h};\cdot)\|^{\Theta_{m+1},\mathrm{Lip}}_{(-\,,\,a,\varepsilon)\ ,\ (-,a\,+\,\Delta d,\varepsilon)} \leq C^e(m)\ \varepsilon^{1/3}$$

$$\forall a \in D = [-d - \Delta d, d]\ . \tag{5.4}$$

For $t \in [0,1]$ let us set $\eta(t) = S^t{}_*(\mathfrak{h})\eta$. Then $\eta(t)$ is a solution of the Cauchy problem

$$\dot{\eta}(t) = \varepsilon\ \mathcal{F}_*(\mathfrak{h}(t))\ \eta(t)\ ,\ \eta(0) = \eta\ ,\ \mathfrak{h}(t) = S^t(\mathfrak{h})\,.$$

By (5.4) for $\mathfrak{h} \in O_m{}^{4c}$ and $a \in D$ we get estimates:

$$\|S_*^t(\mathfrak{h}) - \mathrm{Id}\,\|^{\Theta_{m+1},\mathrm{Lip}}_{(-\,,\,a,\varepsilon)\ ,\ (-,a\,+\,\Delta d,\varepsilon)} \leq tC^e(m)\ \varepsilon^{1/3} \tag{5.5}$$

and

$$\|S_*^t(\mathfrak{h}) - \mathrm{Id} - t\varepsilon\, \mathscr{F}_*(\mathfrak{h})\|^{\Theta_{m+1},\mathrm{Lip}}_{(-,\,a,\varepsilon)\ ,\ (-,a+\Delta d,\varepsilon)} \leq t^2 C_1^e(m)\, \varepsilon^{2/3} \tag{5.6}$$

The first of them results from the identity

$$\eta(t) - \eta = \varepsilon \int_0^t \mathscr{F}_*(\mathfrak{h}(\tau))\, \eta(\tau)\, d\tau$$

and the second one results from the identity

$$\eta(t) - \eta - \varepsilon t\ \mathscr{F}_*(\mathfrak{h})\eta = \varepsilon \int_0^t (\mathscr{F}_*(\mathfrak{h}(\tau))\, \eta(t) - \mathscr{F}_*(\mathfrak{h})\, \eta)\, d\tau\,.$$

Let $\mathfrak{h}(t) = (q(t)\,,\, \xi(t)\,,\, y(t))$ be a solution of (5.1) with $\mathfrak{h}(0) = \mathfrak{h} = (q,\xi,y)$. Then

$$\dot q(\tau) = \varepsilon\, f^{\xi}(q(\tau))\,.$$

So $\Pi_q \circ S^\tau(\mathfrak{h}) = S_q^\tau(q;\theta)$ (i.e. does not depend on ξ and y) and by (2.35)

$$\begin{aligned} |S_q^\tau(q) - q|^{U_m^3,\Theta_{m+1}} &\leq \tau\, C(m)\, \varepsilon^{1/3}\,, \\ |S_q^\tau(q) - q - \tau\,\varepsilon\, f^{\xi}(q)|^{U_m^3,\Theta_{m+1}} &\leq \tau^2\, C_1(m)\, \varepsilon^{2/3} \end{aligned} \tag{5.7}$$

By the first estimate with $\tau = 1$ we get an assertion (2.45).

For $y(t)$ we have an equation

$$\dot{y}(t) = 2\varepsilon\, J\, f^{yy}(q(\tau))y + \varepsilon\, J\, f^{y}(q(\tau)) . \tag{5.8}$$

Let $z(t) = z(t)\,(q;\theta)$ be a solution of (5.8) with zero Cauchy data. Then by (2.36), (2.39), (2.40) and (5.7)

$$\|z(t)\|^{U^4_m,\Theta_{m+1}}_{d-d^0_H + d_1} \le t\, C^e(m)\, \varepsilon^{2/3} \qquad \forall t \in [0,1] . \tag{5.9}$$

Let us substitute into (5.8) $y(t) = z(t) + u(t)$. Then

$$\dot{u} = 2\varepsilon\, J\, f^{yy}(q(\tau))u\ , \ u(0) = y . \tag{5.10}$$

So $u(t) = y + U(t)y$, here $U(t)$ is a linear operator and by (2.39), (5.7)

$$\|U(t)\|^{U^4_m,\Theta_{m+1}}_{d\ ,\ d\ +\ \Delta d} \le t\, C^e(m)\, \varepsilon^{1/3} \qquad \forall t \in [0,1] . \tag{5.11}$$

So $S_m(q,\xi,y) - y = z(1)\,(q;\theta) + U(1)\,(q;\theta)y$ and the estimate (2.47) results from (5.9), (5.11).

The estimate (2.46) results from the equation on $\xi(t)$ and the estimates on $q(t)$, $y(t)$. Now (2.44) results from (2.45) – (2.47) .

The transformation $S_m = S^t|_{t=1}$ is canonical as a shift along the trajectories of Hamiltonian flow (see [1], Theorem 2.4). For to investigate the transformed hamiltonian $\mathcal{H}_m \circ S_m$ we start with an analysis of the quadratic term $\mathfrak{A}(t) = \frac{1}{2} < A_m(q(t))\, y(t)\, , y(t) >$ with $y(t) = z(t) + u(t) = z(t)\,(q;\theta) + y + U(t)\,(q,\theta)y$. It is equal to a sum of terms of zero order, first order and second order on y:

$$\mathfrak{A}(t) = \mathfrak{A}_0(t) + < \mathfrak{A}^y(t), y > + < \mathfrak{A}^{yy}(t)y, y > , \tag{5.12}$$

$$\mathfrak{A}_0(t)\,(q;\theta) = \frac{1}{2} < A_m(q(t);\theta)\, z(t), z(t) > ,$$

$$\mathfrak{A}^y(t)\,(q;\theta) = (I + U(t))^* A_m(q(t))\, z(t) ,$$

$$\mathfrak{A}^{yy}(t)\,(q;\theta) = \frac{1}{2}\,(I + U(t))^* A_m(q(t))\,(I + U(t)) .$$

