A REDUCTION METHOD FOR ATYPICAL REPRESENTATIONS OF CLASSICAL LIE SUPERALGEBRAS

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Abstract. We introduce a reduction method for studying representations of classical Lie superalgebras with atypical central character. We show that the atypical quotient of universal enveloping algebra has a non-trivial Jacobson radical. The factor by this radical has a new center, which is calculated for $sl\,(1|n)$ and psl $(2|2)$. Using this center we obtain new character formulae, generalization of Borel-Weil-Bott and Beilinson-Bernstein localization theorems.

Contents

1. Introduction

In this paper we suggest a way to treat representations of classical Lie superalgebras with an atypical central character. Let $\mathfrak{g}$ be a classical Lie superalgebra (see [9]), U be its universal enveloping algebra and Z be the center of U. By Schur's lemma an irreducible representation of $\mathfrak{g}$ admits a central character $\chi \in \operatorname{Spec} Z$, therefore the structure of the ring Z should play an important role in representation theory of $\mathfrak{g}$. In contrast with classical Lie algebras, where Z is isomorphic to a polynomial ring, for Lie superalgebras Z is not Noetherian. However, for generic points $\chi \in \operatorname{Spec} Z$, called *typical*, the situation is completely under control. V. Kac was first to show that all theorems about finite-dimensional Lie algebra representations can be

Date: October 30, 2001.

easily generalized to typical representations of classical Lie superalgebras. I. Penkov generalized this in [15] by proving the following theorem. If $\chi \in \operatorname{Spec} Z$ is a generic typical central character then the category of all $\mathfrak{g}$-modules which admit the central character χ is equivalent to the category of all modules over its even part $\mathfrak{g}_0$ which admit a certain central character χ_0. He proves this theorem by extending $\mathcal{D}$-module localization theorem of Beilinson and Bernstein to representations of classical Lie superalgebras with generic typical central characters. Recently, M. Gorelik reproved this theorem by purely algebraic methods in [7]. She managed to get rid off the assumption that χ is generic.

I. Penkov also described a certain class of atypical irreducible representations which he called generic. He was able to describe these representations in terms of its $\mathfrak{g}_0$-components. This description generalizes Bernstein-Leites formula introduced in [3].

People still know very little about atypical general representations. To mention few things, finite-dimensional modules are not completely reducible, and the category of finite-dimensional modules is described only for $sl\,(1|n)$ in [4]. Character formulae for finite-dimensional modules are very complicated. Primitive ideals are classified (see[12]), but the relations between them are unknown. Geometric facts such as Borel-Weil-Bott theorem or Beilinson-Bernstein theorem do hold in atypical case.

Here is what we suggest to do. Let $\chi \in \operatorname{Spec} Z$ be atypical, and U^χ denote the quotient $U/(\operatorname{Ker}\chi)$. In contrast with a typical χ the algebra U^χ has a non-trivial Jacobson radical J^χ. Let us study the representations of $\bar{U}^\chi = U^\chi/J^\chi$. This algebra $\bar{U}^\chi$ has a new center $Z\left(\bar{U}^\chi\right)$ which we are still unable to describe in general. However, this center is sufficiently large and we conjecture that $\operatorname{Spec} Z\left(\bar{U}^\chi\right)$ separates almost all finite-dimensional representation. Probably it is possible to prove Penkov's theorem for generic $\varphi \in \operatorname{Spec} Z\left(\bar{U}^\chi\right)$ and the analog of Beilinson-Bernstein theorem.

In this paper we fulfill this program for $\mathfrak{g} = sl\,(1|n)$ and $psl\,(2|2)$. We obtain several new results for $sl\,(1|n)$ including analogs of Penkov and Beilinson-Bernstein theorem and characters formulae for infinite-dimensional highest weight representations. We also do some initial steps for $sl\,(m|n)$ for $m > 1$. We did not discuss the case of $\operatorname{osp}(m|2n)$ here, but we believe that all initial steps can be done for this superalgebra as well. The method can be used successfully for representation theory of Lie superalgebra $P\,(n)$, but this is a subject of another paper.

It would be interesting to give an interpretation of our approach in terms of the orbit method. Indeed, the situation we deal resembles pretty much the solvable case. Typical central characters correspond to generic values of invariants in $\mathfrak{g}^*$, i.e. to generic orbits. Atypical central characters should probably correspond to singular levels of invariants. In the solvable case these submanifolds may contain infinite families of orbits, which correspond to the new center phenomenon in super case.

The author would like to thank I. Penkov, I. Musson and especially M. Gorelik for stimulating discussions and interest.

2. Preliminaries

Let $\mathfrak{g}$ denote one of the following superalgebras $gl(m|n)$, $sl(m|n)$ or $psl(n|n)$, U be the universal enveloping algebra of $\mathfrak{g}$, Z be the center of U. Recall that $\mathfrak{g}$ is Z graded algebra: $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, and naturally U is also $\mathbb{Z}$-graded: $U = U_{-mn} \oplus \cdots \oplus U_{mn}$.

From now on we fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_0$ and denote the set of roots with respect to $\mathfrak{h}$ by Δ. Recall that $\mathfrak{g}$ admits a non-degenerate invariant symmetric form $B(x, y) = \text{str}\, xy$. It induces a non-degenerate form on $\mathfrak{h}^*$ which we denote by $(,)$. If $\mathfrak{b}_0$ is a Borel subalgebra of $\mathfrak{g}_0$, $\mathfrak{h} \subset \mathfrak{b}_0$, then there are finitely many Borel subalgebras $\mathfrak{b} \subset \mathfrak{g}$ with $\mathfrak{b}_0 \subset \mathfrak{b}$. All such Borel subalgebras can be obtained from one of them by odd reflections (see [16]). The Borel subalgebra $\mathfrak{b}_d = \mathfrak{b}_0 + \mathfrak{g}_1$ is called distinguished. By $\mathfrak{b}_d^{\text{op}}$ we denote the Borel subalgebra $\mathfrak{b}_0 + \mathfrak{g}_{-1}$. For any subalgebra $\mathfrak{l} \subset \mathfrak{g}$ invariant under the adjoint action of $\mathfrak{h}$, we denote by $\Delta(\mathfrak{l})$ the set of roots of $\mathfrak{l}$.

Put $\rho_0 = \frac{1}{2}\sum_{\alpha\in\Delta(\mathfrak{b}_0)} \alpha$, $\rho_1 = \frac{1}{2}\sum_{\alpha\in\Delta(\mathfrak{b}_1)} \alpha$, $\rho_{\mathfrak{b}} = \rho_0 - \rho_1$ for a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$. A choice of $\mathfrak{b}$ determines the triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ and the decomposition $U = U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}^+)$. The Harish-Chandra projection $h_{\mathfrak{b}}$ is defined as the natural map $U \to U/(\mathfrak{n}^- U + U\mathfrak{n}^+) = U(\mathfrak{h}) = S(\mathfrak{h})$. The restriction of $h_{\mathfrak{b}}$ to Z is an injective homomorphism $Z \to S(\mathfrak{h})$. The image $h_{\mathfrak{b}}(Z)$ is described in [10] and [17] as follows. Consider $S(\mathfrak{h})$ as the polynomial ring on $\mathfrak{h}^*$. If W is the Weyl group of $\mathfrak{g}_0$, define an action of W on $\mathfrak{h}^*$ by $\mu^w = w(\mu + \rho_{\mathfrak{b}}) - \rho_{\mathfrak{b}}$, and let $S^W(\mathfrak{h})$ denote the subring of W-invariant polynomial in $S(\mathfrak{h})$. Then

(2.1)
$$h_{\mathfrak{b}}(Z) = \left\{p \in S^W(\mathfrak{h}) \mid p(\mu + t\alpha) = p(\mu) \text{ for any } \mu \in \mathfrak{h}^*, \alpha \in \Delta_1, (\alpha, \mu + \rho) = 0\right\}$$

Let $\chi : Z \to \mathbb{C}$ be an arbitrary character. There exists $\mu \in \mathfrak{h}^*$ such that $\chi(z) = \chi_\mu(z) := (h_{\mathfrak{b}}(z))(\mu)$. Let $m_\chi = \{\mu \in \mathfrak{h}^* \mid \chi = \chi_\mu\}$. The following proposition is an easy consequence of the description of $h_{\mathfrak{b}}(Z)$.

Lemma 2.1. *There exist a set of odd roots $\alpha_1, \dots, \alpha_k \in \Delta(\mathfrak{g}_1)$ and a weight $\mu \in \mathfrak{h}^*$ such that $(\alpha_i, \alpha_j) = 0$ and $m_\chi = \cup_{w\in W} l_\chi^w$, where $l_\chi = \mu + \mathbb{C}\alpha_1 + \cdots + \mathbb{C}\alpha_k$.*

If $k = 0$ the central character χ is called *typical*. If $k > 0$ the central character χ is called *atypical*, and the number k is called *the degree of atypicality* of χ.

Let $U^\chi = U/(\text{Ker}\,\chi)$. For a typical χ the structure of U^χ is well understood. More precisely, the following theorem is proved in [15] (see also [7]). Let Z^0 be the center of $U(\mathfrak{g}_0)$, for a central character $\varphi : Z^0 \to \mathbb{C}$ define $U^\varphi(\mathfrak{g}_0) = U(\mathfrak{g}_0)/(\text{Ker}\,\varphi)$. If $\chi\colon Z \to \mathbb{C}$ is typical, one can find $\varphi : Z^0 \to \mathbb{C}$ such that the algebras U^χ and $U^\varphi(\mathfrak{g}_0)$ are Morita equivalent. Moreover, φ can be chosen such that U^χ is a matrix algebra over $U^\varphi(\mathfrak{g}_0)$. Thus, the study of representation of $\mathfrak{g}$ admitting a typical central character can be reduced to the similar problem for $\mathfrak{g}_0$.

The goal of this paper is to generalize the above result for atypical central characters.

3. The Jacobson radical of U^{χ}

In this section we show that for atypical central character χ the algebra U^{χ} has a non-trivial Jacobson radical and give some description of this radical.

We will use the notations of Lemma 2.1. Since the roots $\alpha_1, \dots, \alpha_k$ are mutually orthogonal one can find a Borel subalgebra $\mathfrak{b}$ so that $\alpha_1, \dots, \alpha_k$ are simple positive roots. Then

$$\mathfrak{p} = \mathfrak{g}_{-\alpha_1} \oplus \dots \oplus \mathfrak{g}_{-\alpha_k} \oplus \mathfrak{b}$$

is a parabolic subalgebra of $\mathfrak{g}$. We put $\rho = \rho_{\mathfrak{b}}$. Since $(\rho, \alpha_i) = 0$, $(\mu, \alpha_i) = 0$ for any $\mu \in l_{\chi}$, and therefore μ defines one dimensional $\mathfrak{p}$-module C_{μ}. Put

$$F(\mu) = U \otimes_{U(\mathfrak{p})} C_{\mu}.$$

Here are some properties of a $\mathfrak{g}$-module $F(\mu)$.

Lemma 3.1. *(a) The center Z acts by scalar operators on $F(\mu)$, and $z_{|F(\mu)} = \chi_{\mu}(z)\,\mathrm{Id}$.*

(b) $F(\mu)$ is free $U(\mathfrak{n}^-)$-module.

(c) $F(\mu)$ admits a $\mathfrak{g}_0$-module filtration with subquotients isomorphic to $\mathfrak{g}_0$-Verma modules $M^0(\nu)$, where $\nu = \mu - \gamma$ and γ runs the set of sums of non-repeating roots from $\Delta(\mathfrak{n}_1^+) \setminus \{\alpha_1, \dots, \alpha_k\}$.

(d) Let $l'_{\chi} = \{\mu \in l_{\chi} \mid (\mu, \alpha) \notin \mathbb{Z}, \alpha \in \Delta(\mathfrak{g}_0)\}$. If $\mu \in l'_{\chi}$, then $F(\mu)$ splits into direct sum of $\mathfrak{g}_0$-Verma modules $M^0(\nu)$, and $F(\mu)$ is an irreducible $\mathfrak{g}$-module.

(e) $\cap_{\mu \in l} \mathrm{Ann}\, F(\mu) = \cap_{\mu \in l'} \mathrm{Ann}\, F(\mu)$.

