CLASSIFICATION OF SUPERTRANSITIVE 2-WEBS ON SURFACES

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Abstract. We obtain a complete classification of supertransitive 2-webs on a closed oriented surface of genus $p \geq 1$ in terms of asymptotic directions of leaves on the universal covering surface and represent all topological classes of 2-webs by means of compatible pairs of transversal perfect geodesic laminations.

1. Introduction

The problem of topological (i.e., up to a homeomorphism) classification of foliations on surfaces goes back to the works of Poincaré. As any classification problem, it consists of the following three parts: (1) finding a set of invariants; (2) showing completeness of this set of invariants; (3) choosing canonical representatives of the equivalence classes. Poincaré answered all these questions for transitive foliations without singularities on the torus. Namely, he introduced a topological invariant (the Poincaré rotation number), proved that it (up to an integer-valued unimodular matrix) is a complete invariant of such foliations, and showed that each class of topological equivalence contains a unique foliation whose leaves are geodesics of the flat metric (see [3] and the references therein).

In the higher genus case A. Weil and D. V. Anosov suggested to look at the asymptotic directions of flow lines on the covering hyperbolic plane, see [1]. Following this idea a new topological invariant of flows and foliations on surfaces (the homotopic rotation class) was introduced in the works [3], [4], [10]. This invariant, in particular, leads to necessary and sufficient conditions of topological equivalence of supertransitive foliations on surfaces. The classes of topological equivalence can be described in terms of geodesic lamination, see [2].

The web theory is a classical area of geometry and is mainly devoted to solving local problems. However, 2-webs also naturally appear in the theory of dynamical systems on surfaces as pairs of stable and unstable foliations of Anosov and pseudo-Anosov diffeomorphisms. The topological equivalence of these webs is clearly a necessary condition for topological conjugacy of diffeomorphisms. In the present paper we give a topological classification of supertransitive webs on closed orientable surfaces of genus $p \geq 1$. These results were partially (Theorem 3) announced in the note [6].

2. Definitions and formulation of results

A *foliation* F of a closed oriented surface M of genus $p \geq 1$ with a set of singularities Λ is a partition of M which is locally (on neighbourhoods of all points from $M \setminus \Lambda$) homeomorphic to the standard linear foliation of the plane $\mathbb{R}^2 \cong \mathbb{C}$ consisting of horizontal lines $\Im w = Const$. A singularity $O \in \Lambda$ is called a *topological saddle* with $\nu \geq 1$, $\nu \neq 2$ separatrices if in terms of a local complex coordinate z on a neighbourhood of $O \cong 0$ it is the image of the standard linear foliation under the correspondence

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- $z^\nu = w^2$ if ν is odd,
- $z^{\nu/2} = w$ if ν is even.

The *index* of a saddle with ν separatrices is equal to $1 - \nu/2$.

Definition. A foliation F of a surface M of genus p is called *supertransitive*, if the following conditions are satisfied:

1) the foliation F has a dense leaf;
2) the set Λ of singularities of F is either empty (if $p = 1$) or consists of topological saddles of negative index;
3) there are no separatrices joining two saddles (which may coincide).

The singular set of a supertransitive foliation is finite.

Denote by $\overline{M}$ the universal cover of the surface M, which is either the Euclidean plane $\mathbb{R}^2$ (when $p = 1$), or the hyperbolic plane $\mathbf{H}$ with the absolute $\mathbf{E}$ (when $p > 1$), and by π the natural projection of $\overline{M}$ onto M. The fundamental group $G = \pi_1(M)$ acts on $\overline{M}$ by isometries, and $M = \overline{M}/G$. We may then lift a foliation F from M to the *covering foliation* $\overline{F}$ of $\overline{M}$.

Foliations F and F' are *topologically equivalent* if $F' = \varphi(F)$ for a homeomorphism $\varphi : M \to M$. Any such homeomorphism φ lifts (uniquely, up to translations) to a *covering homeomorphism* $\overline{\varphi} : \overline{M} \to \overline{M}$ which establishes topological equivalence of the covering foliations $\overline{F}$ and $\overline{F}'$. In turn, $\overline{\varphi}$ gives rise to an automorphism $\overline{\varphi}_G(\gamma) = \overline{\varphi}\gamma\overline{\varphi}^{-1}$ of the fundamental group G, and any automorphism of G can be obtained in this way from a certain homeomorphism of M. The automorphism $\overline{\varphi}_G$ is determined by φ up to inner automorphisms induced by translations of $\overline{M}$ (in particular, when $p = 1$ and $G \cong \mathbb{Z}^2$ this automorphism is unique). Further, for $p = 1$ the automorphism $\overline{\varphi}_G$ induces a homeomorphism $\overline{\varphi}_P$ of the projective space $PH_1(M, \mathbb{R})$ over the real homology group $H_1(M, \mathbb{R}) \cong \mathbb{R}^2$, and for $p > 1$ it induces a homeomorphism $\overline{\varphi}_{\mathbf{E}}$ of the absolute $\mathbf{E}$.

