EXCEPTIONAL AND RIGID SHEAVES ON SURFACES WITH ANTICANONICAL CLASS WITHOUT BASE COMPONENTS

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Abstract

The paper consists of three parts. In the first part we discuss types of stability. In particular, the concept of stability with respect to a nef divisor is introduced. The structure of rigid and superrigid vector bundles on smooth projective surfaces with nef anticanonical class is studied in the second part. In particular, we prove that any superrigid bundle has the unique exceptional filtration. In the last part we give a constructible description of exceptional bundles on these surfaces.

Introduction.

This paper contains generalisation of the theory of rigid ($\mathrm{Ext}^1(E,E)=0$) and exceptional ($\mathrm{Ext}^0(E,E)=\mathbb{C}$, $\mathrm{Ext}^i(E,E)=0$ for $i>0$) sheaves on Del Pezzo surfaces , originally described in [11]. The main objects of this paper are rigid and exceptional sheaves on a smooth projective surface S such that $-K_S$ is nef.

If $K_S^2>0$ then such surfaces can be obtained from $\mathbb{P}^2$ by successive blowing up of successive at most 8 points and therefore are the natural extension of Del Pezzo surface class.

For the first time the exceptional sheaves appeared in [6] for the description the possible Chern classes which a stable bundle on $\mathbb{P}^2$. Besides was proved in [7] that any rigid bundle on the projective plane is a direct sum of exceptional bundles .

The author proved the same statement for all Del Pezzo surfaces ([11]). But if $-K_S$ is nef then there exist indecomposable and nonsimple rigid bundle; and their structure is described in terms of exceptional collections. The description of this structure is the goal of the second part. The information about superrigid bundles gives a convenient method to study the exceptional sheaves.

The theory of exceptional bundles on Del Pezzo surfaces uses the stability with respect to the anticanonical class. Throughout this paper all surfaces have the nef anticanonical class. The following question is very interesting in this context. Is there a sufficient notion of stability with respect to nef divisor, and which slope axioms are sufficient for constructing the having meaning stability theory? For example, when does the Garder–Narasimhan filtration exist? The answer to this question is the subject of the first part.

Finally in the last part of this paper we prove the constructibility of exceptional bundles on smooth projective surfaces S over $\mathbb{C}$ with nef anticanonical class and $K_S^2>0$. (Here by constructibility we mean that any exceptional bundle can be obtained from a finite fixed collection of exceptional sheaves by a finite procedure.)

Notations.

Let X be a complete algebraic manifold over $\mathbb{C}$;

$r(F), c_1(F), c_2(F), \ldots$ denote the rank and Chern classes of a coherent sheaf F on X;

$\mathcal{O}_X$ or $\mathcal{O}$ denote the trivial line bundle on X;

$\mathcal{O}_Y$ (for a closed submanifold Y in X) denote the structure sheaf of submanifold Y, which we sometimes consider as a sheaf on X;

$\mathcal{O}_X(D)$ or $\mathcal{O}(D)$ denote the line bundle which corresponds to divisor D;

K_X denote the canonical class of X;

$F(D)$ denote the tensor product of F and $\mathcal{O}(D)$;

F^* denote the dual sheaf, that is the sheaf of local homomorphisms $\mathcal{H}om_{\mathcal{O}_X}(F, \mathcal{O}_X)$;

$\mathrm{Hom}(E, F)$ denote the space of global maps from E to F;

$h^i(E, F)$ denote the dimension of space $\mathrm{Ext}^i(E, F)$;

$\chi(E, F)$ is the Euler characteristic of any two sheaves, which equals $\sum(-1)^i h^i(E, F)$;

$\chi(E)$ is the Euler characteristic of a sheaf, which equals $h^i(\mathcal{O}_X, E)$;

We denote the direct sum $\bigoplus_{i=1}^{k} F_i$ of k copies of F ($\forall i\ F_i = F$) by kF or $V \otimes F$ (where V is a vector space over $\mathbb{C}$ and $\dim V = k$).

We identify a bundle with the sheaf of its local sections. Sometimes we will arrange a long cohomology sequence associated to an exact triple into a table. For example, the application of functor $\mathrm{Ext}^\cdot(F, \cdot)$ to the exact triple

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

gives

k	$\mathrm{Ext}^k(F, A)$	$\rightarrow$	$\mathrm{Ext}^k(F, B)$	$\rightarrow$	$\mathrm{Ext}^k(F, C)$
0	*		?		*
1	0		?		0
2	*		?		*

This table calculates $\mathrm{Ext}^1(F, B)$. In particular, $\mathrm{Ext}^1(F, B) = 0$.

1 Axioms of Stability.

1.1 Definitions and Simple Properties.

The Gieseker and the Mumford–Takemoto stabilities are well known. Recently the notion of the vector stability with respect to a collection of polarizations was introduced in ([21]). All these theories use a slope and similar properties of stable and semistable sheaves. In this section we introduce several slope axioms and obtain the basic properties of stable sheaves.

DEFINITION. Let γ be a function from the set of all torsion-free sheaves on a complete complex algebraic manifold X to $\mathbb{R}^n$ with the lexicographic order. Assume that γ satisfies the following axioms:

SLOPE.1. For any exact triple of torsion-free sheaves

$$0 \longrightarrow F \longrightarrow E \longrightarrow G \longrightarrow 0$$

the following equivalences hold

$$\gamma(F) < \gamma(E) \iff \gamma(E) < \gamma(G),$$

$$\gamma(F) > \gamma(E) \iff \gamma(E) > \gamma(G),$$

$$\gamma(F) = \gamma(E) \iff \gamma(E) = \gamma(G);$$

SLOPE.2. For any two torsion-free sheaves $F \subset E$ the condition $r(F) = r(E)$ implies that

$$\gamma(F) \leq \gamma(E),$$

then we say that γ is a *slope function* and $\gamma(E)$ is a *γ-slope* of E or simply a *slope* of E if it causes no confusion . If $\gamma(E) \in \mathbb{R}^n$ then γ is called a *slope vector function*.

DEFINITION. A torsion-free sheaf E on an algebraic manifold X is said to be *γ-(semi)stable* or simply *(semi)stable* if for any its subsheaf F with $r(F) < r(E)$ the following inequality holds $\gamma(F) < \gamma(E)$ ($\gamma(F) \leq \gamma(E)$ resp.). A subsheaf which contradicts (semi)stability is called *destabilizing*.

1.1.1 REMARK. *1. The torsion-free sheaves of rank 1 are stable with respect to any slope, since they have no rank 0 subsheaves.*

2. Due to the lexicographic order on $\mathbb{R}^n$ the function $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ is the slope if and only if all γ_i satisfy the slope axioms.

3. For the slopes $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ and $\gamma' = (\gamma_1, \gamma_2, \ldots, \gamma_n, \gamma_{n+1}, \ldots, \gamma_m)$ the following statements are true

a) a γ-stable sheaf is γ'-stable;

b) a γ'-semistable sheaf is γ-semistable.

1.1.2 LEMMA. *A torsion-free sheaf E on a manifold X is (semi)stable if for any its subsheaf F such that E/F has no torsion one has*

$$\gamma(F) < \gamma(E) \qquad (\gamma(F) \leq \gamma(E) resp.).$$

1.1.3 LEMMA. *A torsion-free sheaf E is stable (semistable) if and only if the slope of any its torsion free quotient G satisfies the inequality:*

$$\gamma(E) < \gamma(G) \qquad (\gamma(E) \leq \gamma(G) \text{resp.}).$$

The proof follows from 1.1.2 and SLOPE.1.

To study the stability properties we need the following

1.1.4 REMARK. *By the definition of stability and* SLOPE.2 *we obtain that for any pair of torsion-free sheaves $F \subset E$ the semistability of E implies the inequality $\gamma(F) \leq \gamma(E)$ (without the rank condition). If E is stable then*

$$\gamma(F) = \gamma(E) \quad \Longrightarrow \quad r(E) = r(F).$$

Similarly, if G is a quotient of a semistable sheaf E then $\gamma(E) \leq \gamma(G)$. If E is stable then

$$\gamma(G) = \gamma(E) \quad \Longrightarrow \quad E = G.$$

1.1.5 LEMMA. *Let E and F be semistable sheaves. Suppose $\gamma(E) > \gamma(F)$ then*

$$\mathrm{Hom}(E, F) = 0.$$

1.1.6 LEMMA. *Let E and F be semistable sheaves with $\gamma(E) = \gamma(F)$ and let $\varphi : E \longrightarrow F$ be a nontrivial morphism. Then*
a) E is stable $\Longrightarrow$ φ is an injection;
b) F is stable $\Longrightarrow$ φ is an epimorphism at a generic point.

1.1.7 LEMMA. *A stable sheaf is simple that is any its endomorphism has the form $\lambda \cdot \mathrm{id}$.*

1.1.8 LEMMA. *Let $0 \longrightarrow E \xrightarrow{i} G \longrightarrow F \longrightarrow 0$, be an exact sequence of torsion-free sheaves such that $\gamma(E) = \gamma(G) = \gamma(F)$. Then G is semistable if and only if both E and F are semistable. In particular, for any complex finite-dimensional vector space V and a divisor D on X the bundle $V \otimes \mathcal{O}_X(D)$ is semistable.*

1.2 The Harder-Narasimhan Filtration.

The aim of this section is to construct the well-known canonical filtration of a torsion-free sheaf. This filtration is trivial when a sheaf is semistable. Let us recall the main definitions and notations.

The $Gr(E) = (G_n, G_{n-1}, \ldots, G_1)$ means that the sheaf E has a filtration:

$$0 = E_{n+1} \subset E_n \subset \cdots \subset E_2 \subset E_1 = E$$

and $E_i/E_{i+1} = G_i$. The sheaves E_i are called *terms of filtration* and G_i are *quotients of filtration*. Note that $G_n = E_n$ (since $E_{n+1} = 0$).

DEFINITION. A filtration of a torsion-free sheaf E

$$Gr(E) = (G_n, G_{n-1}, \ldots, G_1)$$

is called *the Harder-Narasimhan filtration* if all quotients G_i are semistable and their slopes satisfy the inequality:

$$\gamma(G_{i+1}) > \gamma(G_i) \qquad (i = 1, 2, ..., n-1).$$

To construct this filtration we need another slope axiom and several lemmas.

1.2.1 LEMMA. *Let E be a torsion-free sheaf on X and $\mathcal{G}$ be the set of all torsion-free quotients of E. Then there exists γ_0 such that $\gamma(G) \geq \gamma_0$ for all $G \in \mathcal{G}$.*

PROOF. Let us choose an ample divisor A on X. Then Serre's theorem ([22]) implies that there exists a natural number n such that the sheaf $E(nA)$ is generated by its global sections. Hence we have the short exact sequence:

$$0 \longrightarrow F \longrightarrow \mathrm{H}^0(E(nA)) \otimes \mathcal{O} \xrightarrow{can} E(nA) \longrightarrow 0.$$

Therefore E is the quotient of the semistable bundle

$$\mathrm{H}^0(E(nA)) \otimes \mathcal{O}(-nA).$$

If G is a torsion-free quotient of E then there exists an epimorphism

$$\mathrm{H}^0(E(nA)) \otimes \mathcal{O}(-nA) \longrightarrow G \longrightarrow 0.$$

Now the result follows from 1.1.3 .

1.2.2 LEMMA. *Suppose that a slope function γ satisfies the axiom:*

SLOPE.3. *Let γ_0 be a value of the function γ and $M = \{G_1, G_2, G_3, \ldots \}$ be an ordered set of torsion-free sheaves with $r(G_i) \leq r$ for all i. Then the condition*

$$\gamma(G_i) > \gamma(G_{i+1}) \geq \gamma_0 \qquad i = 1, 2, 3, \ldots$$

implies that M is finite.

Then each torsion-free sheaf E has the quotient G with the minimal slope $\gamma(G)$, i.e. for an other torsion-free quotient Q of E we have: $\gamma(Q) \geq \gamma(G)$.

The proof follows from SLOPE.3 and the previous lemma.

1.2.3 PROPOSITION. *If a slope function γ satisfies the axioms* SLOPE.1 — SLOPE.3 *then there exists the Harder-Narasimhan filtration torsion-free sheaf E*

$$Gr(E) = (G_n, G_{n-1}, \ldots, G_1).$$

Moreover, if $Gr(E) = (G'_m, G'_{m-1}, \ldots, G'_1)$ is another filtration with the semistable quotients and the inequalities $\gamma(G'_i) > \gamma(G'_{i-1})$ hold for all $i = 2, 3, ..., m$ then $m = n$ and $G'_i = G_i$.

PROOF OF THE EXISTENCE. A semistable sheaf has the trivial filtration. Suppose that E is not semistable. Denote by G_1 the torsion-free quotient of $E_1 = E$ with the minimal γ-slope and of maximal rank. Let E_2 be the corresponding subsheaf in E:

$$0 \longrightarrow E_2 \longrightarrow E_1 \longrightarrow G_1 \longrightarrow 0.$$

If E_2 is not semistable then let us choose a torsion-free quotient G_2 of E_2 with the minimal γ-slope and the maximal rank. Denote by E_3 the corresponding subsheaf in E_2, etc. Note that all G_i are semistable by construction. Let us check the inequality $\gamma(G_i) < \gamma(G_{i+1})$ with the help of the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \uparrow & & & & & & \\
 & & G_{i+1} & & & & 0 & & \\
 & & \uparrow & & & & \uparrow & & \\
0 & \longrightarrow & E_{i+1} & \longrightarrow & E_i & \longrightarrow & G_i & \longrightarrow & 0 \\
 & & \uparrow & & {\scriptstyle id_E}\uparrow & & \uparrow & & \\
0 & \longrightarrow & E_{i+2} & \longrightarrow & E_i & \longrightarrow & Q & \longrightarrow & 0 \\
 & & \uparrow & & & & \uparrow & & \\
 & & 0 & & & & G_{i+1} & & \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array} .
$$

We have that Q is the torsion-free quotient of E_i. The codition $r(Q) > r(G_i)$ implies that $\gamma(Q) > \gamma(G_i)$. Finally, using the axiom SLOPE.1, we get $\gamma(G_{i+1}) > \gamma(G_i)$. This concludes the proof of existence.

1.2.4 LEMMA. *Let E, F be sheaves on X and $Gr(E) = (G_n, G_{n-1}, \ldots, G_1)$ be a filtration of E. Then*

a) $\mathrm{Ext}^k(G_i, F) = 0 \quad \forall i \qquad \Longrightarrow \qquad \mathrm{Ext}^k(E, F) = 0;$

b) $\mathrm{Ext}^k(F, G_i) = 0 \quad \forall i \qquad \Longrightarrow \qquad \mathrm{Ext}^k(F, E) = 0.$

PROOF. This lemma can be proved with the help of cohomological long exact sequences.

1.2.5 COROLLARY. *Let $Gr(E) = (G_n, G_{n-1}, \ldots, G_1)$ be the Harder-Narasimhan filtration of a sheaf E and let F be a semistable sheaf. Then*

a) $\gamma(F) < \gamma(G_1) \qquad \Longrightarrow \qquad \mathrm{Hom}(E, F) = 0;$

b) $\gamma(F) > \gamma(G_n) \qquad \Longrightarrow \qquad \mathrm{Hom}(F, E) = 0.$

The proof follows easily from the lemmas 1.2.4 , 1.1.5 and the definition of the Harder-Narasimhan filtration.

1.2.6 LEMMA. *If a sheaf E has a filtration $Gr(E) = (G_n, G_{n-1}, \ldots, G_1)$ then G_n is a subsheaf of E and $Gr(E/G_n) = (G_{n-1}, \ldots, G_1)$.*

PROOF. Since the last quotient of the filtration concides with its last term we get

$G_n \subset E$. Now the statment follows from the following commutative diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & E_{i+1}/G_n & \longrightarrow & E_i/G_n & \longrightarrow & G_i & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & E_{i+1} & \longrightarrow & E_i & \longrightarrow & G_i & \longrightarrow & 0 . \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & G_n & \longrightarrow & G_n & \longrightarrow & 0 & & \\ & & \uparrow & & \uparrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

PROOF OF THE UNIQUENESS OF THE HARDER-NARASIMHAN FILTRATION.
Let

$$Gr(E) = (G_n, G_{n-1}, \dots, G_1) = (G'_m, G'_{m-1}, \dots, G_1)$$

be two Harder-Narasimhan filtrations. Suppose $\gamma(G_1) \neq \gamma(G'_1)$. For example, let $\gamma(G_1) > \gamma(G'_1)$. Then by corollary 1.2.5 and the semistability of G'_1 one gets $\mathrm{Hom}(E, G'_1) = 0$. This contradicts to the existence of an epimorphism: $E \longrightarrow G'_1 \longrightarrow 0$. In the same way we get $\gamma(G_n) = \gamma(G'_m)$.

Denote by E'_i the terms of the second filtration. Let us show by induction on i that G_n is a subsheaf in E'_i. For $i = 1$, there is nothing to prove.

By the inductive hypothesis we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E'_{i+1} & \longrightarrow & E'_i & \longrightarrow & G'_i & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & {\scriptstyle\varphi_i}\uparrow & & \\ & & 0 & \longrightarrow & G_n & \longrightarrow & G_n & \longrightarrow & 0 . \\ & & & & \uparrow & & & & \\ & & & & 0 & & & & \end{array}$$

By the snake lemma, $\ker \varphi_i \subset E'_{i+1}$. On the other hand, the slopes of semistable sheaves G_n, G'_m and G'_i satisfy the conditions: $\gamma(G_n) = \gamma(G'_m) > \gamma(G'_i)$ if $i < m$. Hence, $\varphi_i = 0$ and $\ker \varphi_i = G_n$ (1.1.5).

Thus, $G_n \subset E'_{i+1}$ for $i < m$. In particular, $G_n \subset G'_m$.

In the same way we obtain that $G'_m \subset G_n$. Therefore, $G'_m = G_n$.

It follows from lemma 1.2.6 that

$$Gr(E/G_n) = (G_{n-1}, \dots, G_1) = (G'_{m-1}, \dots, G'_1).$$

Moreover, these are the Harder-Narasimhan filtrations of E/G_n. Now the uniqueness of the Harder-Narasimhan filtration follows easily by induction on the rank of E.

1.3 Examples of Slopes and Types of Stability.

The motivation of the above slope axioms are the properties of the following well-known slopes.

The slope of a bundle on a curve: $\mu(E) = \frac{\deg E}{r(E)}$ where deg$E$ is the degree of the determinant of bundle;

the Mumford-Takemoto slope with respect to an ample divisor A on an n-dimensional manifold X: $\mu_A(E) = \frac{c_1(E)\cdot A^{n-1}}{r(E)}$;

the Gieseker slope w.r.t. an ample divisor A: $\gamma_A(E,n) = \frac{\chi(E(nA))}{r(E)}$.

Let us check that these slopes and the slope $\mu_H(E) = \frac{c_1(E)\cdot H^{n-1}}{r(E)}$ where H is nef, indeed satisfy the above slope axioms. By definition a divisor A is *nef* if the number $D \cdot A^{n-1}$ is nonnegative for any effective divisor D on$X$.

We see that all the above examples of slopes except for γ_A have the form $\gamma = d/r$ where d is an $\mathbb{Z}$-valued additive function on $K_0(X)$ and r is the rank function.

1.3.1 LEMMA. *Any slope function γ of the form $\gamma = d/r$ satisfies the axioms* SLOPE.1 *and* SLOPE.3.

PROOF. For any exact triple of torsion free sheaves

$$0 \longrightarrow F \longrightarrow E \longrightarrow G \longrightarrow 0$$

we have that $\gamma(E) = \frac{d(F)+d(G)}{r(F)+r(G)}$. Note that the sign of the determinant

$$\begin{vmatrix} d(F) & d(G) \\ r(F) & r(G) \end{vmatrix}$$

corresponds to the comparison sign between the fractions: $\frac{d(F)}{r(F)} \quad \frac{d(G)}{r(G)}$. Besides,

$$\begin{vmatrix} d(F) & d(G) \\ r(F) & r(G) \end{vmatrix} = \begin{vmatrix} (d(F)+d(G)) & d(G) \\ (r(F)+r(G)) & r(G) \end{vmatrix}.$$

This implies that γ satisfies the first axiom.

To check the axiom SLOPE.3 note that $|\gamma(G_1) - \gamma(G_2)| \geq 1/r^2$ if the ranks of torsion-free sheaves G_1 and G_2 do not exceed r.

1.3.2 COROLLARY. *Let $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n)$ be a vector function of $K_0(X)$ such that each γ_i has the form d_i/r where d_i is an $\mathbb{Z}$-valued additive function on $K_0(X)$ and r is the rank function. If the values of γ are lexicographically comparable then γ satisfies the axioms* SLOPE.1 *and* SLOPE.3.

As in the case of the Gieseker slope γ_A it is a polynomial of the degree dim$X$ with rational coefficients. So far as the inequality $\gamma_A(E,n) > \gamma_A(F,n)$ holds true if it holds for sufficiently large n, then the comparison γ_A-slopes is equivalent to the lexicographic ordering of the coefficients of the polynomials.

The Hilbert polynomial $\chi(E(nA))$ is an additive function. Hence the Geaseker slope satisfies the axiom SLOPE.1 (see the proof of lemma 1.3.1).

