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PERIOD INTEGRALS ASSOCIATED TO AN AFFINE DELSARTE TYPE HYPERSURFACE

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Abstract. We calculate the period integrals for a special class of affine hypersurfaces (deformed Delsarte hypersurfaces) in an algebraic torus by the aid of their Mellin transforms. A description of the relation between poles of Mellin transforms of period integrals and the mixed Hodge structure of the cohomology of the hypersurface is given. By interpreting the period integrals as solutions to Pochhammer hypergeometric differential equation, we calculate concretely the irreducible monodromy group of period integrals that correspond to the compactification of the affine hypersurface in a complete simplicial toric variety. As an application of the equivalence between oscillating integral for Delsarte polynomial and quantum cohomology of a weighted projective space $\mathbb{P}_{\mathbf{B}}$, we establish an equality between its Stokes matrix and the Gram matrix of the full exceptional collection on $\mathbb{P}_{\mathbf{B}}$.

0. Introduction

In this note we propose a simple method to calculate concretely period integrals associated to an affine non-compact hypersurface for which the number of terms participating in its defining equation is larger than the dimension of the ambient algebraic torus by two (deformed Delsarte hypersurface). A monomial deformation of a Fermat type polynomial belongs to this class. As for the historical reason of the naming, see Remark 2.1. As an important example of the polynomial under consideration, we point out the following Landau-Ginzburg potential familiar in the mirror symmetry setting,

$$f_0(x) = \sum_{j=1}^{n} x_j + \prod_{j=1}^{n} \frac{1}{x_j}.$$

We establish an expression of the position of poles of the Mellin transform with the aid of the mixed Hodge structure of cohomology groups associated to an hypersurface Z_f defined by a $\Delta-$regular polynomial [2] (See §3, Proposition 3.1). The trial to relate the asymptotic behaviour of a period integral with the Hodge structure of the algebraic variety goes back to [36] where Varchenko established the equivalence of the asymptotic Hodge structure and the mixed Hodge structure in the sense of Deligne-Steenbrink for the case of plane curves and (semi-)quasihomogneous singularities.

In this note, we illustrate the utility of this approach in taking the example of a hypersurface in a torus defined by so called simpliciable polynomial (see Definition 2.2).

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For this class of hypersurfaces we can present period integrals as solutions to so called Pochhammer hypergeometric equation (6.3) by using their Mellin transform. The solution space to this equation has reducible monodromy, but it is possible to subtract a solution subspace with irreducible monodromy (Theorem 6.14) that corresponds to period integrals of the compactified quasi-smooth hypersurface $\bar{Z}_f$ in a complete simplicial toric variety $\mathbb{P}_{\Delta(f)}$ ([4, Definition 3.1]).

In [2, Theorem 14.2] , [30, Theorem 8], [22, 3.4] authors gave interpretations of period integrals of affine hypersurfaces as A-hypergeometric functions that form a vector space with dimension equal to the volume of the Newton polyhedron of f . They did not, however, discuss the reducibility/irreducibility of the global monodromy. In fact our restriction on number of terms of the polynomial f is motivated by the fact that in this setting the A- HG system is reduced an univariable Pochhammer HG equation whose monodromy can be concretely calculated (see Theorem 6.14). If we consider a Laurent polynomial with more terms than considered here, it is necessary to deal with a multivariable holonomic system that makes the calculation of monodromy far more difficult as it would presume the monodromy as a representation of the fundamental group of the holomorphic domain of A-HGF .

The relation between the reducible monodromy group of period integrals and the Stokes matrix of corresponding oscillating integrals has been discussed in [31, Theorem 1.1, 1.2], [34, Theorem 5.1]. As it is shown in §4 of this note Batyrev's quotient ring R_f^+ (1.10) is well adapted to the description of the space of oscillating integrals (4.5).

In [19, 4.2], relying on the Stanley-Reisner ring method [30, Theorem 6], the author subtracts period integrals of compact complete intersection from those of affine complete intersection treated in [34]. R.P. Horja discusses the analytic continuation of these period integrals and confirms a correspondence predicted by Kontsevich in connection with homological mirror symmetry conjecture. None the less he did not give a complete description of (irreducible) global monodromy group of period integrals. Our matrices (6.23), (6.24) in Lemma 6.12 (irreducible monodromy) and (7.2), (7.3) (reducible monodromy) give a global monodromy representation of period integrals.

In [8] authors studied the irreducible monodromy acting on the structure sheaf of an affine complete intersection treated in [34] that they subtracted from the vector space with reducible monodromy action. This subtraction procedure to get an irreducible monodromy representation has been tried in [17] before. Our note has genetic similarity with [8] even though the methods used are quite different. We use the Mellin transform of period integrals, while Corti-Golyshev relied uniquely on [10] to prove their main theorems [8, Theorem 1.1, 1.3] that correspond to our Proposition 6.4, Remark 6.5, Theorem 6.8.

The contents of this note can be summarised as follows. In §1 we give a review on the mixed Hodge structure of the cohomology associated to an affine hypersurface according to [2]. In §2 under the main **Assumption** imposed on $f(x)$, principal integral data γ (2.8) and $\mathbf{B}$ (2.11) are introduced. In §3 we establish Proposition 3.1 that calculates concretely the Mellin transform of period integrals. In §4 we write down a differential equation with irregular singularities satisfied by oscillating integrals associated to a Laurent polynomial (or Landau-Ginzburg potential) f_0. In §5 we examine the relation between a Mellin transform based filtration of period integrals and Kashiwara-Malgrange filtration on the

Brieskorn lattice by [13]. In the core part of the note §6, a concrete representation of monodromy group for period integrals in terms of Pochhammer HGF and its Hermitian invariant are described. In §7 we discuss Dubrovin's conjecture [14] on the Stokes matrix for the quantum cohomology of the weighted projective space $\mathbb{P}_{\mathbf{B}}$ that has been discussed in §4.

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1. Hodge structure of the cohomology group of a hypersurface in a torus

In this section we review fundamental notions on the Hodge structure of the cohomology group of a hypersurface in a torus after [2], [10].

Let Δ be a convex $n-$dimensional convex polyhedron in $\mathbb{R}^n$ with all vertices in $\mathbb{Z}^n$. Let us define a ring $S_\Delta \subset \mathbb{C}[u, x^{\pm}] := \mathbb{C}[u, x_1^{\pm}, \cdots, x_n^{\pm}]$ of the Laurent polynomial ring as follows:

$$S_\Delta := \mathbb{C} \oplus \bigoplus_{\frac{\alpha}{k} \in \Delta, \exists k \geq 1} \mathbb{C} \cdot u^k x^\alpha. \tag{1.1}$$

We denote by $\bar{\Delta}(f)$ the convex hull of $\{0\} \in \mathbb{Z}^{n+1}$ and the set $\{(1, \alpha) \in \mathbb{Z}^{n+1}; \alpha \in supp(f)\}$ that we call the Newton polyhedron of a Laurent polynomial (further simply called polynomial) $F(u, x) = uf(x) - 1$. For $\bar{\Delta}(f)$, we introduce the following Jacobi ideal:

$$J_{f,\Delta} = \left\langle \theta_u F, \theta_{x_1} F, \cdots, \theta_{x_n} F \right\rangle \cdot S_{\Delta(f)}. \tag{1.2}$$

with $\theta_u = u\frac{\partial}{\partial u}$, $\theta_{x_i} = x_i \frac{\partial}{\partial x_i}, i = 1, \cdots, n$. Let τ be a $\ell-$dimensional face of $\Delta(f)$, the convex hull of $\{\alpha \in \mathbb{Z}^n; \alpha \in supp(f)\}$ in $\mathbb{R}^n$, and define

$$f_\tau(x) = \sum_{\alpha \in \tau \cap supp(f)} a_\alpha x^\alpha, \tag{1.3}$$

where $f(x) = \sum_{\alpha \in supp(f)} a_\alpha x^\alpha$. The Laurent polynomial $f(x)$ is called Δ regular, if $\Delta(f) = \Delta$ and for every $\ell-$dimensional face $\tau \subset \Delta(f)$ $(\ell > 0)$ the polynomial equations:

$$f_\tau(x) = \theta_{x_1} f_\tau = \cdots = \theta_{x_n} f_\tau = 0,$$

have no common solutions in $\mathbf{T}^n = (\mathbb{C}^\times)^n$.

Proposition 1.1. ([2, Theorem4.8].) *Let f be a Laurent polynomial such that $\Delta(f) = \Delta$. Then the following conditions are equivalent.*

1) The elements $uf, u\theta_{x_1} f, \cdots, u\theta_{x_n} f$ gives rise to a regular sequence in S_Δ

2) For the Jacobi ideal $J_{f,\Delta}$ (1,2), the following equality holds

$$dim(\frac{S_\Delta}{J_{f,\Delta}}) = n!vol(\Delta).$$

3) f is $\Delta-$regular.

For a $\Delta-$regular polynomial f, we shall further denote the space $\frac{S_\Delta}{J_{f,\Delta}}$ by R_f,

$$R_f = \frac{S_\Delta}{J_{f,\Delta}}. \tag{1.4}$$

It is possible to introduce a filtration on S_Δ, namely $u^i x^\alpha \in S_k$ if and only if $i \leq k$ and $\frac{\alpha}{k} \in \Delta$. Consequently we have an increasing filtration;

$$\mathbb{C} \cong \{0\} = S_0 \subset S_1 \subset \cdots \subset S_n \subset \cdots ,$$

that induces a decreasing filtration on R_f so that the decomposition

$$R_f = \bigoplus_{i=0}^{n} R_f^i$$

holds where R_f^i the $i-$the homogeneous part of R_f.

It is worthy to remark here that the filtration on R_f ends up with $n-$th term.

Let us recall the notion of Ehrhart polynomial:

Definition 1.2. *Let Δ be an $n-$dimensional convex polytope. Denote the Poincaré series of graded algebra S_Δ by*

$$P_\Delta(\mathsf{t}) = \sum_{k \geq 0} \ell(k\Delta)\mathsf{t}^k,$$

$$Q_\Delta(\mathsf{t}) = \sum_{k \geq 0} \ell^*(k\Delta)\mathsf{t}^k,$$

where $\ell(k\Delta)$ (resp.$\ell^*(k\Delta)$) represents the number of integer points in $k\Delta$. (resp. interior integer points in $k\Delta$.) Then

$$\Psi_\Delta(\mathsf{t}) = \sum_{k=0}^{n} \psi_k(\Delta)\mathsf{t}^k = (1-\mathsf{t})^{n+1}P_\Delta(\mathsf{t}),$$

$$\Phi_\Delta(\mathsf{t}) = \sum_{k=0}^{n} \varphi_k(\Delta)\mathsf{t}^k = (1-\mathsf{t})^{n+1}Q_\Delta(\mathsf{t}),$$

are called Ehrhart polynomials which satisfy

$$\mathsf{t}^{n+1}\Psi_\Delta(\mathsf{t}^{-1}) = \Phi_\Delta(\mathsf{t}).$$

Let $\mathbf{T}^n = (\mathbb{C} \setminus \{0\})^n = Spec\ \mathbb{C}[x_1^{\pm}, \cdots, x_n^{\pm}]$ and $\mathbf{T}^{n+1} = (\mathbb{C} \setminus \{0\})^{n+1} = Spec\mathbb{C}[u^{\pm}, x_1^{\pm}, \cdots, x_n^{\pm}]$. Further, the main object of our study will be the cohomology group of the complement to the hypersurface $Z_f := \{x \in \mathbf{T}^n; f(x) = 0\}$ i.e. $\mathbf{T}^n \setminus Z_f \cong \{(u,x) \in \mathbf{T}^{n+1}; -uf(x) + 1 = 0\}$. The primitive part $PH^n(\mathbf{T}^n \setminus Z_f)$ is defined by the following exact sequence

$$0 \to H^n(\mathbf{T}^{n+1}) \to H^n(Z_{-uf+1}) \to PH^n(Z_{-uf+1}) \to 0. \tag{1.5}$$

We consider its Hodge filtration

$$0 = F^{n+1}PH^n(\mathbf{T}^n \setminus Z_f) \subset \cdots \subset F^1PH^n(\mathbf{T}^n \setminus Z_f) = F^0PH^n(\mathbf{T}^n \setminus Z_f) = PH^n(\mathbf{T}^n \setminus Z_f).$$

Theorem 1.3. *([2, Theorem 6.9]) For the primitive part $PH^n(\mathbf{T}^n \setminus Z_f)$ of $H^n(\mathbf{T}^n \setminus Z_f)$, the following isomorphism holds;*

$$\frac{F^i PH^n(\mathbf{T}^n \setminus Z_f)}{F^{i+1}PH^n(\mathbf{T}^n \setminus Z_f)} \cong R_f^{n+1-i}. \quad i \in [1; n+1] \tag{1.6}$$

Furthermore

$$dim R_f^{n+1-i} = \psi_{n+1-i}(\Delta),$$

for $i \leq n$.

From here on we shall further use the notation $i \in [m_1; m_2] \Leftrightarrow i \in \{m_1, \cdots, m_2\}$ for two integers $m_1 < m_2$.

Denote by $I_\Delta^{(\ell)}$ $(0 \leq \ell \leq n+1)$ the homogeneous ideal of S_Δ generated as a $\mathbb{C}$ vector space by all monomials $u^k x^\alpha$ such that $\frac{\alpha}{k}$ is located in Δ but not on any face $\Delta' \subset \Delta$ with codimension ℓ.

Thus we obtain the increasing chain of homogeneous ideals in S_Δ

$$0 = I_\Delta^{(0)} \subset I_\Delta^{(1)} \subset \cdots \subset I_\Delta^{(n)} \subset I_\Delta^{(n+1)} = S_\Delta^+, \tag{1.7}$$

where S_Δ^+ is the maximal homogeneous ideal in S_Δ.

Denote by $\mathcal{R}$ the $\mathbb{C}$ linear mapping

$$\mathcal{R} : S_\Delta \to \Omega^n(\mathbf{T}^n \setminus Z_f)$$

defined as

$$\mathcal{R}(u^k x^\alpha) = \frac{(-1)^k (k-1)! x^\alpha}{f(x)^k} \frac{dx}{x^{\mathbf{1}}} \tag{1.8}$$

with $dx = \bigwedge_{i=1}^n dx_i$, $x^{\mathbf{1}} = \prod_{i=1}^n x_i$. Further we shall also use the notation $\omega_0 = \frac{dx}{x^{\mathbf{1}}}$.

We introduce a decreasing $\mathcal{E}$- filatration on S_Δ

$$\mathcal{E} : \cdots \supset \mathcal{E}^{-k} \supset \cdots \supset \mathcal{E}^{-1} \supset \mathcal{E}^0$$

where $\mathcal{E}^{-k}$ denotes the subspace spanned by monomials $u^\ell x^\alpha \in S_\Delta$ with $\ell \leq k$.