Since we are dealing with one-dimensional foliations, any point on a non-compact leaf splits it into two *rays* (half-leaves). Let us take such a ray R in a supertransitive foliation F, lift it to a ray $\overline{R}$ of the covering foliation $\overline{F}$, and assume that the closure of $\overline{R}$ does not contain singularities (the rays R and $\overline{R}$ are then called *non-singular*). In spite of some differences, there is a lot in common in the asymptotic behaviour of $\overline{R}$ in the parabolic ($p = 1$) and the hyperbolic ($p > 1$) cases.

$\boxed{p = 1}$ There exists a limit

$$\lim \frac{y(m)}{x(m)} = \mu \ ,$$

as m goes to infinity along $\overline{R}$ (here $x(m), y(m)$ are the coordinates of m). The number μ is irrational and is called the *Poincaré rotation number.* [More rigorously, one should consider it as the point from the projective space $PH_1(M, \mathbb{R})$ represented by the vector $\left(\begin{smallmatrix} 1 \\ \mu \end{smallmatrix}\right)$.] The rotation number is the same for all rays, and its common value is called the *Poincaré rotation number* of the foliation F. If foliations F and F' are topologically equivalent by means of a homeomorphism φ, then their rotation numbers (as elements of $PH_1(M, \mathbb{R})$) satisfy the relation

$$\mu' = \overline{\varphi}_P \mu \ , \tag{1}$$

[Or, $\mu' = (c+d\mu)/(a+b\mu)$, where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{Z})$ is the matrix representing the automorphism $\overline{\varphi}_G$.] Conversely, if the rotation numbers of two foliations F, F' satisfy the relation $\mu' = \psi_P \mu$ for a certain automorphism ψ of G, then F and F' are topologically equivalent.

In particular, any supertransitive foliation of the torus is topologically equivalent to a *linear foliation*, i.e., to one whose leaves are geodesics of a flat metric.

$\boxed{p>1}$ The ray $\overline{R}$ converges to a boundary point $\sigma(\overline{R}) \in \mathbf{E}$ which is *irrational* in the sense that its stabilizer in G is trivial. The G-orbit

$$\mu(R) = G \cdot \sigma(\overline{R}) \subset \mathbf{E}$$

is called the *homotopic rotation class* of the ray R. If foliations F and F' are topologically equivalent by means of a homeomorphism φ, then in complete analogy with formula (1) the homotopic rotation classes μ, μ' of the rays R and $R' = \varphi(R)$, respectively, are connected as

$$\mu' = \overline{\varphi}_{\mathbf{E}}\mu \tag{2}$$

(note that since $\mu(R)$ is G-invariant, the right-hand side of (2) does not depend on the choice of the covering homeomorphism $\overline{\varphi}$). Conversely, if the homotopic rotation classes μ, μ' of two non-singular rays R, R' in the foliations F, F', respectively, satisfy the relation $\mu' = \psi_{\mathbf{E}}\mu$ for a certain automorphism ψ of the fundamental group G, then F and F' are topologically equivalent.

These facts are crucial for our exposition. For references see [2], [3].

Definition. A *supertransitive 2-web* on a surface is a pair of supertransitive foliations (F_1, F_2) such that they have a common singular set and are topologically transversal at all non-singular points.

Two supertransitive 2-webs (F_1, F_2) and (F_1', F_2') on M are *topologically equivalent* if there is a homeomorphism $\varphi : M \to M$ which maps foliations F_i, $i = 1, 2$ to the corresponding foliations F_i'. In Section 3 we prove

Theorem 1. *Any supertransitive 2-web (F_1, F_2) on the torus $\mathbb{T}^2$ is topologically equivalent to a linear supertransitive 2-web (i.e., one which consists of linear foliations).*

Theorem 1 readily implies

Theorem 2. *Two supertransitive 2-webs (F_1, F_2) and (F_1', F_2') on the torus $\mathbb{T}^2$ are topologically equivalent if and only if there exists an automorphism ψ of the fundamental group $G \cong \mathbb{Z}^2$ such that the rotation numbers μ_i, μ_i' of the foliations F_i, F_i', respectively, satisfy the relation $\mu_i' = \psi\mu_i$.*

There is no immediate analogue of linear webs in the higher genus case. However, in Section 4 we prove the following analogue of Theorem 2:

Theorem 3. *Two supertransitive 2-webs (F_1, F_2) and (F_1', F_2') on a surface M of genus $p > 1$ are topologically equivalent if and only if there exists an automorphism ψ of the fundamental group G and non-singular rays R_i, R_i' of foliations F_i, F_i', respectively, such that their homotopic rotation classes satisfy the relation $\mu_i' = \psi_{\mathbf{E}}\mu_i$, $i = 1, 2$.*

