An invariant trace formula for the universal covering group of SL(2, $\mathbb{R}$)

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Introduction

The aim of the present article is to prove the Selberg trace formula for Hecke operators and automorphic forms of arbitrary real weight. Regardless of the extensive literature on the subject and the far-reaching generalizations in the work of J. Arthur there is, to my knowledge, no publication covering this case.

We include automorphic forms with respect to a finite-dimensional unitary representation of a lattice Γ in the universal covering group G of $\mathrm{SL}(2,\mathbb{R})$ and express the trace formula in an invariant form, namely, in terms of irreducible characters of G. For this purpose the Fourier transform of weighted orbital integrals, obtained by J. Arthur, R. Herb and P. Sally, jr., is explicitly calculated in Propositions 7 and 8. It is the lack of an analogous result for $\mathrm{SL}(2,\mathbb{Q}_p)$ that prevents us from proving as explicit a trace formula in the adèlic case. So we treat Hecke operators in a somewhat old-fashioned way, which allows us, however, to consider non-congruence lattices at the same time.

Our point of view is a representation-theoretic one, thus the trace formula appears as an identity between invariant distributions on Harish-Chandra's L^1-Schwartz space of G. Nevertheless, traditional parametrizations and notations (like $s = \frac{1}{2} + ir$ and $h(r)$) facilitate an immediate comparison with the classical picture presented, e.g., in D. Hejhal's books. The statement of our main result (Theorems 13 and 14 in section 4) uses only notations introduced in sections 1 and 3.

Part of this publication was prepared during a stay at the University of Toronto, to which I would like to express my gratitude for its hopitality.

1 Irreducible representations and intertwining operators

Let us first fix some notations. The group $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm I\}$ will throughout be denoted by G', the symbol G being reserved for the universal covering Lie group of G', the main object of our considerations. We shall view the elements of G as homotopy classes x of paths connecting the identity of G' with some element x' (the image of x under the canonical projection $G \to G'$). The product of x_1, $x_2 \in G$ is defined as usual by $(x_1 \circ t_1)(x_2 \circ t_2)$, where $t = (t_1, t_2)\colon [0,1] \to [0,1]^2$ connects $(0,0)$ with $(1,1)$; here we have neglected the distinction between homotopy classes and their representatives.

For θ, u, $v \in \mathbb{R}$, we denote the matrices

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad \begin{pmatrix} e^{u/2} & 0 \\ 0 & e^{-u/2} \end{pmatrix}, \quad \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix},$$

taken $\mathrm{mod}(\pm I)$, by k'_θ, a'_u and n'_v, respectively. They form subgroups K', A' and N' of G'. Recall that any morphism of linearly connected topological groups lifts to a homomorphism of their universal covering groups. If we apply this to the embeddings of K', A' and N' into G', the universal covering groups K, A and N become one-parameter subgroups of G. There are unique parametrizations $\theta \mapsto k_\theta$, $u \mapsto a_u$, and $v \mapsto n_v$ for them such that the images k'_θ, a'_u and n'_v in G' are just the elements introduced above.

Passing to the universal cover is functorial with respect to direct products of manifolds. Therefore the Cartan decomposition $G' = K' \exp_{G'} \mathfrak{s}$ and the Iwasawa decomposition $G' = K'A'N'$ yield the corresponding decompositions $G = K\exp_G\mathfrak{s}$ (providing the parametrization of G used in [27]) and $G = KAN$. Here $\mathfrak{s} \subset \mathfrak{g}$ consists of the symmetric (2×2)-matrices. The kernel of the projection $G \to G'$ is the centre Z of G consisting of all k_θ with $\theta \in \pi\mathbb{Z}$.

The aforementioned parametrizations carry the Lebesgue measure du resp. dv from $\mathbb{R}$ to A and N. We fix Haar measures on K by $\mathrm{vol}(K/Z) = 1$ and on G by

$$\int_G f(x)\,dx = \int_N \int_A \int_K f(na_u k)e^{-u}\,dk\,da_u\,dn$$

for $f \in C_0(G)$. Together with the counting measure on Z this fixes a Haar measure on G', for which

$$\int_{G'} g(x'(i))\,dx' = \int_{\mathcal{H}} g(x+iy)y^{-2}\,dx\,dy$$

for $g \in C_0(\mathcal{H})$, where $z \mapsto x'(z)$ is the linear fractional action of G' on the complex upper half-plane $\mathcal{H}$.

Viewing elements D of the universal enveloping algebra $\mathfrak{G}$ of $\mathfrak{g}_{\mathbb{C}}$ as distributions on G with support $\{1\}$, we may unambiguously interpret an expression of the type $f(D_1 x_1 \ldots D_n x_n D_{n+1})$ with sufficiently smooth f as the convolution $D_1 * \delta_{x_1} * \cdots * D_n * \delta_{x_n} * D_{n+1}$ evaluated on f. The identity $xDx^{-1} = \mathrm{Ad}(x)D$ reduces this to the case $n = 1$, i.e., to $f(D_1\,;\,s_1;D_2)$ in Harish-Chandra's notation.

Let us define the spaces of rapicly decreasing smooth functions on G as

$$\mathcal{C}^p(G) = \{f \in C^\infty(G)\colon |f(D_1 k_{\theta_1} a_u k_{\theta_2} D_2)| \leq Ce^{-|u|/p}(1+|u|+|\theta_1+\theta_2|)^{-n} \\ \forall n \in \mathbb{N} \text{ and } D_1, D_2 \in \mathfrak{G}\}.$$

These are L^p-functions in view of

$$\int_G f(x)\,dx = 2\pi \int_{Z\backslash K} \int_K \int_0^\infty f(k'^{-1} a_u k k')\sinh u\,du\,dk\,dk'.$$

The same definition applies to G', where $|\theta_i| \leq \pi$, say. $\mathcal{C}^p(G')$ is then Harish-Chandra's familiar L^p-Schwartz space.

Next we turn to the representation theory of G. As usual, we put

$$\pi(f) = \int_G f(x)\pi(x)\,dx$$

for $f \in L^1(G)$ and bounded representations π of G. Now suppose that π has a central character $\varepsilon \in \hat{Z}$. One can then form

$$\pi(f') = \int_{Z\backslash G} f'(x)\pi(x)\,dx$$

for all $f' \in L^1(G,\varepsilon)$, i.e., such that $f'(z^{-1}x) = \varepsilon(z)f'(x)$ and $|f'| \in L^1(Z\backslash G)$. This generalizes the lift of a representation of G' to G, which corresponds to the case when ε is the trivial character of Z. If $f \in L^1(G)$, one immediately checks that $\pi(f) = \pi(f_\varepsilon)$, where

$$f_\varepsilon(x) = \sum_{z \in Z} \varepsilon(z) f(zx)$$

lies in $L^1(G, \varepsilon)$. Inversely, $f \in C^1(G)$ can be recovered as

$$f(x) = \int_{\hat{Z}} f_\varepsilon(x)\, d\varepsilon,$$

the Plancherel measure $d\varepsilon$ being so normalized that $\mathrm{vol}(\hat{Z}) = 1$. This partial Fourier transform with respect to Z extends to a unitary isometry

$$L^2(G) \to \int_{\hat{Z}} L^2(G, \varepsilon)\, d\varepsilon,$$

thereby in a certain sense reducing harmonic analysis of $L^2(G)$ to that of $L^2(G, \varepsilon)$.

We shall only be concerned with such representations π of G which decompose into a finite sum of Z-isotypical components. Then $\pi(f)$ is determined by finitely many f_ε's. One is tempted to consider the sum of the latter instead of f, leaving aside unnecessary information. However, such sums are not contained in $L^1(G)$, and one would have to introduce a certain space of almost Z-periodic functions on G. We shall avoid this technicality, as it seems not to yield any additional information.

Let us now fix notations for induced representations. The unitary characters of K are $\phi_m(k_\theta) = e^{im\theta}$ with weight $m \in \mathbb{R}$ ($\cong \mathfrak{k}^*$). As we use induction from the left, the space $\mathcal{H}_\varepsilon$ of $\mathrm{Ind}_Z^K(\varepsilon)$ will consist of classes of functions ϕ on K satisfying $\phi(zk) = \varepsilon(z)\phi(k)$ and $|\phi| \in L^2(Z\backslash K)$. A basis of $\mathcal{H}_\varepsilon$ is formed by all ϕ_m with m in $\mathbb{R}_\varepsilon = \{m \in \mathbb{R} \colon \phi_m|_Z = \varepsilon\}$, which is a coset mod 2. Inversely, $m \in \mathbb{R}$ determines $\varepsilon_m \in \hat{Z}$ by $m \in \mathbb{R}_{\varepsilon_m}$. Incidentally, this identifies $\hat{Z} \cong \mathbb{R}/2\mathbb{Z}$ and the Plancherel measure $d\varepsilon_m = \frac{1}{2}dm$.

Given $s \in \mathbb{C}$, we extend $\phi \in \mathcal{H}_\varepsilon$ to G by

$$\phi_s(na_uk) = e^{su}\phi(k) \qquad \text{for } n \in N,\ k \in K.$$

These functions constitute the Hilbert space $\mathcal{H}_{\varepsilon,s} \cong \mathcal{H}_\varepsilon$ in which the representation $\pi_{\varepsilon,s}$, induced from the parabolic subgroup $P = NAZ$, acts as

$$(\pi_{\varepsilon,s}(x)\phi_s)(y) = \phi_s(yx).$$

It is unitary (principal series) iff $s = \frac{1}{2} + ir$, $r \in \mathbb{R}$ ($\cong \mathfrak{a}^*$). By the isomorphism $\mathcal{H}_{\varepsilon,s} \cong \mathcal{H}_\varepsilon$ we let $\pi_{\varepsilon,s}$ act on $\mathcal{H}_\varepsilon$. The action of the Lie algebra $\mathfrak{g}_\mathbb{C}$ can easily be calculated. If we put

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and $E_\pm = H \pm i(X+Y)$, then $\phi_{m,s}(Xx) = 0$,

$$\phi_{m,s}(na_u k_\theta E_\pm) = \phi_{m,s}(a_u(e^{\pm 2i\theta} E_\pm)k_\theta) = e^{\pm 2i\theta}\phi_{m,s}(a_u(H \mp i(X-Y))k_\theta)$$
$$= e^{\pm 2i\theta}\left(2\frac{\partial}{\partial u} \mp i\frac{\partial}{\partial\theta}\right)\phi_{m,s}(a_u k_\theta) = (2s \pm m)\phi_{m\pm 2,s}(na_u k_\theta),$$

hence

$$\begin{aligned}\pi_{\varepsilon,s}(E_\pm)\phi_m &= (2s \pm m)\phi_{m\pm 2},\\ \pi_{\varepsilon,s}(X-Y)\phi_m &= im\phi_m,\\ \pi_{\varepsilon,s}(\omega) &= s(s-1)\mathrm{Id}\end{aligned}$$

with the Casimir element

$$\omega = \tfrac{1}{4}(H^2 + 2XY + 2YX) = \tfrac{1}{8}(E_+E_- + E_-E_+ - 2(X-Y)^2).$$

We see that the elements $\phi_{m'}$, $m' \in \pm\{m, m+2, m+4, \dots\}$, span a $\mathfrak{g}_{\mathbb{C}}$-invariant subspace $\mathcal{H}^K_{\pm m}$ of $\mathcal{H}_{\varepsilon_{\pm m}, m/2}$. If $m > 0$, it can be completed to the Hilbert space $\mathcal{H}_{\pm m}$ of a unitary representation $\pi_{\pm m}$ of G (see [27]). One may describe the scalar product of $\mathcal{H}_{\pm m}$ with the help of intertwining operators, which we shall now introduce.

Besides P, we also consider the group $\bar{P} = \bar{N}AZ$, where $\bar{N} = \{\bar{n}_v : v \in \mathbb{R}\}$, $\bar{n}_v = \exp_G(vY)$. Comparing the integral formula for the Bruhat decomposition

$$\int_{Z\backslash G} f(x)\,dx = \frac{1}{\pi}\int_N\int_A\int_{\bar{N}} f(na_u\bar{n})e^{-u}\,d\bar{n}\,da_u\,dn$$

with that for the Iwasawa decomposition, we obtain for ϕ, $\psi \in \mathcal{H}_\varepsilon$

$$(\phi,\psi) = \frac{1}{\pi}\int_{\bar{N}} \phi_s(\bar{n})\overline{\psi_{1-\bar{s}}(\bar{n})}\,d\bar{n}.$$

We shall sometimes write $\phi_{P,s}$ instead of ϕ_s etc., because one may induce representations also from $\bar{P}$ by defining

$$\phi_{\bar{P},s}(\bar{n}a_u k) = e^{-su}\phi(k) \qquad \text{for } \bar{n} \in \bar{N},\ k \in K.$$

Again, $\pi_{\bar{P},\varepsilon,s}$ acts in $\mathcal{H}_{\bar{P},\varepsilon,s} \cong \mathcal{H}_\varepsilon$ by right translations, and

$$(\phi,\psi) = \frac{1}{\pi}\int_N \phi_{\bar{P},s}(n)\overline{\psi_{\bar{P},1-\bar{s}}(n)}\,dn.$$

Given $s \in \mathbb{C}$ with $\Re s > \frac{1}{2}$, we define the bounded operator $J_{\bar{P}P}(\varepsilon,s)$ in $\mathcal{H}_\varepsilon$ by

$$(J_{\bar{P}P}(\varepsilon,s)\phi)_{\bar{P},1-s}(x) = \frac{1}{\pi}\int_{\bar{N}} \phi_{P,s}(\bar{n}x)\,d\bar{n}$$

(cf. [14, p. 130]). This is an intertwining operator:

$$J_{\bar{P}P}(\varepsilon,s)\pi_{P,\varepsilon,s}(x) = \pi_{\bar{P},\varepsilon,1-s}(x)J_{\bar{P}P}(\varepsilon,s).$$

Interchanging P and $\bar{P}$, one obtains $J_{P\bar{P}}(\varepsilon,s)$ with the analogous property. The integral formula for the Bruhat decomposition implies that

$$\int_{ZA\backslash G} f(x)\,dx = \frac{1}{\pi}\int_N\int_{\bar{N}} f(n\bar{n})\,d\bar{n}\,dn = \frac{1}{\pi}\int_{\bar{N}}\int_N f(\bar{n}n)\,dn\,d\bar{n}.$$

Putting $f=\phi_{P,s}\overline{\psi_{\bar{P},\bar{s}}}$, we get $(\phi, J_{P\bar{P}}(\varepsilon,s)\psi)=(J_{\bar{P}P}(\varepsilon,s)\phi,\psi)$, i.e.,

$$J_{\bar{P}P}(\varepsilon,s)^* = J_{P\bar{P}}(\varepsilon,\bar{s}).$$

It is easy to calculate $J_{\bar{P}P}$ explicitly. Since $\bar{n}_v = a_u n_v k_\theta$ with $e^{-u}=1+v^2$ and $e^{2i\theta}=\frac{1-iv}{1+iv}$ (which is easy to check in G' and lifts to G), we obtain

$$\phi_{m,s}(\bar{n}_v) = (1+v^2)^{-s}\left(\frac{1-iv}{1+iv}\right)^{m/2}$$

(continuous branch with value 1 at $v=0$),

$$\begin{aligned} j_m(s) := \frac{1}{\pi}\int_{\bar{N}}\phi_{m,s}(\bar{n})\,d\bar{n} &= \frac{1}{\pi}\int_{-\infty}^{\infty}(1+iv)^{-s-m/2}(1-iv)^{-s+m/2}\,dv \\ &= \frac{2^{2-2s}\Gamma(2s-1)}{\Gamma(s+\frac{m}{2})\Gamma(s-\frac{m}{2})} = \frac{1}{\sqrt{\pi}}\cdot\frac{\Gamma(s)\Gamma(s-\frac{1}{2})}{\Gamma(s+\frac{m}{2})\Gamma(s-\frac{m}{2})}, \end{aligned}$$

and, by the intertwining property,

$$J_{\bar{P}P}(\varepsilon_m,s)\phi_m = j_m(s)\phi_m.$$

Thereby the restriction of $J_{\bar{P}P}(\varepsilon,s)$ to the subspace $\mathcal{H}_\varepsilon^K$ of K-finite elements extends meromorphically to $s\in\mathbb{C}$.

Now we introduce the meromorphic function $\mu(\varepsilon,s)$ by

$$\mu(\varepsilon,s)J_{P\bar{P}}(\varepsilon,1-s)J_{\bar{P}P}(\varepsilon,s) = \mathrm{Id}$$

(cf. [14], p. 141). One deduces from $\mu(\varepsilon,s)j_m(1-s)j_m(s)=1$ with the help of the reflection formula for the Γ-function that

$$\mu(\varepsilon_m,s) = \pi\left(s-\tfrac{1}{2}\right)\frac{\sin 2\pi s}{\cos\pi m-\cos 2\pi s}.$$

We define the normalized intertwining operators as

$$R_{\bar{P}P}(\varepsilon,s) = j_{m_\varepsilon}(s)^{-1}J_{\bar{P}P}(\varepsilon,s),$$

where $m_\varepsilon\in\mathbb{R}_\varepsilon$, $|m_\varepsilon|\le 1$. Explicitly,

$$R_{\bar{P}P}(\varepsilon,s)\phi_m = \left(\prod_{k=1}^{[\frac{|m|+1}{2}]}\frac{s-1-\frac{|m|}{2}+k}{s+\frac{|m|}{2}-k}\right)\phi_m.$$

From the obvious properties

$$R_{P\bar{P}}(\varepsilon, 1-s)R_{\bar{P}P}(\varepsilon, s) = \mathrm{Id}, \qquad R_{\bar{P}P}(\varepsilon, s)^* = R_{P\bar{P}}(\varepsilon, \bar{s})$$

one sees that $R_{\bar{P}P}(\varepsilon, \frac{1}{2} + ir)$ is a unitary intertwining operator between unitary representations.

Let W denote the normalizer of A in K. Then $[W : Z] = 2$, and $W' = W/Z$ is the Weyl group of A' in G'. Left translation by $w \in W$, $w \notin Z$, produces an intertwining operator $\mathcal{H}_{\bar{P},\varepsilon,1-s} \to \mathcal{H}_{P,\varepsilon,1-s}$. Composed with $R_{\bar{P}P}(\varepsilon, s)$, this gives an intertwining operator $\mathcal{H}^K_{P,\varepsilon,s} \to \mathcal{H}^K_{P,\varepsilon,1-s}$, namely,

$$(R_P(\varepsilon, s)\phi)_{1-s}(x) = \phi_{m_\varepsilon}(w)(R_{\bar{P}P}(\varepsilon, s)\phi)_{1-s}(w^{-1}x),$$

where m_ε is as above. The multiplication with $\phi_{m_\varepsilon}(w)$ makes $R_P(\varepsilon, s)$ independent of w; however, for $\varepsilon = \varepsilon_1$ we get two operators $R^{\pm}_P(\varepsilon_1, s)$ depending on the sign of m_ε (i.e., a chamber in $i\mathfrak{k}$). Note that

$$R_P(\varepsilon, s)\phi_m = (-1)^{(m-m_\varepsilon)/2} R_{\bar{P}P}(\varepsilon, s)\phi_m.$$

One checks that, for $0 < m \leq 1$,

$$R^{\pm}_P(\varepsilon_{\pm m}, 1 - \tfrac{m}{2})\colon \mathcal{H}^K_{\varepsilon_{\pm m},1-m/2}/\mathcal{H}^K_{\pm(m-2)} \to \mathcal{H}^K_{\pm m}$$

is a pre-unitary intertwining operator, if the scalar products in the two spaces are given by

$$(\phi, \psi)_{\pm m} = (R^{\pm}_P(\varepsilon_{\pm m}, 1 - \tfrac{m}{2})\phi, \psi) \qquad \text{and} \qquad (\phi, \psi)_{\pm m} = (R^{\pm}_P(\varepsilon_{\pm m}, \tfrac{m}{2})\phi, \psi),$$

respectively (the superscript $\pm$ being ignored for $m \neq 1$). A similar assertion is true for $m > 1$, if one considers $\lim_{s \to 1-m/2}(s - 1 + \frac{m}{2})R_P(\varepsilon_{\pm m}, s)$ and $\lim_{s \to m/2}$ $(s - \frac{m}{2})^{-1}R_P(\varepsilon_{\pm m}, s)$. Completing the pre-Hilbert spaces, one gets two realizations for the unitary representations $\pi_{\pm m}$ (discrete series). Similarly we can unitarize $\pi_{\varepsilon,s}$ for real s (complementary series) provided the scalar product

$$(\phi, \psi)_{\varepsilon,s} = (R_P(\varepsilon, s)\phi, \psi)$$

is positive definite. This is the case iff $s \in (\frac{|m|}{2}, 1 - \frac{|m|}{2})$, where $m \in \mathbb{R}_\varepsilon$, $|m| \leq 1$. There are two extremal cases (exactly those obtained by lifting from $\mathrm{SL}(2, \mathbb{R})$): For the trivial character ε_0 of Z, the complementary series exists for $0 < s < 1$, while for the alternating character $\varepsilon_1 : Z \to \{\pm 1\}$ there is no complementary series, and $\pi_{\varepsilon_1,1/2} = \pi_1 \oplus \pi_{-1}$. If we denote the corresponding orthoprojections by $p_\pm$, then continuation in s gives $R^{\pm}_P(\varepsilon_1, \frac{1}{2}) = \pm(p_+ - p_-)$.

Here is a complete list of the irreducible unitary representations of G (cf. [27]):

(1) the principal series of representations $\pi_{\varepsilon,s}$ with $\Re s = \frac{1}{2}$, $(\varepsilon, s) \neq (\varepsilon_1, \frac{1}{2})$;
(2) the complementary series of representations $\pi_{\varepsilon,s}$ with $s \in \left(\frac{|m|}{2}, 1 - \frac{|m|}{2}\right)$, $s \neq \frac{1}{2}$, where $m \in \mathbb{R}_\varepsilon$, $|m| < 1$;
(3) the discrete series of representations π_m with $|m| > 1$;
(4) the limit of discrete series representations π_1 and π_{-1};
(5) the pseudo-discrete series of representations π_m with $0 < |m| < 1$;
(6) the one-dimensional trivial representation π_0.

The operator $R_P(\varepsilon, s)$ is a unitary equivalence between $\pi_{\varepsilon,s}$ and $\pi_{\varepsilon,1-s}$ if they belong to the principal or complementary series. Any other two representations in this list are non-equivalent to each other.