Theorem 1.4. *([2, Theorem 7.13, 8.2], [30, Theorem 7], [22, Theorem 4.1, 4.2])*

1) There exists the following commutative diagram

$$\begin{array}{ccc} S_\Delta & \longrightarrow & \frac{S_\Delta}{\mathcal{D}_u S_\Delta + \sum_{i=1}^n \mathcal{D}_{x_i} S_\Delta} \\ \downarrow{\scriptstyle \mathcal{R}} & & \downarrow{\scriptstyle \rho} \\ \Omega^n(\mathbf{T}^n \setminus Z_f) & \longrightarrow & H^n(\mathbf{T}^n \setminus Z_f) \end{array}$$

with $\mathcal{D}_u(g) = e^{uf}\theta_u(e^{-uf}g)$, $\mathcal{D}_{x_i}(g) = e^{uf}\theta_{x_i}(e^{-uf}g)$ and ρ an isomorphism. In particular we have the following isomorphism

$$\rho^+ : R_f^+ \to PH^n(\mathbf{T}^n \setminus Z_f), \tag{1.9}$$

for

$$R_f^+ = \frac{S_\Delta^+}{\mathcal{D}_u S_\Delta + \sum_{i=1}^n \mathcal{D}_{x_i} S_\Delta}, \tag{1.10}$$

such that

$$R_f \cong \mathbb{C} \oplus R_f^+. \tag{1.11}$$

2) The weight filtration on $H^n(\mathbf{T}^n \setminus Z_f)$ is given by a increasing filtration

$$0 = W_n \subset W_{n+1} \subset \cdots \subset W_{2n} = H^n(\mathbf{T}^n \setminus Z_f). \tag{1.12}$$

For $1 \leq i \leq n-1$ the subspace W_{n+i} equals $\rho(\mathcal{I}^{(i)})$ where $\mathcal{I}^{(i)}$ is the image of the ideal $I_\Delta^{(i)}$ in the space (1.10). While the remaining cases are described by $\rho(\mathcal{I}^{(n+1)}) = W_{2n-2} = W_{2n-1}$, $\rho(\mathcal{I}^{(n+2)}) = H^n(\mathbf{T}^n \setminus Z_f)$.

3) The graded quotient of R_f^+ with respect to the $\mathcal{E}$- filtration is given by

$$Gr_{\mathcal{E}}^i R_f^+ = \mathcal{E}^{-i}(R_f^+)/\mathcal{E}^{-i+1}(R_f^+) = R_f^i. \tag{1.13}$$

for $i \in [0; n]$. In particular $R_f^+ = R_f^+ \cap \mathcal{E}^{-n}$.

The exact sequence

$$0 \rightarrow H^n(\mathbf{T}^n) \rightarrow H^n(Z_{-uf+1}) \rightarrow^{res} H^{n-1}(Z_f) \rightarrow 0$$

gives rise to the isomorphisms

$$res(F^i PH^n(Z_{-uf+1})) = F^{i-1} PH^{n-1}(Z_f), \quad i \in [1; n+1]$$

and

$$res(W_j PH^n(Z_{-uf+1})) = W_{j-2} PH^{n-1}(Z_f), \quad j \in [n+1; 2n] \tag{1.14}$$

([2, Proposition 5.3]).

2. Simpliciable polynomial

Let us consider a Laurent polynomial satisfying conditions of Proposition 1.1

$$F(x) = \sum_{i \in [1; n+2]} a_i x^{\alpha(i)}. \tag{2.1}$$

Here $\alpha(i)$ denotes the multi-index

$$\alpha(i) = (\alpha_1^i, \cdots, \alpha_n^i) \in \mathbf{M}$$

for an integer lattice $\mathbf{M} \cong \mathbf{Z}^n$. Further we impose the following conditions on the polynomial $F(x)$

Assumption The point $\alpha(n+2) \in \mathbf{M}$ is located in the interior of the convex hull of $\{\alpha(i)\}_{i=1}^{n+1}$ that is an $n-$dimensional simplex.

This assumption means that the interior point fan Σ defined by the Newton polyhedron $\Delta(F)$ is a complete simplicial fan. The defining equation of an affine variety Z_F defined in a torus $\mathbf{T}^n$ is determined up to multiplication by a monomial x^m, $m \in \mathbf{M}$. Therefore one can always assume one of terms participating in the expression (2.1) to be a constant. The convention $\alpha(n+1) = 0 \in \mathbf{M}$ fixes the index of the constant term.

The main object of our further study is the polynomial $f(x) \in \mathbb{C}[x^{\pm 1}][s]$ depending on a parameter $s \in \mathbb{C}$,

$$f(x) = \sum_{j=1}^{n} x^{\alpha(j)} + 1 + s x^{\alpha(n+2)}. \tag{2.2}$$

Further we use the convention $\alpha(n+1) = 0$. To recover the situation in §1 we need to put

$$f_0(x) = x^{-\alpha(n+2)} f(x) - s. \tag{2.3}$$

Indeed a polynomial $uF(x)$ with non-zero coefficients can be reduced to the form $uf(x)$ by the aid of a torus $\mathbf{T}^{n+1}$ action on the variables (x, u).

Remark 2.1. A polynomial that depends on $n-$variables and contains n monomials is called of Delsarte type. Jean Delsarte established a formula counting points over a finite field on the hypersurface defined by a polynomial of this class [11]. T.Shioda found an algorithm to calculate explicitly the Picard number of this kind of surface ($n = 3$) [29]. Delsarte surface began to draw attention of geometers in connection with the mirror symmetry conjecture and detailed studies of its Néron-Severi lattice. One can consider Z_f defined for (2.2) as a one dimensional deformation of a Delsarte type hypersurface.

Let us introduce new variables $T_1, \cdots, T_{n+2}$:

$$T_j = u x^{\alpha(j)}, \ j \in [1; n],$$

$$T_{n+2} = u s x^{\alpha(n+2)}, T_{n+1} = u. \tag{2.4}$$

In making use of these notations , we have the relation

$$\begin{gathered} log\ T_j = log\ u + < \alpha(j), log\ x >, \ j \in [1; n], \\ log\ T_{n+1} = log\ u, \ \ log\ T_{n+2} = log\ u + < \alpha(n+2), log\ x > + log\ s. \\ Log\ \Xi :=^t (\log\ x_1, \cdots, \log\ x_n, , \log\ s, \log\ u). \end{gathered} \tag{2.5}$$

We can rewrite the relation (2.5) with the aid of a matrix $\mathsf{L} \in End(\mathbf{Z}^{n+2})$, as follows:

$$Log\ T = \mathsf{L} \cdot Log\ \Xi. \tag{2.6}$$

where

$$\mathsf{L} = \begin{bmatrix} \alpha_1^1 & \cdots & \alpha_n^1 & 0 & 1 \\ \vdots & \cdots & \vdots & \vdots & 1 \\ & & & & \\ \alpha_1^n & \cdots & \alpha_n^n & 0 & 1 \\ 0 & \cdots & 0 & 0 & 1 \\ & & & & \\ \alpha_1^{n+2} & \cdots & \alpha_n^{n+2} & 1 & 1 \end{bmatrix}, \tag{2.7}$$

We denote the determinant of the matrix (2.7) by

$$\gamma = det(\mathsf{L}). \tag{2.8}$$

We remark here that a map similar to (2.5), (2.6) has been introduced in the proof of the main theorem in [29] where a relation of Delsarte surfaces to Fermat surfaces is established.

Definition 2.2. *We call a polynomial $f(x)$ simpliciable if $det(\mathsf{L}) = \gamma \neq 0$.*

A Laurent polynomial $f(x)$ is simpliciable if and only if its Newton polyhedron $\Delta(f)$ has the dimension of the ambient torus $\mathbf{T}^n$ that is equal to n. A polynomial $F(x)$ satisfying the above **Assumption** is simpliciable. Further we shall assume that the determinant γ of the matrix L is positive for a simpliciable $f(x)$ in such a way that $\gamma = n!vol(\Delta(f))$. This assumption is always satisfied without loss of generality, if we permute certain row vectors of the matrix, which evidently corresponds to the change of names of vertices $\alpha(j)$.

Lemma 2.3. *Let $f(x)$ be a simplicial polynomial. For the simplex polyhedron $\tau_q \in \mathbb{R}^n$ defined as $\langle \alpha(1), \overset{q}{\overset{\vee}{\cdots}}, \alpha(n+1), \alpha(n+2) \rangle$, $q \in [1, n]$, the following equality holds*

$$B_q = n!vol(\tau_q). \tag{2.9}$$

Especially,

$$\sum_{q=1}^{n+1} B_q = |\mathbf{B}| = \gamma = (-1)^{n+1}\chi(Z_f), \tag{2.10}$$

here $\chi(Z_f)$ denotes the Euler-Poincaré characteristic of the affine hypersurface Z_f. Further we shall use the notation

$$\mathbf{B} = (B_1, \cdots, B_{n+1}). \tag{2.11}$$

Proof. The derivation of positive integers $B_1, \cdots, B_{n+1}$ is based on the calculation of $n+1$ minors of the matrix L obtained in removing the $(n+2)-$nd column. To establish the last equality, we recall the Theorem 2 of [21] or Theorem 1 of [26] on the Euler characteristic and the volume of Newton polyhedron.

□

Remark 2.4. *If a polynomial $f(x)$ is simpliciable it has n tuple of linearly independent vectors from $supp(f)$. Such a polynomial with generic coefficients is $\Delta(f)$-regular.*

3. Mellin transforms

In this section we proceed to calculation of the Mellin transform of the period integrals associated to the hypersurface $Z_f = \{x \in \mathbf{T}^n; f(x) = 0\}$ defined by a simpliciable polynomial f (2.2).

First of all we consider the period integral taken along the fibre for $u^k x^{\mathbf{J}} \in R_f^+$ $\rho^+(u^k x^{\mathbf{J}}) \in PH^n(\mathbf{T}^n \setminus Z_f)$ (see Theorem 1.4) as follows,

$$I_{u^k x^{\mathbf{J}}, \mathsf{t}\delta}(s) := \int_{\mathsf{t}\delta(s)} \frac{(k-1)! x^{\mathbf{J}} \omega_0}{f(x)^k} \tag{3.1}$$

where $\mathsf{t}\delta(s) \in H_n(\mathbf{T}^n \setminus Z_f)$ is a cycle obtained after the application of t : Leray's coboundary (or tube) operator to a $n-1$ cycle $\delta(s) \in H_{n-1}(Z_f)$. Leray's coboundary operator can be defined as a S^1 bundle construction over the cycle $\delta(s)$ ([15], Part II).

The Mellin transform of $I_{u^k x^{\mathbf{J}},\mathsf{t}\delta}(s)$ is defined by the following integral:

$$M_{u^k x^{\mathbf{J}},\delta}(z) := \int_{\Pi} s^z I_{u^k x^{\mathbf{J}},\mathsf{t}\delta}(s) \frac{ds}{s}. \tag{3.2}$$

Here Π stands for a semi-real axis of the form $\Pi = \{s \in \mathbf{T}; Arg\ s = \alpha\}$ for some fixed $\alpha \in [0, 2\pi)$ that avoids ramification loci of $I_{u^k x^{\mathbf{J}},\mathsf{t}\delta}(s)$.

First of all we recall the fact

$$\int_{\mathbb{R}_+} u^k e^{-uf(x)} \frac{du}{u} = \frac{(k-1)!}{f(x)^k}$$

for $\Re(f(x)) > 0$, $k \geq 1$. On the Leray coboundary $\mathsf{t}\delta(s) \subset \{x \in \mathbf{T}^n; \mid f(x) \mid = \epsilon, \epsilon > 0\}$ the argument $Arg\ (f(x))$ moves on the circle S^1. Thus we introduce a fibre product along S^1

$$\mathbf{T} \times_{S^1} \mathsf{t}\delta(s) := \bigcup_{\theta \in S^1} \{(u, x) \in (e^{-i\theta}\mathbb{R}_+, \mathsf{t}\delta(s)); f(x) = \epsilon e^{i\theta}\}$$

in order to define the integral

$$\int_{\mathbf{T}\times_{S^1}\mathsf{t}\gamma(s)} e^{-uf(x)} x^{\mathbf{J}} du \wedge \omega_0$$

properly as a function in $s \in \mathbf{T}$. In fact the integrand function has neither branching points nor poles on the $\mathbb{C}_u$ plane and, in general, the turn of the integration path $e^{-i\theta}\mathbb{R}_+$ gives a natural analytic continuation beween integrals $\int_{\mathbb{R}_+} e^{-uT} du$ for $T > 0$ and $\int_{e^{-i\theta}\mathbb{R}_+} e^{-uT} du$ for $Arg\ T \in (-\pi/2 + \theta, \pi/2 + \theta)$. Now we consider the following $(n+2)-$ dimensional chain

$$\tilde{\Gamma} := (\mathbf{T} \times_{S^1} \mathsf{t}\delta(s)) \times_{\Pi} \Pi = \{(u, x, s); (u, x) \in \mathbf{T} \times_{S^1} \mathsf{t}\delta(s), s \in \Pi\}. \tag{3.3}$$

In fact this gives an equivariant fibration over $S^1 \times \Pi$. The movement of (θ, s) inside $S^1 \times \Pi$ provokes no monodromy of the fibre.

We deform the integral (3.2) in making use of the relation (2.6):

$$\begin{aligned} M_{u^k x^{\mathbf{J}},\tilde{\Gamma}}(z) &= \int_{\tilde{\Gamma}} e^{-uf(x)} x^{\mathbf{J}} u^k s^z \frac{du}{u} \wedge \omega_0 \wedge \frac{ds}{s} \\ &= \frac{1}{\gamma} \int_{\mathsf{L}_*(\tilde{\Gamma})} e^{-\Psi(T)} \prod_{q=1}^{n+2} T_q^{\mathcal{L}_q(\mathbf{J},z,k)} \prod_{q=1}^{n+2} \bigwedge \frac{dT_q}{T_q}, \end{aligned} \tag{3.4}$$

with

$$\Psi(T) = T_1(x, u) + \cdots + T_{n+1}(u) + T_{n+2}(x, s, u) = uf(x) \tag{3.5}$$

where each term $T_i(x, u)$, $i \in [1; n]$ represents a monomial term (2.4) of variables x, u of the polynomial (3.5) while $T_{n+1}(u) = u$. Here the exponents $\mathcal{L}_q(\mathbf{J}, z, k)$ denote linear functions of components that shall be concretely given in (3.7).

In the following proposition we denote by $\mathcal{L}_q(\mathbf{J}, z, k)$ the inner product of $(\mathbf{J}, z, k)$ with the $q-$th column vector of L^{-1}.

Proposition 3.1. *1) The Mellin transform $M_{u^k x^{\mathbf{J}},\tilde{\Gamma}}$ of the period integral associated to the simpliciable polynomial $f(x)$ has the following form.*

$$M_{u^k x^{\mathbf{J}},\tilde{\Gamma}} = g_{\tilde{\Gamma}}(z) \prod_{q=1}^{n+2} \Gamma(\mathcal{L}_q(\mathbf{J}, z, k)), \tag{3.6}$$

where $g_{\tilde{\Gamma}}(z)$ is a polynomial $e^{\frac{2\pi i z}{\gamma}}$ with $\gamma = n!vol(\Delta(f))$. The function $\mathcal{L}_q(\mathbf{J}, z, k), q \in [1; n+2]$ linear in $(\mathbf{J}, z, k)$ with coefficients in $\frac{1}{\gamma}\mathbb{Z}$ is given by

$$\mathcal{L}_q(\mathbf{J}, z, k) =^t (\mathbf{J}, z, k) w_q = \frac{< v_q, \mathbf{J} > - B_q z + C_q k}{\gamma}, \tag{3.7}$$

where w_q is the $q-th$ column vector of the matrix $(\mathsf{L})^{-1}$.