Let us now recall the notion of a one-dimensional *$C^{r,l}$-lamination* $\mathcal{L}$ with support N (notation: *supp* $\mathcal{L} = N$) on a surface M, where $0 \le r, l \le \infty$ are integers, and $N \subset M$ is a closed set. A lamination is a decomposition of N into a union of pairwise disjoint curves L_α (called *leaves*), each of which is either a simple closed curve, or a curve without self-intersections. We require that for each $m \in N$ there are a neighbourhood $m \in U \subset M$ and a C^l-diffeomorphism $\varphi : U \to \mathbb{R}^2$ such that φ maps every connected component of the intersections $U \cap L_\alpha$ onto a horizontal line in $\mathbb{R}^2$, and the restriction of φ onto $U \cap L_\alpha$

is a C^r-diffeomorphism (by a C^0-diffeomorphism we mean a homeomorphism). Pairs (U, φ) are called *laminated charts*. Given any chart $(U(m), \varphi_m)$ and a leaf L_α, connected components of the intersection $U(m) \cap L_\alpha$ are called *local leaves*. A lamination is called *geodesic* if its leaves are geodesics of a certain Riemannian metric on M.

Definition. A geodesic lamination $\mathcal{L}$ of a surface M is called *perfect* if:

1) any leaf $L \subset \mathcal{L}$ is dense in the support $N = supp\ \mathcal{L}$;
2) $M \setminus N$ consists of a finite number of domains homeomorphic to an open disk.

We now return to a supertransitive foliation F on a surface M of genus $p \geq 2$. Any leaf l of the covering foliation $\overline{F}$ which is not a separatrix of some singularity has exactly two different boundary points $\sigma^{\pm}(l) \in E$ which are irrational. Denote by g the geodesic on H joining these two points, see Fig. 1.

For a singularity $\overline{O}$ of the foliation $\overline{F}$ let $l_1, \ldots, l_k$, $k \geq 3$ be its separatrices numbered counterclockwise. Each of these separatrices l_i converges to a boundary point $\sigma(l_i) \in \mathbf{E}$. Denote by g_i the geodesics joining $\sigma(l_i)$ and $\sigma(l_{i+1})$ (addition mod k), see Fig. 1.

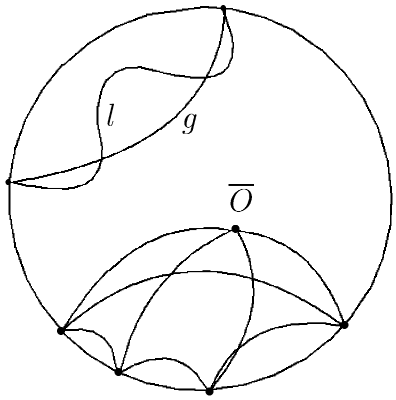

FIGURE 1. Geodesics on H associated with leaves of the foliation $\overline{F}$

Let now $\overline{\Omega}$ be the union of all such geodesics associated with leaves and singularities of the foliation $\overline{F}$, and let $\Omega = \pi(\overline{\Omega})$.

Theorem 4. *The set Ω is closed and carries a perfect geodesic $C^{\infty,0}$-lamination $\mathcal{L}$.*

We shall say that the lamination $\mathcal{L}$ is *associated* with the foliation F. Note that, obviously, the lamination $\mathcal{L}$ does not change if we apply to F a homeomorphism homotopic to identity. In Section 5 we give an intrinsic definition of a *compatible* pair of transversal perfect geodesic laminations and prove

Theorem 5. *For any supertransitive 2-web (F_1, F_2) on a closed oriented surface of genus $p > 1$ the associated perfect geodesic laminations are transversal and form a compatible pair. Conversely, any such pair uniquely (up to a homeomorphism homotopic to identity) determines a supertransitive 2-web on the surface.*

3. Classification of supertransitive 2-webs on torus

Proof of Theorem 1. Step 1. Since the foliations F_i $(i = 1, 2)$ are supertransitive, their Poincaré rotation numbers μ_i are irrational (see Section 2). Let us show that $\mu_1 \neq \mu_2$. Suppose on the contrary that $\mu_1 = \mu_2 = \mu$.

For $i = 1, 2$ denote by l_i the leaf of the foliation $\overline{F}_i$ passing through the origin O of coordinate system on the Euclidean plain $\mathbb{R}^2$. Note that the leaves l_i have no other intersection points (otherwise the domain bounded by the leaf segments between two adjacent intersection points would contain a thorn type singularity in contradiction with the assumption that all singularities have negative index — we shall use this argument on several occasions).

As foliations F_1, F_2 are supertransitive, we may assume without loss of generality that one of them (say, F_1) is linear, so that l_1 is the line $y = \mu x$. By choosing orientations on the leaves l_i, we may split them into positive (l_i^+) and negative (l_i^-) rays with origin O.