Proof. (a) follows from the fact that $F(\mu)$ is a highest weight module with highest weight $\mu \in l_{\chi}$. To prove (b) and (c) let

$$\mathfrak{m} = \oplus_{\alpha \in \Delta(\mathfrak{n}^-) \setminus \{-\alpha_1, \dots, -\alpha_k\}} \mathfrak{g}_{\alpha}.$$

Then $\mathfrak{m}$ is a subalgebra of $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{m}$, $\mathfrak{m}_0 = \mathfrak{n}_0^-$ and both statement follow from the following isomorphisms of vector spaces

$$U \cong U(\mathfrak{m}) \otimes U(\mathfrak{p}),\ U(\mathfrak{m}) \cong U(\mathfrak{n}_0^-) \otimes S(\mathfrak{m}_1),\ F(\mu) \cong U(\mathfrak{n}_0^-) \otimes S(\mathfrak{m}_1) \otimes C_{\mu}.$$

To prove (d) note that in case $\mu \in l'_{\chi}$, all $M^0(\nu)$ are irreducible and

$$\mathrm{Ext}^1_{\mathfrak{g}_0}\left(M^0(\nu), M^0(\nu')\right) = 0$$

for $\nu \neq \nu'$. Since $\mathfrak{h}$ acts semisimply on $F(\mu)$, $F(\mu)$ could not contain self-extensions of M. Therefore $F(\mu)$ is a semisimple $\mathfrak{g}_0$-module.

Now we will show that $F(\mu)$ is an irreducible $\mathfrak{g}$-module. For this we will use the odd reflection method. Let $L(\nu)$ denote an irreducible highest weight module with $\mathfrak{b}$-highest weight ν. Suppose $\nu \in l'_{\chi}$. Then the $\mathfrak{b}_d$-highest weight of $L(\nu)$ is equal to $\nu + \delta$, where δ equals the sum of all roots from $\Delta(\mathfrak{g}_1) \setminus \Delta(\mathfrak{p})$. In the same way $\mathfrak{b}_d^{\mathrm{op}}$-highest weight of $L(\nu)$ is $\nu + \delta'$, where δ' is the sum of all odd roots from $\Delta(\mathfrak{g}_{-1}) \setminus \Delta(\mathfrak{p})$. We call the *length* of $\mathfrak{g}$-module M the maximal positive p such that

$S^p(\mathfrak{g}_1)\,M \neq 0$[1]. The above implies that for any $\nu \in l'_\chi$ the length of $L(\nu)$ is $mn-k$. On the other hand the length of $F(\mu)$ also equals $mn-k$. Assume that $L(\nu)$ is some irreducible submodule of $F(\mu)$. Then $\chi_\nu = \chi_\mu$, and ν is as in (c). One can check easily that $\nu \in l'_\chi$, and therefore the lengths of $L(\nu)$ and $F(\mu)$ are equal. Hence $M^0(\mu) \cap L(\nu) \neq 0$. But $M^0(\mu)$ is an irreducible $\mathfrak{g}_0$-module, therefore $M^0(\mu) \subset L(\nu)$. Thus, $F(\mu) = L(\nu)$, and therefore $F(\mu)$ is irreducible.

Finally, (e) follows from Zariski density of l'_χ in l_χ. □

Let $J^\chi = \cap_{\mu \in l} \operatorname{Ann}_{U^\chi} F(\mu)$, and $\bar{U}^\chi = U^\chi / J^\chi$.

Theorem 3.2. *The ideal J^χ coincides with the Jacobson radical of U^χ.*

Proof. The inclusion $\operatorname{rad} U^\chi \subseteq J^\chi$ follows from Lemma 3.1 (d) and (e). To show the inverse inclusion we will use the result of Musson (see [12]) that every primitive ideal of U coincides with the annihilator of some irreducible highest weight module. Let $L_d(\lambda)$ denote the irreducible $\mathfrak{b}_d$-highest weight module with highest weight λ. We have to show that $J^\chi L_d(\lambda) = 0$ whenever $L_d(\lambda)$ admits central character χ. Let δ be the sum of all roots from $\Delta(\mathfrak{g}_1) \setminus \Delta(\mathfrak{p})$, $l_d = l_\chi + \delta$. One can check easily using Lemma 2.1 that $L_d(\lambda)$ admits the central character χ if and only if $w(\lambda+\rho_0) - \rho_0 \in l_d$ for some $w \in W$. We have to show that $J^\chi L_d(\lambda) = 0$ for all such λ. First, let $\lambda \in l_d$. Then $L_d(\lambda)$ is a subquotient of $F(\lambda - \delta)$, and therefore $J^\chi L_d(\lambda) = 0$. Let now $\nu = w(\lambda+\rho_0) - \rho_0 \in l_d$. Then ν and λ induce the same central character $\varphi : Z^0 \to \mathbb{C}$, and the Verma modules $M^0(\lambda)$, $M^0(\nu)$ are both faithful $U^\varphi(\mathfrak{g}_0)$-modules by Duflo's theorem. Let $(\bar{U}^\chi)^+$ be the image of $U(\mathfrak{g}_0 \oplus \mathfrak{g}_1)$ in $\bar{U}^\chi$. Then $U^\varphi(\mathfrak{g}_0)$ is a quotient algebra of $(\bar{U}^\chi)^+$ by some ideal I^+, and therefore the $U^\varphi(\mathfrak{g}_0)$-module $M^0(\lambda)$ can be made an $(\bar{U}^\chi)^+$-module such that $\mathfrak{g}_1$ acts trivially on $M^0(\lambda)$. Then the $\bar{U}$-module $M = \bar{U}^\chi \otimes_{(\bar{U}^\chi)^+} M^0(\lambda)$ is a highest weight module with $\mathfrak{b}_d$-highest weight λ and $J^\chi M = 0$. Since $L_d(\lambda)$ is a quotient of M, $J^\chi L_d(\lambda) = 0$. □

Now we plan to study the structure of the algebra $\bar{U}^\chi$. Start with the following simple observations.

Corollary 3.3. *The natural map $\mathfrak{g} \to \bar{U}^\chi$ is injective. $\bar{U}^\chi$ inherits the $\mathbb{Z}$-grading of U and $\bar{U}^\chi = \bar{U}^\chi_{-s} \oplus \cdots \oplus \bar{U}^\chi_s$, where $s = mn-k$. In particular, if $x \in U_i$ for $i > s$, then its image in $\bar{U}^\chi$ is zero.*

It is more convenient to study the larger algebra $\bar{U}$ defined in the following way. Let $S = \{\mu \in \mathfrak{h}^* \mid (\mu, \alpha_1) = \cdots = (\mu, \alpha_k) = 0\}$. Put $J = \cap_{\mu \in S} \operatorname{Ann} F(\mu)$. Define $\bar{U}$ as the quotient U/J. One can see immediately that all statements of Corollary 3.3 are true for $\bar{U}$ as well. Every irreducible $\mathfrak{g}$-module with degree of atypicality k is an $\bar{U}$-module.

[1]If z_0 is a central element of $\mathfrak{g}_0$ and z_0 acts semi-simply on a cyclic $\mathfrak{g}$-module M, then the length of M is the number of distinct eigenvalues of z_0 on M.

4. The case $\mathfrak{g} = sl(1|2)$

Let now $\mathfrak{g} = sl(1|2)$. Choose a basis of $\mathfrak{h}$ as follows:

$$z_0 = \text{diag } (2,1,1),\ h = \text{diag}\,(0,1,-1)$$

According to this choice of a basis we numerate weights $\mu \in \mathfrak{h}^*$ by pairs (a,b) where $a = \mu(z_0)$ and $b = \mu(h)$. Assume furthermore that $\mathfrak{b} = \mathfrak{b}_d$. There exists exactly one atypical central character χ, and l_χ is given by the equation $a = -b$. In this case $l_\chi = S$ and $\bar{U} = \bar{U}^\chi$.

Let Ω_0 be the Casimir operator of $sl(2) \subset \mathfrak{g}$, Ω be the Casimir operator in $\text{osp}(1|2) \subset sl(1|2)$[2]. Finally let $L = X_{-\alpha}X_\alpha + X_{-\beta}X_\beta$ where α and β are two roots of $\Delta(\mathfrak{g}_1)$. Note that z_0, Ω_0, Ω and L lie in the centralizer of $\mathfrak{g}_0$. By an abuse of notations we denote the images of these elements in $\bar{U}$ by the same letters.

The following lemma is a very easy calculation.

Lemma 4.1. *If $\mu \in l_\chi$, then $F(\mu)$ considered as an $\text{osp}(1|2)$-module is isomorphic to Verma module with highest weight b. Furthermore as an $sl(2)$-module $F(\mu)$ is isomorphic to the sum of two Verma $sl(2)-$ modules with highest weights b and $b-1$.*

Corollary 4.2. *Ω acts by the scalar $b(b+1)$ on $F(\mu)$, and therefore belongs to the center $Z(\bar{U})$ of $\bar{U}$.*

Lemma 4.3. *The following relations hold in $\bar{U}$:*

$$z_0^2 + \Omega_0 = 2\Omega$$

$$(\Omega - \Omega_0)^2 = z_0^2$$

$$z_0 - 2L = \Omega - \Omega_0$$

Proof. Since $F(\mu)$ for $\mu \in l_\chi$ forms a faithful system of modules it is sufficient to check the relations on $F(\mu)$. This can be done easily using Lemma 4.1. □

In the similar way the following lemma can be proved.

Lemma 4.4. *$C = -z_0^3 - \frac{3}{2}z_0(\Omega - \Omega_0) - \frac{1}{2}z_0 - \frac{1}{2}\Omega$ belongs to $Z(\bar{U})$, and acts on $F(\mu)$ by $b^2(b+1)$.*

Let $\bar{U}(b)$ be the quotient of $\bar{U}$ by the ideal $(\Omega - b(b+1), C - b^2(b+1))$, $U^0(b)$ denote the quotient of $U(gl(2))$ by the ideal $(\Omega_0 - b(b+2), z_0 + b)$ and $U'(b)$ denote the quotient of $U(\text{osp}(1|2))$ by the ideal $(\Omega - b(b+1))$.

Theorem 4.5. *Let $b \neq 0, -1$, then $\bar{U}(b)$ is Morita equivalent to $U^0(b)$. If $b \neq -\frac{1}{2}$, then $\bar{U}(b)$ is isomorphic to $U'(b)$.*

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[2]We normalize Ω and Ω_0 so that Ω acts as $a(a+1)$ on the irreducible $\text{osp}(1|2)$-module with highest weight a and Ω_0 acts as $a(a+2)$ on the irreducible $sl(2)$-module with highest weight a.

Proof. Let M be a $\bar{U}(b)$-module and $M^{\mathfrak{g}_1}$ be the space of $\mathfrak{g}_1$-invariant vectors in M. Clearly $M^{\mathfrak{g}_1}$ is a $\mathfrak{g}_0$-module, $M^{\mathfrak{g}_1} \neq 0$, and $LM^{\mathfrak{g}_1} = 0$. The relations of Lemma 4.3 after substituting $L = 0$ imply one of the following

1. z_0 acts on $M^{\mathfrak{g}_1}$ by the scalar $-b$, Ω_0 acts on $M^{\mathfrak{g}_1}$ by the scalar $b^2 + 2b$;
2. z_0 acts on $M^{\mathfrak{g}_1}$ by the scalar $b + 1$, Ω_0 acts on $M^{\mathfrak{g}_1}$ by the scalar $b^2 - 1$.

However, the condition $C = b^2(b+1)$ excludes the second case for $b \neq 0, -1$ and therefore $M^{\mathfrak{g}_1}$ is a $U^0(b)$-module. Furthermore, $M^{\mathfrak{g}_1} = \operatorname{Ker}(z + b)$. Define a functor Inv from the category of $\bar{U}(b)$-modules to the category of $U^0(b)$-modules by putting $\operatorname{Inv} M = M^{\mathfrak{g}_1} = \operatorname{Ker}(z + b)$. This functor is clearly faithful and exact. It is easy to see that $\operatorname{Inv} F(\mu) = M^0(\mu)$. Let $\bar{U}^+(b)$ denote the subalgebra in $\bar{U}(b)$ generated by $\mathfrak{g}_0$ and $\mathfrak{g}_1$. Every $U^0(b)$-module N can be equipped with the structure of $\bar{U}^+(b)$-module by letting $\mathfrak{g}_1$ act by zero. Put

$$\operatorname{Ind} N = \bar{U}(b) \otimes_{\bar{U}^+(b)} N.$$

Then $(\operatorname{Ind} N)^{\mathfrak{g}_1} = \operatorname{Ker}(z_0 + b) = N$. Therefore the functors Ind and Inv establish the equivalence of the categories of $U^0(b)$-modules and $\bar{U}(b)$-modules.