2) The $n+2$ linear functions $\mathcal{L}_q(\mathbf{J}, z, k)$ are classified into the following three groups.

$$\mathcal{L}_{n+2}(\mathbf{J}, z, k) = \frac{\gamma}{\gamma} z = z. \tag{3.8}$$

There exists unique index q such that $w_q = (v_q, -B_q, \gamma)/\gamma$ for some $v_q \in \mathbb{Z}^n$, and $B_q > 0$. We fix such q to be $n+1$.

$$\mathcal{L}_{n+1}(\mathbf{J}, z, k) = \frac{< v_{n+1}, \mathbf{J} > - B_{n+1} z}{\gamma} + k. \tag{3.9}$$

For q such that $w_q = (v_q, -B_q, 0)/\gamma$ for some $v_q \in \mathbb{Z}^n$, and $B_q > 0$,

$$\mathcal{L}_q(\mathbf{J}, z, k) = \frac{< v_q, \mathbf{J} > - B_q z}{\gamma}. \tag{3.10}$$

For these vectors we have the following equalities:

$$\sum_{q=1}^{n+1} w_q = (0, -1, 1), \quad \sum_{q=1}^{n+1} v_q = 0. \tag{3.11}$$

Proof. 1) The definition of the $\Gamma-$ function can be formulated as follows;

$$\int_{\bar{\mathbb{R}}_+} e^{-T} T^\sigma \frac{dT}{T} = (-1 + e^{2\pi i \sigma}) \int_{\mathbb{R}_+} e^{-T} T^\sigma \frac{dT}{T} = (-1 + e^{2\pi i \sigma}) \Gamma(\sigma),$$

for the unique nontrivial cycle $\bar{\mathbb{R}}_+$ turning once around $T = 0$ that begins and returns to $\Re T \to +\infty$. We apply it to the integral (3.4) and get (3.6). We consider an action on the chain $C_a = \bar{\mathbb{R}}_+$ or $\mathbb{R}_+$ on the complex T_a plane, $\lambda : C_a \to \lambda(C_a)$ defined by the relation,

$$\int_{\lambda(C_a)} e^{T_a} T_a^{\sigma_a} \frac{dT_a}{T_a} = \int_{(C_a)} e^{T_a} (e^{2\pi\sqrt{-1}} T_a)^{\sigma_a} \frac{dT_a}{T_a}.$$

In particular $\lambda(\mathbb{R}_+) = \mathbb{R}_+ + \bar{\mathbb{R}}_+$. By means of this action the chain $\mathsf{L}_*(\tilde{\Gamma})$ turns out to be homologous to an integer coefficients linear combination of chains

$$\prod_{q=1}^{n+1} \lambda^{j_q}(\bar{\mathbb{R}}_+) \lambda^{j_{n+2}}(\mathbb{R}_+) \quad or \quad \prod_{q=1}^{n+2} \lambda^{j_q}(\bar{\mathbb{R}}_+), \tag{3.12}$$

with $j_q \in \mathbb{Z}$. This explains the presence of the factor $g_{\tilde{\Gamma}}(z)$ in (3.6) that is a polynomial in the exponential functions $e^{2\pi\sqrt{-1}j_q\mathcal{L}_q(\mathbf{J},\mathbf{z},k)}$, $q \in [1; n+2]$.

Cramer's formula explains the origin of the coefficient B_q in (3.7) that is an $n \times n$minor of L.

The point 2) is reduced to the linear algebra based on Lemma 2.3. □

Corollary 3.2. *The Newton polyhedron admits the following representation by the aid of linear functions defined in (3.9), (3.10):*

$$\Delta(f) = \{\beta \in \mathbb{R}^n; 0 \leq \mathcal{L}_q(\beta, 0, 1) \leq 1\} \tag{3.13}$$

for $q \in [1; n+1]$.

Proof. After the definition of vectors $v_1, \cdots, v_{n+1}$ we can argue as follows.

For a vector $\vec{i}$ on the hyperplane $\langle 0, \alpha(1), \overset{q}{\vee}\cdots, \alpha(n)\rangle$, $q \in [1; n]$ the scalar product $\langle v_q, \vec{i}\rangle$ vanishes while $\langle v_q, \alpha(q)\rangle = \gamma$.

For a vector $\vec{i}$ from the hyperplane $\langle \alpha(1), \cdots, \alpha(n)\rangle$ not passing through the origin we have scalar products $\langle v_{n+1}, \vec{i}\rangle = \langle v_{n+1}, \alpha(q)\rangle = -\gamma$, $q \in [1; n]$. □

Corollary 3.3. *A monomial* $u^\ell x^{\mathbf{J}} \in \mathbb{C}[u, x^{\pm}]$ *belongs to* S_Δ *if and only if the following* $n-$*tuple of inequalities are satisfied,*

$$0 \leq \mathcal{L}_q(\mathbf{J}, 0, \ell) \leq 1$$

for $q \in [1; n]$.

Corollary 3.4. *Under the above situation, the Mellin inverse of* $M_{u^k x^{\mathbf{J}},\delta}(s)$ *with properly chosen periodic entire function* $g(z)$ *with period* γ *gives (3.1)*

$$I_{u^k x^{\mathbf{J}},\mathfrak{t}\delta}(s) = \int_{\check{\Pi}} g(z)\Gamma(z)\prod_{q=1}^{n+1}\Gamma\big(\mathcal{L}_q(\mathbf{J}, z, k)\big)s^{-z}dz. \tag{3.14}$$

Here the integration path $\check{\Pi}$ *enclosing all poles of* $\Gamma(z) : \mathbb{Z}_{\leq 0}$ *has the initial (resp. terminal) asymptotic direction* $e^{-(\pi/2+\epsilon)i}$ *(resp.* $e^{(\pi/2+\epsilon)i}$*) for some small* ϵ*. This Mellin-Barnes integral defines a convergent analytic function in* $-\pi < arg\ s < \pi$, $0 < |s| < \epsilon$.

Proof. In applying the Stirling's formula

$$\Gamma(z+1) \sim (2\pi z)^{\frac{1}{2}} z^z e^{-z}, \ \ \Re\, z \to +\infty,$$

to the integrand of (3.1), we take into account the relation (2.10). Here we remind us of the formula $\Gamma(z)\Gamma(1-z) = \frac{\pi}{sin\ \pi z}$. As for the choice of the periodic function $g(z)$ one makes use of Nörlund's technique [25]. In this way we can choose such $g(z)$ that the integrand is of exponential decay on $\check{\Pi}$. Theorem on the Mellin inverse transform [25, §2.14] states that the Mellin-Barnes integral (3.1) for properly chosen $g(z)$ recovers the integral $I_{u^k x^{\mathbf{J}},\mathfrak{t}\delta}(s)$. □

In general it is a difficult task to find concrete periodic function $g(z)$ that corresponds to $I_{u^k x^{\mathbf{J}},\mathfrak{t}\delta}(s)$ for a cycle $\delta \in H^{n-1}(Z_f)$. The question how to choose $g(z)$ is a *desideratum* in the study of period integrals by means of Mellin transforms. Matsubara-Heo makes a proposal to establish a correspondence between Pochhammer type cycles and $\Gamma-$series solutions to A-HG equation [24, section 5].

Example 3.5. Let us illustrate the above procedures by a simple example.

$$f(x) = x_1^3 x_2^{-1} + x_1^3 x_2^3 + s x_1^2 x_2 + 1.$$

$$\mathsf{L} = \begin{bmatrix} 3 & -1 & 0 & 1 \\ 3 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 1 \end{bmatrix},$$

$$(\mathsf{L})^{-1} = \frac{1}{12} \begin{bmatrix} 3 & 1 & -4 & 0 \\ -3 & 3 & 0 & 0 \\ -3 & -5 & -4 & 12 \\ 0 & 0 & 12 & 0 \end{bmatrix}, \quad \gamma = det(\mathsf{L}) = 12.$$

We have

$$\mathcal{L}_1(\mathbf{J}, z, k) = \frac{i_1 + 3i_2 - 5z}{12}, \mathcal{L}_2(\mathbf{J}, z, k) = \frac{3i_1 - 3i_2 - 3z}{12},$$

$$\mathcal{L}_3(\mathbf{J}, z, k) = \frac{-4i_1 - 4z}{12} + k, \mathcal{L}_4(\mathbf{J}, z) = \frac{12z}{12}.$$

Let us denote by $\alpha(1) = (3,3)$,$\alpha(2) = (3,-1)$,$\alpha(3) = (0,0)$, $\alpha(4) = (2,1)$. Then we have

$$B_1 = vol(\tau_1) = 2!vol(\alpha(2), \alpha(3), \alpha(4)) = 5.$$

Similarly $B_2 = vol(\tau_2) = 3$, $B_3 = vol(\tau_3) = 4$.

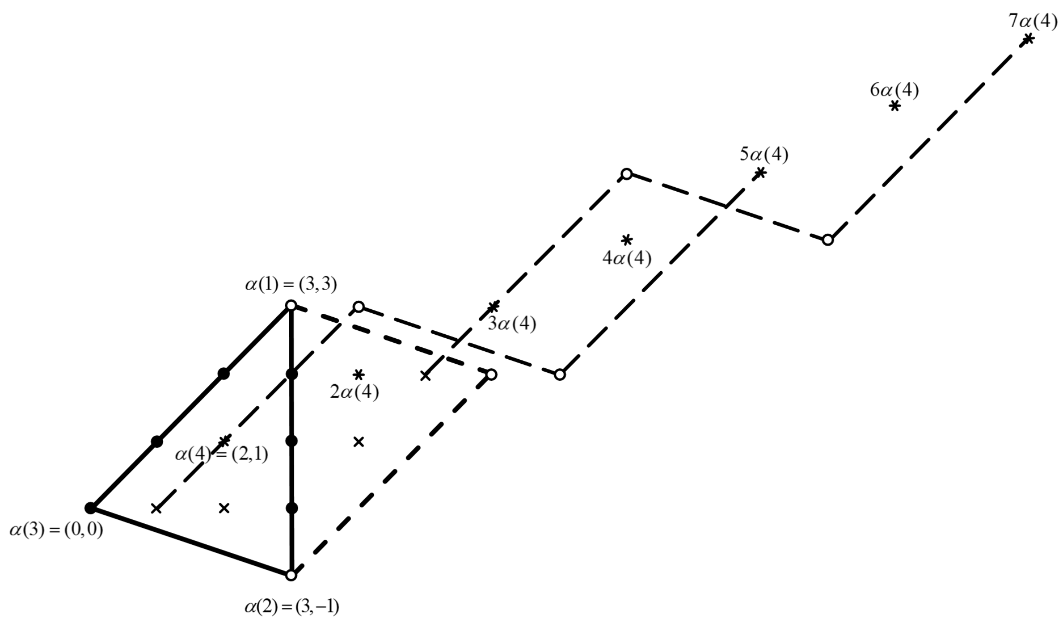
It is worthy noticing that $h.c.f.\mathbf{B} = 1$. Thus we have $\gamma = |\mathbf{B}| = 2!vol(\Delta(f)) = 12$.

We can look at the base representatives of R_f^+ with the following support points:

$$\{(i_1, 0, 1)_{i_1=1}^3, (i_2, 1, 1)_{i_2=1}^3, (i_3, 2, 1)_{i_3=2}^3, (4, 1, 2), (i_3, 2, 2)_{i_3=4}^5\}.$$

We have $dim(R_f^+) = 11$, $R_f \cong \mathbb{C} \oplus R_f^+$.

Later we see (Proposition 6.4) that the set of vectors in $(\frac{1}{12}\mathbb{Z})^3$ given by $(\mathcal{L}_1(\mathbf{J}, 0, \ell)$, $\mathcal{L}_2(\mathbf{J}, 0, \ell)$, $\mathcal{L}_3(\mathbf{J}, 0, \ell))$, $(\ell, \mathbf{J})$ of the above list of support points of base elements of R_f^+ coincides with $(\frac{B_1 k}{\gamma}, \frac{B_2 k}{\gamma}, \frac{B_3 k}{\gamma}) = (\frac{5k}{12}, \frac{3k}{12}, \frac{4k}{12})$, $k \in [1; 11]$ modulo $\mathbb{Z}^3$.



$$\bar{\lambda}(u^3x^{3\alpha(4)}) = \bar{\lambda}(ux_1^3)$$
$$\bar{\lambda}(u^5x^{5\alpha(4)}) = \bar{\lambda}(ux_1)$$
$$\bar{\lambda}(u^7x^{7\alpha(4)}) = \bar{\lambda}(u^2x_1^5x_2^2)$$
$$R_f : \bullet, \times \qquad\qquad * : k\alpha(4)$$

Example 3.6. Now we consider the following (Laurent) polynomial in three variables.

$$f(x) = x_1x_2x_3(x_1 + x_2 + x_3 + s + (x_1x_2x_3)^{-1}).$$

$$\mathsf{L} = \begin{bmatrix} 2 & 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 & 1 \\ 1 & 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(\mathsf{L})^{-1} = \frac{1}{4}\begin{bmatrix} 3 & -1 & -1 & 0 & -1 \\ -1 & 3 & -1 & 0 & -1 \\ -1 & -1 & 3 & 0 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}, \quad \gamma = det(\mathsf{L}) = 4.$$

We have $\mathcal{L}_1(\mathbf{J}, z, k) = \frac{3i_1-i_2-i_3-z}{4}$, $\mathcal{L}_2(\mathbf{J}, z, k) = \frac{-i_1+3i_2-i_3-z}{4}$, $\mathcal{L}_3(\mathbf{J}, z, k) = \frac{-i_1-i_2+3i_3-z}{4}$

$$\mathcal{L}_4(\mathbf{J}, z, k) = \frac{-i_1 - i_2 - i_3 - z}{4} + k, \quad \mathcal{L}_5(\mathbf{J}, z, k) = \frac{4z}{4}.$$

In this case we have $B_1 = \cdots = B_4 = 1$ and $R_f^+ \cong \oplus_{k=1}^3 \mathbb{C}(x_1x_2x_3)^k$, $R_f \cong \mathbb{C} \oplus R_f^+$

4. Oscillating integrals

Assume $\tilde{f}(x) = f_0(x) + 1$ such that $supp(f_0) \not\ni \{0\} \in \Delta(f_0)$ and $\tilde{f}$ be a $\Delta(f_0)-$regular polynomial, $Z_{\tilde{f}}$ non singular. In this situation we consider the deformation Z_{f_0+s} of Z_f. For generic value of $s \in \mathbb{C}$ a smooth afffine variety Z_{f_0+s} is topologically equivalent to Z_f.

In choosing the coefficients of f_0 in a generic position, we may assume that the critical points of f_0 i.e. those of $\tilde{f}$ are of Morse type singularities $c_1, \cdots, c_\gamma$ with $\gamma = n!vol(\Delta(f_0))$. We construct Lefschetz thimble associated to each critical point c_j as follows.

