Embedding of hyperbolic Coxeter groups into products of binary trees and aperiodic tilings

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Abstract

We prove that a finitely generated, right-angled, hyperbolic Coxeter group Γ can be quasiisometrically embedded into the product of n binary trees, where n is the chromatic number of Γ. As application we obtain certain strongly aperiodic tilings of the Davis complex of these groups.

1 Introduction

We consider a finitely generated, right-angled Coxeter group Γ , i.e. a group Γ together with a finite set of generators S, such that every element of S has order two and that all relations in Γ are consequences of relations of the form $st = ts$, where $s, t \in S$.

We prove embedding results of the Cayley graph $C(\Gamma, S)$ of hyperbolic right angled Coxeter groups into products of the binary tree $T = T_{\{0,1\}}$. On graphs and trees we consider always the simplicial metric, hence every edge has length 1. On a product of trees we consider the l_1-product metric, i.e. the distance is equal to the sum of the distances in the factors.

Theorem 1.1. *Let Γ be a finitely generated, hyperbolic, right-angled Coxeter group with chromatic number n. Then there exists a quasiisometric embedding $\psi : C(\Gamma, S) \to T \times \cdots \times T$ (n-factors).*

We can apply Theorem 1.1 for special Coxeter groups operating cocompactly on the hyperbolic plane $\mathbb{H}^2$, e.g. the group generated by the reflections at the edges of the regular right-angled geodesic hexagon. This group has chromatic number 2 and we obtain:

Corollary 1.2. *There exists a quasiisometric embedding of the hyperbolic plane into the product of two binary trees.*

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Combining this with a result of Brady and Farb [BF] we get the following higher dimensional version:

Corollary 1.3. *For every integer n there exists a quasiisometric embedding $\psi : \mathbb{H}^n \to T \times \cdots \times T$ of the hyperbolic space $\mathbb{H}^n$ into the $2(n-1)$-fold product of binary trees.*

It is an open problem, if for $n \geq 3$ there is a quasiisometric embedding of $\mathbb{H}^n$ into the n-fold product of binary trees. There are partial results in this direction. In [BS2] it is shown that there exists a quasiisometric embedding of $\mathbb{H}^n$ into an n-fold product of locally infinite trees. On the other hand a recent construction of Januszkiewicz and Swiatkowski [JS] shows for every n the existence of a right angled Gromov hyperbolic Coxeter group with virtual cohomological dimension and coloring number equal to n. Combining Theorem 1.1 with that result we obtain:

Corollary 1.4. *For every n there exits a Gromov hyperbolic group Γ_n with virtual cohomological dimension n which admits a quasiisometric embedding into the product of n binary trees.*

Corollary 1.4 can be used to determine the hyperbolic rank (compare [BS1]) of a product of trees:

Corollary 1.5. *The hyperbolic rank of the product of n metric trees is $(n-1)$.*

Corollary 1.6. *The hyperbolic rank of the product of n free groups is equal to $(n-1)$.*

With the methods of the proof we also obtain strongly aperiodic tilings of the Davis complex of right-angled, hyperbolic Coxeter groups.

Theorem 1.7. *For every finitely generated, right-angled and hyperbolic Coxeter group Γ the Davis complex X admits a strongly aperiodic tiling Φ with finitely many tiles. In addition, every limit tiling of Φ is also strongly aperiodic.*

In dimensions 2,3 and 4 there are right-angled reflection groups operating with compact fundamental domain on the hyperbolic space. Two dimensional examples come from the regular right-angled p-gons, $p \geq 5$ in the hyperbolic plane. In dimension 3 there exists a right-angled regular hyperbolic dodecahedron (compare also [A]). In dimension 4 there exists e.g. a right-angled hyperbolic 120-cell ([PV],[D2],[C]). Thus we obtain

Corollary 1.8. *The hyperbolic spaces $\mathbb{H}^2, \mathbb{H}^3$ and $\mathbb{H}^4$ admit strongly aperiodic tilings such that every limit tiling is also strongly aperiodic*

A strongly aperiodic tiling of $\mathbb{H}^2$ was recently constructed by Goodman-Strauss [G] (this tiling has even a finite strongly aperiodic set of tiles). Existence of aperiodic tilings in $\mathbb{H}^3$ and $\mathbb{H}^4$ follows from the results of Block-Weinberger [BW]. The aperiodicity of their tilings follow from the fact that the tilings are unbalanced. We construct strictly balanced aperiodic tilings of the Davis complex. Indeed we develop a general method how to modify tilings of the Davis complex to obtain strictly balanced tilings. For a precise statement see Theorem 6.18.

The structure of the paper is as follows. After the preliminaries we prove the main result for the two-colored case in section 3. In section 4 we generalize the argument to the n-colored case. In section 5 we prove the corollaries of this result. In section 6 we discuss aperiodic tilings of the Davis complex.

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2 Preliminaries

2.1 Right-angled Coxeter Groups

In this section we review the necessary facts from the theory of right-angled Coxeter groups.

A Coxeter matrix $(m_{s,t})_{s,t\in S}$ is a symmetric $S \times S$ matrix with 1 on the diagonal and with all other entries nonnegative integers different from 1. A Coxeter matrix defines a Coxeter group Γ generated by the index set S with relations $(st)^{m_{s,t}} = 1$ for all $s, t \in S$. Here we use the convention that $\gamma^0 = 1$ for all elements, thus if $m_{s,t} = 0$ then there is no relation between s and t. A Coxeter group Γ is *finitely generated*, if S is finite. The group Γ is called *right-angled*, if all entries of the corresponding Coxeter matrix are 0,1,2. The Coxeter matrix of a right angled Coxeter group is completely described by a graph with vertex set S where we connect two vertices s and t iff $m(s,t) = 2$.

Let Γ be a right-angled Coxeter group with generating set S. The nerve $N = N(\Gamma, S)$ is the simplicial complex defined in the following way: the vertices of N are the elements of S. Two different vertices s, t are joined by an edge, if and only if $m(s,t) = 2$. In general (k+1) different vertices $s_1, \ldots, s_{k+1}$ span a k-simplex, if and only $m(s_i, s_k) = 2$ for all pairs of different $i, j \in \{1, \ldots, k+1\}$.

There is a simple characterization of hyperbolic right-angled Coxeter groups. A *square* in S is a collection of four elements $t_1, t_2, s_1, s_2 \in S$, such that s_i commutes with t_1 and t_2 for $i = 1, 2$, but s_1 and s_2 as well as t_1 and t_2 do not commute. Thus these elements are the vertices of a square in the nerve. The following characterization is known as the Siebenmann no-square condition:

Lemma 2.1. *A right-angled Coxeter group* Γ *with generating set* S *is hyperbolic if and only if* S *contains no square.*

The *chromatic number* of a graph is the minimal number of colors needed to color the vertices in such a way that adjacent vertices have different colors. The chromatic number of a simplicial complex is the chromatic number of its 1-dimensional skeleton.

Assume that the chromatic number of the nerve $N(\Gamma)$ of a right-angled Coxeter group Γ equals n and let $c : N^{(0)} \to \{1, \ldots, n\}$ be a corresponding coloring map. Then we can write S as a disjoint union $S = \bigsqcup_c S_c$ where S_c are the elements of color c. Elements of a given color do not commute. We prefer to write $S = A \bigsqcup B \bigsqcup C \ldots$, where $A \subset S$ are the elements with color a.

Right-angled Coxeter groups have a very simple deletion law. By the following two operations every word W in the generators S can be transformed to a reduced word and two reduced words representing the same element can be transformed by means only the second operation [Br] :

(i) delete a subword of the form ss, $s \in S$

(ii) replace a subword st by ts if $m_{s,t} = 2$.

This deletion rule has the following consequences:

Lemma 2.2. **(a)** *If* W *and* W' *represent the same element then the lenghts of* W *and* W' *are either both even or both odd.*

(b) *Let* W *and* W' *be reduced representations of the same element* $\gamma \in \Gamma$, *then* W *and* W' *are formed from the same set of letters and they have the same length.*

We now investigate some properties of the Cayley graph $C(\Gamma, S)$ of Γ with respect to the generating set S for a right-angled Coxeter group. Let $\gamma \in \Gamma$ and let W be a reduced representation of γ. The length of W is denoted by $\ell(\gamma)$ and called the norm of γ. This is well defined by (b). If a is a color and $\gamma \in \Gamma$, then we denote with $\ell_a(\gamma)$ the number of letters with color a in a reduced representation of γ. This is well defined and we clearly have $\sum_c \ell_c(\gamma) = \ell(\gamma)$.

On Γ we consider the distance function $d(\gamma, \beta) = \ell(\gamma^{-1}\beta)$. Let γ and β be elements of Γ which are neighbors in the Cayley graph and let W be a reduced word representing γ. Then there exists a generator $s \in S$ such that β has the representation Ws. It follows from (b) that $\ell(\gamma) \neq \ell(\beta)$. Thus an edge in the Cayley graph connects two elements with different norm. As usual one can define geodesics in the Cayley graph. A geodesic between two points $\alpha, \beta \in \Gamma$ is given by a sequence $\alpha = \gamma_0, \dots, \gamma_k = \beta$ with $d(\gamma_i, \gamma_j) = | i - j |$.

Let α, β and γ be elements of Γ. We say that γ *lies between* α *and* β if $d(\alpha, \gamma) + d(\gamma, \beta) = d(\alpha, \beta)$. The Cayley graphs of right angled Coxeter groups have the following property, which says that any three points in Γ span a tripod:

Lemma 2.3. *Let $\alpha, \beta, \gamma \in \Gamma$, then there exists $\delta \in \Gamma$ such that δ lies between γ_i and γ_j for any choice of distinct elements $\gamma_i, \gamma_j \in \{\alpha, \beta, \gamma\}$.*

Proof. By the Γ-invariance of the metric it suffices to show this result for the case that $\gamma = 1$. Let $\alpha, \beta \in \Gamma$ and consider a geodesic path $\alpha = \alpha_0, \dots, \alpha_k = \beta$ from α to β and consider the sequence of norms $n_0 = \ell(\alpha_0), \dots, n_k = \ell(\alpha_k)$. Note that by the properties discussed above $| n_i - n_{i+1} | = 1$. Assume that there is a subsequence $\alpha_{i-1}, \alpha_i, \alpha_{i+1}$ with $n_{i-1} < n_i > n_{i+1}$. Then one can represent α_i in a reduced way as $W_1 t = W_2 s$ where W_1 is a reduced word representing α_{i-1} and W_2 a reduced word representing α_{i+1} and $s, t \in S$. Since by the deletion law one can transform $W_1 t$ into $W_2 s$ by means of operations of type (b), we see that $st = ts$ and that one can represent α_i by a reduced word of the form $Wst = Wts$ where Ws represents α_{i-1} and Wt represents α_{i+1}. Replace now α_i by the element α_i' represented by W and we obtain a new geodesic sequence between α and β such that for the corresponding sequence of norms we have $n_{i-1} > n_i' < n_{i+1}$. Applying this procedure several times we obtain a geodesic path from α to β, such that the sequence of the norms has no local maximum any more and hence only a global minimum n_{i_0}. The corresponding element $\delta = \alpha_{i_0}$ lies between α and β but also between 1 and α resp. 1 and β.

