Hilbert 4-th Problem, Radon Transform and Symplectic Geometry

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1. Introduction.

The Hilbert 4th problem [H] deals with the description of geometries in which geodesics are straight lines. The problem was entitled as "the problem of lines as shortest curves between two points." Originally formulated quite broadly, this problem can be split into several parts. Some parts were solved as soon as 1903 by H.Hamel [Ha], followed by the work of P.Funk in 1930 [Fu], J.Douglas [D] in 1942, and finally by a series of works by H.Busemann [Bu] on the geometry of geodesics. Later appeared a book by A.V.Pogorelov exclusively devoted to the Hilbert 4-th problem.

The Hilbert problem and its generalization that we deal with can be formulated as follows. To describe all Finsler metrics [1]) or more general Lagrangians whose extremals are straight lines. More generally this problem can be formulated in this form: to describe all k-dimensional Lagrangians in $\mathbf{R}^n$ such that all k-dimensional planes are their extremals (of course they have other extremals too).

The solution that is given here uses the Radon transform connected with a pair of Grassmannians (section 6).

First, we describe Finsler metrics in $\mathbf{R}^n$ whose extremals are straight lines. They are solutions of a system of PDE's and they can also be described as the image under the Radon transform of positive measures on the space of hyperplanes in $\mathbf{R}^n$.

We describe all k-dimensional Lagrangians in $\mathbf{R}^n$ that have all k-dimensional planes among their extremals. These Lagrangians are solutions of a system of PDE's and this system coincides with the lower order terms of the Euler-Lagrange equations.

These Lagrangians are the images under the Radon transform of the measures on the space of $(n-k)$-planes in $\mathbf{R}^n$. This Radon transform, first

[1]) Let us remind the definition of a Finsler metric. It is a function $F(x,v)$ of a point x in the domain in $\mathbf{R}^n$ and a tangent vector $v \in T_x$ that is smooth in x, v, that is positive homogeneous in v: $F(x, \lambda v) = \lambda F(x,v)$ for all $\lambda \in \mathbf{R}$, and such that the indicatrix of F defined as $Ind_x F = \{v \in T_x | F(x,v) = 1\}$ is a twice differentiable closed *convex!* hypersurface in T_x. The Riemannian metric is a special example of a Finsler metric when $F(x,v) = \sqrt{quadratic\ form\ in\ v}$

defined in [GS1] is different from the classical Radon transform, but a system of PDE's for k-Lagrangians can be interpreted as the standard F.John's system of PDE's describing the image of this special Radon transform.

The k-dimensional Lagrangians introduced here are analogues of closed differential forms. Other analogues of differential forms and De Rham complex were studied by M. Baranov and A.S.Shvarts [BS].

2. Even and Odd Densities and the Crofton Formula.

Consider in $\mathbf{R}^n$ a k-dimensional manifold M^k. Suppose M^k has the following parametrization: $M^k = \{x(s_1, \ldots, s_k) \in \mathbf{R}^n$, where $s \in \Omega^k \in \mathbf{R}^k\}$ We want a functional

$$S[M^k] = \int_{\Omega^k} L(x(s), \tfrac{\partial x}{\partial s_1}, \ldots, \tfrac{\partial x}{\partial s_k}) ds_1 \ldots ds_k$$

to be independent on the parametrization of M^k. Then after a change of parametrization $s \to t$, L must change in the following way

$$L(x(t), \tfrac{\partial x}{\partial t_1}, \ldots, \tfrac{\partial x}{\partial t_k}) = \det(\tfrac{\partial s_i}{\partial t_j}) L(x(s), \tfrac{\partial x}{\partial s_1}, \ldots, \tfrac{\partial x}{\partial s_k}) \quad \text{or}$$

$$L(x(t), \tfrac{\partial x}{\partial t_1}, \ldots, \tfrac{\partial x}{\partial t_k}) = \left|\det(\tfrac{\partial s_i}{\partial t_j})\right| L(x(s), \tfrac{\partial x}{\partial s_1}, \ldots, \tfrac{\partial x}{\partial s_k}).$$

In the first case L is called an odd k-density, in the second case L is called an even k-density. The main example of odd k-densities are differential k-forms. An example of an even k density is a k-dimensional volume element.