Definition 4.1. *For a fixed complex number $u \in \mathbb{C}^\times$ and $j \in [1; \gamma]$, we consider a path T_j^- on $\mathbb{C}_s$ that starts from $s_j = -f_0(c_j)$ and $\Re\,(us) \to +\infty$. For a 1-parameter deformation of a vanishing cycle $\delta_j \in H_{n-1}(Z_{f_0+s})$ with δ_j vanishing at s_j, the cycle $\Gamma_j := \{(s, \delta_j); s \in T_j^-\}$ of the relative homology $H_n(\mathbf{T}^n, \Re\,(uf_0) > 0; \mathbb{Z})$ is called a Lefschetz thimble associated to c_j.*

The set $\mathcal{U} \subset \mathbb{C}$ of generic values of u is defined by the condition

$$Arg\ u \neq Arg(s_i - s_j) \pm \frac{\pi}{2} \tag{4.1}$$

for all distinctive critical vlues $s_i \neq s_j$. The open set $\mathcal{U}$ consists of open sectors (fans) within the limiting half-lines of the form $Arg\ u = Arg(s_i - s_j) \pm \frac{\pi}{2}$. It is known that for a generic value of u the Lefschetz thimbles $\{\Gamma_1, \cdots, \Gamma_\gamma\}$ form a basis of the relative homology group $H_n(\mathbf{T}^n, \Re\,(uf_0) >> 0; \mathbb{Z})$, (see [27, 1.5], [28, (4.4)]).

Now we introduce the oscillating integral with the phase $f_0(x)$ in the following way,

$$J_{g,\Gamma}(s, u) = \int_\Gamma e^{-uf_0(x)} g(s, x, u)\omega_0 \tag{4.2}$$

for a Laurent polynomial $g(s, x, u) \in \mathbb{C}[s, u, x_1^\pm, \cdots, x_n^\pm]$. Let $C_{i,j}$ be an oriented curve that presents a union of two non-compact non self-intersecting curves each of which are located inside of a sector of $\mathcal{U}$ near infinity. For example we can take a union $C_{i,j} = C_{i,j}^+ - C_{i,j}^-$ of a curve $C_{i,j}^+$ with the asymptote $Arg\ u = Arg(s_i - s_j) + \frac{\pi}{2} + \epsilon$ and a curve $C_{i,j}^-$ with the asymptote $Arg\ u = Arg(s_i - s_j) + \frac{\pi}{2} - \epsilon$. For such a curve C, we can define the Laplace transform of $J_{g,\Gamma}(u)$

$$L_C(J_{g,\Gamma})(s) = \int_C e^{-us} J_{g,\Gamma}(s, u)\frac{du}{u} = \int_C (\int_\Gamma e^{-u(f_0(x)+s)} g(s, x, u)\omega_0)\frac{du}{u} \tag{4.3}$$

if $\Re\ u(f_0+s)|_\delta > 0$ near the boundary of $C \times \delta$ at infinity. We recall the notation $\omega_0 = \frac{dx}{x^{\mathbf{1}}}$. For example, in the case of $C = C_{i,j}$ we can choose a Lefschetz thimble $\Gamma \in \mathbb{Z}\Gamma_i + \mathbb{Z}\Gamma_j + \sum_\ell \mathbb{Z}\Gamma_\ell$ where the summation is taken over Γ_ℓ such that $Arg\ s_\ell \in [Arg\ s_i, Arg\ s_j]$. The Laplace transform $L_{C_{i,j}}(J_{g,\Gamma})(s)$ is well defined in a sector $\{s \in \mathbb{C}; -\pi + \epsilon - Arg\ (s_i - s_j) < Arg\ s < -\epsilon - Arg\ (s_i - s_j), \epsilon > 0\}$.

It is not simple to consider $L_{C_{i,j}}(J_{g,\Gamma})(s)$ outside the given sector even though it is possible by extending analytically the Laplace transform to an open subset of $\mathbb{C}$. This requires a detailed study of the asymptotic behaviour of $J_{g,\Gamma}(s,u)$ from which [2] preferred to abstain.

In [1] the authors derived properties of the oscillating integrals from the period integrals by means of Laplace transform (4.6). In particular they establish a relation between the Stokes matrix of oscillating integrals and the monodromy of period integrals. This procedure was followed in [31] to verify Dubrovin's conjecture [14] for the quantum cohomology of projective space.

Now we look at a $\mathbb{C}[s]$ module $S_{\Delta(f_0)}[s]$ with the basis $\{u^k x^\alpha\}$, $\alpha \in k\Delta(f_0)$. In this situation thanks to Theorem 1.4 the $\mathbb{C}[s]$ module

$$H^{n-1}_{DR}(s) \cong \frac{S_{\Delta(f_0)}[s]}{\mathcal{D}_u S_{\Delta(f_0)}[s] + \sum_{i=1}^n \mathcal{D}_{x_i} S_{\Delta(f_0)}[s]} \tag{4.4}$$

represents the $(n-1)$th relative de Rham cohomology group $H^{n-1}(Z_{f_0+s})$ (compare with [2, §11]). Here the operators $\mathcal{D}_u$ is defined for $f = f_0 + s$ as in Theorem 1.4.

The following is a straightforward consequence of the Stokes' theorem and the fact that the integrand function $e^{-u(f_0(x)+s)}g(s,x,u) = 0$ at the infinity boundary of $\delta \times C$.

Lemma 4.2. *For $h(s,x,u) \in \mathcal{D}_u S_{\Delta(f_0)}[s] + \sum_{i=1}^n \mathcal{D}_{x_i} S_{\Delta(f_0)}[s]$ the Laplace transform of the oscillating integral $L_C(J_{h,\Gamma})(s)$ vanishes identically.*

The vanishing of the Laplace transform for s in an open set means that of $J_{h,\Gamma}(s,u)$ itself. Thus for the fixed value $s = 1$, $\tilde{f}(x) = f_0(x) + 1$ the following oscillating integral vanishes

$$\int_\Gamma e^{-u\tilde{f}(x)}\tilde{g}(x,u)\omega_0$$

for $\tilde{g}(x,u) \in \mathcal{D}_u S_{\Delta(f_0)} + \sum_{i=1}^n \mathcal{D}_{x_i} S_{\Delta(f_0)}$.

In summary we came to the following conclusion:

- The space (1.10) is well adapted to study non-trivial oscillating integrals like

$$J_{\tilde{g},\Gamma}(u) = \int_\Gamma e^{-u\tilde{f}(x)}\tilde{g}(x,u)\omega_0 \tag{4.5}$$

  and their Laplace transforms.
- The space (4.4) is well adapted to study non-trivial oscillating integrals $J_{g,\Gamma}(s,u)$ (4.2) and their their Laplace transforms.

The Brieskorn lattice G_0 defined in [13, 2.c] gives a proper space to examine the oscillating integrals (4.2) for a fixed u and $s \in \mathbb{C}$. It would correspond to $\frac{S^+_\Delta}{\sum_{i=1}^n \mathcal{D}_{x_i} S_\Delta}$ after our notation. As the parameter "u" is fixed to a "Planck constant" there is no room to consider the Laplace transform (4.3).

For $\eta \in \mathbf{T}$ such that $\gamma \cdot arg\ \eta \not\equiv 0 (mod\ 2\pi)$ we consider the integration path $C(\eta)$ that shall be taken as the contour from ∞ along a parallel to the direction $arg\ s = arg\ \eta$ sufficiently far away on the left, turning around all the singular loci (6.4) in the anticlockwise sense and back to ∞ along a parallel to the direction $arg\ s = arg\ \eta$

sufficiently far away on the right [1, Theorem 2]. For such a path $C(\eta)$ and $\Re(u\eta) \geq 0$ the Laplace transform of $I_{g,\mathfrak{t}\delta}(s)$ (5.5) can be defined as follows

$$\int_{C(\eta)} e^{us} I_{g,\mathfrak{t}\delta}(s) ds \tag{4.6}$$

that is equal to an oscillating integral $J_{g,\Gamma}(u)$ for a Lefschetz thimble Γ associated to the vanishing cycle δ (see [28, 3.3]). In (4.6) the path $C(\eta)$ can be homotopically deformed into a union of paths turning around the singular points that correspond to the cycle δ.

Though $C(\eta)$ can be deformed into a zero chain inside the relative homology $H_1(\mathbf{T}, \infty; \mathbb{Z})$, such a deformation is prohibited for (4.6). In fact in the course of a deformation of $C(\eta)$ into a zero chain, the integral would diverge due to the same reason as explained to define the curve $C_{i,j}$ in (4.3). The inverse Laplace transform (4.6) shall be defined for $C(\eta)$ with a single asymptotic direction $\eta \in \mathbf{T}$, $\gamma \cdot arg\ \eta \not\equiv 0 (mod\ 2\pi)$ tending to the infinity. This consideration suggests that $J_{g,\Gamma_j}(u) = J_{g,\Gamma_j}(0, u)$, $j \in [1; \gamma]$ form a $\mathbb{C}$ vector space of dimension γ and not of dimension $\bar{\gamma} = W_{n-1}(H^{n-1}(Z_f))$ discussed in Proposition 6.4, Remark 6.5.

Now we assume $f_0(x)$ to be a Laurent polynomial like in (2.3). The HG equation (6.3) for $I_{u,\mathfrak{t}\delta}(s) = I_{u^1 x^0,\mathfrak{t}\delta}(s)$ becomes

$$R_{(1,0)}(s, \vartheta_s) I_{u,\mathfrak{t}\delta}(s) = 0,$$

with

$$R_{(1,0)}(s, \vartheta_s) = (-\vartheta_s)_\gamma - s^\gamma \prod_{q=1}^{n+1} (\frac{B_q \vartheta_s}{\gamma})_{B_q} \tag{4.7}$$

where

$$(\alpha)_m = \alpha(\alpha+1)\cdots(\alpha+m-1), \tag{4.8}$$

the Pochhammer symbol. On applying the integration by parts we establish the differential equation with irregular singularities at $u = \infty$ for $J_{u^2,\Gamma}(u)$:

$$[u^\gamma - \prod_{q=1}^{n+1} (\frac{-B_q \vartheta_u}{\gamma})_{B_q}] J_{u^2,\Gamma}(u) = 0 \tag{4.9}$$

for every $\Gamma \in H_n(\mathbf{T}^n, \Re(uf_0) >> 0; \mathbb{Z})$. By means of the change of variables $e^{t_1} = (zu)^\gamma$ for the quantization parameter z, the equation (4.9) is transformed into

$$[e^{t_1} - z^\gamma \prod_{q=1}^{n+1} (-B_q \frac{\partial}{\partial t_1})_{B_q}] \tilde{J}(t_1, z) = 0 \tag{4.10}$$

that coincides with the equation for the J function of the weighted projective space $\mathbb{P}_{\mathbf{B}}$ [7, Corollary 1.8], [34, (5.1)]. Thus we can further develop arguments related to Stokes phenomena of solutions to (4.10) in following [34]. See §7 Weighted projective space $\mathbb{P}_{\mathbf{B}}$.

5. Filtration of period integrals

Now we can state the relationship between the Hodge structure of the $PH^n(\mathbf{T}^n \setminus Z_f)$ (1.5) and the poles of the Mellin transform (3.6) .

We recall the notation: under the situation described in §1, the mixed Hodge structure of $PH^n(\mathbf{T}^n \setminus Z_f)$ is defined as follows:

$$Gr_F^p Gr_q^w PH^n(\mathbf{T}^n \setminus Z_f) = \frac{(F^p \cap W_q) + W_{q-1}}{(F^{p+1} \cap W_q) + W_{q-1}}.$$

Theorem 5.1. *1) Let $u^k x^{\mathbf{J}} \in R_f^+$ be a monomial representative such that $\rho^+(u^k x^{\mathbf{J}}) \in Gr_F^{n-k} Gr_{n+1}^w PH^n(\mathbf{T}^n \setminus Z_f)$, $0 \leq k \leq n$. Then the following inequalities hold*

$$0 < \mathcal{L}_q(\mathbf{J}, 0, k) < 1$$

for $q \in [1; n+1]$. The poles of Mellin transform (3.6) located on the positive real axis $\mathbb{R}_{>0}$ are included in the infinite set with semi-group structure called poles of positive direction,

$$\frac{\gamma}{B_q}\left(\mathcal{L}_q(\mathbf{J}, 0, k) + \mathbb{Z}_{\geq 0}\right); q \in [1; n+1], \tag{5.1}$$

while poles on the negative real axis are included in $\mathbb{Z}_{\leq 0}$ i.e. each period integral is holomorphic at $s = 0$.

2) For a monomial satisfying $\rho^+(u^k x^{\mathbf{J}}) \in Gr_F^{n-k} Gr_{n+1+r}^w PH^n(\mathbf{T}^n \setminus Z_f)$, $k \in [0; n], r \in [1; n-1]$, there exist r indices $q_1, \cdots, q_r, r \in [1; n+1]$ such that $\mathcal{L}_{q_j}(\mathbf{J}, 0, k) = 0$ for $j \in [1; r]$ but no such $r+1$ tuple of indices $q_1, \cdots, q_{r+1}$ exists. In other words, the Mellin transform

$$\frac{\prod_{q=1}^{n+1} \Gamma(\mathcal{L}_q(\mathbf{J}, z, k))}{\Gamma(1-z)} \tag{5.2}$$

of the period integral $I_{u^k x^{\mathbf{J}}, \mathfrak{t}\delta}(s)$ has pole of order r at $z = 0$.

Proof of the theorem can be achieved by a combination of Theorem 1.4 and the Proposition 3.1, Corollary 3.2. We remember here that the $\Gamma(z)$ has simple poles at $z = 0, -1, -2, \cdots$.

We shall compare our result of Theorem 5.1, 1) with Kashiwara-Malgrange filtration on the Gauss-Manin system defined by Douai-Sabbah [13, Theorem 4.5].

First of all we remark the isomorphism between $S_{\Delta(f)}$ defined for (2.2) and $S_{\Delta(f_0)}$ for (2.3). In [13] the Gauss-Manin system associated to a Laurent polynomial whose Newton polyhedron contains the origin as an interior point. To adapt the situation to [13] we need to make a transition from f to f_0.

$$S_{\Delta(f)} \to S_{\Delta(f_0)}, \quad u^k x^\alpha \mapsto u^k x^{\alpha - k\alpha(n+2)}. \tag{5.3}$$

The support of $S_{\Delta(f)}$ (resp. $S_{\Delta(f_0)}$) is contained in a cone $\mathsf{C}\Delta(f) := \sum_{j=1}^{n+1} \mathbb{R}_{\geq 0}(1, \alpha(j))$ (resp.$\mathsf{C}\Delta(f_0) := \sum_{j=1}^{n+1} \mathbb{R}_{\geq 0}(1, \alpha_0(j))$ where $\alpha_0(j) := \alpha(j) - \alpha(n+2), j \in [1; n+1]$.) We shall use the notation motivated by the isomorphism (5.3)

$$\pi : (k, \alpha) \mapsto (k, \tilde{\pi}(\alpha)),$$

with
$$\tilde{\pi}(\alpha) = \alpha - k\alpha(n+2) = \alpha + [\mathcal{L}_{n+1}(\alpha, 0, 0)]\alpha(n+2).$$
Here $[\rho]$ means the maximal integer smaller than ρ. It is worthy noticing that
$$u^k x^\alpha \in S_{\Delta(f)} \Leftrightarrow k = -[\mathcal{L}_{n+1}(\alpha, 0, 0)]$$
while for $u^k x^\alpha \in S_{\Delta(f_0)}$ the book keeping index k cannot be recovered from α. The isomorphism (5.3) induces an isomorphism between quotient spaces R_f and R_{f_0} defined in (1.4). We recall here that R_{f_0} is obtained from $S_{\Delta(f_0)}$ by the equivalence relations,
$$uf_0 \equiv 0, \;\; (d_j(k, \alpha', f_0) + ux^{\alpha_0(j)})u^k x^{\alpha'} \equiv 0$$
with $d_j(k, \alpha', f_0) \in \mathbb{Q} \setminus \{0\}$, $u^k x^{\alpha'} \in S_{\Delta(f_0)}$. Compare with Lemma 6.2.