<u>Lemma 5.2</u>. The following estimates are valid:

$$\|\mathfrak{A}^{yy}(1) - \mathfrak{A}^{yy}(0) - \frac{\varepsilon}{2} f^{\xi}(q) \cdot \nabla_q A_m(q) -$$

$$- [J\, A_m(q)\, , \varepsilon\, f^{yy}(q)]\, \|^{U_m^4, \Theta_{m+1}}_{d\, ,\, d-d_H^0} \le C^e(m)\, \varepsilon^{2/3} , \tag{5.13}$$

$$\|\mathfrak{A}^y(1) - \mathfrak{A}^y(0) - J\, A_m(q) f^y(q)\|^{U_m^4, \Theta_{m+1}}_{d-d_H^0} \le C^e(m)\varepsilon , \tag{5.14}$$

$$|\mathfrak{A}_0(1)|^{U_m^4, \Theta_{m+1}} \le C^e(m)\, \varepsilon^{4/3} . \tag{5.15}$$

Proof. By the definition of $\mathfrak{A}^{yy}(t)$ we get an equality:

$$< (\mathfrak{A}^{yy}(1) - \mathfrak{A}^{yy}(0))\, y,y > = \frac{1}{2}\int_0^1 \frac{d}{dt} < A_m(q(t))\, u(t)\, ,\, u(t) > dt =$$

$$= \int_0^t < B(t)\, u(t)\, ,\, u(t) > + \frac{1}{2} < (\varepsilon\, f^{\xi}(q(t)) \cdot \nabla A_m(q(t)))\, u(t)\, ,\, u(t) > dt \tag{5.16}$$

with $B(t) = [J\, A_m(q(t))\, ,\, \varepsilon\, f^{yy}(q(t))]$. By (5.7) and (2.40)

$$\|B(t)\|^{U^4_m, \Theta_{m+1}}_{d\,,\, d-d^0_H} \le C^e(m)\, \varepsilon^{1/3}\, , \tag{5.17}$$

$$\|B(t) - B(0)\|^{U^4_m, \Theta_{m+1}}_{d\,,\, d-d^0_H} \le t\, C^e(m)\, \varepsilon^{2/3}\, . \tag{5.18}$$

By (5.7) and (2.9), (2.10)

$$\|\, \Big|^t_{\tau=0} (\varepsilon\, f^{\xi}(q(\tau)) \cdot \nabla A_m(q(\tau)))\|^{U^4_m, \Theta_{m+1}}_{d\,,\, d-d^0_H} \le t\, C(m)\, \varepsilon^{2/3}\, . \tag{5.19}$$

Now we may replace the integrand in (5.16) by its value at $t = 0$ and get the estimate (5.13) by (5.18), (5.19) and (5.11).

For to prove (5.14) we rewrite $< (\mathfrak{A}^{y}(1) - \mathfrak{A}^{y}(0))\, ,\, y >$ as follows:

$$< (\mathfrak{A}^y(1) - \mathfrak{A}^y(0)) , y > = \int_0^1 \frac{d}{dt} < A_m(q(t))\, z(t) , u(t) > dt =$$

$$= \int_0^1 (< (\tfrac{d}{dt} A_m(q(t))) + A_m\,(q(t))\,(2\,\varepsilon\, J\, f^{yy}(q(t))z + \varepsilon\, J\, f^y(q(t)) , u(t) > +$$

$$+ < A_m\,(q(t))\, z(t) , 2\,\varepsilon\, J\, f^{yy}(q(t))u(t) >)\, dt .$$

If $\|y\|_{-d + d_H^0} \leq 1$ then by (5.12), (5.8) and estimates on f^{yy}, f^y this integral differs from $< J\, A_m(q)\, \varepsilon\, f^y(q) , y >$ by $C^e(m)\, \varepsilon$, as stated in (5.14).

The last estimate of the lemma results from (5.10). ∎

By (5.12) $\nabla(\mathfrak{A}(1) - \mathfrak{A}(0)) = (\mathfrak{A}^y(1) - \mathfrak{A}^y(0)) + 2(\mathfrak{A}^{yy}(1) - \mathfrak{A}^{yy}(0))\, y$. So we have the following consequence from this lemma:

Corollary 5.3. For $\mathfrak{h} \in O^c_{m+1}$

$$\|\nabla_y(\mathfrak{A}(1) - \mathfrak{A}(0) - \tfrac{1}{2} < \varepsilon\, f^{\xi}(q) \cdot \nabla_q\, A_m(q)\, y , y > -$$

$$- < [J\, A_m(q) , \varepsilon\, f^{yy}(q)]\; y , y > - < J\, A_m(q)\, f^y(q) , y >)\|^{U^c_{m+1}, \Theta_{m+1}}_{d - d_H^0} \leq \varepsilon .$$

Let $S_m(\mathfrak{h}) = \mathfrak{h} + \varepsilon \mathfrak{h}^1 = (\tilde{q}, \tilde{\xi}, \tilde{y})$. We write the transformed hamiltonian as follows:

$$\mathscr{H}_m(S_m(\mathfrak{h};\theta);\theta) = (H_m'(\mathfrak{h};\theta) + \varepsilon < \Delta h^{yy}(q;\theta)\, y,y >)$$

$$
\begin{aligned}
&+[\tfrac{1}{2} < A_m(\tilde{q})\,\tilde{y},\tilde{y} > - \tfrac{1}{2} < A_m(q)\,y,y > - \\
&- < [J\,A_m(q)\,,\,\varepsilon\, f^{yy}(q)]\;y,y > - \tfrac{1}{2} < \varepsilon\, f^{\xi}(q)\cdot \nabla_q A_m\;y,y > - \\
&- < J\,A_m(q)\,f^{y}(q)\,,\,y >]_1 + \varepsilon\,[(\xi^1 - \mathscr{g}^{\xi})\cdot A_{m+1}]_2 + \\
&+ \varepsilon\,[h^q - \frac{\partial}{\partial\omega} f^q]_3 + \varepsilon\,[(h^{\xi} - \frac{\partial}{\partial\omega} f^{\xi})\cdot \xi]_4 - \\
&- \varepsilon\,[< \frac{\partial}{\partial\omega} f^y - A_m\,J\,f^y - h^y\,,\,y >]_5 - \\
&- \varepsilon\,[< (\frac{\partial}{\partial\omega} f^{yy} - [J\,A_m\,,\,f^{yy}] - h^{yy} - \tfrac{1}{2} f^{\xi}\cdot \nabla_q A_m + \Delta h^{yy})\;y,y >]_6 + \\
&+ [(\varepsilon H_{2m} + \varepsilon H_{3m} + H^3)(\mathfrak{h} + \varepsilon\mathfrak{h}^1) - (\varepsilon H_{2m} + \varepsilon H_{3m} + H^3)(\mathfrak{h})]_7 + \\
&+ \varepsilon\,[H_{3m}]_8 + H^3
\end{aligned}
\tag{5.20}
$$

We denote by $\Delta_j H$ the functional in the brackets $[\cdot]_j$ (together with the preceding factor).