Now let us explain what is a Crofton density or Crofton Lagrangian. It is an even density that has some additional properties. Let us replace a manifold M^k by its Crofton function. It is a function on the set $H_{n,n-k}$ of $(n-k)$-planes in $\mathbf{R}^n$. The value of the function $\mathrm{Crof}_{M^k}(\xi)$ on the $(n-k)$-plane ξ is equal to the number of intersection points of M^k and ξ. Crofton functions carry almost all information about the original manifold M^k.

Crofton Lagrangians (or Crofton densities) are such densities for which the Crofton formula is valid: i.e. there exists a measure $\mu(\xi)d\xi$ on the set $H_{n,n-k}$ of $(n-k)$-planes such that for every manifold $M^k \in \mathbf{R}^n$

$$\int_{M^k} L = \int_{H_{n,k}} \mathrm{Crof}_{M^k}(\xi)\mu(\xi)d\xi.$$

For example, according to the classical Crofton formula of the integral geometry an element of k-dimensional nonoriented volume in $\mathbf{R}^n$ is an even

Crofton k-density. A Crofton density depends on the measure $\mu(\xi)d\xi$ on $H_{n,n-k}$. We shall write an explicit formula for a Crofton density by means of a special kind of $(n(n-k))$-dimensional Radon transform of the measure μ.

Example. Consider an even 1-density L in $\mathbf{R}^2$ given by the formula

$$L(x_1, x_2; v_1, v_2) = \frac{2(x_1v_1 + x_2v_2)^2 - (v_1^2 + v_2^2)(x_1^2 + x_2^2 + 1)}{(x_1^2+x_2^2+1)^2((v_1^2+v_2^2)(x_1^2+x_2^2+1)-(x_1v_1+x_2v_2)^2)^{\frac{1}{2}}}.$$

Let us parametrize lines in $\mathbf{R}^2$ by the equation $x_2 = ax_1 + b$. In coordinates a, b the corresponding dual measure $\mu(a, b)dadb$ is

$$\mu(a,b) = \frac{1}{2}|v_1|\frac{\partial^2}{\partial v_2^2}L(x_1,\ x_2;\ v_1, v_2)\ \Big|_{\substack{v_2 = av_1 \\ x_2 = ax_1 + b}} = \frac{-1 - a^2 + 2b^2}{2(1 + a^2 + b^2)^{\frac{5}{2}}}.$$

The extremals of the problem $\int L(x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t))dt$ are straight lines.

3. System of PDE for Lagrangians solving Hilbert's Problem.

Theorem 1. *If a function $L(x;\ \mathbf{v}_1, ..., \mathbf{v}_k)$ of a point $x \in \mathbf{R}^n$ and k-tangent vectors $\mathbf{v}_i = (v_i^1, ..., v_i^n) \in T_x\mathbf{R}^n$ is an even Crofton density then it satisfies for all x and all $\mathbf{v}_1, ..., \mathbf{v}_k$: $|\mathbf{v}_1| + ... + |\mathbf{v}_k| \neq 0$ the following equations:*

$$\left(\sum_{s=1}^{n} v_i^s \frac{\partial}{\partial v_j^s}\right)L = \delta_{ij}L, \qquad i, j = 1, \ldots, k, \tag{1}$$

and $$\left(\sum_{p=1}^{k}\sum_{i=1}^{n} v_p^i \frac{\partial^2}{\partial v_p^l \partial x^i}\right)L = \frac{\partial}{\partial x^l}L, \qquad l = 1, \ldots, k. \tag{2}$$

If L is a function of a point $x \in \mathbf{R}^n$ and k-tangent vectors $\mathbf{v}_i = (v_i^1, ..., v_i^n) \in T_x\mathbf{R}^n$ which is even in $\mathbf{v}_i$ and satisfies (1) and (2) then it is a Crofton even k-density.