For the fundamental parallelepiped
$$\Pi_{\Delta(f_0)} = \{\sum_{j=1}^{n+1} t_j \pi(1, \alpha(j)); 0 \le t_j < 1, j \in [1; n+1]\}$$
we denote by $Rep(R^+_{f_0})$ the representative polynomials of $R^+_{f_0}$ whose support is located in $\Pi_{\Delta(f_0)}$. It is known that the generating function of $Rep(R^+_{f_0})$ is given by the Ehrhart polynomial $\Psi_{\Delta(f_0)}(t) = \Psi_{\Delta(f)}(t)$ from Definition 1.2.

Now we introduce the grading on $S_{\Delta(f)}$ as follows;
$$\tilde{L}(k, \alpha) = \frac{\gamma}{B_q} \mathcal{L}_q(\alpha, 0, k) = \frac{< v_q, \alpha > + k\gamma\delta_{n+1,q}}{B_q} \tag{5.4}$$
that is associated to the poles of positive direction (5.1).

The grading on $S_{\Delta(f_0)}$ under the guise of that in [13, 4.a] can be defined as follows. Let
$$\Delta_q(f_0) = \big\langle \alpha_0(1), \overset{\overset{q}{\vee}}{\cdots}, \alpha_0(n+1) \big\rangle, \;\; q \in [1; n+1]$$
be a $(n-1)$ dimensional simplex face of $\Delta(f_0)$. The n dimensional cone
$$\mathsf{C}\Delta_q(f_0) := \bigcup_{\tilde{\alpha} \in \Delta_q(f_0)} \mathbb{R}_{\ge 0}(1, \tilde{\alpha})$$
is obtained as its coning with the apex at the origine. Let $L_q(k, \tilde{\alpha})$ be a linear function satisfying the following conditions:
$$L_q(k, \tilde{\alpha}) = 0 \; for \; \forall (k, \tilde{\alpha}) \in \mathsf{C}\Delta_q, \;\; L_q(k, 0) = -k \; for \; \forall k \in [1; n]$$
in such a way that
$$\mathsf{C}\Delta(f_0) = \bigcap_{q \in [1; n+1]} \{(r, \beta) \in \mathbb{R}_{\ge 0} \times \mathbb{R}^n; L_q(r, \beta) \le 0\}.$$
Under this situation we have

Lemma 5.2. *For every $(k, \alpha) \in \bigcup_{\ell \in \mathbb{Z}_{\ge 0}} (\ell, \ell\Delta(f)) \subset \mathsf{C}\Delta(f)$ we have the equality,*
$$L_q(\pi(k, \alpha)) + \tilde{L}_q(k, \alpha) = 0, \;\; \forall q \in [1, n+1].$$

Proof. For $q \in [1;n]$ the equality below is valid,

$$L_q(\pi(k,\alpha)) = L_q(k, \alpha - k\alpha(n+2)) = -\frac{< v_q, \alpha - k\alpha(n+2) >}{B_q} - k = -\frac{< v_q, \alpha >}{B_q}$$

as $< v_q, \alpha(n+2) >= B_q$ by (2.9).

For $q = n+1$,

$$L_{n+1}(\pi(k,\alpha)) + \tilde{L}_{n+1}(k,\alpha) = -\frac{< v_{n+1}, \alpha > -k < v_{n+1}, \alpha(n+2) >}{B_{n+1}} - k + \frac{< v_{n+1}, \alpha > +k\gamma}{B_{n+1}}.$$

We recall that $v_{n+1} = -\sum_{q=1}^{n} v_q$ by (3.11) and see that $- < v_{n+1}, \alpha(n+2) >= \gamma - B_{n+1}$. □

In order to introduce a filtration defined in terms of Mellin transforms, first we remark that due to (4.4) for every $h \in S_{\Delta(f)}$ the following decomposition holds:

$$I_{h,\mathfrak{t}\delta}(s) = \sum_{u^\ell x^\alpha \in Rep(R_f^+)} \tilde{h}_{\ell,\alpha}(s) I_{u^\ell x^\alpha, \mathfrak{t}\delta}(s), \tag{5.5}$$

for $\tilde{h}_{\ell,\alpha}(s) \in \mathbb{C}[s]$. Here $Rep(R_f^+)$ denotes representative polynomials of R_f^+ whose support is located in the fundamental parallelepiped $\Pi_{\Delta(f)}$. We introduce a filtration $\mathcal{M}_\beta$ on the set of period integrals (4.4) defined for $\beta \in \mathbb{Q}$ with the aid of the notion of poles of positive direction analogous to (5.1) as follows :

$$\mathcal{M}_\beta := \{I_{h,\mathfrak{t}\delta}(s); \text{minimum of poles of positive direction of } M_{h,\mathfrak{t}\delta}(z) \geq \beta\}. \tag{5.6}$$

In a similar manner we introduce

$$\mathcal{M}_{>\beta} := \{I_{h,\mathfrak{t}\delta}(s); \text{minimum of poles of positive direction of } M_{h,\mathfrak{t}\delta}(z) > \beta\}. \tag{5.7}$$

The non-trivial filtration $\mathcal{M}_\beta \neq \emptyset$ means that $\beta \leq \beta_0 := max_{q\in[1;n+1]} \frac{\gamma}{B_q}$. This filtration is decreasing i.e. $\beta_1 \leq \beta_2 \leq \beta_0 \Rightarrow \mathcal{M}_{\beta_1} \supset \mathcal{M}_{\beta_2}$.

Now we introduce the following rational number defined for $h \in S_{\Delta(f)}$.

$$\beta(h) := min_{q\in[1;n+1]} min_{(\ell,\alpha)\in\Pi_{\Delta(f)}, \tilde{h}_{\ell,\alpha}\not\equiv 0} \left(\tilde{L}_q(\ell,\alpha) - deg\ \tilde{h}_{\ell,\alpha}\right) \tag{5.8}$$

Here the notation is the same as in (5.5). From the definition of the Melin transform (3.2) and the grading (5.4), we see that the inequality $\beta(h) \geq \beta$ entails $I_{h,\mathfrak{t}\delta}(s) \in \mathcal{M}_\beta$ for every $\delta(s) \in H_{n-1}(Z_f)$.

In view of Corollary 3.1, Theorem 5.1 we get a result analogous to [13, Lemma 4.11].

Proposition 5.3. *The filtration (5.6), (5.7) satisfies the following three properties.*

1) Let us define a positive integer $r(\beta)$ for $\beta \leq \beta_0$ by

$$r(\beta) = max_{u^\ell x^\alpha \in Rep(R_f^+)} r(\ell, \alpha; \beta),$$

where

$$r(\ell,\alpha;\beta) = \sharp\{q \in [1; n+1]; \frac{\gamma}{B_q}\mathcal{L}_q(\alpha, 0, \ell) = \beta \text{ for some } \beta \in [0, \frac{\gamma}{B_q}]\}.$$

With this notation we have

$$(\vartheta_s + \beta)^{r(\beta)} \mathcal{M}_\beta \subset \mathcal{M}_{>\beta}.$$

In other words, the action of $\vartheta_s + \beta$ on $\mathcal{M}_\beta/\mathcal{M}_{>\beta}$ is nilpotent for every $\beta \leq \beta_0$. We remark also that $r(\beta) = r(\beta - m)$ for $m \in \mathbb{Z}_{\geq 0}$.

2) $\partial_s \mathcal{M}_\beta \subset \mathcal{M}_{\beta+1}$ for $\beta \leq \beta_0 - 1$.

3) $s\mathcal{M}_\beta \subset \mathcal{M}_{\beta-1}$ for $\beta \leq \beta_0$.

6. Hypergeometric group associated to the period integrals

For a monomial reperesentative $u^k x^{\mathbf{J}} \in R_f^+$ we introduce two differential operators of order $\gamma = n!vol_n(\Delta(f)) = |\chi(Z_f)|$ with the aid of the Pochhammer symbol (4.8);

$$P(\vartheta_s) = \prod_{j=0}^{\gamma-1} \big(-\vartheta_s\big)_\gamma \tag{6.1}$$

$$Q_{k,\mathbf{J}}(\vartheta_s) = (-1)^\gamma \prod_{q=1}^{n+1} \prod_{j=0}^{B_q-1} \big(\mathcal{L}_q(\mathbf{J}, -\vartheta_s, k)\big)_{B_q}. \tag{6.2}$$

with $\vartheta_s = s\frac{\partial}{\partial s}$. We have the following theorem as a corollary to the Proposition 3.1 and Corollary 3.2.

Theorem 6.1. *The period integral $I_{u^k x^{\mathbf{J}}, \mathbf{t}\delta}(s)$ is annihilated by the differential operator*

$$R_{(k,\mathbf{J})}(s, \vartheta_s) = P(\vartheta_s) - s^\gamma Q_{k,\mathbf{J}}(\vartheta_s), \tag{6.3}$$

with regular singularities at

$$\{s \in \mathbb{P}^1; (\prod_{q=1}^{n} B_q^{B_q})(\frac{s}{\gamma})^\gamma = 1, s = 0, \infty\}. \tag{6.4}$$

In other words

$$[P(\vartheta_s) - s^\gamma Q_{k,\mathbf{J}}(\vartheta_s)] I_{u^k x^{\mathbf{J}}, \mathbf{t}\delta}(s) = 0. \tag{6.5}$$

It is worthy to remark that the operator $R_{(k,\mathbf{J})}(s, \vartheta_s)$ is a pull-back of the Pochhammer hypergeometric operator of order γ for $\vartheta_t = t\frac{\partial}{\partial t}$,

$$\tilde{R}_{(k,\mathbf{J})}(t, \vartheta_t) = P(\gamma\vartheta_t) - tQ_{k,\mathbf{J}}(\gamma\vartheta_t), \tag{6.6}$$

by the Kummer covering $t = s^\gamma$. In certain cases, the monodromy representation of the kernel of the operator (6.6) turns out to be reducible ([5, Proposition 2.7]). To extract the solution subspace with irreducible monodromy from the kernel of (6.6) we introduce the following $\gamma-$tuples of rational numbers contained in $[0, 1)$.

$$C^+ = \{0, \frac{1}{\gamma}, \cdots, \frac{(\gamma-1)}{\gamma}\}.$$

$$C^-(k, \mathbf{J}) = \bigcup_{q=1}^{n+1} \bigcup_{0 \leq j \leq B_q - 1} < \frac{1}{B_q}(j + 1 + k\delta_{q,n+1} + \frac{< v_q, \mathbf{J} >}{\gamma}) >, \tag{6.7}$$

where $\delta_{q,n+1} = 1$ iff $q = n + 1$. For these multi-sets we define

$$C^0(k, \mathbf{J}) = C^+ \cap C^-(k, \mathbf{J})$$

as a multi-set intersection (repetetive appearance of the same element for several times is accepted). Here we used the notation $<\rho>=\rho-[\rho]$ that means the fractional part of $\rho\in\mathbb{Q}$. The symbol $[\rho]$ stands for the maximal integer smaller than ρ.

After **Assumption** in §2, the Newton polyhedron of $f|_{s=0}$ is the same as that of $f|_{s\neq 0}$ and both of them are $\Delta(f)$ regular. In fact the variety $f(x)=0$ in $\mathbb{C}^n$ becomes singular as $s\to 0$, but $Z_f\subset \mathbf{T}^n$ remains to be smooth for $s=0$ (a Delsarte hypersurface). This argument justifies the following calculation of R_f^+ for the case $s=0$.

Lemma 6.2. *The denominator of R_f^+ in (1.10) gives rise to equivalence relations as follows*

$$(c(k,j)x^{\alpha(j)}-1)u^{k+1}x^{\beta}\equiv 0,\ (b(k,j)u-1)u^kx^{\beta}\equiv 0$$

with $u^kx^\beta\in S_\Delta$ and some non-zero constants $\{b(k,j),c(k,j)\}_{j=1}^n$.

The following is a direct consequence of Lemma 6.2 in view of Proposition 3.1 and Corollary 3.2.

Lemma 6.3. *The positive integer $\sharp|C^0(k,\mathbf{J})|$ is well defined on the equivalence class of $u^kx^{\mathbf{J}}\in R_f^+$ i.e. if $u^\ell x^{\mathbf{J}'}\equiv u^kx^{\mathbf{J}}\in R_F^+$ then $\sharp|C^0(k,\mathbf{J})|=\sharp|C^0(\ell,\mathbf{J}')|$.*

Thanks to Lemma 6.3 we can define a positive integer $\bar\gamma=\sharp|C^+\setminus C^0(1,0)|=\sharp|C^-(1,0)\setminus C^0(1,0)|$. Then "the irreducible part" of the kernel of (6.6) has a monodromy representation equivalent to that of the kernel of the following Pochhammer hypergeometric operator of order $\bar\gamma$ by virtue of [5, Corollary 2.6]:

$$\bar R_{(1,0)}(t,\vartheta_t)=\prod_{\alpha^+\in C^+\setminus C^0(1,0)}(\vartheta_t+\alpha^+)-t\prod_{\alpha^-\in C^-(1,0)\setminus C^0(1,0)}(\vartheta_t+\alpha^-+1). \tag{6.8}$$

We consider the set of monomials S_Δ^{mon} in S_Δ. We shall further study the map $\bar\lambda$: $S_\Delta^{mon}\to\frac{1}{\gamma}[0,\gamma)^{n+1}\cap\mathbb{Q}^{n+1}$ given by

$$\bar\lambda(u^\ell x^{\mathbf{J}})=(<\mathcal{L}_1(\mathbf{J},0,\ell)>,\cdots,<\mathcal{L}_{n+1}(\mathbf{J},0,\ell)>). \tag{6.9}$$

We recall here Corollary 3.3 which tells us that $\mathcal{L}_q(\mathbf{J},0,\ell)$ being independent of $\ell\in\mathbb{Z}_{\geq 0}$ equals to its fractional part $<\mathcal{L}_q(\mathbf{J},0,\ell)>$,$q\in[1;n]$. By virtue of Theorem 5.1 we see that

$$0\leq\mathcal{L}_{n+1}(\mathbf{J},0,\ell)<1 \tag{6.10}$$

for the monomial representative $u^\ell x^{\mathbf{J}}\in R_f^+$. Among these $(n+1)$ tuple of linear functions the relation

$$\sum_{q=1}^{n+1}\mathcal{L}_q(\mathbf{J},0,\ell)=\ell \tag{6.11}$$

holds.

Further we assume that

$$h.c.f.\mathbf{B}=1 \tag{6.12}$$

Especially if one of B_q's is equal to 1, this condition is satisfied.

To examine the map $\bar{\lambda}$ on R_f^+ we shall further use the isomorphism (1.11) to identify R_f with $\mathbb{C} \oplus R_f^+$.