By a simple extension of Harish-Chandra's regularity theorem to groups with infinite centre, the character $\Theta(f) = \operatorname{tr}\pi(f)$ of any irreducible unitary representation π of G is a regular distribution given by integration against an analytic function (which we also denote by Θ) on the set G_{reg} of regular elements of G (i.e., those whose images in $\mathrm{SL}(2,\mathbb{R})$ have different eigenvalues):

$$\Theta(f) = \int_{G_{\mathrm{reg}}} \Theta(x) f(x)\, dx \qquad \text{for } f \in \mathcal{C}^1(G).$$

Obviously, $\Theta(f)$ depends only on f_ε, where ε is the central character of π. We shall therefore restrict our attention to functions $f' \in \mathcal{C}^1(G,\varepsilon)$, for which we put

$$\Theta(f') = \operatorname{tr}\pi(f') = \int_{Z\backslash G_{\mathrm{reg}}} \Theta(x) f'(x)\, dx.$$

For a first reading, one may now pass to section 3.

2 Harmonic analysis on G

Harish-Chandra's invariant integrals associated to the two Cartan subgroups AZ and K of G are tempered distributions (by what we mean continuous linear functionals on $\mathcal{C}^2(G,\varepsilon)$ for any $\varepsilon \in \hat{Z}$). They are defined for $f \in \mathcal{C}^2(G,\varepsilon)$, $z \in Z$, $u \neq 0$, $\theta \notin \pi\mathbb{Z}$ as

$$F_f^A(a_u z) = \left|e^{u/2} - e^{-u/2}\right| \int_{AZ\backslash G} f(x^{-1} a_u z x)\, dx,$$

$$F_f^K(k_\theta) = \left(e^{i\theta} - e^{-i\theta}\right) \int_{K\backslash G} f(x^{-1} k_\theta x)\, dx.$$

F_f^K is a smooth function on $K_{\mathrm{reg}} = K - Z$, and the transformed expression

$$F_f^A(a_u z) = e^{u/2} \int_{Z\backslash K} \int_N f(k^{-1} a_u n z k)\, dn\, dk$$

extends to an even Schwartz function on A (cf. [11], sect. 17, Theorem 5, [28], section 8.8). Our next task will be to express these invariant integrals in terms of the irreducible characters.

Lemma 1. *Let $1/p \geq \max(\Re s, 1 - \Re s)$. If $f \in \mathcal{C}^p(G,\varepsilon)$, then $\pi_{\varepsilon,s}(f)$ is an integral operator in $\mathcal{H}_\varepsilon$ with smooth kernel*

$$K_{f,\varepsilon,s}(x,y) = \int_N \int_A f(x^{-1} a_u n y) e^{su}\, da_u\, dn,$$

which belongs to $\mathcal{H}_{(\varepsilon,\bar{\varepsilon}),(s,1-s)}$ in an obvious notation. This operator is of trace class, its trace equals

$$\Theta_{\varepsilon,s}(f) = \int_{-\infty}^{\infty} F_f^A(a_u)e^{(s-1/2)u}\,du$$

and is a continuous linear functional on $C^p(G,\varepsilon)$. *For fixed* p *and* f, $\Theta_{\varepsilon,s}(f)$ *is a smooth function of* s *in the corresponding strip, holomorphic in its interior and rapidly decreasing together with all derivatives as* $|\Im s| \to \infty$, *uniformly in* $\Re s$.

Proof. We have

$$|f(Dx^{-1}a_u n_v yE)| \le \nu(f)e^{-|u|/p+(\operatorname{sgn} u-1)u/2}(1+|u|)^{-n}(1+v^2)^{-1/p}(\log(1+v^2))^{-n},$$

where ν is a continuous seminorm on $C^p(G,\varepsilon)$ depending on x, $y \in G$, $n \in \mathbb{N}$ and D, $E \in \mathfrak{G}$ (cf. [28], Lemma 41). Thus the integral defining $(D \otimes E)K_{f,\varepsilon,s}$ is absolutely convergent, and a variation of Theorem 1 of [19], ch. VII, implies that $\pi_{\varepsilon,s}$ is of the trace class. $\Theta_{\varepsilon,s}$ is obtained by integration over the diagonal $x = y \in Z\backslash K$, its properties follow from the same estimate as above and the fact that $\Theta_{\varepsilon,s}(\omega^n f) = s^n(s-1)^n\Theta_{\varepsilon,s}(f)$. □

Obviously, it would have been enough to take f in the larger space $C_n^p(G,\varepsilon)$ obtained by completing with respect to the seminorms for fixed exponent of decay n only, provided n is sufficiently large.

By Fourier inversion we obtain

Lemma 2. *For* $f \in C^2(G,\varepsilon)$,

$$F_f^A(a_u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \cos(ru)\Theta_{\varepsilon,\frac{1}{2}+ir}(f)\,dr.$$

In view of $F_f^A(z^{-1}a_u) = \varepsilon(z)F_f^A(a_u)$ for the given f, this contains the full information. Incidentally, the Weyl integration formula

$$\int_G f(x)\,dx = \frac{1}{2}\sum_{z\in Z}\int_A |e^{u/2} - e^{-u/2}|^2 \int_{AZ\backslash G} f(x^{-1}a_u zx)\,dx\,da_u$$
$$+ \int_K (e^{i\theta} - e^{-i\theta})^2 \int_{K\backslash G} f(x^{-1}k_\theta x)\,dx\,dk_\theta$$

shows that

$$\Theta_{\varepsilon,s}(a_u z) = \varepsilon(z)\frac{e^{(s-1/2)u} - e^{-(s-1/2)u}}{|e^{u/2} - e^{-u/2}|}, \qquad \Theta_{\varepsilon,s}(k_\theta) = 0.$$

Now we turn to the discrete series characters. As a convergent geometric progression in the space of distributions on K_{reg}, the character of π_m for $m \neq 0$ is

$$\Theta_m(k_\theta) = \sum_{n=0}^{\infty} e^{(m+2n\operatorname{sgn} m)\theta} = -\operatorname{sgn}(m)\frac{e^{(m-\operatorname{sgn} m)\theta}}{e^{i\theta} - e^{-i\theta}}.$$

From Harish-Chandra's regularity theorem and his matching conditions, generalized to groups with infinite centre, one can deduce that the only extension of Θ_m to G as a tempered invariant K-finite eigendistribution of ω is given by

$$\Theta_m(a_u z) = \varepsilon(z)\frac{e^{-(|m|-1)|u|/2}}{\left|e^{u/2} - e^{-u/2}\right|}$$

(cf.[28]). Alternatively, it has been proved in [27] that this is in fact the value of Θ_m on AZ. With the help of the Weyl integration formula we can write, for $f \in \mathcal{C}^2(G, \varepsilon_m)$, $|m| \leq 1$:

$$\Theta_m(f) = \frac{1}{2}\int_A e^{-(|m|-1)|u|/2} F_f^A(a_u)\, da_u + \operatorname{sgn}(m)\int_{Z\backslash K} e^{i(m-\operatorname{sgn} m)\theta} F_f^K(k_\theta)\, dk_\theta.$$

In the particular case $\varepsilon = \varepsilon_1$ we put $\Theta_{\varepsilon_1} = (\Theta_1 - \Theta_{-1})/2$, whence

$$\Theta_{\varepsilon_1}(f) = \int_{Z\backslash K} F_f^K(k_\theta)\, dk_\theta.$$

Given any $f \in \mathcal{C}^2(G, \varepsilon)$, we thus know the Fourier coefficients of F_f^K and may set up its Fourier series:

$$F_f^K(k_\theta) = \delta_{\varepsilon,\varepsilon_1}\Theta_{\varepsilon_1}(f) + \sum_{n \in \mathbb{R}_\varepsilon + 1} \operatorname{sgn}(n) e^{-in\theta}\Theta_{n+\operatorname{sgn} n}(f) + \frac{1}{2}\int_A b_\varepsilon(\theta, u) F_f^A(a_u)\, da_u,$$

where δ is the Kronecker symbol, and

$$b_\varepsilon(\theta, u) = -\sum_{n \in \mathbb{R}_\varepsilon + 1} \operatorname{sgn}(n) e^{-in\theta - |nu|/2}.$$

Note that $i\dfrac{\partial b_\varepsilon}{\partial \theta} = 2\dfrac{\partial c_\varepsilon}{\partial u}$, where

$$c_\varepsilon(\theta, u) = \operatorname{sgn}(u)\sum_{n \in \mathbb{R}_\varepsilon + 1} e^{-in\theta - |nu|/2}.$$

Applying the summation formula for geometric progressions, we obtain

Lemma 3. *Let $f \in \mathcal{C}^2(G, \varepsilon)$, $m \in \mathbb{R}_\varepsilon$ with $|m| \leq 1$, $\theta \notin \pi\mathbb{Z}$. Then*

$$F_f^K(k_\theta) = \delta_{\varepsilon,\varepsilon_1}\Theta_{\varepsilon_1}(f) + \sum_{n \equiv m+1(2)} \operatorname{sgn}(n) e^{-in\theta}\Theta_{n+\operatorname{sgn} n}(f) + \frac{1}{2}\int_{-\infty}^{\infty} b_\varepsilon(\theta, u) F_f^A(a_u)\, du,$$

$$i\frac{d}{d\theta}F_f^K(k_\theta) = \sum_{n \equiv m+1(2)} |n| e^{-in\theta}\Theta_{n+\operatorname{sgn} n}(f) - \int_{-\infty}^{\infty} c_\varepsilon(\theta, u)\frac{d}{du}F_f^A(a_u)\, du,$$

where

$$b_\varepsilon(\theta,u)=\begin{cases}\dfrac{e^{-i(m-1)\theta}\cosh\frac{m+1}{2}u-e^{-i(m+1)\theta}\cosh\frac{m-1}{2}u}{\cosh u-\cos 2\theta}, & \text{if } |m|<1,\\ \dfrac{\sin 2\theta}{\cosh u-\cos 2\theta}, & \text{if } \varepsilon=\varepsilon_1,\end{cases}$$

$$c_\varepsilon(\theta,u)=\frac{e^{-i(m-1)\theta}\sinh\frac{m+1}{2}u-e^{-i(m+1)\theta}\sinh\frac{m-1}{2}u}{\cosh u-\cos 2\theta}.$$

(When we integrated by parts, the boundary term vanished, since $c_\varepsilon(\theta,0)=0$.) Inserting the formula for F_f^A from Lemma 2, we obtain an expression for F_f^K in terms of the irreducible characters. In order to make it explicit, we have to calculate

$$e_\varepsilon(\theta,r)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\cos(ru)b_\varepsilon(\theta,u)\,du.$$

All we need is the identity

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\lambda u}}{\cosh u-\cos 2\theta}\,du=\frac{\sinh(\pi-2\theta)\lambda}{\sin 2\theta\sinh\pi\lambda}$$

for $\theta\in(0,\pi)$, $\Re\lambda>0$, $|\Im\lambda|<1$ (see [19, Lemma VIII 3.2]). By a lengthy but elementary calculation one deduces from it that, for $\theta\in(0,\pi)$, $|m|<1$,

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\cos ru\cosh\frac{m+1}{2}u}{\cosh u-\cos 2\theta}\,du&=\frac{1}{2\pi}\Re\int_{-\infty}^{\infty}\frac{\exp\left(i|r|u+\frac{m+1}{2}u\right)}{\cosh u-\cos 2\theta}\,du\\&=\frac{1}{\sin 2\theta}\cdot\frac{\cosh 2(\pi-\theta)r\ \cos(m+1)\theta-\cosh 2\theta r\ \cos(m+1)(\pi-\theta)}{\cosh 2\pi r+\cos\pi m}.\end{aligned}$$

If we put absolute value on θ, this formula remains valid for $\theta\in(-\pi,0)$. Combining it with its counterpart for $-m$, we obtain a formula for e_ε, which is even valid for $|m|=1$ by continuity in view of $b_{\varepsilon_1}=\lim_{m\nearrow 1}\left(b_{\varepsilon_m}+b_{\varepsilon_{-m}}\right)/2$.

Lemma 4. *Let $f\in C^2(G,\varepsilon)$, $m\in\mathbb{R}_\varepsilon$ with $|m|\le 1$, $|\theta|\in(0,\pi)$. Then*

$$F_f^K(k_\theta)=\delta_{\varepsilon,\varepsilon_1}\Theta_{\varepsilon_1}(f)+\sum_{n\equiv m+1(2)}\operatorname{sgn}(n)e^{-in\theta}\Theta_{n+\operatorname{sgn}n}(f)+\frac{1}{2}\int_{-\infty}^{\infty}e_\varepsilon(\theta,r)\Theta_{\varepsilon,\frac{1}{2}+ir}(f)\,dr,$$

where

$$e_\varepsilon(\theta,r)=i\operatorname{sgn}(\theta)\frac{\cosh 2(\pi-|\theta|)r+e^{-i\pi m\operatorname{sgn}\theta}\cosh 2\theta r}{\cosh 2\pi r+\cos\pi m}.$$

Since F_f^K has the same behaviour under Z-translations as $f\in C^2(G,\varepsilon)$, it is easy to extend e_ε to arbitrary $\theta\notin\pi\mathbb{Z}$ such that Lemma 4 remains valid, namely,

$$e_\varepsilon(\theta,r)=i\operatorname{sgn}(\sin\theta)\frac{\varepsilon(z_+^{-1})\cosh 2(\theta-\theta_-)r+\varepsilon(z_-^{-1})\cosh 2(\theta-\theta_+)r}{\cosh 2\pi r+\cos\pi m},$$

where $z_\pm = k_{\theta_\pm} \in Z$ are the endpoints of the connected component of K_{reg} containing k_θ.

In the Selberg trace formula orbital integrals over any conjugacy class $\{x\}_G = \{y^{-1}xy : y \in G\}$ in G generally occur. In the previous lemmas, only those over conjugacy classes in G_{reg} (i.e., $\{a_u z\}_G$ with $u \neq 0$ and $\{k_\theta\}_G$ with $\theta \notin \pi\mathbb{Z}$) have been expressed in terms of characters. We need to do the same for the remaining classes $\{z\} \in Z$ and $\{n_{\pm 1} z\}_G$, too. This is easily reduced to Lemma 4 in virtue of the formulae

$$F_f^K(k_{\pm 0} z) := \lim_{\theta \to \pm 0} F_f^K(k_\theta z) = \pm 2\pi i \int_{NZ\backslash G} f(x^{-1} n_{\pm 1} z x)\, dx,$$
$$F_f^K(k_{+0} z) - F_f^K(k_{-0} z) = 2\pi i F_f^A(z),$$
$$\lim_{\theta \to 0} i \frac{d}{d\theta} F_f^K(k_\theta z) = 4\pi f(z),$$

where all occurring distributions are tempered. Indeed, since the natural projection $G \to G'$ maps some G-invariant neighbourhood of $\{n_{\pm 1} z\}_G$ to its image diffeomorphically, it suffices to prove these assertions for G', which has been done in [12] (cf. also [19], ch. VIII, section 2, [28], Theorem 47).

As above, we may restrict attention to $z = 1$. We shall also state the formulae obtained by passing to the limit in Lemma 3. Here one has to use the identity of distributions

$$\frac{1}{x \mp i0} = \text{p.v.}\frac{1}{x} \pm \pi i \delta(x).$$

It implies

$$\lim_{\theta \to \pm 0} \frac{e^{m(u/2 - i\theta)}}{e^{u/2 - i\theta} - e^{-(u/2 - i\theta)}} = \text{p.v.}\frac{e^{mu/2}}{e^{u/2} - e^{-u/2}} \pm 2\pi i \delta(u)$$

and the same for $-u$, which sum up to

$$\lim_{\theta \to \pm 0} b_\varepsilon(\theta, u) = \frac{\sinh \frac{mu}{2}}{\sinh \frac{u}{2}} \pm \pi i \delta(u)$$

for $m \in \mathbb{R}_\varepsilon$, $|m| < 1$, while $\lim_{\theta \to \pm 0} b_{\varepsilon_1}(\theta, u) = \pm 2\pi i \delta(u)$.

Lemma 5. *Let $f \in C^2(G, \varepsilon)$, $m \in \mathbb{R}_\varepsilon$ with $|m| \leq 1$. Then*

$$\begin{aligned}
F_f^K(k_{\pm 0}) - \delta_{\varepsilon,\varepsilon_1} \Theta_{\varepsilon_1}(f) - \sum_{n \equiv m+1(2)} \operatorname{sgn}(n) \Theta_{n + \operatorname{sgn} n}(f) \\
= \frac{1}{2} \int_{-\infty}^{\infty} \left(\frac{\sin \pi m}{\cosh 2\pi r + \cos \pi m} \pm i \right) \Theta_{\varepsilon, \frac{1}{2} + ir}(f)\, dr \\
= \pm \pi i F_f^A(1) + \frac{1}{2}(1 - \delta_{\varepsilon,\varepsilon_1}) \int_{-\infty}^{\infty} \frac{\sinh \frac{mu}{2}}{\sinh \frac{u}{2}} F_f^A(a_u)\, du,
\end{aligned}$$

$$\begin{aligned}
4\pi f(1) - \sum_{n \equiv m+1(2)} |n| \Theta_{n + \operatorname{sgn} n}(f) = \int_{-\infty}^{\infty} \frac{r \sinh 2\pi r}{\cosh 2\pi r + \cos \pi m} \Theta_{\varepsilon, \frac{1}{2} + ir}(f)\, dr \\
= - \int_{-\infty}^{\infty} \frac{\cosh \frac{mu}{2}}{\sinh \frac{u}{2}} \cdot \frac{d}{du} F_f^A(a_u)\, du.
\end{aligned}$$

In accordance with the general theory, the Pancherel measure has turned out to be a multiple of

$$\mu(\varepsilon, \tfrac{1}{2} + ir) = \frac{\pi r \sinh 2\pi r}{\cosh 2\pi r + \cos \pi m}.$$

If we insert $f*g^*$, where f, $g \in \mathcal{C}^2(G, \varepsilon)$, we obtain the Plancherel formula (compare, e.g., [10], [14], [19], [20], [23], [28])

$$4\pi(f,g) = \sum_{n \equiv m+1(2)} |n| \langle \pi_{n+\operatorname{sgn} n}(f), \pi_{n+\operatorname{sgn} n}(g) \rangle$$
$$+ \int_{-\infty}^{\infty} \langle \pi_{\varepsilon, \frac{1}{2}+ir}(f), \pi_{\varepsilon, \frac{1}{2}+ir}(g) \rangle \pi^{-1} \mu(\varepsilon, \tfrac{1}{2} + ir)\, dr,$$

where $\langle B, C \rangle = \operatorname{tr}(BC^*)$ is the scalar product for Hilbert-Schmidt operators in the corresponding representation space. This provides a natural invariant decomposition $L^2(G,\varepsilon) = L^2_{\text{dis}}(G,\varepsilon) \oplus L^2_{\text{con}}(G,\varepsilon)$. The notion "discrete series" used above is therefore justified in so far as it consists of representations discretely occurring in $L^2(G,\varepsilon)$ for the pertinent central character $\varepsilon \in \hat{Z}$. One can prove the deeper result that the aforementioned decomposition, intersected with $\mathcal{C}^2(G,\varepsilon)$, yields a decomposition $\mathcal{C}^2(G,\varepsilon) = \mathcal{C}^2_{\text{dis}}(G,\varepsilon) \oplus \mathcal{C}^2_{\text{con}}(G,\varepsilon)$ (see [28], ch. 8, for $\varepsilon = \varepsilon_0$ or ε_1).

We have now finished the necessary harmonic analysis of invariant distributions on G, i.e., such which take the same value on f as on $f^x : y \mapsto f(xyx^{-1})$. Unfortunately, the truncation procedure usually applied in the proof of the trace formula for non-uniform lattices produces certain non-invariant distributions, which can clearly not be expressed in terms of characters only. These are the so-called weighted orbital integrals, which are defined for $f \in \mathcal{C}^2(G,\varepsilon)$, $a_u z \in AZ$, $u \neq 0$, as

$$T_f^A(a_u z) = -\left|e^{u/2} - e^{-u/2}\right| \int_{AZ\backslash G} f(x^{-1} a_u z x)(H(x) + \bar{H}(x))\, dx.$$

Here we use the notation

$$H(n a_u k) = u, \qquad \bar{H}(\bar{n} a_u k) = -u$$

for $n \in N$, $\bar{n} \in \bar{N}$, $k \in K$. Note that $H + \bar{H}$ is left A-invariant and negative. Due to the Iwasawa decomposition it suffices to check the latter on $\bar{N}$, and in fact $H(\bar{n}_v) = -\log(1 + v^2)$.

As well as F_f^A, T_f^A is an even function of u and a tempered distribution. However, while F_f^A satisfies the homogeneous differential equation

$$\frac{d^2}{du^2} F_f^A(a_u z) = F^A_{(\omega+1/4)f}(a_u z)$$

([13], [28], Theorem 17), where ω is the Casimir element introduced in section 1, T_f^A satisfies the inhomogeneous equation

$$\frac{d^2}{du^2} T_f^A(a_u z) = T^A_{(\omega+1/4)f}(a_u z) + \tfrac{1}{2}(\sinh \tfrac{u}{2})^{-2} F_f^A(a_u z)$$

(see [2]). Moreover, it has been shown in [2], [3] (strictly speaking, only for $\varepsilon = \varepsilon_0$ or ε_1) that if $f \in \mathcal{C}^2_{\text{dis}}(G,\varepsilon)$, $m \in \mathbb{R}_\varepsilon$, $u \neq 0$, then T^A_f can be expressed by the discrete series characters:

$$T^A_f(a_u z) = -\left|e^{u/2} - e^{-u/2}\right| \sum_{\substack{n \equiv m+1(2) \\ n \neq 0}} \Theta_{-n-\operatorname{sgn} n}(a_u z)\Theta_{n+\operatorname{sgn} n}(f).$$

In particular, the restriction of T^A_f to $\mathcal{C}^2_{\text{dis}}(G,\varepsilon)$ turns out to be invariant. One can check (which we shall not do here), that the proof applies to the present case, too.