To check SLOPE.3 note that by the Hirzebruch-Riemann-Roch theorem (see. [22]), the Euler characteristic of sheaf on a smooth manifold can be calculated as follows

$$\chi(E) = \deg(ch(E) \cdot td(T_X))_n, \tag{1}$$

where

$\deg(\ldots)_n$ means the component of degree n in the cohomology ring $H^*(X, \mathbb{Q})$ of X;

T_X is the tangent bundle of X;

$$ch(E) = r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) + \frac{1}{24}(c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4) + \cdots;$$

$$td(E) = 1 + c_1/2 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}(c_1c_2) - \frac{1}{720}(c_1^4 - 4c_1^2c_2 - 3c_2^2 - c_2c_3 + c_4) + \cdots.$$

(where c_i are the Chern classes of the sheaf E).

This yields that the denominators of the coefficients of the Hilbert polynomial $\chi(E(nA))$ do not depend of E. After some modifications of the proof of lemma 1.3.1 , one can easily show that the Gieseker slope $\gamma_A(E)$ satisfies the axiom SLOPE.3.

All examples of slopes satisfy the axiom SLOPE.2 to some extent.

1.3.3 LEMMA. *a) For any pair of torsion-free sheaves $F \subset E$ with the same rank on a manifold X and any nef divisor H the inequality $\mu_H(F) \leq \mu_H(E)$ holds. Moreover, in this case the equality $\mu_H(F) = \mu_H(E)$ is possible only if*

$$\operatorname{codim} \operatorname{supp}(E/F) \geq 1.$$

Provided a slope function satisfies this weakend version of axiom SLOPE.2, *we call it the weak slope;*

b) for any pair of torsion free sheaves $F \subset E$ with the same rank on a manifold X and any ample divisor A the following inequality holds $\mu_A(F) \leq \mu_A(E)$. Moreover, in this case the equality is possible only if

$$\operatorname{codim} \operatorname{supp}(E/F) \geq 2.$$

Provided a slope function satisfies this version of axiom SLOPE.2, *we call it the Mumford-Takemoto slope;*

c) for any pair of torsion free sheaves $F \subset E$ with the same rank on a manifold X and any ample divisor A the inequality $\gamma_A(F) \leq \gamma_A(E)$ holds. Moreover, in this case the equality of slopes is equivalent to

$$E = F.$$

Provided a slope function satisfies this version of axiom SLOPE.2 *we call it the Gieseker slope;*

d) the slope μ of bundles on a curve is the Gieseker slope.

PROOF. The number $c_1(E)\cdot D^{n-1}$ (determined by a sheaf E on a n-dimensional manifold and by a divisor D) is called the degree of a sheaf with respect to D and is denoted by $\deg_D(E)$.

Since the ranks of sheaves E and F coincide we see that the comparison of their slopes is equivalent to the comparison of the degree $\deg_D$ and the quotient $Q = E/F$ has the zero rank. Hence $c_1(Q) = c_1(E) - c_1(F)$ is an effective or the zero divisor.

By the definition of a nef divisor $\deg_H(Q) \geq 0$. This proves the first statement of lemma.

If A is ample and $c_1(Q) \neq 0$ then the Nakai-Moyshezon criterion ([22]) implies that $\deg_A(Q) > 0$. This yields the second statement of lemma.

If A is ample then by the Serre theorem ([22]) one hes $\chi(Q(nA)) > 0$ for any nonzero sheaf Q and for sufficiently large n. Therefore the third statement of lemma also holds.

Finally the degree of the effective divisor $c_1(Q)$ on a curve is nonnegative and is equal to zero only if $c_1(Q) = 0$. This completes the proof.

The more precise conditions of SLOPE.2 allow to formulate the following statement which is stronger than lemma 1.1.6 .

1.3.4 LEMMA. *a) Let E and F be semistable sheaves with respect to the Mumford-Takemoto slope γ, $\gamma(E) = \gamma(F)$, and F stable. Then the cokernel of any nonzero morphism $\varphi : E \longrightarrow F$ has the support C such that* $\operatorname{codim} C \geq 2$. *In particular, φ is an epimorphism if E is locally free.*

b) Let E and F be semistable sheaves with respect to the Gieseker slope γ, $\gamma(E) = \gamma(F)$ and F is stable. Then any nonzero map of E to F is an epimorphism.

This lemma can be proved in the same way as 1.1.6 . Nevertheless, let us recall that $\operatorname{Ext}^1(Q, E) = 0$ if E is locally free and $\operatorname{codim}\operatorname{supp}(Q) \geq 2$.

Note that the slope $\gamma = \mu_A(E) = (c_1(E)\cdot A^{n-1})/r(E)$ has the following property:

SLOPE.4. For any torsion-free sheaf E and a divisor D the equalities

$$\gamma(E^*) = -\gamma(E), \qquad \gamma(E(D)) = \gamma(E) + \gamma(\mathcal{O}(D))$$

are satisfied.

1.3.5 LEMMA. *Assume that the slope function γ satisfies the axiom* SLOPE.4; *then a torsion-free sheaf E is (semi)stable if and only if $E(D)$ is (semi)stable; and the γ-(semi)stability of a reflexive sheaf E ($E^{**} = E$) is equivalent to the γ-semistability of the dual sheaf E^*.*

1.3.6 PROPOSITION. *Any sheaf E semistable w.r.t. the Gieseker slope (see 1.2.3) has the filtration by isotypic quotients:*

$$Gr(E) = (G_n, G_{n-1}, \ldots, G_1),$$

where each of G_i has the filtration with isomorphic quotients:

$$Gr(G_i) = (Q_i, Q_i, \ldots, Q_i) \qquad (\gamma(Q_i) = \gamma(G_i) = \gamma(E)).$$

Moreover, this filtration can be constructed in such a way that

$$\mathrm{Hom}(E_i, G_{i-1}) = \mathrm{Hom}(G_i, G_{i-1}) = \mathrm{Hom}(G_{i-1}, G_i) = 0,$$

where E_i are the terms of the filtration.

PROOF. For a stable sheaf this filtration is trivial. If a sheaf $E = E_1$ is semistable then it has a destabilizing torsion-free quotient Q $(\gamma(Q) = \gamma(E_1))$. Ler us choose from the set of all such quotients a sheaf Q_1 of the minimal rank. Obviously, it is stable. Let E_1^1 be the corresponding subsheaf. It follows from the exact sequence:

$$0 \longrightarrow E_1^1 \longrightarrow E_1 \longrightarrow Q_1 \longrightarrow 0,$$

and the equality $\gamma(Q_1) = \gamma(E_1)$ that E_1^1 is semistable and $\gamma(E_1^1) = \gamma(E_1)$ (see SLOPE.1, 1.1.8). If $\mathrm{Hom}(E_1^1, Q_1) = 0$ then $E_2 = E_1^1$ is the second term of the filtration and $G_1 = Q_1$ is its first quotient.

Conversely, there exists an epimorphism; $E_1^1 \longrightarrow Q_1 \longrightarrow 0$ (1.3.4). Denote by E_1^2 the kernel of this epimorphism.

Continuing this procedure we get the semistable subsheaf E_1^k such that

$$\mathrm{Hom}(E_1^k, Q_1) = 0.$$

By definition, put $E_2 = E_1^k$ and $G_1 = E_1/E_2$. By construction, G_1 and E_2 are semistable, $\gamma(E_2) = \gamma(G_1) = \gamma(E_1)$, $Gr(G_1) = (Q_1, Q_1, \ldots, Q_1)$ and $\mathrm{Hom}(E_2, Q_1) = 0$. Now using the lemma 1.2.4 we obtain that $\mathrm{Hom}(E_2, G_1) = 0$.

By the inductive hypothesis we can assume that E_2 has the filtration by isotypic quotients: $Gr(E_2) = (G_n, G_{n-1}, \ldots, G_2)$. Let us show that the filtration

$$Gr(E) = (G_n, G_{n-1}, \ldots, G_2, G_1)$$

satisfies the assertions of the proposition.

It remains to check that $\mathrm{Hom}(G_2, G_1) = \mathrm{Hom}(G_1, G_2) = 0$. Since $\mathrm{Hom}(E_2, G_1) = 0$ and there exists an epimorphism $E_2 \longrightarrow G_2 \longrightarrow 0$ the equality $\mathrm{Hom}(G_2, G_1) = 0$ trivially holds.

Suppose that there exists a nonzero morphism $G_1 \longrightarrow G_2$. Let us recall that $Gr(G_i) = (Q_i, Q_i, \ldots, Q_i)$. Therefore by 1.2.4 $\mathrm{Hom}(Q_1, G_2) \neq 0$ and there exists a nonzero map $\varphi : Q_1 \longrightarrow Q_2$. It follows from (1.1.6 and 1.3.4) that φ is an isomorphism. This implies that $\mathrm{Hom}(Q_2, G_1) \neq 0$. But Q_2 is a quotient of G_2 and Q_1 is a subsheaf of G_1. Thus, $\mathrm{Hom}(G_2, G_1) \neq 0$. This contradiction concludes the proof.

2 Rigid Sheaves.

2.1 Preliminary Information.

We will study sheaves on a smooth complex projective surface S such that $h^1(\mathcal{O}_S) = 0$ and the anticanonical class $H = -K_S$ has no base components. Note that the last condition implies that H is nef. It is known that if S is a smooth projective surface over an algebraically closed field with nef anticanonical class then we have one of the following options

1. $K_S = 0$;

2. $S \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$;

3. $S \cong \mathbb{P}(F)$ where F is a rank 2 vector bundle on an elliptic curve which is an extension of degree zero line bundles;

4. $S \cong \mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$;

5. S is obtained from $\mathbb{P}^2$ by successively blowing up at most 9 points.

Let us recall some general facts which will be used later.

2.1.1 THEOREM. *(The Riemann-Roch formula for surfaces.) The Euler characteristic of two coherent sheaves E and F on a smooth projective surface X is given by the following formula:*

$$\chi(E,F) = r(E)r(F)\Big(\chi(\mathcal{O}_X) + \frac{1}{2}(\mu_H(F) - \mu_H(E)) + q(F) + q(E) - \frac{(c_1(E)\cdot c_1(F))}{r(E)r(F)}\Big),$$

$$\text{where} \qquad \mu_H(E) = \frac{1}{r(E)}(-K_X \cdot c_1(E)), \quad q(E) = \frac{c_1^2(E) - 2c_2(E)}{2r(E)}.$$

Note that in our case $\chi(\mathcal{O}_S) = 1$.

2.1.2 COROLLARY. *Let E, F be two sheaves on a smooth projective regular ($h^1(\mathcal{O}_S) = 0$) surface with $\chi(E,E) = \chi(F,F) = 1$; then*

$$q(E) = \frac{1}{2}\Big(\frac{c_1^2(E)+1}{r^2(E)} - 1\Big) \qquad \text{and}$$

$$\chi(E,F) = \frac{r(E)r(F)}{2}\Big(\mu_H(F) - \mu_H(E) + \frac{1}{r^2(E)} + \frac{1}{r^2(F)} + (\frac{c_1(F)}{r(F)} - \frac{c_1(E)}{r(E)})^2\Big).$$

2.1.3 THEOREM. *(Serre duality) For any coherent sheaves E and F on a smooth projective surface X the following equality*

$$\mathrm{Ext}^k(E,F)^* \cong \mathrm{Ext}^{2-k}(F, E(K_X))$$

holds.

The proof is contained in [22].

2.1.4 LEMMA. *(Mukai) Let X be a smooth projective surface. Then*

1. For any torsion-free sheaf E on X we have

$$h^1(E,E) \geq h^1(E^{**}, E^{**}) + 2\,length(E^{**}/E).$$

2. *a) Suppose the sheaves G_1, G_2 and E on X form the exact sequence*

$$0 \longrightarrow G_2 \longrightarrow E \longrightarrow G_1 \longrightarrow 0,$$

and satisfy the condition $\mathrm{Hom}(G_2, G_1) = \mathrm{Ext}^2(G_1, G_2) = 0$ *then*

$$h^1(E,E) \geq h^1(G_1,G_1) + h^1(G_2,G_2).$$

b) If moreover, $h^1(E,E) = 0$ *then*

$$h^0(E,E) = h^0(G_1,G_1) + h^0(G_2,G_2) + \chi(G_1,G_2),$$

$$h^2(E,E) = h^2(G_1,G_1) + h^2(G_2,G_2) + \chi(G_2,G_1).$$

This lemma follows from the spectral sequence associated with the above exact triple. See the proof in [14] and [11].

2.2 Exceptional Sheaves.

DEFINITION. A sheaf E on a manifold X is called *rigid* whenever

$$\mathrm{Ext}^1(E,E) = 0.$$

The trivial examples of rigid sheaves are exceptional sheaves.

DEFINITION. A sheaf E on a manifold is called *exceptional*, if $\mathrm{Ext}^0(E,E) = \mathbb{C}$ and $\mathrm{Ext}^i(E,E) = 0 \quad \forall i > 0$.

Using the results of S. Mukai ([14]), A. Gorodentsev ([4]), D. Orlov ([11]) and S. Zube ([8]) we provide the initial information about the structure of rigid and exceptional sheaves.

2.2.1 LEMMA. *A rigid sheaf without torsion on a smooth projective surface is locally free.*
This lemma follows from the Mukai lemma (2.1.4).

Recall that we consider a smooth complex projective surface S the anticanonical class $H = -K_S$ of which has no base components. Let G be a sheaf on S. Denote by TG its torsion subsheaf and by T^0G the subsheaf in TG such that $T^1G = TG/T^0G$ has no a torsion subsheaf with 0-dimensional support.

2.2.2 LEMMA. *(Gorodentsev-Orlov) Any sheaves G and F on a surface S satisfy the following conditions:*
a) the inequality

$$h^0(F,G) \geq h^2(G,F)$$

holds whenever the support of T^0G has no common points with the base set of the anticanonical linear system $|H|$;
b) the inequality

$$h^0(G,G) > h^2(G,G)$$

holds provided that there exists a curve $D \in |H|$ such that $D \cap \mathrm{supp} G \neq \emptyset$. In particular, this inequality is satisfied whenever $r(G) > 0$.

2.2.3 COROLLARY. *Let G be a rigid sheaf on S; then its torsion subsheaf TG and torsion-free quotient $G' = G/TG$ are rigid sheaves. Moreover, $T^0 G = 0$.*

2.2.4 LEMMA. *Suppose E is an exceptional torsion sheaf on S then $c_1^2(E) = -1$. Furthermore,*
either $E = \mathcal{O}_e(d)$, where e is some irreducible rational curve with $e^2 = -1$
or one of the components of the support of E has the zero cup product with K_S.

2.2.5 LEMMA. *Suppose that E is an exceptional sheaf on S then the support of its torsion has the zero cup product with K_S.*

Combining 2.2.1 , 2.2.4 and 2.2.5 we can formulate the following proposition.

2.2.6 PROPOSITION. *Suppose that E is an exceptional sheaf on S then we get one of the following options*
1) E is locally free;
2) E has a torsion subsheaf such that $(\mathrm{supp} TE) \cdot K_S = 0$;
3) $E \cong \mathcal{O}_e(d)$ for some rational curve e with $e^2 = -1$;
4) $r(E) = 0$ and the support of E contains an irreducible component C_0 such that $C_0 \cdot K_S = 0$.

2.2.7 COROLLARY. *(Orlov) If $-K_S$ is ample (S is the Del Pezzo surface) then an exceptional sheaf on S either is locally free or has the form $\mathcal{O}_e(d)$ for some rational curve e with $e^2 = -1$.*

Now let us prove the stability of exceptional bundles on S with respect to the anticanonical class $H = -K_S$.

2.2.8 LEMMA. *(S. Zube) Let D be a smooth elliptic curve from $|H|$ and E be an exceptional bundle on S. Then the restriction of E to D ($E' = E|_D$)is a simple bundle, i.e. $\mathrm{Ext}^0(E', E') = \mathbb{C}$.*

PROOF. Consider the exact sequence

$$0 \longrightarrow E^* \otimes E(K_S) \longrightarrow E^* \otimes E \longrightarrow (E^* \otimes E)|_D \longrightarrow 0.$$

By Serre duality,

$$\mathrm{Ext}^k(E, E(K_S))^* \cong \mathrm{Ext}^{2-k}(E, E).$$

Since E is exceptional we obtain

$$\mathrm{Ext}^0(E, E) = \mathbb{C}, \quad \mathrm{Ext}^1(E, E) = \mathrm{Ext}^2(E, E) = 0.$$

Therefore the cohomology table associated with the exact sequence has the form:

k	$\mathrm{Ext}^k(E, E(K_S))$	$\to$	$\mathrm{Ext}^k(E, E)$	$\to$	$\mathrm{Ext}^k(E', E')$
0	0		$\mathbb{C}$		?
1	0		0		?
2	$\mathbb{C}$		0		?

It implies that $\mathrm{Ext}^0(E', E') = \mathbb{C}$.

2.2.9 COROLLARY. *Any exceptional bundle E on S is stable with respect to the slope $\mu_H = (H \cdot c_1(E))/r(E)$ where $H = -K_S$.*

PROOF. By the Zube lemma the restriction of E to an elliptic curve $D \in |-K_S|$ is simple. It is known that simple bundles on an elliptic curve are stable with respect to the slope $\mu(E) = \frac{\deg(E)}{r(E)}$. Besides, $\mu_H(E) = \mu(E')$ where $E' = E|_D$. Now suppose F is a subsheaf of E such that $r(F) < r(E)$ and $\mu_H(F) > \mu_H(E)$. Without loss of generality we can assume that $F' = F|_D$ is locally free. Thus, $\mu(F') > \mu(E')$. This contradicts to the stability of E'. The corollary is proved.

2.3 Exceptional Collections.

The main results about rigid and superrigid sheaves are formulated in terms of exceptional collections. The aim of this section is to study these collections on the surface S.

DEFINITION. An ordered collection $(E_1, E_2, \ldots, E_n)$ of exceptional sheaves is called *exceptional* whenever

$$\mathrm{Ext}^k(E_i, E_j) = 0 \qquad \text{for } i > j \qquad \text{and} \quad k = 0, 1, 2.$$

An exceptional collection (E, F) is called an *exceptional pair*.

By definition an ordered collection is exceptional if and only if all its pair are exceptional. Thus we shall study exceptional pairs on S.

Suppose (E, F) is an exceptional pair on a Del Pezzo surface. It is known that then we have one of the following cases:

a pair (E, F) has the type *hom* (or in other words (E, F) is a *hom*-pair), that is

$$\mathrm{Ext}^i(E, F) = 0 \quad \text{for} \quad i = 1, 2 \qquad \text{and} \quad \mathrm{Hom}(E, F) \neq 0;$$

a pair (E, F) has the type *ext* (or in other words (E, F) is a *ext*-pair), that is

$$\mathrm{Ext}^i(E, F) = 0 \quad \text{for} \quad i = 0, 2 \qquad \text{and} \quad \mathrm{Ext}^1(E, F) \neq 0;$$

a pair (E, F) has the type *zero* (or in other words (E, F) is a zero-pair), that is

$$\mathrm{Ext}^i(E, F) = 0 \quad \text{for} \quad i = 0, 1, 2.$$

There exist exceptional pairs of a new type in our surfaces.

DEFINITION. An exceptional pair (E, F) is called *singular* if

$$\mathrm{Ext}^i(E, F) \neq 0 \quad \text{for } i = 0, 1 \qquad \text{and} \qquad \mathrm{Ext}^2(E, F) = 0.$$

2.3.1 PROPOSITION. *Let (E, F) be an exceptional pair of bundles on the surface S then we have one of the following cases:*

a)	*(E,F) is a* hom-pair	$\Longleftrightarrow$	$\mu_H(E) < \mu_H(F)$;
b)	*(E,F) is an* ext-pair	$\Longleftrightarrow$	$\mu_H(E) > \mu_H(F)$;
c)	*(E,F) is singular or a zero-pair*	$\Longleftrightarrow$	$\mu_H(E) = \mu_H(F)$.

PROOF. Consider the restriction sequence to a smooth elliptic curve:

$$0 \longrightarrow E^* \otimes F(K_S) \longrightarrow E^* \otimes F \longrightarrow (E^* \otimes F)|_D \longrightarrow 0.$$

Denote $E|_D$ and $F|_D$ by E' and F'. Combining the Serre duality and the definition of exceptional pairs, we get $\text{Ext}^i(E, F(K_S))^* \cong \text{Ext}^{2-i}(F, E) = 0$. Hence the cohomology sequence associated with this exact sequence has the form:

k	$\text{Ext}^k(E, F(K_S))$ $\rightarrow$	$\text{Ext}^k(E, F)$ $\rightarrow$	$\text{Ext}^k(E', F')$
0	0	*	*
1	0	*	*
2	0	*	*

That is,

$$\text{Ext}^i(E, F) \cong \text{Ext}^i(E', F') \qquad \forall i.$$

Since E' and F' are stable bundles on the elliptic curve (see the proof of lemma 2.2.8) we obtain that only one of the spaces $\text{Ext}^0(E', F')$ and $\text{Ext}^1(E', F')$ is nonzero whenever

$$\mu_H(E') \neq \mu_H(F') \qquad \left(\mu(E') = \frac{\deg(E')}{r(E')} = \mu_H(E)\right).$$

Moreover, $\text{Ext}^0(E', F') \neq 0$ iff $\chi(E', F') > 0$ and $\text{Ext}^1(E', F') \neq 0$ iff $\chi(E', F') < 0$. In this case $\chi(E', F')$ is the Euler characteristic of two sheaves on an elliptic curve ([10]):

$$\chi(E', F') = r(E')r(F')\big(\mu(F') - \mu(E')\big).$$

Finally, in both cases we have $\text{Ext}^2(E', F') = 0$. This completes the proof.