The density L defines a functional

$$S[M^k] = \int_{\Omega^k} L(x(t), \tfrac{\partial x}{\partial t_1}, \ldots, \tfrac{\partial x}{\partial t_k})dt_1 \ldots dt_k.$$

We want to find its extremal k dimensional surfaces. Let us write general Euler-Lagrange equations for k-dimensional extremals $x^m(t^1, \ldots, t^k)$ of the functional S. They are

$$\left(-\frac{\partial}{\partial x^l}+\sum_{p=1}^{k}\sum_{i=1}^{n}\frac{\partial x^i(t)}{\partial t^p}\frac{\partial^2}{\partial v_p^l\partial x^i}+\sum_{p=1}^{k}\sum_{j=1}^{k}\sum_{i=1}^{n}\frac{\partial^2 x^m(t)}{\partial t^j\partial t^p}\frac{\partial^2}{\partial v_p^s\partial v_j^i}\right)L\left(x(t),\tfrac{\partial x}{\partial t_1},\ldots\right.$$

$$\left.\ldots,\tfrac{\partial x}{\partial t_k}\right)=0,$$

where $s=1,\ldots,k,\ m=1,\ldots,n$.

Theorem 2. *Differential operator* (2) *in the equations for a Crofton k-density coincide with the terms of Euler-Lagrange operator for L which do not contain second derivatives in v:*

$$-\frac{\partial}{\partial x^l}+\sum_{p=1}^{k}\sum_{i=1}^{n}v_p^i\frac{\partial^2}{\partial v_p^l\partial x^i} \tag{3}$$

From the last theorem it is possible to deduce

Theorem 3. *An even k-density satisfies the Crofton formula if and only if all k-planes in $\mathbf{R}^n$ are contained among its extremals.*

4. Finsler metrics in $\mathbf{R}^n$ whose extremals are sraight lines.

We consider 1-Lagrangians $L(x,\dot{x})$ that are even densities. If the indicatrix $Ind_xL=\{v\in T_x : L(x,v)=1\}$ is convex (and twice differentiable) for all x then this Lagrangian is a *Finsler metric*.

Consider an even 1-density for which the Crofton formula is valid. It depends on a measure on the space of hyperplanes in $\mathbf{R}^n$ and can be expressed through this measure by the Radon transform (see section 5 and [GS1]).

Proposition 5. *The indicatrix of the Crofton 1-density with the positive dual function is strictly convex.*

So such density is a Finsler metric whose geodesics are straight lines. For 1-densities the equations for Lagrangians whose geodesics are straight lines can be written in a particularly simple form. One can find them already in the work of Hamel [Ha] and Funk [Fu]. strangely enough, P.Funk who along with Radon is the "father of the Radon transform" did not notice how the Radon transform can be applied to the Hilbert's 4-th problem, although he himself worked on that problem. H.Busemann [Bu2] noticed the connection between the Hilbert's 4-th problem and the classical integral geometry in the sense of Buffon, Crofton, Poincaré and Chern, but not to the Radon transform.

Now let us write the equations for Crofton 1-density.

Theorem 4. *An even 1-density $L(x,v)$ satisfies the Crofton formula if and only if*

$$\frac{\partial^2 L}{\partial x_i \partial v_j} - \frac{\partial^2 L}{\partial x_j \partial v_i} = 0, \quad \text{for} \quad \textit{all} \quad i,j = 1,\dots,n, \quad \textit{and } v \neq 0.$$

This is equivalent to the property that all extremals of L are straight lines.

Remark. These equations can be deduced from equations (2) of the theorem 1 using the homogeneity equations (1).

5. Radon transform for a pair of Grassmannians $G_{n+1,n}$ and $G_{n+1,k}$ and k-Lagrangians.

The Radon transform in the general situation maps a geometric object on the source space (function, differential form, connection) into integrals of this object over some family of submanifolds in the source space. So it is a mapping from geometric objects on the source space into geometric objects on a target space that is the family of submanifolds in the source space.