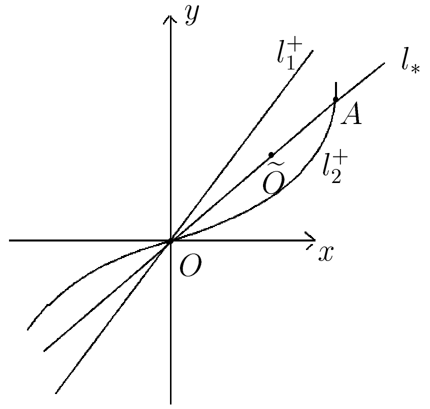

FIGURE 2. The line l_* intersects the curve l_2 twice, which is impossible

Denote by D the open domain which is bounded by the rays l_i^+ and does not contain the ray l_1^-. As the number μ is irrational and foliations F_i are topologically transversal, there is a point $\widetilde{O} \in D$ such that:

a) $\widetilde{O}$ is congruent to O;

b) The line l_* passing through the points O and $\widetilde{O}$ projects onto a simple closed curve $C = \pi(l_*)$ which is not homotopic to zero and is topologically transversal to all leaves of the foliation F_2. Therefore, any leaf of $\overline{F}_2$ has at most one common point with l_* (see the argument at the beginning of the proof).

By the hypothesis the ray l_2^+ has the same asymptotic direction as l_1^+, so that it must intersect l_* at some point $A \neq O$ (see Fig. 2), which is impossible by (b) above. This contradiction proves that $\mu_1 \neq \mu_2$.

Step 2. Since the foliations F_i are supertransitive, they can be both linearized, i.e., there exist linear foliations $\overline{L}_i$ of $\mathbb{R}^2$ (with leaves $y = \mu_i x + c$, where μ_i is the irrational rotation number of F_i) and homeomorphisms $h_i : \mathbb{T}^2 \to \mathbb{T}^2$ such that the associated covering homeomorphisms $\overline{h}_i : \mathbb{R}^2 \to \mathbb{R}^2$ map $\overline{L}_i$ to the covering foliations $\overline{F}_i$ ($i = 1, 2$).

For each $i = 1, 2$ the leaves of the foliation $\overline{L}_i$ are the level sets of the (real additive) cocycle

$$\big((x, y), (x', y')\big) \mapsto (y' - \mu_i x') - (y - \mu_i x)$$

which is the signed distance between the points where the leaves passing through the points (x, y) and (x', y'), respectively, intersect the vertical axis Oy. Denote by χ_i the $\overline{h}_i$-image of this cocycle, so that the leaves of $\overline{F}_i$ are the level sets of χ_i. Clearly, the cocycles χ_i are *G-invariant*, i.e.,

$$\chi_i(gP, gQ) = \chi_i(P, Q) \qquad \forall\, P, Q \in \mathbb{R}^2,\ g \in G\ .$$

Let us put $f_i(P) = \chi_i(O, P)$. Then by G-invariance of χ_i for any $g \in G$

$$f_i(gP) = \chi_i(O, gO) + \chi_i(gO, gP) = f_i(gO) + f_i(P)\ . \tag{3}$$

In particular, $g \mapsto f_i(gO)$ is a group homomorphism. Therefore, the map

$$\theta : g \mapsto \big(f_1(gO), f_2(gO)\big)$$

is also a group homomorphism. We claim that in fact θ is an isomorphism, and its image $\mathcal{G} = \theta(G)$ is a discrete subgroup of $\mathbb{R}^2$. Indeed, if $f_1(gO) = f_2(gO) = 0$, then the points O and gO belong to the same leaf in both foliations $\overline{F}_1$ and $\overline{F}_2$, so that $gO = O$ and $g = \mathrm{id}$. If $\mathcal{G}$ is not a discrete subgroup of $\mathbb{R}^2$, then it is contained in a one-dimensional subgroup of $\mathbb{R}^2$, i.e., there exists a real number $\lambda \neq 0$ such that

$$f_2(gO) = \lambda f_1(gO) \qquad \forall\, g \in G\ . \tag{4}$$

Take a sequence of points P_n going to infinity along a leaf of $\overline{F}_1$, then $\chi_1(P_1, P_n) \equiv 0$, whereas $|\chi_2(P_1, P_n)| \to \infty$, as it follows from the fact that the rotation numbers of F_1 and F_2 are different (look at the linearization of $\overline{F}_1$). Now take a sequence $g_n \in G$ such that the distances from P_n to g_nO are uniformly bounded. Then continuity and G-invariance of the cocycles χ_i implies that $f_1(g_nO)$ remains uniformly bounded, whereas $|f_2(g_nO)| \to \infty$, which contradicts (4). Now for $P \in \mathbb{R}^2$ put