It remains to determine the Fourier transform of T^A_f for f in the complementary subspace $\mathcal{C}^2_{\text{con}}(G,\varepsilon)$. This has been done in [5] for ε_0 and ε_1. Again one can generalize the argument to any $\varepsilon \in \hat{Z}$. We shall not explicate this here but only restate in our parametrization what Lemma 3.2 of [5] will then provide. For this purpose, given $f \in \mathcal{C}^2(G,\varepsilon)$ (which we suppose to be K-finite for simplicity, although this is unnecessary) and $u \neq 0$, we define

$$\begin{aligned} I^P_f(a_u z) &= T^A_f(a_u z) - \frac{\varepsilon(z^{-1})}{2}(1-\delta_{\varepsilon,\varepsilon_1})\,\Theta_{\varepsilon,\frac12}(f) \\ &+ \frac{\varepsilon(z^{-1})}{2\pi}\text{p.v.}\int_{-\infty}^{\infty} e^{-iru}\,\operatorname{tr}\left(\pi_{\varepsilon,\frac12+ir}(f)J_{\bar{P}P}(\varepsilon,\tfrac12+ir)^{-1}J'_{\bar{P}P}(\varepsilon,\tfrac12+ir)\right)\,dr, \end{aligned}$$

where $J'_{\bar{P}P} = \frac{d}{ds}J_{\bar{P}P}$. Note that $I^P_f(a_u z) = \varepsilon(z^{-1})I^P_f(a_u)$. (One may define I^P_f for $f \in \mathcal{C}^2(G)$ by putting a further integration over $\varepsilon \in \hat{Z}$ on the last two terms.) If $f \in \mathcal{C}^p(G,\varepsilon)$ with $p < 2$, we may choose $\sigma < \frac12$ such that the integrand is holomorphic for $\sigma \leq \Re s \leq \frac12$ except for a simple pole at $\frac12$ (if $\varepsilon = \varepsilon_1$), whence

$$I^P_f(a_u z) = T^A_f(a_u z) + \frac{\varepsilon(z^{-1})}{2\pi i}\int_{\Re s=\sigma} e^{(1/2-s)u}\,\operatorname{tr}\left(\pi_{\varepsilon,s}(f)J_{\bar{P}P}(\varepsilon,s)^{-1}J'_{\bar{P}P}(\varepsilon,s)\right)\,ds.$$

Lemma 6. *For $u > 0$, $m \in \mathbb{R}_\varepsilon$ and $f \in \mathcal{C}^2_{\text{con}}(G,\varepsilon)$ satisfying*

$$f(k_\theta x k_{\theta'}) = e^{-im(\theta+\theta')}f(x),$$

one has

$$I^P_f(a_u) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi_{m,r}(u)\Theta_{\varepsilon,\frac12+ir}(f)\,dr,$$

where $\phi_{m,r}$ is a continuous function of at most polynomial growth in the r variable satisfying $\lim_{u\to\infty}\phi_{m,r}(u) = 0$, uniformly on compacta in r. Consequently, $I^P_f(a)$ is an invariant distribution.

(In [5], the right-hand side of (2.8), and therefore that of (2.18), as well as the last three terms of (1.8) should be prefaced with an additional factor $(2\pi)^{-1}$: With $\mu_\chi(\lambda)$ as used on p. 32, the Plancherel measure corresponding to the Haar measure adopted on p. 21 is $(2\pi)^{-1}\mu_\chi(\lambda)$.)

In view of $\Theta_{\varepsilon,\frac12+ir}((\omega+\frac14)f) = -r^2\Theta_{\varepsilon,\frac12+ir}(f)$, I^P_f satisfies the same differential equation as T^A_f. This together with Lemma 2 implies that

$$\phi''_{m,r}(u) = -r^2\phi_{m,r}(u) + \tfrac12\left(\sinh\tfrac{u}{2}\right)^{-2}\cos ru.$$

It is clear that $\phi_{m,r} = (\psi_r + \psi_{-r})/2$, where ψ_r is a solution of

$$\psi_r''(u) = -r^2\psi_r(u) + \tfrac{1}{2}\left(\sinh\tfrac{u}{2}\right)^{-2} e^{iru}$$

on $(0,\infty)$. The substitution $\psi_r(u) = e^{iru}\chi_r(u)$ yields

$$\chi_r''(u) + 2ir\chi_r'(u) = \tfrac{1}{2}\left(\sinh\tfrac{u}{2}\right)^{-2},$$
$$\chi_r'(u) + 2ir\chi_r(u) = \frac{2}{1-e^u} + c_1.$$

Putting now $\chi_r(u) = e^{-2ru}\omega_r(u)$, we obtain

$$\omega_r'(u) = \frac{2e^{2iru}}{1-e^u} + c_1 e^{2iru}.$$

One solution for $c_1 = 0$ is given by

$$-2\int_u^\infty \frac{e^{2irv}}{1-e^v}\,dv = 2\int_0^{e^{-u}} t^{-2ir}(1-t)^{-1}\,dt = 2B_{e^{-u}}(1-2ir,0),$$

where B denotes the incomplete beta function. In general,

$$\omega_r(u) = 2B_{e^{-u}}(1-2ir,0) + c_2e^{2iru} + c_3.$$

Hence

$$\phi_{m,r}(u) = e^{iru}B_{e^{-u}}(1+2ir,0) + e^{-iru}B_{e^{-u}}(1-2ir,0) + c_4e^{iru} + c_5e^{-iru}.$$

The condition on the limit implies $c_4 = c_5 = 0$. In particular, $\phi_{m,r}$ is independent of m, and $\phi_{m,0}(u) = -2\log(1-e^{-u})$. From the integral expression for ω_r we see that

$$\psi_r(u) = \int_u^\infty \frac{e^{iru}}{e^{(u+v)/2}-1}\,dv$$

and, since $\Theta_{\varepsilon,\frac{1}{2}+ir}$ is even in r,

$$\frac{1}{2\pi}\int_{-\infty}^\infty \phi_{m,r}(u)\Theta_{\varepsilon,\frac{1}{2}+ir}(f)\,dr = \int_u^\infty \frac{1}{e^{(u+v)/2}-1}F_f^A(a_v)\,dv$$

by Lemma 2.

In [4], J. Arthur introduces a distribution similar to I_f^P, using $R_{\bar{P}P}$ instead of $J_{\bar{P}P}$. While his distribution is even in u, ours is not: The evenness of T_f^A implies that

$$\begin{aligned} I_f^P(a_u) - I_f^P(a_{-u}) &= \frac{1}{2\pi}\text{p.v.}\int_{-\infty}^\infty e^{-iru}\,\mathrm{tr}\Big(\pi_{\varepsilon,\frac{1}{2}+ir}(f)\big(J_{P\bar{P}}(\varepsilon,\tfrac{1}{2}-ir)J_{\bar{P}P}'(\varepsilon,\tfrac{1}{2}+ir) \\ &\qquad - J_{P\bar{P}}'(\varepsilon,\tfrac{1}{2}-ir)J_{\bar{P}P}(\varepsilon,\tfrac{1}{2}+ir)\big)\Big)\,dr \\ &= \frac{1}{2\pi}\text{p.v.}\int_{-\infty}^\infty e^{iru}\frac{\mu'(\varepsilon,\frac{1}{2}+ir)}{\mu(\varepsilon,\frac{1}{2}+ir)}\Theta_{\varepsilon,\frac{1}{2}+ir}(f)\,dr, \end{aligned}$$

where we have used the intertwining property of $J_{\bar{P}P}$ and of left translation by $w \in W - Z$. Explicitly, for $m \in \mathbb{R}_\varepsilon$,

$$\frac{\mu'(\varepsilon, \frac{1}{2} + ir)}{\mu(\varepsilon, \frac{1}{2} + ir)} = \frac{1}{ir} - 2\pi i \left(\coth 2\pi r - \frac{\sinh 2\pi r}{\cosh 2\pi r + \cos \pi m}\right).$$

In order to express $I_f^P(a_u) - I_f^P(a_{-u})$ in terms of F_f^A, we shall now calculate the Fourier transform of the distribution p.v.$\frac{\mu'}{\mu}$. Using the integral formula for ψ and Lemma 2, we get

$$\frac{1}{2\pi} \int_{\Re s = \sigma} e^{(1/2-s)u} \frac{j_m'(s)}{j_m(s)} \Theta_{\varepsilon,s}(f)\, ds = \int_0^\infty \frac{e^{mv/2} + e^{-mv/2} - e^{v/2} - 1}{e^{v/2} - e^{-v/2}} F_f^A(a_{u+v})\, dv,$$

where $f \in \mathcal{C}^p(G, \varepsilon)$, $\frac{1}{2} < \sigma \leq \frac{1}{p}$, $m \in \mathbb{R}_\varepsilon$, $|m| \leq 1$. This implies that

$$I_f^P(a_u) - I_f^P(a_{-u}) = \int_{-\infty}^\infty \left(\frac{\cosh \frac{mv}{2}}{\sinh \frac{v}{2}} + \frac{\operatorname{sgn} v}{e^{-|v|/2} - 1}\right) F_f^A(a_{u+v})\, dv,$$

valid even for $f \in \mathcal{C}^2(G, \varepsilon)$ by continuity. Combining our results, we obtain the following more explicit variant of Theorem 1.8 of [5].

Proposition 7. *For $u \neq 0$, $f \in \mathcal{C}^2(G, \varepsilon)$ and $m \in \mathbb{R}_\varepsilon$ with $|m| \leq 1$ one has*

$$\begin{aligned}
I_f^P(a_u) &+ \sum_{\substack{n \equiv m+1(2) \\ n \neq 0}} e^{-|nu|/2} \Theta_{n + \operatorname{sgn} n}(f) \\
&= \frac{1}{2\pi} \int_{-\infty}^\infty \left(e^{ir|u|} B_{e^{-|u|}}(1 + 2ir, 0) + e^{-ir|u|} B_{e^{-|u|}}(1 - 2ir, 0)\right) \Theta_{\varepsilon, \frac{1}{2}+ir}(f)\, dr \\
&\quad + \frac{1 - \operatorname{sgn} u}{2} \int_{-\infty}^\infty \sin(ru) \left(\frac{1}{2\pi r} + \coth 2\pi r - \frac{\sinh 2\pi r}{\cosh 2\pi r + \cos \pi m}\right) \Theta_{\varepsilon, \frac{1}{2}+ir}(f)\, dr \\
&= \int_u^\infty \frac{1}{e^{(u+v)/2} - 1} F_f^A(a_v)\, dv - \frac{1 - \operatorname{sgn} u}{2} \int_{-\infty}^\infty \frac{\cos \frac{m(u+v)}{2}}{\sin \frac{u+v}{2}} F_f^A(a_v)\, dv.
\end{aligned}$$

(The singularities occurring in the last two integrals for $u < 0$ cancel each other.)

Now we consider the behaviour of $T_f^A(a_u z)$ as $u \to 0$. Using the Iwasawa decomposition, we get

$$\begin{aligned}
T_f^A(a_u z) &= \left|e^{u/2} - e^{-u/2}\right| \int_{Z\backslash K} \int_N f(k^{-1} n_v^{-1} a_u z n_v k) \log(1 + v^2)\, dn_v\, dk \\
&= e^{u/2} \int_{Z\backslash K} \int_N f(k^{-1} a_u z n_v k) \log\left(1 + (1 - e^{-u})^{-2} v^2\right)\, dn_v\, dk,
\end{aligned}$$

which, unlike the second expression for $F_f^A(a_u z)$, does not extend continuously to $u = 0$. In [2], J. Arthur defines the singular weighted orbital integral as

$$\begin{aligned}
T_f^A(z) &= \lim_{u \to 0} \left(T_f^A(a_u z) + \log(1 - e^{-u})^2 F_f^A(a_u z)\right) \\
&= 2 \int_{Z\backslash K} \int_N f(k^{-1} z n_v k) \log |v|\, dn_v\, dk,
\end{aligned}$$

which also appears in the trace formula. It is easy to check that $T_f^A(z)$ is a tempered distribution, too. Let us define $I_f^P(z)$ by the same formula as before. In order to pass to the limit in Proposition 7, observe that

$$\omega_r(u) + \log(1-e^{-u})^2 = \omega_r(u) - \omega_0(u) = 2\int_u^\infty \frac{e^{(2ir-1)v} - e^{-v}}{1-e^{-v}}\,dv,$$

which for $u \to 0$ gives $2\psi(1) - 2\psi(1-2ir)$ with

$$\psi(s) = \frac{\Gamma'(s)}{\Gamma(s)} = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-st}}{1-e^{-t}}\right) dt$$

($\Re s > 0$). Hence,

$$\lim_{u\to 0}\left(\phi_{m,r}(u) - \log(1-e^{-u})^2\cos ru\right) = 2\psi(1) - \psi(1+2ir) - \psi(1-2ir).$$

On the other hand, integration by parts shows that

$$\int_u^\infty \frac{1}{e^{(u+v)/2}-1} F_f^A(a_v)\,dv - \omega_0(u)F_f^A(a_u)$$
$$= -2\int_u^\infty \log(1-e^{-(u+v)/2})\frac{d}{dv}F_f^A(a_v)\,dv,$$

which also converges as $u \to 0$. So we obtain the following variant of Proposition 4.7 of [5].

Proposition 8. *For $f \in \mathcal{C}^2(G,\varepsilon)$ and $m \in \mathbb{R}_\varepsilon$,*

$$I_f^P(1) + \sum_{\substack{n\equiv m+1(2)\\ n\neq 0}} \Theta_{n+\operatorname{sgn} n}(f) = -2\int_0^\infty \log\left(1-e^{-u/2}\right)\frac{d}{du}F_f^A(a_u)\,du$$
$$= -\frac{1}{2\pi}\int_{-\infty}^\infty (2C + \psi(1+2ir) + \psi(1-2ir))\Theta_{\varepsilon,\frac12+ir}(f)\,dr,$$

where $C = -\psi(1)$ is the Euler-Mascheroni constant.

3 Automorphic forms

Let Γ be a lattice in G and denote its projection on G' by Γ'.

Lemma 9. *Γ' is a lattice in G', and $[Z : \Gamma \cap Z] < \infty$.*

Proof. The Lie algebra of the closure of Γ' is $\mathrm{Ad}(\Gamma)$-invariant and thus $\mathrm{Ad}(G)$-invariant by the Borel density theorem ([24], [6]), i.e., an ideal in $\mathfrak{g}$. But $\mathfrak{g}$ is simple, so if we assume that Γ' is not discrete, then it has to be dense in G'. From this we shall now deduce a contradiction.

Choose a neighbourhood $\mathcal{U}$ of 1 in G and a number $\varepsilon > 0$. Γ' is dense in the open set of elliptic elements in G'. So we can find two elliptic elements $\gamma_1, \gamma_2 \in \Gamma$ (i.e., γ_i is conjugate to some k_{θ_i} with $\theta_i \notin \pi\mathbb{Z}$) with different fixed points z_1, z_2 on the upper half-plane $\mathcal{H}$ such that γ_2' is close to γ_1', and hence $z_2 = gz_1$ for some $g \in \mathcal{U}$. There always are nonzero integers n_1, n_2 with $|n_1\theta_1 - n_2\theta_2| < \varepsilon$. The fixed point of $g\gamma_1 g^{-1}$ is z_2, hence, for ε small enough,

$$\gamma_2^{n_2} \in g\gamma_1^{n_1}g^{-1}\mathcal{U} \subset \mathcal{U}\gamma_1^{n_1}\mathcal{U}^{-1}\mathcal{U}.$$

Γ being discrete, we get $\gamma_2^{n_2} = \gamma_1^{n_1}$ for sufficiently small $\mathcal{U}$, and $z_1 \neq z_2$ then implies $\gamma_1^{n_1} \in Z$. Therefore, $[\Gamma Z : \Gamma] = [Z : \Gamma \cap Z]$ is finite, ΓZ is discrete, and so is $\Gamma' = \Gamma Z/Z$ in contradiction to our assumption. $\square$

Given Γ as above and a unitary representation χ of Γ on a finite dimensional hermitian vector space V, we consider the induced representation $\pi_\chi = \mathrm{Ind}_\Gamma^G(\chi)$. It acts on the Hilbert space $\mathcal{H}_\chi$ consisting of classes of measurable functions $\varphi : G \to V$ which satisfy $\varphi(\gamma x) = \chi(\gamma)\varphi(x)$ (for all $\gamma \in \Gamma$ and a.e. $x \in G$) and $|\varphi| \in L^2(\Gamma\backslash G)$. One may interpret such φ as L^2-sections of the hermitian vector bundle $V \times_\Gamma G$ over $\Gamma\backslash G$ with monodromy χ.

For any $\varepsilon \in \hat{Z}$, let $V_\varepsilon = V(\varepsilon\big|_{\Gamma\cap Z})$ be the ε-isotypical component of χ. Then we may define a unitary representation $\chi_\varepsilon(\gamma z) = \chi(\gamma)\varepsilon(z)\big|_{V_\varepsilon}$ of ΓZ on V_ε. We shall endow $\mathcal{H}_\chi$ with the slightly modified scalar product

$$(\varphi, \psi) = n_\Gamma^{-1} \int_{\Gamma\backslash G} (\varphi(x), \psi(x))\, dx,$$

$n_\Gamma = [Z : \Gamma \cap Z]$, for then the ε-isotypical component $\mathcal{H}_\chi(\varepsilon)$ of π_χ is easily seen to be $\mathcal{H}_{\chi_\varepsilon}$. (Thus one would not restrict generality seriously by assuming that $Z \subset \Gamma$ and $\chi\big|_Z = \varepsilon\,\mathrm{Id}$.) Note that there are only finitely many ε for which $V_\varepsilon \neq \{0\}$ and thus $\mathcal{H}_\chi(\varepsilon) \neq \{0\}$. Decomposing $\pi_\chi\big|_K$ into isotypical components, we obtain the Hilbert direct sum

$$\mathcal{H}_\chi(\varepsilon) = \widehat{\bigoplus_{m \in \mathbb{R}_\varepsilon}} \mathcal{H}_\chi(\phi_m),$$

whose constituents are just the spaces of square integrable automorphic forms of weight m with respect to Γ and χ. While many papers are devoted to the Selberg trace formula for a single weight m, our motivation is to decompose π_χ into a direct integral of irreducible unitary representations of G (in the spirit of [10], say).

To keep later notations shorter, let us agree to write $P_0 = N_0 A_0 Z$ for the standard parabolic subgroup which has been called $P = NAZ$ in section 1. A general parabolic subgroup of G is then $P = k^{-1}P_0 k$ with unipotent radical $N = k^{-1}N_0 k$ and special split component $A = k^{-1}A_0 k$, where $k \in K$ is arbitrary. Conjugation by k transports the Haar measures to P, A and N. For each P, we fix one such $k = k_P$ in the Z-coset of possible ones. If we define $a_{P,u} = k_P^{-1} a_u k_P \in A$ for $u \in \mathbb{R}$, and $H_P(x) = u$ for $x \in N a_{P,u} K$, then $H_P(x) = H(k_P x)$ generalizes $H = H_{P_0}$ and $\bar{H} = H_{\bar{P}_0}$.

A parabolic subgroup P of G is called cuspidal (w.r.t. Γ) if its unipotent radical N contains a nontrivial element of Γ. As one knows, the finitely many cusps of the Riemann surface $\Gamma\backslash\mathcal{H}$ are parametrized by the Γ-conjugacy classes $\{P\}_\Gamma = \{\gamma P \gamma^{-1} : \gamma \in \Gamma/\Gamma \cap P\}$ of cuspidal subgroups. The Iwasawa decomposition $\mathcal{H} \cong G/K \cong NA$ provides a parametrization of the geodesics $nA(i) \subset \mathcal{H}$, $n \in N$, which tend to the boundary point $k_P^{-1}(\infty) \in \mathbb{R} \cup \{\infty\}$. The parameter value is given by the function H_P whose potential surfaces are the N-orbits (horocycles) on $\mathcal{H}$. However, this parametrization determined by the choice of K (or of $i \in \mathcal{H}$) is not adapted to Γ: a geodesic on $\Gamma\backslash\mathcal{H}$ running into a cusp has various lifts to $\mathcal{H}$, from which it inherits different parametrizations.

To rectify this, we replace k_P by $g_P = a_{u_P} k_P$, where $e^{-u_P} = \mathrm{vol}(\Gamma Z \cap N\backslash N)$. Then $\Gamma Z \cap N = \{g_P^{-1} n_v g_P : v \in \mathbb{Z}\}$, and one checks that

$$H_P(x) + u_P = H_{P_0}(g_P x),$$

which is our new parameter. The value 0 now corresponds to the horocycle whose projection on $\Gamma\backslash\mathcal{H}$ has length 1.

Given a cuspidal $P = NAZ$, we denote by V^P the maximal subspace of V on which $\chi\big|_{\Gamma\cap P}$ acts trivially, by pr^P the orthoprojection on V^P and, for every $\varphi \in \mathcal{H}_\chi$, by

$$\varphi^P(x) = \mathrm{vol}(\Gamma \cap N\backslash N)^{-1} \int_{\Gamma\cap N\backslash N} \mathrm{pr}^P \varphi(nx)\, dn$$

its "constant term" along P (convergent in $L^1_{\mathrm{loc}}(G)$). It has the propreties

$$\begin{aligned} \varphi^P(nx) &= \varphi^P(x) \qquad \text{for } n \in N, \\ \varphi^P(\gamma x) &= \chi(\gamma)\varphi^P(x) \qquad \text{for } \gamma \in \Gamma \cap P. \end{aligned}$$

(Since $\Gamma \cap N$ is normal in $\Gamma \cap P$, the measure dn is $\Gamma \cap P$-invariant.) If $\varphi \in \mathcal{H}_\chi(\varepsilon)$, then φ^P takes values in the trivial subspace V_ε^P of $\chi_\varepsilon\big|_{\Gamma Z\cap N}$, which happens to be smaller than $V_\varepsilon \cap V^P$. In such case, the corresponding cusp has been called irregular in [1].

Note that one needs only consider one constant term for each cusp, since

$$\varphi^{\gamma P \gamma^{-1}}(x) = \chi(\gamma)\varphi^P(\gamma^{-1}x).$$

It is therefore useful to fix a (finite) set $\mathfrak{F}$ of representatives for the Γ-conjugacy classes of cuspidal subgroups and to define

$$\varphi^{\mathrm{cst}}(x) = (\varphi^P(g_P^{-1}x))_{P\in\mathfrak{F}},$$

a left N_0-invariant function with values in $V^{\mathrm{cst}} = \bigoplus_{P\in\mathfrak{F}} V^P$. If $\varphi \in \mathcal{H}_\chi(\varepsilon)$, then $\varphi^{\mathrm{cst}}(x) \in V_\varepsilon^{\mathrm{cst}}$ in the obvious sense. The identity $(\pi_\chi(x)\varphi)^P(y) = \varphi^P(yx)$ shows that, heuristically, the constant term operator is something like an intertwining operator. This will be made exact in Proposition 11.