2.3.2 LEMMA. *Let (E, F) be an exceptional pair of bundles on S with $\mu_H(E) = \mu_H(F)$. Let C denote $c_1(F) - c_1(E)$. Then:*

1. $r(E) = r(F)$.
2. $C^2 = -2$ *and* $K_S \cdot C = 0$.
3. *Suppose (E, F) is a singular pair; then*
   (a) *C is a connected curve;*
   (b) $\text{Ext}^0(E, F) = \text{Ext}^1(E, F) = \mathbb{C}$;
   (c) *there exists an exact sequence*

$$0 \longrightarrow E \longrightarrow F \longrightarrow Q \longrightarrow 0,$$

where Q is a torsion sheaf with $c_1(Q) = C$.

PROOF. By the definition of an exceptional pair $\chi(F, E) = 0$. Substituting the discrete invariants of E and F in the Riemann-Roch formula (2.1.2) for exceptional sheaves, we get

$$0 = \frac{1}{r^2(E)} + \frac{1}{r^2(F)} + \Big(\frac{c_1(F)}{r(F)} - \frac{c_1(E)}{r(E)}\Big)^2.$$

From lemma 2.2.8 it follows that the restriction of an exceptional bundle to the elliptic curve $D \in |H|$ is a simple bundle. Moreover,

$$\mu_H(E) = \mu(E|_D).$$

If L is a simple bundle on an elliptic curve; then $r(L)$ and $\deg(L)$ are coprime ([1]). Hence the equality $\mu(E|_D) = \mu(F|_D)$ implies that $r(E) = r(F) = r$.

Hence we obtain

$$0 = \frac{2}{r^2} + \frac{1}{r^2}\Big(c_1(F) - c_1(E)\Big)^2.$$

This means that

$$C^2 = (c_1(F) - c_1(E))^2 = -2.$$

On the other hand,

$$\mu_H(E) = \frac{c_1(E) \cdot H}{r(E)} = \mu_H(F) = \frac{c_1(F) \cdot H}{r(F)}.$$

Therefore, $C \cdot H = 0$, i.e., $C \cdot K_S = 0$. This concludes the proof of the first and the second statements of the lemma.

3. Let (E, F) be a singular pair, i.e., there exists a nonzero map $\varphi : E \longrightarrow F$. Since exceptional bundles on S are μ_H-stable it follows from lemma 1.1.6 that φ is an injection. Moreover, the cokernel of φ has the zero rank. By definition, put $Q = \operatorname{coker}\varphi$. Since the first Chern class is an additive function we get $c_1(Q) = c_1(F) - c_1(E) = C$.

Consider the restriction sequence:

$$0 \longrightarrow E^* \otimes F(K_S) \longrightarrow E^* \otimes F \longrightarrow (E^* \otimes F)|_D \longrightarrow 0.$$

We have the following isomorphisms:

$$\operatorname{Hom}(E, F) \cong \operatorname{Hom}(E', F'); \quad \operatorname{Ext}^1(E, F) \cong \operatorname{Ext}^1(E', F'); \quad \operatorname{Ext}^2(E, F) = 0,$$

where $E' = E|_D$ and $F' = F|_D$.

By assumption, $\operatorname{Hom}(E, F) \neq 0$. Therefore there exists a nonzero map $\varphi' : E' \longrightarrow F'$. Since E' and F' are stable bundles on a curve and $\mu(E') = \mu(F')$ we see that φ' is an isomorphism. Further, all stable bundles are simple (1.1.7) and the canonical class of an elliptic curve is trivial. It follows from Serre duality that $\operatorname{Ext}^1(E', F') = \mathbb{C}$. Thus we have $\operatorname{Ext}^1(E, F) \cong \operatorname{Ext}^0(E, F) = \mathbb{C}$.

Now we show that Q is simple. Let us write the cohomology tables associated with the exact sequence

$$0 \longrightarrow E \longrightarrow F \longrightarrow Q \longrightarrow 0.$$

k	$\operatorname{Ext}^k(E, E)$	$\to$	$\operatorname{Ext}^k(E, F)$	$\to$	$\operatorname{Ext}^k(E, Q)$
0	$\mathbb{C}$		$\mathbb{C}$		?
1	0		$\mathbb{C}$		?
2	0		0		?

;

k	$\mathrm{Ext}^k(F,E)$	$\to$	$\mathrm{Ext}^k(F,F)$	$\to$	$\mathrm{Ext}^k(F,Q)$
0	0		$\mathbb{C}$		?
1	0		0		?
2	0		0		?

;

k	$\mathrm{Ext}^k(Q,Q)$	$\to$	$\mathrm{Ext}^k(F,Q)$	$\to$	$\mathrm{Ext}^k(E,Q)$
0	?		$\mathbb{C}$		0
1	?		0		$\mathbb{C}$
2	?		0		0

.

From the last table it follows that the quotient Q is simple. Hence $C = \mathrm{supp}Q$ is connected. In fact the group of endomorphisms of Q contains projectors unless $\mathrm{supp}Q$ is connected.

2.4 Structure of Rigid Sheaves.

In the paper [11] it was shown that any rigid bundle on Del Pezzo surface is a direct sum of exceptional bundles. At the same time there exist indecomposable rigid bundles E with $\mathrm{Hom}(E,E) \neq \mathbb{C}$ in the case when $H = -K_S$ is nef. For example, consider a -2-curve C on S with $C \cdot K_S = 0$. It can be easily shown that $(\mathcal{O}_S, \mathcal{O}_S(C))$ is an exceptional singular pair. Denote by E a nontrivial extension of $\mathcal{O}_S$ by $\mathcal{O}_S(C)$:

$$0 \longrightarrow \mathcal{O}(C) \longrightarrow E \longrightarrow \mathcal{O} \longrightarrow 0.$$

It can be proved that E is rigid and $\mathrm{Hom}(E,E) \cong \mathbb{C}^2$.

In this section we prove that any rigid bundle on the surface S with the nef anticanonical class and $K_S^2 > 0$ has the similar structure. Unfortunately, the structure of rigid bundles on S with $K_S^2 = 0$ is not known. From now one assume that $K_S^2 > 0$.

DEFINITION. We say that a torsion-free sheaf F has an *exceptional filtration* whenever there exists a filtration of F

$$Gr(F) = (x_n E_n, x_{n-1}E_{n-1}, \ldots, x_1 E_1),$$

where $(E_1, E_2, \ldots, E_n)$ is an exceptional collection of bundles such that $\mu_H(E_i) \leq \mu_H(E_{i+1})$ for $i = 1, 2, \ldots, n-1$.

The aim of this section is to prove the following theorem:

2.4.1 THEOREM. *Let S be a smooth complex projective surface the anticanonical class of which has no base components and $K_S^2 > 0$. Then*

1. *Any torsion free rigid sheaf on S is a direct sum of μ_H-semistable rigid bundles.*
2. *Any indecomposable rigid sheaf without torsion on S is μ_H-semistable.*
3. *Any μ_H-semistable rigid sheaf has an exceptional filtration. Moreover, all pairs of associated exceptional collection have the zero or the singular type.*

We shall use the vector slope

$$\bar{\gamma}(E) = (\mu_H(E), \mu_A(E), \frac{c_1^2(E) - 2c_2(E)}{r(E)})$$

where A is an ample divisor, H is the anticanonical class of S, and $\mu_D(E) = \frac{c_1(E)\cdot D}{r(E)}$ with $D =$ H or A.

It can easily be checked that the stability with respect to this slope is the Gieseker stability 1.3.3 . In particular, any $\bar{\gamma}$-semistable sheaf has the filtration by isotypic quotients (1.3.6).

The slope $\mu_H(E)$ has the Mumford-Takemoto type and satisfy the axiom SLOPE.4.

2.4.2 LEMMA. *Let F be a $\bar{\gamma}$-semistable rigid sheaf on S with $K_S^2 > 0$. Suppose F has a filtration by $\bar{\gamma}$-stable isomorphic one to another quotients:*

$$Gr(F) = (G_n, G_{n-1}, \dots, G_1) \qquad (G_i \cong E \quad \forall i);$$

Then F is the multiple of the exceptional bundle E, i.e $F = nE$.

PROOF. Consider the spectral sequence associated with the filtration of F which converge to the groups $\mathrm{Ext}^k(F, F)$. Its E_1-term has the form:

$$E_1^{pq} = \bigoplus_i \mathrm{Ext}^{p+q}(G_i, G_{p+i}).$$

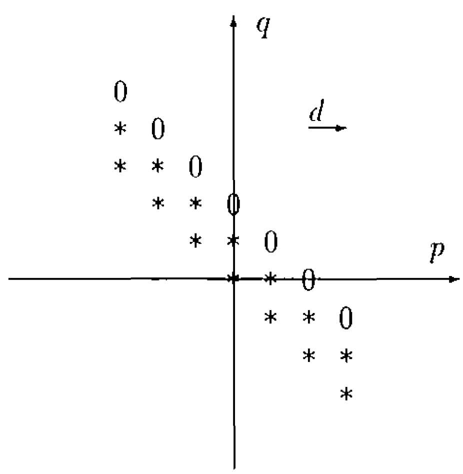
Since the quotients $G_i \cong E$ are $\bar{\gamma}$-stable we see that they are μ_H-semistable (see remark 1.1.1). Hence it follows from lemma 1.3.5 that the sheaf $E(K_S)$ is also μ_H-semistable. On the other hand, the square of the canonical class of our surface is positive. Thus,

$$\mu_H(E(K_S)) = \mu_H(E) + K_S \cdot H < \mu_H(E).$$

Now, using Serre duality and lemma 1.1.5 we have

$$\mathrm{Ext}^2(E, E) = 0.$$

Thus, the E_1-term of the spectral sequence has the form:



This yields that $\mathrm{Ext}^1(G_n, G_1) = E_1^{1-n,n} = E_\infty^{1-n,n}$. On the other hand,

$$E_\infty^{1-n,n} \subset \mathrm{Ext}^1(F, F) = 0.$$

But, $G_i \cong E \quad \forall i$. Consequently E is a torsion-free rigid sheaf. It follows from lemma 2.2.1 that E is locally free. Besides, since E is $\bar{\gamma}$-stable we see that it is simple and $\mathrm{Ext}^2(E, E) = 0$, whereby E is an exceptional bundle.

Finally, since $G_i \cong E \quad \forall i$ and E is exceptional, we have

$$\mathrm{Ext}^1(G_i, G_j) = 0 \quad \forall i, j.$$

This implies the equality:

$$F = \bigoplus_i G_i = nE.$$

2.4.3 LEMMA. *Suppose F is a rigid $\bar{\gamma}$-semistable sheaf on S then F is a direct sum of exceptional bundles.*

PROOF. By proposition 1.3.6 it follows that F has a filtration by isotypic quotients:

$$0 = F_{n+1} \subset F_n \subset \cdots \subset F_2 \subset F_1 = F$$

where $G_i = F_i/F_{i+1}$ are $\bar{\gamma}$-semistable and they have filtrations by isomorphic one to another $\bar{\gamma}$-stable quotients. Besides,

$$\forall i: \qquad \bar{\gamma}(G_i) = \bar{\gamma}(F), \qquad \mathrm{Hom}(F_{i+1}, G_i) = 0.$$

Let us apply the Mukai lemma (2.1.4) to the exact sequence:

$$0 \longrightarrow F_{i+1} \longrightarrow F_i \longrightarrow G_i \longrightarrow 0.$$

It can be proved by induction on i that G_i and F_{i+1} are torsion-free rigid sheaves.

Note that each G_i satisfies the assumption of the previous lemma. Therefore we have $G_i = x_i E_i$, where E_i are exceptional bundles.

Since all G_i are $\bar{\gamma}$-semistable we see that they are μ_H-semistable (1.1.1). Moreover,

$$\forall i: \qquad \bar{\gamma}(G_i) = \bar{\gamma}(F), \qquad \Longrightarrow \qquad \mu_H(G_i) = \mu_H(F).$$

Hence, by the same argument as before, we get $\mathrm{Ext}^2(G_i, G_j) = 0 \quad \forall i, j$. Thus the E_1-term of the spectral sequence associated with the filtration of F has the same form as in the proof of lemma 2.4.2 Therefore,

$$\mathrm{Ext}^1(G_n, G_1) = 0.$$

But in this case the quotients of the filtration of F are different. To complete the proof we need the information about the groups $\mathrm{Ext}^1(G_i, G_j)$ for $i < j$.

Let us recall that $G_i = x_i E_i$, where E_i are the exceptional bundles. Hence,

$$\mathrm{Ext}^1(G_n, G_1) \quad \Longrightarrow \quad \mathrm{Ext}^1(E_n, E_1) = 0. \tag{2}$$

By construction, the bundles E_i are $\bar{\gamma}$-stable and $\bar{\gamma}(E_i) = \bar{\gamma}(F)$ $\forall i$. Since E_i are $\bar{\gamma}$-stable, it follows from lemma 1.3.4 that $E_n \cong E_1$ provided $\mathrm{Hom}(E_n, E_1) \neq 0$ or $\mathrm{Hom}(E_1, E_n) \neq 0$.

Suppose $E_n \cong E_1$ then by (2) we have $\mathrm{Ext}^1(E_1, E_n) = 0$.

Assume that $E_n \not\cong E_1$ then we obtain

$$\mathrm{Ext}^0(E_n, E_1) = \mathrm{Ext}^0(E_1, E_n) = 0. \tag{3}$$

Since $\bar{\gamma}(E_1) = \bar{\gamma}(E_n)$ we get $\mu_H(E_1) = \mu_H(E_n)$.

Combining the μ_H-stability of exceptional bundles on S, Serre duality and the inequality $K_S^2 > 0$, we obtain

$$\mathrm{Ext}^2(E_n, E_1) = \mathrm{Ext}^2(E_1, E_n) = 0.$$

Combining this with (3) we get

$$\chi(E_1, E_n) = -h^1(E_1, E_n); \qquad \chi(E_n, E_1) = -h^1(E_n, E_1) = 0.$$

On the other hand, it follows from the Riemann-Roch theorem for exceptional sheaves (2.1.2) and the equality $\mu_H(E_1) = \mu_H(E_n)$ that $\chi(E_1, E_n) = \chi(E_n, E_1)$. That is,

$$h^1(E_1, E_n) = h^1(E_n, E_1) = 0.$$

Thus we proved that $\mathrm{Ext}^1(E_1, E_n) = 0$. This yields that $\mathrm{Ext}^1(G_1, G_n) = 0$.

Note that the second term of the filtration (i.e. F_2) satisfies the assumptions of our lemma. At the same time

$$Gr(F_2) = (G_n, G_{n-1}, \dots, G_2).$$

By the inductive hypothesis one can assume that

$$F_2 = \bigoplus_{i=2}^{n} G_i.$$

That is the sheaf F is included in the exact sequence:

$$0 \longrightarrow \bigoplus_{i=2}^{n} G_i \longrightarrow F \longrightarrow G_1 \longrightarrow 0.$$

Since $\mathrm{Ext}^1(G_1, G_n) = 0$ we obtain $F = \tilde{F} \oplus G_n$ where $\tilde{F}$ is a $\bar{\gamma}$-semistable rigid sheaf with

$$Gr(\tilde{F}) = (G_{n-1}, G_{n-2}, \dots, G_1).$$

Using the inductive hypothesis again we have $\tilde{F} = \bigoplus_{i=1}^{n-1} G_i$. That is,

$$F = \bigoplus_{i=1}^{n} G_i = \bigoplus_{i=1}^{n} x_i E_i.$$

This completes the proof.

2.4.4 LEMMA. *Any rigid μ_H-semistable sheaf F on the surface S has an exceptional filtration:*

$$Gr(F) = (x_m E_m, x_{m-1} E_{m-1}, \dots, x_1 E_1)$$

such that all exceptional pairs of collection $(E_1, E_2, \dots, E_n)$ *are the singular or the zero-pairs.*

PROOF. Suppose F is $\bar{\gamma}$-semistable; then by the previous lemma, $F = \bigoplus_i x_i E_i$, where E_i are $\bar{\gamma}$-stable exceptional bundles with equal one to another $\bar{\gamma}$-slopes. Without loss of generality it can be assumed that $E_i \ncong E_j$ for $i \neq j$. Using lemma 1.3.4 , we get

$$\mathrm{Ext}^0(E_i, E_j) = 0 \quad \forall i, j. \tag{4}$$

On the other hand, the equality $\bar{\gamma}(E_i) = \bar{\gamma}(E_j)$ yields that $\mu_H(E_i) = \mu_H(E_j)$. Now the equality

$$\mathrm{Ext}^2(E_i, E_j) = 0 \quad \forall i, j \tag{5}$$

can be proved by the standard method for the surface S with $K_S^2 > 0$.

Finally, since F is rigid we have

$$0 = \mathrm{Ext}^1(F, F) = \mathrm{Ext}^1(\bigoplus_i x_i E_i, \bigoplus_i x_j E_j) = \bigoplus_{i,j} \mathrm{Ext}^1(E_i, E_j),$$

i.e., $\mathrm{Ext}^1(E_i, E_j) = 0$.

Combining this with equalities (4) and (5) we see that each pair of bundles in the collection $(E_1, E_2, \dots, E_n)$ is an exceptional zero-pair.

Now we suppose that F is not $\bar{\gamma}$-semistable. Consider its Harder-Narasimhan filtration:

$$Gr(F) = (G_n, G_{n-1}, \dots, G_1)$$

(see proposition 1.2.3).

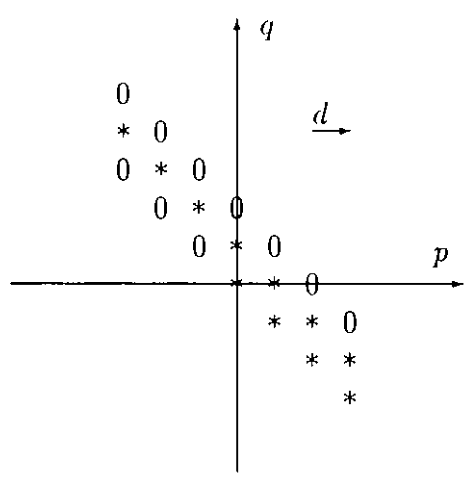
Since G_i are $\bar{\gamma}$-semistable and $\bar{\gamma}(G_i) > \bar{\gamma}(G_{i-1})$ for all i, we get

$$\mathrm{Ext}^0(G_i, G_j) = 0 \quad \forall i > j. \tag{6}$$

Note that lemma 1.1.8 and μ_H-semistability of the sheaf F imply μ_H-semistability of the quotients G_i and the equality $\mu_H(G_i) = \mu_H(F)$. Therefore, as before,

$$\mathrm{Ext}^2(G_i, G_j) = 0 \quad \forall i, j. \tag{7}$$

Combining (6) and (7), we see that the E_1-term of the spectral sequence associated with the Harder-Narasimhan filtration of F has the form:



This spectral sequence converges to the groups $\mathrm{Ext}^k(F,F)$ of the rigid sheaf. Hence,

$$\mathrm{Ext}^1(G_i,G_j)=0 \quad \forall i\geq j. \tag{8}$$

In particular, all G_i are rigid $\bar{\gamma}$-semistable sheaves. By the previous lemma $G_i = \bigoplus_k x_{ik}E_{ik}$, where E_{ik} are exceptional bundles. Besides, any pair (E_{ik},E_{is}) has the zero type.

Combining (6), (7) and (8), we obtain that $\mathrm{Ext}^d(E_{ik},E_{js})=0$ for $i>j$ and $d=0,1,2$. In other words, the set of all bundles E_{ik} can be enumerated in such a way the collection $(E_1,E_2,\dots,E_m)$ is exceptional. It remains to note that all bundles E_i have the μ_H-slope coinciding with $\mu_H(F)$. Thus it follows from 2.3.1 that each pair from this collection is either the singular or the zero-pair.

The plan of the proof of the main theorem is clear now. We consider the spectral sequence associated with the Harder-Narasimhan filtration of a rigid torsion-free sheaf to obtain the information about the groups $\mathrm{Ext}^1(G_i,G_j)$ where G_i are quotients of this filtration. To do this we need the following last statement.

2.4.5 LEMMA. *Let G_1 and G_2 be μ_H-semistable rigid sheaves on the surface S. Suppose $\mu_H(G_2)>\mu_H(G_1)$; then the equality $\mathrm{Ext}^1(G_2,G_1)=0$ implies $\mathrm{Ext}^1(G_1,G_2)=0$.*

PROOF. It follows from lemma 2.4.4 that each of G_i has the exceptional filtration:

$$Gr(G_1)=(x_{n1}E_{n1},\dots,x_{11}E_{11});\quad Gr(G_2)=(x_{m2}E_{m2},\dots,x_{12}E_{12}).$$

Moreover, μ_H-slopes of E_{ij} do not depend on the first index, i.e, $\mu_H(E_{i1})=\mu_H(G_1)$ and $\mu_H(E_{j2})=\mu_H(G_2)$.