$$\overline{\varphi}(P) = \big(f_1(P), f_2(P)\big)\ .$$

Then the map $\overline{\varphi}$ is continuous. Since the leaves of the foliations F_i are precisely the level sets of the functions f_i, and any two leaves of F_1 and F_2 have a single common point, the map $\overline{\varphi}$ is easily seen to be a homeomorphism of $\mathbb{R}^2$. Further, formula (3) implies that

$$\overline{\varphi}(gP) = \overline{\varphi}(P) + \big(f_1(gO), f_2(gO)\big) = \overline{\varphi}(P) + \theta(g)\ ,$$

so that $\overline{\varphi}$ is a covering homeomorphism for a homeomorphism

$$\varphi : \mathbb{T}^2 \cong \mathbb{R}^2/G \to \mathbb{T}^2 \cong \mathbb{R}^2/\mathcal{G}\ .$$

By the construction the $\overline{\varphi}$-images of the foliations $\overline{F}_i$ are linear. □

Two linear webs on $\mathbb{T}^2$ are topologically equivalent if and only if they are connected with an algebraic toral automorphism, which at once implies Theorem 2.

4. Classification of supertransitive 2-webs on hyperbolic surfaces

Proof of Theorem 3. The "only if" part being a direct corollary of the description of the behaviour of homotopic rotation classes under homeomorphisms (Section 2), we only have to prove the "if" part.

Given a point $\overline{x} \in \mathbf{H}$ which does not belong to the common singular set of the foliations $\overline{F}_i$, denote by l_i the leaf of the foliation $\overline{F}_i$ passing through $\overline{x}$, and choose on each of these leafs a half-leaf l_i^* with origin $\overline{x}$. If $\overline{x}$ is a singularity, denote by l_i^* a separatrix of $\overline{x}$ which is a leaf of $\overline{F}_i$. Denote by $\sigma(l_i^*) \in \mathbf{E}$ the corresponding boundary points. The following claim (analogous to Step 1 in the proof of Theorem 1) is very important for what follows:

$$\sigma(l_1^*) \neq \sigma(l_2^*). \tag{5}$$

Indeed, suppose that $\sigma(l_1^*) = \sigma(l_2^*) = \sigma^*$. As the foliations F_i, are transversal and all their singularities have negative index, $\overline{x}$ is the only intersection point of the rays l_i^*. Denote by $D \subset \mathbf{H}$ the open domain bounded by the rays l_i^* (which have a common starting point $\overline{x}$ and a common endpoint at infinity σ^*). Choose a point $\overline{y} \in l_2^*$ such that the leaf $l_1(\overline{y})$ of the foliation $\overline{F}_1$ passing through the point $\overline{y}$ is not a separatrix of any singularity of the foliation $\overline{F}_1$. As the leaf $l_1(\overline{y})$ has only one common point with the curve l_2^*, one of its rays is contained in the domain D, and consequently one of its boundary points coincides with σ^*. The latter is impossible as by [3], [10] two half-leafs of supertransitive foliations with no common points have no common boundary points on the absolute.

Since conditions of Theorem 3 imply topological equivalence of the foliations F_1 and F_1', we may without loss of generality assume that $F_1 = F_1'$. Moreover, by [2], [3], [10] coincidence of the homotopic rotation classes of the foliations F_2 and F_2' implies the following property of covering foliations $\overline{F}_2$ and $\overline{F}_2'$:

1) If l_2 is a leaf of $\overline{F}_2$ which is not a separatrix, then there exists a unique leaf l_2' of $\overline{F}_2'$ which is not a separatrix either and such that the sets of boundary points $\{\sigma^{\pm}(l_2)\}$ and $\{\sigma^{\pm}(l_2')\}$ coincide;
2) If l_2 is a separatrix of $\overline{F}_2$, then there exists a unique separatrix l_2' of $\overline{F}_2'$ such that the boundary points $\{\sigma(l_2)\}$ and $\{\sigma(l_2')\}$ coincide.

Now take a point $\overline{x} \in \mathbf{H}$ and consider the following two cases:

Case I. The point $\overline{x}$ does not belong to the common singular set of the covering foliations $\overline{F}_i, \overline{F}_i'$. Then $\overline{x}$ is the point of transversal intersection of a leaf l_1 of the foliation $\overline{F}_1$ with a leaf l_2 of the foliation $\overline{F}_2$. By using properties (1) and (2) above replace the leaf l_2 with the leaf l_2' of the foliation $\overline{F}_2'$ with the same set of boundary points as the leaf l_2, denote by $\overline{x}'$ the intersection point of l_1 and l_2' (which is easily seen to exist due to formula (5)), and put $\overline{\varphi}(\overline{x}) = \overline{x}'$.

Case II. The point $\overline{x}$ is a singularity of one of the foliations $\overline{F}_i, \overline{F}_i'$. Then put $\overline{\varphi}(\overline{x}) = \overline{x}$.

