Some Estimates for Plane Cuspidal Curves

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Introduction

Let $\bar{D}$ be an irreducible curve in $\mathbf{CP}^2$ of degree d and of genus g. A classical question is what set of singularities may it have. A necessary condition on the set of singularities is given by the genus formula:

$$(d-1)(d-2) = 2g + \sum_{i=1}^{s}(\mu_i + r_i - 1), \tag{1}$$

where μ_i and r_i are respectively the Milnor number and the number of analytical branches at p_i (by $p_1, ..., p_s$ we denote the singular points of $\bar{D}$). Of course, this condition is far from being sufficient.

One of the most powerful methods for obtaining other restrictions (see [**H**]) is applying some variants of the Bogomolov-Miyaoka-Yau (BMY) inequality. This paper is devoted to some computations that allow one to apply the BMY inequality in the following logarithmic form (see [**Miy**,Corollary 1.2]). Suppose that $\bar{\kappa}(\mathbf{CP}^2 - \bar{D}) \geq 0$. (It is the case, for example, when $\bar{D}$ has more than one singular point, or $g \geq 1$ and $d \geq 4$; see [**W**].) Let $s : V \to \mathbf{CP}^2$ be the minimal resolution of the singularities of $\bar{D}$ and $D = \sigma^{-1}(\bar{D})$. Then

$$(K + D)^2 \leq 3e(\mathbf{CP}^2 - \bar{D}), \tag{2}$$

where $K = K_V$ is the canonical divisor and e is the Euler characteristic.

Using the genus formula (1) one can represent the right hand side of (2) as

$$e(\mathbf{CP}^2 - \bar{D}) = (d^2 - 3d + 3) - \sum_{i=1}^{s} \mu_i. \tag{3}$$

After decomposing $\operatorname{Pic} V \otimes \mathbf{Q}$ into a direct sum of pairwise orthogonal summands, one of which corresponds to a generic line and the others correspond to singular points, one can represent the left hand side of (2) as

$$(K + D)^2 = (d-3)^2 + \sum_{i=1}^{s}(K_i + D_i)^2. \tag{4}$$

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It is easy to see that $(K_i + D_i)^2$ depends only on the weighted graph of the minimal resolution of the singularity of $\bar{D}$ at p_i.

Combining (3) and (4), one can write (2) as

$$\sum_{i=1}^{s} \alpha_i \mu_i \leq d^2 - \frac{3}{2} d, \tag{5}$$

where $\alpha_i = 3/2 + (K_i + D_i)^2/2\mu_i$ are local characteristics of the singularities. It is easy to show that $(K_i + D_i)^2 > -\mu_i$ (see Proposition 6.1 below) and hence, $\alpha_i > 1$.

In Theorem 6.2 below we prove that if $\bar{D}$ is analyticaly irreducible (one place) at p_i, then

$$\alpha_i > 1 + \frac{1}{2m_i},$$

where m_i is the multiplicity of $\bar{D}$ at p_i. As a corollary we obtain

Theorem A. *Let $\bar{\kappa}(\mathbf{CP}^2 - \bar{D}) \geq 0$. Suppose that each singularity of $\bar{D}$ is analytically irreducible (one place) and has multiplicity not greater than m. Then*

$$\sum_{i=1}^{s} \mu_i \leq \frac{2m}{2m+1} (d^2 + \frac{3}{2} d).$$

Corollary B. *In the hypothesis of Theorem A*

$$g \geq \frac{d^2 - (9m+3)d}{2(2m+1)} + 1 \qquad .$$

Remarks.

1. For $m = 3$ Theorem A was proved by Yoshihara [**Y1**] by the same method and his paper, as well as the question posed in [**LZ**], inspired us for this work. Similar results are obtained in [**Y2**] by a different method.

2. In [**MS**] it is proved that if $\bar{D}$ as above is rational, then $d < 3m$, which is better than the estimate $d \leq 9m + 2$ given by Corollary B.

Inequality (2) can be strengthened using the Zariski decomposition $K_V + D = H + N$. One of the consequences of the result of [**KNS**] is that $(K + D)^2$ in (2) can be replaced by H^2. (Since $N^2 < 0$, we obtain a stronger inequality.) Projecting H onto the direct summands of $\operatorname{Pic} V \otimes \mathbf{Q}$, we may replace the "local terms" $(K_i + D_i)^2$ in (3) by H_i^2. It follows from Fujita's peeling theory [**F**] that if $\bar{D}$ satisfies some additional conditions then the H_i also depend only on local properties of the singularities. An example of such conditions is given by Theorem 1.2 below.

We compute the $(K_i + D_i)$ and the H_i^2 in terms of discriminants of subgraphs of the resolution graph, and if the singularity is analytically irreducible, in terms of Puiseux characteristic pairs. An analogue of characteristic pairs for an analytically reducible singularity (more than one places) is a notion of a splice diagram introduced by Eisenbud and Neumann [**EN**] (see Remark 2.5). It turns out that the above H_i^2 (but not the $(K_i + D_i)^2$)

are rational functions of the weights of the splice diagrams (we write them explicitly in Corollary 2.4). It is not evident a priori, because though the weighted graph of the minimal resolution (and hence, H_i^2) is uniquely defined by the splice diagram, the process of its reconstruction is rather complicated (it involves solving several diophantine equations, developments into continuous fractions, etc.)