Denote by G'_i the restriction of the sheaves G_i to an elliptic curve $D\in|-K_S|$. It is obvious that the sheaves G'_i have the filtrations:

$$Gr(G'_1)=(x_{n1}E'_{n1},\dots,x_{11}E'_{11}),\qquad Gr(G'_2)=(x_{m2}E'_{m2},\dots,x_{12}E'_{12}),$$

where $E'_{ki}=E_{ki}|_D$. Moreover, since E_{ki} are exceptional bundles we see that E'_{ki} are stable with respect to the standard slope μ on the curve (see lemma 2.2.8). Furthemore,

$$\mu(E'_{ki})=\mu_H(E_{ki})=\mu_H(G_i)=\mu(G'_i).$$

Now by lemma 1.1.8 G'_i are μ-semistable and $\mu(G'_2)>\mu(G'_1)$. Thus from lemma 1.1.5 it follows that $\mathrm{Hom}(G'_2,G'_1)=0$.

Using the last equality and the long cohomology sequence associated with the exact triple

$$0\longrightarrow G_2^*\otimes G_1(K_S)\longrightarrow G_2^*\otimes G_1\longrightarrow (G_2^*\otimes G_1)|_D\longrightarrow 0,$$

we obtain

$$\mathrm{Ext}^1(G_2,G_1(K_s))\subset \mathrm{Ext}^1(G_2,G_1).$$

Now the proof follows from Serre duality.

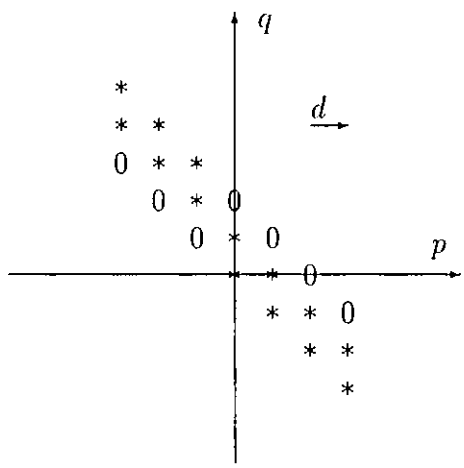
PROOF OF THEOREM 2.4.1 . Let F be any torsion-free rigid sheaf on S. Consider its Harder-Narasimhan filtration by μ_H-semistable quotients

$$Gr(F)=(G_n,G_{n-1},\dots,G_1).$$

It follows from the inequalities $\mu_H(G_i) > \mu_H(G_j)$ for $i > j$ that

$$\mathrm{Hom}(G_i, G_j) = 0 \text{ for } i > j \quad \text{and} \quad \mathrm{Ext}^2(G_j, G_i) = 0 \text{ for } i \geq j.$$

Therefore the E_1-term of the spectral sequence associated with this filtration has the form:



Since the sequence is convergent to the groups $\mathrm{Ext}^i(F, F)$ of the rigid sheaf, we obtain

$$0 = E_\infty^{-1,2} = E_1^{-1,2} = \bigoplus_i \mathrm{Ext}^1(G_i, G_{i-1}),$$

$$0 = E_\infty^{0,1} = E_1^{0,1} = \bigoplus_i \mathrm{Ext}^1(G_i, G_i),$$

That is G_i are rigid μ_H-semistable sheaves and $\mathrm{Ext}^1(G_i, G_{i-1}) = 0$.

By the previous lemma the groups $\mathrm{Ext}^1(G_{i-1}, G_i)$ are also trivial. In particular,

$$\mathrm{Ext}^1(G_1, G_2) = 0.$$

Let F_2 be the first term of the filtration $Gr(F)$, i.e.,

$$0 \longrightarrow F_2 \longrightarrow F \longrightarrow G_1 \longrightarrow 0. \tag{9}$$

Note that $Gr(F_2) = (G_n, \ldots, G_2)$ is also the Harder-Narasimhan filtration and $\mu_H(G_2) > \mu_H(G_1)$. Taking into account corollary 1.2.5 we obtain $\mathrm{Hom}(F_2, G_1) = 0$. In addition, the sheaves F_2 and G_1 have no torsion. Hence we can apply lemma 2.2.2 to these sheaves and get $\mathrm{Ext}^2(G_1, F_2) = 0$.

Now applying the Mukai lemma to (9) we obtain

$$h^1(F, F) \geq h^1(F_2, F_2) + h^1(G_1, G_1).$$

That is the sheaf F_2 is also rigid. The number of its Harder-Narasimhan filtration quotients is less then n. Hence by the inductive hypothesis we have $F_2 = \bigoplus_{i=2}^{n} G_i$, and

$$0 \longrightarrow \bigoplus_{i=2}^{n} G_i \longrightarrow F \longrightarrow G_1 \longrightarrow 0.$$

Let us recal that $\mathrm{Ext}^1(G_1, G_2) = 0$. Therefore, $F = \tilde{F} \oplus G_2$, where $\tilde{F}$ is a torsion-free rigid sheaf. We apply again the inductive hypothesis to the sheaf $\tilde{F}$ to obtain

$$F = \bigoplus_{i=1}^{n} G_i.$$

Thus any rigid sheaf without torsion on S is a direct sum of μ_H-semistable rigid sheaves. They are locally free by 2.2.1 . In particular, if F is indecomposable then it is μ_H-semistable. This concludes the proof of the first and the second theorem statements. The last is equivalent to lemma 2.4.4 .

2.4.6 COROLLARY. *Any torsion-free rigid sheaf on a Del Pezzo surface X is a direct sum of exceptional bundles.*

PROOF. Since the anticanonical class of a Del Pezzo surface is ample we see that exceptional pairs on X cannot be singular (see 2.3.2). On the other hand, we have proved that any indecomposable torsion-free rigid sheaf on S (in particular, on X) has an exceptional filtration. Besides, all pairs in associated exceptional collection are singular or zero. Thus the quotients of the exceptional filtration of any torsion-free rigid sheaf on X are its direct summands.

2.5 Structure of Superrigid Sheaves.

In the previous section we have proved that any μ_H-semistable sheaf has the exceptional filtration and any torsion-free rigid sheaf is a direct sum of μ_H-semistable rigid bundles. Therefore for classifying rigid bundles we need a description of exceptional bundles and collections of ones. This description is the subject of the next part. To start it we need the following theorem.

2.5.1 THEOREM. *Let S be a smooth complex projective surface the anticanonical class H of which has no base components and $H^2 > 0$. Then the following statements hold.*

1. For any exceptional collection of bundles $(E_1, E_2, \ldots, E_n)$ on S such that $\mu_H(E_i) \leq \mu_H(E_{i+1})$ $\forall i$ there exists a superrigid bundle E

$$(\mathrm{Ext}^1(E, E) = \mathrm{Ext}^2(E, E) = 0)$$

such that $Gr(E) = (x_n E_n, x_{n-1} E_{n-1}, \ldots, x_1 E_1)$. We say that this bundle is associated with the exceptional collection.

2. Any superrigid torsion-free sheaf E has the exceptional filtration

$$Gr(E) = (x_n E_n, x_{n-1} E_{n-1}, \ldots, x_1 E_1),$$

i.e. the collection $(E_1, E_2, \dots, E_n)$ *is exceptional and the* μ_H*-slopes of bundles* E_i *satisfy the inequalities:* $\mu_H(E_i) \le \mu_H(E_{i+1}) \quad \forall i$.

3. Suppose a superrigid torsion-free sheaf E *has two exceptional filtrations:*

$$Gr(E) = (x_n E_n, x_{n-1} E_{n-1}, \dots, x_1 E_1) = (y_m F_m, y_{m-1} F_{m-1}, \dots, y_1 F_1);$$

then $m = n$ *and the exceptional collection* $(F_1, F_2, \dots, F_m)$ *can be obtained from* $(E_1, E_2, \dots, E_n)$ *by mutations of neighboring zero-pairs* (E_i, E_{i+1}).

Note that this theorem is obvious provided S is a Del Pezzo surface (see 2.4.6). But if $-K_S$ is nef then the statement is nontrivial and its proof is difficult.

Let us first state and prove several lemmas.

2.5.2 REMARK. *Suppose a sheaf* F *has a filtration* $Gr(F) = (G_n, G_{n-1}, \dots, G_1)$ *such that* $Gr(G_i) = (E_{ik_i}, \dots, E_{i1})$; *then there exists the filtration*

$$Gr(F) = (E_{nk_n}, \dots, E_{n1}, \dots, E_{1k_1}, \dots, E_{11}).$$

And back to front, the neighboring quotients can be "join".

2.5.3 REMARK. *Suppose* $Gr(F) = (G_n, G_{n-1}, \dots, G_1)$ *is a filtration of a sheaf* F *such that* $\mathrm{Ext}^1(G_i, G_{i+1}) = \mathrm{Ext}^1(G_{i+1}, G_i) = 0$ *for certain* i; *then the sheaf* F *has the filtration* $Gr(F) = (G_n, G_{n-1}, \dots, G_i, G_{i+1}, \dots G_1)$.

2.5.4 LEMMA. *Suppose* F *is an indecomposable rigid bundle on* S *with* $K_S^2 > 0$; *then* F *has the following filtrations:*

a) $Gr_R(F) = (Q_n, Q_{n-1}, \dots, Q_1)$ *such that* $\forall i \quad Q_i = \bigoplus y_{is} E_{is}$; E_{is} *are exceptional bundles, the collection* $(E_{11}, \dots, E_{1m_1}, \dots, E_{n1}, \dots, E_{nm_n})$ *is exceptional and for each bundle* $E_{is} \quad (i = 1, \dots, n-1)$ *there is* $E_{i+1,k}$ *such that the pair* $(E_{i,s}, E_{i+1,k})$ *is singular.*

b) $Gr_L(F) = (G_n, G_{n-1}, \dots, G_1)$ *such that* $\forall i \quad G_i = \bigoplus x_{is} E_{is}$; E_{is} *are exceptional bundles, the collection* $(E_{11}, \dots, E_{1k_1}, \dots, E_{n1}, \dots, E_{nk_n})$ *is exceptional and for each bundle* $E_{is} \quad (i = 2, \dots, n)$ *there is* $E_{i-1,l}$ *such that the pair* $(E_{i-1,l}, E_{is})$ *is singular.*

PROOF. Let us construct the first filtration. The second one is constructed similarly.

By the theorem about rigid bundles (2.4.1) the sheaf F has the exceptional filtration

$$Gr(F) = (x_n E_n, x_{n-1} E_{n-1}, \dots, x_1 E_1).$$

Let us subdivide the exceptional collection

$$(E_1, E_2, \dots, E_n),$$

associated with this filtration into subcollections

$$(E_{i_{s-1}+1}, E_{i_{s-1}+2}, \dots, E_{i_s}),$$

where $i_0 = 0$ such that the following conditions hold.

PART.1: *Any pair of each subcollection has the zero type whenever this subcollection contains more than one bundle.*

PART.2: *For the last bundle E_{i_s} of each subcollection there exists E_j in the next subcollection such that the pair (E_{i_s}, E_j) is singular.*

There exists at most one singular pair in this collection since the bundle F is indecomposable. This implies that this partition exists.

Denote by Q_s the direct sum

$$\bigoplus_{j=i_{s-1}+1}^{i_s} x_j E_j$$

of exceptional bundles from the subcollection with the index s. By remark 2.5.2 F has the filtration $Gr(F) = (Q_k, Q_{k-1}, \dots, Q_1)$ where k is the number of all subcollections. Note that this filtration coincides with Gr_R if and only if the collection decomposed into subcollections satisfies the conditions PART.1 and the following

PART.2R: *for any bundle E_i of each subcollection there is a bundle E_j in the next subcollection such that the pair (E_i, E_j) is singular.*

To construct the required collection we shall intermix the bundles of subcollections and move sometimes bundles from subcollection to the next subcollection.

Suppose there is a bundle E_α in the first subcollection such that for all bundles E_β in the second one all (E_α, E_β) are zero pairs. Let us shift E_α to the second subcollection. Since this shift can be realized by permutations of members in neighboring zero-pairs we get that the obtained collection is exceptional. Besides, it satisfy the conditions PART.1 and PART.2. It is clear that after a finite number of such shifts we get the exceptional collection decomposed into subcollections such that for each bundle E_α of the first subcollection there is E_β in the second one such that (E_α, E_β) is a singular pair. Let us mention that one can do the some thing with an arbitrary pair of neighboring subcollections. This process will be called the *displacement*.

Let us do the displacement with each pair of the neighboring subcollections, starting from the first one. The number of the subcollections does not change during the process. The number of the bundles in the first subcollection can only decrease. Two latter subcollections will satisfy the condition PART.2.R. But since we moved the bundles from the left to the right one can find now two neighboring subcollections (with the numbers s and $s+1$, for example) satisfying the following conditions. Any pair (E_i, E_j) with E_i belonging to the s-th subcollection and E_j belonging to the $(s+1)$-th subcollection has the type zero. Moreover, one can guarantee that two latter subcollection satisfy the condition PART.2R only.

Let us unite (if it is necessary) the neighboring subcollections to satisfy the conditions PART.1 and PART.2.

Let us do the displacement with each pair of neighboring subcollection and join all what is possible to join, ets...

This process cannot be repeated ad infinitum. Indeed there exists $k_0 \in \mathbb{N}$ such that for any $k > k_0$ the number of the subcollections will not change after the k-th step consisting of "the displacement and the join". After some successive step the number of bundles in the first subcollection will not change, ets... Thus, the number of the subcollections and the number of the bundles in each subcollection will not change since some moment. That means that any bundle does not go from one subcollection to another. Hence we are done.

2.5.5 LEMMA. *Let F be an indecomposable torsion-free rigid sheaf on the surface S. Assume that*

$$Gr(F) = (x_n E_n, x_{n-1} E_{n-1}, \dots, x_1 E_1) = (y_m F_m, y_{m-1} F_{m-1}, \dots, y_1 F_1)$$

are two exceptional filtrations of F, i.e. the collections

$$(E_1, E_2, \dots, E_n) \quad \text{and} \quad (F_1, F_2, \dots, F_m)$$

are exceptional. Suppose

$$\mu_H(E_i) = \mu_H(F_j) = \mu_H(F) \qquad \forall i, j; \tag{10}$$

then $m = n$ and the collection $(E_1, E_2, \dots, E_n)$ can be obtained from $(F_1, F_2, \dots, F_m)$ by mutations of the neighboring zero pairs.

PROOF. It follows from proposition 2.3.1 that each pair of these collections has the zero or the singular type. Let us show that any such collection can be ordered by the $\bar{\gamma}$-slope using the permutations of the neighboring zero pairs members only. In this case the lemma follows from the uniqueness of the Harder-Narasimhan filtration (1.2.3).

The possibility of such ordering is obtained by induction on the number of the terms in the collection by the following arguments.

Suppose that (E, F) is a singular pair; then it follows from lemma 2.3.2 that ranks of the sheaves E and F coicude and $c_1(F) - c_1(E) = C$ is an effective -2-divisor. (Recall that the $\bar{\gamma}$-slope is the vector

$$(\mu_H, \mu_A, \frac{c_1^2 - 2c_2}{r}),$$

where $\mu_A = \frac{c_1 \cdot A}{r}$, and A is an ample divisor.) Since A is ample, we get $\mu_A(E) < \mu_A(F)$. By assumption we have $\mu_H(E) = \mu_H(F)$. Therefore, $\bar{\gamma}(E) < \bar{\gamma}(F)$.

2.5.6 LEMMA. *Suppose that (E, F) is an exceptional singular pair on S and G is a torsion-free sheaf; then the following implications hold*

a) $\mathrm{Ext}^2(G, E) = 0 \implies \mathrm{Ext}^2(G, F) = 0$;
b) $\mathrm{Ext}^0(G, F) = 0 \implies \mathrm{Ext}^0(G, E) = 0$;
c) $\mathrm{Ext}^0(E, G) = 0 \implies \mathrm{Ext}^0(F, G) = 0$;
d) $\mathrm{Ext}^2(F, G) = 0 \implies \mathrm{Ext}^2(E, G) = 0$;

PROOF. The lemma follows from the cohomology tables associated with the exact triple

$$0 \longrightarrow E \longrightarrow F \longrightarrow Q \longrightarrow 0,$$

where Q is a torsion sheaf. Moreover, since Q has the zero rank and G is torsion free we get that $\mathrm{Hom}(Q, G) = 0$. Furthemore, using Serre duality, we get $\mathrm{Ext}^2(G, Q) = 0$.

2.5.7 LEMMA. *Let F be a μ_H-semistable rigid bundle. Let*

$$Gr(F) = (x_n E_n, x_{n-1} E_{n-1}, \dots, x_1 E_1)$$

be its exceptional filtration and G be a torsion-free sheaf. Then

a) $\mathrm{Ext}^i(G,F)=0\ \forall i \iff \mathrm{Ext}^i(G,E_k)=0\ \forall i,k;$
b) $\mathrm{Ext}^i(F,G)=0\ \forall i \iff \mathrm{Ext}^i(E_k,G)=0\ \forall i,k.$

PROOF. Without loss of generality it can be assumed that F is indecomposable. Consider its filtration $Gr_R(F)=(Q_m,Q_{m-1},\dots,Q_1)$ from lemma 2.5.4 . By 2.5.5 we can assume without loss of generality that

$$Q_j=\bigoplus_{i=s_j}^{s_{j+1}-1} y_iE_i \qquad 1=s_1<s_2<\cdots<s_m<s_{m+1}=n+1.$$

To prove the first statement of our lemma it is sufficient to check the following implication

$$\mathrm{Ext}^i(G,F)=0\ \forall i \implies \mathrm{Ext}^i(G,E_k)=0\ \forall i,k.$$

(The second implication follows from 1.2.4 .) Let us apply the functor $\mathrm{Ext}^{\cdot}(G,\cdot)$ to each of the sequences:

$$0\longrightarrow F_{j+1}\longrightarrow F_j\longrightarrow Q_j\longrightarrow 0,$$

where F_j are terms of the filtration $Gr_R(F)$, $F_1=F$ and $F_n=Q_n$. We see that the spaces $\mathrm{Hom}(G,F_j)=0$ for $j=2,3,\dots,n$. In particular, $\mathrm{Hom}(G,Q_n)=0$.

Step 1. Let us show that $\mathrm{Hom}(G,E_i)=0$ for all i.

By the equality $\mathrm{Hom}(G,Q_n)=0$ we get $\mathrm{Hom}(G,E_i)=0$ for $s_m\le i\le n$. By the construction of the filtration $Gr_R(F)$ for each direct summand E_α of the bundle Q_{n-1} there exists a bundle E_β with $s_m\le\beta\le n$ such that (E_α,E_β) is a singular pair. Applying lemma 2.5.6 to this pair we get $\mathrm{Hom}(G,E_\alpha)=0$.

In the same way using the properties of the filtration $Gr_R(F)$ and 2.5.6 we conclude the first step.

Step 2. Let us check that $\mathrm{Ext}^2(G,E_i)=0$ for all i.

Now let us intermix the bundles in collection $(E_1,E_2,\dots,E_n)$ to obtain the filtration $Gr_L(F)$. Without loss of generality we can assume that

$$G_j=\bigoplus_{i=s_j}^{s_{j+1}-1} y_iE_i \qquad 1=s_1<s_2<\cdots<s_m<s_{m+1}=n+1.$$

are quotients of $Gr_L(F)$.

Applying the functor $\mathrm{Ext}^{\cdot}(G,\cdot)$ to the exact triple

$$0\longrightarrow F_2'\longrightarrow F\longrightarrow G_1\longrightarrow 0,$$

where F_2' is the first term of the filtration $Gr_L(F)$ we get $\mathrm{Ext}^2(G,G_1)=0$. This means that $\mathrm{Ext}^2(G,E_i)=0$ for any direct summand of the bundle G_1.

As before, using the properties of the filtration $Gr_L(F)$ and lemma 2.5.6 , we have

$$\mathrm{Ext}^2(G,E_i)=0\quad\forall i.$$

Thus it is proved that for any quotient E_i of the exceptional filtration of the bundle F the groups $\mathrm{Ext}^0(G,E_i)$ and $\mathrm{Ext}^2(G,E_i)$ are trivial. Hence, $\chi(G,E_i)\le 0\quad\forall i$. Since the Euler characteristic of sheaves is an additive function we have

$$\sum_{i=1}^n x_i\chi(G,E_i)=\chi(G,F)=0.$$

Moreover all x_i are positive integers. Thus, $\chi(G, E_i) = 0\ \forall i$. The first statement of the lemma was proved. Similarly the second statement is proved.

PROOF OF THEOREM 2.5.1

1. First we assume that all pairs of the collection $(E_1, E_2, \dots, E_n)$ have the zero or singular type. The proof is by induction on the number n of bundles in the collection. For $n = 1$, there is nothing to prove.

By the inductive hypothesis there exists a superrigid bundle E' such that

$$Gr(E') = (E_n, E_{n-1}, \dots, E_2).$$

Suppose the pair (E_1, E_i) has the zero type for any i then $E = E' \oplus E_1$ is a superrigid bundle (see 1.2.4).