The constant terms give, of course, no information about the G-invariant subspace

$$\mathcal{H}_\chi^{\mathrm{cus}} = \{\varphi \in \mathcal{H}_\chi : \varphi^P = 0 \text{ for all cuspidal } P\}$$

of cusp forms (which equals $\mathcal{H}_\chi$ if Γ is a uniform lattice). A crucial result is that for every K-finite eigenfunction $\varphi \in \mathcal{H}_\chi$ of the Casimir element ω, every compact set $\Omega \in G$ and every $n \in \mathbb{N}$ one can find a constant C such that

$$\left|(\varphi - \varphi^P)(xy)\right| \le Ce^{-nH_P(x)} \qquad \text{for } H_P(x) > u_0,\ y \in \Omega.$$

Note that, for sufficiently large u_0, $\Gamma \cap G \backslash \{x \in G : H_P(x) + u_P > u_0\}$ projects diffeomorphically on some "neighbourhood" $\mathcal{C}_{P,u_0} \subset \Gamma \backslash G$ of the cusp corresponding to P, and

$$\int_{\mathcal{C}_{P,u_0}} |\varphi(x)|^2 dx = n_{\Gamma,P}^{-1} \int_{\Gamma \cap Z \backslash K} \int_{u_0 - u_P}^{\infty} \int_{\Gamma \cap N \backslash N} |\varphi(n a_{P,u} k)|^2 dn\, e^{-u} du\, dk,$$

where $n_{\Gamma,P} = [\Gamma Z \cap N : \Gamma \cap N]$. The rapid decrease in the cusps is the reason why the restriction of $\pi_\chi(f)$ to $\mathcal{H}_\chi^{\text{cus}}$ is of the trace class for $f \in \mathcal{C}^1(G)$ (see [21]). As a consequence, $\mathcal{H}_\chi^{\text{cus}}$ is contained in $\mathcal{H}_\chi^{\text{dis}}$, the maximal discretely decomposable subspace of $\mathcal{H}_\chi$. Moreover, every irreducible unitary representation of G occurs in $\mathcal{H}_\chi^{\text{cus}}$ with at most finite multiplicity.

We have now an orthogonal decomposition

$$\mathcal{H}_\chi = \mathcal{H}_\chi^{\text{dis}} \oplus \mathcal{H}_\chi^{\text{con}} = \mathcal{H}_\chi^{\text{cus}} \oplus \mathcal{H}_\chi^{\text{res}} \oplus \mathcal{H}_\chi^{\text{con}},$$

where the superscripts "res", "con" will become clear later. The above estimates also entail restrictions on the possible representations which may occur in $\mathcal{H}_\chi^{\text{res}}$, because φ^P must then be square integrable on $\mathcal{C}_{P,u_0}$.

Lemma 10. *The representations of G occurring in $\mathcal{H}_\chi^{\text{res}}$ belong either to the complementary series, or to the pseudo-discrete series, or are trivial. Moreover, if φ is a K-finite element in the isotypical component $\mathcal{H}_\chi^{\text{res}}(\pi_{\epsilon,s})$ and w.l.o.g. $s \in (0, 1/2)$, then $\varphi^P \in \mathcal{H}_{P,\epsilon,s} \otimes V_\epsilon^P$.*

Proof. Let $\varphi \in \mathcal{H}_\chi^{\text{res}}(\pi_{\epsilon,s})$ be K-finite. Clearly, $\pi_\chi(\omega)\varphi = s(s-1)\varphi$. Take a cuspidal P. Replacing Γ by a conjugate subgroup if necessary, we may assume that $P = P_0$. Now, in the notation of section 1, $\omega + \frac{1}{4} = \frac{1}{4}(H-1)^2 + XY$, $\varphi^P((\omega + \frac{1}{4})x) = \frac{1}{4}\varphi^P((H-1)^2 x)$,

$$\left(s - \frac{1}{2}\right)^2 \varphi^P(a_u k) = \left(\frac{d}{du} - \frac{1}{2}\right)^2 \varphi^P(a_u k).$$

This differential equation has the basic solutions e^{su} and $e^{(1-s)u}$ (resp. $ue^{u/2}$ if $s = \frac{1}{2}$). But e^{su} is square integrable on (u_0, ∞) with respect to $e^{-u}du$ iff $\Re s < \frac{1}{2}$. Thereby the principal series is excluded, and the assertion about the complementary series follows.

Let now φ be a vector of weight $\pm m$ in $\mathcal{H}_\chi(\pi_{\pm m})$, $m \geq 0$. Then

$$0 = \varphi^P(a_u E_\mp) = \varphi^P\big(a_u(h \pm i(X - Y))\big) = \left(2\frac{d}{du} - m\right)\varphi^P(a_u),$$

thus $\varphi^P(a_u k) = e^{mu/2}\varphi^P(k)$, which is square integrable iff $m < 1$. $\square$

As for the trivial representation π_0 of G, it is clear that $\mathcal{H}_\chi(\pi_0) \subset \mathcal{H}_\chi^{\text{res}}$ consists of the constant functions with values in the subspace of χ-invariants in V. The multiplicity of the other irreducible representations of G in $\mathcal{H}_\chi$ is hard to determine (except for the discrete series).

A refinement of this question is connected with Hecke operators. We denote by

$$\bar{\Gamma} = \{\xi \in \Gamma : \xi^{-1}\Gamma\xi \text{ is commensurable with } \Gamma\}$$

the commensurator of Γ in G and define the Hecke algebra as

$$\mathfrak{H}(G,\chi) = \{t : \bar{\Gamma} \to \mathrm{End}V \big| \ t(\gamma_1\xi\gamma_2) = \chi(\gamma_1)t(\xi)\chi(\gamma_2) \text{ for } \gamma_1,\ \gamma_2 \in \Gamma,\ \xi \in \bar{\Gamma}; \\ \mathrm{supp}(t)/\Gamma \text{ is finite}\}$$

with operation

$$t_1 * t_2(\xi) = \sum_{\eta \in \bar{\Gamma}/\Gamma} t_1(\eta)t_2(\eta^{-1}\xi) = \sum_{\eta \in \Gamma\backslash\bar{\Gamma}} t_1(\xi\eta^{-1})t_2(\eta),$$

involution $t^*(\xi) = t(\xi^{-1})^*$ and unit element χ (extended by zero to all of $\bar{\Gamma}$). The restriction of t to a double coset $\Gamma\xi\Gamma$ is determined by the value $t(\xi)$, which can be arbitrary in $\mathrm{Hom}_{\Gamma\cap\xi^{-1}\Gamma\xi}(V, V^{\xi})$, if V^{ξ} denotes V endowed with the representation $\chi^{\xi}(\gamma) = \chi(\xi\gamma\xi^{-1})$. $\mathfrak{H}(G,\chi)$ has a $*$-representation

$$(\tau_\chi(t)\varphi)(x) = \sum_{\eta \in \bar{\Gamma}/\Gamma} t(\xi)\varphi(\xi^{-1}x)$$

on $\mathcal{H}_\chi$ commuting with π_χ (see [16] for the case $\chi = 1$). The restriction of $\tau_\chi(t)$ to $\mathcal{H}_\chi(\varepsilon)$ is $\tau_{\chi_\varepsilon}(t_\varepsilon)$, where

$$t_\varepsilon(\xi) = \sum_{z \in Z/\Gamma\cap Z} t(\xi z)\varepsilon(z^{-1})\mathrm{pr}_\varepsilon.$$

One easily recovers t from the t_ε's as $\sum_\varepsilon t_\varepsilon = n_\Gamma t$, using that $\sum_{\varepsilon|_{\Gamma\cap Z}=\varepsilon'} \varepsilon(z) = 0$ if $z \notin \Gamma \cap Z$. For $\chi = 1$ and congruence subgroups Γ, one can take the adèlic point of view, well suited for the study of Hecke operators $\tau_\chi(t)$ (see [17]). One is interested in their traces in the isotypical components $\mathcal{H}_\chi(\pi)$. Some information can be obtained with the help of Eisenstein serties, which will be considered in section 5. In order to state the results, let us introduce certain Dirichlet series which will also appear in the trace formula.

Some preparation is necessary. Take any $w \in W \setminus Z$. The Bruhat decomposition

$$G = P_0 \cup N_0 w P_0 \qquad \text{(disjoint union)},$$

being Z-invariant, is a simple consequence of that of G. For any pair P, Q of cuspidal subgroups and any $x \in G$,

$$\begin{array}{lll} \text{either} & x \in g_Q^{-1} z_{QP}(x) a_u N_0 g_P & \text{with } z_{QP}(x) \in Z \\ \text{or} & x \in g_Q^{-1} N_0 w_{QP}(x) a_u N_0 g_P & \text{with } w_{QP}(x) \in W \setminus Z, \end{array}$$

depending on whether $xPx^{-1} = Q$ or not. Here $H_{QP}(x) = u$ and $z_{QP}(x)$ resp. $w_{QP}(x)$ are uniquely determined. We shall use the notation

$$\begin{aligned}\bar{\Gamma}_{QP}(1) &= \bar{\Gamma} \cap g_Q^{-1} P_0 g_P,\\ \bar{\Gamma}_{QP}(w) &= \bar{\Gamma} \cap g_Q^{-1} N_0 w P_0 g_P.\end{aligned}$$

For any $\xi_0 \in \bar{\Gamma}_{QP}(1)$, the map $\xi \mapsto \xi\xi_0$ induces an injection $\bar{\Gamma}_{QP}(1)/\Gamma \cap P \xrightarrow{\sim} \bar{\Gamma} \cap P/\Gamma \cap P \to \bar{\Gamma}/\Gamma$.

Given $t \in \mathfrak{H}(G,\chi)$, $\varepsilon \in \hat{Z}$ and $s \in \mathbb{C}$, let us consider the finite sum

$$c_{QP}(t,1,\varepsilon,s) = \sum_{\xi \in \bar{\Gamma}_{QP}(1)/\Gamma\cap P} \varepsilon(z_{QP}(\xi)^{-1}) e^{-sH_{QP}(\xi)} t(\xi) \mathrm{pr}_\varepsilon^P$$

and the Dirichlet series

$$\tilde{c}_{QP}(t,w,\varepsilon,s) = \pi n_{\Gamma,Q}^{-1} \sum_{\xi \in \Gamma\cap U\backslash\bar{\Gamma}_{QP}(w)/\Gamma\cap P} \phi_m(w_{QP}(\xi)^{-1}) e^{-sH_{QP}(\xi)} \mathrm{pr}_\varepsilon^Q t(\xi) \mathrm{pr}_\varepsilon^P,$$

both with values in $\mathrm{Hom}(V_\varepsilon^P, V_\varepsilon^Q)$, where U denotes the unipotent radical of Q and $m \in \mathbb{R}_\varepsilon$, $|m| \le 1$. If $m = \pm 1$, we indicate its sign as $\tilde{c}_{QP}^{\pm}$. We may replace Γ by ΓZ and t by t_ε without changing the value of $\tilde{c}_{QP}$; then the coefficient $n_{\Gamma,Q} = [\Gamma Z \cap U : \Gamma \cap U]$ vanishes. We put $c_{QP}(t,w,\varepsilon,s) = j_m(s)\tilde{c}_{QP}(t,w,\varepsilon,s)$ with $j_m(s)$ as in section 1. Clearly, the maps c_{QP}, $\tilde{c}_{QP}$ for all Q, $P \in \mathfrak{F}$ combine to linear operators c, $\tilde{c}$ in $V_\varepsilon^{\mathrm{cst}}$.

Proposition 11. *The series $c(t,w,\varepsilon,s)$ is absolutely convergent for $\Re s > 1$ and extends to a meromorphic function on $\mathbb{C}$, whose singularities, except for a finite set $S_\varepsilon \subset \left(\frac{1}{2}, 1 - \frac{|m|}{2}\right]$ of simple poles, lie in the half-plane $\Re s < \frac{1}{2}$. (Here $m \in \mathbb{R}_\varepsilon$ with $|m| \le 1$.) Moreover,*

$$\begin{gathered} c(t,1,\varepsilon,s)^* = c(t^*,1,\varepsilon,1-\bar{s}), \qquad c(t,w,\varepsilon,s)^* = c(t^*,w,\varepsilon,\bar{s}),\\ c(\chi,1,\varepsilon,s) = \mathrm{Id}, \qquad c(t_1,1,\varepsilon,s)c(t_2,1,\varepsilon,s) = c(t_1 * t_2,1,\varepsilon,s),\\ c(t_1,1,\varepsilon,1-s)c(t_2,w,\varepsilon,s) = c(t_1,w,\varepsilon,s)c(t_2,1,\varepsilon,s) = c(t_1 * t_2,w,\varepsilon,s),\\ c(t_1,w,\varepsilon,1-s)c(t_2,w,\varepsilon,s) = c(t_1 * t_2,1,\varepsilon,s), \end{gathered}$$

where t, t_1, $t_2 \in \mathfrak{H}(G,\chi)$. In particular, $c(\chi,w,\varepsilon,s)$ is unitary for $\Re s = \frac{1}{2}$, self-adjoint for real s, and an involution for $s = \frac{1}{2}$.

For $\sigma \in S_\varepsilon$, let $V(\chi,\varepsilon,1-\sigma)$ be the range of the self-adjoint operator $q(\chi,\varepsilon,\sigma) = \mathrm{Res}_{s=\sigma}\, c(\chi,w,\varepsilon,s)$ endowed with the scalar product

$$\big(q(\chi,\varepsilon,\sigma)v, q(\chi,\varepsilon,\sigma)w\big)_{\chi,\varepsilon,1-\sigma} = \big(q(\chi,\varepsilon,\sigma)v, w\big).$$

Theorem 12. *For any $\varepsilon \in \hat{Z}$, there are isometries*

$$I_\chi^{\rm con} : \mathcal{H}_\chi^{\rm con}(\varepsilon) \longrightarrow \frac{1}{2\pi}\int_0^\infty \mathcal{H}_{\varepsilon,\frac{1}{2}+ir} \otimes V_\varepsilon^{\rm cst} dr \cong L^2(\tfrac{1}{2}+i\mathbb{R}_+, \tfrac{dr}{2\pi}) \hat{\otimes} \mathcal{H}_\varepsilon \otimes V_\varepsilon^{\rm cst},$$

$$I_\chi^{\rm res} : \mathcal{H}_\chi^{\rm res}(\varepsilon) \longrightarrow \mathcal{H}_m \otimes V(\chi,\varepsilon,\tfrac{|m|}{2}) \oplus \bigoplus_{\substack{\sigma\in S_\varepsilon \\ \sigma<1-|m|/2}} \mathcal{H}_{\varepsilon,1-\sigma} \otimes V(\chi,\varepsilon,1-\sigma).$$

The map $I_\chi = I_\chi^{\rm con} \oplus I_\chi^{\rm res}$ is explicitly given by

$$(I_\chi\varphi)(s,x) = \int_{-\infty}^{\infty} \varphi^{\rm cst}(a_u x) e^{-su} du$$

for $\varphi \in \mathcal{H}_\chi(\varepsilon)$, where the integral is understood as a Fourier transform of distributions. I_χ intertwines the representations π_χ and τ_χ of G and $\mathfrak{H}(G,\chi)$ with the direct integral of $\pi_{\varepsilon,s} \otimes c(.,1,\varepsilon,s)$ (resp. $\pi_m \otimes c(.,1,\varepsilon,s)$ for $s = \frac{|m|}{2}$) over $s \in (\frac{1}{2}+i\mathbb{R}_+) \cup (1-S_\varepsilon)$.

$I_\chi\varphi$, being a priori a hyperfunction on $\mathbb{C}$ with values in $\mathcal{H}_\varepsilon \otimes V_\varepsilon^{\rm cst}$, turns out to be the sum of an L^2-function on $\frac{1}{2}+i\mathbb{R}$ (with a certain symmetry explained in Proposition 18) and of some δ-functions located at $1-S_\varepsilon$.

Proposition 11 and Theorem 12 will be proved in section 5.

4 The trace formula

We shall now state the trace formula for a unitary representation χ of a lattice Γ in G. We prefer to do so in the special case when $Z \subset \Gamma$ and $\chi\big|_Z = \varepsilon{\rm Id}$ for some $\varepsilon \in \hat{Z}$. As discussed in the beginning of section 3, this is no serious restriction.

Recall from section 1 that $G_{\rm reg}$ denotes the set of regular elements of G. It is the union of the G-invariant sets $G_{\rm ell}$ and $G_{\rm hyp}$ consisting of all elements conjugate to k_θ with $\theta \notin \pi\mathbb{Z}$ or to $a_u z$ with $u \neq 0$, $z \in Z$, respectively. Put $\bar{\Gamma}_{\rm reg} = \bar{\Gamma} \cap G_{\rm reg}$, $\bar{\Gamma}_{\rm sing} = \bar{\Gamma} - \bar{\Gamma}_{\rm reg}$. While any element of $\bar{\Gamma}_{\rm sing}$ belongs to some cuspidal parabolic subgroup (see section 7), the same is true only for a subset of elements of $\bar{\Gamma}_{\rm reg}$ (disjoint with Γ), which we call $\bar{\Gamma}_{\rm par}$. We shall see in section 6 that any element of $\bar{\Gamma}_{\rm par}$ belongs to exactly two different cuspidal subgroups, i.e., fixes two boundary points of the upper half-plane $\mathcal{H}$. Consequently, $\bar{\Gamma}_{\rm par} \subset G_{\rm hyp}$. Let us subdivide $\bar{\Gamma}_{\rm reg} - \bar{\Gamma}_{\rm par}$ into two subsets $\bar{\Gamma}_{\rm ell}$, $\bar{\Gamma}_{\rm hyp}$ in the obvious way, thus obtaining a disjoint union

$$\bar{\Gamma} = \bar{\Gamma}_{\rm ell} \cup \bar{\Gamma}_{\rm hyp} \cup \bar{\Gamma}_{\rm par} \cup \bar{\Gamma}_{\rm sing}.$$

By our assumption on χ, $\mathcal{H}_\chi = \mathcal{H}_\chi(\varepsilon)$. Recall the invariant decomposition $\mathcal{H}_\chi = \mathcal{H}_\chi^{\rm dis} \otimes \mathcal{H}_\chi^{\rm con}$ and let $p^{\rm dis}$, $p^{\rm con}$ be the corresponding orthoprojections. Put $\pi_\chi^{\rm dis}(f) := p^{\rm dis}\pi_\chi(f)$ and $\tau_\chi^{\rm dis}(t) := p^{\rm dis}\tau_\chi(t)$. Any element of $\mathfrak{H}(G,\chi)$ can be decomposed as $\lambda\chi + t$ with $\lambda \in \mathbb{C}$ and ${\rm supp}\, t \cap \Gamma = \emptyset$. It is convenient to state the trace formula for χ and t separately.

Theorem 13. *Given Γ, χ, ε as above, let $f \in C^1(G,\varepsilon)$, $m \in \mathbb{R}_\varepsilon$. Denote*

$$h(r) = \Theta_{\varepsilon,\frac{1}{2}+ir}(f), \qquad h_n = \Theta_{n+\operatorname{sgn} n}(f)$$

for $|\Im r| \le \frac{1}{2}$, $n \equiv m+1(2)$, $n \neq 0$. If $\varepsilon = \varepsilon_1$, put $h_0 = (\Theta_1(f) - \Theta_{-1}(f))/2$. Then $\pi_\chi^{\mathrm{dis}}(f)$ is of trace class, and its trace equals the following absolutely convergent integral-series:

$$\begin{aligned}
&\dim(V)\frac{\operatorname{vol}(\Gamma\backslash G)}{4\pi}\Big(\int_{-\infty}^{\infty}\frac{r\sinh 2\pi r}{\cosh 2\pi r+\cos\pi m}h(r)\,dr + \sum_n{}^{*}|n|h_n\Big)\\
&+\sum_{\{\gamma\}_\Gamma\subset Z\backslash\Gamma_{\mathrm{ell}}}\frac{\operatorname{tr}\chi(\gamma)}{2i[\Gamma_\gamma{:}Z]\sin\theta}\Big(\frac{1}{2}\int_{-\infty}^{\infty}\frac{\cosh 2(\pi-\theta)r+e^{-i\pi m}\cosh 2\theta r}{\cosh 2\pi r+\cos\pi m}h(r)\,dr\\
&\qquad\qquad+\delta_{\varepsilon,\varepsilon_1}h_0+\sum_n{}^{*}\operatorname{sgn}(n)e^{-in\theta}h_n\Big)\\
&+\sum_{\{\gamma\}_\Gamma\subset Z\backslash\Gamma_{\mathrm{hyp}}}\frac{\operatorname{tr}\chi(\gamma)u}{4\pi[\Gamma_\gamma{:}Z]\sinh\frac{u}{2}}\int_{-\infty}^{\infty}\cos(ru)h(r)\,dr\\
&-\dim(V^{\mathrm{cst}})\Big(\frac{1}{2\pi}\int_{-\infty}^{\infty}\psi(1+2ir)h(r)\,dr+\frac{1}{2}\sum_{n\neq 0}{}^{*}h_n\Big)\\
&+\sum_{\substack{P\in\mathfrak{F}\\ \alpha\in(-\pi,\pi)}}m_\alpha^P\Big(\frac{1}{4\pi}\int_{-\infty}^{\infty}\Big(\frac{\alpha\sin\pi m}{\cosh 2\pi r+\cos\pi m}+2\log\big(2\cos(\tfrac{\alpha}{2})\big)\Big)h(r)\,dr\\
&\qquad\qquad+\frac{\alpha}{2\pi}\Big(\delta_{\varepsilon,\varepsilon_1}h_0+\sum_n{}^{*}\operatorname{sgn}(n)h_n\Big)\Big)\\
&+\frac{1}{4\pi}\mathrm{p.v.}\int_{-\infty}^{\infty}\operatorname{tr}\big(\tilde{c}(\chi,w,\varepsilon,\tfrac{1}{2}-ir)(\tilde{c}'(\chi,w,\varepsilon,\tfrac{1}{2}+ir)\big)h(r)\,dr\\
&-\frac{1}{4}(1-\delta_{\varepsilon,\varepsilon_1})\operatorname{tr}(c(\chi,w,\varepsilon,\tfrac{1}{2})-\mathrm{Id})h(0)-\frac{1}{2}\delta_{\varepsilon,\varepsilon_1}\operatorname{tr}c^{+}(\chi,w,\varepsilon,\tfrac{1}{2})h_0.
\end{aligned}$$

Here $\sum_n^* = \sum_{n\equiv m+1(2)}$, $\delta_{\varepsilon,\varepsilon_1}$ is the Kronecker symbol, $\psi(s) = \Gamma'(s)/\Gamma(s)$, m_α^P is the multiplicity of $-e^{i\alpha}$ as an eigenvalue of $\chi(g_P^{-1}n_1g_P)$, and each γ (suitably chosen modulo Z) determines $\theta \in (0,\pi)$ or $u > 0$ by $\{\gamma\}_G = \{k_\theta\}_G$ or $\{\gamma\}_G = \{a_u\}_G$, respectively.