In fact the proof of Theorem A does not use the Zariski decomposition, peeling theory (see Sect. 1), Fujita's theorem (Theorem 2.1) etc., and though the terms H_i and N_i are involved in the proofs, they have purely formal meaning and are used just because the formulas for them are convenient for computations. However, using these formulas and Fujita's theorem one can obtain some more strong estimates. For example, if $\bar{D}$ satisfies the hypothesis of Theorem 1.2 and all its singularities are usual cusps ($x^2 + y^3 +$ (higher terms) $= 0$), then $H_i^2 = -1/6$ and one can strengthen the estimates of Theorem A and Corollary B as

$$\sum \mu_i \le \frac{24}{35}(d^2 + \frac{3}{2}d) \quad \text{and} \quad g \ge \frac{11d^2 - 141d}{70} + 1$$

and if $g = 0$, then $d \le 12$ (in fact, in this case $d \le 5$ by [MS]).

Conjecture. *The statements of Theorems 6.2 and A are true without the assumption of analytical irreducibility.*

In Sect.7 we show how to modify the local formulas for the case of a plane affine curve with one place at infinity.

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1. Local peeling and Zariski's decomposition of $K + D$

Let $D = E \cup C$ be a reduced curve with simple normal crossings on a smooth rational projective surface V, such that the dual graph of E is a rational tree, all irreducible components of E are linearly independant in $\operatorname{Pic} V \otimes \mathbf{Q}$ and the intersection form of E is non-degenerate.

As usual, the *dual graph of* E is a weighted graph Γ_E, whose vertices are the irreducible components of E, the edges are the intersection points and the weights are the self-intersection numbers. Besides the dual graph of E we shall also consider the *dual graph of C near E*. It is a partially weighted graph $\Gamma_{C,E}$, whose weighted part coincides with Γ_E and whose non-weighted vertices are the intersection points of E and C (recall that all intersections are transversal). Such a vertex is joined by an edge with that component of E, on which the corresponding intersection point lies. Example: if E and C are respectively a line and a conic in $\mathbf{CP}^2$ in general position, then $\Gamma_{C,E}$ is a linear chain of three vertices, where the middle one has the weight $+1$.

A partially weighted graph Γ will be called *an extended weighted graph* if all its non-weighted vertices are ends. It is clear that the above $\Gamma_{C,E}$ satisfies this definition.

By definition, the *discriminant* $d(\Gamma)$ of a weighted graph Γ is $\det(-A)$, where A is the incidence matrix of Γ (the intersection matrix of E, if $\Gamma = \Gamma_E$). The discriminant of a partially weighted graph is defined to be the discriminant of its weighted part. The discriminant of an empty graph is $+1$. The *valency* of a vertex in a graph is the number of incident edges. The vertex is said to be *an end, linear* or *branch point* (or *node*) *of* Γ, if its valency is $1, 2$ or > 2 respectively.

A subgraph L of a partially weighted graph Γ is called *a twig* if all its vertices are weighted, all but one are linear in Γ, one vertex (which is called *a tip of* L) is an end of Γ and another end of L is incident to a branch point of Γ. This branch point is called the *root* of L (remark: $\operatorname{root}(L) \notin L$). A twig is said to be *admissible* if the weights of all its vertices are ≤ -2. In this case its intersection form is negatively definite. The *inductance* of a twig L is defined as

$$\operatorname{ind}(L) = d(L - \operatorname{tip}(L))/d(L).$$

Denote by $\operatorname{Vect}(E)$ the subspace of $\operatorname{Pic} V \otimes \mathbf{Q}$ spanned by the irreducible components of E, and let $\operatorname{pr}_E : \operatorname{Pic} V \otimes \mathbf{Q} \to \operatorname{Vect}(E)$ be the orthogonal projection (with respect to the intersection form). Denote: $C_E = \operatorname{pr}_E(C)$, $D_E = \operatorname{pr}_E(D)$ and $K_E = \operatorname{pr}_E(K_V)$, where K_V is the canonical divisor of V.

We shall assume, that $\Gamma_{C,E}$ is minimal, i.e. it does not contain linear or end vertices weighted by -1.

By *a local peeling* (or *Fujita decomposition*) of $K + D$ near E we shall mean a decomposition $K_E + D_E = H_E + N_E$, where $\operatorname{supp}(N_E)$ is contained in the union of all twigs of $\Gamma_{C,E}$, and H_E is orthogonal to each irreducible component of N_E.

It is not difficult to prove that if all the twigs are admissible, then the decomposition is uniquely defined [**Fu**].