Let us define now the Radon transform that is used here. Let $\rho(\zeta)$ be a function on a manifold $H_{n,n-k}$ of $(n-k)$-planes in $\mathbf{R}^n$. To each $x \in \mathbf{R}^n$ we associate a variety $H_x = \{\zeta \in H_{n,n-k} | \zeta \text{ passes through } x\}$. To every pair (x, ℓ) where $x \in \mathbf{R}^n$, and ℓ is a k-subspace of a tangent space T_x we associate a pair: a variety H_x and a measure σ_ℓ on H_x which is a $k(n-k)$-dimensional subvariety in $H_{n,n-k}$.

The even Radon transform is given by

$$\phi(x,\ell) = \int\limits_{H_x} \rho(\zeta) d\sigma_\ell.$$

We can write now an explicit formula expressing a Crofton density L in terms of μ using the even Radon transform connected with a pair of Grassmann manifolds $G_{n+1,n}$ and $G_{n+1,k}$. This transform will be defined below. We shall do it in several steps.

1. We regard $\mathbf{R}^n_x$ as an affine part of $G_{n+1,n}$.

2. We go from the measure μ on $H_{n,n-k}$ to the function $\tilde{\mu}$ on a frame manifold $E_{n+1,k}$.

3. For every $x \in \mathbf{R}^n$ and k-tangent vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ in the point x we construct a differential $k(n-k)$-form Ω on the frame manifold $E_{n,k}$. Actually this form is constructed using not x but vectors $\mathbf{u}_1, \dots, \mathbf{u}_n$ which spans n-space from $G_{n+1,n}$ corresponding to the point x.

4. The $k(n-k)$-form Ω can be pulled down from the frame manifold $E_{n,k}$ which is a bundle over $G_{n,k}$ to a Grassmann manifold $G_{n,k}$ and

$$L(x,v) = \int_{G_{n,k}} \Omega_*.$$

Let us repeat this construction in details.

1. Compactification of $\mathbf{R}^n$ and $H_{n,n-k}$ Let us consider $\mathbf{R}^n_x$ as an affine part of $G_{n+1,n}$. Then the set of all $(n-k)$ planes in $\mathbf{R}^n_x$, which we have denoted by $H_{n,n-k}$, can be compactified to the Grassmannian $G_{n+1,n-k+1}$, which is canonically isomorphic to $G_{n+1,k}$. Consider two manifolds of frames: $E_{n+1,k}$, the manifold of k-frames in $\mathbf{R}^{n+1}$ and $E_{n+1,n}$, the manifold of n-frames in $\mathbf{R}^{n+1}$. They are bundles over $G_{n+1,k}$ and $G_{n+1,n}$ correspondingly.

2. Construction of $\tilde{\mu}$. To every measure $\mu(\zeta)d\zeta$ on $H_{n,n-k} \subset G_{n+1,n-k+1}$ we correspond in a canonical way a function $\tilde{\mu}(\mathbf{w})$ of a k-frame $\mathbf{w} = (\mathbf{w}_1, \ldots, \mathbf{w}_k) \in E_{n+1,k}$ that has the property

$$\tilde{\mu}(A\mathbf{w}) = |\det A|^{-n-1}\tilde{\mu}(\mathbf{w}), \text{ for } \quad A \in GL(k,\mathbf{R}). \tag{7}$$