□

2.2 Rooted Trees

Let Ω be a set. We associate to Ω a rooted simplicial tree T_Ω in the following way : the set of vertices is the set of finite sequences $(\omega_1, \dots, \omega_k)$ with $\omega_i \in \Omega$. The empty sequence defines the root vertex and is denoted by $\emptyset$. Two vertices are connected by an edge in T_Ω if their length (as sequences) differ by one and the shorter can be obtained by erasing the last term of the longer. The root vertex has $|\Omega|$ neighbors and every other vertex has $|\Omega| + 1$ neighbors, one ancestor and $|\Omega|$ descendents. Here $|\Omega|$ denotes the cardinality of $|\Omega|$. The set Ω is also called the *label set* of T_Ω. On trees we consider always the simplicial metric, hence every edge has length 1. On a product of trees we consider the l_1-product metric, i.e. the distance is equal to the sum of the distances in the factors.

The tree $T_{\{0,1\}}$ is also called the *binary tree*. If Ω is finite and $|\Omega| \geq 2$, then T_Ω is quasiisometric to the binary tree. This is easily proved using the fact that the elements of Ω can be represented by binary sequences of length $\leq \log_2(|\Omega|) + 1$.

3 Two-colored Coxeter Groups

Since the case of chromatic number 2 is technically much easier, we first give the proof of our main result in this case. Thus we asssume in this section, that Γ is generated by a finite set S which can be decomposed as $S = A \bigsqcup B$ such that elements within A do not commute with each other and the same holds for elements in B.

3.1 Canonical a-presentation

Fix a color, say a. A *reduced left a-representation* of $\gamma \in \Gamma$ is a reduced word

$$W = W_1 a_1 W_2 a_2 \dots W_r a_r W_{r+1}$$

representing γ in which the $W_i = w^i_{r_i} \dots w^i_1$ are words with letters in B and the last entry of W_i does not commute with a_i, i.e. the letters with color a are moved as left as possible. The reduced a-representation of γ is unique and thus we call it also the *canonical a-presentation* of γ. The word W_i in the canonical a-representation is called the *coefficient* of a_i if $i \leq r$ and it is called the *free coefficient* if $i = r+1$. Note that W_i could be empty. Let W be a reduced word, then the reduction to the canonical a-representation permutes the letters of W. We call this permutation the *canonical reduction map* of W.

We now study in general the situation that we have two canonical a-representations U and V such that the composition UV is a reduced word. We choose the notation such that

$$U = U_1 a_1 \cdots U_p a_p U_{p+1},$$

$$V = V_{p+1} a_{p+1} \cdots V_{p+m} a_{p+m} V_{p+m+1}.$$

Let then

$$W = W_1 a_1 W_2 a_2 \dots W_{p+m} a_{p+m} W_{p+m+1}$$

be the canonical a-presentation of the word UV and let $\varphi : UV \to W$ be the canonical reduction map for UV. With this notation $\varphi(a_i) = a_i$ for all $1 \leq i \leq p+m$. Let $U_{p+1} = u_q \dots u_1$ be the free coefficient of U and $V_{p+1} = v_k \dots v_1$ be the coefficient of a_{p+1} in V and let $U_R = \{u \in U | a_{p+1} < \varphi(u)\}$ denote the set of all letters $u \in U$ that are moved by φ right of a_{p+1}. Then

Proposition 3.1. *(1) $\varphi(v_i) = w_i$ for all $i \leq k$, where $W_{p+1} = w_l \dots w_1$ is the coefficient of a_{p+1} in W.*

(2) If $k \geq 1$ then $U_R = \emptyset$ and $UV = W$ is the canonical a-representation.

(3) If $U_R \neq \emptyset$, then U_R are the last $|U_R|$ letters of $u_q \dots u_1$

(4) If Γ is hyperbolic, then there is at most one $u \in U_R$ with $\varphi(u) > a_{p+2}$.

Proof. The first point is obvious. If $k \geq 1$, then (since no element $u_i \in U_{p+1}$ can pass v_1) $U_R = \emptyset$. This proves (2). If $U_R \neq \emptyset$, then $k = 0$ and clearly U_R consists out of the last letters of U_{p+1}, hence (3). To prove the last point assume that there are two elements in U_R which are mapped by Φ to the right of a_{p+2}. Then these two elements commute with a_{p+1} and a_{p+2} and thus this four elements together form a square. Hence Γ is not hyperbolic by Lemma 2.1. □

Consider two elements $\gamma, \overline{\gamma} \in \Gamma$. We investigate the situation, how the canonical a-representations of these elements differ. By Lemma 2.3 $1, \gamma$ and $\overline{\gamma}$ span a tripod. Thus there are words U,V,$\overline{V}$ given in canonical a-presentations, such that UV is a reduced representation of γ and $U\overline{V}$ is a reduced representation of $\overline{\gamma}$. Furthermore $V^{-1}\overline{V}$ is a reduced representation of $\gamma^{-1}\overline{\gamma}$. Let $m = \ell_a(V)$ and $\overline{m} = \ell_a(\overline{V})$.

We then write

$$U = U_1 a_1 \dots U_p a_p U_{p+1}$$

$$V = V_{p+1} a_p \dots V_{p+m} a_{p+m} V_{p+m+1}$$

$$\overline{V} = \overline{V}_{p+1} \overline{a}_p \dots \overline{V}_{p+\overline{m}} \overline{a}_{p+\overline{m}} \overline{V}_{p+\overline{m}+1}$$

Let

$$W = W_1 a_1 \dots W_{p+m} a_{p+m} W_{p+m+1}$$

$$\overline{W} = \overline{W}_1 \overline{a}_1 \dots \overline{W}_{p+\overline{m}} \overline{a}_{p+\overline{m}} \overline{W}_{p+\overline{m}+1}$$

be the canonical a-representations of $\gamma, \overline{\gamma}$. Clearly $W_i a_i = \overline{W}_i \overline{a}_i = U_i a_i$ for $1 \leq i \leq p$.

Lemma 3.2. $W_{p+1} a_{p+1} \neq \overline{W}_{p+1} \overline{a}_{p+1}$.

Proof. Since $V^{-1}\overline{V}$ is reduced, the first letter from V is different from the first letter of $\overline{V}$. We start by considering the case, that the first letters are both of color a, i.e. $V_{p+1} = \emptyset = \overline{V}_{p+1}$. In this case we have $a_{p+1} \neq \overline{a}_{p+1}$ and hence the claim. If (say) $V_{p+1} = \emptyset$ and $\overline{V}_{p+1} \neq \emptyset$, then we get from Proposition 3.1 that $\overline{W}_{p+1}$ is longer then W_{p+1} and hence the claim. If both $V_{p+1} \neq \emptyset$ and $\overline{V}_{p+1} \neq \emptyset$, then by Proposition 3.1 $W_{p+1} = U_{p+1} V_{p+1}$ and $\overline{W}_{p+1} = \overline{U}_{p+1} \overline{V}_{p+1}$ and hence the two word differ at the first entry of V_{p+1} resp. $\overline{V}_{p+1}$. This proves the result.

□

3.2 Alice's diary

Let $\kappa \in \mathbb{N}$. Define $E = (A \cup B \cup \{\emptyset\})$. Let T_a^{dia} be the rooted tree with label set E^κ. We call T_a^{dia} the *diary tree*. We define a map $\psi_a^{\mathrm{dia}} : \Gamma \to T_a^{\mathrm{dia}}$ as follows:

Let $\gamma \in \Gamma$ be given by the canonical a-presentation

$$W = (w_{k_1}^1 \dots w_1^1)a_1 \dots (w_{k_r}^r \dots w_1^r)a_r(w_{k_{r+1}}^{r+1} \dots w_1^{r+1}).$$

By $W|_i$ we denote the i-cut

$$W|_i = (w_{k_1}^1 \dots w_1^1)a_1 \dots a_{i-1}(w_{k_i}^i \dots w_1^i).$$

Recall that the vertices of the tree T_a^{dia} are finite sequences of elements of E^κ. We define $\psi_a^{\mathrm{dia}}(W) = (\alpha_1, \dots, \alpha_i, \dots, \alpha_r)$ as a sequence of length r by induction on i.

Let α_1 be the string of the last κ symbols in the chain

$$\emptyset \dots \emptyset w_{k_1}^1 \dots w_1^1$$

considered as a word in the alphabet E. We assume here that we have enough (say κ) symbols $\emptyset$ in front. We define α_i as the string of the last κ symbols (E-letters) of the word in the alphabet E that is obtained from the word

$$\emptyset \dots \emptyset W|_i$$

by removing the E-letters from $\cup_{j<i}\alpha_j$.

Remark 3.3. One can intuitively describe this "diary" in the following way: Consider the word W as description of a long journey of (say) Alice who wants to write a diary about her trip. Every letter of W represents a day. There are two types of days during this journey. The days of color b where the weather is so fine that Alice has no time to write her diary. Then there are more cloudy days of color a. In the morning of every a-day Alice writes κ pages in her diary: she starts with yesterday and then the day before yesterday etc. Of course she skips the days which were already described earlier in the diary. If there is no day left to describe, she marks on the corresponding pages of her diary the symbol $\emptyset$.

It is not difficult to show and left as an exercise to prove the following: if $\psi_a^{\mathrm{dia}}(W) = (\alpha_1, \dots, \alpha_i, \dots, \alpha_r)$ and α_i contains the symbol $\emptyset$, then one can reconstruct the whole i-cut $W|_i$ from $(\alpha_1, \dots, \alpha_i)$.

Remark 3.4. It is a subtle point that Alice writes her diary in the morning and not in the evening; i.e. that the entry α_i does not contain a_i. This choice of the diary makes it possible to reconstruct from the diary the i-cut $W|_i$.

More generally we have

Proposition 3.5. *Let $W = W_1 a_1 \dots W_{j+r} a_{j+r} W_{j+r+1}$ be a canonical a-representation and assume that*

$$k := \kappa(r+1) - \ell(W_{j+1} a_{j+1} \dots W_{j+r}) \geq 1.$$

Then one of the following holds:

(1) We can reconstruct the subword $W_j a_j$ from the diary entries $(\alpha_j, \dots, \alpha_{j+r})$.

(2) The word $W_j a_j$ has $\geq$ k letters and we can reconstruct from $(\alpha_j, \dots, \alpha_{j+r})$ the last k letters of $W_j a_j$.

Proof. If some entry α_q for some $j \leq q \leq j+r$ contains $\emptyset$, then we can reconstruct the complete q-cut $W|_q$ from the diary entries $(\alpha_1, \dots, \alpha_q)$. In particular we can reconstruct in this case $W_j a_j$ from the diary entries α_s, $j \leq s \leq j+r$.

Thus we can assume that the entries α_q, $j \leq q \leq j+r$ do not contain $\emptyset$ and hence each entry contains κ letters from the word $W_1 a_1 \dots a_{j+r-1} W_{j+r}$. If we cannot reconstruct $W_j a_j$ from these entries, then all entries from α_q, $j \leq q \leq j+r$ are from the word $W_j a_j \dots a_{q-1} W_q$ (if some entry is from $W_{j-1} a_{j-1}$, then the first occuring of this entry is a_{j-1} and in this moment we have all information to reconstruct $W_j a_j$). Thus all $\kappa(r+1)$ diary entries are from $W_j a_j \dots a_{j+r-1} W_{j+r}$, and hence at least k of them from $W_j a_j$. $\square$

We collect the maps ψ_a^{dia} and the corresponding ψ_b^{dia} to a common map

$$\psi^{\mathrm{dia}} : \Gamma \to T_a^{\mathrm{dia}} \times T_b^{\mathrm{dia}}$$

On the product of the trees we consider the l_1-product metric.