Lemma 1.1. *[***Fu***, (6.16)]. $-N_E^2$ is equal to the sum of inductances of all twigs of $\Gamma_{C,E}$.*

Now we describe the relation of the local peeling with the Zariski decomposition of $K + D$. Let us return to the notation from the introduction. We denote also: $E^{(i)} = \sigma^{-1}(p_i)$, $L = \sigma^{-1}(\bar{L})$, where L is a generic line in $\mathbf{P}^2$. Let $H + N$ be the Zariski decomposition of $K + D$ and let $H_i + N_i$ and $H_L + N_L$ be its local Fujita decomposition near $E^{(i)}$, $(i = 1, ..., s)$ and near L, respectively. Clearly, $H_L = (d-3)L$, $N_L = 0$.

As a corollary from [**Fu**, 6.20-6.24] one can obtain the following

Theorem 1.2. *Suppose that D is a rational tree on the surface V such that*

i) $X = V - D$ is affine,

ii) $\bar{\kappa}(X) \geq 0$,

iii) the dual graph $\Gamma(D)$ is minimal and has at least two branching points,

iv) there is no (-1)-curves F in V such that F is not contained in D and $F \cdot D = 1 = F \cdot T$ for some twig T of D.

Then $H = (d-3)L + \sum_i H_i$ and $N = \sum_i N_i$.

2. Computation of H_E^2 via discriminants of branches of the dual graph

All the notation from the previous section is preserved. Let $E_1, ..., E_n$ be irreducible components of E. Denote by $\operatorname{Vect}(E)^*$ the dual space to $\operatorname{Vect}(E)$ and let $\langle , \rangle : \operatorname{Vect}(E)^* \otimes \operatorname{Vect}(E)$ be the natural pairing. Denote by $e^* = \{E_1^*, ..., E_n^*\}$ the base in $\operatorname{Vect}(E)^*$, dual to the base $e = \{E_1, ..., E_n\}$, i.e. $\langle E_i^*, E_j \rangle = \delta_{ij}$. Denote by $A_E : \operatorname{Vect}(E) \to \operatorname{Vect}(E)^*$ the linear operator, defined by the intersection form: $\langle A_E(D_1), D_2 \rangle = D_1 \cdot D_2$. So, the matrix of A_E in the bases e and e^* is just the intersection matrix of E. Recall that we assume the intersection form to be non-degenerate on $\mathrm{Vect}(E)$. Denote A_E^{-1} by $B_E = (b_{ij})$.

Proposition 2.1.

$$b_{ij} = -d(\Gamma_{ij})/d(\Gamma_E),$$

where Γ_{ij} is the subgraph of Γ_E, obtained by deleting the shortest path between E_i and E_j (E_i and E_j also being deleted).

Proof. Apply Cramer's inverse matrix formula.■

Denote by ν_i and $\bar{\nu}_i$ $(i = 1, ..., n)$ the valencies of E_i in Γ_E and $\Gamma_{C,E}$ respectively.

Proposition 2.2.
(*i*) $A_E(K_E + E) = \sum_i(\nu_i - 2)E_i^*$.
(*ii*) $A_E(C_E) = \sum_i(\bar{\nu}_i - \nu_i)E_i^*$.
(*iii*) $A_E(K_E + D_E) = \sum_i(\bar{\nu}_i - 2)E_i^*$.

Proof. (*i*) For each E_i we have $\langle (K_E + E)^*, E_i\rangle = K_E \cdot E_i + E \cdot E_i = (K_E + E_i) \cdot E_i + \nu_i = -2 + \nu_i$.
(*ii*) By the definition $\bar{\nu}_i = \nu_i + C \cdot E_i$.
(*iii*) Add (*i*) and (*ii*). ■

For a branch point E_i of the graph $\Gamma_{C,E}$ put $c_i = \bar{\nu}_i - 2 - \sum 1/d(L)$, where the summation is over all twigs such that $\mathrm{root}(L) = E_i$.

Proposition 2.3. $A_E(H_E) = \sum_{\bar{\nu}_i > 2} c_i E_i^*$.

Proof. If $\bar{\nu}_i = 1$ then E_i belongs to some twig, and by the definition of local peeling $H_E \cdot E_i = 0$.

If $\bar{\nu}_i = 2$ then either E_i lies on a twig and then $H_E \cdot E_i = 0$, or E_i does not intersect any twig and then $E_i \cdot H_E = E_i \cdot (K_E + D_E) = (\bar{\nu}_i - 2) = 0$.

If $\bar{\nu}_i > 2$, denote by $L_1, ..., L_k$ all twigs rooted by E_i, and let $E_{i_1}, ..., E_{i_k}$ be their vertices, incident with E_i. According to [Fu], the coefficient of E_{i_j} in N_E equals $1/d(L_j)$. Hence,

$$E_i \cdot H_E = E_i \cdot (K_E + D_E) - E_i \cdot N_E = (\bar{\nu}_i - 2) - \sum_{j=1}^{k} \frac{1}{d(L_j)} = c_i.$$

■

Corollary 2.4.

$$H_E^2 = \sum b_{ij} c_i c_j, \tag{6}$$

where the summation is over all pairs (i, j), such that $\bar{\nu}_i > 2$ and $\bar{\nu}_j > 2$.