Because $G_{n+1,n-k+1}$ and $G_{n+1,k}$ are canonically isomorphic, every measure μ on $G_{n+1,n-k+1}$ is isomorphic to a measure μ' on $G_{n+1,k}$. We now consider a local coordinate system on $G_{n+1,k}$ given by

$$\begin{pmatrix} c_1^1 & \ldots & c_q^1 & 1 & \ldots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ c_1^k & \ldots & c_q^k & 0 & \ldots & 1 \end{pmatrix},$$

where $q = n-k+1$. In local coordinates $||c_j^i||$ the measure μ' can be written as $\mu'(c_1^1, \ldots, c_k^q) \; dc_1^1 \ldots dc_k^q$, where $\mu'(c_1^1, \ldots, c_k^q)$ is a function of $||c_j^i||$. For a k-frame $\mathbf{w}' = (\mathbf{w}'_1, \ldots, \mathbf{w}'_k) \in E_{n+1,k}$ with $\mathbf{w}'_1 = (c_1^1, ..., c_q^1, 1, \ldots, 0)$, $\ldots$, $\mathbf{w}'_k = (c_1^k, ..., c_q^k, 0, \ldots, 1)$ we define $\tilde{\mu}(\mathbf{w}')$ by the equality

$$\tilde{\mu}(\mathbf{w}') = \mu' \; (c_1^1, \ldots, c_k^q)$$

and we extend $\tilde{\mu}$ for arbitrary $\mathbf{w} \in E_{n+1,k}$ using (7).

3. Construction of a form Ω. Consider now a point $x \in \mathbf{R}^n \subset G_{n+1,n}$. We can look at x as at the n-subspace $L_x \subset \mathbf{R}^{n+1}$. Let $\mathbf{u} = (\mathbf{u}_1, \ldots, \mathbf{u}_n)$ be a basis of L_x. Then any k-frame $\mathbf{w} = (\mathbf{w}_1, \ldots, \mathbf{w}_k)$ in L_x

can be written in the unique way as

$$(\mathbf{w}_1, \ldots, \mathbf{w}_k) = (\mathbf{u}_1, \ldots, \mathbf{u}_n) \begin{pmatrix} t_1^1 & \ldots & t_k^1 \\ \vdots & & \vdots \\ t_1^n & \ldots & t_k^n \end{pmatrix}.$$

So when we fix a basis $\mathbf{u}_1, \ldots, \mathbf{u}_n$ of L_x the components of the matrix $T = ||t_i^j||$ form the local coordinate system on the manifold $E_{n,k}$ of k-frames in L_x. We define differential forms $\sigma_1, \ldots, \sigma_k$ on $E_{n,k}$ as

$$\sigma_i = \sum_{(p_1 \ldots p_n)} (-1)^s t_1^{p_1} t_2^{p_2} \ldots t_k^{p_k} dt_i^{p_{k+1}} \ldots dt_i^{p_n}, \quad i = 1, \ldots, k$$

where s is the sign of the permutation $(p_1, p_2, \ldots, p_n)$. Symbolically σ_i can be written as

$$\sigma_i = \det \begin{pmatrix} t_1^1 & \ldots & t_k^1 & dt_i^1 & \ldots & dt_i^1 \\ \vdots & & \vdots & \vdots & & \vdots \\ t_1^n & \ldots & t_k^n & dt_i^n & \ldots & dt_n^i \end{pmatrix},$$

where we multiply differentials dt_i^j by the exterior product.

For a basis $\mathbf{u}_1, \ldots, \mathbf{u}_n$ of $L_x \in G_{n+1,n}$ and k tangent vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k \in T_x\mathbf{R}^n$ we define a differential form Ω on $E_{n,k}$ as

$$\Omega(\mathbf{u}_1, \ldots, \mathbf{u}_n;\ \mathbf{v}_1, \ldots, \mathbf{v}_k) =$$

$$= \tilde{\mu}((\mathbf{u}_1 \ldots \mathbf{u}_n)T\ ) \left| \sum_{i_1 < \ldots < i_k} p_{i_1 \ldots i_k} T_{i_1 \ldots i_k} \right| \sigma_1 \wedge \ldots \wedge \sigma_k,$$

where $p_{i_1 \ldots i_k} = \det \begin{pmatrix} v_1^{i_1} & \ldots & v_k^{i_1} \\ \vdots & & \vdots \\ v_1^{i_k} & \ldots & v_{i_k}^k \end{pmatrix}$, and $T_{i_1 \ldots i_k} = \det \begin{pmatrix} t_1^{i_1} & \ldots & t_k^{i_1} \\ \vdots & & \vdots \\ t_1^{i_k} & \ldots & t_k^{i_k} \end{pmatrix}$,

$1 \leq i_1 < \ldots < i_k \leq n$.