<u>Lemma 5.4</u>. For $j = 1, \dots, 8$ the following estimates hold:

$$
|\Delta_j H|^{O^c_{m+1},\Theta_{m+1}} \le \tfrac{1}{8}\, C_*(m+1)\,\varepsilon^{\rho+1}
\tag{5.21}
$$

$$
\|\nabla_y\, \Delta_j H\|^{O^c_{m+1},\Theta_{m+1}}_{d\;-\;d^0_H} \le \tfrac{1}{8}\, C_*(m+1)\,\varepsilon^{\frac{2}{3}(\rho+1)}
\tag{5.22}
$$

<u>Proof</u>. We prove more complicated estimates (5.22) only.

j = 1 . The estimate is contained in Corollary 5.3.

j = 2 . For the natural projection $\Pi_y : E^c_{(+,-d+d_H^0,\varepsilon)} \longrightarrow Y_{d-d_H^0}$ we have:

$$\|\Pi_y\|_{(+,-d+d_H^0,\varepsilon)\,,\,d-d_H^0} \le \varepsilon^{-1/3} . \tag{5.23}$$

By (5.6) with $a = -d + d_H^0 \in D$, $t = 1$ and by Lemma 5.1

$$\|(S_m - \mathrm{Id} - \varepsilon\,\mathscr{F})^*(\mathfrak{h})\|^{\Theta_{m+1},\mathrm{Lip}}_{(+\,,\,d_1-d-1,\varepsilon)\,,\,(+,-d+d_H^0,\varepsilon)} \le$$

$$\le C_1^e(m)\,\varepsilon^{2/3} . \tag{5.24}$$

Since

$$\nabla_y(\varepsilon(\xi^1 - \mathscr{F}^\xi)\cdot\Lambda_{m+1}) = \Pi_y \circ (S_m - \mathrm{Id} - \varepsilon\,\mathscr{F})^*(\mathfrak{h})(0,\Lambda_{m+1},0)$$

and $\|(0,\Lambda_{m+1},0)\|_{(+,d_1-d-1,\varepsilon)} \le C\,\varepsilon^{2/3}$, then the estimate (5.22) results from (5.23), (5.24).

j = 3 – 6 . $\Delta_3 H = \dots = \Delta_6 H = 0$.

j = 7 . For arbitrary function H we have an identity:

$$\nabla_y(H(\mathfrak{h} + \varepsilon\mathfrak{h}^1) - H(\mathfrak{h})) = (\nabla_y H(\mathfrak{h})\,|_{\mathfrak{h}=\mathfrak{h}+\varepsilon\mathfrak{h}^1} - \nabla_y H(\mathfrak{h})) +$$

$$+ \Pi_y(\varepsilon\mathfrak{h}^1)^*(\mathfrak{h})\,\nabla_{\mathfrak{h}} H(\mathfrak{h} + \varepsilon\mathfrak{h}^1) .$$

So we have to estimate two terms,

$$\nabla_y H(\mathfrak{h})\,\big|_{\mathfrak{h}=\mathfrak{h}+\varepsilon\mathfrak{h}^1} - \nabla_y H(\mathfrak{h}) \tag{5.25}$$

and

$$\Pi_y(\varepsilon\mathfrak{h}^1)^*(\mathfrak{h})\ \nabla_{\mathfrak{h}} H(\mathfrak{h} + \varepsilon\mathfrak{h}^1)\ , \tag{5.26}$$

for $H = \varepsilon(H_{2m} + \varepsilon H_{3m})$ and for $H = H^3$. Let us denote $\tilde{\mathfrak{h}} = S_m(\mathfrak{h})$ and mention that $\Pi_q \circ (\mathfrak{h}^1{}_*(\mathfrak{h}))\,(0,0,y) \equiv 0$. So

$$\Pi_y \circ (\varepsilon\mathfrak{h}^1)^*(\mathfrak{h})\ \nabla_{\mathfrak{h}} H(\tilde{\mathfrak{h}}) = \Pi_y \circ (\varepsilon\mathfrak{h}^1)^*(\mathfrak{h})\ (0,\nabla_\xi H(\tilde{\mathfrak{h}}),\nabla_y H(\tilde{\mathfrak{h}}))$$

and by (5.23) and (5.5) with $a = -d + d_H^0$

$$\begin{aligned}
&\|\Pi_y \circ (\varepsilon\mathfrak{h}^1)^*(\mathfrak{h})\ \nabla_{\mathfrak{h}} H(\tilde{\mathfrak{h}})\|_{d-d_H^0}^{\Theta_{m+1},\mathrm{Lip}} \le \\
&\le \varepsilon^{-1/3}\|(\varepsilon\mathfrak{h}^1)^*\|^{\Theta_{m+1},\mathrm{Lip}}_{(+\,,-d+d_1-1,\varepsilon)\,,\,(+,-d+d_H^0,\varepsilon)} \times \\
&\times \|(0,\nabla_\xi H(\tilde{\mathfrak{h}}))\,,\,\nabla_y H(\tilde{\mathfrak{h}}))\|^{\Theta_{m+1},\mathrm{Lip}}_{(+\,,-d+d_1-1,\varepsilon)} \le \\
&\le C^e(m)\,(\varepsilon^{2/3}|\nabla_\xi H(\tilde{\mathfrak{h}})|^{\Theta_{m+1},\mathrm{Lip}} + \varepsilon^{1/3}\|\nabla_y H(\tilde{\mathfrak{h}})\|^{\Theta_{m+1},\mathrm{Lip}}_{d-d_1+1})
\end{aligned} \tag{5.27}$$

Let $H = \varepsilon(H_{2m} + H_{3m})$. Then the estimate (5.22) for the term (5.25) results from (2.12), (2.44) and the Cauchy estimate. The estimate for the term (5.26) results from (5.27), (2.11), (2.12) .