Proof. Let $Y = A_E(H_E)$. Then $H_E^2 = \langle B_E Y, Y\rangle$.■

Remark 2.5. One can see that (6) represents H_E^2 as a rational function of $d(\Gamma_E)$ and of the discriminants of the branches of $\Gamma_{C,E}$ at branch points (*a branch* of a graph Γ at a vertex v is a connected component of $\Gamma - v$). This rational function depends only on the topology of $\Gamma_{C,E}$.

In [EN] the notion of *splice diagram* was introduced. A splice diagram is a graph, some of whose ends are marked as arrowheads, all nodes have weights ± 1, and all ends of edges near nodes (more formally speaking, all pairs (n, e) where n is a node and e is an edge, incident to n) are weighted by integers. To each unimodular (with discriminant ± 1) extended weighted graph Γ corresponds a splice diagram Δ which is constructed from Γ as follows: Replace each linear chain of Γ by a single edge, weight all the nodes by the same integer $d(\Gamma)$ and weight each end of edge at a node by the discriminant of the correspoding branch of Γ at the node. It is shown in [EN] that if the intersection form of Γ is negative definite, then the minimal extended graph is uniquely defined by the splice diagram.

For singularities of plane curves, the notion of a splice diagram is a generalization of the notion of Puiseux pairs (see Proposition 5.1). It is easy to see that the right hand side of (6) is a rational function of the weights of the splice diagram.

Remark 2.6. The same arguments applied to a resolution of a singulariry of a plane curve provides us a formula for the Milnor number of the singularity in terms of the weights of the splice diagram:

$$\mu = 1 - \langle B_E Y, Z \rangle = 1 - \sum_{i,j} b_{ij}(\nu_i - 2)(\bar{\nu}_j - \nu_j),$$

where $Y = A_E(K_E + D_E)$, $Z = A_E(C_E)$. Milnor's formula [Mil,p.93] (see also (13) below) can be easily deduced from the above in the same way that (11) is deduced from (9) in Corollary 5.3 below.

3. Some lemmas on discriminants of graphs

The next two lemmas are well known.

Lemma 3.1. *Let L be a linear weighted graph with vertices $v_1, \ldots, v_k$ (v_1 and v_k being the ends of L). Then*

$$d(L - v_1)d(L - v_k) - d(L)d(L - v_1 - v_k) = 1.$$

Proof. Use induction by the length of L and the recurrent formula:

$$d(L) = -w_1 d(L - v_1) - d(L - v_1 - v_2),$$

where w_1 is the weight of v_1. ■

Lemma 3.2. *Let Γ be a weighted tree, and $[v_1 v_2]$ its edge. Denote by Γ_1 and Γ_2 the components of $\Gamma - [v_1 v_2]$, containing v_1 and v_2 respectively. Then*

$$d(\Gamma) = d(\Gamma_1)d(\Gamma_2) - d(\Gamma_1 - v_1)d(\Gamma_2 - v_2).$$

Lemma 3.3. *Let L be a linear extremal chain of Γ with vertices $v_1, ..., v_k$, such that v_k is an end of Γ and v_1 is connected with $\Gamma - L$ by the edge $[v_0 v_1]$. Then*

$$d(\Gamma - L - v_0) = d(\Gamma - v_k)d(L) - d(\Gamma)d(L - v_k).$$

Proof. Denote:

$$d = d(L);\ d_0 = d(L - v_k);\ d' = d(L - v_1);\ d_0' = d(L - v_1 - v_k);$$

$$\Delta = d(\Gamma);\ \Delta_0 = d(\Gamma - v_k);\ A = d(\Gamma - L);\ A' = d(\Gamma - L - v_0).$$

Then, by lemma 3.2 applied to Γ and $\Gamma - v_k$ in the edge $[v_0 v_1]$, we have:

$$\Delta = Ad - A'd';\ \ \Delta_0 = Ad_0 - A'd_0'.$$

Substracting the first equality multiplied by d_0 from the second one multiplied by d, we obtain

$$\Delta_0 d - \Delta d_0 = A'(d_0 d' - dd_0').$$

But by Lemma 3.1 $d_0 d' - dd_0' = 1$. ■

Corollary 3.4. *If $d(L) \neq 0$ and $d(\Gamma) \neq 0$ then*

$$\frac{d(\Gamma - v_k)}{d(\Gamma)} = \frac{d(\Gamma - L - v_0)}{d(L)d(\Gamma)} + \operatorname{ind}(L).$$

Corollary 3.5. *If $d(\Gamma) = 1$ and all the weights of L are ≤ -2, then*

$$d(\Gamma - v_k) = \lceil a \rceil; \qquad \operatorname{ind}(L) = \lceil a \rceil - a,$$

where $a = d(\Gamma - L - v_0)/d(L)$ and by $\lceil a \rceil$ is denoted the minimal integer greater than a.