Finally, to establish the isomorphism between $\bar{U}(b)$ and $U'(b)$, consider the following sequence

$$U(\operatorname{osp}(1|2)) \xrightarrow{i} \bar{U}(b) \xrightarrow{\sigma} F(F(\mu), F(\mu)),$$

where for arbitrary $\mathfrak{g}$-modules M and N, $F(M, N)$ denote the subspace of $\operatorname{Hom}(M, N)$ locally finite as $\operatorname{ad}\mathfrak{g}_0$-module, the first map i is induced by the embedding of $\operatorname{osp}(1|2)$ into $\mathfrak{g}$ and the second map σ is the representation of $\bar{U}(b)$. As it follows from [6] the composite map $\sigma \circ i$ is an isomorphism for $b \neq -\frac{1}{2}$. Therefore $\sigma\left(\bar{U}(b)\right)$ is isomorphic to $U'(b)$. On the other hand, $F(\mu) = \operatorname{Ind} M^0(\mu)$ and $M^0(\mu)$ is a faithful $U^0(b)$-module. Hence $F(\mu)$ is a faithful $\bar{U}(b)$-module and therefore σ is injective. Thus

$$\bar{U}(b) \cong F(F(\mu), F(\mu)) \cong U'(b).$$

□

Remark 4.6. The isomorphism between $U'(b)$ and $\bar{U}(b)$ has its origin in the following geometric fact. If B' is a Borel subgroup of $\operatorname{OSP}(1|2)$ and P is the parabolic subgroup of $\operatorname{SL}(1|2)$ with the Lie algebra $\mathfrak{p}$, then the algebraic supervarieties $\operatorname{OSP}(1|2)/B'$ and $\operatorname{SL}(1|2)/P$ are isomorphic.

Finally let us look what happens in case $b = 0, -1$. Then $\bar{U}(0)$ is isomorphic to $\bar{U}(-1)$, and $z_0^2 = \Omega_0^2 = -\Omega_0$. One can prove similarly to the above theorem that $\bar{U}(0)/(z_0 - \Omega_0) \cong \bar{U}(0)/(z_0 + \Omega_0) \cong U'(0)$. One one can construct the injective homomorphism

$$f : \bar{U}(0) \to F(F(0), F(0)) \oplus F(F(-1), F(-1)) \cong U'(0) \oplus U'(0).$$

One can show that $\bar{U}(0) \cong \operatorname{Im} f = \{(x,y) \in U'(0) \oplus U'(0) \mid p(x) = p(y)\}$, where $p : U'(0) \to \mathbb{C}$ is the projection to the factor by augmentation ideal.

5. The case $\mathfrak{g} = psl(2|2)$

The Lie algebra $\mathfrak{g}_0$ is isomorphic to $sl(2) \oplus sl(2)$. The Cartan subalgebra is 2-dimensional, and we choose a basis $\{h_1, h_2\}$ in it so that h_i is the standard element of a copy of $sl(2)$. We choose a Borel subalgebra $\mathfrak{b}$ so that all simple roots are odd. In this case $\rho_{\mathfrak{b}} = 0$. There is exactly one atypical central character χ, which is induced by the trivial module. In this case $S = l_\chi = \{\mu \in \mathfrak{h}^* \mid \mu(h_1) = -\mu(h_2)\}$, and again $\bar{U} = \bar{U}^\chi$. By $F(a)$ we denote $F(\mu)$ with $\mu(h_1) = -\mu(h_2) = a$. Let Ω_1 and Ω_2 be the Casimir elements of two $sl(2)$-subalgebras of $\mathfrak{g}_0$ and Ω be the Casimir element of $\mathfrak{g}$. Let X_1, X_2, X_3, X_4 be a basis of $\mathfrak{g}_1$ and Y_1, Y_2, Y_3, Y_4 be the dual basis of $\mathfrak{g}_{-1}$. Then the element $L = \sum_{i=1}^4 Y_i X_i = \sum_{i=1}^4 -X_i Y_i$ belongs to the centralizer of $\mathfrak{g}_0$ in U, and we have the relation

$$\Omega_1 - \Omega_2 + 2L = \Omega. \tag{5.1}$$

Note that the image of Ω in U^χ is zero, and therefore $2L = \Omega_2 - \Omega_1$.

Let $M^0(a,b)$ denote $sl(2) \oplus sl(2)$-Verma module with highest weight (a,b). The following lemma is obtained by direct calculation[3].

Lemma 5.1. *$F(a) \cong M^0(a,-a) \oplus 2M^0(a-1,-1-a) \oplus M^0(a-2,-2-a)$ as $\mathfrak{g}_0$-module, the element*

$$C_1 = \frac{(\Omega_1+\Omega_2)^2}{4} + \frac{\Omega_1+\Omega_2}{2} - \frac{(\Omega_1-\Omega_2)^2}{8} \tag{5.2}$$

acts by the scalar $a^2(a^2-1)$ on $F(a)$, and the element

$$C_2 = \frac{(\Omega_1+\Omega_2)^3}{8} + \frac{(\Omega_1+\Omega_2)^2}{4} - 3\frac{(\Omega_1-\Omega_2)^4}{256} + \frac{(\Omega_1-\Omega_2)^2}{16} \tag{5.3}$$

acts by the scalar $a^2(a^2-1)^2$ on $F(a)$.

Thus, the images of C_1 and C_2 in $\bar{U}$ belong to the center $Z(\bar{U})$ of $\bar{U}$. Again we denote their images in $\bar{U}$ by the same letters C_1 and C_2. Now put

$$\bar{U}(a) = \bar{U} / \left(C_1 - a^2(a^2-1), C_2 - a^2(a^2-1)^2\right)$$

and

$$U^0(a) = U(\mathfrak{g}_0) / \left(\Omega_1 - a^2 + 1, \Omega_2 - a^2 + 1\right).$$

Theorem 5.2. *If $a \neq 0, \pm 1$, then $\bar{U}(a)$ and $U^0(a)$ are Morita equivalent.*

The rest of this section will contain the proof of Theorem 5.2.

For any $\mathfrak{g}$-module M we denote by $M^{\mathfrak{g}_{\pm 1}}$ the space of $\mathfrak{g}_{\pm 1}$-invariants. Obviously $M^{\mathfrak{g}_{\pm 1}}$ is $\mathfrak{g}_0$-invariant and $LM^{\mathfrak{g}_{\pm 1}} = 0$.

[3] Use that Ω_1 and Ω_2 act on $M^0(a,b)$ as $a(a+2)$ and $b(b+2)$ correspondingly.

Lemma 5.3. *Let $a \neq 0, \pm 1$ and M be a $\bar{U}(a)$-module, then $M^{\mathfrak{g}_1}$ and $M^{\mathfrak{g}_{-1}}$ are $U^0(a)$-modules.*

Proof. Since $LM^{\mathfrak{g}_{\pm 1}} = 0$, it follows from (5.1) that $\Omega_1 = \Omega_2$ on $M^{\mathfrak{g}_1}$. The relation (5.2) implies that

$$\Omega_1^2 + \Omega_1 = a^2\left(a^2 - 1\right),$$

and hence $\Omega_1 = \Omega_2 = a^2 - 1$ or $\Omega_1 = \Omega_2 = -a^2$. The relation (5.3) implies

$$\Omega_1^3 + \Omega_1^2 = a^2\left(a^2 - 1\right)^2.$$

If $a^2 \neq 0, 1$ we have $\Omega_1 = \Omega_2 = a^2 - 1$. Therefore $M^{\mathfrak{g}_{\pm 1}}$ is $U^0(a)$-module. $\square$

Denote by Inv the functor from the category of $\bar{U}(a)$-modules to the category of $U^0(a)$-modules which sends M to $M^{\mathfrak{g}_1}$. Obviously Inv is faithful and exact on the left.

Lemma 5.4. *Let V be the $\operatorname{ad}\mathfrak{g}$-submodule of $\bar{U}$ generated by L, and $\bar{U}^2$ denote the second term in the filtration of $\bar{U}$ induced by the natural filtration of U. Then $\bar{U}^2 = V \oplus \mathbb{C} \oplus \mathfrak{g}$.*

Proof. By Poincare-Birkhoff-Witt's theorem $U^2 \cong \mathbb{C} \oplus \mathfrak{g} \oplus S^2(\mathfrak{g})$. A direct calculation shows that $S^2(\mathfrak{g}) \cong \mathbb{C}\Omega \oplus L(2,0) \oplus L(0,2)$. Therefore $\bar{U}^2 \cong \mathbb{C} \oplus \mathfrak{g} \oplus L(2,0) \oplus L(0,2)$. Furthermore, $L \in L(2,0) \oplus L(0,2)$, but L does not belong to $L(2,0)$ or $L(0,2)$. Hence L generates V. $\square$

Lemma 5.5. *Let M be a $\bar{U}$-module such that $M = M^{\mathfrak{g}_1} + M^{\mathfrak{g}_{-1}}$. Then M is a trivial $\mathfrak{g}$-module.*

Proof. Since $LM = 0$, we have $S^2(\mathfrak{g}) \subset \operatorname{Ann}_U M$ by Lemma 5.4. Hence $X^2 M = 0$ for any $X \in \mathfrak{g}$. This implies that $\mathfrak{g}M = 0$. $\square$

Now we will construct a functor Ind form the category of $U^0(a)$-modules to the category of $\bar{U}(a)$-modules. Let $\bar{U}^+(a)$ be the subalgebra of $\bar{U}(a)$ generated by $\mathfrak{g}_0$ and $\mathfrak{g}_1$. Every $U^0(a)$-module N becomes an $\bar{U}^+(a)$-module if we put $\mathfrak{g}_1 N = 0$. Put

$$\operatorname{Ind} N = \bar{U}(a) \otimes_{\bar{U}^+(a)} N.$$

Lemma 5.6. *If $a \neq 0, 1, -1$, then for every $\bar{U}^0(a)$-module N, $\operatorname{Inv}(\operatorname{Ind} N) \cong N$, and for every $\bar{U}(a)$-module M, $\operatorname{Ind}(\operatorname{Inv} M) \cong M$.*

Proof. $\operatorname{Ind} N = N \oplus \mathfrak{g}_1 N \oplus S^2(\mathfrak{g}_1) N$. Let $N' = \operatorname{Inv}(\operatorname{Ind} N) \cap (\mathfrak{g}_1 N \oplus S^2(\mathfrak{g}_1) N)$. It suffices to show that $N' = 0$. Indeed let M' be the $\mathfrak{g}$-submodule of $\operatorname{Ind} N$ generated by N'. Then $M' \subset \mathfrak{g}_{-1} N \oplus S^2(\mathfrak{g}_{-1}) N$. Then M' satisfies the condition of Lemma 5.5, and therefore $M' = N'$ is trivial. This is possible only for $a = 0, \pm 1$.

To prove the second statement consider the natural surjection

$$\pi : \operatorname{Ind}(\operatorname{Inv} M) \to M.$$

Let $\operatorname{Ker} \pi = R$. Then $\operatorname{Inv} R \cap M = \{0\}$ and $\operatorname{Inv} R \subset \operatorname{Inv}(\operatorname{Ind} M) = M$. Therefore $\operatorname{Inv} R = \{0\}$, and $R = \{0\}$ because Inv is faithful. $\square$

Corollary 5.7. *If $a \neq 0, \pm 1$, then the functors* Ind *and* Inv *establish the equivalence between the categories of $U^0(a)$-modules and $\bar{U}(a)$-modules.*

6. The center of $\bar{U}$ for $\mathfrak{g} = sl(1|n)$

In this section $\mathfrak{g} = sl(1|n)$, χ is some atypical character, $\mathfrak{b}$ can be always chosen $\mathfrak{b}_d$. We denote the simple roots of $\mathfrak{b}$ by $\alpha, \beta_1, \dots, \beta_{n-1}$, where α is odd and $\beta_1, \dots, \beta_{n-1}$ are even simple roots of $sl(n) \subset \mathfrak{g}_0$. Then

$$l_\chi = \{\mu \in \mathfrak{h}^* \mid (\mu, \alpha) = 0, (\mu, \beta_2) = \mu_2, \dots, (\mu, \beta_{n-1}) = \mu_{n-1}\},$$

where $\mu_2, \dots, \mu_{n-1} \in \mathbb{C}$. We put $\mu_1 = (\mu, \beta_1)$ and use μ_1 as the coordinate on the line l_χ. Note that S is the hyperplane given by the equation $(\mu, \alpha) = 0$, and we use $\mu_1, \dots, \mu_{n-1}$ as coordinates on S.