Lemma 3.6. *The map ψ^{dia} is 1-Lipschitz.*

Proof. Let $\gamma, \gamma' \in \Gamma$ elements with $d(\gamma, \gamma') = 1$. Then $|\ell(\gamma') - \ell(\gamma)| = 1$ and we can assume that $\ell(\gamma') = \ell(\gamma) + 1$. In this case we can represent γ' in a reduced way as Us, where U is a reduced word representing γ. Assume w.l.o.g. that $s \in A$. In this case clearly $\psi_b^{\mathrm{dia}}(\gamma) = \psi_b^{\mathrm{dia}}(\gamma')$. Note that $\psi_a^{\mathrm{dia}}(\gamma')$ has one more entry as $\psi_a^{\mathrm{dia}}(\gamma)$, and all other entries agree. Thus $\psi^{\mathrm{dia}}(\gamma)$ and $\psi^{\mathrm{dia}}(\gamma')$ have distance 1. $\square$

The next Lemma shows that the diary map is a "radial isometry".

Lemma 3.7. *The map ψ^{dia} is a radial isometry, i.e. $|\psi^{\mathrm{dia}}(\gamma)| = \ell(\gamma)$, where $|.|$ denotes the distance from $(root, root) \in T_a^{\mathrm{dia}} \times T_b^{\mathrm{dia}}$.*

Proof. It follows from the definition of ψ_a^{dia}, that the sequence $\psi_a^{\mathrm{dia}}(\gamma)$ has $\ell_a(\gamma)$ terms. It follows that

$$|\psi^{\mathrm{dia}}(\gamma)| = \ell_a(\gamma) + \ell_b(\gamma) = \ell(\gamma).$$

$\square$

A small disadvantage of the diary constructed above is the fact that α_i does not contain the information about the last letter a_i (compare remark 3.4. To obtain this additional information we define an "augmented" diary tree $T_a'^{\mathrm{dia}}$ by the label set $E^\kappa \times A$ and define

$$\psi_a'^{\mathrm{dia}}(\gamma) = \psi_a'^{\mathrm{dia}}(W) = ((\alpha_1, a_1), \ldots, (\alpha_r, a_r))$$

where W is a canonical a-representation of γ.

The augmented diary map $\psi'^{\mathrm{dia}} = \psi_a'^{\mathrm{dia}} \times \psi_b'^{\mathrm{dia}}$ carries some additional information. Also this map is 1-Lipschitz and a radial isometry.

Remark 3.8. The maps ψ^{dia} and ψ'^{dia} are (in most cases) not quasiisometric. Consider reduced words $\overline{A}$ formed from a-letters, and $\overline{B}$ formed from b-letters, such that the first and the last entry of $\overline{B}$ are different. Then also the words $\overline{B}^k$ are reduced for $k \geq 1$. Assume $\ell(\overline{B}) \ll \ell(\overline{A})$. Let $k \in \mathbb{N}$ such that $\ell(\overline{A}) \ll k$ and let also $\kappa \ll k$. Now consider the two reduced words $\gamma = (\overline{B})^k \overline{A}$ and $\overline{\gamma} = (\overline{B})^{k+1}\overline{A}$. The conditions imply that $\psi_a^{\mathrm{dia}}(\gamma) = \psi_a^{\mathrm{dia}}(\overline{\gamma})$ and

$$|\psi_b^{\mathrm{dia}}(\overline{\gamma}) - \psi_b^{\mathrm{dia}}(\gamma)| = \ell(\overline{B}) \ll 2\ell(\overline{A}) + \ell(\overline{B}) = d(\gamma, \overline{\gamma}).$$

Thus ψ^{dia} cannot be quasiisometric. It turns out that the essential point of this example is the periodicity within the words γ and $\overline{\gamma}$. To exclude this periodicity, we use the Morse Thue decoration.

3.3 The Morse Thue decoration

The following sequence was studied by Thue [T] and later independently by Morse [M]

Definition 3.9. *(Morse-Thue Sequence t(i)). Consider the substitution rule* $0 \to 01$ *and* $1 \to 10$*. Then start from* 0 *to perform this substitutions*

$$0 \to 01 \to 0110 \to 01101001 \to \ldots$$

to obtain a nested family of sequences of length 2^k *in the alphabet* $\{0,1\}$*. The resulting limit sequence is called the Morse-Thue sequence.*

The Morse-Thue sequence has the following remarkable property (see e.g. [He]).

Theorem 3.10. *The Morse-Thue sequence contains no string of type* WWW *where* W *is any word in* 0 *and* 1*.*

3.4 The quasiisometry

We will use the Morse-Thue sequence to "decorate" elements of the generating set. To every generator $s \in S$ can be given a decoration 0 or 1. Let $E_a = \{b^0, b^1 | b \in B\} \cup A \cup \{\emptyset\}$. Thus E_a is the alphabet E where we in addition have decorations of the letters of color b. Let T_a be the rooted tree with label set $(E_a)^\kappa \times A$.

We define the final map ψ_a from Γ into the tree T_a in the following way. As above take the canonical a-representation W of an element $\gamma \in \Gamma$ and decorate every element with color b in W by the Morse Thue sequence; i.e. the i-th letter of color b gets the decoration $t(i)$. Then we apply the map $\psi_a'^{\,\mathrm{dia}}(W)$ to the decorated word.

Analagously we define E_b, T_b and a map $\psi_b : \Gamma \to T_b$ by interchanging the roles of the two colors.

Let

$$\psi = (\psi_a, \psi_b) : \Gamma \to T_a \times T_b$$

In the same way as above we see that ψ is 1-Lipschitz and a radial isometry.

Theorem 3.11. *If Γ is 2-colored, right angled and hyperbolic Coxeter group and let $\kappa \geq 160$ then the map ψ is a quasiisometry.*

Proof. To keep the proof more general, define $n = 2$ to be the chromatic number.

We know already that ψ is 1-Lipschitz. To prove the opposite inequality let $\gamma, \overline{\gamma} \in \Gamma$ and let $c := d(\gamma, \overline{\gamma})$.

Claim: If $c \geq 60n$ then $|\psi(\gamma)\psi(\overline{\gamma})| \geq c/6n$.

Clearly the claim implies the desired lower estimate on the distance of image points. To prove the claim, we use the notation as in Lemma 3.2. Hence we have canonical a-representations

$$U = U_1 a_1 \dots U_p a_p U_{p+1}$$

$$V = V_{p+1} a_p \dots V_{p+m} a_{p+m} V_{p+m+1}$$
$$\overline{V} = \overline{V}_{p+1} \overline{a}_p \dots \overline{V}_{p+\overline{m}} \overline{a}_{p+\overline{m}} \overline{V}_{p+\overline{m}+1}$$

such that

$$W = W_1 a_1 \dots W_{p+m} a_{p+m} W_{p+m+1}$$

represents γ and $\overline{\gamma}$ is represented by

$$\overline{W} = \overline{W}_1 \overline{a}_1 \dots \overline{W}_{p+\overline{m}} \overline{a}_{p+\overline{m}} \overline{W}_{p+\overline{m}+1}$$

We can assume without loss of generality that in a reduced representation of $\gamma^{-1}\overline{\gamma}$ the number of letters with color a is larger or equal than the number of letters with color b.

This means that $m + \overline{m} \geq c/n$.

We first show that the claim is true in the case $m < c/3n$ or $\overline{m} < c/3n$: Assume that (say) $m < c/3n$. Since $m + \overline{m} \geq c/n$ we have $\overline{m} - m \geq c/3n$. Since ψ is a radial isometry, we have

$$|\psi_a(\overline{\gamma})| - |\psi_a(\gamma)| = (p + \overline{m}) - (p + m) = \overline{m} - m \geq c/3n$$

and hence $|\psi_a(\gamma)\psi_a(\overline{\gamma})| \geq c/3n$. Thus we can assume for the rest of the proof that $m \geq c/3n$ and $\overline{m} \geq c/3n$.

We will now prove the claim by contradiction. Therefore assume that $|\psi_a(\gamma)\psi_a(\overline{\gamma})| < c/6n$. This assuption together with $m \geq c/3n$ and $\overline{m} \geq c/3n$ implies

$$\psi_a(W|_{p+r}a_{p+r}) = \psi_a(\overline{W}|_{p+r}\overline{a}_{p+r}) \quad \text{for some} \quad r > c/6n \tag{1}$$

In particular the diary entries $\psi_a^{\text{dia}}(W|_{p+r}a_{p+r}) = (\alpha_1, \dots, \alpha_{p+r})$ coincide with the corresponding entries $(\overline{\alpha}_1, \dots, \overline{\alpha}_{p+r})$. Since $c \geq 60n$ by assumption we have $r > 10$ and one computes elementary that also $(r-1) \geq c/10n$ and $(r-2) \geq c/10n$.

Consider now the three cases :

Case 1: $V_{p+1} = \emptyset = \overline{V}_{p+1}$. Then it follows from the fact that $V^{-1}\overline{V}$ is reduced that $a_{p+1} \neq \overline{a}_{p+1}$ and hence $\psi_a(W|_{p+1}a_{p+1}) \neq \psi_a(\overline{W}|_{p+1}\overline{a}_{p+1})$ (by definition of the augmented map $\psi_a'^{\text{dia}}$). This is a contradiction to equation (1).

Case 2: $V_{p+1} = \emptyset$ and $\overline{V}_{p+1} \neq \emptyset$.

In this case $\overline{W}_j = \overline{V}_j$ for all $j \geq p+2$ by Proposition 3.1 (2). In particular

$$\ell(\overline{W}_{p+3}\overline{a}_{p+3} \dots \overline{W}_{p+r}) \leq \ell(\overline{V}) \leq c$$

and hence (since $\kappa \geq 20n$ and $(r-2) \geq c/10n$).

$$k = \kappa(r-2) - \ell(\overline{W}_{p+3}\overline{a}_{p+3} \dots \overline{W}_{p+r}) \geq c$$

By Proposition 3.5 we can reconstruct from $(\overline{\alpha}_{p+2}, \dots, \overline{\alpha}_{p+r})$ either the whole word $\overline{W}_{p+2}\overline{a}_{p+2}$ or at least the last $k \geq c$ letters of this word. Since $\overline{W}_{p+2}\overline{a}_{p+2}$ is a subword of $\overline{V}$ it has length $\leq c$ and hence we can reconstruct the whole word. Then, since $(\overline{\alpha}_{p+2}, \dots, \overline{\alpha}_{p+r}) = (\alpha_{p+2}, \dots, \alpha_{p+r})$ we conclude $\overline{W}_{p+2}\overline{a}_{p+2} = W_{p+2}a_{p+2}$.