Suppose there is an index i such that (E_1, E_i) is singular then $\mathrm{Ext}^k(E_1, E_i) = \mathbb{C}$ for $k = 0, 1$ and

$$\mathrm{Ext}^2(E_1, E_j) = \mathrm{Ext}^k(E_j, E_1) = 0 \quad \forall j, k.$$

Therefore, $\mathrm{Ext}^k(E', E_1) = 0\ \forall k$ and

$$\mathrm{Ext}^0(E_1, E') = V \neq 0, \quad \mathrm{Ext}^1(E_1, E') = W \neq 0, \quad \mathrm{Ext}^2(E_1, E') = 0.$$

Consider the universal extension:

$$0 \longrightarrow E' \longrightarrow E \longrightarrow W \otimes E_1 \longrightarrow 0.$$

Using the cohomology tables let us show that E is superrigid. The first table has the form:

k	$\mathrm{Ext}^k(E_1, E')$	$\rightarrow$	$\mathrm{Ext}^k(E_1, E)$	$\rightarrow$	$W \otimes \mathrm{Ext}^k(E_1, E_1)$
0	V		?		W
1	W		?		0
2	0		?		0

.

Since the extension is universal, we see that the coboundary homomorphism

$$W \rightarrow \mathrm{Ext}^1(E_1, E')$$

is isomorphism. Hence

$$\mathrm{Ext}^1(E_1, E) = \mathrm{Ext}^2(E_1, E) = 0.$$

The next tables have the form:

k	$\mathrm{Ext}^k(E', E')$	$\rightarrow$	$\mathrm{Ext}^k(E', E)$	$\rightarrow$	$W \otimes \mathrm{Ext}^k(E', E_1)$
0	*		?		0
1	0		?		0
2	0		?		0

,

k	$W^* \otimes \mathrm{Ext}^k(E_1, E)$	$\rightarrow$	$\mathrm{Ext}^k(E, E)$	$\rightarrow$	$\mathrm{Ext}^k(E', E)$
0	*		?		*
1	0		?		0
2	0		?		0

.

Thus, E is a superrigid bundle.

Now assume that $(E_1, E_2, \dots, E_n)$ is an arbitrary exceptional collection of bundles such that

$$\mu_H(E_1) \leq \mu_H(E_2) \leq \cdots \leq \mu_H(E_n).$$

Let us subdivede it into subcollections of bundles with equal μ_H- slopes. Since all pairs in obtained subcollections are the singular or the zero pairs we see that there exists superrigid bundles $G_1, G_2, \dots, G_k$ constructed by these subcollections. Moreover, $\mu_H(G_i) < \mu_H(G_{i+1})$. Now let us recall that a pair (E_i, E_j) of bundles has the type *hom* provided $\mu_H(E_i) < \mu_H(E_j)$, i.e. $\mathrm{Ext}^k(E_i, E_j) = 0$ for $k = 1, 2$ and $\mathrm{Ext}^k(E_j, E_i) = 0$ for $k = 0, 1, 2$. This yields that the bundle $E_i \oplus E_j$ is superrigid . Thus, $\bigoplus G_i$ is the required bundle.

2. It follows from theorem 2.4.1 that a torsion-free superrigid sheaf is a direct sum of μ_H-semistable rigid bundles $F = \bigoplus\limits_{j=1}^{m} F_j$. Without loss of generality we can assume that

$$\mu_H(F_j) < \mu_H(F_{j+1}) \quad \forall i.$$

Since F is superrigid we see that $\mathrm{Ext}^k(F_i, F_j) = 0$ for $k = 1, 2$ and for any pair i, j. Besides, it follows from the μ_H-semistability of F_i and the last inequality that $\mathrm{Hom}(F_j, F_i) = 0$ for $j > i$.

Taking into account theorem 2.4.1 we obtain that each F_j has the exceptional filtration

$$Gr(F_j) = (x_{s_j-1} E_{s_j-1}, \dots, x_{s_{j-1}} E_{s_{j-1}}).$$

Using the previous lemma and the already proved fact that $\mathrm{Ext}^k(F_j, F_i) = 0$ for $j > i$ and $k = 0, 1, 2$ we see that the collection of the direct summands of all bundles F_j (with the some order) is exceptional. This concludes the proof of the second statement.

3. Since any torsion-free rigid sheaf is locally free and direct summands are uniquely determined we see that it is sufficient to prove the third statement in the case when F is an indecomposable superrigid bundle. But this case is already settled in 2.5.5 .

This completes the proof of the theorem.

3 Constructibility of Exceptional bundles.

3.1 Introduction to the Helix Theory.

In this section we recall following [19], [5], [2], and [4] the general concepts and facts related to the exceptional sheaves on manifolds (see the definition of exceptional sheaves in 2.2) and exceptional objects in the derived category.

The notion of exceptional bundles was introduced in the paper [6]. The main result of that paper is the description of Chern classes of semistable bundles on $\mathbb{P}^2$. Exceptional bundles appeared there as some kind of boundary points.

Later the exceptional bundles and the exceptional objects in the derived category of sheaves were studied on the Rudakov's seminar in Moscow. It became clear that the exceptional objects (sheaves) organized in the exceptional collections can generate the whole derived category of sheaves. Therefore, there exists a spectral sequence of Beilinson type associated with an exceptional collection. Let us note that for the first time a spectral sequence of such type in the case of $\mathbb{P}^2$ appeared in [7]. But the general result was obtained by A. L. Gorodentsev ([5]) independently on [7] .

The existence of this spectral sequence is the serious reason to study the exceptional sheaves. Moreover, the exceptional bundles are interesting as bundles with the zero-dimensional moduli space.

Another application of the exceptional bundles is the description of moduli spaces of semistable bundles. There exists such description in the case of the projective plane ([6]) and of the smooth 2-dimensional quadric ([21]).

The helix theory is connected with number theory. Namely, A. A. Markov, studied in particular, solutions of the following Diophantian equation:

$$x^2 + y^2 + z^2 = 3xyz. \tag{11}$$

(This equation is called now the Markov equation and its solutions are called the Markov numbers.) It was proved that the Markov numbers coincide with the ranks of the exceptional bundles on $\mathbb{P}^2$ (which form a foundation of a helix).

A. A. Markov formulated the following conjecture:

Any triple of natural solutions of the equation (11) is uniquely determined by its maximal element.

This conjecture can be reformulated in terms of the exceptional bundles in the following way.

Suppose E and F are exceptional bundles on $\mathbb{P}^2$ with equal ranks; then either $E = F(n)$ or $E^* = F(n)$ for some natural n.

More details can be found in [18].

Now let us return to the helix theory. The definition of a helix and the first results about helices appeared in [18]. The definition of a helix is due to A. L. Gorodentsev and A. N. Rudakov. The word "helix" and the idea of considering a helix as an infinite system of bundles with some form of periodicity is due to V. N. Danilov.

Now we following [19], formulate the axioms of the helix theory.

We consider pairs of objects of a category $\mathfrak{U}$ or elements of a set $\mathfrak{U}$.

DEFINITION. A pair (A, B) is called *left admissible* if a certain pair $(L_A B, A)$ is defined. The pair $(L_A B, A)$ is called a *left mutation* of (A, B) and the object $L_A B$ is called a *left shift* of B. Similarly, a pair (A, B) is *right admissible* if a certain pair $(B, R_B A)$ is defined. The pair $(B, R_B A)$ in this case is called a *right mutation* of (A, B) and the object $R_B A$ is a *right shift* of A.

The axioms are the following.

(1L) If (A, B) is left admissible then $(L_A B, A)$ is right admissible and $R_A L_A B = B$.

(1R) If (A, B) is right admissible then $(B, R_B A)$ is left admissible and $L_B R_B A = A$.

(2L) Let (A, B, C) be such a triple that the pairs $(B, C), (A, L_B C)$ and (A, B) are left admissible. Then the pairs $(A, C), (B', L_A C)$ are left admissible where $B' = L_A B$ and $L_A L_B C = L_{B'} L_A C$.

(2R) Let (A, B, C) be such a triple that the pairs $(B, C), (A, B)$ and $(R_B A, C)$ are right admissible. Then the pairs $(A, C), (R_C A, B')$ are right admissible, where $B' = R_C B$ and $R_C R_B A = R_{B'} R_C A$.

The equalities in the axioms (2L) and (2R) are usually called the *triangle equations*.

It will be convenient to denote the object $L_A L_B C$, which appeared in (2L) by $L^{(2)}C$ and also to set $R^{(2)}A = R_C R_B A$. In the same way if $(A_0, A_1, A_2, \ldots, A_n)$ is a system of objects we put $L^{(0)}A_s = A_s, L^{(1)}A_s = L_{A_{s-1}}A_s, \ldots, L^{(i)}A_s = L_{A_{s-i}}L^{(i-1)}A_s$ with the condition that the resulting pairs are left admissible. Analogous notation will be used for the right mutations.

DEFINITION. A collection $\{A_i | \ i \in \mathbb{Z}\}$ will be called a *helix of period* n if for all s the following condition holds:

HEL: The pairs $(A_{s-1}, A_s), (A_{s-2}, L^{(1)}A_s), \ldots, (A_{s-n+1}, L^{(n-2)}A_s)$ are left admissible and $L^{(n-1)}A_s = A_{s-n}$.

Further we shall assume that (1L), (1R), (2L) and (2R) are satisfied. Then HEL is equivalent to

HEL': The pairs $(A_{s-n}, A_{s-n+1}), (R^{(1)}A_{s-n}, A_{s-n+2}), \ldots, (R^{(n-2)}A_{s-n}, A_s)$ are right admissible and $R^{(n-1)}A_{s-n} = A_s$.

Each collection of the form $\{A_i, A_{i+1}, \ldots, A_{i+n-1}\}$ is called a *foundation* of the helix $\{A_i\}$. Note that a helix is uniquely determined by any of its foundations.

A collection $\{B_i | \ i \in \mathbb{Z}\}$ with

$$\begin{array}{lll} B_i = LA_{i+1} & \text{for} & i \equiv m - 1 (mod\ n), \\ B_i = A_{i-1} & \text{for} & i \equiv m (mod\ n), \\ B_i = A_i & \text{for} & i \not\equiv m, m-1 (mod\ n), \end{array}$$

is called a *left mutation* of the helix at A_m and is denoted by L_m.

A collection $\{C_i | \ i \in \mathbb{Z}\}$ with

$$\begin{array}{lll} C_i = RA_{i-1} & \text{for} & i \equiv m + 1 (mod\ n), \\ C_i = A_{i+1} & \text{for} & i \equiv m (mod\ n), \\ C_i = A_i & \text{for} & i \not\equiv m, m+1 (mod\ n), \end{array}$$

is called a *right mutation* of the helix at A_m and is denoted by R_m.

The basic fact about helices is the following statement.

3.1.1 THEOREM. *The right or the left mutation of a helix is a helix.*

All applications of helices are based on this theorem.

Looking at the triangle equations we see that the mutations of helices define the action of the braid group on the set of all helices. One of the main questions in the helix theory is to define the number of orbits of this action. of this action.

Let us return to the exceptional sheaves on surfaces and define the mutations of an exceptional pair of sheaves. (The definition of exceptional pairs and their types can be found in section 2.3.)

LEMMA-DEFINITION. 1. Let (E, F) be an exceptional *hom*-pair of sheaves. Consider the canonical homomorphisms

$$\operatorname{Hom}(E, F) \otimes E \xrightarrow{lcan} F \qquad \text{and} \qquad E \xrightarrow{rcan} \operatorname{Hom}(E, F)^* \otimes F.$$

If $lcan$ is an epimorphism then the pair (E, F) is left admissible and

$$L_E F = \ker(lcan).$$

Besides, the sheaf $L_E F$ is exceptional and the pair $(L_E F, E)$ is also exceptional.

The pair (E, F) is right admissible provided $rcan$ is a monomorphism. In this case,

$$R_F E = \mathrm{coker}(rcan).$$

Besides, the sheaf $R_F E$ and the pair $(F, R_F E)$ are also exceptional.

In both these cases the mutation of the pair (E, F) is called *regular*.

Suppose $lcan$ is a monomorphism then the left shift of F is defined as $L_E F = \mathrm{coker}(lcan)$. (The pair $(L_E F, E)$ is exceptional as well.)

The right shift of E is defined as $R_F E = \ker(rcan)$ whenever $rcan$ is an epimorphism. (In this case the pair $(F, R_F E)$ is exceptional as well.)

2. The *ext*-pair (E, F) is both left and right admissible. The following universal extensions define the mutations of the *ext*-pair.

$$0 \longrightarrow F \longrightarrow L_E F \longrightarrow \mathrm{Ext}^1(E, F) \otimes E \longrightarrow 0,$$

$$0 \longrightarrow \mathrm{Ext}^1(E, F)^* \otimes F \longrightarrow R_F E \longrightarrow E \longrightarrow 0.$$

In this case as before, $L_E F$ and $R_F E$ are exceptional and $(L_E F, E)$, $(F, R_F E)$ are *hom*-pairs.

3. Both the left and the right mutation of a zero-pair is permutation of the entries of the pair.

It follows from this lemma that there are cases when the left or the right mutation of a *hom*-pair is not defined. Moreover, there are not mutations of a singular pair of sheaves.

To overcome these limitations let us pass following ([4]) to the bounded derived category $(D^b(S))$ of sheaves on the surface S . Exceptional objects and collections in this category are defined in the same way as in the basic category of sheaves.

LEMMA-DEFINITION. Let (E, F) be an exceptional pair in $D^b(S)$. Objects $L_E F$ and $R_F E$ which complete the canonical morphisms

$$L_E F \longrightarrow R^{\cdot}\mathrm{Hom}(E, F) \otimes E \longrightarrow F \qquad \text{and} \qquad E \longrightarrow R^{\cdot}\mathrm{Hom}(E, F)^* \otimes F \longrightarrow R_F E$$

up to the distinguished triangles are exceptional just as the pairs $(L_E F, E)$, $(F, R_F E)$,.

The category of sheaves is imbedded into $D^b(S)$ by the morphism δ. Any exceptional sheaf stands exceptional under this imbedding. Mutations in the base and in the derived category are related in the following way. If an exceptional pair of sheaves (E, F) is left admissible then the left shift of $\delta(F)$ in the derived category is quasiisomorphic to $\delta(L_E F)$. That is, it is a complex with a unique nonzero cohomology coincided with $L_E F$, and vice versa. The similar statement holds in the case of the right mutation. Thus we can assume that any exceptional pair of sheaves is both left and right admissible.

3.1.2 THEOREM. *(Gorodentsev-Orlov) Any exceptional object of $D^b(S)$ is quasiisomorphic to an exceptional sheaf provided S is Del Pezzo surface. That is, all mutations of exceptional pairs of sheaves belong to the base category.*

3.1.3 THEOREM. *(Rudakov-Gorodentsev) 1. The mutations of sheaves above defined and the exceptional objects of the derived category satisfy the axioms (1L),(2L),(1R) and (2R).*

2. An exceptional collection remains exceptional whenever some its pair of neighboring sheaves is replaced by a mutation of this pair. This procedure is called the mutation of a collection.

DEFINITION. Let $\sigma = (E_1, E_2, \dots, E_n)$ be an exceptional collection of sheaves or objects of $D^b(S)$. It is *full* provided $D^b(S)$ is generated by σ, i.e. the set of all objects of $D^b(S)$ can be obtained from the elemets of σ by taking the direct sums, tensoring and forming cones of all possible morphisms.

For example, the following collection of line bundles on $\mathbb{P}^2$

$$(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(2))$$

is the full exceptional collection.

3.1.4 THEOREM. *(Bondal) Let $\sigma = (E_1, E_2, \dots, E_n)$ be an exceptional collection of sheaves or objects of the derived category on a manifold X. Then the following statements are true.*

1. If σ is full then its left and right mutations are full collections.

2. The collection of the form

$$\sigma_\infty = \{E_i | i \in \mathbb{Z},\ E_{i+sn} = E_i(-sK_X)\}$$

is a helix of period n if and only if σ is full.

We see that full collections are closely connected with helices.

To write the spectral sequence mentioned at the beginning of this section let us define dual collections.

Let $\sigma = (E_1, E_2, \dots, E_k)$ be an exceptional collection. The following collection $({}^\vee E_{-k}, \dots, {}^\vee E_{-1}, {}^\vee E_0)$, where

$${}^\vee E_0 = E_0, {}^\vee E_{-1} = LE_1, {}^\vee E_{-2} = L^{(2)}E_2, \dots, {}^\vee E_{-k} = L^{(k)}E_k$$

is called the *left dual* to σ. The collection $(E_k^\vee, E_{k-1}^\vee, \dots, E_0^\vee)$, where

$$E_0^\vee = R^{(k)}E_0, E_1^\vee = R^{(k-1)}E_1, E_2^\vee = R^{(k-2)}E_2, \dots, E_k^\vee = E_k$$

is called the *right dual* to σ.

In these notations the following theorem is valid.

3.1.5 THEOREM. *(Gorodentsev) Let Q be an exceptional object belonging to the subcategory generated by an exceptional collection $(E_0, E_1, E_2, \dots, E_k)$. Then there exists a spectral sequence*

$$E^{p,q} \Longrightarrow H^{p+q}(Q),$$

the E_1-term of which has the form

$$E_1^{p,q} = \bigoplus_{r+s=q} \mathrm{Hom}^r_{D^b(S)}(E_{k-p}, Q) \otimes H^s({}^\vee E_{k-p}).$$

In this case we say that the spectral sequence is associated with the left dual collection.

Similarly, one can write the spectral sequence associated with the right dual collection.

3.1.6 COROLLARY. *Let $(E_0, E_1, E_2, \dots, E_k)$ be an exceptional collection of sheaves on the surface S. Suppose the left dual collection belongs to the base category, i.e. each element of the left dual collection is a sheaf; then for any sheaf Q belonging to the category generated by this collection there exists a spectral sequence $E^{p,q}$ with the E_1-term of the form*

$$E_1^{p,q} = \mathrm{Ext}^{q-\Delta_p}(E_{-p}, Q) \otimes {}^\vee E_{-p},$$

where Δ_p is the number of nonregular mutations needed for constructing the sheaf ${}^\vee E_{-p}$. Besides, there exists a spectral sequence $E^{p,q}$ with E_1-term

$$E_1^{p,q} = \mathrm{Ext}^{k-q-\Delta_p}(Q, E_{-p})^* \otimes E_{-p}^\vee,$$

where Δ_p is the number of nonregular mutations needed to construct the sheaf $E_{-p}^\vee$.

Both these sequences converge to Q on the main diagonal, i.e. $E_\infty^{p,q} = 0$ for $p+q \neq 0$ and

$$Gr(Q) = (E_\infty^{0,0}, E_\infty^{-1,1}, \dots, E_\infty^{-n,n}).$$

The helix theory has the following basic open problems.

1. Do there exist full exceptional collections on a given manifold?

2. How many orbits has the action of the braid group on the set of all helices? We say that all helices (or full exceptional collections) are *constructible* provided the orbits are unique.

3. Does an arbitrary exceptional collection belong to a foundation of a helix? In other words, is there a full exceptional collection containing a given exceptional collection? We say that the exceptional sheaves are *constructible* whenever the answer of the second and the last problems are positive.

4. We can consider the action of the braid group on the set of exceptional collections which generate one and the same derived subcategory of $D^b(X)$. How many orbits has this action?

5. Description of stable subgroups of the braid group action.

Full collections were found on $\mathbb{P}^n$, Del Pezzo surfaces, $G(2,4)$. Besides the following result was proved by D. Orlov in [17].

3.1.7 THEOREM. *(Orlov) 1. Let $\mathbb{P}(E) \to M$ be the projectivisation of a vector bundle on a manifold X. Suppose there is a full exceptional collection on X then there exists such collection on $\mathbb{P}(E)$.*

2. Let $\tilde{X}$ be obtained from X by blowing up a smooth regular submanifold Y. Suppose that there exist full exceptional collections on X and on Y then there exists a full exceptional collection on $\tilde{X}$ as well.

In the papers [18], [20], [11] it was proved that all exceptional sheaves and all helices on Del Pezzo surfaces are constructible. The constructibility of helices on the ruled surfaces with the rational base and on $\mathbb{P}^3$ was proved in ([15]).

In the last part of our paper we shall prove the following theorem.

3.1.8 THEOREM. *1. Let σ be an exceptional collection of bundles on a smooth projective surfaces S with anticanonical class without base components and $K_S^2 > 0$. Suppose that the rank of each bundle of this collection is greater than 1 then there is a full exceptional collection τ such that σ is a subcollection of τ. Moreover, τ can be obtained by mutations from the basic full collection. In other words, all helices on S are constructible.*

2. The condition about ranks can be omitted provided $K_S^2 > 1$.

3.2 Restriction of Superrigid Bundles to an Exceptional Curve.

Let us recall that we deal with the surface S the anticanonical class $H = -K_S$ of which has no base components. This means that H is nef.

In the beginning of section 2.4 we restricted the class of considered surfaces by the condition $K_S^2 > 0$. Using the description of surfaces with numerically effective anticanonical class from section 2.1, we see that the surfaces satisfying this condition are the following: $\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$, F_2 or surfaces obtained from $\mathbb{P}^2$ by blowing up at most 8 points.

Furthermore we can assume that the surface S satisfies the following conditions.

1. $K_S^2 > 0$.