Let $Z(\bar{U})$ denote the center of $\bar{U}$. By $\bar{Z}$ we denote the image of Z in $\bar{U}$ under the natural projection. Clearly $\bar{Z} \subseteq Z(\bar{U})$. In this section we describe $Z(\bar{U})$.

To proceed farther we need to recall certain facts from finite-dimensional representation theory of $sl(1|n)$. See [4] for reference. By $L(\lambda)$ we denote the irreducible $\mathfrak{g}$-module with highest weight λ. By $K(\lambda)$ we denote Kac module $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0 + \mathfrak{g}_1)} L^0(\lambda)$, where $L^0(\lambda)$ is the irreducible $\mathfrak{g}_0$-module with trivial action of $\mathfrak{g}_1$. By $K'(\lambda)$ we denote the induced module $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0 + \mathfrak{g}_{-1})} L^0(\lambda)$. Finally by $P(\lambda)$ we denote the projective cover[4] of $L(\lambda)$, which is an indecomposable submodule in $U \otimes_{U(\mathfrak{g}_0)} L^0(\lambda)$. $P(\lambda)$ is also injective, contragredient and has a unique irreducible submodule and unique irreducible quotient isomorphic to $L(\lambda)$. We are interested in the structure of $P(0)$. $P(0)$ has four irreducible subquotient which could be calculated from the following exact sequences

$$\begin{gathered} 0 \to K'(-2\rho_1) \to P(0) \to K'(0) \to 0 \\ 0 \to L(2\rho_1) \to K'(0) \to L(0) \to 0 \\ 0 \to L(0) \to K'(-2\rho_1) \to L(-\alpha) \to 0 \end{gathered} \tag{6.1}$$

Let $X_\beta \in \mathfrak{g}_\beta$ such that $\beta'([X_\beta, X_{-\beta}]) = (\beta', \beta)$ for any $\beta' \in \Delta$. Let

$$X = \prod_{\beta \in \Delta(\mathfrak{g}_1)} X_\beta, \; Y = \prod_{\beta \in \Delta(\mathfrak{g}_{-1})} X_\beta,$$

and

$$X^{\mathrm{ad}} = \prod_{\beta \in \Delta(\mathfrak{g}_1)} \mathrm{ad}\, X_\beta, \; Y^{\mathrm{ad}} = \prod_{\beta \in \Delta(\mathfrak{g}_{-1})} \mathrm{ad}\, X_\beta.$$

The following three lemmas and corollary hold for $sl(m|n)$ with arbitrary m and n. They can be extracted from [5], but we give independent proofs here.

Lemma 6.1. $Y^{\mathrm{ad}}(U(\mathfrak{g}_0)) = YU(\mathfrak{g}_0)$. *In particular* $Y^{\mathrm{ad}}(Z^0) = YZ^0$.

[4] $P(\lambda)$ is projective in the category of all finite-dimensional $\mathfrak{g}$-modules, semi-simple over $\mathfrak{h}$.

Proof. It is obvious that $Y^{\mathrm{ad}}(U(\mathfrak{g}_0)) \subseteq YU(\mathfrak{g}_0)$. To check the equality note that $Y^{\mathrm{ad}} : U(\mathfrak{g}_0) \to YU(\mathfrak{g}_0)$ is a homomorphism of $\mathrm{ad}\, sl(n)$-modules. A simple calculation shows that $Y^{\mathrm{ad}}(U(\mathfrak{h})) = YU(\mathfrak{h})$. Since $U(\mathfrak{h})$ generates $U(\mathfrak{g}_0)$ as $\mathrm{ad}\, sl(n)$-module, we have $Y^{\mathrm{ad}}(U(\mathfrak{g}_0)) = YU(\mathfrak{g}_0)$. □

Recall that h denotes the Harish-Chandra homomorphism. We put $h_u = h(u)$ for $u \in U$.

Lemma 6.2. *Let $Z' = X^{\mathrm{ad}}(YZ^0)$. Then $Z' \subset Z$. For any $x \in Z^0$*

$$h\left(X^{\mathrm{ad}}(Yx)\right) = \Delta(\mu)\, h_x,$$

where $\Delta(\mu) = \prod_{\beta \in \Delta(\mathfrak{g}_1)} (\mu + \rho, \beta)$.

Proof. Let $y \in YZ^0$. Then y is $\mathrm{ad}\,\mathfrak{g}_{-1}$-invariant. If M is the $\mathrm{ad}\,\mathfrak{g}$-submodule generated by y, then obviously M is a quotient of $K'(-2\rho_1)$. From the exact sequence

$$0 \to L(0) \to K'(-2\rho_1) \to L(-\alpha) \to 0$$

it is clear that $X^{\mathrm{ad}}(y) = 0$ or $X^{\mathrm{ad}}(y)$ generates a trivial $\mathrm{ad}\,\mathfrak{g}$-submodule. Thus $X^{\mathrm{ad}}(y) \in Z$. To find $h\left(X^{\mathrm{ad}}(Yx)\right)$ apply $X^{\mathrm{ad}}(Yx)$ to the highest vector v of a Verma module $M(\mu)$. We have

$$X^{\mathrm{ad}}(Yx)\, v = XYxv = \Delta(\mu)\, h_x(\mu)\, v.$$

□

Corollary 6.3. $h(Z') = \Delta(\mu)\, h(Z^0)$.

Lemma 6.4. *Every $z \in Z'$ is contained in some* $\mathrm{ad}\,\mathfrak{g}$*-submodule $N_z \subset U$ isomorphic to $P(0)$.*

Proof. Let $z \in Z'$. Then by Lemma 6.1 $z = X^{\mathrm{ad}} Y^{\mathrm{ad}}(c)$ for some $c \in Z^0$. Let M_z be the $\mathrm{ad}\,\mathfrak{g}$-submodule in U generated by c. Clearly z generates a trivial submodule of M_z. Let M be the injective hull of M_z with injection $i : M_z \to M$. Then $M \cong P(0) \oplus R$ for some injective module R, and $i(z) \in P(0)$. Let $\pi : M \to P(0)$ denote the projection, then $\pi(c) \neq 0$, and therefore $\pi \circ i$ is surjective. Since $P(0)$ is projective, there exist an injective homomorphism $j : P(0) \to M_z$ such that $i \circ j = id$. Put $N_z = j(P(0))$. Then $z \in N_z$. □

Staring from this point all results are proven only for $\mathfrak{g} = sl(1|n)$.

Lemma 6.5. *Let $j : P(0) \to U$ be a non-trivial homomorphism of $\mathfrak{g}$-modules, where $\mathfrak{g}$-action on U is adjoint. Then either j is injective or $j(P(0))$ is a trivial* $\mathrm{ad}\,\mathfrak{g}$*-submodule in U.*

Proof. Let c be an $\mathrm{ad}\,\mathfrak{g}_0$-invariant generator of $j(P(0))$. If $Y^{\mathrm{ad}}(c) \neq 0$, then $Y^{\mathrm{ad}}(c) = Yx$ for some $x \in Z^0$ and, as follows from Lemma 6.2, $X^{\mathrm{ad}} Y^{\mathrm{ad}}(c) \neq 0$. Therefore $j(P(0))$ contains a trivial submodule, and by exact sequences (6.1) j is injective. In the same way it can be proven that if $X^{\mathrm{ad}}(c) \neq 0$, then j is injective as well.

Therefore if j is not injective, then $Y^{\mathrm{ad}}(c) = X^{\mathrm{ad}}(c) = 0$. Then looking at the exact sequences (6.1) one can conclude that $j(P(0)) = \mathbb{C}c$ and therefore $j(P(0))$ is isomorphic to a trivial module. $\square$

Lemma 6.6. *$Z' = Z \cap J$, and therefore $\bar{Z} \cong Z/Z'$.*

Proof. Let $z \in Z'$. Then by Corollary 6.3 $zM(\mu) = 0$ for all atypical $\mu \in \mathfrak{h}^*$. Therefore $z \in \cap_{\mu\in S} \operatorname{Ann} M(\mu) \subset \cap_{\mu \in S} \operatorname{Ann} F(\mu) = J$. Hence $Z' \subset Z \cap J$.

Now suppose $z \in Z \cap J$. Then z acts trivially on $M(\mu)$ for all $\mu \in S$. Hence z acts trivially on $M(\mu)$ for all atypical μ. Therefore $h_z(\mu) = 0$ whenever $(\mu + \rho, \beta) = 0$ for some $\beta \in \Delta(\mathfrak{g}_1)$. Therefore $\Delta(\mu)$ divides $h_z(\mu)$. Then by Corollary 6.3 $z \in Z'$, and we have $Z \cap J \subseteq Z'$. $\square$

Lemma 6.7. *Let $\bar{N}_z$ be the image of N_z in $\bar{U}$ under the natural projection. Then $\bar{N}_z \subset Z(\bar{U})$.*

Proof. As follows from the exact sequences (6.1) $U_n \cap N_z$ and $U_{-n} \cap N_z$ generate the unique maximal submodule R_z of N_z. By Corollary 3.3 $\bar{U}_n = \bar{U}_{-n} = 0$. Therefore $\bar{R}_z = 0$. Thus, the image $\bar{N}_z$ in $\bar{U}$ is either 0 or a trivial $\operatorname{ad}\mathfrak{g}$-module, and hence $\bar{N}_z \subset Z(\bar{U})$. $\square$

Lemma 6.8. *$Z(\bar{U})$ is spanned by $\bar{Z}$ and $\bar{N}_z$ for all $z \in Z'$.*

Proof. Let $z \in Z(\bar{U})$. There exists a homomorphism $j : P(0) \to U$ such that the image of $j(P(0))$ in $\bar{U}$ coincides with $\mathbb{C}z$. Let c be an $\operatorname{ad}\mathfrak{g}_0$-invariant generator of $j(P(0))$ such that $\bar{c} = z$. As follows from Lemma 6.5 either $j(P(0))$ is a trivial $\operatorname{ad}\mathfrak{g}$-submodule in U or j is injective. In the former case $c \in Z$ and $z = \bar{c} \in \bar{Z}$. In the latter case $y = X^{\mathrm{ad}}Y^{\mathrm{ad}}(c) \neq 0$ and $y \in Z'$. Let us choose a generator $u \in U^{\operatorname{ad}\mathfrak{g}_0}$ of N_y such that $X^{\mathrm{ad}}Y^{\mathrm{ad}}(u) = y$. Then $X^{\mathrm{ad}}Y^{\mathrm{ad}}(c-u) = 0$, and hence $c - u \in Z$ by Lemma 6.5. Then $z = \bar{c} = \bar{u} + (\bar{c} - \bar{u})$, and therefore $z \in \bar{Z} + \bar{N}_y$. $\square$

To describe $Z(\bar{U})$ we define the Harish-Chandra homomorphism $\bar{h} : Z(\bar{U}) \to S(S^*) \cong \mathbb{C}[\mu_1, \dots, \mu_{n-1}]$ by $\bar{h}_z(\mu) = h_x(\mu)$ for any $x \in U$ with $\bar{x} = z$ and any $\mu \in \mathfrak{h}^*$. To see that $\bar{h}_z$ does not depend on choice of x note that, since any element $z \in Z(\bar{U})$ acts by the scalar $c_{z,\mu}$ on $F(\mu)$ for any $\mu \in S$, we have $\bar{h}_z(\mu) = c_{z,\mu}$. Since the family $F(\mu)$ is faithful over $\bar{U}$, $\bar{h}$ is an injective homomorphism. Let $\mathcal{A}$ be the image of $Z(\bar{U})$ under the Harish-Chandra homomorphism $\bar{h}$.