We make a similar computation at the place $p+1$. In the same way as above we compute

$$\ell(\overline{W}_{p+2}\overline{a}_{p+2} \dots \overline{W}_{p+r}) \leq \ell(\overline{V}) \leq c$$

and hence (since $\kappa \geq 80n$ and $(r-1) \geq c/10n$)

$$k = \kappa(r-1) - \ell(\overline{W}_{p+2}\overline{a}_{p+2} \dots \overline{W}_{p+r}) \geq 7c.$$

By Proposition 3.5 we can reconstruct from $(\overline{\alpha}_{p+1}, \dots, \overline{\alpha}_{p+r})$ either the whole word $\overline{W}_{p+1}\overline{a}_{p+1}$ or at least the last k letters of this word. Assume for a moment that we can reconstruct the whole word. Then, since $(\overline{\alpha}_{p+1}, \dots, \overline{\alpha}_{p+r}) = (\alpha_{p+1}, \dots, \alpha_{p+r})$ we have $\overline{W}_{p+1}\overline{a}_{p+1} = W_{p+1}a_{p+1}$ in contradiction to Lemma 3.2.

It follows that $\overline{W}_{p+1}\overline{a}_{p+1}$ and $W_{p+1}a_{p+1}$ have at least $k \geq 7c$ letters and the last k letters of the words coincide.

Note that (using the terminology of Proposition 3.1): $\overline{W}_{p+1} = W_{p+1}U_R\overline{V}_{p+1}$. By the last point of Proposition 3.1 and the just proved fact that $\overline{W}_{p+2} = W_{p+2}$ we see

$$|U_R| \leq |W_{p+2}| + 1 = |\overline{W}_{p+2}| + 1 \leq \ell(\overline{V}) \leq c.$$

It follows that $|U_R\overline{V}_{p+1}| \leq 2c$. Since the last $6c$ letters of $\overline{W}_{p+1}$ and W_{p+1} coincide, we can write W_{p+2} in the form MHH and $\overline{W}_{p+2}$ in the form $MHHH$ where $H = U_R\overline{V}_{p+1}$.

Since ψ_a also contains the Morse-Thue decoration of the elements of B and the property of the Morse-Thue sequence, there can not be any subsequence of decorated letters of the form HHH . Thus we obtain a contradiction.

Case 3. $V_{p+1} \neq \emptyset$ and $\overline{V}_{p+1} \neq \emptyset$.

Then $\overline{W}_{p+1} = U_{p+1}\overline{V}_{p+1}$ and $W_{p+1} = U_{p+1}V_{p+1}$. Since $\overline{V}_{p+1} \neq V_{p+1}$ but the last $7c$ letters of $\overline{W}_{p+1}$ and W_{p+1} coincide by the same arguments as in Case 2, we see $\ell(\overline{V}_{p+1}) \neq \ell(V_{p+1})$. Let w.l.o.g. $\ell(\overline{V}_{p+1}) > \ell(V_{p+1})$. It is now completely elementary to see that again W_{p+2} is of the form $MHHP$ and $\overline{W}_{p+2}$ of the form $MHHHP$ where H are the first $\ell(\overline{V}_{p+1}) - \ell(V_{p+1})$ letters of $\overline{V}_{p+1}$. We obtain now a contradiction as in Case 2.

□

4 The n-colored case

In this section we modify the arguments of section 3 to prove the general case. We assume that Γ is a finitely generated right-angled Coxeter group with chromatic number n. Thus the set S of generators has a decomposition $S = A \bigsqcup B \bigsqcup \dots$ into n subsets in such a way that elements of the same color do not commute. Fix a color, say a. A *reduced left a-representation* of $\gamma \in \Gamma$ is a recudes word W representing γ, which has the form

$$W = W_1 a_1 W_1 a_2 \dots W_r a_r W_{r+1}$$

in which for every i no letter x in W_i commutes with a_i and with all letters of W_i to the right of x.

As discussed in 2.1 every set of mutually commutative generators $R \subset S$ defines a simplex in the nerve $N(\Gamma, S)$. For every element $\gamma \in \Gamma$ we define a *right simplex presentation*

$$U = \Delta_q \dots \Delta_1$$

where the letters of each word Δ_i form a simplex. The simplices Δ_i are defined by induction on i. We consider U in reduced presentation and define Δ_1 as the set of all letters from U which can be placed at the very end of the word. So Δ_1 consists of all letters x that commute with all letters to the right. Then U can be represented as $U^1\Delta_1$. Then we apply this procedure to U^1 to obtain Δ_2 and so on. We note that the right simplex presentation is unique in the sense that the sequence of simplices is uniquely determined. Thus, it gives a unique presentation of a group element up to permutation of the letters inside simplices. A right simplex presentation $U = \Delta_q \dots \Delta_1$ is a word in the alphabet S. On the other hand it is a word in the alphabet Σ, the set of all simplices. We will use the same notation for both.

We now define a *canonical a-representation* of $\gamma \in \Gamma$ to be a reduced left a-presentation

$$W = W_1 a_1 W_1 a_2 \dots W_r a_r W_{r+1}$$

where each $W_i = \Delta^i_{r_i} \dots \Delta^i_1$ is in the right simplex presentation.

The word W_i in the left a-representation is called the *coefficient* of a_i if $i \leq r$ and it is called the free coefficient, if $i = r + 1$.

Let Σ_a be the set of nontrivial simplices of $N(\Gamma, S)$ which do not contain letters with color a. We can view W as a word in the alphabet $\mathcal{A} = A \cup \Sigma_a$. Considered as a word in this alphabet it is unique and therefore also called *the* canonical a-representation. Note that considered as a word in S a simplex from Σ_a has length $\leq n - 1$ since it cannot contain letters with the same color.

Let W be a reduced word, a reduction to a canonical a-representation permutes the letters of W. We call this permutation the *canonical reduction map* of W.

As above we study the situation that we have

$$U = U_1 a_1 \cdots U_p a_p U_{p+1},$$

$$V = V_{p+1} a_{p+1} \cdots V_{p+m} a_{p+m} V_{p+m+1}.$$

and

$$W = W_1 a_1 W_2 a_2 \dots W_{p+m} a_{p+m} W_{p+m+1}$$

the canonical a-presentation of the word UV and let $\varphi : UV \to W$ be the canonical reduction map for UV. With this notation $\varphi(a_i) = a_i$ for all $1 \leq i \leq p+m$. Let $U_{p+1} = \Delta^u_q \dots \Delta^u_1$ be the free coefficient of U and $V_{p+1} = \Delta^v_k \dots \Delta^v_1$ be the coefficient of a_{p+1} in V and let $U_R = \{u \in U | a_{p+1} < \varphi(u)\}$ denote the set of all letters $u \in U$ that are moved by φ right of a_{p+1}. Then

Proposition 4.1. *(1) $\varphi(\Delta^v_k) \subset \varphi(\Delta^w_k)$ where $\Delta^w_k \dots \Delta^w_1 = W_{p+1}$ is the coefficient of a_{p+1} of W.*

(2) If Γ is hyperbolic, then the following holds: If $u \in U_R$ and $\varphi(u) > a_{p+2}$, then $u \in \Delta_1^u$.

(3) If Γ is hyperbolic, and $V_{p+1} \neq \emptyset$, then $U_R \subset \Delta_1^u$.

Proof. (1). Let U_{p+1} be the free coefficient of U and V_{p+1} be the first coefficient of V. By the definition of the canonical a-reduction the permutation Φ is a composition $\Phi = \Phi_2 \circ \Phi_1$ of two maps, where Φ_1 moves the a's to the left as far as possible. In other words Φ_1 moves U_R to the right of a_{p+1}. Then Φ_2 forms the simplices $\Delta_k^w \dots \Delta_1^w = W_{p+1}$ out of $\overline{U}_{p+1}V_{p+1}$ where $\overline{U}_{p+1} = U_{p+1} \setminus U_R$. One now checks easily $\Phi(\Delta_k^v) \subset \Phi(\Delta_k^w)$.

(2). Assume that there exists $u \in U_{p+1} \setminus \Delta_1^u$ with $\varphi(u) > a_{p+2}$. Since u cannot go right of all elements of Δ_1^u, there exists $u' \in \Delta_1^u$ which does not commute with u. Then $\varphi(u') > \varphi(u) > a_{p+2}$. If $a_{p+1} \neq a_{p+2}$ then u, u', a_{p+1}, a_{p+2} form a square in contradiction to the hyperbolicity assumption. If $a_{p+1} = a_{p+2}$, then there exists u'' between a_{p+1} and a_{p+2} which does not commute with them. Then u, u', u'', a_{p+1} form a square.

(3). If we assume the contrary, then similar to (2) we obtain a square formed by two elements from U_R, one element from V_{p+1} and a_{p+1}. □

In the same way as in section 3 represent two given elements $\gamma, \overline{\gamma} \in \Gamma$ by means of words

$$U = U_1 a_1 \dots U_p a_p U_{p+1}$$

$$V = V_{p+1} a_p \dots V_{p+m} a_{p+m} V_{p+m+1}$$

$$\overline{V} = \overline{V}_{p+1} \overline{a}_p \dots \overline{V}_{p+\overline{m}} \overline{a}_{p+\overline{m}} \overline{V}_{p+\overline{m}+1}$$

such that γ and $\overline{\gamma}$ are represented by

$$W = W_1 a_1 \dots W_{p+m} a_{p+m} W_{p+m+1}$$

$$\overline{W} = \overline{W}_1 \overline{a}_1 \dots \overline{W}_{p+\overline{m}} \overline{a}_{p+\overline{m}} \overline{W}_{p+\overline{m}+1}$$

Clearly $W_i a_i = \overline{W}_i \overline{a}_i = U_i a_i$ for $1 \leq i \leq p$ and we have in generalization of Lemma 3.2

Lemma 4.2. $W_{p+1} a_{p+1} \neq \overline{W}_{p+1} \overline{a}_{p+1}$.

Proof. Since $V^{-1}\overline{V}$ is reduced, the first letter from V is different from the first letter of $\overline{V}$.

Case 1:$V_{p+1} = \emptyset = \overline{V}_{p+1}$. In this case we have $a_{p+1} \neq \overline{a}_{p+1}$ and hence the claim.

Case 2: $V_{p+1} = \emptyset$ and $\overline{V}_{p+1} \neq \emptyset$.

We can assume in addition that $a_{p+1} = \overline{a}_{p+1}$ since otherwise the result is trivially true. Note that no element from $\overline{V}_{p+1}$ can be moved right to $\overline{a}_{p+1} = a_{p+1}$. Therefore all elements in $U_{p+1}\overline{V}_{p+1}$ which can be moved right of $\overline{a}_{p+1}$ are elements of U_{p+1} and hence elements from U_R in the notation of Proposition 4.1 If follows that $\ell(W_{p+1}) < \ell(\overline{W}_{p+1})$.

Case 3: $V_{p+1} \neq \emptyset$ and $\overline{V}_{p+1} \neq \emptyset$. As above we can assume in addition that $a_{p+1} = \overline{a}_{p+1}$.

We first investigate the case $U_R \neq U_{\overline{R}}$. Assume that there exists $u \in U_R$ such that u is not in $U_{\overline{R}}$. Then there exists a "blocking element" $b \in \overline{V}_{p+1}$ which does not commute with u. The letter b is not contained in V_{p+1}, since $u \in U_R$. Furthermore b is not contained in $U_{\overline{R}}$ since $U_{\overline{R}} \cap \overline{V}_{p+1} = \emptyset$. It follows that the letter b occurs more often in $\overline{W}_{p+1}$ than in W_{p+1}. But this implies the claim.