Using the results of section 2, we may express all items except the p.v. integral in terms of the Fourier transform of $h(r)$, namely $g(u) = F_f^A(a_u)$, and h_n. If $f \in C^p(G,\varepsilon)$, $\frac{1}{p} = \max(\frac{|m|}{2},1)$ and $f(k_\theta x k_{\theta'}) = e^{-im(\theta+\theta')}f(x)$, then $\pi_{\varepsilon,\frac{1}{2}+ir}(f)\phi_{m'} = \delta_{m,m'}h(r)\phi_m$, thus $h_n = h(\frac{in}{2})$ for n between 0 and m, $h_0 = \frac{1}{2}h(0)\operatorname{sgn} m$, and $h_n = 0$ otherwise. In this case, Theorem 13 becomes essentially Theorem 6.2 of [15], cf. also [1], [9], [10], [18], [29].

It remains to consider $t \in \mathfrak{H}(G,\chi)$, $\operatorname{supp} t \cap \Gamma = \emptyset$. First we introduce some notations. For $P = NAZ \in \mathfrak{F}$, let us parametrize the elements of N as $n_{P,v} = g_P^{-1}n_v g_P$, thus providing an isomorphism $\bar{\Gamma}\cap N/\Gamma\cap N \cong \mathbb{Q}/\mathbb{Z}$ (see [16,Lemma 9.3]). If $\xi \in \bar{\Gamma}\cap N$, then ξ commutes with $\Gamma\cap N$, hence $t(\xi)$ intertwines $\chi|_{\Gamma\cap N}$. As above, $n_{P,1}$ is a generator of $\Gamma\cap N$, and the eigenvalues of $\chi(n_{P,1})$ different from 1 are of the form $-e^{i\alpha}$, $\alpha \in (-\pi, \pi)$. Given $\xi \in \bar{\Gamma} \cap N / \Gamma \cap N$, let $v_\xi^\pm = \min\{v > 0 : n_{P, \pm v} \in \xi(\Gamma \cap N)\}$. Clearly, $v_1^\pm = 1$, and $v_\xi^+ + v_\xi^- = 1$ for $\xi \notin \Gamma \cap N$. Let $t_\alpha^\pm(\xi)$ be the restriction of $t(n_{P, v_\xi^\pm})$ to the eigenspace of $\chi|_{\Gamma \cap N}$ corresponding to α. Obviously, these eigenspaces make up the orthogonal complement of V^P.

We shall also need the hypergeometric series

$$\beta(v, z) = v^{-1} F(v, 1; v+1; z) = \sum_{n=0}^{\infty} \frac{z^n}{n+v} = \int_0^1 t^{v-1}(1 - zt)^{-1}\, dt.$$

This series converges absolutely for $|z| < 1$ and conditionally for $|z| = 1$, $z \neq 1$. Incidentally, $z^v \beta(v, z) = B_z(v, 0)$ for $z \in [0, 1)$, where B denotes the incomplete beta-function. Moreover, if $v = \frac{p}{q}$, where $p \leq q$ are positive integers (the only case we need), then the substitution $t = x^q$ yields

$$\beta(\tfrac{p}{q}, z) = -\sum_{w^q = z} w^{-p} \log(1 - w) \qquad \text{for } z \notin [1, \infty),$$

where we take the continuous branch of the logarithm on $\mathbb{C} - (-\infty, 0]$ determined by $\log(1) = 0$.

Theorem 14. *In the situation of Theorem 13, let* $t \in \mathfrak{H}(G, \chi)$, $\operatorname{supp} t \cap \Gamma = \emptyset$. *Then* $\operatorname{tr}(\pi_\chi^{\mathrm{dis}}(f) \tau_\chi^{\mathrm{dis}}(t))$ *equals the following absolutely convergent integral-series:*

$$\sum_{\{\xi\}_\Gamma \subset Z \backslash \bar{\Gamma}_{\mathrm{ell}}} \frac{\operatorname{tr}\chi(\xi)}{2i[\Gamma_\xi : Z] \sin\theta} \left(\frac{1}{2} \int_{-\infty}^{\infty} \frac{\cosh 2(\pi - \theta) r + e^{-i\pi m} \cosh 2\theta r}{\cosh 2\pi r + \cos \pi m} h(r)\, dr \right.$$
$$\left. + \delta_{\varepsilon, \varepsilon_1} h_0 + \sum_n{}^{*} \operatorname{sgn}(n) e^{-in\theta} h_n \right)$$
$$+ \sum_{\{\xi\}_\Gamma \subset Z \backslash \bar{\Gamma}_{\mathrm{hyp}}} \frac{\operatorname{tr}\chi(\xi) u}{4\pi [\Gamma_\xi : Z] \sinh \frac{u}{2}} \int_{-\infty}^{\infty} \cos(ru) h(r)\, dr$$
$$+ \sum_{\{\xi\}_\Gamma \subset Z \backslash \bar{\Gamma}_{\mathrm{par}}} \frac{\operatorname{tr}\chi(\xi)}{2 \sinh \frac{u}{2}} \left[-\sum_{n \neq 0}{}^{*} e^{-|n| u/2} h_n + \int_{-\infty}^{\infty} \left(\frac{e^{iru}}{2\pi} (l + 2B_{e^{-u}}(1 + 2ir, 0)) \right. \right.$$
$$\left. \left. + \frac{\sin ru}{2} \left(\frac{1}{2\pi r} + \coth 2\pi r - \frac{\sinh 2\pi r}{\cosh 2\pi r + \cos \pi m} \right) \right) h(r)\, dr \right]$$
$$- \sum_{P \in \mathfrak{F}} \sum_{\xi \in \bar{\Gamma} \cap N / \Gamma \cap N} \operatorname{tr}(t(\xi) \operatorname{pr}^P) \left[\frac{1}{2} \sum_{n \neq 0}{}^{*} (1 + i \cot \pi v_\xi^+) h_n \right.$$
$$\left. + \int_{-\infty}^{\infty} \frac{1}{4\pi} \left(2\psi(1 + 2ir) + 2C + \psi(v_\xi^+) + \psi(v_\xi^-) + \frac{\pi i \cot \pi v_\xi^+ \sin \pi m}{\cosh 2\pi r + \cos \pi m} \right) h(r)\, dr \right]$$

$$+\sum_{P\in\mathfrak{F}}\sum_{\substack{\xi\in\bar\Gamma\cap N/\Gamma\cap N\\ \alpha\in(-\pi,\pi)}}\Bigg[\Big(\beta(v_\xi^+,-e^{i\alpha})\operatorname{tr}t_\alpha^+(\xi)+\beta(v_\xi^-,-e^{-i\alpha})\operatorname{tr}t_\alpha^-(\xi)\Big)\frac{1}{4\pi}\int_{-\infty}^{\infty}h(r)\,dr$$

$$+\frac{e^{-iv_\xi^+\alpha}\operatorname{tr}t_\alpha^+(\xi)}{2i\sin\pi v_\xi^+}\left(\frac{1}{2}\int_{-\infty}^{\infty}\frac{\sin\pi m}{\cosh 2\pi r+\cos\pi m}h(r)\,dr+\delta_{\varepsilon,\varepsilon_1}h_0+\sum_n{}^{*}\operatorname{sgn}(n)h_n\right)\Bigg]$$

$$+\frac{1}{4\pi}\mathrm{p.v.}\int_{-\infty}^{\infty}\operatorname{tr}\big(\tilde c(t,w,\varepsilon,\tfrac12-ir)(\tilde c'(\chi,w,\varepsilon,\tfrac12+ir)\big)h(r)\,dr$$

$$-\frac{1}{4}\big(1-\delta_{\varepsilon,\varepsilon_1}\big)\operatorname{tr}(c(t,w,\varepsilon,\tfrac12)-(c(t,1,\varepsilon,\tfrac12))h(0)-\frac{1}{2}\delta_{\varepsilon,\varepsilon_1}\operatorname{tr}c^+(t,w,\varepsilon,\tfrac12)h_0.$$

Here $C=-\psi(1)$ denotes the Euler-Mascheroni constant, $P=NAZ$, and each $\xi\in\bar\Gamma_{\mathrm{reg}}$ (suitably chosen modulo Z) determines $\theta\in(0,\pi)$ or $u>0$ by $\{\xi\}_G=\{k_\theta\}_G$ or $\{\xi\}_G=\{a_u\}_G$, respectively. Moreover, each $\xi\in\bar\Gamma_{\mathrm{par}}$ determines $l=H_{QP}(\gamma)$, where $\gamma\in\Gamma_{QP}(w)$ and $P\neq Q$ are the cuspidal subgroups containing ξ.

Theorems 13 and 14 follow from Theorem 25 and the results of section 3.

5 Eisenstein series

We turn now to the harmonic analysis of the orthogonal complement $\mathcal{H}_\chi^{\mathrm{Eis}}=\mathcal{H}_\chi^{\mathrm{res}}\oplus\mathcal{H}_\chi^{\mathrm{con}}$ of $\mathcal{H}_\chi^{\mathrm{cus}}$ in $\mathcal{H}_\chi$. In view of the extensive literature covering this topic for G', our explanations will sometimes be sketchy.

The constant term operator defined for a cuspidal subgroup $P=NAZ$ in section 3, when restricted to compactly supported (mod Γ) functions $\varphi\in\mathcal{H}_\chi$, is a G-equivariant map to the space

$$\mathcal{H}_{P,\chi}=\{\psi:G\to V\,|\,\psi(n\gamma x)=\chi(\gamma)\psi(x)\quad\text{for } n\in N,\ \gamma\in\Gamma\cap P\text{ and a.e. }x\in G;\\ |\psi|\in L^2(N(\Gamma\cap P)G)\}$$

of the induced representation $\pi_{P,\chi}=\operatorname{Ind}_{N(\Gamma\cap P)}^G(\chi|_{\Gamma\cap P})$. Such ψ are automatically V^P-valued. We equip $\mathcal{H}_{P,\chi}$ with the scalar product

$$\begin{aligned}(\psi_1,\psi_2)&=n_\Gamma^{-1}\int_{\Gamma\cap P\backslash G}(\psi_1(x),\psi_2(x))\,dx\\ &=n_\Gamma^{-1}\int_{\Gamma\cap Z\backslash K}\int_{-\infty}^{\infty}(\psi_1(a_{P,u}k),\psi_2(a_{P,u}k))e^{-u-u_P}\,du\,dk.\end{aligned}$$

The formally adjoint of the constant term operator is the series θ_P satisfying

$$(\varphi^P,\psi)=n_\Gamma^{-1}\int_{\Gamma\cap P\backslash G}(\varphi(x),\psi(x))\,dx=(\varphi,\theta_P(\chi,\psi)),$$

which should conveniently be defined more generally as

$$\theta_P(t,\psi,x)=\sum_{\xi\in\bar\Gamma/\Gamma\cap P}t(\xi)\psi(\xi^{-1}x)$$

for any $t \in \mathfrak{H}(G,\chi)$, because then $\tau(t)\theta_P(\chi,\psi) = \theta_P(t,\psi)$. Later we shall see (Proposition 15 with $\phi = 1$) that $\theta_P(t,\psi)$ is a.e. absolutely convergent provided $|\psi(x)|$ is essentially bounded by a multiple of $\exp(\sigma H_P(x))$ with $\sigma > 1$.

It is easy to reduce $\theta_P(t,\psi)$ to the case $t = \chi$: collecting all items with the same $Q = \xi P \xi^{-1}$, we obtain

$$\theta_P(t,\psi,x) = \sum_Q \sum_{\xi \in \bar{\Gamma}_{QP}(1)/\Gamma \cap P} t(\xi)\psi(\xi^{-1}x)$$

in the notation of section 3. With $\bar{\Gamma}_{\gamma Q \gamma^{-1},P}(1) = \gamma \bar{\Gamma}_{QP}(1)$ we obtain

$$\theta_P(t,\psi) = \sum_{Q \in \mathfrak{F}} \theta_Q(\chi, \tau_{QP}(t,1)\psi),$$

where

$$(\tau_{QP}(t,1)\psi)(x) = \sum_{\xi \in \bar{\Gamma}_{QP}(1)/\Gamma \cap P} t(\xi)\psi(\xi^{-1}x)$$

is a finite sum. Clearly, $\tau_{PP}(\chi,1) = \mathrm{Id}$, $\tau_{QP}(\chi,1) = 0$ if P and Q are not Γ-conjugate, and

$$\sum_{Q \in \mathfrak{F}} \tau_{RQ}(t',1)\tau_{QP}(t,1) = \tau_{RP}(t' * t, 1).$$

Dually, we may express the constant terms $(\tau_\xi(t)\varphi)^Q$ by those of $\varphi \in \mathcal{H}_\chi$: the elements of each coset $\xi\Gamma \subset \bar{\Gamma}$ conjugate the cuspidal subgroups of exactly one class $\{P\}_\Gamma$ to Q, thus

$$(\tau_\chi(t)\varphi)(x) = \sum_{P \in \mathfrak{F}} \sum_{\xi \in \bar{\Gamma}_{QP}(1)/\Gamma \cap P} t(\xi)\varphi(\xi^{-1}x).$$

Since for $\xi \in \bar{\Gamma}_{QP}(1)$ we have $\xi^{-1}U\xi = N$ (U the unipotent radical of Q) and $t(\xi)V^P = V^Q$,

$$(\tau_\chi(t)\varphi)^Q = \sum_{P \in \mathfrak{F}} \tau_{QP}(t,1)\varphi^P.$$

Calculating $(\varphi, \theta_P(t,\psi))$ in two ways, we obtain the formally adjoint $\tau_{QP}(t,1)^* = \tau_{PQ}(t^*,1)$.

The functions $\theta_P(\chi,\psi)$ for all cuspidal P and all $\psi \in \mathcal{H}_{P,\chi}$ with compact support mod $(\Gamma \cap P)$ are dense in $\mathcal{H}_\chi^{\mathrm{Eis}}$, as follows from their definition. One can thus describe $\mathcal{H}_\chi^{\mathrm{Eis}}$ in terms of them, provided one knows their scalar product

$$(\theta_P(t,\psi), \theta_Q(\chi,\psi')) = (\theta_P^Q(t,\psi), \psi').$$

In order to calculate θ_P^Q, split $\theta_P(t,\psi)$ into two terms according to $\bar{\Gamma} = \bar{\Gamma}_{QP}(1) \cup \bar{\Gamma}_{QP}(w)$ and take the constant term along Q of each one. The first sum then simplifies to $\tau_{QP}(t,1)\psi$. On the other hand, for $\xi \in \bar{\Gamma}_{QP}(w)$ we have $\xi P \xi^{-1} \cap U =$

$\{1\}$, whence $\Gamma \cap U$ acts freely on $\bar{\Gamma}_{QP}(w)/\Gamma \cap P$ from the left. We split off a summation over $\Gamma \cap U$ and join it with the constant term integration to obtain

$$\theta_P^Q(t,\psi) = \tau_{QP}(t,1)\psi + \tau_{QP}(t,w)\psi,$$

where

$$(\tau_{QP}(t,w)\psi)(x) = \mathrm{vol}(\Gamma \cap U \backslash U)^{-1} \sum_{\xi \in \Gamma\cap U \backslash \bar{\Gamma}_{QP}(w)/\Gamma\cap P} \mathrm{pr}^Q t(\xi) \int_U \psi(\xi^{-1}ux)\, du$$

is absolutely convergent if $\theta_P(t,\psi)$ is so. Calculating the above scalar product in two ways, we get $\tau_{QP}(t,w)^* = \tau_{PQ}(t^*,w)$. Taking the constant term of

$$\tau_\chi(t')\theta_P(t,\psi) = \theta_P(t' * t,\psi) = \sum_{Q\in\mathfrak{F}} \theta_Q(t', \tau_{QP}(t,1)\psi),$$

we see that

$$\sum_{Q\in\mathfrak{F}} \tau_{RQ}(t',1)\tau_{QP}(t,w) = \tau_{RP}(t' * t, w) = \sum_{Q\in\mathfrak{F}} \tau_{RQ}(t',w)\tau_{QP}(t,1).$$

The next step is harmonic analysis of $\mathcal{H}_{P,\chi}$. Since, for ψ_1, $\psi_2 \in \mathcal{H}_{P,\chi}(\varepsilon)$,

$$(\psi_1,\psi_2) = \int_{Z\backslash K}\int_{-\infty}^{\infty} (\psi_1(g_P^{-1}a_u k), \psi_2(g_P^{-1}a_u k))e^{-u}\, du\, dk,$$

left g_P-translation and Fourier transform provide a G-equivariant isomorphism

$$\mathcal{H}_{P,\chi}(\varepsilon) \cong \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{H}_{\varepsilon,\frac{1}{2}+ir} \otimes V_\varepsilon^P dr \cong L^2(\tfrac{1}{2}+i\mathbb{R}, \tfrac{dr}{2\pi}) \hat{\otimes} \mathcal{H}_\varepsilon \otimes V_\varepsilon^P$$

with a direct integral over the principal series. We write the inverse Fourier transform for $\alpha \in L^2(\frac{1}{2}+ir)$, $\phi \in \mathcal{H}_\varepsilon \otimes V_\varepsilon^P$ as

$$(\alpha\otimes\phi)_P^\vee(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \alpha(\tfrac{1}{2}+ir)\phi_{\frac{1}{2}+ir}(g_P x)\, dr.$$

For $\alpha(s)$ extending holomorphically to $\Re s \in [\frac{1}{2},\sigma]$, $\sigma > 1$, with sufficient decay as $|\Im s| \to \infty$, this can be inserted into θ_P, and

$$\theta_P(t,(\alpha\otimes\phi)_P^\vee,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \alpha(\sigma+ir)E_P(t,\phi,\sigma+ir,x)\, dr,$$

provided the Eisenstein series

$$E_P(t,\phi,s,x) = \sum_{\xi\in\bar{\Gamma}/\Gamma\cap P} t(\xi)\phi_s(g_P\xi^{-1}x) = \theta_P(t,\phi_{s,P},x)$$

is absolutely convergent for $\Re s = \sigma$. Here, $\phi_{s,P}(x) = \phi_s(g_P x)$. Let now $C^n(K,\varepsilon)$ be the subspace of n times continuously differentiable functions in $\mathcal{H}_\varepsilon$.

Proposition 15. *For $t \in \mathfrak{H}(G,\chi)$ and $\phi \in C^n(K,\varepsilon) \otimes V_\varepsilon^P$, the Eisenstein series $E_P(t,\phi)$ is absolutely uniformly convergent for (s,x) in compact subsets of $G \times \{s \in \mathbb{C} : \Re s > 1\}$ to a function holomorphic in s and n times continuously differentiable in x. If Q is any cuspidal subgroup, $E_P(t,\phi,s,x)$ is bounded on $\mathcal{C}_{Q,u}$ by a multiple of $\exp(\sigma H_Q(x))$, $\sigma = \Re s$. Furthermore,*

$$\tau_\chi(t')\pi_\chi(xD)E_P(t,\phi,s) = E_P(t' * t, (\pi_{\varepsilon,s}(xD) \otimes Id)\phi, s)$$

for $t' \in \mathfrak{H}(G,\chi)$, $x \in G$, $D \in \mathfrak{G}$ with $\deg D \leq n$, where τ_χ and π_χ have the obvious meaning although $E_P \notin \mathcal{H}_\chi$.

Provided convergence, the last assertion is obvious. It reduces the proof to the case $t = \chi$, $n = 1$. But χ and ϕ are bounded, thus ony $E_P(1,1,s)$ will bother us. Here, e.g., the proof of [18, Theorem 2.1.1] together with the complementary remarks applies. Note that any Dirichlet series is holomorphic in its half-plane of absolute convergence.

One checks that

$$\begin{aligned}
\tau_{QP}(t,1)\phi_{s,P} &= (C_{QP}(t,1,s)\phi)_{s,Q}, \\
\tau_{QP}(t,w)\phi_{s,P} &= (C_{QP}(t,w,s)\phi)_{1-s,Q} \qquad \text{for } \Re s > 1
\end{aligned}$$

with uniquely determined maps $C_{QP} : C^n(K,\varepsilon) \otimes V_\varepsilon^P \to C^n(K,\varepsilon) \otimes V_\varepsilon^Q$, which are G-equivariant and thus preserve the decomposition of $\mathcal{H}_\varepsilon$ by K-types. In order to state the properties of C_{QP} which follow from those of τ_{QP}, we consider

$$E(t,\phi,s) = \sum_{P \in \mathfrak{F}} E_P(t, \mathrm{pr}_P\phi, s)$$

for $\phi \in \mathcal{H}_\varepsilon \otimes V_\varepsilon^{\mathrm{cst}}$ (cf. section 3), where pr_P denotes the projection on the P-component. The C_{QP} then combine to linear operators C in $C^n(K,\varepsilon) \otimes V_\varepsilon^{\mathrm{cst}}$ such that

$$\begin{gathered}
\pi_\chi(x)E(t,\phi,s) = E(t,(\pi_{\varepsilon,s}(x) \otimes \mathrm{Id})\phi, s), \\
\tau_\chi(t')E(t,\phi,s) = E(t' * t, \phi, s) = E(t', C(t,1,s)\phi, s), \\
E^{\mathrm{cst}}(t,\phi,s) = (C(t,1,s)\phi)_s + (C(t,w,s)\phi)_{1-s}, \\
C(t,1,s)^* = C(t^*,1,1-\bar{s}), \qquad C(t,w,s)^* = C(t^*,w,\bar{s}), \\
C(\chi,1,s) = \mathrm{Id}, \qquad C(t',1,s)C(t,1,s) = C(t' * t,1,s) \\
C(t',1,1-s)C(t,w,s) = C(t' * t,w,s) = C(t',w,s)C(t,1,s).
\end{gathered}$$

We shall not indicate the dependence of E and C on $\varepsilon \in \hat{Z}$ but rather consider them as Z-equivariant maps defined on the space $\sum_\varepsilon C^n(K,\varepsilon) \otimes V_\varepsilon^{\mathrm{cst}}$ of functions $\phi : K \to V^{\mathrm{cst}}$.

We may now insert $\psi_1 = (\alpha \otimes \phi)_P^\vee$ and $\psi_2 = (\beta \otimes \phi')_Q^\vee$, where $\phi \in \mathcal{H}_\varepsilon \otimes V_\varepsilon^P$, $\phi' \in \mathcal{H}_\varepsilon \otimes V_\varepsilon^Q$, into the scalar product formula for θ_P, θ_Q and express the latter by Eisenstein series. Suppose that α, β are holomorphic and rapidly decreasing for $1-\sigma < \Re s < \sigma$, $\sigma > 1$. Fourier inversion shows that

$$(\phi''_{s,Q}, (\beta \otimes \phi')^{\vee}_Q) = (\phi'', \phi')\overline{\beta(1-\bar{s})}$$

for any $\phi'' \in \mathcal{H}_\varepsilon \otimes V_\varepsilon^Q$. Some calculation now gives

$$\begin{aligned}&\big(\theta_P(t, (\alpha \otimes \phi)^{\vee}_P), \theta_Q(\chi, (\beta \otimes \phi')^{\vee}_Q)\big)\\ &= \frac{1}{2\pi i}\int_{\Re s=\sigma} \big(\alpha(s)\overline{\beta(1-\bar{s})}(C_{QP}(t,1,s)\phi, \phi') + \alpha(s)\overline{\beta(\bar{s})}(C_{QP}(t,w,s)\phi,\phi')\big)\, ds.\end{aligned}$$

Thus, as usual, the next task is analytic continuation of E and C to $\Re s = \frac{1}{2}$.