Let W' be the subgroup of W which preserves α. Clearly W' is isomorphic to S_{n-1}, and $S^w = S$ for any $w \in W'$. Let $\mathcal{R}$ denote the ring of W'-invariant polynomials on S, $\mathcal{R}'$ be the subring of polynomials in $\mathcal{R}$ constant on all lines l_χ. In other words polynomials from $\mathcal{R}'$ do not depend on μ_1.

Lemma 6.9. *(a) Let $r \colon S(\mathfrak{h}) \to S(S^*)$ be the restriction homomorphism. Then $r \circ h(Z^0) = \mathcal{R}$.*

(b) $\mathcal{A} \subset \mathcal{R}$.

(c) $\bar{h}(\bar{Z}) = \mathcal{R}'$.

Proof. (a) follows from the fact that $h(Z^0) = S^W(\mathfrak{h})$ and $r(S^W(\mathfrak{h})) = \mathcal{R}$. (*b*) follows from the very definition of $\bar{h}$. Indeed choose $\operatorname{ad}\mathfrak{g}_0$-invariant $x \in U$ such that $\bar{x} = z$. Then $\bar{h}_z = r(h_x)$. Since x is $\operatorname{ad}\mathfrak{g}_0$-invariant, h_x is W-invariant and $r(h_x)$ is W'-invariant. (*c*) follows from the formula (2.1). ☐

Now we will describe $\bar{h}(\bar{N}_z)$. The following polynomial plays a special role

$$\Theta(\mu) = \prod_{k=1}^{n-1}(\mu + \rho_0, \beta_1 + \cdots + \beta_k) = \prod_{i=1}^{n-1}(i + \mu_1 + \cdots + \mu_i) = \prod_{\gamma \in \Delta(\mathfrak{g}_1)\backslash\alpha}(\mu + \rho, \gamma).$$

Lemma 6.10. *Let $z \in Z'$, $z = X^{\mathrm{ad}}(Yu)$ for some $u \in Z^0$ and $p_z \in \bar{h}(\bar{N}_z)$. Then*

$$p_z(\mu - \alpha) - p_z(\mu) = a\Theta(\mu)\, h_u(\mu) \tag{6.2}$$

for some $a \in \mathbb{C}$.

Proof. The idea is to look at the zeros of $p_z(\mu) - p_z(\mu - \alpha)$. Suppose $p_z(\mu) = p_z(\mu - \alpha)$. Let $c \in U^{\mathrm{ad}\,\mathfrak{g}_0}$ be the generator of N_z such that $Y^{\mathrm{ad}}(c) = Yu$. Then $\bar{h}_c$ is proportional to p_z, and c acts by the same constant ν(proportional to $p_z(\mu)$) on $F(\mu)$ and $F(\mu - \alpha)$. We will use the exact sequence

$$0 \to F(\mu - \alpha) \to M(\mu) \to F(\mu) \to 0.$$

Put $c' = c - \nu\,\mathrm{Id}$. Then $c'M(\mu) \subset F(\mu - \alpha)$ and $(c')^2 M(\mu) = 0$. There are two cases

1. c$' = 0$ on $M(\mu)$;
2. c$' \neq 0$ on $M(\mu)$.

Recall that $Y^{\mathrm{ad}}(c) = Y^{\mathrm{ad}}(c') = Yu$. Therefore in the first case we must have $Yu(M(\mu)) = 0$. Let $M^0(\mu)$ be the $\mathfrak{g}_0$ submodule of $M(\mu)$ generated by a highest vector. Then $u(M^0(\mu)) = 0$, i.e. $h_u(\mu) = 0$.

Consider the second case, assuming $h_u(\mu) \neq 0$. In this case c' is a non-zero element of $\mathrm{Hom}_{\mathfrak{g}_0}(F(\mu), F(\mu - \alpha))$. Lemma 3.1 (*c*) implies that for some distinct $\gamma_1, \ldots, \gamma_s \in \Delta(\mathfrak{b}_1)\backslash\alpha$ and some distinct $\gamma'_1, \ldots, \gamma'_{s-1} \in \Delta(\mathfrak{b}_1)\backslash\alpha$

$$\mathrm{Hom}_{\mathfrak{g}_0}\left(M^0(\mu - \gamma_1 - \cdots - \gamma_s), M^0(\mu - \alpha - \gamma'_1 - \cdots - \gamma'_{s-1})\right) \neq 0.$$

Therefore for some $w \in W$

$$w(\mu - \gamma_1 - \cdots - \gamma_s + \rho_0) = \mu - \alpha - \gamma'_1 - \cdots - \gamma'_{s-1} + \rho_0.$$

This is possible if and only if for some γ_j and some $u \in W$

$$u(\mu - \gamma_j + \rho_0) = \mu - \alpha.$$

Let $\gamma_j = \alpha + \beta_1 + \cdots + \beta_k$. Then $(\mu - \alpha + \rho_0, \beta_1 + \cdots + \beta_k) = 1$, or using $(\alpha, \beta_1) = -1$, $(\alpha, \beta_i) = 0$ for $i > 1$, $(\mu + \rho_0, \beta_1 + \cdots + \beta_k) = 0$.

Thus we obtained that the zeros of the polynomial $p_z(\mu-\alpha)-p_z(\mu)$ are the zeros of $\Theta(\mu)\,h_u(\mu)$. The condition $Y^{\mathrm{ad}}(c)=Yu$ implies $\deg\bar{h}_c=\deg h_u+n$. Since $\bar{h}_c$ and p_z are proportional we obtain

$$\deg\left(p_z(\mu-\alpha)-p_z(\mu)\right)=\deg h_u+n-1=\deg\Theta(\mu)\,h_u(\mu).$$

Hence the polynomials $p_z(\mu-\alpha)-p_z(\mu)$ and $\Theta(\mu)\,h_u(\mu)$ must be proportional. $\square$

Remark 6.11. Note that the equation (6.2) in coordinates $\mu_1,\dots,\mu_{n-1}$ looks as follows

$$p_z(\mu_1,\dots,\mu_{n-1})-p_z(\mu_1+1,\dots,\mu_{n-1})=a\Theta(\mu)\,h_u(\mu)$$

Theorem 6.12. *Let (Θ) be the ideal in $\mathcal{R}$ generated by the polynomial Θ. Then*

$$\mathcal{A}=\{p\in\mathcal{R}\mid p(\mu-\alpha)-p(\mu)\in(\Theta)\}$$

Proof. The inclusion

$$\mathcal{A}\subseteq\{p\in\mathcal{R}\mid p(\mu-\alpha)-p(\mu)\in(\Theta)\}$$

follows from Lemma 6.8, Lemma 6.9 (b) and Lemma 6.10.

To show the inverse inclusion let $p\in\mathcal{R}$ such that $p(\mu-\alpha)-p(\mu)=q(\mu)\,\Theta(\mu)$ for some $q(\mu)\in\mathcal{R}$. By Lemma 6.9 (a) $q=h_u$ for some $u\in Z^0$. Let $z=X^{\mathrm{ad}}(Yu)$. Choose $p_z\in\bar{h}\left(\bar{N}_z\right)$ satisfying the equation (6.2) with $a=1$. Then

$$p_z(\mu-\alpha)-p_z(\mu)=p(\mu-\alpha)-p(\mu).$$

Therefore $p-p_z\in\mathcal{R}'$. Since $p_z\in\mathcal{A}$, $p\in\mathcal{A}$. $\square$

Let S_Θ be the set of zeros of the polynomial $\Theta(\mu)$.

Corollary 6.13.

$$\mathcal{A}=\{p\in\mathcal{R}\mid p(\mu)=p(\mu-\alpha)\ \text{ for all } \mu\in S_\Theta\}$$

Corollary 6.14. *Let $S'=S\setminus(S_\Theta\cup S_\Theta-\alpha)$. Then the natural map* $\operatorname{Spec}\mathcal{R}\to\operatorname{Spec}\mathcal{A}$ *is an isomorphism when restricted to S'.*

Proof. Let $\mu\in S'$, $\nu\in S$ and $\mu\neq\nu$. Then by Corollary 6.13 there exists $p\in\mathcal{A}$ such that $p(\mu)=0$ and $p(\nu)\neq 0$ $\square$

7. On structure of $\bar{U}$-modules for $\mathfrak{g}=sl(1|n)$

We start with the following simple observation.

Theorem 7.1. *Let M be a finite-dimensional $\bar{U}$-module. Then M is completely reducible.*

Proof. Let $L(\lambda)$ and $L(\mu)$ be irreducible finite-dimensional $\bar{U}$-modules with highest weights $\lambda, \mu \in S$. It is sufficient to show that $\mathrm{Ext}^1_{\bar{U}}(L(\mu), L(\lambda)) = 0$. Consider some extension E over U. We have the following exact sequence

$$0 \to L(\mu) \to E \to L(\lambda) \to 0.$$

Since this sequence does not split λ and μ must be compatible with respect to Bruhat order. Therefore we have three cases: $\lambda = \mu$, $\lambda > \mu$ and $\lambda < \mu$. However, as it is shown in [4] , there is no self-extensions of finite-dimensional representations. Thus, the first case is impossible. If $\lambda > \mu$ then E is isomorphic to $K(\lambda)$. If $\lambda < \mu$, let us consider the contragredient exact sequence

$$0 \to L'(\lambda) \to E' \to L'(\mu) \to 0.$$

Note that $L'(\lambda) \cong L(\lambda)$ and $L'(\mu) \cong L(\mu)$, and therefore E' is isomorphic to $K(\mu)$.

Thus, our task is to show that a Kac module $K(\lambda)$ is not a $\bar{U}$-module, i.e. $JK(\lambda) \neq 0$. Indeed, $U_{-n}K(\lambda) \neq 0$ because $K(\lambda)$ is free over $U(\mathfrak{g}_{-1})$. But $U_{-n} \subset J$ by Corollary 3.3. Contradiction. □

Our next step is to generalize Penkov's theorem for $\bar{U}$. For any $\mu \in S$ let $\phi_\mu : Z(\bar{U}) \to \mathbb{C}$ be the character defined by $\phi_\mu(z) = \bar{h}_z(\mu)$ for any $z \in Z(\bar{U})$. The weight μ also defines the character $\varphi_\mu : Z^0 \to \mathbb{C}$ by $\varphi_\mu(z) = h_z(\mu)$ for any $z \in Z^0$. Let $\bar{U}^\mu = \bar{U}/(\mathrm{Ker}\,\phi_\mu)$ and $U^\mu(\mathfrak{g}_0) = U(\mathfrak{g}_0)/(\mathrm{Ker}\,\varphi_\mu)$. We will prove the following theorem.

Theorem 7.2. *If $\mu \in S'$, then $\bar{U}^\mu$ and $U^\mu(\mathfrak{g}_0)$ are Morita equivalent.*

The proof of this theorem is similar to ones given for examples in Sections 4 and 5. We will construct two functors

$$\mathrm{Inv}: \bar{U}^\mu - \mathrm{mod} \to U^\mu(\mathfrak{g}_0) - \mathrm{mod}$$

$$\mathrm{Ind}: U^\mu(\mathfrak{g}_0) - \mathrm{mod} \to \bar{U}^\mu - \mathrm{mod}$$

which establish the equivalence between the categories of $\bar{U}^\mu$-modules and $U^\mu(\mathfrak{g}_0)$-modules.