2. There exists a blowing down of S onto $\mathbb{P}^2$.

Suppose S is obtained from $\mathbb{P}^2$ in the following way

$$S \xrightarrow{\sigma_d} S_{d-1} \xrightarrow{\sigma_{d-1}} \cdots \xrightarrow{\sigma_2} S_1 \xrightarrow{\sigma_1} S_0 = \mathbb{P}^2,$$

where σ_i is a blow up of a point $p_{i-1} \in S_{i-1}$ and $d \leq 8$. By definition, put $e_i = \sigma_i^{-1}(p_{i-1})$.

It is clear that e_i are exceptional -1-curves for all i and e_d is irreducible. We see that e_d is a smooth rational curve.

It is known that the divisors $h, e_1, \dots, e_d$ generate the group $\mathrm{Pic}(S)$ (here h is the preimage of a line on $\mathbb{P}^2$). Besides

$$he_i = e_ie_j = 0 \quad \text{for} \quad i \neq j \quad \text{and} \quad e_i^2 = -1.$$

$$K_S = -3h + \sum_{i=1}^{d} e_i.$$

3.2.1 REMARK. *The divisor h is numerically effective.*

PROOF. In fact, a line on $\mathbb{P}^2$ has no base points. Hence its preimage is base set free as well. Therefore, the cup product h with any curve on S is nonnegative.

3.2.2 LEMMA. *Let E and F be exceptional bundles on S with the equal μ_H-slopes and let $e = e_d$ be an irreducible exceptional curve. Suppose $E = F$ or $c_1(E) - c_1(F) = C$ is a -2-curve; then*

either $\mathrm{Ext}^2(E, F(-e)) = 0$

or $K_S^2 = 1$ *and* (E, F) *is an exceptional pair of the form* $\big(\mathcal{O}_S(D), \mathcal{O}_S(D+e+K_S)\big)$,

where D is some divisor of $\mathrm{Pic}(S)$.

PROOF. By Serre duality theorem,

$$\mathrm{Ext}^2(E, F(-e))^* \cong \mathrm{Hom}(F, E(e + K_S)).$$

Suppose $K_S^2 > 1$ then

$$\mu_H(E(e + K_S)) = \mu_H(E) - K_S \cdot e - K_S^2 < \mu_H(E) = \mu_H(F).$$

Hence the equality $\mathrm{Hom}(F, E(e + K_S)) = 0$ follows from the μ_H-stability of exceptional bundles on S and 1.1.5 .

Now suppose $K_S^2 = 1$ and $\mathrm{Hom}(F, E(e+K_S)) \neq 0$. It follows from the equality $\mu_H(F) = \mu_H(E(e + K_S))$ and 1.1.6 that there exists an exact triple:

$$0 \longrightarrow F \longrightarrow E(e + K_S) \longrightarrow Q \longrightarrow 0, \qquad (12)$$

where Q is a torsion sheaf. Denote by r the rank of the bundles E and F. (Let us recall that $r(E) = r(F)$). We get

$$c_1(Q) = c_1(E) - c_1(F) + r(e + K_S) = C + r(e + K_S).$$

Recall that the first Chern class of a torsion sheaf must be "nonnegative", i.e. either effective or trivial. Assume that $E = F$ then one gets $c_1(Q) = r(e + K_S)$. But this is impossible since this divisor is ineffective.

Assume that $E \neq F$. Then, by the assumptions of the lemma $C = c_1(E) - c_1(F)$ is -2-divisor such that $C \cdot K_S = 0$ (recall that $\mu_H(E) = \mu_H(F)$ and $r(E) = r(F)$). Such divisors were described by Yu. I. Manin in [12]. Using his results we can state that if $C = ah - \sum_{i=1}^{d} b_i e_i$ then $|a| \leq 3$. Moreover $C = 3h - e_j - \sum_{i=1}^{d} e_i$ whenever $a = 3$.

We assume that sequence (12) exists. In this case the divisor $C + r(e + K_S)$ is effective. Whereby, the cup product $h \cdot (C + r(e+K_S))$ is nonnegative (3.2.1). Thus, $C = 3h - e_j - \sum_{i=1}^{d} e_i$ and $r = 1$.

We have $C + r(e + K_S) = 2e_d - 2e_j$ (recall that $e = e_d$). The curve e_d is irreducible. Hence, $2e_d - 2e_j$ is ineffective if $j \neq d$. Therefore, $e_d = e_j$ and $C = -K_S - e$. Thus the pair (E, F) is equal to

$$\big(\mathcal{O}_S(D), \mathcal{O}_S(D + e + K_S)\big).$$

This concludes the proof.

3.2.3 COROLLARY OF THE PROOF. *Suppose C is -2-divisor with $C \cdot K_S = 0$ and $e = e_d$; then the divisor $C + e + K_S$ is nonpositive.*

3.2.4 LEMMA. *Let (E, F) be an exceptional pair of bundles on S with $\mu_H(E) < \mu_H(F)$ and $e = e_d$ be the irreducible exceptional curve. Then*

$$\mathrm{Ext}^2(E, F(-e)) = 0.$$

PROOF. By Serre duality theorem we have

$$\mathrm{Ext}^2(E, F(-e))^* \cong \mathrm{Hom}(F, E(e + K_S)).$$

But,

$$\mu_H(E(e + K_S)) = \mu_H(E) + 1 - K_S^2 \leq \mu_H(E) < \mu_H(F)$$

and the proof follows from the μ_H-stability of exceptional bundles on S and lemma 1.1.5 .

3.2.5 LEMMA. *Suppose that E and F are rigid μ_H-semistable bundles on S. Assume that they have exceptional filtrations*

$$Gr(E) = (x_n E_n, x_{n-1}E_{n-1}, \ldots, x_1 E_1), \qquad Gr(F) = (y_m F_m, y_{m-1}F_{m-1}, \ldots, y_1 F_1).$$

In addition we assume that the following conditions hold.

1. $\mathrm{Ext}^k(F, E) = 0 \quad \forall k = 0, 1, 2.$

2. $\mu_H(E) < \mu_H(F) < \mu_H(E) + K_S^2$.

3. Provided $K_S^2 = 1$ the exceptional collections $(E_1, E_2, \ldots, E_n)$, $(F_1, F_2, \ldots, F_m)$ have no pairs of the form $\big(\mathcal{O}_S(D), \mathcal{O}_S(D + e + K_S)\big)$ where $D \in \mathrm{Pic}(S)$ and $e = e_d$ is an irreducible exceptional curve.

Then the restrictions of E and F to e have the form

$$E' = E|_e = \alpha\mathcal{O}_e(s-1) \oplus \beta\mathcal{O}_e(s), \quad F' = F|_e = \gamma\mathcal{O}_e(s-1) \oplus \delta\mathcal{O}_e(s) \oplus \epsilon\mathcal{O}_e(s+1)$$

where $\alpha, \beta, \gamma, \delta, \epsilon$ are nonnegative integer with $\gamma\epsilon = 0$.

PROOF. It follows from the assumptions of the lemma and 2.4.1 that all pairs (E_i, E_j) and (F_i, F_j) for $i < j$ are exceptional singular or zero pairs. By proposition 2.3.2 we have

$$\mu_H(E_i) = \mu_H(E_j), \ \mu_H(F_i) = \mu_H(F_j)$$

and the differencies of the first Chern classes

$$c_1(E_j) - c_1(E_i) \text{ and } c_1(F_j) - c_1(F_i)$$

are -2-divisors. Since among these pairs there are no pairs of the kind

$$\big(\mathcal{O}_S(D), \mathcal{O}_S(D + e + K_S)\big)$$

(in the case $K_S^2 = 1$), it follows from lemma 3.2.2 that

$$\mathrm{Ext}^2(E_i, E_j(-e)) = \mathrm{Ext}^2(E_i, E_i(-e)) = \mathrm{Ext}^2(F_i, F_i(-e)) = \mathrm{Ext}^2(F_i, F_j(-e)) = 0$$

for any pair i, j. Thus the equalities

$$\mathrm{Ext}^2(E, E(-e)) = \mathrm{Ext}^2(F, F(-e)) = 0$$

follow from 3.2.2

Since the bundles E and F are rigid using the exact triples

$$0 \longrightarrow E^* \otimes E(-e) \longrightarrow E^* \otimes E \longrightarrow (E^* \otimes E)|_e \longrightarrow 0,$$

$$0 \longrightarrow F^* \otimes F(-e) \longrightarrow F^* \otimes F \longrightarrow (F^* \otimes F)|_e \longrightarrow 0$$

we get $\mathrm{Ext}^1(E', E') = \mathrm{Ext}^1(F', F') = 0$.

By the Grothendieck theorem [16] any bundle on a projective line (in particular, E' and F' on e) is a direct sum of line bundles. From the rigidity of E' and F' and Bott's formula, which calculating the cohomology of line bundles on the projective line ([3]) we obtain

$$E' = E|_e = \alpha\mathcal{O}_e(s-1) \oplus \beta\mathcal{O}_e(s), \quad F' = F|_e = \gamma\mathcal{O}_e(s'-1) \oplus \delta\mathcal{O}_e(s').$$

Using the first and the second conditions of the lemma let us show that

$$s \le s' \le s+1.$$

Note that it follows from the condition 1 and proposition 2.3.1 that each pair (E_i, F_j) is exceptional. Besides,

$$\mu_H(E_i) < \mu_H(F_j) < \mu_H(E_i) + K_S^2.$$

Applying lemma 3.2.4 to the pairs (E_i, F_j) we get $\mathrm{Ext}^2(E_i, F_j(-e)) = 0$. This means that $\mathrm{Ext}^2(E, F(-e)) = 0$.

By virtue of the inequalities on the μ_H-slopes and 2.3.1 the pairs (E_i, F_j) have the type *hom*. In particular $\mathrm{Ext}^1(E_i, F_j) = 0$. Hence we have $\mathrm{Ext}^1(E, F) = 0$. Now it follows from the long exact cohomology sequence associated with the restriction sequence to the exceptional curve

$$0 \longrightarrow E^* \otimes F(-e) \longrightarrow E^* \otimes F \longrightarrow (E^* \otimes F)|_e \longrightarrow 0$$

that $\mathrm{Ext}^1(E', F') = 0$.

By Serre duality and the first condition of the lemma we get

$$\mathrm{Ext}^k(E, F(K_S)) = 0 \qquad \text{for} \qquad k = 0, 1, 2.$$

The second condition of the lemma yields the inequality

$$\mu_H(F(K_S)) < \mu_H(E) < \mu_H(F(K_S)) + K_S^2.$$

Repeating the reasoning for rigid bundles $F(K_S)$ and E we get that $\mathrm{Ext}^1(F(K_S)|_e, E') = 0$.

Note that $F(K_S)|_e = F'(-1)$, i.e. $\mathrm{Ext}^1(F'(-1), E') = 0$. Now the inequality ($s \le s' \le s+1$) follows from Bott's formula. This completes the proof.

3.2.6 COROLLARY. *Assume that an ordered collection of μ_H-semistable rigid bundles*

$$(E_1, E_2, \ldots, E_m)$$

satisfies the following conditions.

1. $\mathrm{Ext}^k(E_i, E_j) = 0$ *for* $j > i$, $k = 0, 1, 2$.

2. $\mu_H(E_1) < \mu_H(E_2) < \cdots < \mu_H(E_m) < \mu_H(E_1) + K_S^2$.

3. *Provided* $K_S^2 = 1$ *the exceptional collections corresponding to the exceptional filtrations of all* E_i *have no pairs of the form* $\big(\mathcal{O}_S(D), \mathcal{O}_S(D + e + K_S)\big)$ *where* $e = e_d$ *is an irreducible exceptional curve.*

Then there is a number i *such that*

$$\big(E_i \oplus \cdots \oplus E_m \oplus E_1(-K_S) \oplus \cdots \oplus E_{i-1}(-K_S)\big)|_e = \alpha\mathcal{O}_e(s) \oplus \beta\mathcal{O}_e(s+1).$$

PROOF. We say that an ordered pair of rigid μ_H-semistable bundles (E, F) on S *has the zero type of decomposition* whenever

$$(E \oplus F)|_e = \alpha\mathcal{O}_e(s) \oplus \beta\mathcal{O}_e(s+1).$$

It *has the first type of decomposition* whenever

$$E|_e = \alpha\mathcal{O}_e(s) \oplus \beta\mathcal{O}_e(s+1), \qquad F|_e = \gamma\mathcal{O}_e(s+1) \oplus \delta\mathcal{O}_e(s+2)$$

with $\alpha \cdot \delta \neq 0$.

From the previous lemma it follows that each pair from our collection has either the zero or the first type of decomposition.

We see that the above statement holds provided the pair (E_1, E_i) has the zero type of decomposition for all i.

In the opposite case denote by i the minimal number such that the pair (E_1, E_i) has the first type of decomposition. Note that in this case $\forall j < i \leq k$ the pair (E_j, E_k) has the first type of decomposition and the pair (E_s, E_l) has the zero type of decomposition whenever $i \leq s < l$ or $s < l \leq i$. Besides if a pair (E, F) has the first type of decomposition then the pair $(F, E(-K_S))$ has the zero type of decomposition.

Thus, each pair of the collection

$$\big(E_i \oplus \cdots \oplus E_m \oplus E_1(-K_S) \oplus \cdots \oplus E_{i-1}(-K_S)\big)$$

has the zero type of a decomposition. This completes the proof.

3.3 Equivalence of Collections and the Key Exact Sequence.

DEFINITION. We say that an exceptional collection $\sigma = (E_1, E_2, \ldots, E_k)$ (of sheaves or of objects in $D^b(S)$) on S is *constructible* whenever there is a full exceptional collection $(E_1, \ldots, E_k, E_{k+1}, \ldots, E_n)$ containing σ such that it is obtained from the basic collection

$$\sigma_0 = (\mathcal{O}_{e_1}(-1), \ldots, \mathcal{O}_{e_d}(-1), \mathcal{O}_S, \mathcal{O}_S(h), \mathcal{O}_S(2h))$$

by mutations. Here h is the preimage of a line on $\mathbb{P}^2$ and e_i are the blow up divisors. (It follows from [17] that the basic collection is exceptional and full.)

We say that an exceptional collection σ is *equivalent* to an exceptional collection τ whenever the following condition holds. The collection σ is constructible if and only if τ is constructible.

3.3.1 LEMMA. a) *Suppose an exceptional collection* σ *is obtained from an exceptional collection* τ *by mutations; then these collections are equivalent.*

b) An exceptional collection $(E_1, E_2, \ldots, E_k)$ is equivalent to the following collections:

$$(E_k(K_S), E_1, \ldots, E_{k-1}) \qquad \text{and} \qquad (E_2, \ldots, E_k, E_1(-K_S)).$$

PROOF. a) Assume that an exceptional collection $\sigma = (E_1, E_2, \ldots, E_k)$ is obtained from $\tau = (F_1, F_2, \ldots, F_k)$ by mutations. Since all mutations of collections are invertible (see the axioms of the helix theory), we can assume that τ is also obtained from σ by mutations. Suppose σ is constructible, i.e. there exists a full exceptional collection $\sigma' = (E_1, \ldots, E_k, E_{k+1}, \ldots, E_n)$ obtained from the basic collection by mutations. Then the exceptional collection $\tau' = (F_1, \ldots, F_k, E_{k+1}, \ldots, E_n)$ is also obtained from the basic collection by mutations. Therefore τ' is full (3.1.4). Besides the basic collection and τ' belong to one and the same orbit of the braid group action. Thus τ is also constructible.

b) In order to prove the second statement it is sufficient to check that the collections $\sigma = (E_1, E_2, \ldots, E_k)$ and $(E_2, \ldots, E_k, E_1(-K_S))$ are equivalent. Suppose σ is constructible and $\sigma_1 = (E_1, \ldots, E_k, E_{k+1}, \ldots, E_n)$ is a full exceptional collection corresponding to σ. By theorem 3.1.4 if follows from the fact that the collection σ_1 is full that it is a foundation of a helix and $E_1(-K_S) = R^{n-1}E_1$. That is, the collection

$$\sigma_2 = (E_2, \ldots, E_k, E_{k+1}, \ldots, E_n, E_1(-K_S))$$

is equivalent to σ_1. Now we shift each of the sheaves $E_n, E_{n-1}, \ldots, E_{k+1}$ to the right by $E_1(-K_S)$ to obtain the full collection

$$\tau_1 = (E_2, \ldots, E_k, E_1(-K_S), RE_{k+1}, RE_{k+2}, \ldots, RE_n),$$

equivalent to σ_2. Thus τ is constructible as well.

Since all operations are invertible we see that the collection σ is equivalent to τ.

NOTATION. Let $\sigma = (E_1, E_2, \ldots, E_k)$ be an exceptional collection of bundles. Denote

$$\mu_-(\sigma) = min\{\mu_H(E_i)\}, \qquad \mu_+(\sigma) = max\{\mu_H(E_i)\}.$$

3.3.2 LEMMA. *For any exceptional collection of bundles $\sigma = (E_1, E_2, \ldots, E_k)$ there exists an exceptional collection of bundles $\tau = (F_1, F_2, \ldots, F_k)$ equivalent to σ such that*

$$\mu_-(\sigma) \le \mu_-(\tau) = \mu_H(F_1) \le \ldots \le \mu_H(F_k) = \mu_+(\tau) \le \mu_+(\sigma).$$

Furthermore we say that exceptional collection of bundles $(F_1, F_2, \ldots, F_n)$ is a *hom-collection* whenever

$$\mu_H(F_1) \le \mu_H(F_2) \le \cdots \le \mu_H(F_n).$$

PROOF. Let s be the minimal number such that $\mu_H(E_s) > \mu_H(E_{s+1})$. It follows from proposition 2.3.1 that the exceptional pair (E_s, E_{s+1}) has the type *ext*. Consider the left mutation of this pair

$$0 \longrightarrow E_{s+1} \longrightarrow L_{E_s}E_{s+1} \longrightarrow E_s \otimes \operatorname{Ext}^1(E_s, E_{s+1}) \longrightarrow 0.$$

Since the sheaves E_s and E_{s+1} are locally free we get that $L_{E_s}E_{s+1}$ is also locally free. By the μ_H-stability of exceptional bundles we have

$$\mu_H(E_{s+1}) < \mu_H(L_{E_s}E_{s+1}) < \mu_H(E_s).$$

Now suppose that $\mu_H(L_{E_s}E_{s+1}) < \mu_H(E_{s-1})$ then we apply the left mutation of this *ext*-pair, etc...

It is clear that after a finite number of mutations we shall obtain an exceptional collection σ' equivalent to the original one and such that

$$\mu_-(\sigma) \leq \mu_-(\sigma') < \mu_+(\sigma') \leq \mu_+(\sigma).$$

Moreover, if we denote by s' the minimal number such that

$$\mu_H(E'_{s'}) > \mu_H(E'_{s'+1})$$

then $s' > s$. This implies the necessary statement.

3.3.3 LEMMA. *For any exceptional collection of bundles* $\sigma = (E_1, E_2, \ldots, E_k)$ *there exists an exceptional hom-collection of bundles* $\tau = (F_1, F_2, \ldots, F_k)$ *equivalent to* σ *such that*

$$\mu_+(\tau) - \mu_-(\tau) < K_S^2.$$

PROOF. By definition, put $\Delta\mu(\sigma) = \mu_+(\sigma) - \mu_-(\sigma)$. Assume that $\Delta\mu(\sigma) > K_S^2$. By lemma 3.3.2 , without loss of generality it can be assumed that σ is a *hom*-collection. We have

$$\mu_-(\sigma) = \mu_H(E_1), \qquad \mu_+(\sigma) = \mu_H(E_n).$$

Since $\Delta\mu(\sigma) > K_S^2$ and $\mu_H(E_1(-K_S)) = \mu_H(E_1) + K_S^2$ we see that there exists a number s such that

$$\mu_H(E_{s-1}) \leq \mu_H(E_1(-K_S)) < \mu_H(E_s).$$

The collection $\sigma_1 = (E_s, \ldots, E_n, E_1(-K_S), \ldots, E_{s-1}(-K_S))$ is equivalent to σ and it has the following limits of the μ_h-slope:

$$\mu_-(\sigma_1) = \mu_H(E_1(-K_S)) = \mu_H(E_1) + K_S^2,$$
$$\mu_+(\sigma_1) = max\{\mu_H(E_{s-1}(-K_S)), \mu_H(E_n)\}.$$

Suppose $\mu_+(\sigma_1) = \mu_H(E_{s-1}(-K_S))$ then $s > 1$ and

$$\Delta\mu(\sigma_1) = \mu_H(E_{s-1}) - \mu_H(E_1) \leq K_S^2.$$

Hence ordering the collection σ_1 as in 3.3.2 we obtain the *hom*-collection σ_2 equivalent to the original one and such that $\Delta\mu(\sigma_2) \leq K_S^2$.

Suppose $\mu_+(\sigma_1) = \mu_H(E_n) = \mu_+(\sigma)$ then ordering the collection σ_1 by the μ_H-slopes we obtain the *hom*-collection σ_3 equivalent to the original one and such that $\Delta\mu(\sigma_3) \leq \Delta\mu(\sigma) - K_S^2$.

Repeating this operation several times we finally obtain a *hom*-collection equivalent to σ with $\Delta\mu \leq K_S^2$.