Under this convention we see that the map

$$\bar{\lambda}|_{R_f^+} : R_f^+ \rightarrow \frac{1}{\gamma}[0, \gamma)^{n+1} \cap \mathbb{Q}^{n+1}$$

is injective. In fact from Corollary 3.2, Lemma 6.2 and a formula in Definition 1.2, it follows that the condition $\bar{\lambda}(u^\ell x^{\mathbf{J}}) \equiv \bar{\lambda}(u^{\ell'} x^{\mathbf{J}'})$ *mod* $\mathbb{Z}^{n+1}$ yields

$$u^\ell x^{\mathbf{J}} \equiv const. u^{\ell'} x^{\mathbf{J}'} \;\; in \; R_f^+.$$

for some non-zero constant *const*.

Being motivated by [8] we introduce the following two sets of rational numbers located in $[0, 1)^{n+1}$;

$$Im(\bar{\lambda}|_{R_f})^0 = Im(\bar{\lambda}|_{R_f}) \cap (\mathbb{Q}^\times)^{n+1}.$$

$$(\mathbb{Z}/\gamma)^0 = \{0, \cdots, \gamma - 1\} \setminus \{k \in [0; \gamma - 1]; \gamma \mid kB_q, \exists q \in [1; n+1]\} \tag{6.13}$$
$$\cong \left\{ (< \frac{kB_1}{\gamma} >, \cdots, < \frac{kB_{n+1}}{\gamma} >); kB_q \not\equiv 0 \; mod \; \gamma, \; \forall q \in [1; n+1] \right\}.$$

We will also make use of the following γ-tuple of monomials from S_Δ^{mon};

$$\mathcal{K} = \{ux^{\alpha(n+2)}, \cdots, u^\gamma x^{\gamma\alpha(n+2)}\}.$$

The set $(\mathbb{Z}/\gamma)^0$ may seem to be dependent on the choice of $\mathbf{B}$ or that of $\alpha(n+2)$ according to its definition (6.13). In fact it is independent of this choice (See remark 6.6). For these sets we establish the following

Proposition 6.4. *Under the assumption (6.12) the order $\bar{\gamma}$ of the differential operator (6.8) is equal to the following quantities. 1) $\sharp Im(\bar{\lambda}|_{R_f})^0$, 2) $\sharp(\mathbb{Z}/\gamma)^0$, 3) the dimension of the space $W_{n-1}(H^{n-1}(Z_f))$, i.e. $\bar{\gamma}$ is independent of the choice of $\alpha(n+2)$ under the condition (6.12).*

4) Furthermore $\bar{\gamma} = \sharp|C^+ \setminus C^0(\ell, \mathbf{J})| = \sharp|C^-(\ell, \mathbf{J}) \setminus C^0(\ell, \mathbf{J})|$ for every $u^\ell x^{\mathbf{J}} \in R_f^+$.

Proof. 1) First we show the equality $\bar{\gamma} = \sharp(\mathbb{Z}/\gamma)^0$. The definition $\bar{\gamma} = \sharp|C^+ \setminus C^0(1, 0)|$ entails

$$\bar{\gamma} = \gamma - \sharp|\bigcup_{q=1}^{n+1} \bigcup_{0 \leq j \leq B_q - 1} (\frac{j}{B_q}) \bigcap_{0 \leq i \leq \gamma - 1} (\frac{i}{\gamma})|.$$

It is easy to verify that the RHS of the above expression is equal to $\sharp(\mathbb{Z}/\gamma)^0$ under the condition (6.12).

2) In order to see that $\sharp Im(\bar{\lambda}|_{R_f})^0 = \sharp(\mathbb{Z}/\gamma)^0$ we examine the image of the map

$$\bar{\lambda}|_{\mathcal{K}} : \mathcal{K} \rightarrow \frac{1}{\gamma}[0, \gamma)^{n+1} \cap \mathbb{Q}^{n+1}.$$

Under the condition (6.12) the map $\bar{\lambda}|_{\mathcal{K}}$ is injective. The injectivity of $\bar{\lambda}|_{\mathcal{K}}$ can be shown in two steps. If one of B_q is equal to 1,

$$(\frac{jB_1}{\gamma}, \cdots, \frac{jB_{n+1}}{\gamma}) \equiv 0 \quad mod \; \mathbb{Z}^{n+1} \tag{6.14}$$

if and only if $j \equiv 0 \; mod \; \gamma$. In general suppose that B_q admits the prime decomposition $B_q = \prod_{j=1}^{m} p_i^{r_i(q)}$ with $r_i(q) \geq 0$. In a similar manner assume $\gamma = \prod_{j=1}^{m} p_i^{g_i}$. The minimal number j such that jB_q is divided by γ for every q can be expressed as

$$j = \prod_{i=1}^{m} p_i^{g_i - min_q r_i(q)}.$$

The condition (6.12) means that $min_q r_i(q) = 0$. Therefore $j = \gamma$. This means that $\bar{\lambda}|_{\mathcal{K}}$ is injective. In fact if (6.12) is not satisfied $\bar{\lambda}|_{\mathcal{K}}$ is neither injective nor surjective onto $Im(\bar{\lambda}|_{\mathcal{R}_f})^0$.

Further we show that for every $k \in [1; \gamma - 1]$ we find unique monomial representative $u^\ell x^{\mathbf{J}} \in Rep(R_f^+)$ (5.5) such that

$$u^\ell x^{\mathbf{J}} \equiv const. u^k x^{k\alpha(n+2)} \;\; in \; R_f^+ \tag{6.15}$$

for some non-zero constant $const$.

By Lemma 6.2 for a $\Delta(f)$ regular Laurent polynomial f we have

$$\mathcal{L}_{n+1}(k, 0, k\alpha(n+2)) - 1 = \mathcal{L}_{n+1}(k-1, 0, k\alpha(n+2) - \alpha(i))$$

for some $i \in \{1, \cdots, n\}$ satisfying $u^{k-1} x^{k\alpha(n+2) - \alpha(i)} \in S_\Delta$. By induction we find $r \geq 0$ such that

$$0 \leq \mathcal{L}_{n+1}(k, 0, k\alpha(n+2)) - r = \mathcal{L}_{n+1}\left(k - r, 0, k\alpha(n+2) - \sum_{i \in \mathbb{I}_r} \alpha(i)\right) < 1$$

and $0 < k - r \leq n$ for an index set $\mathbb{I}_r \subset \{1, \cdots, n\}^r$. Here $u^{k-r} x^{k\alpha(n+2) - \sum_{i \in \mathbb{I}_r} \alpha(i)} \in S_\Delta$ while for every $i' \in \{1, \cdots, n\}$, $u^{k-r-1} x^{k\alpha(n+2) - \sum_{i \in \mathbb{I}_r} \alpha(i) - \alpha(i')} \not\in S_\Delta$. That is to say this descending process starting from $u^k x^{k\alpha(n+2)}$ stops at certain monomial representative of the quotient ring R_f^+ (see figure of Example 3.5). In this way the desired monomial representative $u^\ell x^{\mathbf{J}} \in R_f^+$ with $\ell = k - r$, $\mathbf{J} = k\alpha(n+2) - \sum_{i \in \mathbb{I}_r} \alpha(i)$ is obtained. The equality $\bar{\lambda}(u^\ell x^{\mathbf{J}}) = \bar{\lambda}(u^k x^{k\alpha(n+2)})$ follows immediately from Corollary 3.2, Lemma 6.2. This shows that the images of two maps $\bar{\lambda}|_{R_f}$ and $\bar{\lambda}|_{\mathcal{K}}$ coincide. The uniqueness of the required monomial $u^\ell x^{\mathbf{J}}$ follows from the injectivity of $\bar{\lambda}|_{\mathcal{K}}$ and that of $\bar{\lambda}|_{R_f}$. In short

$$Im(\bar{\lambda}|_{R_f})^0 = \bigcup_{k \in \{0, \cdots, \gamma-1\}} \bar{\lambda}(u^k x^{k\alpha(n+2)}) \bigcap (\mathbb{Q}^\times)^{n+1}.$$

We see that $\bar{\lambda}(\alpha(n+2), 0, 1) = (\frac{B_1}{\gamma}, \cdots, \frac{B_{n+1}}{\gamma})$ as the point $(1, \alpha(n+2))$ represents the weighted barycenter $(1, \sum_{q=1}^{n+1} \frac{B_q \alpha(q)}{\gamma})$ (cf. Proposition 3.1. 3), Corollary 3.2 and its proof). That is to say, $Im(\bar{\lambda}|_{R_f})^0$ coincides with the second half of (6.13). This shows the equality $\sharp Im(\bar{\lambda}|_{R_f})^0 = \sharp(\mathbb{Z}/\gamma)^0$.

3) The image of the map $\bar{\lambda}|_{R_f}$ admits the following representation:

$$\bigcup_{u^\ell x^{\mathbf{J}} \in R_f^+} (\left\langle \frac{< v_1, \mathbf{J} >}{\gamma} \right\rangle, \cdots, \left\langle \frac{< v_{n+1}, \mathbf{J} >}{\gamma} \right\rangle) \bigcap (\mathbb{Q}^\times)^{n+1}.$$

This means that only the monomials $u^\ell x^{\mathbf{J}} \in I^{(1)}_\Delta$ contribute to $Im(\bar\lambda|_{R_f})^0$. By virtue of Theorem 1.4, 2) and (1.14) we conclude $\sharp Im(\bar\lambda|_{R_f})^0 = dim W_{n-1}(H^{n-1}(Z_f))$.

4) The argument in 2) shows that $\frac{k<v_q,\alpha(n+2)>}{\gamma} = \frac{kB_q}{\gamma}$ for every $q \in [1; n+1], k \in \mathbb{Z}$. For $u^\ell x^{\mathbf{J}} \in R_f^+$ such that

$$u^\ell x^{\mathbf{J}} \equiv const. u^k x^{k\alpha(n+2)} \ \ in\ R_f^+$$

we calculate

$$C^-(\ell, \mathbf{J}) = \bigcup_{q=1}^{n+1} \bigcup_{0 \le j \le B_q - 1} \left\langle \frac{1}{B_q}(j + 1 + k\delta_{q,n+1}) + \frac{k}{\gamma} \right\rangle.$$

This set is nothing but a $\frac{k}{\gamma}$ shift of $C^-(k, 0)$ introduced in (6.7). This yields $\sharp C^0(\ell, \mathbf{J}) = \sharp C^0(k, 0) = \sharp C^0(1, 0)$ and $\sharp|C^+ \setminus C^0(\ell, \mathbf{J})| = \sharp|C^+ \setminus C^0(1, 0)| = \bar\gamma$. $\square$

Remark 6.5. We remark here that the space $W_{n-1}(H^{n-1}(Z_f)) \cong W_{n-1}(H^{n-1}(Z_{f_0}))$ of a pure Hodge structure is known to be isomorphic to $PH^{n-1}(\bar Z_{f_0})$ where $\bar Z_{f_0}$ is a compactification of Z_{f_0} defined in a complete simplicial toric variety $\mathbb{P}_{\Delta(f_0)} = Proj S_{\Delta(f_0)}$ ([4, Proposition 11.6]). Thus we have $\bar\gamma = dim(PH^{n-1}(\bar Z_{f_0}))$.

Authors like L.Borisov-R.P.Horja [6, Corollary 5.12], J.Stienstra [30] studied the $\bar\gamma$ dimensional space embedded into $H^{n-1}(Z_f)$. They investigate this space as solution space of GKZ A-hypergeometric functions.

The positive integer $\gamma - \bar\gamma$ can be interpreted as the dimension of period integrals of the affine variety Z_f originated from the homology of the ambient space $\mathbf{T}^n$. In the exact sequence seen from Theorem 1.4 2),

$$0 \to \bigcup_{2 \le j \le n+2} \rho(\mathcal{I}^{(j)}) \to H^n(\mathbf{T}^n \setminus Z_f) \to \rho(\mathcal{I}^{(1)}) \to 0$$

the term second from left can be interpreted as the dual to "relations" (of rank $\gamma - \bar\gamma$) among period integrals. The quotient by these "relations" would lead to period integrals originated from $W_{n-1}(H^{n-1}(Z_f))$. In principle these "relations" can be read off in comparing monodromy representation matrices (6.23), (6.24) in Lemma 6.12 and (7.2), (7.3) after proper base change of solution spaces $Ker\ \tilde R_{(1,0)}(t, \vartheta_t) \supset Ker\ \bar R_{(1,0)}(t, \vartheta_t)$.

Remark 6.6. *The arguments developed in the proof of Proposition 6.4 2), 3), show that* $\bar X_0(\mathsf{t})$ *does not depend on the choice of* $\alpha(n+2)$ *under the condition (6.12). In other words the set* $(\mathbb{Z}/\gamma)^0$ *is determined in a way independent of the choice of the deformation term* $sx^{\alpha(n+2)}$ *in (2.2).*

Example 6.7. *We recall the Example 3.5. A simple examination of the set of rational vectors* $(\frac{B_1k}{\gamma}, \frac{B_2k}{\gamma}, \frac{B_3k}{\gamma}) = (\frac{5k}{12}, \frac{3k}{12}, \frac{4k}{12}), k \in \{0, \cdots, 11\}$ *gives us* $(\mathbb{Z}/12)^0 = \{1, 2, 5, 7, 10, 11\}$. *In fact there are 6 monomials in* $R_f \cap I^{(1)} : \{ux_1, ux_1^2, ux_1x_2, u^2x_1^4x_2, u^2x_1^4x_2^2, u^2x_1^5x_2^2\}$.

Let us introduce orderings on the sets of rational numbers $C^+ \setminus C^0(1, 0)$

$$0 < \alpha_1^+ < \alpha_2^+ < \cdots < \alpha_{\bar\gamma}^+,$$

and on $C^-(1, 0) \setminus C^0(1, 0)$,

$$0 = \alpha_1^- \le \alpha_2^- \le \cdots \le \alpha_{\bar\gamma}^-.$$

After [8, 1.2] we define the following integer for $k \in (\mathbb{Z}/\gamma)^0$,

$$p(k) = \sharp\{i; \alpha_i^- < \frac{k}{\gamma}\} - j \tag{6.16}$$

where the index j is determined by the relation $\alpha_j^+ = \frac{k}{\gamma}$. We can state the following proposition corresponding to [8, Proposition 1.5].

Proposition 6.8. *For $k \in (\mathbb{Z}/\gamma)^0$ we define the integer $\ell \in [1; n]$ satisfying (6.15). Then we have*

$$\sharp\, p^{-1}(n-\ell) = h^{\ell, n+1-\ell}(PH^n(\mathbf{T}^n \setminus Z_f)) = dim\ Gr_F^{\ell} Gr_{n+1}^w(PH^n(\mathbf{T}^n \setminus Z_f)).$$

Proof. We present our proof in view of typographical errors in [8, 1.2]. By definition

$$p(k) = \sharp|(C^-(1,0) \setminus C^0(1,0)) \cap [0, \frac{k}{\gamma})| - \sharp|(C^+ \setminus C^0(1,0)) \cap [0, \frac{k}{\gamma})|$$

$$= \sharp|C^-(1,0) \cap [0, \frac{k}{\gamma})| - \sharp|C^+ \cap [0, \frac{k}{\gamma})| = \sum_{q=1}^{n+1}(1 + [\frac{kB_q}{\gamma}]) - (k+1).$$

By (2.10) this is equal to

$$n - \sum_{q=1}^{n+1}(\frac{kB_q}{\gamma} - [\frac{kB_q}{\gamma}]).$$

By (6.11) we have $\sum_{q=1}^{n+1}\left\langle \frac{kB_q}{\gamma} \right\rangle = \ell$ for $u^\ell x^{\mathbf{J}}$ satisfying (6.15). Taking Theorem 5.1, 1) into account we obtain the desired result. □

From the proof of Proposition 6.4, 2) we see $\bar{\lambda}(\alpha(n+2), 0, 1) = (\frac{B_1}{\gamma}, \cdots, \frac{B_{n+1}}{\gamma})$ and the following consequence can be derived from Proposition 6.8.