Proposition 16. *Let ϕ, $\phi' \in \mathcal{H} \otimes V_\varepsilon^{\mathrm{cst}}$ be K-finite, t, $t' \in \mathfrak{H}(G,\chi)$, $m \in \mathbb{R}_\varepsilon$ with $|m| \le 1$. Then $E(t,\phi,s,x)$ and $c(s) = (C(t,w,s)\phi,\phi')$ extend to meromorphic functions on $\mathbb{C}$ with the same singularities (for generic $x \in G$), which, except for a finite set $S_\varepsilon \subset \left(\frac{1}{2}, 1-\frac{|m|}{2}\right]$ of simple poles, lie in the half-plane $\Re s < \frac{1}{2}$. Moreover,*

$$E(t,\phi,s) = E(\chi, C(t,w,s)\phi, 1-s),$$

$$C(t',w,1-s)C(t,w,s) = C(t' * t, 1, s),$$

$$\left| c(s)a^{s-1/2} \prod_{\sigma \in S_\varepsilon} \frac{s-\sigma}{1-s-\sigma}\right| \le 1$$

for some $a > 0$ and all s with $\Re s \ge \frac{1}{2}$.

Now the aforementioned properties of E and C extend to all regular s. In particular, $C(\chi, w, s)$ is unitary for $\Re s = \frac{1}{2}$, self-adjoint for regular $s \in \mathbb{R}$, and is an involution for $s = \frac{1}{2}$. Being a K-finite eigenfunction of ω, $E(t,\phi,s)$ is analytic on G. We shall see later (Lemma 19) that $S_\varepsilon - \{1 - \frac{|m|}{2}\}$ is independent of the weights occurring in ϕ.

It suffices to prove the proposition for t, $t' = \chi$ and ϕ, ϕ' of the form $\phi_{m'} \otimes v$, with $m' \in \mathbb{R}_\varepsilon$ and $v \in V_\varepsilon^P$. In this case, proofs are given in [25], [15, pp. 130, 156; 296f, 299; 374, 380]. One may also adapt the elegant proof from [7], [8].

All proofs depend on the Maaß-Selberg relations, which also play a role in the proof of the trace formula. They are connected with the notion of truncation. Given a truncation parameter $u \in \mathbb{R}$, let $\chi_{P,u}$ be the characteristic function of $\{x \in G : H_P(x) + u_P > u\}$, which projects on $\mathcal{C}_{P,u} \subset \Gamma\backslash G$ for large u. By the choice of u_P,

$$\chi_{P,u}(\gamma x) = \chi_{\gamma^{-1}P\gamma,u}(x).$$

We have seen a similar invariance for the constant terms, and thus the truncation operator Λ_u, defined for $\varphi \in \mathcal{H}_\chi$ as

$$\Lambda_u\varphi = \varphi - \sum_P \chi_{P,u}\varphi^P,$$

yields an element of $\mathcal{H}_\chi$. Here the sum over all cuspidal subgroups P is locally finite, whence we may apply Λ_u to more general functions $\varphi : G \to V$ with the same Γ-equivariance but which are only locally integrable, say. For large u, the $\chi_{P,u}$ even have pairwise disjoint support, and if we view φ as a section of the bundle$V \times_\Gamma G$, then Λ_u simply replaces φ by $\varphi - \varphi^P$ over each $\mathcal{C}_{P,u}$.

A special case of the Maaß-Selberg relations is the following

Lemma 17. *Let $\phi, \phi' \in \mathcal{H}_\varepsilon \otimes V_\varepsilon^{\mathrm{cst}}$ be K-finite, $t \in \mathfrak{H}(G,\chi)$, and take $s \in \mathbb{C}$ such that s and $1-s$ are regular points for E. Then $\Lambda_u E(t,\phi,s) \in \mathcal{H}_\chi$, and*

$$\begin{aligned}(\Lambda_u E(t,\phi,s), \Lambda_u E(\chi,\phi',1-\bar{s}))& \\ = 2u(C(t,1,s)\phi,\phi') - (C'(\chi,w,1-s)C(t,w,s)\phi,\phi')& \\ + \frac{1}{2s-1}\left(e^{(2s-1)u}(C(\chi,w,1-s)C(t,1,s)\phi,\phi') - e^{(1-2s)u}(C(t,w,s)\phi,\phi')\right).&\end{aligned}$$

Sketch of proof. One quickly obtains the identity

$$\begin{aligned}\int_{\Gamma\backslash G} [(\Lambda_u\varphi(\omega x), \Lambda_u\varphi'(x)) - (\Lambda_u\varphi(x), \Lambda_u\varphi'((\omega x))]\, dx& \\ = -\sum_{P\in\mathfrak{F}} \int_{\Gamma\cap P\backslash G} \left[(\varphi^P(\omega x), {\varphi'}^P(x)) - (\varphi^P(x), {\varphi'}^P(\omega x))\right]\chi_{P,u}(x)\, dx&\end{aligned}$$

for smooth φ, $\varphi' \in \mathcal{H}_\chi$, one of them with compact support modulo Γ. Integration by parts transforms the right-hand side to the integral over $k \in \Gamma \cap Z\backslash K$ of

$$\left(\frac{d}{du}(e^{-u/2}\varphi^{\mathrm{cst}}(a_u k)), e^{-u/2}\varphi'^{\mathrm{cst}}(a_u k)\right) - \left(e^{-u/2}\varphi^{\mathrm{cst}}(a_u k), \frac{d}{du}(e^{-u/2}\varphi'^{\mathrm{cst}}(a_u k))\right)$$

The resulting identity extends to K-finite eigenfunctions of ω in view of the rapid decrease of $\Lambda_u\varphi$ on each $\mathcal{C}_{P,u}$. In particular, for ϕ, ϕ' as in the lemma and regular s, $\bar{s}' \in \mathbb{C}$ we obtain

$$\begin{aligned}&(s(s-1) - s'(s'-1))(\Lambda_u E(t,\phi,s), \Lambda_u E(\chi,\phi',\bar{s}'))\\ &= (s-s')\left(e^{(s+s'-1)u}(C(t,1,s)\phi,\phi') - e^{(1-s-s')u}(C(\chi,w,s')C(t,w,s)\phi,\phi')\right)\\ &\quad + (s+s'-1)\left(e^{(s-s')u}(C(\chi,w,s')C(t,1,s)\phi,\phi') - e^{(s'-s)u}(C(t,w,s)\phi,\phi')\right).\end{aligned}$$

Dividing both sides by $s(s-1)-s'(s'-1) = (s+s'-1)(s-s')$ and letting $s' \to 1-s$, we prove the lemma. □

In the formulae connecting the series θ_P (resp. their scalar products) with E (resp. C), we now move the line of integration to $\Re s = \frac{1}{2}$, taking care of the residues at S_ε. Applying an approximation argument to $\alpha \otimes \phi$, one proves as usual

Proposition 18. *Fix $\varepsilon \in \hat{Z}$ and, for $\sigma \in S_\varepsilon$, let $\mathcal{H}(\chi,\varepsilon,\sigma)$ be the Hilbert space obtained from $\mathcal{H}_\varepsilon^K \otimes V_\varepsilon^{\mathrm{cst}}$ by factoring out the null space of the scalar product*

$$(\Phi,\Phi')_\sigma = (\operatorname*{Res}_{s=\sigma} C(\chi,w,s)\Phi,\Phi')$$

and then completing. Then there are isometries

$$\begin{aligned}&\mathcal{I}_\chi^{\mathrm{con}} : \{\Phi \in L^2(\tfrac{1}{2}+i\mathbb{R}, \tfrac{dr}{4\pi})\hat{\otimes}\mathcal{H}_\varepsilon \otimes V_\varepsilon^{\mathrm{cst}} : \Phi(1-s) = C(\chi,w,s)\Phi(s)\} \to \mathcal{H}_\chi^{\mathrm{con}}(\varepsilon)\\ &\mathcal{I}_\chi^{\mathrm{res}} : \bigoplus_{\sigma\in S_\varepsilon} \mathcal{H}(\chi,\varepsilon,\sigma) \to \mathcal{H}_\chi^{\mathrm{res}}(\varepsilon),\end{aligned}$$

which are given by

$$\mathcal{I}_\chi^{\text{con}}\Phi = \frac{1}{4\pi}\int_{-\infty}^{\infty} E(\chi, \Phi(\tfrac{1}{2}+ir), \tfrac{1}{2}+ir)\,dr,$$
$$\mathcal{I}_\chi^{\text{res}}\Phi = \sum_{\sigma\in S_\varepsilon} \operatorname*{Res}_{s=\sigma} E(\chi, \Phi_\sigma, s)$$

for compactly supported Φ. $\mathcal{I}_\chi^{\text{con}}$ *and* $\mathcal{I}_\chi^{\text{res}}$ *intertwine the representations of* G *and* $\mathfrak{H}(G,\chi)$ *defined by* $\pi_{\varepsilon,s}\otimes Id$ *and* $C(.,1,s)$ *with the representations* π_χ *and* τ_χ *in* $\mathcal{H}_\chi(\varepsilon)$.

From Schur's lemma it is clear that the intertwining operators C_{QP} should be multiples of R_{P_0}. We can make this explicit. Some calculation using the notations introduced in connection with Proposition 11 allows us to specialize the formulas for τ_{QP} as follows:

$$(C_{QP}(t,1,s)\phi)_s(x) = \sum_{\xi\in\tilde\Gamma_{QP}(1)/\Gamma\cap P} t(\xi)\phi_s(a_{-H_{QP}(\xi)}z_{QP}(\xi)^{-1}x),$$
$$(C_{QP}(t,w,s)\phi)_{1-s}(x) = n_{\Gamma,Q}^{-1}\sum_{\xi\in\Gamma\cap U\backslash\tilde\Gamma_{QP}(w)/\Gamma\cap P} \operatorname{pr}^Q t(\xi) \cdot \int_{N_0}\phi_s(a_{-H_{QP}(\xi)}w_{QP}(\xi)^{-1}nx)\,dn.$$

Comparing this formula with the formula for R_{P_0}, we obtain

Lemma 19. *With the notations of Proposition 11,*

$$C(t,1,s)\big|_{\mathcal{H}_\varepsilon\otimes V_\varepsilon^{\text{cst}}} = Id\otimes c(t,1,\varepsilon,s),$$
$$C(t,w,s)\big|_{\mathcal{H}_\varepsilon\otimes V_\varepsilon^{\text{cst}}} = R_{P_0}^{\pm}(\varepsilon,s)\otimes c^{\pm}(t,w,\varepsilon,s),$$

where the second formula is an equality of convergent integral-series for $\Re s > 1$, *and the superscript* $\pm$ *is ignored for* $\varepsilon\neq\varepsilon_1$.

Proof of Proposition 11. Restricting the second formula in the preceding lemma to $\phi_m\otimes V_\varepsilon^P$ with $m\in\mathbb{R}_\varepsilon$, $|m|\le 1$, we get

$$C_{QP}(t,w,s)(\phi_m\otimes v) = \phi_m\otimes c_{QP}^{\operatorname{sgn} m}(t,w,\varepsilon,s)v.$$

The functional equations for c follow from those of C. $\square$

Proof of Theorem 12. Applying I_χ to the formulae of Proposition 11, we obtain by Fourier inversion, using the invariance of Φ,

$$(I_\chi\mathcal{I}_\chi^{\text{con}}\Phi)(\tfrac{1}{2}+ir) = \Phi(\tfrac{1}{2}+ir)_{\frac{1}{2}+ir}$$

and, as a Fourier transform of distributions,

$$(I_\chi\mathcal{I}_\chi^{\text{res}}\Phi)(s) = \sum_\sigma \operatorname*{Res}_{z=\sigma} C(\chi,w,z)\Phi_\sigma\int_{-\infty}^{\infty} e^{(\sigma-s)u}du = \sum_\sigma \operatorname*{Res}_{z=\sigma} C(\chi,w,z)\Phi_\sigma\delta(s-\sigma).$$

The assertions about I_χ^{con} follow immediately.

As for the residual part, we have now the commutative diagram of isomorphisms

$$\begin{tikzcd}
\mathcal{H}_\chi^{\text{res}}(\varepsilon) \arrow[rr, "I_\chi^{\text{res}}"] & & \bigoplus_\sigma \operatorname{Res}_{s=\sigma} C(\chi, w, s)(\mathcal{H}_\varepsilon \otimes V_\varepsilon^{\text{cst}}) \\
& \bigoplus_\sigma \mathcal{H}(\chi, \varepsilon, \sigma) \arrow[ul, "\mathcal{I}_\chi^{\text{res}}"] \arrow[ur, "\oplus \operatorname{Res} C"'] &
\end{tikzcd}$$

It remains to identify the range of $\operatorname{Res} C$ and to transfer the scalar product to it. Since $R_{P_0}(\varepsilon, s)$ is holomorphic on $\left(\frac{1}{2}, 1 - \frac{|m|}{2}\right]$, taking residues in the formula in Lemma 19 affects only the second factor, which gives $q(\chi, \varepsilon, \sigma)$. The range of $R_{P_0}(\varepsilon, s)$ has been determined in section 1, and the assertions about scalar products are easily checked. $\square$

6 The geometric side—regular part

In the remaining sections we shall prove the trace formula under the assumption that $Z \subset \Gamma$, $\chi\big|_Z = \varepsilon \mathrm{Id}$. Let $f \in C_0(G, \varepsilon)$, $t \in \mathfrak{H}(G, \chi)$. One calculates that

$$(\pi_\chi(f)\tau_\chi(t)\varphi)(x) = \int_{\Gamma\backslash G} K_{f,t}(x,y)\varphi(y)\,dy,$$

where

$$K_{f,t}(x,y) = \sum_{\xi \in Z\backslash\Gamma} f(x^{-1}\xi y)t(\xi)$$

is absolutely uniformly convergent on compact sets and slowly increasing ([21, ch. 8] applied to $|f|$ and the finitely many cosets in $\Gamma\backslash\operatorname{supp}(t)$). Given a Γ-cuspidal $P = NAZ$, we similarly obtain

$$(\pi_{P,\chi}(f)\tau_{PP}(t)\psi)(x) = \int_{\Gamma\cap P\backslash G} K_{P,f,t}(x,y)\psi(y)\,dy$$

in the notations of section 5, where

$$K_{P,f,t}(x,y) = \operatorname{vol}(\Gamma \cap N\backslash N)^{-1} \int_{\Gamma\cap N\backslash N} \sum_{\xi \in Z\backslash\Gamma\cap P} f(x^{-1}\xi n y)t(\xi)\operatorname{pr}^P dn.$$

Being obtained from a subseries of $K_{f,t}$, integrated over a compact set, $K_{P,f,t}$ is slowly increasing on $\mathcal{C}_{P,u_0}$, too. When restricted to the diagonal $x = y$, $K_{P,f,t}$ will turn out to be the leading term of $K_{f,t}$ on $\mathcal{C}_{P,u_0}$ as $u_0 \to \infty$. The trace formula will be obtained by integrating the trace in V of

$$K_{f,t}(x,x) - \sum_P \chi_{P,u_0}(x)K_{P,f,t}(x,x)$$

over $\Gamma\backslash G$, where the sum is taken over all cuspidal P (which makes it Γ-invariant). Here we interpret

$$\int_{\Gamma\backslash G} \varphi(x)\,dx = \lim_{u_1\to\infty} \int_{\Gamma\backslash G} \varphi(x)\Big(1 - \sum_P \chi_{P,u_1}(x)\Big)\,dx.$$

The disjoint union $\bar\Gamma = \bar\Gamma_{\mathrm{reg}} \cup \bar\Gamma_{\mathrm{sing}}$ gives an obvious decomposition

$$K_{f,t} = K_{f,t}^{\mathrm{reg}} + K_{f,t}^{\mathrm{sing}}, \qquad K_{P,f,t} = K_{P,f,t}^{\mathrm{reg}} + K_{P,f,t}^{\mathrm{sing}}.$$

In the present section we shall obtain a geometric expression for the regular part, deferring the singular part to section 7.

To handle the contribution from $\bar\Gamma_{\mathrm{reg}} \cap P$, $P = NAZ \in \mathfrak{F}$ (which is absent in the case $t = \chi$), we need some preparation.

Lemma 20. *For $u \neq 0$,*

$$\sum_{\{\xi\}_{\Gamma\cap N} \subset \bar\Gamma \cap a_{P,u}N} \operatorname{tr} t(\xi) = \left|e^{-u} - 1\right| \sum_{\xi \in \bar\Gamma \cap a_{P,u}N/\Gamma\cap N} \operatorname{tr}(t(\xi)\mathrm{pr}^P).$$

Note that $a_{P,u}N = \{a_{P,u}\}_N$ and that $\#(\mathrm{supp}(t) \cap P/\Gamma \cap N) \leq \#(\mathrm{supp}(t)/\Gamma) < \infty$. Thus the right-hand side is finite and vanishes but for finitely many u.

Proof. Assume for simplicity that $P = P_0$ and multiply the right-hand side by

$$\left|e^{-u} - 1\right|^{-1} \int_N g(a_u n)\,dn = \int_N g(n^{-1} a_u n)\,dn,$$

where g is in the Schwartz space $\mathcal{S}(a_u N) \cong \mathcal{S}(\mathbb{R})$. Rewrite the result as

$$\mathrm{vol}(\Gamma \cap N\backslash N)^{-1} \int_{\Gamma\cap N\backslash N} \sum_{\xi \in \bar\Gamma \cap a_u N/\Gamma\cap N} \operatorname{tr}(t(\xi)\mathrm{pr}^P) \int_N g(n^{-1}\xi n' n)\,dn'\,dn$$

with a futile integration over $\Gamma \cap N\backslash N$. If the Fourier transform of g has suitable compact support, this equals

$$\int_{\Gamma\cap N\backslash N} \sum_{\xi \in \bar\Gamma \cap a_u N} \operatorname{tr}(t(\xi)\mathrm{pr}^P) g(n^{-1}\xi n)\,dn$$

by the Poisson summation formula (cf. [16], Lemma 7.9), or else

$$\sum_{\{\xi\}_{\Gamma\cap N} \subset \bar\Gamma \cap a_u N} \operatorname{tr}(t(\xi)\mathrm{pr}^P) \int_N g(n^{-1}\xi n)\,dn.$$

After replacing ξ by a_u in the integral we may divide it out, since it is nonzero for suitable g.

Thus it only remains to show that the left-hand side of the asserted identity does not change if we replace $t(\xi)$ by $t(\xi)\mathrm{pr}^P$. It is obvious at least that we may replace

$t(\xi)$ by $t(\xi\gamma)$ for any $\gamma \in \Gamma \cap N$, since this only permutes the $(\Gamma \cap N)$-conjugacy classes in $\bar{\Gamma} \cap a_u N$. Linear combination now yields

$$\sum_{\{\xi\}_{\Gamma\cap N} \subset \bar{\Gamma}\cap a_{P,u}N} \operatorname{tr}(t(\xi)p(\chi(\gamma_0) - \mathrm{Id})) = 0$$

for any polynomial p over $\mathbb{C}$, where γ_0 is a generator of $\Gamma \cap N$. Choosing p with $p(0) = 0$ and $p(\lambda - 1) = 1$ for every eigenvalue $\lambda \neq 1$ of $\chi(\gamma_0)$, we get $p(\chi(\gamma_0) - \mathrm{Id}) = \mathrm{pr}^P - \mathrm{Id}$. □

Multiplying both sides of the identity in Lemma 20 by an integral as in the proof and transforming the right-hand side in the same way, we see that

$$\int_{\Gamma\cap N\backslash N} \sum_{\xi \in \bar{\Gamma}_{\mathrm{reg}}\cap AN} g(n^{-1}\xi n) \operatorname{tr} t(\xi)\, dn = \int_{\Gamma\cap N\backslash N} \sum_{\xi\in\bar{\Gamma}_{\mathrm{reg}}\cap AN} g(\xi n) \operatorname{tr}(t(\xi)\mathrm{pr}^P)\, dn,$$

now for any $g \in C_0(AN)$. Applied to $g(y) = f(x^{-1}yx)$, this shows, after integration over a compact set, that

$$\int_{\Gamma\cap P\backslash G} \Big(\sum_{\xi\in Z\backslash\bar{\Gamma}_{\mathrm{reg}}\cap P} f(x^{-1}\xi x) \operatorname{tr} t(\xi) - \operatorname{tr} K^{\mathrm{reg}}_{P,f,t}(x,x) \Big) \chi_{P,u_0,u_1}(x)\, dx = 0,$$

where $\chi_{P,u_0,u_1} = \chi_{P,u_0} - \chi_{P,u_1}$. One can check that both terms depend linearly on $u_1 - u_0$. In view of $K^{\mathrm{reg}}_{\gamma^{-1}P\gamma,f,t}(x,x) = K^{\mathrm{reg}}_{P,f,t}(\gamma x, \gamma x)$ and the similar property of χ_{P,u_0}, summing the last equation over $P \in \mathfrak{F}$ gives

$$\int_{\Gamma\backslash G} \sum_P \Big(\sum_{\xi\in Z\backslash\bar{\Gamma}_{\mathrm{reg}}\cap P} f(x^{-1}\xi x) \operatorname{tr} t(\xi) - \operatorname{tr} K^{\mathrm{reg}}_{P,f,t}(x,x) \Big) \chi_{P,u_0,u_1}(x)\, dx = 0.$$

Letting $u_1 \to \infty$, we see that

$$\int_{\Gamma\backslash G} \sum_P \operatorname{tr} \Big(\sum_{\xi\in Z\backslash\bar{\Gamma}_{\mathrm{reg}}\cap P} f(x^{-1}\xi x) t(\xi) - K^{\mathrm{reg}}_{P,f,t}(x,x) \Big) \chi_{P,u_0}(x)\, dx = 0,$$

the integral being at least conditionally convergent in the obvious sense.

The regular part of the geometric side of the trace formula now becomes

$$\begin{aligned}
&\int_{\Gamma\backslash G} \operatorname{tr} \Big(K^{\mathrm{reg}}_{f,t}(x,x) - \sum_P \chi_{P,u_0}(x) K^{\mathrm{reg}}_{P,f,t}(x,x) \Big)\, dx \\
&\qquad = \sum_{\{\xi\}_\Gamma \subset Z\backslash\bar{\Gamma}_{\mathrm{reg}}} \operatorname{tr} t(\xi) \int_{\Gamma_\xi\backslash G} f(x^{-1}\xi x) \Big(1 - \sum_{P\ni\xi} \chi_{P,u_0}(x) \Big)\, dx,
\end{aligned}$$

where Γ_ξ denotes the centralizer of ξ in Γ.