4. The case of a contractible graph

We use here the notation from Sect.2 and 3. If the graph of E is contractible, i.e. if E can be blown down by a birational morphism $\sigma_E : V \to \bar{V}$ such that $(\sigma_E)|_{V-E}$ is an isomorpfism and $\sigma_E(E)$ is a single smooth point on $\bar{V}$, then the formula for $(K_E + E)^2$ and hence, for H_E^2 can be essentially simplified: the summation over two indices can be replaced by a summation over a single index.

Proposition 4.1. *If E can be blown down to a smooth point, then*

$$(K_E + E)^2 = -2 - \sum_{i=1}^{n} b_{ii}(\nu_i - 2). \tag{7}$$

Remark 4.2. Since the intersection form on $\operatorname{Vect}(E)$ is negative definite, all the b_{ii} are negative.

Proof. Since V can be obtained from $\bar{V}$ by means of successive blow-ups, we shall use the induction with respect to the number of irreducible components of E. If E is irreducible with $E^2 = -1$, it is clear that both sides of (7) are equal to -4. Now assume that the proposition is valid for a pair (V, E) and let us prove it for $(\tilde{V}, \tilde{E})$, where $\tilde{V}$ is the result of a blowing-up $\sigma : \tilde{V} \to V$ of a point p on E, and $\tilde{E} = \sigma^{-1}(E)$. Denote the irreducible components of E and $\tilde{E}$ by $E_1, ..., E_n$ and $\tilde{E}_0, ..., \tilde{E}_n$ respectively, where $\tilde{E}_i$ is the proper transform of E_i for $i \geq 1$ and $\tilde{E}_0 = \sigma^{-1}(p)$ is the exceptional curve of σ. Let $\tilde{K}_{\tilde{E}}, \tilde{B}_{\tilde{E}} = (\tilde{b}_{ij})$ etc. be given as before with (V, E) replaced by $(\tilde{V}, \tilde{E})$.

We shall show that the blow-up changes both sides of (7) by the same quantity. Indeed,

$$\tilde{K}_{\tilde{E}} = \sigma^*(K_E) + \tilde{E}_0, \quad \tilde{E} = \sigma^*(E) - (\tilde{\nu}_0 - 1)\tilde{E}_0.$$

hence,

$$\tilde{K}_{\tilde{E}} + \tilde{E} = \sigma^*(K_E + E) - (\tilde{\nu}_0 - 2)\tilde{E}_0,$$

and since $\sigma^*(K_E + E)$ is orthogonal to $\tilde{E}_0$, we have

$$(\tilde{K}_{\tilde{E}} + \tilde{E})^2 = (\sigma^*(K_E + E))^2 + (\tilde{\nu}_0 - 2)^2 \tilde{E}_0^2 = (K_E + E)^2 - (\tilde{\nu}_0 - 2)^2.$$

Thus, $(K_E + E)^2$ decreases by 1 if p is a smooth point of E (case 1), and does not change if p is the intersection point of two components of E (case 2).

Now, let us see what happens with the right hand side of (7) in both cases.

Case 1. Without loss of generality we may assume that $p \in E_1$. Then we have: $\tilde{b}_{00} = b_{11} - 1$, $\tilde{b}_{ii} = b_{ii}$ for $i \geq 1$, $\tilde{\nu}_0 = 1$, $\tilde{\nu}_1 = \nu_1 + 1$, $\tilde{\nu}_i = \nu_i$ for $i \geq 2$. Indeed, to prove the first equality, it is enough to note that $-\tilde{b}_{00}$ is the determinant of the matrix $-A_E$ with $-a_{11}$ replaced by $-(a_{11} + 1)$, and the determinant of the complementary minor is exactly $-b_{11}$. The second equality is just the invariance of the discriminant under blowing up. The others equalities are trivial. Thus,

$$-\sum_{i=0}^{n} \tilde{b}_{ii}(\tilde{\nu}_i - 2) = (b_{11} - 1) - b_{11}(\nu_1 - 1) - \sum_{i=2}^{n} b_{ii}(\nu_i - 2) = -1 - \sum_{i=1}^{n} b_{ii}(\nu_i - 2).$$

Case 2. Without loss of generality we may assume that $p = E_1 \cap E_2$. Then we have: $\tilde{b}_{ii} = b_{ii}$ for $i \geq 1$, $\tilde{\nu}_0 = 2$, $\tilde{\nu}_i = \nu_i$ for $i \geq 1$. (As above, the first equality is just the invariance of the discriminant under blowing up.) The invariance of the right hand side of (7) is a trivial consequence of these equalities. ∎