Let $H = H^3$. The term (5.25) is equal to

$$\int_0^\varepsilon \frac{d}{d\tau} \nabla_y H^3(\mathfrak{h} + \tau\mathfrak{h}^1) d\tau = \int_0^\varepsilon (\nabla_y H_3)_*(\mathfrak{h} + \tau\mathfrak{h}^1)\mathfrak{h}^1 d\tau$$

and its $\|\cdot\|_{d-d_H^0}^{\Theta_{m+1},\mathrm{Lip}}$ –norm is estimated above by

$$|(\nabla_y H_3)_*(\tilde{\mathfrak{h}})|_{E^c_{d+d_H-d_H^0},\, Y^c_{d-d_H^0}}^{\Theta_{m+1},\mathrm{Lip}} \, \|\varepsilon\mathfrak{h}^1\|_{d+d_H-d_H^0}^{\Theta_{m+1},\mathrm{Lip}} .$$

The first factor is no greater than $C\,\varepsilon^{2/3}$ by (1.24), (1.24′) and Cauchy estimate. The second one is no greater than ε^ρ by (2.44) as $d_H \le d_1 - 1$. So the term (5.25) is estimated.

The estimate for the term (5.26) results from (5.27), (1.24), (1.24′) and Cauchy estimate because $d - d_1 + 1 \le d - d_H$.

<u>j = 8</u> . The estimate contains in Lemma 2.1, item c) . ∎

By the equation (5.20) and Lemma 5.4 hamiltonian $\mathcal{H}_m(S_m(\mathfrak{h};\theta);\theta)$ has a form (2.6) with

$$A_{m+1}(q;\theta) = A_m(q;\theta) + 2\varepsilon\, \Delta h^{yy}(q;\theta)\,. \tag{5.28}$$

Lemma 2.4 is proved.

6. Proof of Statement 1.2.

By the definitions of maps $\sum^0$ and $\sum^{\varepsilon_0}$, for $\mathfrak{h} = (q,0,0) \in T_0^n$
$\sum^0(\mathfrak{h};\theta) = \Pi_{\mathcal{Y}}(q,0,0;\theta) = (q,0,0) \in \mathcal{Y}$ and $\sum^{\varepsilon_0}(q;\theta) = \sum^0_\infty(q,0,0;\theta)$. So we have to prove that

$$|\sum^0_\infty - \Pi_{\mathcal{Y}}|^{T_0^n \times \Theta_{\varepsilon_0}, \mathrm{Lip}}_{E_{d_c}} \le C\,\varepsilon_0 \tag{6.1}$$

By the proof of Theorem 1.1 the map $\sum^0_\infty$ is equal to

$$\sum^0_\infty(\mathfrak{h}\,;\theta) = S_0(\cdot\,;\theta) \circ S_1(\cdot\,;\theta) \circ \ldots \circ S_{m-1}(\cdot\,;\theta) \circ \sum^m_\infty(\mathfrak{h}\,;\theta) \tag{6.2}$$

and

$$|\sum^m_\infty - \Pi_{\mathcal{Y}}|^{O^c_\infty \times \Theta_{\varepsilon_0}, \mathrm{Lip}}_{E_{d_c}} \le 3\,\varepsilon_m{}^\rho \tag{6.3}$$

(Lemma 2.5). The r.h.s. in (6.3) is smaller than ε_0 if $m \ge m(\rho)$. So for to prove (6.1) it is enough to check that

$$|S_j - \Pi_{\mathcal{Y}}|^{O^c_{j+1} \times \Theta_{j+1}, \mathrm{Lip}}_{E_{d_c}} \le C\,\varepsilon_0 \qquad \forall j \le m(\rho) \tag{6.4}$$

In a similar way,

$$\omega'(\omega,I) = \omega + \varepsilon_0 h_0^{0\xi} + \varepsilon_1 h_1^{0\xi} + \dots \tag{6.5}$$

(see (2.56)) and $|\varepsilon_j h_j^{0\xi} + \varepsilon_{j+1} h_{j+1}^{0\xi} + \dots| \le C(j)\,\varepsilon_j^{1/3}$ (see (2.60) with $m = j$, $p = \infty$). So for to get (1.34) we have to prove that

$$|\varepsilon_j h_j^{0\xi}| \le C\,\varepsilon_0 \qquad \forall j \le m(\rho) \tag{6.6}$$

(we increase $m(\rho)$ if there is need in it).

For to prove (6.4), (6.6) we shall improve the constants in the r.h.s. of the estimates of Lemmas 2.1, 2.2. For this end we define independent on ε_0 domains $Q_m{}^c$, $Q_m{}^{jc}$ instead of $O_m{}^c$, $O_m{}^{jc}$: $Q_m{}^{jc} = O(\mathbb{T}^n \times \{0\} \times \{0\}, \delta_m{}^j, \mathcal{Y}_d{}^c)$, $Q_m{}^c = Q_m{}^{0c}$ (see (2.4)).

We shall prove by induction the following statement. Hamiltonian $\mathscr{H}_m$ (see (2.6)) may be written down in the domain $Q_m{}^c$ in the following way:

$$\mathscr{H}_m = H_{0m}(\mathfrak{h};\theta) + \varepsilon_0\, H_{(m)}(\mathfrak{h};\theta) + H^3(\mathfrak{h};\theta)\,. \tag{6.7}$$

Here the function H^3 is the same as in (1.24), $H_{(m)} \in \mathscr{A}_{\Theta_m}^{R}(Q_m{}^c;\mathbb{C})$ and

$$|H_{(m)}|^{Q_m{}^c,\Theta_m} \le C_m\,,\ \|\nabla_y H_{(m)}\|_{d-d_H^0}^{Q_m{}^c,\Theta_m} \le C_m \tag{6.8}$$

By (2.6) and (2.7) we see that $\varepsilon_0 H_{(m)} = \varepsilon_m H_m$ on $O_m{}^c$. So $\varepsilon_0 H_{(m)}$ is an analytical continuation of $\varepsilon_m H_m$ on the domain $Q_m{}^c$.