Let $p : U_0 \to U(\mathfrak{g}_0)$ be the projection along $U_0 \cap U\mathfrak{g}_1$. If κ is the $\mathfrak{g}_0$-invariant projection $F(\mu) \to M^0(\mu)$ then

$$p(x)\kappa v = \kappa x v \text{ for all } x \in U_0,\ v \in F(\mu).$$

Therefore p induces $\bar{p} : \bar{U}_0 \to \bar{U}(\mathfrak{g}_0)$, where

$$\bar{U}(\mathfrak{g}_0) = U(\mathfrak{g}_0) / \cap_{\nu \in S} \mathrm{Ann}_{U(\mathfrak{g}_0)} M^0(\nu).$$

Note that p maps the space of $\mathrm{ad}\,\mathfrak{g}_0$-invariants of U to Z^0. Therefore $\bar{p}$ maps $\mathrm{ad}\,\mathfrak{g}_0$-invariants of $\bar{U}$ to $\widetilde{Z} = Z^0/\cap_{\nu \in S} \mathrm{Ker}\,\varphi_\nu$. In particular, $\bar{p}(Z(\bar{U})) \subset \widetilde{Z}$. Using the identity $h = h \circ p$ one can easily check that

$$\bar{h}(z) = r \circ h \circ \bar{p}(z) \text{ for any } z \in Z(\bar{U}). \tag{7.1}$$

Recall that r is the restriction homomorphism $S(\mathfrak{h}) \to S(S^*)$.

Lemma 7.3. *Let $\mu \in S'$. Then $\widetilde{Z}\bar{p}(\operatorname{Ker}\phi_\mu) = \operatorname{Ker}\varphi_\mu$.*[5]

Proof. By Lemma 6.9 (a) the map $r \circ h \colon Z^0 \to \mathcal{R}$ is surjective. Obviously

$$(\operatorname{Ker} r \circ h) \cap Z^0 = \cap_{\mu \in S} \operatorname{Ker}\varphi_\mu.$$

Thus $r \circ h : \widetilde{Z} \to \mathcal{R}$ is a well-defined isomorphism. By (7.1) $r \circ h\left(\bar{p}\left(Z\left(\bar{U}\right)\right)\right) = \mathcal{A}$. Let $\mathcal{I}_\mu \subset S(S^*)$ be the maximal ideal of the point $\mu \in S$. Clearly $r \circ h \circ \bar{p}(\operatorname{Ker}\phi_\mu) = \mathcal{I}_\mu \cap A$ and $r \circ h(\operatorname{Ker}\varphi_\mu) = \mathcal{I}_\mu \cap \mathcal{R}$. It suffices to show that $\mathcal{I}_\mu \cap \mathcal{R} = \mathcal{R}(\mathcal{I}_\mu \cap \mathcal{A})$. But this follows from Corollary 6.14. $\square$

Lemma 7.4. *Let $\mu \in S'$ and M be an $\bar{U}^\mu$-module. Let $M^{\mathfrak{g}_1} = \operatorname{Ker}\mathfrak{g}_1$. Then*
(a) $M^{\mathfrak{g}_1}$ is an $U^\mu(\mathfrak{g}_0)$-module.
(b) If $z_0 = \operatorname{diag}(n, 1, \dots, 1)$, $\xi = \mu(z_0)$, then $M^{\mathfrak{g}_1} = \operatorname{Ker}(z_0 - \xi\,\mathrm{id})$.

Proof. (a)For any $z \in Z\left(\bar{U}\right)$, $m \in M^{\mathfrak{g}_1}$ we have $zm = \bar{p}(z)\,m$. Since $\operatorname{Ker}\phi_\mu$ annihilates $M^{\mathfrak{g}_1}$, then $\widetilde{Z}\bar{p}(\operatorname{Ker}\phi_\mu)$ annihilates $M^{\mathfrak{g}_1}$. By Lemma 7.3 $\widetilde{Z}\bar{p}(\operatorname{Ker}\phi_\mu) = \operatorname{Ker}\varphi_\mu$. Thus $\operatorname{Ker}\varphi_\mu$ annihilates $M^{\mathfrak{g}_1}$, i.e. $M^{\mathfrak{g}_1}$ is $U^\mu(\mathfrak{g}_0)$-module.

(b) Since $\varphi_\mu(z_0) = \xi$, we have $M^{\mathfrak{g}_1} \subseteq \operatorname{Ker}(z_0 - \xi\,\mathrm{id})$. On the other hand M is a quotient of $M' = U \otimes_{U(\mathfrak{g}_0 + \mathfrak{g}_1)} M^{\mathfrak{g}_1}$, $\operatorname{Ker}(z_0 - \xi\,\mathrm{id})$ in M' coincides with $M^{\mathfrak{g}_1}$. Therefore $M^{\mathfrak{g}_1} = \operatorname{Ker}(z_0 - \xi\,\mathrm{id})$. $\square$

Corollary 7.5. *Let $\operatorname{Inv}\colon \bar{U}^\mu - \mod \to U^\mu(\mathfrak{g}_0) - \mod$ be the functor defined by $\operatorname{Inv}(M) = M^{\mathfrak{g}_1}$. Then Inv is always faithful and exact for $\mu \in S'$.*

Proof. Faithfulness is obvious, and exactness follows from Lemma 7.4 (b). $\square$

Lemma 7.6. *Let $s : U(\mathfrak{g}_0) \to \bar{U}^\mu$ be the composition of embedding $U(\mathfrak{g}_0) \to U$ and the natural projection $U \to \bar{U}^\mu$. Then $\operatorname{Ker}s \subset \operatorname{Ker}\varphi_\mu$.*

Proof. Let M be some faithful $\bar{U}^\mu$-module. Consider the homomorphism $\pi : \bar{U}^\mu \to \operatorname{End}(M)$. Then

$$\operatorname{Ker}s = \operatorname{Ker}\pi \circ s = \operatorname{Ann}_{U(\mathfrak{g}_0)} M \subset \operatorname{Ann}_{U(\mathfrak{g}_0)} M^{\mathfrak{g}_1} \subset \operatorname{Ker}\varphi_\mu.$$

$\square$

Let $\left(\bar{U}^\mu\right)^+$ denote the image of $U(\mathfrak{g}_0 + \mathfrak{g}_1)$ in $\bar{U}^\mu$ under the natural projection. By the above Lemma every $U^\mu(\mathfrak{g}_0)$-module N has the unique structure of $\left(\bar{U}^\mu\right)^+$-module such that $\mathfrak{g}_1 N = 0$. Let

$$\operatorname{Ind} N = \bar{U}^\mu \otimes_{\left(\bar{U}^\mu\right)^+} N.$$

Then Ind is a faithful functor from $U^\mu(\mathfrak{g}_0) - \mod$ to $\bar{U}^\mu - \mod$.

Lemma 7.7. *Let $\mu \in S'$, N be any $U^\mu(\mathfrak{g}_0)$-module. Then $\operatorname{Inv}(\operatorname{Ind} N) \cong N$.*

[5]Note that since $\mu \in S$, $\varphi_\mu(\cap_{\nu \in S}\operatorname{Ker}\varphi_\nu) = 0$. Therefore φ_μ induces the character on $\widetilde{Z}$.

Proof. Immediately follows from Lemma 7.4 (b). $\square$

Corollary 7.8. *Functors* Inv *and* Ind *establish the equivalence of the category of $\bar{U}^{\mu}$-modules and the category $U^{\mu}(\mathfrak{g}_0)$-modules.*

Corollary 7.8 implies Theorem 7.2. It also implies

Corollary 7.9. *For any $\mu \in S'$, $F(\mu)$ is a faithful $\bar{U}^{\mu}$-module.*

Remark 7.10. Note that in the same way one can prove that the category of right $\bar{U}^{\mu}$-modules and the category of right $U^{\mu}(\mathfrak{g}_0)$-modules are equivalent. Indeed, define functors Rinv: $\mathrm{mod} - \bar{U}^{\mu} \to \mathrm{mod} - U^{\mu}(\mathfrak{g}_0)$ and Rind: $\mathrm{mod} - U^{\mu}(\mathfrak{g}_0) \to \mathrm{mod} - \bar{U}^{\mu}$ in the following way

$$\mathrm{Rinv}(M) = \{m \in M \mid m\mathfrak{g}_{-1} = 0\}, \ \mathrm{Rind}(N) = N \otimes_{\left(\bar{U}^{\mu}\right)^{-}} \bar{U}^{\mu},$$

where $\left(\bar{U}^{\mu}\right)^{-}$ denotes the image of $U(\mathfrak{g}_0 \oplus \mathfrak{g}_{-1})$ under the natural projection $U \to \bar{U}^{\mu}$. Then the functors Rinv and Rind establish equivalence of $\mathrm{mod} - \bar{U}^{\mu}$ and $\mathrm{mod} - U^{\mu}(\mathfrak{g}_0)$.

Let M and N be $\mathfrak{g}_0$-modules. Recall that $F(M,N)$ denotes the maximal subspace of $\mathrm{Hom}(M,N)$ on which $\mathrm{ad}\,\mathfrak{g}_0$ acts locally finitely. The classical result of Joseph (see [8]) claims that $U^{\mu}(\mathfrak{g}_0)$ is isomorphic to $F(M^0(\mu), M^0(\mu))$. The next theorem is an analog of this result for $\bar{U}^{\mu}$.

Theorem 7.11. *Let $\mu \in S'$. Then the natural map $\pi \colon \bar{U}^{\mu} \to F(F(\mu), F(\mu))$ is an isomorphism.*

Proof. Theorem 7.2 implies that $F(\mu)$ is a faithful $\bar{U}^{\mu}$-modules, and therefore π is injective. To show surjectivity note that $F(F(\mu), F(\mu))$ is a left $\bar{U}^{\mu}$-module. Therefore by Theorem 7.2

$$F(F(\mu), F(\mu)) = \mathrm{Ind}(\mathrm{Inv}\, F(F(\mu), F(\mu))) = \mathrm{Ind}\, F\left(F(\mu), M^0(\mu)\right).$$

Therefore it suffices to show that $F(F(\mu), M^0(\mu)) \subset \mathrm{Im}\,\pi$. On the other hand, $F(F(\mu), M^0(\mu))$ is a right $\bar{U}^{\mu}$-module, and hence by Remark 7.10

$$F\left(F(\mu), M^0(\mu)\right) = \mathrm{Rind}\left(\mathrm{Rinv}\ F\left(F(\mu), M^0(\mu)\right)\right).$$

Therefore it suffices to show that Rinv $F(F(\mu), M^0(\mu)) \subset \mathrm{Im}\,\pi$. Let κ denote the $U(\mathfrak{g}_0)$-invariant projection $F(\mu) \to M^0(\mu)$. It is clear that
Rinv $F(F(\mu), M^0(\mu)) = F(M^0(\mu), M^0(\mu))\,\kappa$.

Let $K \in S^{n-1}(\mathfrak{g}_1) \otimes S^{n-1}(\mathfrak{g}_{-1})$ be $\mathrm{ad}\,\mathfrak{g}_0$-invariant element, defined uniquely up to proportionality. One can check easily that for a highest vector $v \in F(\mu)$, Kv is proportional to v and is not zero for $\mu \in S'$. Therefore $\pi(K)$ is proportional to κ and hence

$$F\left(M^0(\mu), M^0(\mu)\right)\kappa = \pi(U(\mathfrak{g}_0)\,K).$$

$\square$

8. Degenerate case

Theorem 7.2 is not true if $\mu \notin S'$. Here we give a generalization which works for all μ.

Let $\mu \in S$, $S_\mu = \{\nu \in S \mid \phi_\nu = \phi_\mu\}$ and $T_\mu = \cup_{w \in W} S_\mu^w$. By Corollary 6.13, $S_\mu = \{\mu\}$ iff $\mu \in S'$.

Recall that $\mu \in \mathfrak{h}^*$ is *integer* if $(\mu, \beta_i) \in \mathbb{Z}$, *dominant* if $(\mu + \rho, \beta_i) > 0$ and *regular* if $(\mu + \rho, \beta_i) \neq 0$ for all $i \leq n-1$. The irreducible highest weight module $L(\mu)$ is finite-dimensional iff μ is integer dominant (see [9]).

Lemma 8.1. *(a) For any $\nu \in S$ there exists $\mu \in S \backslash S_\Theta$ and $k \geq 0$ such that $S_\nu = S_\mu = \{\mu + j\alpha \mid j = 0, \dots, k\}$.*

(b) T_μ contains at most one regular W-orbit.

(c) T_μ contains at most one integer dominant weight.

Proof. (a) easily follows from Corollary 6.13.

(b) Note that all weights from S_Θ are not regular. If S_μ contains more than one element, then as follows from (a) and Corollary 6.13 $S_\mu \backslash \{\mu\} \subset S_\Theta$, and hence all weights from $S_\mu \backslash \{\mu\}$ are not regular. Thus S_μ contains at most one regular weight, and therefore T_μ contains at most one regular orbit.