5. The analytically irreducible case: Computation of H_E^2 via Puiseux pairs

Let $\bar{C}$ be a germ of an analytically irreducible curve at the origin in $\mathbf{C}^2$, and let

$$x = t^m, \qquad y = a_n t^n + a_{n+1} t^{n+1} + ..., \qquad a_n \neq 0,$$

be its local analytic parametrization. Put: $d_1 = m$, $m_1 = n$;

$$d_i = \gcd(d_{i-1}, m_{i-1}), \qquad m_i = \min\{\, j \mid a_j \neq 0, d_i \nmid j \,\}, \; i > 1;$$

denote by h an integer such that $d_h \neq 1$, $d_{h+1} = 1$. Thus, m_i and d_i are defined for $i = 1, ..., h$, and for $i = 1, ..., h+1$ respectively, and

$$0 < m_1 < m_2 < ... < m_h, \qquad m = d_1 > d_2 > ... > d_{h+1} = 1.$$

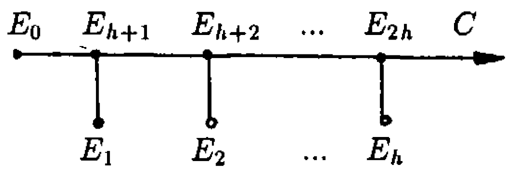
Let $q_1 = m_1$, $q_i = m_i - m_{i-1}$ for $i = 2, ..., h$, and let

$$r_i = (q_1 d_1 + ... + q_i d_i)/d_i, \ \ i = 1, ..., h. \tag{8}$$

After changing the coordinates, if necessary, we may assume that $m < n$, and under this assumption the sequence $(m; m_1, m_2, ..., m_h)$ is uniquely defined and is called *the Puiseux characteristic sequence of the singularity of* $\bar{C}$ *at* 0 (see [A], [Mil]).

Now, we shall describe the relations between the Puiseux characteristic sequence and the resolution graph. Let $\sigma : V \to \mathbf{C}^2$ be the minimal resolution of the singularity of the curve $\bar{C}$ at the origin, $E = \sigma^{-1}(0)$ and C be the proper transform of $\bar{C}$.

Proposition 5.1. *(see [EN]). a) The dual graph* $\Gamma_{C,E}$ *of* C *near* E *is the following:*



(here the vertices of valency 2 are not shown).

b) Denote by R_i, D_i *and* S_i *the connected components of* $\Gamma_{C,E} - E_{h+i}$ *which are to the left, to the bottom and to the right of the node* E_{h+i}*, respectively. Denote by* Q_i *the linear chain between* E_{h+i-1} *and* E_{h+i} *(excluding* E_{h+i-1} *and* E_{h+i}*). Then*

$$d(R_i) = \frac{r_i}{d_{i+1}}, \quad d(D_i) = \frac{d_i}{d_{i+1}}, \quad d(S_i) = 1, \quad d(Q_i) = \frac{q_i}{d_{i+1}}.$$

■

Remark. This proposition explains why splice diagram can be considered as a generalization of Puiseux characteristic sequence (see Remark 2.5).

Let μ be the Milnor number of the singularity of $\bar{C}$ at 0. Introduce for E and C the notation from Sect.2 and 3. Denote by $\lceil a \rceil$ the smallest integer greater than a.

Proposition 5.2.

i)

$$2\mu + H_E^2 = -\frac{d_1}{r_1} + \sum_{i=1}^{h} \frac{r_i}{d_{i+1}} \left(\frac{d_i}{d_{i+1}} - \frac{d_{i+1}}{d_i} \right);$$

ii)

$$N_E^2 = \frac{d_1}{r_1} - \lceil \frac{d_1}{r_1} \rceil + \sum_{i=1}^{h} \left(\frac{r_i}{d_i} - \lceil \frac{r_i}{d_i} \rceil \right); \tag{9}$$

iii)

$$2\mu + (K_E + D_E)^2 = -\lceil \frac{d_1}{r_1} \rceil + \sum_{i=1}^{h} (\frac{r_i d_i}{d_{i+1}^2} - \lceil \frac{r_i}{d_i} \rceil).$$

Proof. It is known that $\mu = -C_E \cdot (K_E + C_E + E) + 1$. Hence,

$$2\mu + H_E^2 = 2\mu + (K_E + C_E + E)^2 - N_E^2 = (K_E + E)^2 + 2 - C_E^2 - N_E^2. \qquad (10)$$

Let the irreducible components of E with $\bar{\nu}_i \neq 2$ be numerated as in the above diagram (recall that ν_i and $\bar{\nu}_i$ are the valencies of E_i in Γ_E and in $\Gamma_{C,E}$ respectively). Then $\nu_0 = ... = \nu_h = 1, \nu_{h+1} = ... = \nu_{2h-1} = 3, \nu_{2h} = 2$.

According to Proposition 4.1,

$$(K_E + E)^2 + 2 = -\sum_{i=1}^{n} b_{ii}(\nu_i - 2) = \sum_{i=1}^{h} b_{ii} - \sum_{i=h+1}^{2h-1} b_{ii}.$$

From Propositions 2.1 and 5.1 we have

$$b_{h+i,h+i} = -\frac{r_i}{d_{i+1}} \cdot \frac{d_i}{d_{i+1}}, \quad i = 1, ..., h.$$

Denote the twigs of $\Gamma_{C,E}$ by $L_0, ..., L_h$, where E_i is the tip of L_i), so that $L_i = D_i$ for $i = 1, ..., h$. By Propositions 2.1, 5.1 and Corollary 3.4 we have

$$-b_{00} = \frac{d_1}{r_1} + \mathrm{ind}(L_0); \quad -b_{ii} = \frac{r_i}{d_i} + \mathrm{ind}(L_i), \quad i = 1, ..., h.$$