For $m = 0$ the representation (6.7) coincides with the initial one (see (1.23), (1.24)). Let us suppose that the statement is true for some $0 \le m \le m(\rho) - 1$. We denote the terms $\varepsilon_m H_m$, $\varepsilon_m h^q$, $\varepsilon_m h^{1\xi}$ etc. in the decomposition (2.16) by $\varepsilon_0 H_{(m)}$, $\varepsilon_0 h^q{}_{(m)}$, $\varepsilon_0 h^{1\xi}_{(m)}$ etc. and denote the coefficients $\varepsilon_m f^q$, $\varepsilon_m f^\xi$ etc. of the hamiltonian $\varepsilon_m F$ by $\varepsilon_0 f^q_{(m)}$, $\varepsilon_0 f^\xi_{(m)}$ etc. By repeating the proof of Lemma 2.1 we have for $h^q_{(m)}$, $h^{1\xi}_{(m)}$ etc. the estimates of the items a), b) of Lemma 2.1 with r.h.s. replaced by C_m (we don't controll the rate of increase on m).

In particular,

$$\varepsilon_m |h^{0\xi}_m| = \varepsilon_0 |h^{0\xi}_{(m)}| \le \varepsilon_0 C_m . \tag{6.9}$$

For $H^3_{(m)}$ we have an estimate of the form (6.8).

By repeating the proof of Lemma 2.2 we get for $f^q_{(m)}$, $f^\xi_{(m)}$ etc. the estimates of form (2.35) – (2.40) with the r.h.s. replaced by C^1_m. So after the analytical continuation into domain $Q_m{}^{3c}$ the vector–field of equation (2.28) is no larger than $C^2_m \varepsilon_0$. So S_m may be (analytically) continued to a map from $Q_m{}^{4c}$ into $Q_m{}^{3c}$ and for this continuation the estimates of the item a), Lemma 2.4, are true with r.h.s. replaced by $C^3_m \varepsilon_0$ (and with Q^c_{m+1,d_c} in the notations of the norms). In particular

$$|S_m - \Pi_{\mathscr{Y}}|_{E_{d_c}}^{Q^c_{m+1,d_c} \times \Theta_{m+1},\mathrm{Lip}} \le C_m^3 \, \varepsilon_0 . \tag{6.10}$$

Hence the transformed hamiltonian $\mathscr{H}_m \circ S_m$ may be continued to domain Q^c_{m+1,d_c} and has there a form (6.7) with $m := m + 1$.

Now the estimates (6.4) and (6.6) result from (6.9), (6.10) with

$m = 0 , 1 , \dots , m(\rho)$.

7. Final remarks.

7.1 On the decomposition $\varepsilon_0 H^0 = \varepsilon_0 H_0 + H^3$

The assumption $d_H^0 \le 0$ (see (1.25)) means that the quadratic on y term of the perturbation $\varepsilon_0 H^0$ is determined by a bounded operator. This assumption may be somewhat changed. Indeed, the only part of the proof where we need the assumption is § 4 because for to solve the homological equations (2.32), (2.33), we perform non–autonomous change of varibable in the phase–space Y of a form

$$y(q) = W(q)\,\tilde{y}(q)\ ,\ q \in \mathbf{T}^n \tag{7.1}$$

The operator $W(q)$ is diagonal, $W(q) = \mathrm{diag}\,(\exp i\, W_j(q))$, W_j is an analytical function, $W_j \sim \Delta\,\lambda_j$,

$$\Delta\,\lambda_j(q) = \lambda_j'(q) - \int \lambda_j^1(q)\, dq/(2\pi)^n \tag{7.2}$$

(see (3.13)). By our estimates (see (2.10)) $\Delta\,\lambda_j(q)$ is of order $j^{d_H^0}$ $(j \longrightarrow \infty)$. So if $d_H^0 > 0$ then the change of variable (7.1) is unbounded in any complex neighborhood of $\mathbf{T}^n$, and our proof is spoiled.

Let $\Delta\,\lambda_{j(m)}(q)$ be $\Delta\,\lambda_j(q)$ corresponding to the iteration number m . Then by (5.27) and (3.7)

$$\Delta\,\lambda_{j(m+1)}(q) = \Delta\,\lambda_{j(m)} + \tfrac{1}{2} \sum_{\sigma=\pm} \langle h^{yy}\, \varphi_j^\sigma\, , \varphi_j^\sigma \rangle +$$

$$+ \text{"a term of order } \Delta \lambda_{j(m)} \text{"} .$$

So what we need indeed is not the boundedness of the quadratic term, but the boundedness of the eigenvalues of the operator $J\, h^{yy}$:

$$\frac{1}{2} \sum_{\sigma=\pm} < h^{yy} \varphi_j^\sigma , \varphi_j^\sigma > \leq C \qquad \forall j \in \mathbb{N} . \tag{7.3}$$

So it is enough to assume the assumption (7.3) for the quadratic part of H_0 (see (1.23)) and to check that (7.3) holds for hamiltonian $\mathscr{H}_{m+1}$ provided it holds for $\mathscr{H}_m$, $m = 0 , 1 , 2 , \ldots$. A not complicated analysis of the terms $\Delta_k H$ in (5.20) shows that the last statement is true if $d_H^0 \leq \frac{1}{2}(d_1 - 1)$. So we have a version of the theorem. We formulate it in a case $d_1 > 1$ only:

Statement 7.1. The assertions of Theorem 1.1 are true if $d_H > 0$, $d_1 > 1$ and instead of (1.25) the following two assumptions hold for some $\delta > 0$:

$$d_H \leq d_1 - 1 , \quad 0 \leq d_H^0 \leq \tfrac{1}{2} (d_1 - 1) ,$$

$$\Big| \sum_{\sigma=\pm} < (\nabla_y H_0(\cdot,0,0;\cdot,\varepsilon_0))_* |_Y \varphi_j^\sigma , \varphi_j^\sigma > \Big|^{U(\delta),Lip} \leq K_1$$

$$\forall j = 1 , 2 , 3 , \ldots .$$

7.2 On the reducibility of variational equations.

In the statement of Theorem 1.1 we made no use of the estimates (2.9). (2.10), (2.24) on the quadratic on y part of hamiltonian $\mathscr{H}_m$. These estimates allow us to prove that the variational equations for (1.19) along solutions $z^{\varepsilon_0}(t)$ are reducible to the constant coefficient ones (this reducibility is a typical by–product of KAM—procedure; see [A1], § 5.5.10).