(c) follows from (b). ☐

Lemma 8.2. *Let $\mu \in S \backslash S_\Theta$.*

(a) $\widetilde{Z}\bar{p}(\operatorname{Ker}\phi_\mu) = \prod_{j=0}^k \operatorname{Ker}\varphi_{\mu+j\alpha}$

(b) If M is a $\bar{U}^\mu$-module then $\operatorname{Inv} M = M_0' \oplus \dots \oplus M_k'$, *where M_j' is a $U^{\mu+j\alpha}(\mathfrak{g}_0)$-module.*

Proof. (a) The proof is similar to the proof of Lemma 7.3. Recall that $r \circ h : \widetilde{Z} \to \mathcal{R}$ is an isomorphism. By Corollary 6.13 and Lemma 8.1 (a)

$$\prod_{j=0}^{k} (\mathcal{I}_{\mu+j\alpha} \cap \mathcal{R}) = \mathcal{R}(\mathcal{I}_\mu \cap \mathcal{A}) = r \circ h\left(\widetilde{Z}\bar{p}(\operatorname{Ker}\phi_\mu)\right).$$

Thus

$$\widetilde{Z}\bar{p}(\operatorname{Ker}\phi_\mu) = (r \circ h)^{-1} \prod_{j=0}^{k} (\mathcal{I}_{\mu+j\alpha} \cap \mathcal{R}) = \prod_{j=0}^{k} \operatorname{Ker}\varphi_{\mu+j\alpha}.$$

(b) follows from (a). ☐

Corollary 8.3. *The irreducible highest weight $\mathfrak{g}$-module $L(\lambda)$ is an $\bar{U}^\mu$-module iff $\lambda \in T_\mu$.*

Proof. Follows from Lemma 8.2 (b) and $\operatorname{Inv} L(\lambda) = L^0(\lambda)$. ☐

Corollary 8.4. *The category of $\bar{U}^\mu$-modules has at most one up to an isomorphism irreducible finite dimensional module.*

Let $\mu \in S \backslash S_\Theta$. Put $z_\mu = \prod_{i=0}^{n-1} (z_0 - \mu(z_0) - i)$ and $Y^\mu = \bar{U}^\mu / (z_\mu)$.

Lemma 8.5. *(a)If $\mu \in S'$, then $Y^\mu = \bar{U}^\mu$.*

(b)If $\mu \in S \backslash S_\Theta$ and M is a Y^μ-module, then $\operatorname{Inv} M = \operatorname{Ker}(z_0 - \mu(z_0) \operatorname{id})$ *and* $\operatorname{Inv} M$ *is a $U^\mu(\mathfrak{g}_0)$-module.*

Proof. (a) If $\mu \in S'$, then $\operatorname{Inv} M$ is a $U^\mu(\mathfrak{g}_0)$-module, and $M \cong \operatorname{Ind}(\operatorname{Inv} M)$. Since $(z_0 - \mu(z_0)) \operatorname{Inv} M = 0$, $z_\mu M = 0$.

(*b*) By Lemma 8.2 (*b*) $\operatorname{Inv} M = M_0' \oplus \dots \oplus M_k'$. But $z_\mu(M_j') \neq 0$ for $j > 0$. Hence $\operatorname{Inv} M = M_0'$ is a $U^\mu(\mathfrak{g}_0)$-module. Clearly, $M_0' = \operatorname{Ker}(z_0 - \mu(z_0) \operatorname{id})$. □

One can see immediately from Lemma 8.5 (*b*) that $\operatorname{Inv}\colon Y^\mu - \operatorname{mod} \to U^\mu(\mathfrak{g}_0) - \operatorname{mod}$ is an exact faithful functor. One can define a right adjoint functor $\operatorname{Ind}\colon U^\mu(\mathfrak{g}_0) - \operatorname{mod} \to Y^\mu - \operatorname{mod}$ by putting

$$\operatorname{Ind} N = Y^\mu \otimes_{(Y^\mu)^+} N,$$

where $(Y^\mu)^+$ is the image of $U(\mathfrak{g}_0 \oplus \mathfrak{g}_1)$ in Y^μ and $(Y^\mu)^+$-action on N is induced by $\mathfrak{g}_1 N = 0$.

Lemma 8.2 (*b*) implies $\operatorname{Inv}(\operatorname{Ind} N) = N$ and therefore the following theorem is true.

Theorem 8.6. *Let $\mu \in S \backslash S_\Theta$. Then Y^μ is Morita equivalent to $U^\mu(\mathfrak{g}_0)$. Functors* Inv *and* Ind *establish the equivalence between the category of Y^μ-modules and the category of $U^\mu(\mathfrak{g}_0)$-modules.*

9. Category $\mathcal{O}$ and character formulae

In this section we obtain some formulae for characters of irreducible highest weight $sl(1|n)$-modules $L(\lambda)$ (possibly infinite-dimensional). Note that if λ is typical then

$$L(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0 \oplus \mathfrak{g}_1)} L^0(\lambda),$$

and therefore

$$\operatorname{ch} L(\lambda) = \operatorname{ch} L^0(\lambda) \prod_{\alpha \in \Delta(\mathfrak{g}_1)} \left(1 + e^{-\alpha}\right)$$

We assume now that λ is atypical. Then $L(\lambda)$ is an $\bar{U}$-module and therefore an $\bar{U}^\mu$-module for some $\mu \in S \backslash S_\Theta$. We are able to obtain a formula for $\operatorname{ch} L(\lambda)$ if $\lambda = \mu^w$. For example this formula works for arbitrary regular λ.

Denote by $\mathcal{O}(\mathfrak{g}_0)$ the category of $\mathfrak{g}_0$-modules, which are finitely generated, $\mathfrak{h}$-semisimple and locally finite over $\mathfrak{b}_0$ (see [2]). Let $\mathcal{O}^\mu(\mathfrak{g})$ denote the category of Y^μ-modules which belong to the category $\mathcal{O}(\mathfrak{g}_0)$ as $\mathfrak{g}_0$-modules. Finally, let $\mathcal{O}^\mu(\mathfrak{g}_0)$ be the subcategory of $\mathcal{O}(\mathfrak{g}_0)$ admitting central character φ_μ. The following lemma easily follows from Theorem 8.6.

Lemma 9.1. *If $\mu \in S \backslash S_\Theta$ then $\mathcal{O}^\mu(\mathfrak{g})$ and $\mathcal{O}^\mu(\mathfrak{g}_0)$ are equivalent. For any $\lambda = \mu^w$,* $\operatorname{Ind} L^0(\lambda) = L(\lambda)$.

If $\mu \in S \backslash S_\Theta$, then $\operatorname{Inv} F(\mu) = M^0(\mu)$ and $\operatorname{Ind} M^0(\mu) = F(\mu)$. Let now $\lambda = \mu^w$ and $\alpha' = w(\alpha)$[6]. Let $\mathfrak{b}'$ be the Borel subalgebra such that $\mathfrak{b}_0 \subset \mathfrak{b}'$ and α' is a simple root. Let $\mathfrak{p}' = \mathfrak{b}' \oplus \mathfrak{g}_{-\alpha'}$ and $\lambda' = \lambda - \sum_{\beta \in \Delta(\mathfrak{g}_1) \backslash \Delta(\mathfrak{p}'_1)} \beta$. We put

$$F(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}')} C_{\lambda'}.$$

Let us start with calculating the character of $F(\lambda)$. By Lemma 3.1

$$\operatorname{ch} F(\lambda) = \operatorname{ch} M^0(\lambda') \prod_{\beta \in \Delta(\mathfrak{p}'_1)} \left(1 + e^{-\beta}\right).$$

Furthermore

$$\operatorname{ch} M^0(\lambda') = e^{\lambda' + \rho_0} \prod_{\beta \in \Delta(\mathfrak{b}_0)} \left(e^{\beta/2} - e^{-\beta/2}\right),$$

and using

$$\prod_{\beta \in \Delta(\mathfrak{p}'_1)} \left(1 + e^{-\beta}\right) = \frac{\prod_{\beta \in \Delta(\mathfrak{b}'_1)} \left(1 + e^{-\beta}\right)}{\left(1 + e^{-\alpha'}\right)}$$

and

$$\prod_{\beta \in \Delta(\mathfrak{b}'_1)} \left(1 + e^{-\beta}\right) = e^{\rho_{\mathfrak{b}'} - \rho_0} \prod_{\beta \in \Delta(\mathfrak{b}'_1)} \left(e^{\beta/2} + e^{-\beta/2}\right)$$

we obtain

$$\operatorname{ch} F(\lambda) = \frac{e^{\lambda' + \rho_{\mathfrak{b}'}} \prod_{\beta \in \Delta(\mathfrak{b}'_1)} \left(e^{\beta/2} + e^{-\beta/2}\right)}{\left(1 + e^{-\alpha'}\right) \prod_{\beta \in \Delta(\mathfrak{b}_0)} \left(e^{\beta/2} - e^{-\beta/2}\right)}.$$

Finally using $\lambda' + \rho_{\mathfrak{b}'} = \lambda + \rho$, we obtain

$$\operatorname{ch} F(\lambda) = \frac{e^{\lambda + \rho} \prod_{\beta \in \Delta(\mathfrak{g}_1)} \left(e^{\beta/2} + e^{-\beta/2}\right)}{\left(1 + e^{-\alpha'}\right) \prod_{\beta \in \Delta(\mathfrak{b}_0)} \left(e^{\beta/2} - e^{-\beta/2}\right)}. \tag{9.1}$$

Lemma 9.2. *If $\mu \in S \backslash S_\Theta$ and $\lambda = \mu^w$, then $F(\lambda) \cong \operatorname{Ind} M^0(\lambda)$.*

Proof. Note that $F(\lambda)$ is a $\mathfrak{b}'$-highest weight module, i.e. it is generated by a $\mathfrak{b}'$-highest vector v of weight λ'. Let

$$u = \prod_{\beta \in \Delta(\mathfrak{g}_1) \backslash \Delta(\mathfrak{p}'_1)} X_\beta v, \; v' = \prod_{\beta \in \Delta(\mathfrak{g}_1) \backslash \Delta(\mathfrak{p}'_1)} X_{-\beta} u.$$

[6]Since $\mu \notin S_\Theta$, α' is defined uniquely.

Then u is $\mathfrak{b}$-highest vector, v' is proportional to v and the condition $(\lambda+\rho,\beta)\neq 0$ for $\beta\neq\alpha'$ insures that $v'\neq 0$. Therefore u also generates $F(\lambda)$, and $F(\lambda)$ is a $\mathfrak{b}$-highest weight module with highest weight λ. Since λ is atypical, $F(\lambda)$ is a $\bar{U}$-module. Let

$$F'(\lambda)=\left\{m\in F(\lambda)\mid zm=\phi_\mu(z)\,m, z\in Z\left(\bar{U}\right)\right\}.$$

Then $F'(\lambda)$ is $\mathfrak{g}$-invariant and $u\in F'(\lambda)$. Therefore $F'(\lambda)=F(\lambda)$. Hence $F(\lambda)$ is a $\bar{U}^\mu$-module. It follows from (9.1) that $z_\mu(F(\lambda))=0$. Thus $F(\lambda)$ is a Y^μ-module. Furthermore,

$$\operatorname{Inv}F(\lambda)=\operatorname{Ker}(z_0-\mu(z_0)\operatorname{id})=\operatorname{Ker}(z_0-\lambda(z_0)\operatorname{id})=M^0(\lambda).$$

Therefore $\operatorname{Ind}M^0(\lambda)=F(\lambda)$ by Theorem 8.6. $\square$

Let M be an irreducible module, and N be a module with finite Jordan-Holder series. Then $[N:M]$ denotes the multiplicity of M in N. Lemma 9.2 and Theorem 8.6 imply that for any $\mu\in S\backslash S_\Theta$ and $\lambda,\nu\in\mu^W$

$$[F(\lambda):L(\nu)]=\left[M^0(\lambda):L^0(\nu)\right].$$

The last formula implies

Corollary 9.3. *If $\mu\in S\backslash S_\Theta$, $\lambda=\mu^w$ and*

$$\operatorname{ch}L^0(\lambda)=\sum_\nu a_{\lambda,\nu}\operatorname{ch}M^0(\nu),$$

then

$$\operatorname{ch}L(\lambda)=\sum_\nu a_{\lambda,\nu}\operatorname{ch}F(\nu).$$

10. Geometric realization of $\bar{U}$ and localization theorem

First, we assume that $\mathfrak{g}=sl(m|n)$ for arbitrary m and n and keep the notations of Sections 2 and 3. One can see immediately that $\mathfrak{p}/[\mathfrak{p},\mathfrak{p}]$ is isomorphic to $\bar{\mathfrak{h}}=\mathfrak{h}/(\mathfrak{h}\cap[\mathfrak{p},\mathfrak{p}])$. There is the canonical isomorphism identifying $\bar{\mathfrak{h}}^*$ with S.