It is clear that $C_E \cdot E_i = \delta_{2h,i}$, i.e. $A_E(C_E) = E_{2h}^*$. Hence,

$$C_E^2 = \langle B_E E_{2h}^*, E_{2h}^* \rangle = b_{2h,2h} = -r_h d_h.$$

To complete the proof of i) and ii), we put all these formulas into (10) and apply Lemma 1.1 and Corollary 3.5; iii) is the sum of i) and ii). ■

Corollary 5.3.

$$2\mu + H_E^2 = -\frac{d_1}{q_1} + \sum_{i=1}^{h} q_i(d_i - \frac{1}{d_i}). \qquad (11)$$

Proof. Put (8) into (9)(i) and change the order of summation:

$$2\mu + H_E^2 = -\frac{d_1}{q_1} + \sum_{i=1}^{h}\sum_{j=1}^{i} \frac{q_j d_j}{d_i d_{i+1}} (\frac{d_i}{d_{i+1}} - \frac{d_{i+1}}{d_i})$$

$$= -\frac{d_1}{q_1} + \sum_{j=1}^{h} q_j d_j \sum_{i=j}^{h} (\frac{1}{d_{i+1}^2} - \frac{1}{d_i^2}) = -\frac{d_1}{q_1} + \sum_{j=1}^{h} q_j d_j (1 - \frac{1}{d_j^2}).$$

■

Corollary 5.4. *If the analytically irreducible singularity of $\tilde{C}$ at 0 has only one Puiseux characteristic pair (i.e. if the above m, n are relatively prime), then*

$$-H_E^2 = (m-2)(n-2) + (m-n)^2/mn.$$

Indeed, in this case $\mu = (m-1)(n-1)$, see [**Mil**, p. 95], and $h = 1, d_1 = m, q_1 = m_1 = r_1 = n$.

6. Estimates of H_E^2 via the Milnor number and multiplicity

The first estimate (Proposition 6.1 below) is quite elementary and in its proof we use nothing (except definitions) from the above part of the paper. This estimate holds for any singularity of a plane curve. The second estimate (Proposition 6.2 below) is stronger and is based on the computations involving the Puiseux characteristic sequence (see Sect.5). We prove the second estimate for irreducible singularities only.

Let $\bar{C}$ be a germ of a curve at the origin in $\mathbf{C}^2$ (not necessary analytically irreducible), and let $\sigma : V \to \mathbf{C}^2$, $E = \sigma^{-1}(0)$, be the minimal resolution of its singularity, $D = E + C$, where C is the proper transform of $\bar{C}$. We use the same notation as in Sect.2 and 3. Let μ be the Milnor number of $\bar{C}$ at 0.

Proposition 6.1.

$$-\mu < (K_E + D_E)^2 < H_E^2 \leq 0.$$

Proof. $H_E^2 \leq 0$ By the negative definiteness of the intersection form on $Vect(E)$, $H_E^2 \leq 0$. Since $N_E^2 < 0$, it suffices to prove that $\mu + (K_E + D_E)^2 > 0$. Note that

$$\mu+(K_E+D_E)^2 = (-C_E\cdot(K_E+D_E)+1)+(K_E+D_E)^2 = (K_E+E)(K_E+C_E+E)+1. \quad (12)$$

Since the minimal resolution of the singularity is a composition of blow-ups, it suffices to prove that the right hand side of (12) does not decreases under a blow-up (as in the proof of Proposition 4.1).

Let $\sigma : \tilde{V} \to V$, $\tilde{E} = \sigma^{-1}(E)$ be a blow-up at the point $p \in E$ with the exceptional curve $E_0 = \sigma^{-1}(p)$. Denote by $\tilde{C}$ the proper transform of C and by α and ν the multiplicities at p of C and E respectively. Then we have:

$$\tilde{C}_{\tilde{E}} = \sigma^*(C_E) - \alpha E_0, \qquad \tilde{K}_{\tilde{E}} = \sigma^*(K_E) + E_0, \qquad \tilde{E} = \sigma^*(E) - (\nu - 1)E_0.$$

Hence,

$$(\tilde{K}_{\tilde{E}} + \tilde{E})(\tilde{K}_{\tilde{E}} + \tilde{E} + \tilde{C}_{\tilde{E}}) - (K_E + E)(K_E + E + C_E) = (\nu - 2)(\alpha + \nu - 2)E_0^2.$$

Since $\nu = 1$ or 2, this difference is either 0 or $\alpha - 1$. But since the resolution is minimal, all blow-ups are done at points of C. So, $\alpha \geq 1$. ■

Now we assume that $\bar{C}$ is analytically irreducible at 0 and use the notation from Sect.5. Clearly, under the assumption that $m < n$, m is the multiplicity of $\bar{C}$ at the origin.