The variational equations for $\delta z_0 = (\delta q_0, \delta\xi_0, \delta y_0) \in E_d$ along the solution $z = z^{\varepsilon_0}(t)$ have a form:

$$\delta\dot{q}_0 = \varepsilon_0(\nabla_\xi H_0(z))_* \delta z_0 \, , \; \delta\dot{\xi}_0 = -\varepsilon_0(\nabla_q H_0(z))_* \delta z_0 \, ,$$
$$\delta\dot{y}_0 = J(A(\omega)\ \delta y_0 + \varepsilon_0(\nabla_y H_0(z))_* \delta z_0) \, . \tag{7.4}$$

Let us denote by $T^n_{\varepsilon_0} = T^n_{\varepsilon_0}(\omega,I) \equiv \Sigma^{\varepsilon_0}_{(\omega,I)}(\mathbf{T}^n)$ the invariant tori constructed in Theorem 1.1.

<u>Theorem 7.2</u>. Let under the assumption of Theorem 1.1 $d_H \le 0$. Then there exists an analytical mapping $\Phi_1 : T^n_{\varepsilon_0} \longrightarrow \mathscr{L}(E_d, E_d)$ such that the substitution $\delta z_0 = \Phi_1(z(t))\ \delta\mathfrak{h}$, $\delta\mathfrak{h} = (\delta q, \delta\xi, \delta y) \in E_d$, transforms solutions of (7.4) into solutions of equations

$$\delta\dot{q} = 0 \, , \; \delta\dot{\xi} = 0 \, , \; \delta\dot{y} = J\ \overline{A}_\infty(\theta)\ \delta y \, . \tag{7.5}$$

Here $\overline{A}_\infty(\theta)\ \varphi^\pm_j = \overline{\lambda}_j(\theta)\ \varphi^\pm_j \quad \forall j$ and $|\overline{\lambda}_j(\theta) - \lambda_j(\omega)| \le \varepsilon_0^\rho\ j^{d^0_H} \quad \forall \rho < \frac{1}{3}$.

The change of variables Φ_1 is constructed in two steps:

1. The substitution

$$z^{\varepsilon_0}(t) = \Sigma^0_\infty(\mathfrak{h}_\infty(t)) \ , \ \mathfrak{h}_\infty(t) = (q + \omega' t, 0, 0)$$

$$\delta z_0 = \Sigma^0_\infty(\mathfrak{h}_\infty(t))_* \delta\mathfrak{h}_\infty$$

transforms solutions of (7.4) into solutions of equations

$$\delta\dot{q}_\infty = 0 \ , \ \delta\dot{\xi}_\infty = 0 \ , \ \delta\dot{y}_\infty = J\, A_\infty(q_\infty(t))\ \delta y_\infty \tag{7.5}$$

2. The equation for δy_∞ in (7.5) may be reduced to the constant–coefficient one via a substitution $\delta y_\infty = W(q_\infty)\ \delta y$, $W = \mathrm{diag}\,(\exp\,(iW_j(q_\infty)))$; see § 3.

We omit the details.

Appendix A. Interpolation theorem.

Let X_1 be a real Hilbert space with a Hilbert basis $\{\eta_j \mid j \in \mathbb{Z}_0\}$ (i.e. $< \eta_j , \eta_k >_{X_1} = \delta_{j,k}$) . Let X_2 be a dense subspace of X_1 with the Hilbert basis $\{\chi_j^{-1} \eta_j\}$, $\chi_j \geq C$ $\forall j$. Then for $0 \leq \tau \leq 1$ the interpolation space $[X_2 , X_1]_\tau$ is a Hilbert space with the Hilbert basis $\{\chi_j^{-1+\tau} \eta_j \mid j \in \mathbb{Z}_0\}$. In particular if $X_1 = Y_a$, $X_2 = Y_b$, $b > a$, and Y_a , Y_b are spaces from the scale $\{Y_s\}$ as in § 1, then for the conditions (1.21)

$$[X_2 , X_1]_\tau = [Y_b , Y_a]_\tau = Y_{\tau a+(1-\tau)b}$$

(one has to take $\eta_j = \varphi_j^+$ for $j > 0$ and $\eta_j = \varphi_{-j}^-$ for $j < 0$). The norms in the spaces are equivalent:

$$K^{-1} \|y\|_{\tau a+(1-\tau)b} \leq |y|_{[Y_b , Y_a]_\tau} \leq K \|y\|_{\tau a+(1-\tau)b}$$

For complexifications X_1^c and X_2^c of the spaces X_1 , X_2 we set by definition

$$[X_2^c , X_1^c]_\tau = [X_2 , X_1]_\tau \otimes_{\mathbb{R}} \mathbb{C}$$

(i.e. an interpolation of complexifications is equal to the complexification of interpolation). So $[Y_b^c , Y_a^c]_\tau = Y^c_{\tau a+(1-\tau)b}$.

<u>Theorem A1</u> (interpolation theorem). Let a linear operator $L : Y^c_\infty \longrightarrow Y^c_{-\infty}$ may be continued to continuous maps $Y_{s_0}^c \longrightarrow Y_{l_0}^c$ and $Y_{s_1}^c \longrightarrow Y_{l_1}^c$. Then

$\forall\tau \in [0,1]$ it may be continued to the continuous map $Y_{s_\tau}^c \longrightarrow Y_{l_\tau}^c$, $s_\tau = \tau s_0 + (1-\tau) s_1$, $l_\tau = \tau l_0 + (1-\tau) l_1$, and

$$\|L\|_{s_\tau, l_\tau} \leq C \max \{\|L\|_{s_0, l_0}, \|L\|_{s_1, l_1}\} .$$

For a general formulation of the theorem and for a proof see [LM, RS2].

Corollary A2. Let a linear continuous operator $Y_s^c \longrightarrow Y_l^c$ be symmetric with respect to the pairing $< \cdot , \cdot >$ (i.e. $L \in \mathscr{L}^s(Y_s^c, Y_l^c)$). Then $\forall\tau \in [0,1]$ $L \in \mathscr{L}^s(Y_{s_\tau}^c, Y_{l_\tau}^c)$, $s_\tau = \tau(s+1) - 1$, $l_\tau = \tau(s+1) - 1$, and $\|L\|_{s_\tau, l_\tau} \leq C \|L\|_{s,l}$.