Now let us assume that $\Delta\mu(\sigma) = K_S^2$. Denote by s the minimal number such that $\mu_H(E_s) < \mu_H(E_{s+1})$. Consider the equivalent collection

$$\tau = (E_{s+1}, \dots, E_n, E_1(-K_S), \dots, E_s(-K_S)).$$

By the choice of s we have

$$\mu_+(\tau) = \mu_H(E_n) = \mu_H(E_1(-K_s)) = \dots = \mu_H(E_s(-K_S)) = \mu_+(\sigma),$$

$$\text{and} \qquad \mu_-(\tau) = \mu_H(E_{s+1}) > \mu_-(\sigma).$$

In other words, τ is the *hom*-collection with $\Delta\mu(\tau) < K_S^2$. This completes the proof.

3.3.4 LEMMA. *Let $\sigma = (E_1, E_2, \dots, E_k)$ be an exceptional collection of bundles on a surface S with $K_S^2 \geq 1$. Assume in addition that in the case $K_S^2 = 1$ this collection has no pairs of the form $(\mathcal{O}_S(D), \mathcal{O}_S(D + e + K_S))$ where D is some divisor and $e = e_d$ is the exceptional rational curve.*

Then there exists an exceptional hom-collection $(F_1, F_2, \dots, F_k)$ equivalent to σ such that the superrigid bundle F $\left(Gr(F) = (x_n F_n, \dots, x_1 F_1)\right)$ associated with it is included in the exact sequence

$$0 \to G \to F \to \mathrm{Hom}(F, \mathcal{O}_e(-1))^* \otimes \mathcal{O}_e(-1) \to 0, \tag{13}$$

where G is a superrigid bundle with

$$\mathrm{Ext}^k(G, \mathcal{O}_e(-1)) = 0 \qquad \forall k.$$

PROOF. By lemma 3.3.3 there exists a *hom*-collection $\tau = (E'_1, E'_2, \dots, E'_k)$ equivalent to the original one and such that $\mu_+(\tau) - \mu_-(\tau) < K_S^2$.

We will split this collection into groups of bundles with equal μ_H-slopes. Using these groups we construct a set of superrigid μ_H-semistable bundles $\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_m$ (see theorem 2.5.1).

We have $\mathrm{Ext}^k(\mathcal{E}_j, \mathcal{E}_i) = 0$ for $j > i$, $k = 0, 1, 2$ and

$$\mu_H(\mathcal{E}_1) < \mu_H(\mathcal{E}_2) < \dots < \mu_H(\mathcal{E}_m) < \mu_H(\mathcal{E}_1) + K_S^2.$$

By corollary 3.2.6 there exists a number i such that

$$(\mathcal{E}_i \oplus \dots \oplus \mathcal{E}_m \oplus \mathcal{E}_1(-K_S) \oplus \dots \mathcal{E}_{i-1}(-K_S))|_e = \alpha\mathcal{O}_e(d-1) \oplus \beta\mathcal{O}_e(d).$$

Hence there is a number j such that the superrigid bundle $\bar{F}$ associated with the *hom*-collection

$$\tau' = (E'_j, \dots, E'_k, E'_1(-K_S), \dots, E'_{j-1}(-K_S))$$

satisfies the condition

$$\bar{F}|_e = \alpha\mathcal{O}_e(d-1) \oplus \beta\mathcal{O}_e(d).$$

Thus the superrigid bundle F constructed from the exceptional *hom*-collection

$$\tau'' = \tau'(dK_S) = (E'_j(dK_S), \dots, E'_k(dK_S), E'_1((d-1)K_S), \dots, E'_{j-1}((d-1)K_S))$$

restricts to the curve e in the following way:

$$F|_e = \alpha\mathcal{O}_e(-1) \oplus \beta\mathcal{O}_e.$$

One can easily show that the collection τ'' is equivalent to the original σ.

The following equalities can be obtained by direct calculations.

$$h^i(F, \mathcal{O}_e(-1)) = \begin{cases} \alpha & \text{for} \quad i = 0 \\ 0 & \text{for} \quad i > 0 \end{cases},$$

$$h^i(\mathcal{O}_e(-1), F) = \begin{cases} \beta & \text{for} \quad i = 1 \\ 0 & \text{for} \quad i \neq 1 \end{cases}.$$

Consider the canonical map:

$$F \stackrel{rcan}{\longrightarrow} \mathrm{Hom}(F, \mathcal{O}_e(-1))^* \otimes \mathcal{O}_e(-1).$$

Since the restriction of this map to the curve e is an epimorphism we see that exact sequence (13) is valid.

The sheaf G from this sequence, as a subsheaf of a bundle has no torsion. In order to calculate its cohomology let us consider the cohomology tables corresponding to the exact sequence (13). Denote by $\mathcal{L}$ the torsion sheaf $\mathcal{O}_e(-1)$.

k	$\mathrm{Hom}(F,\mathcal{L}) \otimes \mathrm{Ext}^k(\mathcal{L},\mathcal{L})$	$\rightarrow$	$\mathrm{Ext}^k(F,\mathcal{L})$	$\rightarrow$	$\mathrm{Ext}^k(G,\mathcal{L})$
0	$\mathrm{Hom}(F,\mathcal{L}) \otimes \mathbb{C}$		$\mathrm{Hom}(F,\mathcal{L})$		?
1	0		0		?
2	0		0		?

k	$\mathrm{Hom}(F,\mathcal{L}) \otimes \mathrm{Ext}^k(\mathcal{L},F)$	$\rightarrow$	$\mathrm{Ext}^k(F,F)$	$\rightarrow$	$\mathrm{Ext}^k(G,F)$
0	0		*		?
1	*		0		?
2	0		0		?

k	$\mathrm{Ext}^k(G,G)$	$\rightarrow$	$\mathrm{Ext}^k(G,F)$	$\rightarrow$	$\mathrm{Hom}(F,\mathcal{L})^* \otimes \mathrm{Ext}^k(G,\mathcal{L})$
0	?		*		0
1	?		0		0
2	?		0		0

This concludes the lemma proof.

The idea of the construction of the exact sequence (13) on Del Pezzo surface with an exceptional bundle like F belongs to D. O. Orlov ([11]).

3.4 Category Generated by a Pair.

In the previous section we constructed starting from an exceptional collection σ of bundles the *hom*-collection $\tau = (F_1, F_2, \ldots, F_k)$ equivalent to σ and such that the superrigid bundle associated with τ is included in the exact sequence (13). In the next section we shall shwo using double induction that this sequence implies the constructibility of the collection τ.

Here we check the base of one of the inductions. Namely we prove the following proposition.

3.4.1 PROPOSITION. *Suppose that a superrigid sheaf F on the surface S is included in the exact sequence*

$$0 \longrightarrow y_1 G_1 \longrightarrow F \longrightarrow y_0 G_0 \longrightarrow 0,$$

where (G_0, G_1) is an exceptional pair y_i are positive integer and G_1 is a bundel; then

1. *If G_0 is locally free then F has a unique (up to permutations of quotients) exceptional filtration*

$$Gr(F) = (x_1 F_1, x_0 F_0) \qquad (x_i \geq 0).$$

Moreover,

(a) *the pair (F_0, F_1) is obtained from the pair (G_0, G_1) by mutations,*
(b) $r(F_0) + r(F_1) \geq r(G_0) + r(G_1)$ *whenever* $x_0 \cdot x_1 \neq 0$,
(c) $r(F_0) \geq r(G_0) + r(G_1)$ *provided* $x_1 = 0$,
(d) *the equality of the sums of ranks holds if and only if $F_i = G_i$ for $i = 1, 2$.*

2. *Assume that $G_0 = \mathcal{O}_e(-1)$ for the exceptional rational curve $e = e_d$ then*

$$F = x_1 F_1 \oplus x_0 F_0 \qquad (x_i \geq 0).$$

Moreover,

(a) *the exceptional pair (F_0, F_1) is obtained from the pair (G_0, G_1) by mutations,*
(b) $r(F_0) + r(F_1) \geq r(G_0) + r(G_1)$ *whenever* $x_0 \cdot x_1 \neq 0$,
(c) $r(F_0) \geq r(G_0) + r(G_1)$ *provided* $x_1 = 0$,
(d) *F_0 is locally free.*

To prove this proposition, we need several lemmas.

3.4.2 LEMMA. *Let A and B be sheaves on a manifold X and let $\varphi : V \otimes A \longrightarrow W \otimes B$ be a morphism of sheaves. Then*

1. *The canonical map $lcan : \mathrm{Hom}(A, B) \otimes A \longrightarrow B$ is an epimorphism provided that φ is also an epimorphism.*

2. *The canonical map $rcan : A \longrightarrow \mathrm{Hom}(A, B)^* \otimes B$ is a monomorphism provided that φ is also a monomorphism.*

PROOF. In view of symmetry of statements it is sufficient to check the first of them. At first consider the case of the one-dimensional space W, i.e.

$$\varphi : V \otimes A \longrightarrow B \longrightarrow 0.$$

Recall that the canonical map $lcan$ is determined by the element of $\mathrm{Hom}(A, B)^* \otimes \mathrm{Hom}(A, B)$ corresponding to the identical morphism $\mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}(A, B)$. Denote by $lcan$ this element as well. Let us define a line map $\psi : V \longrightarrow \mathrm{Hom}(A, B)$ such that

$$\psi^* \otimes id_{\mathrm{Hom}(A,B)} : lcan \to \varphi.$$

This leads to the following commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}(A,B)\otimes A & \xrightarrow{lcan} & B & & \\ {\scriptstyle \psi\otimes id_A}\uparrow & & {\scriptstyle id_B}\uparrow & & , \\ V\otimes A & \xrightarrow{\varphi} & B & \longrightarrow & 0 \end{array}$$

This diagram shows that *lcan* is an epimorphism.

Now suppose

$$\varphi : V\otimes A \longrightarrow W\otimes B \longrightarrow 0.$$

Then

$$\mathrm{Hom}(A, W\otimes B)\otimes A \xrightarrow{lcan} W\otimes B \longrightarrow 0.$$

We see that there is a commutative diagram

$$\begin{array}{ccccc} 0 & & 0 & & \\ \uparrow & & \uparrow & & \\ \mathrm{Hom}(A,B)\otimes A & \xrightarrow{lcan} & B & \longrightarrow & 0\ , \\ {\scriptstyle \pi_*\otimes id_A}\uparrow & & {\scriptstyle \pi}\uparrow & & \\ \mathrm{Hom}(A, W\otimes B)\otimes A & \xrightarrow{lcan} & W\otimes B & \longrightarrow & 0 \end{array}$$

where π is a projection $W\otimes B \longrightarrow B \longrightarrow 0$.

3.4.3 LEMMA. *Let F be a rigid sheaf and (A,B) be an exceptional hom-pair of sheaves on the surface S. Then the following statements hold.*

1. *If the sequence*

$$0 \longrightarrow F \longrightarrow xA \longrightarrow yB \longrightarrow 0 \tag{14}$$

is exact for positive integers x and y then

(a) *the left mutation of the pair (A,B) belongs to the basic category and it is regular;*

(b) *either $F = wA \oplus zL_AB$ or there exists an exact sequence*

$$0 \longrightarrow F \longrightarrow zL_AB \longrightarrow wA \longrightarrow 0$$

for some nonnegative integers z and w.

2. *If the sequence*

$$0 \longrightarrow xA \longrightarrow yB \longrightarrow F \longrightarrow 0 \tag{15}$$

is exact for positive integers x and y then

(a) *the right mutation of the pair (A,B) belongs to the basic category and it is regular;*

(b) *either $F = wB \oplus zR_BA$ or there exists an exact sequence*

$$0 \longrightarrow wB \longrightarrow zR_BA \longrightarrow F \longrightarrow 0$$

for some nonnegative integers z and w.

PROOF. Since the statements of the lemma are dual it is sufficient to check the first of them.

The regularity of the left mutation of the *hom*-pair (A, B) follows from its definition, sequence (14) and lemma 3.4.2 . Note that in this case the pair $(L_A B, A)$ has also the *hom*-type.

Sequence (14) yields that the sheaf F belongs to the category generated by the pair (A, B). Therefore there exists a spectral sequence $E^{p,q}$ (3.1.6) converging to F on the principal diagonal. Its E_1-term has the form:

$$\begin{array}{lcl} E_1^{-1,1} = \mathrm{Ext}^1(B,F)\otimes L_A B & \xrightarrow{d} & E_1^{0,1} = \mathrm{Ext}^1(A,F)\otimes A \\ E_1^{-1,0} = \mathrm{Ext}^0(B,F)\otimes L_A B & \xrightarrow{d} & E_1^{0,0} = \mathrm{Ext}^0(A,F)\otimes A \end{array}.$$

The exact sequence (14) and the fact that the pair (A, B) is exceptional imply that the group $\mathrm{Ext}^0(B, F)$ is trivial. Hence the spectral sequence splits into two exact triples:

$$0 \longrightarrow C \longrightarrow \mathrm{Ext}^1(B,F)\otimes L_A B \longrightarrow \mathrm{Ext}^1(A,F)\otimes A \longrightarrow 0,$$

$$0 \longrightarrow \mathrm{Hom}(A,F)\otimes A \longrightarrow F \longrightarrow C \longrightarrow 0.$$

Assume that $\mathrm{Hom}(A, F) \neq 0$. Consider the cohomology table corresponding to the first of these triples:

k	$V\otimes \mathrm{Ext}^k(A,A)$	$\to$	$W\otimes \mathrm{Ext}^k(L_A B,A)$	$\to$	$\mathrm{Ext}^k(C,A)$
0	$*$		$*$		?
1	0		0		?
2	0		0		?

where $\mathrm{Ext}^1(A,F)^* = V$, $\mathrm{Ext}^1(B,F)^* = W$. The first and the second columns are filled in using the properties of the pair $(L_A B, A)$.

From the table the equality $\mathrm{Ext}^1(C, A) = 0$ follows. This means that

$$F = C \oplus \mathrm{Hom}(A,F)\otimes A.$$

Since F is rigid we get $\mathrm{Ext}^1(A,C) = 0$ and $\mathrm{Ext}^1(A,A) = 0$. Hence, $\mathrm{Ext}^1(A,F) = 0$. Thus,

$$C = \mathrm{Ext}^1(B,F)\otimes L_A B$$

and F is the direct sum of multiplicities of the sheaves A and $L_A B$.

Assume that $\mathrm{Hom}(A, F) \neq 0$ then the spectral sequence degenerates into the exact triple

$$0 \longrightarrow F \longrightarrow \mathrm{Ext}^1(B,F)\otimes L_A B \longrightarrow \mathrm{Ext}^1(A,F)\otimes A \longrightarrow 0.$$

This concludes the proof.

3.4.4 LEMMA. *Let (E_0, E_1) be an exceptional ext-pair of sheaves on a manifold X with $\chi(E_0, E_1) < -1$. Assume in addition that for each positive integer n the following sheaves are determined:*

$$E_{n+1} = R_{E_n} E_{n-1}, \qquad E_{-(n+1)} = L_{E_{-n}} E_{1-n}.$$

Suppose that for a given sheaf F and for any positive integer n there exist positive integers $x_n, y_n z_n, w_n$ such that the following exact sequences

$$0 \longrightarrow F \longrightarrow x_n E_{-(n+1)} \longrightarrow y_n E_{-n} \longrightarrow 0,$$

$$0 \longrightarrow z_n E_n \longrightarrow w_n E_{(n+1)} \longrightarrow F \longrightarrow 0$$

are valid then the Euler characteristic $\chi(F, F)$ is nonpositive.

PROOF. Denote by e_n the images of E_n in $K_0(X)$. The module $K_0(X)$ inherits the bilinear form $\chi(\cdot,\cdot)$. Denote it by $(\cdot,\cdot)$.

Using the assumptions of the lemma we get

$$(e_0, e_0) = (e_1, e_1) = 1, \qquad (e_1, e_0) = 0, \qquad (e_0, e_1) = -h < -1.$$

By definition of the mutations of an *ext*-pair we get

$$e_{-1} = e_1 + he_0, \qquad e_2 = he_1 + e_0.$$

It follows from the exact sequences and our assumptions that all pairs (E_n, E_{n+1}) for $n \in \mathbb{Z}^*$ have the *hom*-type and both mutations of these pairs (except for the left mutation of (E_1, E_2) and the right one of (E_{-1}, E_0)) are regular (3.4.2).

The following formulae are easily obtained from the definition of mutations of *ext*- and *hom*-pairs.

$$e_{-n} = he_{1-n} - e_{2-n} \qquad (n > 1),$$

$$e_n = he_{n-1} - e_{n-2} \qquad (n > 2),$$

$$\forall n \in \mathbb{Z}: \qquad (e_n, e_n) = 1, \qquad (e_{n+1}, e_n) = 0$$

$$\text{and for} \quad n \neq 0 \qquad (e_n, e_{n+1}) = h.$$

Denote by x_n and x_{n-1} coordinates of the vector e_n $(n > 0)$ with respect to the basis $\{e_1, e_0\}$: $\quad e_n = x_n e_1 + x_{n-1} e_0$. The recurrence relations

$$x_0 = 0, \qquad x_1 = 1, \qquad x_{n+1} = hx_n - x_{n-1}$$

are proved by induction on n.

Note that the vectors e_{-n} $(n > 0)$ are expressed through the same numbers, namely

$$e_{-n} = x_{n-1} e_1 + x_n e_0.$$

Let V be a 2-dimensional vector space over $\mathbb{Q}$ generated dy e_0, e_1. Let us choose an affine map U in $\mathbb{P}(V)$ containing the image of e_0 as the origin.

$$xe_1 + ye_0 \quad \rightsquigarrow \frac{x}{y} e_1 + e_0.$$

We preserve the notations for the images of e_n on U. Let us calculate the coordinates l_+ and l_- of limit points $e_{+\infty} = \lim_{n\to\infty} e_n$; $\quad e_{-\infty} = \lim_{n\to\infty} e_{-n}$ on U.

$$l_+ = \lim_{n\to\infty} \frac{x_n}{x_{n-1}} = h - \lim_{n\to\infty} \frac{x_{n-2}}{x_{n-1}} = h - l_- = h - 1/l_+.$$

Hence, l_+ and l_- are the roots of the equation $l^2 - hl + 1 = 0$, i.e.

$$l_\pm = \frac{h \pm \sqrt{h^2-4}}{2}$$

(by assumption, $h \geq 2$). Taking into account the exact triples from the assumptions we see that the point f on U corresponding to the sheaf F has the coordinate $x \in [l_-, l_+]$.

On the other hand, the sign of $\chi(F,F)$ is determined by the sign of $(e_0 + xe_1)^2 = x^2 - hx + 1$. Now the proof follows from the inequality

$$x^2 - hx + 1 \leq 0 \qquad \text{for} \qquad x \in [l_-, l_+].$$

3.4.5 COROLLARY. *Under the same assumptions as in the previous lemma we have* $r(E_n) \geq r(E_0) + r(E_1)$ *(for* $n \neq 0$ *and* $n \neq 1$*). Moreover,* $r(E_n) > r(E_0) + r(E_1)$ *for* $n \neq 0$ *and* $n \neq 1$ *whenever both* E_0 *and* E_1 *has a positive rank.*

PROOF. In fact, we see that the image of the sheaf E_n in $K_0(X)$ has the form: $e_n = ae_0 + be_1$ for some positive integers a and b. Thus our statement follows from the additivity of the rank function.

PROOF OF THE PROPOSITION 3.4.1. Suppose that G_i are locally free. If the pair (G_0, G_1) has the zero- or *hom*-type then $h^1(G_0, G_1) = 0$ and $F = y_0G_0 \oplus y_1G_1$.

If the pair (G_0, G_1) is singular then $\mu_H(G_0) = \mu_H(G_1)$ and the proof follows from the uniqueness of the exceptional filtration (2.5.1).

Now, suppose G_0 is a torsion sheaf and G_1 is a bundle then the pair (G_0, G_1) is necessarily the *ext*-pair.