4. Pull down of Ω to $G_{n,k}$. We can see that when we evaluate form Ω on $k(n-k)$ tangent vectors to $E_{n,k}$, components of these tangent vectors which are tangent to the fibers of the bundle $E_{n,k} \to G_{n,k}$ does not play any role. So Ω can be pulled down to $G_{n,k}$.

Theorem 5. *Every Crofton k-density $L(x;\ \mathbf{v}_1, \ldots, \mathbf{v}_k)$ in $\mathbf{R}^n$ can be represented as*

$$L(x;\ \mathbf{v}_1, \ldots, \mathbf{v}_k) = \int_{G_{n,k}} \Omega_*(\mathbf{u}_1, \ldots, \mathbf{u}_n;\ \mathbf{v}_1, \ldots, \mathbf{v}_k),$$

where Ω_* is the form Ω pulled down to $G_{n,k}$ and $\mathbf{u}_1 \ldots \mathbf{u}_n$ is any n-frame spanning the space $L_x \in G_{n+1,n}$ corresponding to the point $x \in \mathbf{R}^n$.

6. Symplectic forms on the manifold of oriented lines in $\mathbf{R}^n$.

Let H denote the manifold of hyperplanes on $\mathbf{R}^n$. Let P be a two-dimensional plane in $\mathbf{R}^n$. Let us denote by $L(P)$ the space of nonoriented lines lying on P.

A smooth measure μ on H induces a smooth measure μ_P on $L(P)$ for every 2 dimensional affine subspace $P \subset \mathbf{R}^n$.

Let H_P be the set of all hyperplanes in $\mathbf{R}^n$ that intersect P in a line. The set H_P is an open dense subset of H. We consider the natural projection

$$\pi_P : H_P \longrightarrow L(P)\ .$$

If U is a Borel subset of $L(P)$ we set $\mu_P(U) := \mu(\pi_P^{-1}(U))$.

Since the map π_P is a submersion the measure μ_P is smooth if the measure μ is smooth.

Theorem 6. *A Crofton 1-density in $\mathbf{R}^n$ with dual measure μ defines a Finsler metric if and only if the induced measures μ_P are positive for all 2-dimensional affine subspaces $P \subset \mathbf{R}^n$.*

Let us remark that the positivity of the induced measures μ_P does not imply the positivity of the measure μ.

Let us denote by $\tilde{L}(P)$ the space of oriented lines lying on P. As a double cover of L_P, $\tilde{L}(P)$ carries a unique measure $\tilde{\mu}_P$ which induces the measure μ_P on L_P.

Theorem 7. *Let $\tilde{L}(\mathbf{R}^n)$ be the space of oriented lines in $\mathbf{R}^n$, and let*

$$i_P : \tilde{L}(P) \longrightarrow \tilde{L}(\mathbf{R}^n)$$

be the inclusion map. There exists a unique closed 2-form ω on $\tilde{L}(\mathbf{R}^n)$ such that $|i_P^*\omega| = \tilde{\mu}_P$ for all 2-dimensional affines subspaces $P \subset \mathbf{R}^n$. Moreover, the induced measures μ_P are positive if and only if the form ω is symplectic.

Definition. *Let γ_x denote the set of all oriented lines passing through* *a point $x \in \mathbf{R}^n$, and let*

$$j_x : \gamma_x \longrightarrow \tilde{L}(\mathbf{R}^n)$$

be the inclusion map. A 2-form ω is said to satisy condition (A) *if $j_x^* \omega = 0$ for all $x \in \mathbf{R}^n$.*

Proposition. *A 2-form ω on $\tilde{L}(\mathbf{R}^n)$ satisfies condition* (A) *if and only if there exists a smooth signed measure μ on H such that for every 2-dimensional affine subspace $P \subset \mathbf{R}^n$ we have that $|i_P^* \omega| = \tilde{\mu}_P$.*

Theorem 8. *There is bijection between the set of Finsler metrics on $\mathbf{R}^n$ whose geodesics are straight lines (so called Desarguesian Finsler metrics) and the set of symplectic forms on $\tilde{L}(\mathbf{R}^n)$ which satisfy condition* (A).