Corollary 6.9. *We consider a Mellin transform (up to constant multiplication of variable t) of solutions to (6.8),*

$$M^0_{ux^{\alpha(n+2)}}(\gamma z) = \frac{\prod_{q=1}^{n+1} \Gamma(\frac{B_q(1-\gamma z)}{\gamma})}{\Gamma(1-\gamma z)}$$

by choosing a suitable $g(z)$ in (3.6). Then the integer (6.16) for $k \in (\mathbb{Z}/\gamma)^0$ can be expressed by

$$p(k) = -\sum_{\frac{1}{\gamma} \le z_i \le \frac{k+1}{\gamma}} Res_{z=z_i} dlog M^0_{ux^{\alpha(n+2)}}(\gamma z),$$

where the residues are taken on the set of poles $\{z_i\}_{i=1}^{m(k)}$ with

$$m(k) = \sharp|\left((C^+ \cup C^-(1,0)) \setminus C^0(1,0)\right) \cap [0, \frac{k}{\gamma})|.$$

This means that we can read off the Hodge filtration of $W_{n+1}(H^n(\mathbf{T}^n \setminus Z_f)$ (equivalently that of $W_{n-1}(H^{n-1}(Z_f))$ or $PH^{n-1}(\bar{Z}_f)$) from the poles of $\frac{dlogM^0_{ux^{\alpha(n+2)}}(\gamma z)}{dz}$.

Further we examine the kernel of the operator $\bar{R}_{(1,0)}(t,\vartheta_t)$ (6.8) and monodromy actions on it. We define characteristic polynomials of the local monodromy of $Ker\ \bar{R}_{(1,0)}(t,\vartheta_t)$ at $t=\infty$ (an anticlockwise turn around $t=\infty$)

$$\bar{X}_\infty(\mathsf{t}) = \prod_{\alpha^-\in C^-(1,0)\setminus C^0(1,0)} (\mathsf{t} - e^{2\pi\sqrt{-1}\alpha^-}) = \frac{\prod_{q=1}^{n+1}(\mathsf{t}^{B_q}-1)}{\varphi(\mathsf{t})}. \tag{6.17}$$

at $t=0$(an anticlockwise turn around $t=0$)

$$\bar{X}_0(\mathsf{t}) = \prod_{\alpha^+\in C^+\setminus C^0(1,0)} (\mathsf{t} - e^{2\pi\sqrt{-1}\alpha^+}) = \frac{(\mathsf{t}^{\gamma}-1)}{\varphi(\mathsf{t})}. \tag{6.18}$$

Here

$$\varphi(\mathsf{t}) = h.c.f.(\prod_{q=1}^{n+1}(\mathsf{t}^{B_q}-1), (\mathsf{t}^{\gamma}-1)), \tag{6.19}$$

a polynomial of degree $\gamma - \bar{\gamma}$. Thus we have $deg\bar{X}_\infty(\mathsf{t}) = deg\bar{X}_0(\mathsf{t}) = \bar{\gamma}$ in view of Proposition 6.4.

Proposition 6.10. *Two degree $\bar{\gamma}$ polynomials $\bar{X}_0(\mathsf{t})$, $\bar{X}_\infty(\mathsf{t})$ are of integer coefficients. More precisely*

$$\bar{X}_0(\mathsf{t}) \in \mathbb{Q}(e^{2\pi\sqrt{-1}/\gamma})[\mathsf{t}] \cap \mathbb{Z}[\mathsf{t}], \quad \bar{X}_\infty(\mathsf{t}) \in \mathbb{Q}(e^{2\pi\sqrt{-1}/B_1}, \cdots, e^{2\pi\sqrt{-1}/B_{n+1}})[\mathsf{t}] \cap \mathbb{Z}[\mathsf{t}].$$

To prove this Proposition, we prepare the following lemma.

Lemma 6.11. *Consider two cyclotomic polynomials $P_0(\mathsf{t}) = \prod_{i=1}^{p}(\mathsf{t}^{A_i}-1)$ and $Q_0(\mathsf{t}) = \prod_{j=1}^{q}(\mathsf{t}^{B_j}-1)$ such that the multi-set of roots of $Q_0(\mathsf{t})$ is contained in the multi-set of roots of $P_0(\mathsf{t})$. Then the rational function $P_0(\mathsf{t})/Q_0(\mathsf{t})$ is a polynomial with integer coefficients.*

Proof. It is clear that $P_0(\mathsf{t})/Q_0(\mathsf{t})$ is a polynomial. We consider the function $P_0(\mathsf{t})/Q_0(\mathsf{t})$ for $|t|<1$ and expand the denominator into a convergent power series in making use of the relation

$$\frac{1}{1-\mathsf{t}^{B_i}} = \sum_{m=0}^{\infty} \mathsf{t}^{mB_i}.$$

The obtained convergent power series expression of $P_0(\mathsf{t})/Q_0(\mathsf{t})$ has integer coefficients. But, in fact, it is a polynomial so the power series breaks down within a finite number of terms. □

Proof. (of Proposition 6.10)

The proof is reduced to a precise formulathat can be established by an inductive manner; For an ordered set

$$\mathbf{k} = \{q_1, \cdots, q_k\} \subset \{1, \cdots, n+1\} \tag{6.20}$$

we introduce a rational function

$$\varphi_{\mathbf{k}}(\mathsf{t}) = \prod_{r=1}^{|\mathbf{k}|} \left(\frac{(\mathsf{t}^{C^{(r)}_{q_1,\cdots,q_r}} - 1)}{(\mathsf{t}-1)} \right)^{(-1)^{r-1}}, \quad 1 \leq k = |\mathbf{k}| \leq n+1$$

that is in fact a polynomial from $\mathbb{Z}[\mathsf{t}]$ due to Lemma 6.11. Here the exponents shall be intepreted as follows: $C^{(1)}_q = h.c.f.(B_q, \gamma)$ and $C^{(r)}_{q_1,\cdots,q_r} = h.c.f(C^{(1)}_{q_1}, \cdots, C^{(1)}_{q_r})$, for $r = 2, \cdots, n+1$. We shall remark that $C^{(n+1)}_{1,\cdots,n+1} = 1$ by assumption (6.12). We then have a formula o for the polynomial $\varphi(\mathsf{t})$ (6.19)

$$\varphi(\mathsf{t}) = (\mathsf{t}-1) \prod_{\mathbf{k}} \varphi_{\mathbf{k}}(\mathsf{t}). \tag{6.21}$$

where the index $\mathbf{k}$ runs over all ordered sets (6.20) such that $\{\mathsf{t}; \varphi_{\mathbf{k}}(\mathsf{t}) = 0\} \cap \{\mathsf{t}; \varphi_{\mathbf{k}'}(\mathsf{t}) = 0\} = \emptyset \iff \mathbf{k} \neq \mathbf{k}'$. We apply Lemma 6.11 to (6.17), (6.18), (6.19) in taking into account the formula (6.21).

□

For the polynomials introduced in (6.17),(6.18) we define two vectors $(\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_{\bar\gamma})$, $(\mathcal{B}_1, \mathcal{B}_2, \cdots, \mathcal{B}_{\bar\gamma}) \in \mathbb{Z}^{\bar\gamma}$, after the following relations:

$$\bar{X}_\infty(\mathsf{t}) = \mathsf{t}^{\bar\gamma} + \mathcal{A}_1 \mathsf{t}^{\bar\gamma-1} + \mathcal{A}_2 \mathsf{t}^{\bar\gamma-2} + \cdots + \mathcal{A}_{\bar\gamma},$$

$$\bar{X}_0(\mathsf{t}) = \mathsf{t}^{\bar\gamma} + \mathcal{B}_1 \mathsf{t}^{\bar\gamma-1} + \mathcal{B}_2 \mathsf{t}^{\bar\gamma-2} + \cdots + \mathcal{B}_{\bar\gamma}.$$

An examination of the elements of $\alpha^- \in C^-(1,0) \setminus C^0(1,0)$ leads us to conclude that $X_\infty(\mathsf{t})$ is a product of $(\mathsf{t}-1)^n$ and a factor vanishing on a set of non-real complex numbers with rational arguments symmetrically located with respect to the real axis. This means that $\mathcal{A}_{\bar\gamma} = (-1)^n$. From the symmetry of the set $(C^+ \setminus C^0(1,0)) \setminus \{1/2\}$ with respect to $1/2$ it follows that $\mathcal{B}_{\bar\gamma} = 1$. In other words, for every $r \leq \frac{\sharp|(C^+ \setminus C^0(1,0)) \setminus \{1/2\}|}{2}$ and index set $\mathcal{I}(r) \subset \{1, \cdots, \sharp|(C^+ \setminus C^0(1,0)) \setminus \{1/2\}|\}$ such that $\sharp\mathcal{I}(r) = r$ we can find an unique index set $\mathcal{I}'(r) \subset \{1, \cdots, \sharp|(C^+ \setminus C^0(1,0)) \setminus \{1/2\}|\} \setminus \mathcal{I}(r)$, $\sharp\mathcal{I}'(r) = r$ so that

$$\sum_{j \in \mathcal{I}(r)} \alpha^+_j + \sum_{j' \in \mathcal{I}'(r)} \alpha^+_{j'} = r. \tag{6.22}$$

A theorem due to A.H.M. Levelt (see [23], [5]) tells us that the global monodromy representation of the solution space $Ker\ \bar{R}_{(1,0)}(t, \vartheta_t)$ with irreducible monodromy can be recovered from polynomials (6.17),(6.18).

Lemma 6.12. *The global monodromy group $\bar{H}_{\gamma,\mathbf{B}}$ of the solution space $Ker\ \bar{R}_{(1,0)}(t, \vartheta_t)$ is generated by*

$$h_\infty = \begin{pmatrix} 0 & 0 & \cdots & 0 & (-1)^{n+1} \\ 1 & 0 & \ddots & 0 & -\mathcal{A}_{\bar\gamma-1} \\ 0 & 1 & \ddots & 0 & -\mathcal{A}_{\bar\gamma-2} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -\mathcal{A}_1 \end{pmatrix}, \tag{6.23}$$

$$(h_0)^{-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & \ddots & 0 & -\mathcal{B}_{\bar{\gamma}-1} \\ 0 & 1 & \ddots & 0 & -\mathcal{B}_{\bar{\gamma}-2} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -\mathcal{B}_1 \end{pmatrix}. \tag{6.24}$$

Here h_0 (resp. h_∞) corresponds to the monodromy action around a loop turning anticlockwise around $t = 0$ (resp. $t = \infty$). The monodromy action around a point $t = 1$ is given by $h_1 = (h_0 h_\infty)^{-1}$.

Proposition 6.13. *For $u^\ell x^{\mathbf{J}} \in R_f^+$ such that $\bar{\lambda}(u^\ell x^{\mathbf{J}}) = \bar{\lambda}(u^k x^{k\alpha(n+2)})$ we have the following relation between corresponding differential operators (6.6):*

$$\bar{R}_{(\ell,\mathbf{J})}(t, \vartheta_t) = \bar{R}_{(k,k\alpha(n+2))}(t, \vartheta_t) = \bar{R}_{(1,0)}(t, \vartheta_t + \frac{k}{\gamma}).$$

The monodromy representation of the solution space to the equation

$$\bar{R}_{(\ell,\mathbf{J})}(t, \vartheta_t)u(t) = 0$$

is equivalent to that for $ker\bar{R}_{(1,0)}(t, \vartheta_t)$ up to exponent shifts

$$\alpha^+ \to \alpha^+ + \frac{k}{\gamma}, \quad \alpha^- \to \alpha^- + \frac{k}{\gamma}.$$

The proof follows from the representation of the set $C^-(\ell, \mathbf{J})$ in this situation obtained in the proof of Proposition 6.4, 4).

Let us denote by $\omega^i, i = 0, 1, 2, \cdots, \gamma - 1$ the non-zero singular points of the equation (6.4).

Theorem 6.14. *There is a $\bar{\gamma}$ dimensional subspace (i.e. $Ker\ \bar{R}_{(1,0)}(s^\gamma, \vartheta_s/\gamma)$) of the solution space $ker R_{(1,0)}(s, \vartheta_s)$ ($k = 1, \mathbf{J} = 0$ in (6.3)) whose global monodromy group $\bar{H}_{\gamma,\mathbf{B}}$ is given by generators*

$$M_{\omega^0} = h_1 = (h_0 h_\infty)^{-1}, M_\infty = h_\infty^\gamma, M_{\omega^i} = h_\infty^{-i} h_1 h_\infty^i (i = 1, 2, \cdots, \gamma - 1),$$

for the matrices h_0, h_∞, h_1 defined in Lemma 6.12. Here M_{ω^i} denotes the monodromy action around the point $\omega^i \in \mathbb{P}^1_s$.

In particular $\bar{H}_{\gamma,\mathbf{B}}$ is a subgroup of $GL(\bar{\gamma}, \mathbb{Z})$ and $h_1^2 = id$ for n : odd.

Proof. The monodromies of the solutions annihilated by $\bar{R}_{(1,0)}(t, \vartheta_t)$ are given by h_0, (resp. h_1, h_∞) after Lemma 6.12 at $t = 0$, (resp. $t = 1, \infty$). Let us think of a $\gamma-$leaf covering $\tilde{\mathbb{P}}^1_t$ of $\mathbb{P}^1_s$ that corresponds to the Kummer covering $s^\gamma = t$.

For the solution space $Ker\ \bar{R}_{(1,0)}(s^\gamma, \vartheta_s/\gamma)$ its monodromy can be described as follows. In lifting up the path around $t = 1$ the first leaf of $\tilde{\mathbb{P}}^1_s$, the monodromy h_1 is sent to the conjugation with a path around $t = \infty$. That is to say we have $M_{\omega^1} = h_\infty^{-1} h_1 h_\infty$. For other leaves the argument is similar (Reidemeister-Schreier method).

The monodromy around $s = 0$ would be h_0^γ but in fact this is an identity matrix in view of (6.18). This fact matches Theorem 5.1, 1) stating that all the period integrals (3.1) are holomorphic near $s = 0$.

The statement that it is the subgroup of $GL(\bar{\gamma}, \mathbb{Z})$ follows from Proposition 6.10. $\square$

An element h of the monodromy group $\bar{H}_{\gamma,\mathbf{B}}$ acts naturally on the space of $\bar{\gamma}\times\bar{\gamma}$-matrices by

$$X \mapsto h^T \cdot X \cdot \bar{h},$$

where h^T is the transpose of h and $\bar{h}$ the complex conjugate to h . The following is a corollary of Proposition 6.8 and Theorem 6.14:

Corollary 6.15. *There exists a non degenerate Hermitian invariant* $\bar{\mathfrak{X}}$ *such that*

$$h^T \cdot \bar{\mathfrak{X}} \cdot h = \bar{\mathfrak{X}}$$

for every $h \in \bar{H}_{\gamma,\mathbf{B}} \subset GL(\bar{\gamma}, \mathbb{Z})$.