Now it is easy to verify (in the same way as in [3]) that $\overline{\varphi} : \mathbf{H} \to \mathbf{H}$ is a covering homeomorphism of some homeomorphism $\varphi : M \to M$, that the latter is homotopic to the identity and that it establishes topological equivalence of the webs (F_1, F_2) and (F_1', F_2'). $\square$

5. Foliations, webs and geodesic laminations

Proof of Theorem 4. We shall split our proof (inspired by the proof of Theorem 1.1 in [4]) into a number of claims.

Claim 1. For any geodesic associated with a leaf or with a singularity of $\overline{F}$ (see Fig. 1) its projection to M is not closed and has no self-intersections.

Let g be the geodesic associated with a leaf l of $\overline{F}$. Since the boundary points of g are irrational, the projection $\pi(g)$ is not closed. In order to prove that $\pi(g)$ has no self-intersections, let us assume the contrary. Then there is an element $\gamma \in G$ such that $\gamma(g) \cap g \neq \varnothing$. Since the leaves $\gamma(l)$ and l have the same boundary points as the geodesics $\gamma(g)$ and g, respectively, the former also intersect, which is impossible. The same argument applies to the geodesics associated with singularities of $\overline{F}$ as well.

Claim 2. For any two geodesics g, g' as in Claim 1 above either $\pi(g) \cap \pi(g') = \varnothing$ or $\pi(g) = \pi(g')$.

If not, then there exist elements $\gamma, \gamma' \in G$ such that the intersection of the geodesics $\gamma(g)$ and $\gamma'(g')$ consists of a single point. Hence, the pair of boundary points of the geodesic $\gamma(g)$ is divided by the pair of boundary points of the geodesic $\gamma'(g')$. For notational simplicity me may assume $\gamma = \gamma' = \mathrm{id}$. Returning to the definition of geodesics associated with a leaf or with a singularity of $\overline{F}$ (see Fig. 1), we may now take the curve l (resp., l'), which is either a leaf of $\overline{F}$ or a union of two separatrices issued from a common singularity and has the same boundary points as the geodesic g (resp., g'). Then l and l' have a non-empty intersection without being equal, which is impossible.

Claim 3. The set Ω is closed.

Let $x_n \in \Omega$ be a sequence converging to a point $a \in M$. Then we may lift the points x_n and a, respectively, to points $\overline{x}_n, \overline{a} \in \mathbf{H}$ in such a way that $\overline{x}_n \to \overline{a}$. Denote by $g_n \subset \overline{\Omega}$ the unique geodesic going through the point $\overline{a}_n$ (see Claim 2). By passing to a subsequence, we may assume without loss of generality that the geodesics g_n converge pointwise to a limit geodesic g, which goes through the point $\overline{a}$ and that the endpoints $\sigma^{\pm}(g) \in \mathbf{E}$ of g are limits of the corresponding endpoints $\sigma^{\pm}(g_n) \in \mathbf{E}$ of the geodesics g_n. Further, in view of Claim 2 we may also assume (passing again to a subsequence) that all projected geodesics $\pi(g_n)$ are pairwise disjoint.

Denote by l_n the curve obtained by joining the endpoints $\sigma^{\pm}(g_n)$ either with a leaf of $\overline{F}$ or with a union of two separatrices issued from a common singularity (cf. the proof of Claim 2). Since the lower topological limit λ_* of the sequence of connected sets $\{l_n\}$ is not empty (it contains the points $\sigma^{\pm}(g)$), its upper topological limit λ^* is necessarily connected in the closed ball $\mathbf{H} \cup \mathbf{E}$ (see [9, Sections 29 and 47] for definitions and properties of lower and upper topological limits). If $\lambda^* \subset \mathbf{E}$, then the interior of λ^* is non-empty (as λ^* is connected and contains two distinct points $\sigma^{\pm}(g)$). By [12] there is an element $\gamma \in G$ such that the translations $\gamma\sigma^{\pm}(g_1)$ both belong to this interior, so that a certain curve l_n intersects the curve $\gamma(l_1)$, which is impossible.

Therefore, we only have to consider the situation when there exists a point $\overline{b} \in \lambda^* \cap \mathbf{H}$. Then there is a curve l which contains the point $\overline{b}$ and is either a leaf of $\overline{F}$ (see Fig. 3a) or a union of two separatrices issued from the same saddle point (see Fig. 3b).