7. Nonlocal differentials and the Hilbert transform for 1-Lagrangians.

We shall explain now in what sense even Crofton 1-densities can be viewed as "differentials" of functions. Let f be a function in $\mathbf{R}^n$. The usual differential df, which depends on a point x from $\mathbf{R}^n$ and a tangent vector v from $T_x\mathbf{R}^n$ can be written as $(df)(x,v) = \int_{-\infty}^{+\infty} \delta'(t) f(x - vt)dt$. The *nonlocal (or even) differential* of a function f is

$$(d^o f)(x,v) = \int_{-\infty}^{+\infty} \frac{1}{t^2} f(x - vt)dt = \int_0^{+\infty} \frac{f(x+vt) + f(x-vt) - 2f(x)}{t^2} dt.$$

We suppose that f is such that this integral converges at infinity.

Theorem 9. *Let $f(x)$ be a rapidly decreasing function in the Schwartz space $S(\mathbf{R}^n)$. Let f be the Radon transform of a function $F(\alpha)$:*

$$f(x_1, ..., x_n) = \int_{-\infty}^{+\infty} F(\alpha_1, ..., \alpha_{n-1},\ x_n - (\alpha_1 x_1 + ... + \alpha_{n-1} x_{n-1}))d\alpha_1 ... d\alpha_{n-1}$$

Let $L(x,v) = d^0 F$ be an even differential of the function F. Then

(1) *$L(x,v)$ is a Crofton density in $\mathbf{R}^n$.*

(2) *Let $D^o_{\alpha_n} F(\alpha_1, \ldots, \alpha_n)$, be the "even partial derivative" of F defined as*

$$D^o_{\alpha_n} F(\alpha_1, \ldots, \alpha_n) = \int_{-\infty}^{\infty} \frac{1}{t^2} F(\alpha_1, \ldots, \alpha_n - t)dt.$$

Then the the dual measure for L *is* $\mu(\alpha)d\alpha$ *where* $\mu(\alpha_1, \ldots, \alpha_n) = D^o_{\alpha_n} F(\alpha_1, \ldots, \alpha_n)$.

Let us remind that an *even 1-density* is a function $L(x, v)$ of a point $x \in \mathbf{R}^n$ and a tangent vector v at the point x such that $L(x, \lambda v) = |\lambda| L(x, v)$ for every $\lambda \in \mathbf{R}$. An *odd 1-density* is a function $\theta(x, v)$ of a point $x \in \mathbf{R}^n$ and a tangent vector v such that $\theta(x, \lambda v) = \lambda\theta(x, v)$ for every $\lambda \in \mathbf{R}$. Differential 1-forms give a specific example of odd 1-densities, they are linear in v.

The 1-density $(HL)(x, v)$

$$(HL)(x,v) = P.V.\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{L(x - vt, v)}{t} dt = \frac{1}{\pi}\int_{0}^{+\infty} \frac{L(x + vt, v) - L(x - vt, v)}{t} dt$$

is called the *Hilbert transform* of a 1-density L. We suppose that L satisfy some growth conditions such that its Hilbert transform exists.

The Hilbert transform of an even 1-density is an odd 1-density. The Hilbert transform of an odd 1-density is an even 1-density.