Thus we can assume that $U_R = U_{\overline{R}}$. Now by Lemma 4.1(3) $U_R = U_{\overline{R}} \subset \Delta_1^u$ which implies that we can write U_{p+1} in a reduced representation as $U_{p+1} = U'U_R = U'U_{\overline{R}}$. Hence the word W_{p+1} represents the same group element as $U'V_{p+1}$ and $\overline{W}_{p+1}$ the same element as $U'\overline{V}_{p+1}$. Since V_{p+1} represents a different group element as $\overline{V}_{p+1}$ we obtain the result.

□

We can now define the diary map. Let $\kappa \in \mathbb{N}$. Define $E_a = (\mathcal{A} \cup \{\emptyset\})$. Then E_a^κ is the label set T_a^{dia}, the *diary tree.*

In the same way as in section 3 we define the diary map

$$\psi_a^{\mathrm{dia}}(W) = (\alpha_1, \ldots, \alpha_i, \ldots, \alpha_r)$$

as a sequence of length $\ell_a(W)$. As above we obtain

Proposition 4.3. *Let $W = W_1 a_1 \ldots W_{i+m} a_{i+m} W_{i+m+1}$ be a canonical a-representation and assume that*

$$k := \kappa m - \ell_{\mathcal{A}}(W_{i+1} a_{i+1} \ldots W_{i+m}) \geq 1,$$

where $\ell_{\mathcal{A}}$ denotes the length of a word in the alphabet $\mathcal{A}$. Then one of the following holds:

1. *we can reconstruct $W_i a_i$ from the diary entries $(\alpha_{i+1}, \ldots, \alpha_{i+m})$*
2. *The word $W_i a_i$ has $\geq k$ $\mathcal{A}$-letters and we can reconstruct from $(\alpha_{i+1}, \ldots, \alpha_{i+m})$ the last k $\mathcal{A}$-letters of $W_i a_i$.*

We collect the maps ψ_c^{dia} to a common map $\psi^{\mathrm{dia}} : \Gamma \to \prod_c T_c^{\mathrm{dia}}$. and we also define the augmented diary map. As in chapter 3 we see that the diary map is Lipschitz and a radial isometry. Finally we decorate the simplices from Σ_a by the Morse Thue sequence in the following way. Given a word W in the canonical a-representation, we decorate for every color $b \neq a$ every S-letter with color b by the Morse-Thue sequence, i.e. the i-th letter of color b gets the decoration $t(i)$. Since within a given color the order of the elements cannot change, this is well defined. Thus a simplex is now a simplex of decorated S-letters. To this decorated sequence we apply the augmented diary map to obtain the map ψ_a and finally we collect the maps to get

$$\psi = \prod_c \psi_c : \Gamma \to \prod_c T_c$$

Now we can prove Theorem 1.1 in general. The proof is essentially as in section 3. There are however some significant differences in the discussion of Case 2 and Case 3.

The idea is to reduce the argument to the case of two colors. Therefore we use the following reduction. Let W be a word given in the canonical a-decomposition and let b be a color different from a. We define a word W^b in the alphabet $\{b^o, b^1 | b \in B\} \cup A \cup \{\star\}$ (where $\star$ is some additional symbol) in the following way: Replace every decorated simplex $\Delta \in \Sigma_a$ from W by the decorated letter with color b, if Δ contains a letter with color b. If Δ does not contain a letter with color b then replace Δ by $\star$.

Lemma 4.4. *Let G and H be words given in the canonical a-representation and assume that GH is reduced. Assume that the first $\mathcal{A}$ letter of H contains a letter of color b. Then $(GH)^b$ can be obtained from G^bH^b by removing certain numbers of the letter $\star$ from G^b.*

Proof. Let $H = \Delta_l \dots \Delta_1$ be the canonical a-decomposition of H. Let b be a color that occurs in Δ_l. Since GH is reduced, there is no letter of color b in G which is mapped by the canonical reduction map right to a letter of color b in Δ_l. The result follows now easily. $\square$

We now proceed with the proof of the main theorem in the n-colored case.

Case 2: $V_{p+1} = \emptyset$ and $\overline{V}_{p+1} \neq \emptyset$.

Similar in chapter 3 we conclude that $\overline{W}_{p+1}a_{p+1}$ and $W_{p+1}a_{p+1}$ have the last $k \geq 7c$ letters in common. These letters are decorated simplices. We furthermore see that in this case we can write $\overline{W}_{p+1} = W_{p+1}H$ where $H = U'\overline{V}_{p+1}$ and $U' = U_R \setminus \overline{U}_R$. Let $H = \Delta_l \dots \Delta_1$ be the canonical a-decomposition of H. Let b be a color that occurs in Δ_l. This implies that there is no letter with color b in $\overline{U}_R$. Let $\Delta_l^w \dots \Delta_1^w$ be the last simplices of W_{p+1}. Since the last $\mathcal{A}$-letters of W_{p+1} and $\overline{W}_{p+1} = W_{p+1}H$ coincide we have

$$\Delta_i \subset \Delta_i^w \tag{2}$$

by Proposition 4.1 (1).

We use now the reduction to the 2-colored case. By Lemma 4.4 we obtain $(W_{p+1}H)^b$ from $W_{p+1}^b H^b$ by deleting some $\star$-letters from W_{p+1}^b.

Because of relation (2) and since $\overline{U}_R$ contains no letter of color b, we see that the last $k \geq 7c$ letters of W_{p+1}^b coincide with H^b. It follows from Lemma 4.4 that there are words $Q, H_1, H_2 \in \mathcal{A}$, such that $W_{p+1} = QH_1$,

$\overline{W}_{p+1} = QH_1H_0$ such that $H_0^b = H^b$ and H_1^b can be obtained from H^b by deleting some $\star$-letters. In particular the first letter of H_1^b has color b. Again, since the last letters of $W_{p+1}^b = (QH_1)^b$ coincide with the last letters from $\overline{W}_{p+1}^b = (QH_1H_0)^b$ we see that $\overline{W}_{p+1}^b = \sigma H_2^b H_1^b H^b$ where H_2^b is obtained from H_1^b by deleting some $\star$-letters. Thus the sequence of simplices with a vertex of color b has a subsequence of period 3 in contradiction to the properties of the decoration.

Case 3: $V_{p+1} \neq \emptyset$ and $\overline{V}_{p+1} \neq \emptyset$.

We first show, that the length of V_{p+1} differs from the length of $\overline{V}_{p+1}$ where the length is measured in the alphabet $\mathcal{A}$.

Assume to the contrary that

$$V_{p+1} = \Delta_l^v \dots \Delta_1^v$$

and

$$\overline{V}_{p+1} = \Delta_l^{\overline{v}} \dots \Delta_1^{\overline{v}}$$

are of the same length.

Since $V\overline{V}^{-1}$ is reduced, $\Delta_l^{\overline{v}} \cap \Delta_l^v = \emptyset$. Let $x \in \Delta_l^{\overline{v}}$ and $y \in \Delta_l^v$ letters (of S) .

Since the last $7c$ $\mathcal{A}$-letters of W_{p+1} and $\overline{W}_{p+1}$ coincide, we see $\Delta_l^w = \Delta_l^{\overline{w}}$ and by Proposition 4.1 (1) $\Delta_l^{\overline{v}} \subset \Delta_l^w$ and $\Delta_l^v \subset \Delta_l^w$. Thus the letters x, y must also occur in U in a way, that the letter x can be moved in W to Δ_l^w and y in $\overline{W}$ to $\Delta_l^{\overline{w}}$. Since $U\overline{V}$ is reduced, there exists $z \in U$ right to x which does not commute with x. Since this x can be moved to Δ_l^v we see that also z can be moved right to Δ_l^v which implies that y commutes with z. In the same way we see that there exists $u \in U$ right of y which does not commute with y but commutes with x. The letters x, y, z, u form a square in contradiction to the hyperbolicity condition.

We therefore can assume w.l.o.g. that $\overline{V}_{p+1}$ is longer than V_{p+1}. Let b be a color which accurs in the first $\mathcal{A}$-letter of $\overline{V}_{p+1}$. With arguments as in case 2 one shows now that $\overline{W}_{p+1}^b$ is obtained from a word of the form $M^b H_2^b H_1^b H_0^b P^b$ by removing some $\star$-letters. Here H_i^b is obtained from H^b by removing $\star$-letters and H is formed by the first $\ell_{\mathcal{A}}(\overline{V}_{p+1}) - \ell_{\mathcal{A}}(V_{p+1})$ $\mathcal{A}$-letters of $\overline{V}_{p+1}$. We obtain a contradiction in a similar way.

5 Proof of the Corollaries

Proof. (of Corollary 1.2)

Consider the right angled Coxeter group Γ given by the generator set $S = \{s_1, \dots, s_6\}$ and relations $s_i s_{i+1} = s_{i+1} s_i$ (indices mod 6). This group acts discretely on the hyperbolic plane such that a Dirichlet fundamental domain is bounded by the regular right angled hexagon in $\mathbb{H}^2$. By Theorem 1.1. we can embed the Cayley graph of Γ quasiisometrically into the product of two binary trees. Since $\mathbb{H}^2$ is quasiisometric to this Cayley graph, we obtain the result. $\square$

Proof. (of Corollary 1.3)

By a result of Brady and Farb [BF] there exists a quasiisometric embedding of the hyperbolic space $\mathbb{H}^n$ into the $(n-1)$-fold product of hyperbolic planes. Actually the proof in [BF] gives a bilipschitz embedding (compare [F, section 2]). Combining Corollary 1.2 with this result, we are done.

$\square$

Proof. (of Corollary 1.4)

A recent result of Januszkiewicz and Swiatkowski [JS] shows the existence of a Gromov-hyperbolic right angled Coxeter group Γ_n of arbitrary given n, such that the virtual cohomological dimension of Γ_n is n. The construction of these groups imply that they have chromatic number n. By Theorem 1.1 Γ_n can be quasiisometrically embedded into the product of n binary trees.

$\square$

Proof. (of Corollary 1.5)

We recall the definition of the hyperbolic rank of a metric space. Given a metric space M consider all locally compact Gromov-hyperbolic subspaces Y quasiisometrically embedded into M. Then $\mathrm{rank}_h(M) = \sup_Y \dim \partial_\infty Y$ is called the hyperbolic rank. (Compare [BS1] for a discussion of this notion). Let Γ_n as in the proof of Corollary 1.4 above. Γ_n can be embedded in a bilipschitz way into the n-fold product T^n of the binary tree T. Since the virtual cohomological dimension of Γ_n is n, we have $\dim \partial_\infty \Gamma_n = (n-1)$ by [BM]. Thus $\mathrm{rank}_h(T^n) \geq (n-1)$ by the definition of the hyperbolic rank. The opposite inequality $\mathrm{rank}_h(T^n) \leq (n-1)$ follows from standard topological considerations.

$\square$

6 Strongly aperiodic tilings of the Davis Complex

As an application of the methods developed above, we construct certain aperiodic tilings of the Davis complex of X

We recall the definition of tiling of a metric space from [BW]. Let X be a metric spcace. A set of tiles $(\mathcal{T}, \mathcal{F})$ is a finite collection of compact n-dimensional complexes $t \in \mathcal{T}$ and a collection of subcomplexes $f \in \mathcal{F}$ of dimension $< n$, together with an opposition function $o : \mathcal{F} \to \mathcal{F}$, $o^2 = id$. A space X is tiled by the set $(\mathcal{T}, \mathcal{F})$ if

(1) $X = \cup_\lambda t_\lambda$ where each t_λ is isometric to one of the tiles in $\mathcal{T}$;

(2) $t_\lambda \setminus \cup_{f \in t_\lambda} = Int(t_\lambda)$ in X for every λ;

(3) If $Int(t_\lambda \cup t_{\lambda'}) \neq Int(t_\lambda) \cup Int(t_{\lambda'})$ then t_λ and $t_{\lambda'}$ intersect along $f \in t_\lambda$ and $o(f) \in t_{\lambda'}$;

(4) There are no free faces of t_λ.