Theorem 6.2.

$$\mu + H_E^2 > \mu + (K_E + D_E)^2 \geq \mu/m,$$

where equality holds if and only if $m = 2$.

Proof. Milnor's formula for μ in terms of Puiseux pairs [**Mil**, p.93] (see also Remark 2.5 above) is obviously equivalent to

$$\mu = -d_1 + 1 + \sum_{i=1}^{h} q_i(d_i - 1). \quad (13)$$

Substracting (13) from (11), we obtain

$$\mu + H_E^2 = d_1(1 - \frac{1}{q_1}) - 1 + \sum_{j=1}^{h} q_j(1 - \frac{1}{d_j}),$$

and hence

$$\mu + (K_E + D_E)^2 - \frac{\mu}{m} = d_1(1 - \frac{1}{q_1}) - \frac{1}{d_1} + N_E^2 + \sum_{j=1}^{h} q_j(1 - \frac{d_j}{d_1})(1 - \frac{1}{d_j}) \qquad (14)$$

(recall, that $m = d_1$). It is clear that the last sum in (14) is positive.

Let, as above, L_i be the twig of $\Gamma_{C,E}$ with the tip E_i (see the diagram in Proposition 5.1). Since $\mathrm{ind}(L_i) < 1$, we have

$$N_E^2 > -\mathrm{ind}(L_0) - h$$

and by Corollary 3.5

$$-\mathrm{ind}(L_0) - d_1/q_1 = -\lceil d_1/q_1 \rceil = -1.$$

Thus, the expression in the right hand side of (14) is greater than $m - \frac{1}{m} - 1 - h$ (denote this quantity by a). Since $m = d_1$ and $d_i/d_{i+1} \geq 2$ we have $h \leq \log_2 m$. Hence, $a > 0$ for $m \geq 4$. If $m = 3$, then $h = 1$ and $a = 2/3 > 0$.

To complete the proof, it remains to consider the case $m = 2$. In this case $q_1 = n$ is odd, hence, $\lceil n/m \rceil = \lceil n/2 \rceil = (n+1)/2$, $\mu = n - 1$, and by (14)

$$\mu + (K_E + D_E)^2 = \lceil n/2 \rceil - 1 = (n-1)/2 = \mu/2.$$

■

Remark 6.3. The estimate in Theorem 6.2 is sharp in the following sense. For any positive integer m and for any $\epsilon > 0$ there exists a curve $\bar{C}$ with an analytically irreducible singularity at 0 of multiplicity m, such that

$$\mu + H_E^2 < (1 + \epsilon)\mu/m.$$

Indeed, consider the curve $x^m = y^n$, where n is big enough and relatively prime with m.

7. Plane affine curves with one place at infinity

The Puiseux expansions of an analytically irreducible singularity is very similar to that of an affine curve with one place at infinity (see [**A**, **NR**]). All the formulas from Sect.5 can be easily modified for this case. These modifications are nothing more than changing signs at several places. Here we just reproduce the answers because the proofs are the same as in the former case.

Let $\bar{D}$ be the closure in $\mathbf{CP}^2$ of an algebraic curve in $\mathbf{C}^2$ with one place at infinity. Denote by L the projective line at infinity (i.e. $\mathbf{C}^2 \cup L = \mathbf{CP}^2$). Let $\sigma : V \to \mathbf{CP}^2$ be the resolution of the singularity of $\bar{D}$ at infinity, $E = \sigma^{-1}(L)$, $D = \sigma^{-1}(\bar{D}) = C + E$, where C is the proper transform of $\bar{D}$.

As in Sect.5 we define the Puiseux characteristic sequence as follows. Let

$$x = t^{-m}, \qquad y = a_{-n}t^{-n} + a_{-n+1}t^{-n+1} + \dots, \qquad a_{-n} \neq 0,$$

be a local analytic parametrization of the branch of $\bar{D}$ with the centrum at L. Put: $d_1 = m$, $m_1 = -n$; and define d_i, q_i, r_i, h by the same formulas as at the beginning of Sect.5. Then we have

$$m_1 < 0; \quad m_1 < m_2 < \dots < m_h; \quad d_1 > d_2 > \dots > d_{h+1} = 1$$

and, according to [**A**] and [**NR**, Corollary 6.4], $r_i < 0$ for $i = 1, \dots, h$.

Proposition 7.1. *The dual graph of C near E is the same as in Proposition 5.1. The discriminants of its subgraphs are:*

$$d(R_i) = -\frac{r_i}{d_{i+1}}, \quad d(D_i) = \frac{d_i}{d_{i+1}}, \quad d(S_i) = 1, \quad d(Q_i) = \frac{q_i}{d_{i+1}}.$$

■

Let $\mu_\infty = 2\pi_a(C) = C(C + K) + 2$ be "the Milnor number of C at infinity".

Proposition 7.2.

$$2\mu_\infty - H_E^2 = \frac{d_1}{r_1} - \sum_{i=1}^{h} \frac{r_i}{d_{i+1}}\left(\frac{d_i}{d_{i+1}} - \frac{d_{i+1}}{d_i}\right).$$

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