The evaluation of the terms for the various conjugacy classes $\{\xi\}_\Gamma$ depends on the type of ξ. If ξ doesn't belong to any cuspidal P, then

$$\int_{\Gamma_\xi\backslash G} f(x^{-1}\xi x)\,dx = \begin{cases} \dfrac{\mathrm{vol}(\Gamma_\xi\backslash G_\xi)}{2\sinh\frac{u}{2}} F_f^A(a_u) \\ \dfrac{\mathrm{vol}(\Gamma_\xi\backslash G_\xi)}{2i\sin\theta} F_f^K(k_\theta), \end{cases}$$

depending on whether ξ is conjugate to a_u with $u > 0$ or to k_θ.

Let us turn to the remaining case $\xi \in P$. We may choose the representative ξ in such a way that $P \in \mathfrak{F}$ and $\xi \in AN$. From [16, Lemma 8.1] it follows that there exist $Q \in \mathfrak{F}$ and $\gamma \in \Gamma$ such that $\xi \in \gamma^{-1}Q\gamma \neq P$ and hence $\gamma \in \Gamma_{QP}(w)$. Moreover, since $G_\xi = Zn_\xi^{-1}An_\xi$ for a unique $n_\xi \in N$, $\Gamma_\xi = G_\xi \cap P \cap \Gamma = Z$. In this situation, the lemma just cited tells us that $P \neq Q$. If $Q = N'A'Z$ is the Langlands decomposition of Q, then $G_{\gamma\xi\gamma^{-1}} = Zn_\xi'^{-1}A'n_\xi'$ for a unique $n_\xi' \in N'$. Thus, in the notation of Proposition 11,

$$\gamma = n_\xi'^{-1} g_Q^{-1} w_{QP}(\gamma) a_l g_P n_\xi,$$

where $l = H_{QP}(\gamma)$ is uniquely determined by ξ, even independently of the choice of P in view of $H_{PQ}(\gamma^{-1}) = H_{QP}(\gamma)$.

The contribution from $\{\xi\}_\Gamma$ now becomes $\mathrm{tr}\, t(\xi)$ times

$$\int_{Z\backslash G} f(x^{-1}\xi x)(1 - \chi_{P,u_0}(x) - \chi_{\gamma^{-1}Q\gamma,u_0}(x))\,dx$$
$$= \int_{Z\backslash K}\int_N f(k^{-1}n^{-1}a_{P,u}nk)\int_{n_\xi^{-1}An_\xi}((1 - \chi_{P,u_0}(a) - \chi_{Q,u_0}(\gamma a n_\xi^{-1} n))\,da\,dn\,dk,$$

where $a_{P,u} = n_\xi \xi n_\xi^{-1}$. The inner integral equals

$$\int_A (1 - \chi_{P,u_0}(a) - \chi_{Q,u_0}(g_Q^{-1} w_{QP}(\gamma) a_l g_P a n))\,da.$$

Here the first characteristic function is nonzero for $H(g_P a) > u_0$, and the second one for $\bar{H}(a_l g_P a n) > u_0$. So the integral equals $2u_0 + l + \bar{H}(k_P n)$, and the contribution from $\{\xi\}_\Gamma$ can be transformed to

$$\frac{\mathrm{tr}\, t(\xi)}{2\sinh\left|\frac{u}{2}\right|}((2u_0 + l)F_f^A(a_u) + T_f^A(a_u)).$$

If f is smooth and K-finite, Proposition 6 implies

$$2T_f^A(a_u) = I_f^{P_0}(a_u) + I_f^{P_0}(a_{-u}) + \left(1 - \delta_{\varepsilon,\varepsilon_1}\right)\Theta_{\varepsilon,\frac{1}{2}}(f)$$
$$- \frac{1}{2\pi}\mathrm{p.v.}\int_{-\infty}^{\infty}\left(e^{-irH_P(\xi)} + e^{-irH_P(\xi)}\right)\mathrm{tr}\left(\pi_{\varepsilon,\frac{1}{2}+ir}(f)J(\varepsilon,\tfrac{1}{2}+ir)^{-1}J'(\varepsilon,\tfrac{1}{2}+ir)\right)dr,$$

where $J = J_{\bar{P}_0 P_0}$. The terms containing spectral data or the truncation parameter u_0 have to be rewritten so that they will finally cancel against analogous terms on the spectral side. First of all, by [16, Lemma 8.1], $\{\xi\}_\Gamma \cap P = \{\xi\}_{\Gamma\cap P}$, and the only cuspidals in $\mathfrak{F}$ which intersect $\{\xi\}_\Gamma$ are P and Q. Thus,

$$\sum_{\{\xi\}_\Gamma \subset Z\backslash\Gamma_{\mathrm{par}}} = \frac{1}{2}\sum_{P\in\mathfrak{F}}\sum_{\{\xi\}_{\Gamma\cap N}\subset\Gamma_{\mathrm{reg}}\cap AN}.$$

Using Lemma 20, we finally we obtain

Lemma 21. *For K-finite $f \in C_0^\infty(G,\varepsilon)$ and $t \in \mathfrak{H}(G,\chi)$,*

$$\int_{\Gamma\backslash G} \operatorname{tr}\Big(K_{f,t}^{\mathrm{reg}}(x,x) - \sum_P \chi_{P,u_0}(x) K_{P,f,t}^{\mathrm{reg}}(x,x)\Big)\,dx$$

$$= \sum_{\{\xi\}_\Gamma \subset Z\backslash\bar\Gamma_{\mathrm{hyp}}} \frac{\operatorname{tr} t(\xi)\operatorname{vol}(\Gamma_\xi\backslash G_\xi)}{2\sinh\frac{u}{2}} F_f^A(a_u) + \sum_{\{\xi\}_\Gamma\subset Z\backslash\bar\Gamma_{\mathrm{ell}}} \frac{\operatorname{tr} t(\xi)\operatorname{vol}(\Gamma_\xi\backslash G_\xi)}{2i\sin\theta} F_f^K(k_\theta)$$

$$+ \sum_{\{\xi\}_\Gamma\subset Z\backslash\bar\Gamma_{\mathrm{par}}} \frac{\operatorname{tr} t(\xi)}{4\sinh\frac{u}{2}}\big(2lF_f^A(a_u) + I_f^{P_0}(a_u) + I_f^{P_0}(a_{-u})\big)$$

$$+ \sum_{P\in\mathfrak{F}} \sum_{\xi\in\bar\Gamma_{\mathrm{reg}}\cap AN/\Gamma\cap N} \operatorname{tr}(t(\xi)\mathrm{pr}^P)\Big(\frac{1}{4}e^{-H_P(\xi)/2}(1-\delta_{\varepsilon,\varepsilon_1})\Theta_{\varepsilon,\frac12}(f)$$

$$+ \frac{1}{2\pi}\mathrm{p.v.}\int_{-\infty}^{\infty} e^{-(1/2+ir)H_P(\xi)} \operatorname{tr}\big(\pi_{\varepsilon,\frac12+ir}(f)(u_0\mathrm{Id} - \tfrac12 J(\varepsilon,\tfrac12+ir)^{-1}J'(\varepsilon,\tfrac12+ir))\big)\,dr\Big),$$

where $P = NAZ$, and ξ (suitably chosen mod Z) determines θ, u and l as follows: $\{\xi\}_G = \{k_\theta\}_G$ or $\{\xi\}_G = \{a_u\}_G$, $u > 0$, and $l = H_{QP}(\gamma)$, where $\gamma \in \Gamma_{QP}(w)$ and $P \neq Q$ are the cuspidal subgroups containing ξ.

7 The geometric side—singular part

In this section we shall calculate the singular part of the geometric side of the trace formula. As before, let $f \in C_0^\infty(G,\varepsilon)$, $t \in \mathfrak{H}(G,\chi)$ with χ as in section 6. Since $\bar\Gamma_{\mathrm{sing}} = \bigcup_P(\bar\Gamma \cap ZN)$, where P runs through all cuspidal subgroups and N is the unipotent radical of P ([16, Proposition 7.2]), the singular part can be transformed as

$$\int_{\Gamma\backslash G} \operatorname{tr}\Big(K_{f,t}^{\mathrm{sing}}(x,x) - \sum_P \chi_{P,u_0}(x)K_{P,f,t}^{\mathrm{sing}}(x,x)\Big)\,dx - \operatorname{vol}(\Gamma\backslash G)f(1)\operatorname{tr} t(1)$$

$$= \sum_{P\in\mathfrak{F}} \lim_{u_1\to\infty}\int_{\Gamma\cap P\backslash G}(1-\chi_{P,u_1}(x))\Big(\sum_{\substack{\xi\in\bar\Gamma\cap N\\ \xi\neq 1}} f(x^{-1}\xi x)\operatorname{tr} t(\xi)$$

$$-\chi_{P,u_0}(x)\operatorname{vol}(\Gamma\cap N\backslash N)^{-1}\sum_{\xi\in\bar\Gamma\cap N/\Gamma\cap N}\operatorname{tr}(t(\xi)\mathrm{pr}^P)\int_N f(x^{-1}nx)\,dn\Big)\,dx$$

$$= \sum_{P\in\mathfrak{F}}\Big(J_{f,t}^P(1) + \lim_{u_1\to\infty}\int_{\Gamma\cap P\backslash G}(1-\chi_{P,u_1}(x))\sum_{\substack{\xi\in\bar\Gamma\cap N\\ \xi\neq1}} f(x^{-1}\xi x)\operatorname{tr}(t(\xi)(\mathrm{Id}-\mathrm{pr}^P))dx\Big)$$

whith

$$J_{f,t}^P(s) = \sum_{\xi\in\bar\Gamma\cap N/\Gamma\cap N}\operatorname{tr}(t(\xi)\mathrm{pr}^P)\int_{NZ\backslash G}\Big(\operatorname{vol}(\Gamma\cap N\backslash N)\sum_{\substack{\gamma\in\Gamma\cap N\\ \xi\gamma\neq1}} f(x^{-1}\xi\gamma x)$$

$$-\chi_{P,u_0}(x)\int_N f(x^{-1}nx)\,dn\Big)e^{(1-s)(H_P(x)+u_P)}dx,$$

the exterior sum being finite. With the help of the Poisson summation formula one can prove (see [16, Proposition 10.2]) that the integral over $NZ\backslash G \cong A \times Z\backslash K$ is absolutely convergent to a holomorphic function for $\Re s > 0$. Moreover, the second term of the integrand is integrable for $\Re s > 1$:

$$\int_{Z\backslash K} \int_{-\infty}^{\infty} \chi_{P,u_0}(a_{P,u}) \int_N f(k^{-1} a_{P,-u} n a_{P,u} k)\, dn\, e^{(1-s)(u+u_P)-u} du\, dk$$
$$= \int_{Z\backslash K} \int_{-\infty}^{\infty} \chi_{P,u_0}(a_{P,u}) \int_N f(k^{-1} n k)\, dn\, e^{(1-s)(u+u_P)} du\, dk = \frac{e^{(1-s)u_0}}{(s-1)} F_f^A(1).$$

Hence the first term of $J_{f,t}^P(s)$, namely,

$$Y_{f,t}^P(s) = \int_{\Gamma\cap P\backslash G} \sum_{\substack{\xi\in\bar{\Gamma}\cap N\\ \xi\neq 1}} f(x^{-1}\xi x)\, \mathrm{tr}(t(\xi)\mathrm{pr}^P) e^{(1-s)(H_P(x)+u_P)} dx,$$

also has a meromorphic continuation to $\Re s > 0$, whose only possible pole is a simple one at $s = 1$ with residue

$$\operatorname*{Res}_{s=1} Y_{f,t}^P(s) = \sum_{\xi\in\bar{\Gamma}\cap N/\Gamma\cap N} \mathrm{tr}(t(\xi)\mathrm{pr}^P) F_f^A(1).$$

Clearly, $J_{f,t}^P(1)$ equals the difference of the constant terms in the two Laurent series:

$$J_{f,t}^P(1) = \lim_{s\to 1} \frac{d}{ds}\left((s-1) Y_{f,t}^P(s)\right) + u_0 \sum_{\xi\in\bar{\Gamma}\cap N/\Gamma\cap N} \mathrm{tr}(t(\xi)\mathrm{pr}^P) F_f^A(1).$$

We may rewrite

$$Y_{f,t}^P(s) = \sum_{\substack{\xi\in\bar{\Gamma}\cap N\\ \xi\neq 1}} \mathrm{tr}(t(\xi)\mathrm{pr}^P) \int_{Z\backslash K} \int_{-\infty}^{\infty} f(k^{-1} a_{P,-u} \xi a_{P,u} k) e^{-s(u+u_P)} du\, dk$$
$$= \sum_{\substack{n_{P,v}\in\bar{\Gamma}\\ v\neq 0}} \mathrm{tr}(t(n_{P,v})\mathrm{pr}^P) \int_{Z\backslash K} \int_{-\infty}^{\infty} f(k^{-1} a_{-u} n_v a_u k) e^{-su} du\, dk$$
$$= \sum_{\substack{n_{P,v}\in\bar{\Gamma}\\ v\neq 0}} \mathrm{tr}(t(n_{P,v})\mathrm{pr}^P) |v|^{-s} \int_{Z\backslash K} \int_{v'/v>0} f(k^{-1} n_{v'} k) |v'|^{s-1} dv'\, dk.$$

If we define

$$\zeta_{t,\pm}^P(s) = \sum_{\substack{n_{P,v}\in\bar{\Gamma}\\ \pm v>0}} \mathrm{tr}(t(n_{P,v})\mathrm{pr}^P) |v|^{-s}$$

and recall from section 2 that

$$F_f^K(k_{\pm 0}) = \pm 2\pi i \int_{Z\backslash K} \int_{\pm v>0} f(k^{-1} n_v k)\, dv\, dk,$$

we get another expression

$$2\pi i \operatorname*{Res}_{s=1} Y^P_{f,t}(s) = \operatorname*{Res}_{s=1} \zeta^P_{t,+}(s) F^K_f(k_{+0}) - \operatorname*{Res}_{s=1} \zeta^P_{t,-}(s) F^K_f(k_{-0})$$

for the residue of $Y^P_{f,t}$. Comparing both, we see that

$$\operatorname*{Res}_{s=1} \zeta^P_{t,\pm}(s) = \sum_{\xi \in \tilde{\Gamma}\cap N/\Gamma\cap N} \operatorname{tr}(t(\xi)\mathrm{pr}^P).$$

It is now easy to calculate that

$$\lim_{s\to 1} \frac{d}{ds}\big((s-1)Y^P_{f,t}(s)\big) = \frac{1}{2} \sum_{\xi \in \tilde{\Gamma}\cap N/\Gamma\cap N} \operatorname{tr}(t(\xi)\mathrm{pr}^P) T^A_f(1)$$
$$+ \frac{1}{2\pi i} \lim_{s\to 1} \frac{d}{ds}(s-1)\big(\zeta^P_{t,+}(s) F^K_f(k_{+0}) - \zeta^P_{t,-}(s) F^K_f(k_{-0})\big).$$

We may express $\zeta^P_{t,\pm}$ in terms of the generalized zeta-function

$$\zeta(s,v) = \sum_{n=0}^{\infty} (n+v)^{-s},$$

namely,

$$\zeta^P_{t,\pm}(s) = \sum_{\xi \in \tilde{\Gamma}\cap N/\Gamma\cap N} \operatorname{tr}(t(\xi)\mathrm{pr}^P)\zeta(s, v^{\pm}_{\xi}),$$

where $v^{\pm}_{\xi}$ is as in section 4. The well-known formula

$$\lim_{s\to 1}\left(\zeta(s,v) - \frac{1}{s-1}\right) = -\psi(v) \qquad \text{for } \Re v > 0$$

implies that

$$J^P_{f,t}(1) = \sum_{\xi \in \tilde{\Gamma}\cap N/\Gamma\cap N} \operatorname{tr}(t(\xi)\mathrm{pr}^P)$$
$$\cdot \left(u_0 F^A_f(1) + \frac{1}{2} T^A_f(1) - \frac{1}{2\pi i}\big(\psi(v^+_{\xi}) F^K_f(k_{+0}) - \psi(v^-_{\xi}) F^K_f(k_{-0})\big)\right).$$

The remaining constituent of the singular part, besides $\operatorname{vol}(\Gamma\backslash G) f(1) \operatorname{tr} t(1) +$

$\sum_{P\in\mathfrak{F}} J^P_{f,t}(1)$, is the sum over all P of

$$\lim_{u_1\to\infty}\int_{\Gamma\cap P\backslash G}(1-\chi_{P,u_1}(x))\sum_{\substack{\xi\in\bar\Gamma\cap N\\ \xi\neq 1}} f(x^{-1}\xi x)\operatorname{tr}(t(\xi)(\mathrm{Id}-\mathrm{pr}^P))\,dx$$

$$=\lim_{u_1\to\infty}\int_{Z\backslash K}\int_{u_1}^{\infty}\sum_{\substack{\xi\in\bar\Gamma\cap N\\ \xi\neq 1}}\operatorname{tr}(t(\xi)(\mathrm{Id}-\mathrm{pr}^P))f(k^{-1}a_{P,-u}\xi a_{P,u}k)e^{-(u+u_P)}du\,dk$$

$$=\lim_{u_1\to\infty}\int_{Z\backslash K}\int_{u_1}^{\infty}\sum_{\substack{n_{P,v}\in\bar\Gamma\\ v\neq 0}}\operatorname{tr}(t(n_{P,v})(\mathrm{Id}-\mathrm{pr}^P))f(k^{-1}a_{-u}n_v a_u k)e^{-u}du\,dk$$

$$=\lim_{u_1\to\infty}\int_{Z\backslash K}\int_{-\infty}^{\infty}\sum_{\substack{n_{P,v}\in\bar\Gamma\\ v'/v>e^{-u_1}}}\operatorname{tr}(t(n_{P,v})(\mathrm{Id}-\mathrm{pr}^P))|v|^{-1}f(k^{-1}n_{v'}k)\,dv'\,dk$$

$$=\frac12\sum_{\substack{n_{P,v}\in\bar\Gamma\\ v\neq 0}}\operatorname{tr}(t(n_{P,v})(\mathrm{Id}-\mathrm{pr}^P))\big(|v|^{-1}F_f^A(1)+(2\pi i v)^{-1}(F_f^K(k_{+0})+F_f^K(k_{-0}))\big),$$

where we were able to pass to the limit under the integral by dominated convergence. With the notations of section 4 our expression becomes

$$\frac12\sum_{\substack{\xi\in\bar\Gamma\cap N/\Gamma\cap N\\ \alpha\in(-\pi,\pi)}}\Bigg(\Bigg(\operatorname{tr}t_\alpha^+(\xi)\sum_{n=0}^{\infty}\frac{(-e^{i\alpha})^n}{n+v_\xi^+}+\operatorname{tr}t_\alpha^-(\xi)\sum_{n=0}^{\infty}\frac{(-e^{-i\alpha})^n}{n+v_\xi^-}\Bigg)F_f^A(1)$$
$$+\frac{1}{2\pi i}\operatorname{tr}t_\alpha^+(\xi)\sum_{\substack{n=-\infty\\ n+v_\xi^+\neq 0}}^{\infty}\frac{(-e^{i\alpha})^n}{n+v_\xi^+}(F_f^K(k_{+0})+F_f^K(k_{-0}))\Bigg).$$

The first two series can be expressed in terms of β, for the last one we need

Lemma 22. *If $\alpha\in(-\pi,\pi)$, $v\in\mathbb{R}$, $v\notin\mathbb{Z}$, then*

$$\sum_{n=-\infty}^{\infty}\frac{(-e^{i\alpha})^n}{n+v}=\frac{\pi e^{-iv\alpha}}{\sin\pi v}.$$

Consequently,

$$\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{(-e^{i\alpha})^n}{n}=-i\alpha.$$

Proof. Substituting t^{-1} for t in the second term, we get, for $0<\Re v<1$,

$$\beta(v,-e^{i\alpha})+e^{-i\alpha}\beta(1-v,-e^{-i\alpha})=\int_0^\infty t^{v-1}\big(1+e^{i\alpha}t\big)^{-1}dt$$
$$=e^{-iv\alpha}\int_C t^{v-1}\big(1+t\big)^{-1}dt=e^{-iv\alpha}B(v,1-v).\qquad\square$$

Let us collect the results of this section.

Lemma 23. *For K-finite $f \in C_0^\infty(G,\varepsilon)$ and $t \in \mathfrak{H}(G,\chi)$,*

$$\int_{\Gamma\backslash G} \operatorname{tr}\Big(K_{f,t}^{\mathrm{sing}}(x,x) - \sum_P \chi_{P,u_0}(x) K_{P,f,t}^{\mathrm{sing}}(x,x)\Big)\, dx = \operatorname{vol}(\Gamma\backslash G) f(1) \operatorname{tr} t(1)$$

$$+ \sum_{P\in\mathfrak{F}} \sum_{\xi\in\bar\Gamma\cap N/\Gamma\cap N} \operatorname{tr}(t(\xi)\mathrm{pr}^P)\Big(\frac{1}{2} I_f^{P_0}(1) - \frac{1}{2\pi i}\big(\psi(v_\xi^+)F_f^K(k_{+0}) - \psi(v_\xi^-)F_f^K(k_{-0})\big)\Big)$$

$$+ \sum_{\substack{P\in\mathfrak{F}\\ \alpha\in(-\pi,\pi)}} \Bigg[\frac{1}{2} \sum_{\substack{\xi\in\bar\Gamma\cap N/\Gamma\cap N\\ \xi\notin\Gamma\cap N}} \Big(\big(\beta(v_\xi^+, -e^{i\alpha}) \operatorname{tr} t_\alpha^+(\xi) + \beta(v_\xi^-, -e^{-i\alpha}) \operatorname{tr} t_\alpha^-(\xi)\big) F_f^A(1)$$

$$+ \frac{e^{-iv_\xi^+\alpha} \operatorname{tr} t_\alpha^+(\xi)}{2i \sin \pi v_\xi^+}\big(F_f^K(k_{+0}) + F_f^K(k_{-0})\big)\Big)$$

$$- \frac{\operatorname{tr} t(1)}{\dim V} m_\alpha^P \big(\log(2\cos\tfrac{\alpha}{2}) F_f^A(1) + \frac{\alpha}{4\pi}\big(F_f^K(k_{+0}) + F_f^K(k_{-0})\big)\big)\Bigg]$$

$$+ \sum_{P\in\mathfrak{F}} \sum_{\xi\in\bar\Gamma\cap N/\Gamma\cap N} \operatorname{tr}(t(\xi)\mathrm{pr}^P)\Big(\frac{1}{4}(1-\delta_{\varepsilon,\varepsilon_1})\Theta_{\varepsilon,\frac12}(f)$$

$$+ \frac{1}{2\pi}\mathrm{p.v.}\int_{-\infty}^{\infty} \operatorname{tr}\big(\pi_{\varepsilon,\frac12+ir}(f)(u_0 \mathrm{Id} - \frac{1}{2} J(\varepsilon, \tfrac12 + ir)^{-1} J'(\varepsilon, \tfrac12 + ir))\big)\, dr\Big),$$

where m_α^P is the multiplicity of $-e^{i\alpha}$ as an eigenvalue of $\chi(n_{P,1})$.