Remark 6.1. Strictly speaking the tiling is given by a collection $\{\varphi_\lambda\}$ of specified isometries $\varphi_\lambda : t_\lambda \to t$, where $t \in \mathcal{T}$, such that for an $n-1$-dimensional face $\sigma \subset t_\lambda$ we have $\varphi_\lambda(\sigma) \in \mathcal{F}$. Point (3) says the following: if $\sigma = t_\lambda \cap t_{\lambda'}$ is the "common" face, then $o(\varphi_\lambda(\sigma)) = \varphi_{\lambda'}(\sigma)$

Let g be an isometry of X preserving the decomposition structure, i.e. every t_λ is mapped by g to some $t_{\lambda'}$. Then g induces a new tiling $\{g^*\varphi_\lambda\}$ with the same tiling set $(\mathcal{T}, \mathcal{F})$ by $g^*\varphi_\lambda = \varphi_{\lambda'} \circ g$. We say that g is an isometry of the tiling, if $\{g^*\varphi_\lambda\} = \{\varphi_\lambda\}$. We call a tiling $\{\psi_\lambda\}$ a *limit tiling* of $\{\varphi_\lambda\}$, if there exists a sequence g_i of decomposition preserving isometries of X, such that for every λ there exists $i_0 = i_0(\lambda) \in \mathbb{N}$, such that $\psi_\lambda = g_i^*\varphi_\lambda$ for all $i \geq i_0(\lambda)$.

Definition 6.2. *(1) A tiling is called aperiodic, if the isometry group of the tiling does not act cocompactly on X.*

(2) A tiling is called strongly aperiodic if it has a trivial group of isometries.

Theorem 6.3. *For every finitely generated, right-angled and hyperbolic Coxeter group Γ the Davis complex X admits a strongly aperiodic tiling with finitely many tiles. In addition every limit tiling is also strongly aperiodic.*

Since in dimensions $2, 3, 4$ there are right-angled reflection groups with compact fundamental domain we obtain

Corollary 6.4. *The hyperbolic spaces $\mathbb{H}^2, \mathbb{H}^3, \mathbb{H}^4$ admit strongly aperiodic tilings such that all limit tilings are also and strongly aperiodic.*

Before we prove this result, we recall the basic facts concerning the Davis complex.

6.1 The Davis Complex

Let Γ be a right angled Coxeter group with generating set S and let $N = N(\Gamma, S)$ be the nerve of (Γ, S). By N' we denote the barycentric subdivision of N. The cone $C = \operatorname{Cone} N'$ over N' is called a *chamber* for Γ. The Davis complex [D1] $X = X(\Gamma, S)$ is the image of a simplicial map $q : \Gamma \times C \to X$ defined by the following equivalence relation on the vertices: $a \times v_\sigma \sim b \times v_\sigma$ provided $a^{-1}b \in \Gamma_\sigma$. Here σ is a simplex in N, Γ_σ is the subgroup of Γ generated by the vertices of σ, v_σ is the barycenter of σ. We identify C with the image $q(1 \times C)$ as a subset of X. The group Γ acts simplicially on X by

$\gamma q(\alpha \times x) = q(\gamma\alpha \times x)$ and the orbit space is equal to the chamber C. Thus the Davis complex X is obtained by gluing the chambers γC , $\gamma \in \Gamma$ along the boundaries. Note that X admits an equivariant cell structure with the vertices $X^{(0)}$ equal the cone points of the chambers and with the 1-skeleton $X^{(1)}$ isomorphic to the Cayley graph of Γ.

The generators $s \in S$ and their conjugates $r = \gamma s \gamma^{-1}$, $\gamma \in \Gamma$ are called *reflections.* A *mirror (or wall)* of a reflection $r \in \Gamma$ is the set of fixed points $M_r \subset X$ of r acting on the Davis complex X. Note that $M_{\gamma s \gamma^{-1}} = \gamma M_s$. A wall is indexed by the corresponding generator s. Thus the wall γM_s has the index s. If $s \in A$ then we say that the mirror has *S-color a.* We use the notation S-color, because we will define other colorings of the walls later. Wall correspond to reflections, thus we speak also of the S-color of a reflection r.

We call a boundary wall of a chamber γC a face in X. Thus every face in X occurs in exactly two adjacent chambers. The faces are pieces of walls.

Let $\mathcal{M}$ be the set of mirrors of the Davis complex, and let $\mathcal{M}_a$ be the mirrors of S-color a.

Lemma 6.5. *For every generator s in a right-angled Coxeter group Γ we have $M_s = \{q(w \times x) \mid w \in Z_s, x \in St(s, N')\}$, where Z_s is the centralizer of s in Γ.*

Proof. " $\supset$ " : Let $w \in Z_s$ and $x \in St(s, N')$, i.e. x is an affine combination $x = \sum_{s\in\sigma} x_\sigma v_\sigma$ one easily computes

$$sq(w \times x) = wq(s \times x) = wq(1 \times x) = q(w \times x).$$

" $\subset$ " : Let $z \in M_s$. Then $z = q(g \times x)$ for some $g \in \Gamma$ and $x \in cone(N')$. The condition $s(z) = z$ can be rewritten as $q(sg \times x) = q(g \times x)$. Hence $g^{-1}sg \in \Gamma_\sigma$ for some simplex σ of N and $x \in \cap_{v\in\sigma} St(v, N')$. By the deletion law $s \in \Gamma_\sigma$, since the number of s in $g^{-1}sg$ is odd. Hence $s \in \sigma$ and $x \in St(s, N')$.

Let $s_1 \dots s_k$ be a reduced presentation of $g^{-1}sg$. We note that all $s_i \in \sigma$. Since the group is rightangled, Γ_σ is commutative and hence all s_i are different. Let $u_1 \dots u_l$ be a reduced presentation of g. Note that $s_j \neq u_i$ for every $u_i \neq s$, since u_i appears even number times in the word $u_l \dots u_1 s u_1 \dots u_l$. Hence $g^{-1}sg = s$, i.e. $g \in Z_s$. $\square$

Lemma 6.6. *Different mirrors of the same S-color are disjoint.*

Proof. Let $a, a' \in A$ and let $M = M_a$ and $M' = M_{a'}$ be the corresponding mirrors. Assume that $\gamma_1(M) \cap \gamma_2(M') \neq \emptyset$. Therefore $gM \cap M' \neq \emptyset$ where $g = \gamma_2^{-1}\gamma_1$. Let $x \in gM \cap M'$. By Lemma 6.5 we have $x = q(w \times y) = q(gu \times z)$, where $w \in Z_{a'}$, $y \in St(t, N')$ and $u \in Z_a$, $z \in St(s, N')$. Hence $y = z = \sum_{a,a'\in\sigma} y_\sigma v_\sigma$. In particular a, a' are in a common simplex and hence commute. Thus $a = a'$ since different elements in A do not commute.

Since $q(w \times y) = q(gu \times z)$, we have $w^{-1}gu \in \Gamma_\sigma$ for some simplex σ with $a \in \sigma$. Thus $w^{-1}gu \in Z_a$ which implies $g \in Z_a$ and $gM_a = M_a$.

□

The following Lemma is useful

Lemma 6.7. *If a group G acts on the Davis complex X of a Coxeter group Γ in such a way that Γ takes walls to walls and respects the indices of the walls. Then G is a subgroup of Γ.*

Proof. Let $g \in G$ and aplly g to the base chamber $C \subset X$. Since g preserves the walls of X, also gC is a chamber, i.e. $gC = \gamma C$ as a set. Since g and γ are isometries and both respect the indices of the walls, we clearly have $g = \gamma \in \Gamma$. □

6.2 A tiling by color of the Davis Complex

Let X be the Davis complex of a right-angled Coxeter group Γ.

By a *coloring* of mirrors by finitely many colors we mean a function $\Phi : \mathcal{M} \to F$, where F is a finite set. The coloring induces a coloring of the faces of all chambers, and chambers are glued together respecting the coloring.

Proposition 6.8. *Every coloring $\Phi : \mathcal{M} \to F$ with a finite F defines a tiling $(\mathcal{T}, \mathcal{F})$ of the Davis complex X with $o(f) = f$ for all $f \in \mathcal{F}$.*

Proof. The set of tiles $\mathcal{T}$ is the set of chambers with all posible colorings of their faces. The set of faces $\mathcal{F}$ is the set of possible colored faces of the chambers. Set $o(f) = f$. Then all conditions hold. □

In this situation we can identify $\mathcal{F}$ with F. We call such a tiling as *tiling by color*.

We note that for any tiling by color $\Phi : \mathcal{M} \to F$ there are limit tilings. We can describe them as follows. The group Γ acts (from the right) on the compact space $F^{\mathcal{M}}$ of F-sequences indexed by $\mathcal{M}$. Every limit point Φ' of the sequence $\Gamma(\Phi) \subset F^{\mathcal{M}}$ is a limit for some sequence $g_i \in \Gamma$, $g_i(\Phi) \to \Phi'$. Then Φ' is the limit tiling for $\{g_i\}$.

It is easy to construct a tiling by color of the Davis complex which is strongly aperiodic. For that it suffices to paint all walls of the base chamber by different colors and all remaining wall by another color. Clearly, in this example every limit tiling is Γ-periodic. In this section we give a strongly aperiodic tilings by color of the Davis complex such that all of its limit tilings are strongly aperiodic.

The proof of Theorem 1.1 gives a coloring of the walls of the Davis complex in the following way. Let M_r be a wall, where $r = \gamma a \gamma^{-1}$ is a reflection for (say) an element $a \in A \subset S$. We assume that $\ell(\gamma a) = \ell(\gamma) + 1$.

(If not we replace γ by γa). Now we define the color of $\Phi(M_r)$ as the last entry in the sequence $\psi_a(\gamma a)$. We have to show that this is well defined: if we can also write $r = \beta a \beta^{-1}$ with some β and $\ell(\beta)a = \ell(\beta) + 1$, then $\beta^{-1}\gamma$ commutes with a and hence does not contain (in a reduced representation) a letter of A. Note that $\gamma a = \beta a \beta^{-1}\gamma$. By the definition of ψ_a we see that $\psi_a(\gamma a) = \psi_a(\beta a \beta^{-1}\gamma) = \psi_a(\beta a)$, where the last inequality holds since a reduced word representing $\beta^{-1}\gamma$ does not contain letters with color a.

We investigate what information this coloring gives. We call two faces of the tiling adjacent, if the two faces are different but are contained in a common simplex.