Theorem 10. *The Hilbert transform of a closed 1-form with rapidly decreasing coefficients is the Crofton 1-density. The Hilbert transform of a rapidly decreasing Crofton 1-density* L *is a closed 1-form. ("Rapid decreasing of* L*" means that* $L(x - vt, v)$ *is a rapidly decreasing function of* t *for any fixed* v *and* x*). Let* f *be a rapidly decreasing function,* $\omega = df$ *is its differential and* $L = \frac{1}{\pi} d^o f$ *is its nonlocal differential (which is the Crofton 1-density). Then* $(HL)(x, v) = \omega(x, v)$ *and* $(H\omega)(x, v) = L(x, v)$*. So the Hilbert transform maps even differentials into odd ones and vice versa.*

We can summarize properties of Crofton 1-densities on the following diagram:

$$\begin{tikzcd}
F \arrow[r, "\text{Radon transf.}"] \arrow[d, "D^o_{\alpha_n}"'] & f \arrow[r, equal] \arrow[d, "d^o"'] & f \arrow[d, "d"'] & F \arrow[l, "\text{Radon transf.}"'] \arrow[d, "D_{\alpha_n}"] \\
\mu \arrow[r, "\text{Even Radon tr.}"] & d^o f \arrow[r, leftrightarrow, "\text{Hilbert tr.}"] \arrow[d, equal] & df \arrow[d, equal] & \nu \arrow[l, "\text{Odd Radon tr.}"'] \\
& \underset{\text{Crofton 1-density}}{\varphi} & \underset{\text{Closed 1-form}}{\omega} &
\end{tikzcd}$$

8. Crofton 1-Lagrangians on the hyperbolic plane.

Consider the hyperbolic plane as an upper half-plane $\{(x,y) \mid y > 0\}$ with the metric $\frac{dx^2+dy^2}{y^2}$. Take $z = x+iy$. The upper half-plane is a model of the Lobachevsky geometry, where Lobachevsky lines are vertical rays $\{(x,y) \mid x = x_0, y > 0\}$ and half circles $\{(x,y) \mid (x-x_0)^2+y^2 = R^2, y > 0\}$.

Let us parametrize "lines" on the hyperbolic plane. To each half circle $\{(x,y) \mid (x-x_0)^2+y^2 = R^2, y > 0\}$ we correspond a point $\zeta = \zeta_1 + i\zeta_2$ where $\zeta_1 = x_0$ and $\zeta_2 = R$. Thus we parametrize almost all "lines" except vertical rays.

The set of all lines which come through the point (x,y) forms a branch of a hyperbola $(x-\zeta_1)^2+y^2 = \zeta_2^2$ on the dual plane ζ_1, ζ_2.

Let us as usual define even and odd 1-densities and also define Crofton densities with respect to Lobachevsky lines. For example the length element $\frac{dx^2+dy^2}{y^2} = \frac{v_1^2+v_2^2}{y^2}$ will be a Crofton 1-density.

Theorem 11. *Even Crofton densities on the Lobachevsky plane satisfy the following equation*

$$\left(\frac{\partial^2}{\partial x\partial v_2} - \frac{\partial^2}{\partial y\partial v_1} - \frac{(v_1^2+v_2^2)}{yv_1}\frac{\partial^2}{\partial v_2\partial v_2}\right)L = 0, \quad \text{for} \qquad v \neq 0.$$

Crofton densities have the following integral representation through their dual measures on the set of Lobachevskian lines:

$$L(x,y;v_1,v_2) = \int_{-\infty}^{\infty} \mu(\zeta_1, ((x-\zeta_1)^2+y^2)^{1/2})\frac{|yv_2+(x-\zeta_1)v_1|}{((x-\zeta_1)^2+y^2)^{1/2}}d\zeta_1.$$

Extremals of arbitrary Crofton density are Lobachevsky lines.

8. Crofton Lagrangians as analogs of closed differential k-forms.

In the class of odd densities, densities for which the oriented analog of the Crofton formula is valid are exactly closed differential k-forms [GS1]. They have analogous representation through Radon transform. The PDE's for the image of this Radon transform are exactly conditions of closedness of the form.

Crofton k-densities give a good example of the integro-geometrictex prepr functional. There were several attempts recently to use integro-geometric functionals other then area in the functional integral in QFT. For example, Ambartsumian-Savvidi [S1] used for this Steiner functional of the classical integral geometry.

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