Proof. We have equalities: $\|L\|_{-l,-s} = \|L^*\|_{-l,-s} = \|L\|_{s,l}$. Here L^* is the operator, conjugate to L with respect to pairing $< \cdot , \cdot >$. Now the assertion results from Theorem A1 with $s_0 = s$, $s_1 = -l$, $l_0 = l$, $l_1 = -s$. ■

Appendix B. Some estimates for Fourier series.

Let B be a Banach space with a norm $\|\cdot\|$, B^c be the complexification of B, $M = \{\mu\}$ be a metric space, $\xi > 0$ and

$$G \in \mathscr{A}_M^R(U(\xi); B^c), \ \|G\|^{U(\xi),M} \leq 1 . \tag{B1}$$

Let us write a Fourier series for G :

$$G(q;\mu) = \sum_{s \in \mathbb{Z}^n} \widehat{G}(s;\mu)\, e^{is\cdot q} . \tag{B2}$$

Lemma B1. For every $s \in \mathbb{Z}^n$

$$\|\widehat{G}(s;\cdot)\|^{M,Lip} \leq e^{-\xi |s|} . \tag{B3}$$

and

$$\widehat{G}(s,\mu) = \overline{\widehat{G}(-s,\mu)} \qquad \forall s , \forall \mu \tag{B4}$$

An "almost inverse" statement is true:

Lemma B2. If (B3), (B4) are true $\forall s \in \mathbb{Z}^n$ and $0 < \Delta < \xi$ then the series (B2) converges $\forall q \in U(\xi - \Delta)$, the map G is analytic and

$$G \in \mathscr{A}^R_M(U(\xi - \Delta) ; B^c) , \|G\|^{U(\xi-\Delta),M} \leq 4^n \Delta^{-n}$$

Lemma B3. If (B1) takes place, $0 < 2\Delta < \xi < 1$ and

$$R_{M_*} G(q) = \sum_{|s| \geq M_*} \widehat{G}(s;\mu)\, e^{is\cdot q} ,$$

then

$$\|R_{M_*}G\|^{U(\xi-2\Delta),M} \leq C(n)\,\Delta^{-n-1}\,e^{-\frac{3}{4}M_*\Delta}\ .$$

The prooves of the lemmas given in [A, § 4.2] for $B = \mathbb{R}^n$, are valid for arbitrary Banach space B.

Appendix C. Lipschitz homeomorphisms of Borel sets.

Let $\Omega \subset \mathbb{R}^n$ be a bounded Borel subset and $\Lambda : \Omega \longrightarrow \mathbb{R}^n$ be a Lipschitz map of a form $\Lambda(a) = a + \Lambda_1(a)$,

$$\mathrm{Lip}\,\Lambda_1 \leq \mu < 1\ . \tag{C1}$$

So

$$\mathrm{Lip}\,\Lambda \leq 1 + \mu\ . \tag{C2}$$

Theorem C1. If (C1) takes place than the inverse map Λ^{-1} is well–defined and

$$\mathrm{Lip}\,\Lambda^{-1} \leq (1-\mu)^{-1}\ . \tag{C3}$$

For arbitrary Borel set $\Omega' \subset \Omega$

$$(1-\mu)^n \,\mathrm{mes}\,\Omega' \leq \mathrm{mes}\,\Lambda(\Omega') \leq (1+\mu)^n \,\mathrm{mes}\,\Omega \tag{C4}$$

Proof. The first statement is evident. Indeed, if $\Lambda(x_j) = y_j$, $j = 1\ , 2$, then

$(x_1 - x_2) + (\Lambda_1 x_1 - \Lambda_1 x_2) = y_1 - y_2$ and by (C1) $|x_1 - x_2|^2 \le \mu |x_1 - x_2|^2 + |x_1 - x_2|\,|y_1 - y_2|$. So $|x_1 - x_2| \le (1-\mu)^{-1}\,|y_1 - y_2|$ and (C3) is proved.

For to prove (C4) let us continue Λ to a Lipschitz map $\Lambda^c : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with the same Lipschitz constant (Kirszbraun's theorem, see [F]). Let mes $\Omega' = a$. Then the upper measure of Ω' is equal to a, too. So $\forall \varepsilon > 0$ the set Ω' may be covered by a countable set of balls $B_j \subset \mathbb{R}^n$, radius of B_j is equal to r_j, and

$$V_1 \sum_{j=1}^{\infty} r_j^n \le (1+\varepsilon)\, a$$

(V_1 is the measure of 1–ball in $\mathbb{R}^n$). As $\operatorname{Lip} \Lambda^c = \operatorname{Lip} \Lambda \le (1+\mu)$, then $\Lambda(B_j)$ is contained in a ball of radius $(1+\mu)\, r_j$. As $\Lambda(\Omega') \subset \cup\, \Lambda(B_j)$, then

$$\text{mes}\, \Lambda(\Omega') \le V_1 \sum (1+\mu)^n r_j^n \le (1+\mu)^n (1+\varepsilon)\, \text{mes}\, \Omega' .$$

The second inequality in (C4) is proved because $\varepsilon > 0$ may be chosen arbitrarily small.

For to prove the first inequality we have to consider the map Λ^{-1} and to use (C3).

List of notations

1. Constants.

$C, C_1, C_2, \ldots$ – positive constants which arrive in estimates. They are independent on ε and m and are different in different parts of the text.

$K, K_1, \ldots$ – constants which characterize initial data in theorems;

m – a number of iteration;

$C(m), C_1(m), \ldots$ – functions of m of the form $C_1 m^{C_2}$;

$C_{*j}, C_{*j}(m)$ – fixed constants and fixed functions of the form $C(m)$;

$C_1^e(m), C_2^e(m), \ldots$ – functions of m of the form $\exp C(m)$;

$$e(m) = \frac{1^{-2} + 2^{-2} + \ldots + m^{-2}}{2(1^{-2} + 2^{-2} + \ldots)}, \qquad e(m) < \tfrac{1}{2} \quad \forall m;$$

$$\varepsilon_m = \varepsilon_0^{(1+\rho)^m}, \; 0 < \rho < 1/3;$$

$$\delta_m = \delta_0(1 - e_m), \; \delta_m > \tfrac{1}{2}\delta_0 \; \forall m;$$

$$\delta_m^j = (1 - \tfrac{j}{6})\,\delta_m + \tfrac{j}{6}\,\delta_{m+1}, \; 0 \le j \le 5$$