Definition 6.9. *A radial gallery of faces is a sequence $F_0, \ldots, F_k$ of faces, such that there is a geodesic $\gamma_0, \ldots, \gamma_k$ in Γ starting from $\gamma_0 = 1$, such that F_i is a face of the chamber $\gamma_i C$.*

Lemma 6.10. *Let Φ be the above tiling, and let F be a face. Then we can reconstruct from the properties of the coloring Φ a radial gallery $F_0, \ldots, F_k$ of faces with $F = F_k$.*

Proof. Let F be the face between the chambers γC and $\gamma a C$ for some generator $a \in A \subset S$, such that $\ell(\gamma a) = \ell(\gamma) + 1$. Now the coloring gives as information the last entry of $\psi_a(\gamma a)$. This last entry contains in particular the diary entry, which itself consits of the last κ entries (in the alphabet $\mathcal{A}$) of the word W which is the canonical a-representation of γ. The first of this κ entries is a (decorated) simplex $\Delta \subset N(\Gamma, S)$. Note that by definition $s \in \Delta$ if and only if $\ell(\gamma s) < \ell(\gamma)$. Choose some $s \in \Delta$ and let F' be the face between γs and γ. Inductively we will construct a gallery $F, F', F'', \ldots$ which finally will stop at the base chamber. Note that the diary entries of all faces of the base chamber C are κ times the symbol $\emptyset$ and a diary entry is of the form $(\emptyset, \ldots, \emptyset)$ if and only if this entry comes from a face of the basechamber. Thus the color tells us, when to stop. In this way we obtain the desired gallery. $\square$

Remark 6.11.
(1) One easily sees from the proof, that one can reconstruct from the coloring all possible geodesics from γC to C.
(2) Two different faces of the gallery belong to different walls.

Now it is almost immediate that the tiling Φ is strongly aperiodic. Note that as mentioned in the proof of Lemma 6.10 the faces of the base chamber are characterized by coloring Φ. Hence every isometry g of the tiling has to fix the base chamber C. Since $g \in \Gamma$ by Lemma 6.7, we see $g = 1$.

We now consider the case of limit tilings. Indeed we need some additional properties for the coloring. We have to make the coloring in a way that "nearby" walls of the same S-color have different colors in Φ.

To make this precise we define

Definition 6.12. *Let M_{r_1} and M_{r_2} be two different mirrors with the same S-color (say) a. Then the distance is defined to be the number*

$$d(M_{r_1}, M_{r_2}) = \inf\{d(\gamma_1, \gamma_2) + 1 \mid \exists \gamma_1, \gamma_2 \in \Gamma,\ a_1, a_2 \in A \text{ with } r_i = \gamma_i a_i \gamma_i^{-1}\}$$

Remark 6.13. We can view the mirrors M_{r_i} as subsets of the Cayley graph such that M_{r_i} is the set of midpoints of the edges with endpoints γ_i and $\gamma_i s_i$ where $\gamma_i s_i \gamma_i^{-1}$ is a representation of r_i. Since by Lemma 6.6 two different mirrors with the same color do not intersect, the distance defined above is exactly the distance of the mirrors considered as subsets of the Cayley graph.

The following result is essential for our construction

Proposition 6.14. *For every $D \geq 0$ there exists a map $f_D : \mathcal{M} \to G$ where G is a finite set such that for two reflections r_j, $j = 1, 2$ which are conjugates of the same element $s \in S$ the following holds: $f_D(M_{r_1}) = f_D(M_{r_2})$ implies $M_{r_1} = M_{r_2}$ or $d(M_{r_1}, M_{r_2}) \geq D$.*

Proof. Let $\alpha_1, \ldots, \alpha_k \in \Gamma$ be the set of nontrivial elements with $\ell(\alpha_i) \leq 2D + 2$. Since a Coxeter group Γ is residually finite, there exists a finite group G and a homomorphism $\sigma : \Gamma \to G$ such that $\sigma(\alpha_i) \neq 1$ for all $i = 1, \ldots, k$. For a reflection r we define $f_D(M_r) = \sigma(r)$. Let r_1, r_2 be as in the assumption and let $d(M_{r_1}, M_{r_2}) < D$. Then there exits $s \in S$ and $\gamma_j \in \Gamma$ such that $r_j = \gamma_j s \gamma_j^{-1}$ and $d(\gamma_1, \gamma_2) \leq D$. Let $\tau = \gamma_1^{-1}\gamma_2$, hence $\ell(\tau) \leq D$. By assumption $r_1^{-1} r_2 \in ker(\sigma)$. Since $r_1^{-1} r_2 = \gamma_1 s \tau s \tau^{-1} \gamma_1^{-1}$ we have $s\tau s\tau^{-1} \in ker(\sigma)$. Since $\ell(s\tau s\tau^{-1}) \leq 2D + 2$ we have by construction that $s\tau s\tau^{-1}$ is trivial and hence τ commutes with s which implies that $r_1 = r_2$. ☐

Proposition 6.15. *For every finitely generated, right-angled and hyperbolic Coxeter group Γ the Davis complex X admits a tiling Ψ by colors with finitely many tiles such that Ψ and every limit tiling of Ψ is also strongly aperiodic.*

Proof. We have to construct a coloring of the walls $\mathcal{M}$ by finitely many colors, such that the corresponding tiling is strongly aperiodic and also all limit tiling are strongly periodic. First we associate to every wall M_r the "color" $\Phi(M_r)$. This gives already information about

- the S-color of M_r
- the diary entries in order that Lemma 6.10 holds.
- the Morse Thue decoration.

We need in addition a second "decoration", which we will call the *distance decoration*, of the walls with the following property: If M and M' are walls with the same S-color and the same distance decoration, then

$$M = M' \text{ or } d(M, M') > 2C_h + 3,$$

where C_h is the hyperbolicity constant of Γ. It follows directly from Proposition 6.14 that we can find such a decoration with a finite decorating set.

If we combine the coloring Φ with this additional decoration, we obtain a coloring $\Psi : \mathcal{M} \to F$, where F is some finite set. Thus the induced tiling has again only finitely many tiles. As already shown above the tiling is strongly aperiodic and we have to consider a limit tiling. In the case of a limit tiling Lemma 6.10 then implies that given a face F_0 of the tiling, we can construct a gallery $F_0, F_{-1}, F_{-2} \dots$ starting from F_0. We can assume that this gallery does not end, since we are in a limit case. Note that this gallery can be considered as a geodesic $\nu : \{0, 1, 2, \dots\} \to C(\Gamma, S)$, where every point $\nu(i)$ correpond to a face of the tiling or an edge in $C(\Gamma, S)$. Assume that there is an isometry g of the tiling. Then $g\nu$ is also a geodesic in $C(\Gamma, S)$. Since ν and $g\nu$ are limits of isometric images of geodesics segments starting at the base chamber, the geodesics ν and $g\nu$ are asymptotic. Thus there exists a "shift constant" k (depending on g) such that $d(\nu(i), g\nu(i+k)) \leq C_h$ for all i large enough, where C_h is the hyperbolicity constant of Γ. By shifting the initial point of the geodesic and maybe interchanging the role of ν and $g\nu$, we can assume w.l.o.g. that $k \geq 0$ and $d(\nu(i), g\nu(i+k)) \leq C_h$ for all $i \geq 0$. There exists some S-color (say) a, such that this color occurs infinitely often in this sequence. Now fix in addition a possible decoration. Let $j_0, j_1, \dots$ be the subsequence such that the $\nu(j_i)$ are the faces of S-color a and this given decoration. Note that this sequence may be finite. By Remark 6.11 (2) the walls belonging to the faces $\nu(j_i)$ are different. By the properties of the distance decoration there is a sequence k_i, with $k_i \leq j_i \leq k_{i+1}$, such that the faces $\nu(k_i)$ and $\nu(k_{i+1})$ have distance $\geq C_h$ from the walls belonging to the face $\nu(j_i)$ and these points lie on different sides of this wall. By triangle inequality also the points $g\nu(k_i)$ and $g\nu(k_{i+1})$ are on different sides of this wall which implies that $g\nu$ has to cross the wall. Thus the geodesic $g\nu$ also crosses all the faces with S-color a and the given decoration which occur on the geodesic ν. This argument holds for all possible decorations. Hence all walls of S-color a, which are intersected by ν are also intersected by $g\nu$ and (since walls of the same S-color do not intersect) in the same order. Thus the sequence of these faces on ν is a shift these faces on $g\nu$. By the properties of the Morse Thue decoration, there are no propper shifts. This implies that g leaves these walls invariant. This implies $k = 0$ and hence the displacement of g is $\leq C_h$ on $\nu(i)$. Thus there is a chamber of X which is moved by g at most by distance $C_h + 1$. Since g preserves also the distance decoration, all the walls of this chamber are invariant under g which implies that g fixes the chamber and hence (since the coloring also contains the information of the element $s \in S$ of the wall M_r, where r is a conjugate of s), g is the identity.

$\square$

6.3 Balanced Tilings

Let $(\mathcal{T}, \mathcal{F})$ be a set of tiles. A function $w : \mathcal{F} \to \mathbf{Z}$ is called *a weight function* if $w(o(f)) = -w(f)$ for every $f \in \mathcal{F}$. We recall a definition from [BW].

Definition 6.16. *A finite set of tiles $(\mathcal{T}, \mathcal{F})$ is unbalanced if there is a weight function w such that $\sum_{f\in t} w(f) > 0$ for all $t \in \mathcal{T}$.*

It is called semibalanced if $\sum_{f\in t} w(f) \geq 0$ for all $t \in \mathcal{T}$.

We call a set of tiles *strictly balanced* if for every nontrivial weight function w there are tiles t_+ and t_- such that $\sum_{f\in t_+} w(f) > 0$ and $\sum_{f\in t_-} w(f) < 0$. A tiling called *unbalanced (strictly balanced)* if the corresponding set of tiles is unbalanced (strictly balanced).

In [BW] aperiodic tilings of some nonamenable metric spaces (such as the Davis complexes of hyperbolic Coxeter groups) are constructed where the aperiodicity follows from the fact that they are unbalanced. Here we show, how one can modify a tiling Φ by color of the Davis complex in a way, that the new tiling is strictly balanced and also all limit tilings are strictly balanced. We start from a tiling by color, i.e. a function $\Phi : \mathcal{M} \to F$ where F is a finite set.

We now associate in addition to every wall an orientation. A wall divides the Davis complex into two components. Roughly speaking the orientation says which of the components is left and which is right. The orientation of the walls defines a new tiling $(\mathcal{T}', \mathcal{F}')$ where the new faces have in addition a sign $+$ or $-$. Thus $\mathcal{F}' = \mathcal{F}_+ \cup \mathcal{F}_-$, where $\mathcal{F}_+$ and $\mathcal{F}_-$ are copies of $\mathcal{F}$. The face $f \in t_\lambda$ has sign $+$, if $Int(t_\lambda)$ is left of the wall and sign $-$, if $Int(t_\lambda)$ is right of the wall. In this case the opposition function $o : \mathcal{F}' \to \mathcal{F}'$ maps f_+ to f_-. Geometrically this means that we deform all faces of given color and given sign in the same direction by the same pattern. For the faces of the same color but of opposit sign we take the opposite deformation. We call such tiling a *geometric resolution* of a tiling by color. This new tiling is not any more a tiling by color.

Note that in the Davis complex a wall has a canonical orientation, by deciding that the base chamber is in the left component. Thus we can indicate the choosen orientation itself by a sign. A wall gets the sign $+$, if the orientation of the wall is the canonical one and $-$ otherwise.

The following is obvious.

Lemma 6.17. *(1)Assume that a coloring $\Phi : \mathcal{M} \to F$ defines a strongly aperiodic tiling of X. Then any geometric resolution is also strongly aperiodic.*

(2) If every limit tiling of Φ is strongly aperiodic, then every limit tiling of any geometric resolution is strongly aperiodic.