The signature (σ^+, σ^-) *of* $\mathfrak{X}$ *is given by 1)* $|\sigma^+ - \sigma^-| = 0$ *for* n *: even, 2)*

$$|\sigma^+ - \sigma^-| = \tau(\bar{Z}_f)$$

that is the index of the variety $\tau(\bar{Z}_f)$ *for* n *: odd.*

Proof. To see the existence of a non degenerate Hermitian invariant $\bar{\mathfrak{X}}$ we apply [5, Theorem 4.3] to our situation. It would be enough to recall the condition (6.22) for $\bar{X}_0(\mathsf{t})$ and an analogous symmetry condition for the roots of $\bar{X}_\infty(\mathsf{t})$. It is also possible to repeat the argument [34, §3] that can be applied to our situation almost verbatim.

In combining Proposition 6.8 formulated for the value (6.16) and [5, Theorem 4.5] we see that the signature (σ^+, σ^-) of the generating quadratic invariant is given by

$$|\sigma^+ - \sigma^-| = \sum_{\ell=1}^{n} (-1)^\ell h^{\ell, n+1-\ell},$$

while $h^{n+1,0} = h^{0,n+1} = 0$ [2, Proposition 5.3]. The symmetry of Hodge numbers $h^{p,q} = h^{q,p}$ establishes the result 1). The result 2) is nothing but the definition of the index $\tau(\bar{Z}_f)$. $\square$

Remark 6.16. Corollary 6.15 means that for n :even (the special eigenvalue of $h_1 = 1$) the monodromy group $\bar{H}_{\gamma,\mathbf{B}}$ is isomorphic to a subgroup of $O(\frac{\bar{\gamma}}{2}, \frac{\bar{\gamma}}{2})$ up to real conjugate isomorphism. In other words it is isomorphic to a subgroup of $Sp(\frac{\bar{\gamma}}{2}, \mathbb{C})$ up to a complex conjugate isomorphism. For n : odd (the special eigenvalue of $h_1 = -1$) it is isomorphic to a subgroup of $O(\bar{\gamma}, \mathbb{C})$ up to complex conjugate isomorphism. This result can be obtained by means of an application of [5, Proposition 6.1] to our situation. Again $\bar{H}_{\gamma,\mathbf{B}}$ is self dual in the sense of [5] thanks to the condition (6.22) for $\bar{X}_0(\mathsf{t})$ and an analogous symmetry condition for the roots of $\bar{X}_\infty(\mathsf{t})$.

7. Weighted projective space $\mathbb{P}_\mathbf{B}$

Let $\mathbf{N}$ be the dual lattice to $\mathbf{M}$ introduced in (2.1). We define the polar polyhedron $\Delta^o(f_0) \subset \mathbf{N}_\mathbb{R}$ of the Newton polyhedron $\Delta(f_0)$

$$\Delta^o(f_0) = \{v \in \mathbf{N}_\mathbb{R}; < v, \alpha > \geq -1, \forall \alpha \in \Delta(f_0)\}.$$

Lemma 7.1. *We denote by* $\mathfrak{A}_j$ *the* $j-th$ *column vector of the upper* $n \times (n+2)$ *part of the matrix* L^{-1} *inverse to* L *(2.7). The polar polyhedron* $\Delta^o(f_0)$ *is represented as the convex* *hull of vectors*

$$\{\frac{\gamma\mathfrak{A}_1}{B_1},\cdots,\frac{\gamma\mathfrak{A}_{n+1}}{B_{n+1}}\}.$$

The lemma can be seen from the following relations that hold for every $j \in [1; n+1]$,

$$< \alpha_0(i), \frac{\gamma\mathfrak{A}_j}{B_j} >= -1 + \frac{\gamma}{B_j}\delta_{i,j}, \forall i \in [1;n], \;\; < \alpha_0(n+1), \frac{\gamma\mathfrak{A}_j}{B_j} >= -1 + \frac{\gamma}{B_{n+1}}.$$

The normal fan of $\Delta^o(f_0)$ is generated by cones over the proper faces of $\Delta(f_0)$ [9, Lemma 3.2.1]. Every cone of the interior point fan of $\Delta(f_0)$ is generated by $(n-k)-$tuples of $\{\alpha_0(j)\}_{j=1}^{n+1}$ satisfying

$$\sum_{j=1}^{n+1} B_j \alpha_0(j) = 0 \tag{7.1}$$

for $k \in [0; n-1]$. In fact we have seen that $\alpha(n+2) = \sum_{j=1}^{n+1} \frac{B_j\alpha(j)}{\gamma}$ during the proof of Proposition 6.4, 2).

This means that the toric variety $\mathbb{P}_{\Delta^o(f_0)}$ is nothing but the weighted projective space $\mathbb{P}_{\mathbf{B}}$ under the condition (6.12). It is known that the toric variety $\mathbb{P}_{\mathbf{B}}$ is a Fano variety if and only if $\frac{\gamma}{B_j} \in \mathbb{Z}, \forall j \in [1; n+1]$, [9, Lemma 3.5.6]. Lemma 7.1 yields that if the $\Delta^o(f_0)$ is an integral polyhedron then the weighted projective space $\mathbb{P}_{\mathbf{B}}$ is a Fano variety. This means that $\Delta(f_0)$ is a reflexive polytope. See [3, Theorem 4.1.9].

Now we examine the relation between the Stokes matrix for the oscillating integral (4.2), (4.9) and the Gram matrix of the full exceptional collection on $\mathbb{P}_{\mathbf{B}}$.

First we recall that the monodromy group $H_{\gamma,\mathbf{B}}$ in $GL(\gamma, \mathbb{Z})$ of $Ker\ \tilde{R}_{1,0}(t, \vartheta_t)$ (6.6) is generated by two matrices ([34, Theorem 1.1])

$$H_\infty = \begin{pmatrix} 0 & 0 & \dots & 0 & (-1)^n \\ 1 & 0 & \dots & 0 & -\mathfrak{B}_{\gamma-1} \\ 0 & 1 & \dots & 0 & -\mathfrak{B}_{\gamma-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -\mathfrak{B}_1 \end{pmatrix} \tag{7.2}$$

and

$$H_0^{-1} = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \tag{7.3}$$

where

$$\prod_{q=1}^{n+1}(\mathsf{t}^{B_q} - 1) = \mathsf{t}^\gamma + \mathfrak{B}_1\mathsf{t}^{\gamma-1} + \mathfrak{B}_2\mathsf{t}^{\gamma-2} + \cdots + (-1)^{n+1} \tag{7.4}$$

is the characteristic polynomial of the monodromy at infinity. It is worthy noticing that $H_{\gamma,\mathbf{B}}$ admits a reducible monodromy representation and the Levelt type theorem [5, Theorem 3.5] cannot be directly applied to $\tilde{R}_{1,0}(t,\vartheta_t)$. The validity of the monodromy representation (7.2), (7.3) is based on the existence of a vector v that is cyclic with respect to H_0 satisfying

$$H_0^i v = H_\infty^{-i} v, \qquad i \in [1;\gamma-1]. \tag{7.5}$$

See [34, Proposition 2.3].

Let $(\mathcal{E}_i)_{i=1}^{\gamma}$ be the full strong exceptional collection on $D^b\,\mathrm{coh}\,\mathbb{P}_{\mathbf{B}}$ given as

$$(\mathcal{E}_1,\ldots,\mathcal{E}_\gamma) = (\mathcal{O},\ldots,\mathcal{O}(\gamma-1)),$$

and $(\mathcal{F}_1,\ldots,\mathcal{F}_\gamma)$ be its right dual exceptional collection characterised by the condition

$$\mathrm{Ext}^k(\mathcal{E}_{\gamma-i+1},\mathcal{F}_j) = \begin{cases} \mathbb{C} & i=j, \text{ and } k=0,\\ 0 & \text{otherwise.}\end{cases}$$

In other words

$$\chi(\mathcal{E}_{\gamma-i+1},\mathcal{F}_j) = \delta_{ij}$$

where

$$\chi(\mathcal{E},\mathcal{F}) = \sum_k (-1)^k \dim \mathrm{Ext}^k(\mathcal{E},\mathcal{F}) \tag{7.6}$$

is the Euler form. Note that $\mathcal{F}_1 = \mathcal{O}_{\mathbf{P_B}}(-1)[n]$ and $\mathcal{F}_\gamma = \mathcal{E}_1 = \mathcal{O}_{\mathbf{P_B}}$.

We construct a hypersurface Y of weighted degree $\gamma = |\mathbf{B}|$ in $\mathbb{P}_{\mathbf{B}}$ by means of a "transposition" of the Newton polyhedron $\Delta(f_0)$. First we consider a $n\times(n+1)$ matrix defined by columns $\alpha_0(j) = \alpha(j)-\alpha(n+2), j\in[1;n+1]$ for (2.2) whose rows we denote by $\mathfrak{b}(i), i\in[1;n]$;

$$[\alpha_0(1),\cdots,\alpha_0(n+1)] = \begin{bmatrix} \mathfrak{b}(1)\\ \vdots \\ \mathfrak{b}(n)\end{bmatrix}. \tag{7.7}$$

The polynomial with generic coefficients below, constructed from (7.7), is weighted homogenous with respect to the weight system $w(y_q) = B_q, q\in[1;n+1]$ with weight $|\mathbf{B}| = \gamma$;

$$f^T(y) = y^{\mathbf{1}}(\sum_{i=1}^{n} b_i y^{\mathfrak{b}(i)} + b_{n+1}). \tag{7.8}$$

This can be seen from the relations $<\mathfrak{b}(i),\mathbf{B}>=0, \forall i\in[1;n]$, $<\mathbf{1},\mathbf{B}>=\gamma$.

Let $Y\subset\mathbb{P}_{\mathbf{B}}$ a hypersurface defined by a weighted polynomial of weight $\gamma=|\mathbf{B}|$;

$$Y = \{y\in\mathbb{P}_{\mathbf{B}}; f^T(y)=0\}.$$

If it is smooth, then it is a Calabi-Yau manifold. Under the standard definition of the Poincaré polynomial $P_Y(\mathtt{t})$ of a weighted homogenous hypersurface ([12, 3.4]) the following equality is established;

$$P_Y(\mathsf{t}) = \frac{(1-\mathsf{t}^{\gamma})}{\prod_{q=1}^{n+1}(1-\mathsf{t}^{B_q})} = (-1)^n \frac{X_\infty(\mathsf{t})}{X_0(\mathsf{t})} = (-1)^n \frac{\bar{X}_\infty(\mathsf{t})}{\bar{X}_0(\mathsf{t})}. \tag{7.9}$$

In considering the derived restrictions $\{\bar{\mathcal{F}}_i\}_{i=1}^{\gamma}$ of $\{\mathcal{F}_i\}_{i=1}^{\gamma}$ to Y that split-generate the derived category $D^b \operatorname{coh} Y$ of coherent sheaves on Y, we see that the Stokes matrix for (4.9), (4.10) is given by

$$S_{ij} = (\sigma_i, \sigma_j) = \chi(\bar{\mathcal{F}}_i, \bar{\mathcal{F}}_j) = \chi(\mathcal{F}_i, \mathcal{F}_j) + (-1)^{n-1}\chi(\mathcal{F}_j, \mathcal{F}_i) = \chi(\mathcal{F}_i, \mathcal{F}_j)$$

for $i < j$, $S_{ii} = 1$ and $S_{ij} = 0$ for $i > j$. These numbers are given as intersection number of vanishing cycles used to define Lefschetz thimbles in Definition 4.1. The $\gamma \times \gamma$ matrix

$$\mathfrak{X} = \big(\chi(\bar{\mathcal{F}}_i, \bar{\mathcal{F}}_j)\big)_{i,j=1}^{\gamma} = \big(S_{ij} + (-1)^{n-1}S_{ji}\big)_{i,j=1}^{\gamma} \tag{7.10}$$

corresponds to the Hermitian invariant of the monodromy group $H_{\gamma,\mathbf{B}}$ ([34, Proposition 4.1]) satisfying,

$$h^T \cdot \mathfrak{X} \cdot \bar{h} = \mathfrak{X}$$

for every $h \in H_{\gamma,\mathbf{B}} \subset GL(\gamma, \mathbb{Z})$. We recall that we can assume $h = \bar{h} \in H_{\gamma,\mathbf{B}}$ by virtue of (7.2), (7.3), (7.4). The space of Hermitian invariants of $H_{\gamma,\mathbf{B}}$ is one dimensional and generated by (7.10).

In summary, in applying [34, Theorem 5.1] to our situation we get the following.

Theorem 7.2. *1) The Stokes matrix $(S_{ij})_{i,j=1}^{\gamma}$ for the quantum cohomology of the weighted projective space $\mathbb{P}_{\mathbf{B}}$ is equivalent to that for the oscillating integral (4.9).*

2) This Stokes matrix is given by the Gram matrix of the full exceptional collection $(\mathcal{F}_i)_{i=1}^{\gamma}$ on $\mathbb{P}_{\mathbf{B}}$ with respect to the Euler form;

$$S_{ij} = \chi(\mathcal{F}_i, \mathcal{F}_j). \tag{7.11}$$

This generalises a conjecture proposed by Dubrovin [14] for Fano manifolds that has been proven first by D.Guzzetti [18] for the case of the projective space (see [31] also). H.Iritani [20, Remark 4.13] mentions the correspondence between Lefschetz thimbles Γ_i and exceptional collection of coherent sheaves $\mathcal{F}_i$ for the case of a weighted projective space. This is a consequence of the assertion that there exist $\mathcal{G}_1, \cdots, \mathcal{G}_\gamma$ in the Grothendieck group $K(\mathbb{P}_{\mathbf{B}})$ such that $\chi(\mathcal{G}_i, \mathcal{G}_j) = S_{ij}$ [20, Theorem 4.11, Corollary 4.12]. Thus the above Theorem 7.2 can be considered as a concrete realisation of Iritani's theorem. In view of the formulation of Gamma conjectures [16, Definition 4.6.1] it would be desirable to give a newly adapted version of the above Theorem 7.2.

Remark 7.3. *The situation explained in this section can be summarised into a diagram as follows:*

$$\begin{array}{ccc} \gamma = rank H^n(\mathbf{T}^n \setminus Z_{f_0}), f_0 : LG\ potential & \Longleftrightarrow & \mathbb{P}_{\mathbf{B}}, rank\ K(\mathbb{P}_{\mathbf{B}}) = \gamma \\ \bar{\gamma} = rank PH^{n-1}(\bar{Z}_{f_0}), Delsarte : \bar{Z}_{f_0} \subset \mathbb{P}_{\Delta(f_0)} & \Longleftrightarrow & C.Y. : Y \subset \mathbb{P}_{\mathbf{B}}, rank\ \iota_* K(\mathbb{P}_{\mathbf{B}}) = \bar{\gamma}. \end{array}$$

For $\iota : Y \hookrightarrow \mathbb{P}_{\mathbf{B}}$ the inclusion we denoted with $\iota_* K(\mathbb{P}_{\mathbf{B}})$ the subgroup of $K(\mathbb{P}_{\mathbf{B}})$ generated by $\{[\iota_* \mathcal{O}_Y(i)]\}_{i \in \mathbb{Z}}$. The correspondence " $\Longleftrightarrow$ " indicates mirror symmetry in certain *sense (Givental' $I = J$ mirror* [7], [20]*, Batyrev dual polytope mirror* [3] *or expected homological mirror symmetry).*

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