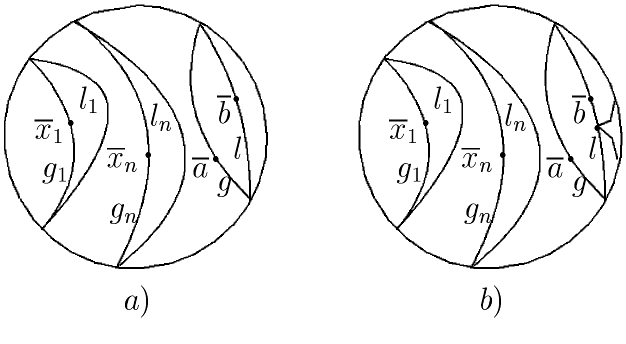

FIGURE 3. Topological limit of the sequence $\{l_n\}$

Let us show that the sets of endpoints $\{g^-, g^+\}$ and $\{l^-, l^+\}$ coincide. Indeed, suppose that $l^+ \neq g^{\pm}$, choose a neighbourhood U of l^+ such that $g^{\pm} \notin U$, and take a ray $R \subset l$ converging to l^+.

Since the foliation F is transitive and all its singularities have negative index it is not difficult to construct a non-trivial (not homotopic to zero) simple closed curve c which is topologically transversal to all leaves of F (for example, see [7, Theorem 4]). Let us partition of the lift of c to the hyperbolic plane $\mathbf{H}$ into a countable set $\mathcal{C}$ of simple curves such that any curve $c \in \mathcal{C}$ has exactly two boundary points which lay on the absolute $\mathbf{E}$ and the closures in $\mathbf{H} \cup E$ of curves from $\mathcal{C}$ are pairwise disjoint.

As all singularities of F have negative index, the intersection of any curve l_n with any curve from $\mathcal{C}$ is either empty or consists of a single point. The projection $\pi(R)$ intersects the transversal c in a countable number of points, which means that the ray R going to infinity intersects a countable number of curves $c_i \in \mathcal{C}$. By [3], these curves converge to the point l^+, so that $c_n \subset U$ for all sufficiently big n. As for any $n \in \mathbb{Z}^+$ the curve c_n divides $\mathbf{H}$ into parts with no common points, continuous dependence of leaves of the foliation $\overline{F}$ on initial conditions implies that for all sufficiently big n one of the boundary points $g_n^\pm$ of the curves l_n also lies in U, which is impossible. Therefore, we have proved that the sets $\{g^-, g^+\}$ and $\{l^-, l^+\}$ coincide. Thus, the geodesic g belongs to the set $\overline{\Omega}$, which means that $\overline{a} \in \overline{\Omega}$ and $a = \pi(\overline{a}) \in \Omega$. Therefore, the set Ω is closed.

Claim 4. Any geodesic in Ω is dense.

Lift a fixed geodesic g from Ω to a geodesic $\overline{g}$ in $\mathbf{H}$, and let l be the associated curve with the same endpoints as $\overline{g}$. Take a point $x \in \Omega$, and lift it to a point $\overline{x} \in \overline{\Omega}$. Denote by $\overline{g}_0 \subset \overline{\Omega}$ the geodesic passing through $\overline{x}$, and by l_0 the associated curve with the same endpoints.

By [7] the projection $\pi(l)$ is dense in M, so that there is a sequence $l_n = \gamma_n l$ of translations of l whose topological limit contains the curve l_0. Then the topological limit of the sequence g_n of the associated geodesics with the same endpoints as l_n is the geodesic $\overline{g}_0$. Consequently, any point of $\overline{g}_0$ (in particular, $\overline{x}$) can be approximated by a sequence of points from the translations $\gamma_n \overline{g}$.

Claim 5. The complement $M \setminus \Omega$ consists of a finite number of domains homeomorphic to an open disk.

For any singularity O_i of the foliation F take a lift $\overline{O}_i \in \mathbf{H}$, and denote by $D_i \subset \mathbf{H}$ the open domain bounded by all geodesics associated with $\overline{O}_i$. Then any such set D_i is wandering with respect to the group G. Indeed, if a translation γD_i, $\gamma \in G \setminus \{\mathrm{id}\}$ intersects D_i, then at least one of the boundary geodesics of γD_i intersects at least one of the boundary geodesics of D_i without coinciding with it (because the endpoints of all boundary geodesics are irrational), which is impossible. Thus, the projection $\pi(D_i)$ is homeomorphic to an open disk, and by the construction $M \setminus \Omega$ is the union of domains $\pi(D_i)$ corresponding to all singularities of F. The theorem is proven. □

In the same way as in [5], the description of the complement $M \setminus \Omega$ from Claim 5 implies

Corollary. *Let μ be the Riemannian area of the hyperbolic metric on M. Then $\mu(\Omega) = 0$.*

Proof. By the Gauss–Bonnet formula

$$\mu(\pi(D_i)) = \mu(D_i) = \pi(k_i - 2) = -2\pi J_i \ ,$$

where k_i is the number of separatrices of the singularity O_i, and $J_i = 1 - k_i/2$ is its index. Therefore, by the Euler–Poincaré formula

$$\sum_i \mu(\pi(D_i)) = -2\pi \sum_i J_i = -2\pi(2 - 2p) \ ,$$

where p is the genus of M. On the other hand, again by the Gauss–Bonnet formula, $\mu(M) = 4\pi(p - 1)$. □

Consider now a supertransitive 2-web (F_1, F_2). The pair of geodesic laminations $\mathcal{L}_i$ associated with the foliations F_i then has the following properties:

1) The sets $M \setminus \Omega_i$, $i = 1, 2$ have the same number of connected components equal to the number of (common) singularities of the foliations F_i.