We formulate and prove the main result of this section.

Theorem 6.18. *For every coloring $\Phi : \mathcal{M} \to F$ with the property that walls of the same color do not intersect, there is a strictly balanced geometric resolution of this tiling. In addition also every limit tiling of this resolution is strictly balanced.*

Proof. In the first step of the proof we construct a strictly balanced geometric resolution of Φ. Consider the set of mirrors $\mathcal{M}_f = \Phi^{-1}(f)$ of the same color f. Since walls of the same color do not intersect by assumption, they are ordered by level from the base chamber. (The level is defined by induction. If one removes the mirrors $\mathcal{M}_f$ from X, the space is devided into components. Mirrors that bound the component of the base chamber are of level one. Then drop mirrors of level one and repeat the procedure to get new mirrors of level one and call them of level two and so on). We give the mirrors signs in alternate fasion by the level: $+ - + - + - \dots$. Thus, elements of $\mathcal{M}_f$ have in alternate ways the $+$-orientation and the $-$-orientation.

The choosen orientation induces a geometric resolution of the tiling. We show that this resolution is strictly balanced.

Let $w : F_+ \cup F_- \to \mathbb{Z}$ be a nontrivial weight function with $w(f_+) = -w(f_-)$. We show that there is a chamber C_+ such that

$$\sum_{M \in C_+} w(M) > 0$$

and there is a chamber C_- such that

$$\sum_{M \in C_-} w(M) < 0.$$

We show the first.

Since the function w is nontrivial there exists a face f^0 which is the common face of two adjacent complexes t_λ and $t_{\lambda'}$ such that $w(f^0) \neq 0$. (By a slight abuse of notation we here identify the "real face" $\sigma = t_\lambda \cap t_{\lambda'} \subset X$ with the color $\Phi(\varphi_\lambda(\sigma)) = \Phi(\varphi_{\lambda'}(\sigma)) \in \mathcal{F}$.)

Now there are four cases corresponding to the parity of the sign of $w(f^0_+)$ and the orientation of the mirror M_{f^0} defined by the face f^0. All this cases are treated in a similar way. We discuss only one, and to be fair not the easiest of these cases: let $w(f^0_+) > 0$ and the orientation of the wall M_{f^0} be negative.

By the convention how we use the words "left" and "right" this means that the base chamber is right of the wall M_{f^0}. Now $f^0 = t_{\lambda'} \cap t_\lambda$ where we assume that t_λ is right, and $t_{\lambda'}$ is left of the wall. Thus f^0 considered as a face of t_λ has sign $-$ and considered as a face of $t_{\lambda'}$ has sign $+$. Now we choose some number $k \in \mathbb{N}$ such that every geodesic from $Int(t_\lambda)$ to $Int(C)$ intersects $\leq k$ mirrors. (i.e. k is larger than the combinatorial distance of $t_{\lambda'}$ and the base chamber.)

Now consider any color f. Then there are three possibilities:

(a) $w(f_+) > 0$ and $w(f_-) < 0$; then we call f an *even* color for w,

(b) $w(f_+) < 0$ and $w(f_-) > 0$; then we call f an *odd* color for w.

(c) $w(f_+) = w(f_-) = 0$; then we call f a 0-color for w.

Let

$$\mathcal{M}_{ev}^{2k} = \{M_f | f \neq f^0,\ f \ \text{ even for } w \text{ and level of } f = 2k\}$$

$$\mathcal{M}_{odd}^{2k+1} = \{M_f | f \neq f^0,\ f \ \text{ odd for } w \text{ and level of } f = 2k+1\}$$

$$\mathcal{M}_0^{k+1} = \{M_f | f \ \text{ 0-color for } w \text{ and level of } f = k+1\}$$

Claim 1: *The set of mirrors* $\mathcal{M}_{ev}^{2k} \cup \mathcal{M}_{odd}^{2k+1} \cup \mathcal{M}_0^k \cup \{M_{f^0}\}$ *bound a bounded set* D *containing the chamber* $t_{\lambda'}$.

Clearly this set of mirrors bounds a convex set in the Hadamard space X. If the component containing $t_{\lambda'}$ is unbounded, then there is a ray from $t_{\lambda'}$ to the visual boundary which does not intersect any of our mirrors. Since we have only finitly many colors, there is a color f such that this ray intersects infinitely many mirrors of this color. By the choice of k the first of these intersected mirrors has level $\leq k+1$ and mirrors of the same color and level are intersected at most twice by convexity. Thus one of these intersected mirrors is contained in our set of mirrors. Contradiction.

Claim 2: *If* f *occurs as face of a tile* $t_\mu \subset D$ *such that* $f \subset \partial D$*, then* $w(f) \geq 0$.

To prove Claim 2 we consider the cases

(i) If $f = f^0$, then f as a face of t_μ has the orientation $+$ since t_μ lies on the same side of M_{f^0} as $t_{\lambda'}$. Thus $w(f) = w(f_+^0) > 0$.

(ii) If f is a 0-face , then anyway $w(f) = 0$.

(iii) Let f be an even color for w. Then f is contained in a wall of $\mathcal{M}_{ev}^{2k}$. By the choice of k the tile t_μ is on the same side of this wall as the base chamber. The wall has level $2k$ hence orientation $+$, thus the basechamber and hence t_μ is on the left side of this wall. This implies that f considered as a face of t_μ gets the sign $+$. Hence $w(f) = w(f_+) > 0$ since f is an even color for w.

(iv) A similar argument aplies for f, if f is an odd color for w. This proves Claim 2.

According to the Claim 1 we have $D = \cup_{i=1}^k C_i$ where $C_1, \ldots, C_k$ is a finite collection of chambers. Then

$$\sum_{i=1}^{k} \sum_{f' \in C_i} w(f') = \sum_{f' \in \partial D} w(f') \geq 0.$$

by Claim 2. Since also f_+^0 is in the last set of faces we see that the expression is indeed > 0. Therefore, $\sum_{f' \in C_i} w(f') > 0$ for some i.

To obtain the chamber C_-, we make a similar construction.

This finishes the proof of the first step. Thus we have constructed a strictly balanced geometric resolution of Φ.

Actually the proof of the first step shows more: If we choose for any given color f an orientations of the walls $\mathcal{M}_f$ in alternate way $+-+-\dots$ *or* $-+-+\dots$ (and maybe for different colors in a different way), then the resulting geometric resolution is strictly balanced. (This more general result follows from some obvious modifications of the above proof). Let us call such a choice of orientations an *allowed* orientation of walls. The levels of walls depend on the base chamber. If we define levels with respect to a different chamber, they are changed. The change has the property, that either all parities of the levels are preserved or all are changed. Thus, whether an orientation of the walls is allowed (or not), does not depend on the choice of the basechamber. As a consequence we have the following: if the orientation of a tiling by color $\{\varphi_\lambda\}$ is allowed, then also the orientation of $\{g^*\varphi_\lambda\}$ is allowed. Thus also all limit tilings of the tiling constructed in step 1 are strictly balanced.

$\square$

In twodimensional jigsaw tiling puzzles a geometric resolution is usually realized by adding rounded tabs out on the sides of the pieces with a corresponding blank cut into the intervening sides to receive the tabs of adjacent pieces. This procedure destroyes the convexity of the pieces. We show that in the case of the hyperbolic plane $\mathbb{H}^2$ we can modify this construction to obtain aperiodic and strictly balanced tilings with convex tiles. Compare also the papers [MaMo], [Mo].

Theorem 6.19. *(1) For every $n \geq 3$ there is a strictly balanced strongly aperiodic tiling of $\mathbb{H}^2$ by convex $2n$-gons with finitely many tiles such that every limit tiling is strongly aperiodic.*

(2) For every $n \geq 3$ there is a finite set of tiles $(\mathcal{T}, \mathcal{F})$ that consists of convex $2n$-gons with a strongly aperiodic tiling of $\mathbb{H}^2$ with strongly aperiodic limit tilings such that every $(\mathcal{T}, \mathcal{F})$-tiling of $\mathbb{H}^2$ is aperiodic.

Proof. (1) Identify $\mathbb{H}^2$ with the Davis complex for the right-angled Coxeter group generated by the reflections at a regular right angled $2n$-gon. This is a 2-colored group. We call these S-colors a and b. We fix a strongly aperiodic tiling Φ by color as constructed above and an orientation of the walls in order that the geometric resolution of the tiling is strictly balanced. We now define a modification of the tiling defined by Φ. Consider a vertex of the Davis complex. This is a point where some a-wall intersects some b-wall. Thus due to the map Φ the point correspond to a unique pair of colors $(a_i, b_j) \in F \times F$, where F is the finite image of Φ and a_i (resp. b_j) indicates that the corresponding S-color is a (resp. b). The additional information about the orientation of the walls give a well defined lokal coordinate system around the vertex (by deciding that the positive quadrant is the quadrant which is right to both walls, and by deciding that the a-wall correspond to the first and the b-wall to the second coordinate.) We now move the vertex by a small amount using these local coordinates. This move should only depend on (a_i, b_j) and not on the vertex. E.g. we can choose a small $\mu(a_i, b_j) > 0$ and move the point to a distance $\mu(a_i, b_j)$ into the direction of the diagonal of the positive quadrant. After this deformation we obtain a finite number of new convex tiles, which (for generic deformations) only allow tilings of $\mathbb{H}^2$ compatible with the matching rules defined by Φ and the orientation of the walls. Roughly speaking the matching rules are now encoded in the length of the sides and the angles of the tiles. Our original aperiodic tiling is deformed to an aperiodic tiling with the desired properties.

(2) We start with the tiling by color $\Phi : \mathcal{M} \to F$ defined earlier and take an unbalanced geometric resolution. One of the way to do it is to assign + for passing through a wall from the component that contains the base chamber. Then for every chamber C the faces whose walls do not separate C and the base chamber obtain the sign +, all other -. In view of hyperbolicity the number of + is greater than - for every C. We define a weight function by sending a positive face to $+1$ and a negative face to -1. This geometric realization has all required properties except the last one by (1). Since the set of tiles $(\mathcal{T}, \mathcal{F})$ is unbalanced, in view of the following proposition any other tiling by $(\mathcal{T}, \mathcal{F})$ is aperiodic. □

Proposition 6.20. *Let $(\mathcal{T}, \mathcal{F})$ denote the set of tiles of a geometric realization of a tiling by color of the Davis complex X of a Coxeter group Γ supplied with a left Γ-invariant metric. Suppose that the set of tiles $(\mathcal{T}, \mathcal{F})$ is unbalanced. Then any $(\mathcal{T}, \mathcal{F})$-tiling of X is aperiodic.*

Proof. This proposition can be derived formally from Proposition 4.1 [BW]. Since the proof there has some omissions we present a proof below.

Let G be a group of isometries of a $(\mathcal{T}, \mathcal{F})$-tiling of X which acts cocompactly. Since the group of isometries of X is a matrix group, by Selberg Lemma G contains a torsion free subgroup G' of finite index. Then the orbit space X/G' is compact and admits a $(\mathcal{T}, \mathcal{F})$-tiling (Note that by taking X/G as in [BW] we cannot always obtain a tiling because of free faces). Then we obtain a contradiction:

$$0 < \sum_{t \in G/G'} \sum_{f \in t} w(f) = \sum_{f \in X/G'} w(f) + w(o(f)) = 0.$$

□

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