2. Linear spaces and maps.

Y , Z – Hilbert spaces with norms $\|\cdot\|_Y$, $\|\cdot\|_Z$ and inner products $<\cdot,\cdot>_Y$, $<\cdot,\cdot>_Z$;

$\{Y_s \mid s \in \mathbb{R}\}$ – a scale of Hilbert spaces Y_s , $|\cdot|_{Y_s} = \|\cdot\|_s$, $Y_0 = Y$, $Y_{s_1} \subset Y_{s_2}$ for $s_1 \geq s_2$, Y_s and Y_{-s} are conjugate with respect to the pairing $<\cdot,\cdot> = <\cdot,\cdot>_Y$;

$\{\lambda_j^{(-s)} \varphi_j^{\pm} \mid j \in \mathbb{N}\}$ – a Hilbert basis of Y_s , $\lambda_j^{(-s)} = (\lambda_j^{(s)})^{-1} >$ $\forall j , \forall s$;

Y^c , $Y_s^{\,c}$ – complexifications of Y , Y_s , the scalar product $<\cdot,\cdot>$ in Y is continued to a complex–bilinear pairing $Y_s^{\,c} \times Y_{-s}^{\,c} \longrightarrow \mathbb{C}$, $s \in \mathbb{R}$;

$\mathscr{L}(Y_s^{\,c}; Y_l^{\,c})$ a space of linear continuous operators from $Y_s^{\,c}$ to $Y_l^{\,c}$ provided with the operator norm $\|\cdot\|_{s,l}$;

$\mathscr{L}^s(Y_s^{\,c}; Y_l^{\,c})$ – operators from $\mathscr{L}(Y_s^{\,c}; Y_l^{\,c})$ symmetric with respect to $<\cdot,\cdot>$

3. Sets and domains

$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}_0^{\,s} = \mathbb{Z}^s \setminus \{0\}$, $\mathbb{Z}_0 = \mathbb{Z} \setminus \{0\}$;

$O(Q,\delta,M)$ – δ–neighborhood of a subset Q of a metric space M ;

$O(\delta, Z) = O(0, \delta, Z)$ for a Banach space Z ;

$\mathfrak{A} \subset \mathbb{R}^n$ – a set of parameters a ;

$\mathfrak{A}(a_0, \delta) = \{a \in \mathfrak{A} \subset \mathbb{R}^n \mid |a - a_0| \leq \delta\}$;

Ω_0 – a set of frequencies vectors $(\omega_1, \dots, \omega_n)$;

$\mathcal{J}$ – a set of actions $(I_1, \dots, I_n)$;

$\Theta_j = \{\theta = (\omega, I)\}$, $j = 0, 1, \dots$ – subsets of $\Omega_0 \times \mathcal{J}$;

$\Theta[I] = \{\omega \in \Omega \mid (\omega, I) \in \Theta\}$ for a $\Theta \subset \Omega \times \mathcal{J}$ and arbitrary $I \in \mathcal{J}$;

$\mathcal{Y}_s = \mathbf{T}^n \times \mathbb{R}^n \times Y_s$, $\mathcal{Y} = \mathcal{Y}_0$, tangent space to $\mathfrak{h} \in \mathcal{Y}_s$ is identified with $E_s = \mathbb{R}^n \times \mathbb{R}^n \times Y_s$;

$\mathcal{Y}_s{}^c = (\mathbb{C}^n / 2\pi \mathbb{Z}^n) \times \mathbb{C}^n \times Y_s{}^c$;

$U(\delta) = \{\xi \in \mathbb{C}^n / 2\pi \mathbb{Z}^n \mid |\operatorname{Im} \xi| < \delta\}$;

$O^c(\xi_0, \xi_1, \xi_2; \mathcal{Y}_s{}^c) = U(\xi_0) \times O(\xi_1, \mathbb{C}^n) \times O(\xi_2, Y_s{}^c)$;

$U_m = U(\delta_m)$, $O_m{}^c = O^c(\delta_m, \varepsilon_m{}^{2/3}, \varepsilon_m{}^{1/3}; \mathcal{Y}_d{}^c)$;

$$O_m^{jc} = O^c(\delta_m^{\ j}, (2^{-j}\varepsilon_m)^{2/3}, (2^{-j}\varepsilon_m)^{1/3}; \mathscr{Y}_d^{\ c}), \ 0 \le j \le 5;$$

$$O_m = O_m^{\ c} \cap \mathscr{Y}_d;$$

4. Maps and functions

For a map $G : Q_1 \longrightarrow Q_2$ (Q_j is a metric space with a distance dist_j, $j = 1,2$)

$$\mathrm{Lip}\, G = \sup_{x_1 \neq x_2} \frac{\mathrm{dist}_2(G(x_1), G(x_2))}{\mathrm{dist}_1(x_1, x_2)};$$

$|G|_{Q_2}^{Q_1,\mathrm{Lip}} = \max\{\sup_{q \in Q_1} |G(q)|_{Q_2}, \mathrm{Lip}\, G\}$ if $G : Q_1 \longrightarrow Q_2$ and Q_2 is a Banach space;

$\mathscr{A}^R(O_1^{\ c}; O_2^{\ c})$ is the set of Frechet complex–analytical mappings from $O_1^{\ c} \subset B_1^{\ c}$ to $O_2^{\ c} \subset B_2^{\ c}$ which map $O_1^{\ c} \cap B_1$ into B_2;

$\mathscr{A}^R_M(O_1^{\ c}; O_2^{\ c})$ is the set of mappings $G : O_1^{\ c} \times M \longrightarrow O_2^{\ c}$ such that $G(\cdot\,; m) \in \mathscr{A}^R(O_1^{\ c}; O_2^{\ c})$ $\forall m \in M$ and

$$|G|_{B_2}^{O_1^{\ c}; M} = \sup_{b \in O_1^{\ c}} |G(b; \cdot)|_{B_2}^{M,\mathrm{Lip}} < \infty;$$

$\langle \bar{J}\, dz, dz \rangle_Z$ is a 2–form in a Hilbert space Z, $\langle \bar{J}\, dz, dz \rangle_Z [z_1, z_2] =$
$= \langle \bar{J}\, dz, dz \rangle_Z$.

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