Consider the algebraic flag supervariety $X=G/P$, where $P\subset G$ is a supergroup corresponding to Lie supealgebra $\mathfrak{p}$. For definition and properties of flag supervarieties see [11, 14] and [13]. Let $\mathfrak{p}_x\subset\mathfrak{g}$ be the Lie superalgebra of the stabilizer of the point $x\in X$.

Let $\mathcal{O}_X$ be the structure sheaf on X, $\mathcal{T}_X$ be the sheaf of vector fields on X and $\mathcal{D}_X$ be the sheaf of differential operators. The action of G on X defines the natural homomorphism $a:\mathfrak{g}\to\mathcal{T}_X$. Define a Lie algebra sheaf $\mathfrak{g}_X$ and a ring sheaf U_X by

$$\mathfrak{g}_X=\mathcal{O}_X\otimes\mathfrak{g},\ U_X=\mathcal{O}_X\otimes U,\ [g,f]=a(g)\,f\text{ for }f\in\mathcal{O}_X,\ g\in\mathfrak{g}.$$

Let

$$\mathcal{N}_X=\{\xi\in\mathfrak{g}_X\mid\xi_x\in[\mathfrak{p}_x,\mathfrak{p}_x]\text{ for all }x\in X\},$$

and put $\bar{\mathcal{D}}_X=U_X/\mathcal{N}_X$.

Lemma 10.1. *The sheaf $\bar{\mathcal{D}}_X$ is locally isomorphic to $S\left(\bar{\mathfrak{h}}\right)\otimes\mathcal{D}_X$.*

Proof. It is sufficient to show that $\bar{\mathcal{D}}_X(V)\cong S\left(\bar{\mathfrak{h}}\right)\otimes\mathcal{D}_X(V)$ for a big Schubert cell $V\subset X$. The natural map $u\colon\bar{\mathcal{D}}_X(V)\to S\left(\bar{\mathfrak{h}}\right)\otimes\mathcal{D}_X(V)$ is obviously injective. To prove its surjectivity consider the $\mathfrak{h}$-invariant Lie sublgebra $\mathfrak{m}$ such that $\mathfrak{g}=\mathfrak{m}\oplus\mathfrak{p}$. Since $\mathfrak{m}$ acts transitively on V and the map $\mathcal{O}_X(V)\otimes U(\mathfrak{m})\to\mathcal{D}_X(V)$ is surjective. Then the map $\mathcal{O}_X(V)\otimes U(\mathfrak{m}+\mathfrak{h})\to S\left(\bar{\mathfrak{h}}\right)\otimes\mathcal{D}_X(V)$ is also surjective, and that implies the surjectivity of u. □

Consider now two natural maps

$$\gamma\colon U\to\Gamma\left(\bar{\mathcal{D}}_X\right)\text{ and }\tau:S\left(\bar{\mathfrak{h}}\right)\to\Gamma\left(\bar{\mathcal{D}}_X\right).$$

Lemma 10.2. $\operatorname{Ker}\gamma=J$.

Proof. Let M be the irreducible $\bar{\mathcal{D}}_X(V)$-module with support at $x\in X$ such that $\mathfrak{p}_x=\mathfrak{p}$. The map γ defines a $\mathfrak{g}$-module structure on M and one can check that

$$M\cong U\otimes_{U(\mathfrak{p})}\left(S\left(\bar{\mathfrak{h}}\right)\otimes C_{-\sigma}\right),$$

where $\sigma=2\rho+\alpha_1+\cdots+\alpha_k$. Since M is a faithful $\bar{\mathcal{D}}_X(V)$-module

$$\operatorname{Ker}\gamma=\operatorname{Ann}_U M=\cap_{\mu\in S}\operatorname{Ann}_U F(\mu)=J.$$

□

Let $\bar{\gamma}$ be the natural injection $\bar{U}\to\Gamma\left(\bar{\mathcal{D}}_X\right)$. One can see that $\bar{\gamma}\left(Z\left(\bar{U}\right)\right)\subset\tau\left(S\left(\bar{\mathfrak{h}}\right)\right)$ and that

$$\bar{\gamma}(z)=\tau\left(\bar{h}_z\right)\text{ for any }z\in Z\left(\bar{U}\right),\tag{10.1}$$

where $\bar{h}:Z\left(\bar{U}\right)\to S\left(\bar{\mathfrak{h}}\right)$ is the Harish-Chandra homomorphism. Hence it is possible to define the homomorphism $\widetilde{\gamma}:S\left(\bar{\mathfrak{h}}\right)\otimes_{Z\left(\bar{U}\right)}\bar{U}\to\Gamma\left(\bar{\mathcal{D}}_X\right)$.

Now take an arbitrary $\mu\in S$. Let $\mathcal{I}_\mu\subset S\left(\bar{\mathfrak{h}}\right)$ be the maximal ideal of $\mu\in S$, and $\mathcal{D}_X^\mu$ be the quotient sheaf $\bar{\mathcal{D}}_X/\tau\left(\mathcal{I}_{\mu+\sigma}\right)\bar{\mathcal{D}}_X$. It is obvious that $\mathcal{D}_X^\mu$ is a twisted differential operators sheaf on X, i.e. $\mathcal{D}_X^\mu$ is locally isomorphic to $\mathcal{D}_X$.

For the rest of the section we assume that $\mathfrak{g}=sl(1|n)$. It is clear that the homomorphism $\widetilde{\gamma}$ can be pushed down to

$$\widetilde{\gamma}_\mu\colon\bar{U}^\mu\to\Gamma\left(\mathcal{D}_X^\mu\right).$$

Theorem 10.3. *Let $\mathfrak{g}=sl(1|n)$ and $\mu\in S'$. Then $\bar{U}^\mu$ is isomorphic to $\Gamma\left(\mathcal{D}_X^\mu\right)$.*

Proof. We want to show that $\widetilde{\gamma}_\mu$ is an isomorphism. Let M be the irreducible $\mathcal{D}_X^\mu(V)$-module with the support at $x\in V$ such that $\mathfrak{p}_x=\mathfrak{p}$. Denote by θ the homomorphism $\Gamma\left(\mathcal{D}_X^\mu\right)\to\operatorname{End}(M)$. Since M is a faithful $\Gamma\left(\mathcal{D}_X^\mu\right)$-module, $\operatorname{Ker}\theta=0$. The composition $\theta\circ\widetilde{\gamma}_\mu$ induces a $\bar{U}^\mu$-module structure on M. It is clear that M is isomorphic to $F(\mu)$ as $\bar{U}^\mu$-module. By Corollary 7.9 $F(\mu)$ is a faithful $\bar{U}^\mu$-module, and hence $\widetilde{\gamma}_\mu$ is injective. On the other hand, $\Gamma\left(\mathcal{D}_X^\mu\right)$ is a G-module, and therefore

$\theta\left(\Gamma\left(\mathcal{D}_X^{\mu}\right)\right) \subseteq F\left(M, M\right)$. Now using Theorem 7.11 we obtain $\theta \circ \widetilde{\gamma}_{\mu}\left(\bar{U}^{\mu}\right) = F\left(M, M\right)$. Thus, both θ and $\widetilde{\gamma}_{\mu}$ are isomorphisms. □

Furthermore, in case of $sl\left(1|n\right)$ one can formulate Beilinson-Bernstein localization theorem for atypical modules.

Corollary 10.4. *Let $\mathfrak{g} = sl\left(1|n\right)$, $\mu \in S'$ and $\left(\mu, \beta_i\right) \notin \mathbb{Z}^+$. Then the category of $\bar{U}^{\mu}$-modules is equivalent to the category of $\mathcal{D}_X^{\mu}$-modules.*

Proof. Follows from Beilinson-Bernstein theorem [1] and Theorem 7.2. □

Finally note that, in case when $\mu \in S$ is integer, $\mathcal{D}_X^{\mu}$ is the sheaf of differential operators on some invertible sheaf $\mathcal{O}_X^{\mu}$. It is interesting that the analog of Borel-Weil-Bott theorem on X holds even if $\mu \notin S'$.

Let $\mu \in S$ be some integer weight, and $\mathcal{O}\left(\mu\right)$ be the sheaf of sections of the linear bundle $G/P \otimes_P C_{-\mu}$ on X. Then the following theorem is true.

Theorem 10.5. *(a) If S_{μ} does not contain regular weights, then $H^i\left(X, \mathcal{O}\left(\mu\right)\right) = 0$ for all i.*

(b) Let $\lambda \in T_{\mu}$ be the unique dominant weight and $\mu = \lambda^w + j\alpha$ for some $j \geq 0$ and $w \in W$, then $H^i\left(X, \mathcal{O}\left(\mu\right)\right) = 0$ for for all $i \neq l\left(w\right) + j$ and $H^{l(w)+j}\left(X, \mathcal{O}\left(\mu\right)\right) \cong L\left(\lambda\right)^*$.

Proof. (a) Note that $H^i\left(X, \mathcal{O}\left(\mu\right)\right)^*$ is a finite dimensional $\bar{U}^{\mu}$-module. If S_{μ} does not contain regular weights, then $\bar{U}^{\mu}$ does not have finite-dimensional modules, and therefore the cohomology must vanish.

(*b*) Since all $H^i\left(X, \mathcal{O}\left(\mu\right)\right)^*$ are finite-dimensional, Corollary 8.4 implies that $H^i\left(X, \mathcal{O}\left(\mu\right)\right)^*$ is completely reducible and must be isomorphic to a direct sum of several copies of $L\left(\lambda\right)^*$.

Let $X_0 = G_0/B_0$ be the underlying flag variety and $\mathcal{O}_{X_0}\left(\upsilon\right)$ denotes the invertible sheaf induced by the infinitesimal character $-\upsilon$ on X_0. The structure sheaf $\mathcal{O}_X$ has a filtration by $\mathcal{O}_{X_0}$-modules with irreducible quotients $\mathcal{O}_{X_0}\left(\gamma\right)$ where γ runs the set of weights of $\Lambda\left(\mathfrak{m}_1\right)$. (We recall that $\mathfrak{m}$ is the $\mathfrak{h}$-invariant subalgebra such that $\mathfrak{m} \oplus \mathfrak{p} = \mathfrak{g}$). This filtration induces the filtration of $\mathcal{O}_X$-module $\mathcal{O}\left(\mu\right)$ with quotients $\mathcal{O}_{X_0}\left(\mu + \gamma\right)$. Thus, by usual Borel-Weil-Bott theorem for X_0, all $\mathfrak{g}_0$-components which appear in $H^i\left(X, \mathcal{O}\left(\mu\right)\right)^*$ have highest weight $\left(\mu + \gamma\right)^u$, where $u \in W$ with $l\left(u\right) = i$. Thus, we have to find all $u \in W$ and γ such that

$$\mu + \gamma = \lambda^u.$$

Substituting $\mu = \lambda^w + j\alpha$ we obtain $\lambda^w + j\alpha + \gamma = \lambda^u$, or $\lambda + w^{-1}\left(j\alpha + \gamma\right) = \lambda^{w^{-1}u}$. Let $s = w^{-1}u$. Then $\lambda - \lambda^s = w^{-1}\left(j\alpha + \gamma\right)$. That implies

$$w^{-1}\left(j\alpha + \gamma\right) = j\beta_i + \left(j-1\right)\beta_{i+1} + \cdots + \beta_{i+j-1},\ s = \left(i+j-1, \ldots, i\right).$$

Then $l\left(s\right) = j$, $l\left(u\right) = l\left(w\right) + j$, that implies (*b*). □

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