The geometric side of the trace formula is obtained by adding the formulae from Lemmas 21 and 23. The last terms of these formulae may be written as sums over $\bar\Gamma_{\mathrm{reg}} \cap P/\Gamma\cap P$ and $\bar\Gamma_{\mathrm{sing}} \cap P/\Gamma \cap P$, respectively, and since

$$c_{PP}(t,1,\varepsilon,s) = \sum_{\xi\in\bar\Gamma\cap P/\Gamma\cap P} \varepsilon(z_{PP}(\xi)^{-1}) e^{-sH_P(\xi)} t(\xi)\mathrm{pr}^P,$$

they combine to $\frac14 \operatorname{tr} c(t,1,\varepsilon,\frac12)(1-\delta_{\varepsilon,\varepsilon_1})\Theta_{\varepsilon,\frac12}(f)$ plus

$$\frac{1}{2\pi}\mathrm{p.v.}\int_{-\infty}^{\infty} \operatorname{tr} c(t,1,\varepsilon,\tfrac12 + ir) \operatorname{tr}\big(\pi_{\varepsilon,\frac12+ir}(f)(u_0\mathrm{Id} - \frac12 J(\varepsilon,\tfrac12+ir)^{-1} J'(\varepsilon,\tfrac12+ir))\big)\, dr,$$

where $J = J_{\bar P_0 P_0}$. This term will occur on the spectral side as well, thus will be canceled.

8 The spectral side

Suppose that $f \in C_0^\infty(G,\varepsilon)$ is K-finite and $t \in \mathfrak{H}(G,\chi)$ as before. Our aim in this section is to obtain an expression for

$$\int_{\Gamma\backslash G} \operatorname{tr}\Big(K_{f,t}(x,x) - \sum_P \chi_{P,u_0}(x) K_{P,f,t}(x,x)\Big)\, dx$$

in spectral terms, where the integral is at least conditionally convergent as explained in the beginning of section 6.

By our assumption on χ, $\mathcal{H}_\chi = \mathcal{H}_\chi(\varepsilon)$. As one knows, $\pi_\chi^{\mathrm{dis}}(f)\tau_\chi^{\mathrm{dis}}(t)$ is of trace class (we shall see this for more general f in the proof of Theorem 25 below), while $\pi_\chi^{\mathrm{con}}(f)\tau_\chi^{\mathrm{con}}(t) := p^{\mathrm{con}}\pi_\chi(f)\tau_\chi(t)$ has an explicit description by Proposition 18:

$$\pi_\chi^{\mathrm{con}}(f)\tau_\chi^{\mathrm{con}}(t) = \mathcal{I}^{\mathrm{con}} C(t,1,.)\pi_{\varepsilon,.}^{\mathrm{cst}}(f)(\mathcal{I}^{\mathrm{con}})^*,$$

where, for shortness, $\pi_{\varepsilon,s}^{\mathrm{cst}}(x) = \pi_{\varepsilon,s}(x)\otimes \mathrm{Id}_{V_\varepsilon^{\mathrm{cst}}}$. Using the fact that $C(t,1,.)\pi_{\varepsilon,.}^{\mathrm{cst}}(f)$ is an integral operator in $\mathcal{H}_\varepsilon \otimes V_\varepsilon^{\mathrm{cst}}$ with kernel

$$K_{f,t,\varepsilon,s} = \big(C(t,1,s)\otimes \mathrm{Id}\big)\,\big(K_{f,\varepsilon,s}\otimes \mathrm{Id}_{V_\varepsilon^{\mathrm{cst}}}\big)$$

belonging to $\mathcal{H}_{(\varepsilon,\bar\varepsilon),(s,1-s)} \otimes \mathrm{End}\, V_\varepsilon^{\mathrm{cst}} \cong (\mathcal{H}_{\varepsilon,s}\otimes V_\varepsilon^{\mathrm{cst}})\ \hat\otimes\ (\mathcal{H}_{\varepsilon,s}\otimes V_\varepsilon^{\mathrm{cst}})^*$ (cf. Lemma 2), one shows by a formal manipulation [16, Prop. 5.2] that $\pi_\chi^{\mathrm{con}}(f)\tau_\chi^{\mathrm{con}}(t)$ is an integral operator with kernel

$$K_{f,t}^{\mathrm{con}}(x,y) = \frac{1}{4\pi i}\int_{\Re s=1/2} \mathbf{E}(t\otimes\chi,(s,1-s)K_{f,t,\varepsilon,s},(x,y))\,ds,$$

where $\mathbf{E}(t_1\otimes t_2,(s_1,s_2),\phi_1\otimes\phi_2,(x_1,x_2)) = E(t_1,\phi_1,s_1,x_1)\otimes E(t_2,\phi_2,s_2,x_2)$ is the Eisenstein series for $\Gamma\times\Gamma\backslash G\times G$. Since f is K-finite, $K_{f,t,\varepsilon,s}$ is a finite linear combination of terms $\phi\otimes\phi'$, and Lemma 17 implies that

$$\begin{aligned}
&\int_{\Gamma\backslash G} (\Lambda_{u_1}\otimes\Lambda_{u_1})\,\mathrm{tr}\,\mathbf{E}(t\otimes\chi,(s,1-s),K_{f,t,\varepsilon,s},(x,x))\,dx\\
&= 2u_1\,\mathrm{tr}\big(C(t,1,s)\pi_{\varepsilon,s}^{\mathrm{cst}}(f)\big) - \mathrm{tr}\big(C'(\chi,w,1-s)C(t,w,s)\pi_{\varepsilon,s}^{\mathrm{cst}}(f)\big)\\
&+\frac{1}{2s-1}\left(e^{(2s-1)u_1}\,\mathrm{tr}\big(C(t,w,1-s)\pi_{\varepsilon,1-s}^{\mathrm{cst}}(f)\big) - e^{(1-2s)u_1}\,\mathrm{tr}\big(C(t,w,s)\pi_{\varepsilon,s}^{\mathrm{cst}}(f)\big)\right).
\end{aligned}$$

Inserting this, we get

$$\begin{aligned}
\int_{\Gamma\backslash G} (\Lambda_{u_1}\otimes\Lambda_{u_1})\,\mathrm{tr}\,K_{f,t}^{\mathrm{con}}(x,x)\,dx &= \frac{u_1}{2\pi}\int_{-\infty}^{\infty} \mathrm{tr}\big(C(t,1,\tfrac12+ir)\pi_{\varepsilon,\frac12+ir}^{\mathrm{cst}}(f)\big)\,dr\\
&\quad -\frac{1}{4\pi}\int_{-\infty}^{\infty} \mathrm{tr}\big(C'(\chi,w,\tfrac12-ir)C(t,w,\tfrac12+ir)\pi_{\varepsilon,\frac12+ir}^{\mathrm{cst}}(f)\big)\,dr\\
+\frac{1}{4\pi}\int_{-\infty}^{\infty}\left(\cos(2ru_1)\,\mathrm{tr}\,\frac{\eta(\varepsilon,-r)-\eta(\varepsilon,r)}{2ir}\right. &+ \left.\frac{\sin 2ru_1}{2r}\,\mathrm{tr}\big(\eta(\varepsilon,r)+\eta(\varepsilon,-r)\big)\right)dr,
\end{aligned}$$

where, for shortness, $\eta(\varepsilon,r) = C(t,w,\frac12+ir)\pi_{\varepsilon,\frac12+ir}^{\mathrm{cst}}(f)$, an operator of finite rank.

The other contribution to $K_{f,t}$ is

$$K_{f,t}^{\mathrm{dis}} = K_{f,t} - K_{f,t}^{\mathrm{con}},$$

which is, of course, the kernel of $\pi_\chi^{\mathrm{dis}}(f)\tau_\chi^{\mathrm{dis}}(t)$. Thus,

$$\int_{\Gamma\backslash G} \mathrm{tr}\,K_{f,t}^{\mathrm{dis}}(x,x)\,dx = \mathrm{tr}\big(\pi_\chi^{\mathrm{dis}}(f)\tau_\chi^{\mathrm{dis}}(t)\big).$$

It remains to consider $K_{P,f,t}$. Denoting again $\chi_{P,u_0,u_1} = \chi_{P,u_1} - \chi_{P,u_0}$, then

$$\begin{aligned}\int_{\Gamma\backslash G} \sum_P \chi_{P,u_0,u_1}(x) \operatorname{tr} K_{P,f,t}(x,x)\,dx &= \sum_{P\in\mathfrak{F}} \int_{\Gamma\cap P\backslash G} \chi_{P,u_0,u_1}(x) \operatorname{tr} K_{P,f,t}(x,x)\,dx \\ &= \sum_{P\in\mathfrak{F}} \operatorname{tr}\big(\chi_{P,u_0,u_1}\tau_{PP}(t,1)\pi_{P,\chi}(f)\big).\end{aligned}$$

Here we have to take the trace of an operator in $\bigoplus_{P\in\mathfrak{F}} \mathcal{H}_{P,\chi}$. After Fourier transform (cf. section 5) the corresponding operator in $L^2(\frac{1}{2}+i\mathbb{R}, \frac{dr}{2\pi}) \hat{\otimes} \mathcal{H}_\varepsilon \otimes V_\varepsilon^{\text{cst}}$ is convolution with the Fourier transform of χ_{P,u_0,u_1} times multiplication with $C(t,1,s)\pi_{\varepsilon,s}^{\text{cst}}(f)$. Integrating the kernel of this operator over the diagonal, we get

$$\frac{u_1-u_0}{2\pi}\int_{-\infty}^{\infty} \operatorname{tr}\big(C(t,1,\tfrac{1}{2}+ir)\pi^{\text{cst}}_{\varepsilon,\frac{1}{2}+ir}(f)\big)\,dr$$

(cf. [16, Lemma 6.3]). Combining our results, we now obtain the spectral side of the trace formula:

$$\begin{aligned}&\int_{\Gamma\backslash G} \operatorname{tr}\Big(K_{f,t}(x,x) - \sum_P \chi_{P,u_0}(x)K_{P,f,t}(x,x)\Big)\,dx = \int_{\Gamma\backslash G} \operatorname{tr} K^{\text{dis}}_{f,t}(x,x)\,dx \\ &\quad+ \lim_{u_1\to\infty}\int_{\Gamma\backslash G}\Big(\big(\Lambda_{u_1}\otimes\Lambda_{u_1}\big)\operatorname{tr} K^{\text{con}}_{f,t}(x,x) - \sum_P \chi_{P,u_0,u_1}(x)\operatorname{tr} K_{P,f,t}(x,x)\Big)\,dx \\ &= \operatorname{tr}\big(\pi_\chi^{\text{dis}}(f)\tau_\chi^{\text{dis}}(t)\big) - \frac{1}{4\pi}\int_{-\infty}^{\infty}\operatorname{tr}\big(C'(\chi,w,\tfrac{1}{2}-ir)C(t,w,\tfrac{1}{2}+ir)\pi^{\text{cst}}_{\varepsilon,\frac{1}{2}+ir}(f)\big)\,dr \\ &\quad+ \frac{1}{4}\operatorname{tr}\big(C(t,w,\tfrac{1}{2})\pi^{\text{cst}}_{\varepsilon,\frac{1}{2}}(f)\big) + \frac{u_0}{2\pi}\int_{-\infty}^{\infty}\operatorname{tr}\big(C(t,1,\tfrac{1}{2}+ir)\pi^{\text{cst}}_{\varepsilon,\frac{1}{2}+ir}(f)\big)\,dr\end{aligned}$$

(cf. [16, Prop. 6.4]). Differentiation of the functional equation for C yields

$$C(\chi,w,1-s)C'(\chi,w,s) = C'(\chi,w,1-s)C(\chi,w,s).$$

Thus we may transform one of the integrands as

$$\begin{aligned}\operatorname{tr}\big(C'(\chi,w,1-s)C(t,w,s)\pi^{\text{cst}}_{\varepsilon,s}(f)\big) &= \operatorname{tr}\big(C(\chi,w,1-s)C'(\chi,w,s)\pi^{\text{cst}}_{\varepsilon,s}(f)C(t,1,s)\big) \\ &= \operatorname{tr}\big(C(t,w,1-s)C'(\chi,w,s)\pi^{\text{cst}}_{\varepsilon,s}(f)\big).\end{aligned}$$

With the help of Lemma 19 we may express C in terms of R and c or in terms of J and $\tilde{c}$:

$$C(t,w,s)\big|_{\mathcal{H}_\varepsilon\otimes V_\varepsilon^{\text{cst}}} = J^\pm_{P_0}(\varepsilon,s)\otimes\tilde{c}^\pm(t,w,\varepsilon,s),$$

where $J^\pm_{P_0}$ is connected with $J_{\bar{P}_0P_0}$ like $R^\pm_{P_0}$ is with $R_{\bar{P}_0P_0}$. (In the adèlic picture, C splits into a tensor product of local intertwining operators at all places of $\mathbb{Q}$, and J is just the contribution from the infinite place.)

Lemma 24. *If $f \in C_0^\infty(G,\varepsilon)$ is K-finite, $t \in \mathfrak{H}(G,\chi)$, then*

$$\int_{\Gamma\backslash G} \operatorname{tr}\Big(K_{f,t}(x,x) - \sum_P \chi_{P,u_0}(x) K_{P,f,t}(x,x)\Big)\,dx$$
$$= -\frac{1}{4\pi}\text{p.v.}\int_{-\infty}^{\infty} \operatorname{tr}\big(\tilde{c}(t,w,\varepsilon,\tfrac12 - ir)\tilde{c}'(\chi,w,\varepsilon,\tfrac12+ir)\big)\Theta_{\varepsilon,\frac12+ir}(f)\,dr$$
$$+ \operatorname{tr}\big(\pi_\chi^{\mathrm{dis}}(f)\tau_\chi^{\mathrm{dis}}(t)\big) + \frac14 \operatorname{tr} c^+(t,w,\varepsilon,\tfrac12)\big((1-\delta_{\varepsilon,\varepsilon_1})\Theta_{\varepsilon,\frac12}(f) + 2\delta_{\varepsilon,\varepsilon_1}\Theta_{\varepsilon_1}(f)\big)$$
$$+ \frac{1}{2\pi}\text{p.v.}\int_{-\infty}^{\infty} \operatorname{tr} c(t,1,\varepsilon,\tfrac12+ir)\operatorname{tr}\big(\pi_{\varepsilon,\frac12+ir}(f)(u_0\mathrm{Id} - \tfrac12 J(\varepsilon,\tfrac12+ir)^{-1}J'(\varepsilon,\tfrac12+ir))\big)\,dr.$$

Comparing the geometric side (Lemmas 21 and 23) with the spectral side of the trace formula just derived, we see that the terms containing the truncation parameter u_0 and the logarithmic derivative of the intertwining operator J cancel. All other terms are invariant distributions not depending on the choice of K.

Theorem 25. *Let $f \in \mathcal{C}^1(G,\varepsilon)$ and $t \in \mathfrak{H}(G,\chi)$, where $Z \subset \Gamma$, $\chi\big|_Z = \varepsilon\mathrm{Id}$. Then we have the following equality of absolutely convergent integral-series:*

$$\operatorname{vol}(\Gamma\backslash G) f(1)\operatorname{tr} t(1) + \sum_{\{\xi\}_\Gamma \subset Z\backslash\Gamma_{\mathrm{par}}} \frac{\operatorname{tr} t(\xi)}{4\sinh\frac{u}{2}}\big(2lF_f^A(a_u) + I_f^{P_0}(a_u) + I_f^{P_0}(a_{-u})\big)$$
$$+ \sum_{\{\xi\}_\Gamma \subset Z\backslash\Gamma_{\mathrm{hyp}}} \frac{\operatorname{tr} t(\xi)\operatorname{vol}(\Gamma_\xi\backslash G_\xi)}{2\sinh\frac{u}{2}} F_f^A(a_u) + \sum_{\{\xi\}_\Gamma \subset Z\backslash\Gamma_{\mathrm{ell}}} \frac{\operatorname{tr} t(\xi)\operatorname{vol}(\Gamma_\xi\backslash G_\xi)}{2i\sin\theta} F_f^K(k_\theta)$$
$$+ \sum_{P\in\mathfrak{F}}\sum_{\xi\in\Gamma\cap N/\Gamma\cap N} \operatorname{tr}(t(\xi)\mathrm{pr}^P)\Big(\frac12 I_f^{P_0}(1) - \frac{1}{2\pi i}\big(\psi(v_\xi^+)F_f^K(k_{+0}) - \psi(v_\xi^-)F_f^K(k_{-0})\big)\Big)$$
$$+ \sum_{\substack{P\in\mathfrak{F}\\ \alpha\in(-\pi,\pi)}} \Bigg[\frac12 \sum_{\substack{\xi\in\Gamma\cap N/\Gamma\cap N\\ \xi\notin\Gamma\cap N}} \Big(\big(\beta(v_\xi^+,-e^{i\alpha})\operatorname{tr} t_\alpha^+(\xi) + \beta(v_\xi^-,-e^{-i\alpha})\operatorname{tr} t_\alpha^-(\xi)\big)F_f^A(1)$$
$$+ \frac{e^{-iv_\xi^+\alpha}\operatorname{tr} t_\alpha^+(\xi)}{2i\sin\pi v_\xi^+}\big(F_f^K(k_{+0}) + F_f^K(k_{-0})\big)\Big)$$
$$- \frac{\operatorname{tr} t(1)}{\dim V} m_\alpha^P\Big(\log(2\cos\tfrac{\alpha}{2})F_f^A(1) + \frac{\alpha}{4\pi}\big(F_f^K(k_{+0}) + F_f^K(k_{-0})\big)\Big)\Bigg]$$
$$= -\frac{1}{4\pi}\text{p.v.}\int_{-\infty}^{\infty} \operatorname{tr}\big(\tilde{c}(t,w,\varepsilon,\tfrac12 - ir)\tilde{c}'(\chi,w,\varepsilon,\tfrac12+ir)\big)\Theta_{\varepsilon,\frac12+ir}(f)\,dr$$
$$+ \frac14(1-\delta_{\varepsilon,\varepsilon_1})\operatorname{tr}(c(t,w,\varepsilon,\tfrac12) - c(t,1,\varepsilon,\tfrac12))\Theta_{\varepsilon,\frac12}(f) + \frac12\delta_{\varepsilon,\varepsilon_1}\operatorname{tr} c^+(t,w,\varepsilon,\tfrac12)\Theta_{\varepsilon_1}(f)$$
$$+ \operatorname{tr}\big(\pi_\chi^{\mathrm{dis}}(f)\tau_\chi^{\mathrm{dis}}(t)\big),$$

where u, θ, l depend on the corresponding ξ as explained in Lemmas 21 and 23, and the notations are as there.

Proof. We shall show that all terms are absolutely convergent for $f \in \mathcal{C}_n^1(G,\varepsilon)$ (cf. the remark after Lemma 1). Since the formula is true for K-finite $f \in C_0^\infty(G,\varepsilon)$, it extends by continuity.

We have seen in section 2 that $F_f^A(a)$, $F_f^K(k)$, $F_f^K(k_{\pm 0})$ and$I_f^{P_0}(a)$ are tempered distributions. Moreover, F_f^A is a continuous map from $\mathcal{C}^2(G,\varepsilon)$ to the Schwartz space on A, whence $\Theta_{\varepsilon,\frac{1}{2}+ir}$ is continuous from $\mathcal{C}^2(G,\varepsilon)$ to the Schwartz space on $\frac{1}{2}+i\mathbb{R}$ by Lemma 2. With the estimate from Proposition 16 this implies that the integral on the spectral side is a tempered distribution, too. Note that the embedding $\mathcal{C}_n^1(G,\varepsilon) \to \mathcal{C}^2(G,\varepsilon)$ is continuous.

Next we show that $\pi_\chi^{\text{dis}}(f) = \pi_\chi^{\text{cus}}(f) + \pi_\chi^{\text{res}}(f)$ is of the trace class for $f \in \mathcal{C}_n^1(G,\varepsilon)$. This is well-known for $\pi_\chi^{\text{cus}}(f)$ (see [OW]). By Theorem 12, $\mathcal{H}_\chi^{\text{res}}$ is a finite direct sum of spaces G-isomorphic to $\mathcal{H}_{\varepsilon,s}$ or its invariant subspace for some $s \in [0,1]$, so our assertion follows from Lemma 1.

As $\#(\Gamma\backslash \operatorname{supp} t) < \infty$, all sums except that over $\bar{\Gamma}_{\text{hyp}}$ are finite. The remaining one equals

$$\int_{\Gamma\backslash G} \sum_{\xi\in\bar{\Gamma}_{\text{hyp}}} f(x^{-1}\xi x)\operatorname{tr} t(\xi)\, dx,$$

which is absolutely convergent for $f \in C_0^\infty(G,\varepsilon)$ by the way it arose. The methods of [22], Lemma 11.4, would only imply convergence for $f \in \mathcal{C}^p(G,\varepsilon)$, $0 < p < 1$, so we proceed differently. Since all the other terms extend to continuous linear functionals on $\mathcal{C}_n^1(G,\varepsilon)$, so does this one. In particular, let t_0 be the characteristic function of any double Γ-coset in $\bar{\Gamma}$ and $f_0 \in \mathcal{C}_n^1(G')$ a universal majorant for all elements of $\mathcal{C}_n^1(G,\varepsilon)$ (for its existence, see [22], p. 95). Then, for any monotonely increasing sequence $\{f_k\}_{k=1}^\infty \subset C_0^\infty(G')$ of positive functions tending to f_0 in $\mathcal{C}_n^1(G')$,

$$\int_{\Gamma\backslash G} \sum_{\xi\in\bar{\Gamma}_{\text{hyp}}} f_k(x^{-1}\xi x) t_0(\xi)\, dx \le \nu(f_k),$$

where ν is a continuous seminorm on $\mathcal{C}_n^1(G')$. By the theorem of B. Levi, the same is true for $k = 0$, which was to be proved. $\square$

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