2) For each connected component $D_1 \subset M \setminus \Omega_1$ there is exactly one connected component $D_2 \subset M \setminus \Omega_2$ such that one can lift D_1 and D_2 to geodesic polygons d_1, d_2 with alternating vertices on the absolute (see Fig. 4).

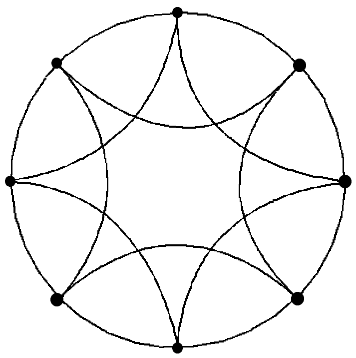

FIGURE 4. Polygons d_1, d_2

Definition. Two transversal perfect geodesic laminations $(\mathcal{L}_1, \mathcal{L}_2)$ with supports Ω_1, Ω_2, respectively, are *compatible* if conditions 1) and 2) above are satisfied.

Proof of Theorem 5. The first part of the Theorem is an immediate corollary of Theorems 3 and 4.

To prove the second part (conversely), let us introduce (following [8], [11] where it was used for constructing a pseudo-Anosov diffeomorphism determined by a given supertransitive 2-web) the equivalence relation $\sim$ on the surface M such that $x \sim y$ if either x and y belong to the closure of the same connected component of one of the sets $M \setminus (\Omega_1 \cup \Omega_2)$, $\Omega_1 \setminus \Omega_2$, $\Omega_2 \setminus \Omega_1$ or $x = y$.

Denote by $\widehat{M} = M/\sim$ the associated quotient space, by $q : \widehat{M} \to M$ the corresponding projection, and set $\widehat{F_i} = q(\Omega_i)$. Then $\widehat{M}$ is a closed surface homeomorphic to the M, $\widehat{F_i}$ are supertransitive foliations, and $(\widehat{F_1}, \widehat{F_2})$ is a supertransitive $2 - web$ on $\widehat{M}$ (see [8], [11] for details). $\square$

REFERENCES

1. D. V. Anosov, *Flows on closed surfaces and behavior of trajectories lifted to the universal covering plane*, J. Dynam. Control Systems **1** (1995), 125–138.
2. S. Kh. Aranson, *Qualitative properties of foliations on closed surfaces*, J. Dynam. Contral Systems **6** (2000), 127–157.
3. S. Kh. Aranson, V. Z. Grines, *Certain invariants of dynamical systems on two-dimensional manifolds (necessary and sufficient conditions for the topological equivalence of transitive systems),* Mat. Sb. (N.S.) **90** (1973), 372–402 (Russian); *English transl.:* Math USSR – Sb. **19** (1973), 365–393.
4. S. Kh. Aranson, V. Z. Grines, *The representation of minimal sets of flows on two-dimensional manifolds by geodesic lines,* Izv. Akad. Nauk SSSR, Ser. Mat. **42** (1978), 104–129 (Russian); *English transl.:* Math. USSR Izvestia, **12** (1978), 103–124.
5. S. Kh. Aranson, V. Z. Grines, *Dynamical systems with minimal entropy on two-dimesinal manifolds.* Gorkii Gos. Univ., 1980 (Russian); *English transl.:* Selecta. Math. Sov. **2** (1982), 123–158.

6. S. Kh. Aranson, V. Z. Grines, *On topological classification of supertransitive 2-webs on closed surfaces of Euler negative characteristic*, Webs and Quasigroups, Tver State Univ., Tver, 2000, 85–91.
7. S. Kh. Aranson, E. V. Zhuzhoma, *On the structure of quasiminimal sets of foliations on surfaces*, Mat. Sb. **185** (1994), no 8, 31-62 (Russian); *English transl.:* Russian Acad. Sci. Sb. Mat. **82** (1995), 397–424.
8. A. J. Casson, S. A. Bleiler, *Automorphisms of Surfaces after Nielsen and Thurston,* Cambrige University Press, Cambridge, 1988.
9. K. Kuratowski, *Topology,* vols. 1 and 2, Academic Press, New York – London, 1966, 1968.
10. G. Levitt, *Foliations and laminations on hyperbolic surfaces,* Topology **22** (1983), 119–135.
11. R. T. Miller, *Geodesic laminations from Nielsen's viewpoint,* Adv. Math. **45** (1982), 189–212.
12. J. Nielsen, *Untersuhungen zur Topologie der geschlossenen zweiseitigen Flächen,* Acta Math. **50** (1927), 189–358.

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