Thus, let (G_0, G_1) be an *ext*-pair. Following traditions take

$$G_{n+1} = R_{G_n}G_{n-1} \qquad \text{and} \qquad G_{-n} = L_{G_{1-n}}G_{2-n}.$$

STEP 1. *One of the following possibilities holds*

$$F = x_1G_1 \oplus x_2G_2,$$

$$0 \longrightarrow x_1G_1 \longrightarrow x_2G_2 \longrightarrow F \longrightarrow 0.$$

Consider the spectral sequence converging to F and constructing by the right dual collection $(G_1^\vee, G_0^\vee)$ (3.1.6). (Recall that $G_1^\vee = G_1$ and $G_0^\vee = R_{G_1}G_0 = G_2$.) Since the right mutation of the pair (G_0, G_1) is nonregular we get

$$\Delta_0 = 1 \quad \text{and} \quad E_1^{0,q} = \mathrm{Ext}^{-q}(F, G_0)^* \otimes G_2.$$

In addition, we do not mutations to obtain the sheaf $G_1^\vee$. Hence, $\Delta_1 = 0$ and

$$E_1^{-1,q} = \mathrm{Ext}^{1-q}(F, G_1)^* \otimes G_1.$$

Thus we see that the E_1-term of the spectral sequence has the form

$$\begin{array}{rcl} E_1^{-1,1} = \mathrm{Ext}^0(F,G_1)^* \otimes G_1 & \xrightarrow{d} & 0 \\ E_1^{-1,0} = \mathrm{Ext}^1(F,G_1)^* \otimes G_1 & \xrightarrow{d} & E_1^{0,0} = \mathrm{Ext}^0(F,G_0)^* \otimes G_2 \\ E_1^{-1,-1} = \mathrm{Ext}^2(F,G_1)^* \otimes G_1 & \xrightarrow{d} & E_1^{0,-1} = \mathrm{Ext}^1(F,G_0)^* \otimes G_2 \\ 0 & \xrightarrow{d} & E_1^{0,-2} = \mathrm{Ext}^2(F,G_0)^* \otimes G_2 \end{array}.$$

Using the cohomology tables corresponding to the exact sequence from our assumptions:

$$0 \longrightarrow y_1 G_1 \longrightarrow F \longrightarrow y_0 G_0 \longrightarrow 0,$$

let us calculate the groups $\mathrm{Ext}^k(F, G_i)$.

k	$y_0\mathrm{Ext}^k(G_0, G_1)$	$\to$	$\mathrm{Ext}^k(F, G_1)$	$\to$	$y_1\mathrm{Ext}^k(G_1, G_1)$
0	0		?		*
1	*		?		0
2	0		?		0

k	$y_0\mathrm{Ext}^k(G_0, G_0)$	$\to$	$\mathrm{Ext}^k(F, G_0)$	$\to$	$y_1\mathrm{Ext}^k(G_1, G_0)$
0	*		?		0
1	0		?		0
2	0		?		0

Whereby, the spectral sequence splits into two exact triplies:

$$0 \longrightarrow \mathrm{Ext}^1(F, G_1)^* \otimes G_1 \longrightarrow \mathrm{Ext}^0(F, G_0)^* \otimes G_2 \longrightarrow C \longrightarrow 0,$$

$$0 \longrightarrow C \longrightarrow F \longrightarrow \mathrm{Ext}^0(F, G_1)^* \otimes G_1 \longrightarrow 0.$$

Now as in the proof of lemma 3.4.3 using the first of these triples it is easily shown that $\mathrm{Ext}^1(G_1, C) = 0$. Therefore, if $\mathrm{Ext}^0(F, G_1) \neq 0$ then F is a direct sum. In the opposite case the sheaf F is included in the exact sequence.

STEP 2. *One of the following possibilities holds*

$$F = x_{-1}G_{-1} \oplus x_0 G_0,$$

$$0 \longrightarrow F \longrightarrow x_{-1}G_{-1} \longrightarrow x_0 G_0 \longrightarrow 0.$$

This step is checked in the same way as the first one by using the spectral sequence associated with the left dual collection (G_{-1}, G_0).

STEP 3. *The sheaf F is decomposed into the direct sum:*

$$F = x_{n-1}G_{n-1} \oplus x_n G_n$$

for some $n \in \mathbb{Z}$ and nonnegative integers x_{n-1}, x_n. (That is $F_0 = G_{n-1}$ and $F_1 = G_n$ in the formulation of the proposition.)

Using the first two steps and lemma 3.4.3 it can be stated that for any $n > 0$ the following exact triples

$$0 \longrightarrow x_n G_n \longrightarrow x_{n+1}G_{n+1} \longrightarrow F \longrightarrow 0,$$

$$0 \longrightarrow F \longrightarrow x_{-n}G_{-n} \longrightarrow x_{1-n}G_{1-n} \longrightarrow 0$$

hold unless $F = x_{n-1}G_{n-1} \oplus x_n G_n$.

Let us show that these triples contradict the assumptions.

Suppose $h^1(G_0, G_1) > 1$ then it follows from these sequences and lemma 3.4.4 that $\chi(F, F) \leq 0$. This contradicts the fact that F is rigid.

Suppose $h^1(G_0, G_1) = 1$ then the series of the exceptional sheaves G_n is formed by G_0, G_1, G_2. In fact in this case both the right and the left mutation of the *ext*-pair (G_0, G_1) is described by the sequence

$$0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_0 \longrightarrow 0.$$

Whence, $L_{G_0}G_1 = G_2$ and $R_{G_2}G_1 = G_0$. Hence there are exact triples:

$$0 \longrightarrow y_1G_1 \longrightarrow F \longrightarrow y_0G_0 \longrightarrow 0,$$

$$0 \longrightarrow x_2G_2 \longrightarrow x_0G_0 \longrightarrow F \longrightarrow 0.$$

Since G_0 is indecomposable, it follows from the second sequence that $h^1(F, G_2) \neq 0$. We apply the functor $\mathrm{Ext}^{\cdot}(\cdot, G_2)$ to the first triple to obtain

$$y_0\mathrm{Ext}^1(G_0, G_2) \longrightarrow \mathrm{Ext}^1(F, G_2) \longrightarrow y_1\mathrm{Ext}^1(G_1, G_2).$$

Since the pair (G_2, G_0) is exceptional we get $\mathrm{Ext}^1(G_0, G_2) = 0$. Besides, since (G_1, G_2) is a *hom*-pair, we obtain $\mathrm{Ext}^1(G_1, G_2) = 0$. Thus, $h^1(F, G_2) = 0$. This contradiction proves the 3-th step.

STEP 4. *Suppose G_1 is a bundle and $G_0 = \mathcal{O}_e(-1)$ then F is locally free or F_0 is a bundle and $F_1 = \mathcal{O}_e(-1)$.*

By assumptions the sheaf F is included in the exact triple:

$$0 \longrightarrow y_1G_1 \longrightarrow F \longrightarrow y_0\mathcal{O}_e(-1) \longrightarrow 0.$$

Since F is rigid we see that F is locally free whenever F has no torsion (2.2.1). Therefore its direct summands are locally free as well.

Assume that F has a torsion TF. Since G_1 is locally free we obtain the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & y_1G_1 & \longrightarrow & F' & \longrightarrow & Q & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & y_1G_1 & \longrightarrow & F & \longrightarrow & y_0\mathcal{O}_e(-1) & \longrightarrow & 0\ , \\
 & & \uparrow & & \uparrow & & \varphi\uparrow & & \\
 & & 0 & \longrightarrow & TF & \longrightarrow & TF & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where F' is torsion free. Since TF is a subsheaf of $y_0\mathcal{O}_e(-1)$ and the curve e is isomorphic to the projective line we get

$$TF \cong \bigoplus_i z_i\mathcal{O}_e(s_i).$$

Hence,

$$Q \cong \left[\bigoplus_j w_j \mathcal{O}_e(d_j) \right] \oplus T^0,$$

where T^0 is a torsion sheaf with a zero-dimensional support.

Consider the upper row of the above diagram. Assume that $T^0 \neq 0$. Since G_1 is locally free and the support of T^o is zero-dimensional, we get $\mathrm{Ext}^1(T^0, y_1G_1) = 0$. Hence T^0 is the direct summand of F'. But this contradicts to the fact that F' has no torsion. For the same reason,

$$\mathrm{Ext}^1(\mathcal{O}_e(d_j), G_1) \neq 0 \qquad \forall j.$$

Let us show that this yields the inequality $d_j \leq -1$.

Indeed, by assumption, $(\mathcal{O}_e(-1), G_1)$ is an exceptional pair. Therefore it is easily follows from the calculation of cohomology that $(G_1)|_e = r(G_1)\mathcal{O}_e$. Thus,

$$\mathrm{Ext}^1(\mathcal{O}_e(d_j), G_1)^* \cong \mathrm{Ext}^1(G_1, \mathcal{O}_e(d_j) \otimes K_S) = r(G_1)\mathrm{Ext}^1(\mathcal{O}_e, \mathcal{O}_e(d_j - 1)) \neq 0$$

and the inequalities $d_j \leq -1$ hold for all j.

On the other hand, Q is a quotient of $y_0\mathcal{O}_e(-1)$. Hence, $d_j \geq -1 \quad \forall j$. From these inequalities it follows that $d_j = -1 \quad \forall j$ and $Q = w\mathcal{O}_e(-1)$.

We see that $TF = z\mathcal{O}_e(-1)$.

Now, the exact sequence

$$0 \longrightarrow y_1G_1 \longrightarrow F' \longrightarrow w\mathcal{O}_e(-1) \longrightarrow 0$$

implies that $\mathrm{Ext}^1(F', \mathcal{O}_e(-1)) = 0$, i.e. $F = F' \oplus z\mathcal{O}_e(-1)$.

By the previous step $F = x_{n-1}G_{n-1} \oplus x_nG_n$. Therefore $F = x_{-1}G_{-1} \oplus x_0\mathcal{O}_e(-1)$ or $F = x_{-1}G_{-1} \oplus x_0\mathcal{O}_e(-1) \oplus x_1G_1$. Since $(\mathcal{O}_e(-1), G_1)$ is the *ext*-pair and F is a superrigid sheaf we see that the last relation is impossible. On the other hand, F' is locally free, as rigid sheaf without torsion (2.2.1). Thus the sheaf $x_0F_0 = x_{-1}G_{-1} = F'$ is locally free as well.

STEP 5. *$r(F_0) + r(F_1) > r(G_0) + r(G_1)$ for $x_0 \cdot x_1 \neq 0$, and $r(F_0) \geq r(G_0) + r(G_1)$ for $x_1 = 0$.*

Since F_0 and F_1 are direct summands of a superrigid sheaf we obtain that the pair (F_0, F_1) is exceptional and it has the *hom*-type. Therefore it does not coincides with the pair (G_0, G_1). In view of this the first inequality follows from corollary 3.4.5.

Suppose $F = x_0F_0$ then $F_0 \neq G_0, G_1$. By the same argument, $r(F_0) > r(G_0) + r(G_1)$. The equality of ranks is possible here only if $F_0 = G_1$ and $G_0 = \mathcal{O}_e(-1) = 0$. This completes the proof.

3.5 Proof of the Main Theorem.

It follows from lemma 3.3.4 that for any exceptional collection of bundles on the surface S satisfying the conditions of the main theorem there is a *hom*-collection

$$\tau = (F_0, F_1, F_2, \ldots, F_k)$$

equivalent to the original one such that the superrigid bundle F associated with τ is included in the exact sequence (13):

$$0 \longrightarrow G \longrightarrow F \longrightarrow \mathrm{Hom}(F, \mathcal{O}_e(-1))^* \otimes \mathcal{O}_e(-1) \longrightarrow 0,$$

where G is a superrigid bundle with $\mathrm{Ext}^k(G, \mathcal{O}_e(-1)) = 0 \quad \forall k = 0,1,2$. (Further we denote by $\mathcal{B}(F_0, F_1, F_2, \ldots, F_k)$ the superrigid bundle associated with the *hom*-collection $(F_0, F_1, F_2, \ldots, F_k)$.)

In particular, we see that $G|_e = s\mathcal{O}_e$. Therefore there exists a superrigid bundle G' on the surface S' obtained from S by blowing down the curve e $(\sigma : S \longrightarrow S')$ such that $\sigma^*(G') = G$.

Since G' is superrigid we see that there exists its exceptional filtration:

$$Gr(G') = (y_n G'_n, y_{n-1} G'_{n-1}, \ldots, y_1 G'_1).$$

Using the induction on the number of blow up divisors on S we can assume that the exceptional collection of bundles $({G'}_1, {G'}_2, \ldots, {G'}_n)$ is constructible. That is it includes in some full exceptional collection obtained from the basic collection

$$\Big(\mathcal{O}_{e_1}(-1), \ldots, \mathcal{O}_{e_{d-1}}(-1), \mathcal{O}_S, \mathcal{O}_S(h), \mathcal{O}_S(2h)\Big)$$

by mutations. (Note that $K_{S'}^2 = K_S^2 + 1 > 1$. Therefore the constructibility of the collection $({G'}_1, {G'}_2, \ldots, {G'}_n)$ does not depend on the ranks of the G'_j (see theorem 3.1.8)).

Let us recall that the base of the induction, i.e. the case of the projective plane, has been settled in the paper [18].

Since $\sigma^*(G') = G$ we obtain that the bundle G has the exceptional filtration

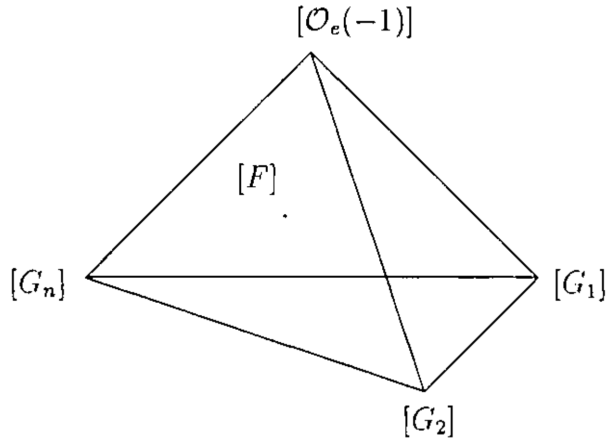
$$Gr(G) = (y_n G_n, y_{n-1} G_{n-1}, \ldots, y_1 G_1),$$

where $G_i = \sigma^*(G'_i)$. Moreover, the collection $\tau' = (\mathcal{O}_e(-1), G_1, \ldots, G_n)$ is exceptional (the triviality of the groups $\mathrm{Ext}^k(G_i, \mathcal{O}_e(-1))$ follows from the fact that $G_i|_e = s_i\mathcal{O}_e$). Furthermore, the constructibility of the collection $({G'}_1, {G'}_2, \ldots, {G'}_n)$ implies the constructibility of τ'.

Now to prove theorem 3.1.8 it suffices to show that the collection τ is included in an exceptional collection obtained from τ' by mutations.

Let us illustrate the procedure of this inclusion for the projectivisation of $K_0(S) \otimes \mathbb{Q} = K$. Assign to each sheaf E on S the vector $[E]$ in K. It is obvious that vectors corresponding to sheaves from an exceptional collection are linearly independent. Recall that the nonsingular bilinear form $(\cdot, \cdot)$ is well-defined on K . It corresponds to the Euler characteristic of sheaves $\chi(E, F)$. Since all exceptional sheaves satisfy the equation $\chi(E, E) = 1$, we see that the corresponding vectors are nonproportional. Let us consider the projectivisation of K. In this case, the vectors corresponding to sheaves of an exceptional collection are projected to vertices of some simplex.

The key exact sequence implies that the vector $[F]$ maps onto the simplex with the vertices $[\mathcal{O}_e(-1)], [G_1], \ldots, [G_n]$.



Let us project the point $[F]$ to the edge $([\mathcal{O}_e(-1)], [G_1])$. Note that this projection corresponds to a superrigid sheaf, and the exceptional pair (G_0'', G_1'') associated with it is obtained by mutations of the pair $(\mathcal{O}_e(-1), G_1)$. As the result, we get a smaller simplex containing $[F]$. Next let us project $[F]$ to the face $([G_1'], [G_2], \ldots, [G_n])$, etc... It remains to show that this process is finite.

Let us prove two statements about projections.

3.5.1 LEMMA. *Let*

$$0 \longrightarrow G \longrightarrow F \longrightarrow E \longrightarrow 0 \tag{16}$$

be an exact sequence of superrigid sheaves on the surface S. Let

$$Gr(E) = (y_k E_k, y_{k-1} E_{k-1}, \ldots, y_1 E_1),$$

$$Gr(G) = (y_m G_m, y_{m-1} G_{m-1}, \ldots, y_{k+1} G_{k+1})$$

be exceptional filtrations of E and G such that the collection

$$(E_1, \ldots, E_k, G_{k+1}, \ldots, G_m)$$

is exceptional. Let us split the filtration of the sheaf G into two groups

$$0 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 0, \tag{17}$$

where G' and G'' are the sheaves with the exceptional filtrations

$$Gr(G') = (y_m G_m, y_{m-1} G_{m-1}, \ldots, y_{s+1} G_{s+1}),$$

$$Gr(G'') = (y_s G_s, y_{s-1} G_{s-1}, \ldots, y_{k+1} G_{k+1}).$$

Then

1. *G' and G'' are superrigid;*
2. $\mathrm{End}(G') \cong \mathrm{Hom}(G', F)$*;*
3. $\mathrm{Ext}^i(G', F) = 0$ *for* $i > 0$*;*
4. $\mathrm{Ext}^2(F, G') = 0$*;*
5. *There is an exact sequence:*

$$0 \longrightarrow G' \longrightarrow F \longrightarrow E' \longrightarrow 0, \tag{18}$$

where E' is a superrigid sheaf included in the exact triple

$$0 \longrightarrow G'' \longrightarrow E' \longrightarrow E \longrightarrow 0. \tag{19}$$

Besides, $\mathrm{Ext}^i(G', E') = 0 \quad \forall i$.

PROOF. By the definition of an exceptional collection $\mathrm{Ext}^k(G_j, G_i) = 0$ for $j > i$ and all k. Therefore, $\forall k: \quad \mathrm{Ext}^k(G', G'') = 0$ (1.2.4). Hence it follows from lemma 2.2.2 that $\mathrm{Ext}^2(G'', G') = 0$. We apply the Mukai lemma to exact sequence (17) to obtain that G' and G'' are rigid. Since the collection $(G_{k+1}, ..., G_m)$ is exceptional we see that $\mathrm{Ext}^2(G_i, G_j) = 0$ for any pair i, j. This implies $\mathrm{Ext}^2(G', G') = \mathrm{Ext}^2(G'', G'') = 0$. Thus the first statement holds.

We saw that $\mathrm{Ext}^k(G', G'') = 0 \quad \forall k$. Whence, using exact triple (17) and the fact that G' is superrigid we have

$$\mathrm{Hom}(G', G) \cong \mathrm{End}(G') \qquad \text{and} \qquad \mathrm{Ext}^i(G', G) = 0 \quad \text{for } i > 0.$$

Besides, in view of the definition of the sheaf G' and the fact that the collection

$$(E_1, \ldots, E_k, G_{k+1}, \ldots, G_m)$$

is exceptional the following identities are valid.

$$\mathrm{Ext}^i(G', E) = 0 \quad \forall i; \qquad \mathrm{Ext}^2(E, G') = \mathrm{Ext}^2(G, G') = 0.$$

Consider two cohomology tables corresponding to sequence (16).

k	$\mathrm{Ext}^k(G', G)$	$\to$	$\mathrm{Ext}^k(G', F)$	$\to$	$\mathrm{Ext}^k(G', E)$
	$\mathrm{End}(G')$		?		0
	0		?		0
	0		?		0

k	$\mathrm{Ext}^k(E, G')$	$\to$	$\mathrm{Ext}^k(F, G')$	$\to$	$\mathrm{Ext}^k(G, G')$
	*		?		*
	*		?		*
	0		?		0

The statements 2, 3 and 4 follow from these tables.

Exact triples (16) and (17) give the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \uparrow & & & & & & \\
 & & G'' & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & G & \longrightarrow & F & \longrightarrow & E & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & G' & \longrightarrow & F & \longrightarrow & E' & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & G'' & & \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}
$$

It yields exact sequences (18) and (19).

Now in order to prove the lemma it remains to check that the sheaf E' is superrigid and for all i $\mathrm{Ext}^i(G', E') = 0$. All these facts follow from the following cohomology tables associated with sequence (18)

k	$\mathrm{Ext}^k(G', G')$	$\to$	$\mathrm{Ext}^k(G', F)$	$\to$	$\mathrm{Ext}^k(G', E')$
	$\mathrm{End}(G')$		$\mathrm{Hom}(G', F)$		?
	0		0		?
	0		0		?

k	$\mathrm{Ext}^k(F, G')$	$\to$	$\mathrm{Ext}^k(F, F)$	$\to$	$\mathrm{Ext}^k(F, E')$
	*		*		?
	*		0		?
	0		0		?

k	$\mathrm{Ext}^k(E', E')$	$\to$	$\mathrm{Ext}^k(F, E')$	$\to$	$\mathrm{Ext}^k(G', E')$
	?		*		0
	?		0		0
	?		0		0

This completes the proof.

The dual statement can be proved by the same argument.

3.5.2 LEMMA. *Under the assumptions of the previous lemma let us split the filtration of the sheaf E into two groups:*

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0,$$

where E' and E'' are sheaves with the exceptional filtrations

$$Gr(E') = (y_k E_k, y_{k-1} E_{k-1}, \dots, y_{s+1} E_{s+1}), \qquad Gr(E'') = (y_s E_s, y_{s-1} E_{s-1}, \dots, y_1 E_1).$$

Then

1. E' and E'' are superrigid;
2. $\mathrm{End}(E'') \cong \mathrm{Hom}(F, E'')$;
3. $\mathrm{Ext}^i(F, E'') = 0$ for $i > 0$;
4. $\mathrm{Ext}^2(E'', F) = 0$;
5. There exists an exact sequence:

$$0 \longrightarrow G' \longrightarrow F \longrightarrow E'' \longrightarrow 0,$$

where G' is a superrigid sheaf included in the exact triple:

$$0 \longrightarrow G \longrightarrow G' \longrightarrow E' \longrightarrow 0.$$

Moreover, $\mathrm{Ext}^i(G', E'') = 0 \quad \forall i$.

3.5.3 Remark.

1. Lemma 3.5.1 is also valid provided $E = y_1\mathcal{O}_e(-1)$ for the exceptional rational curve $e = e_d$;
2. lemma 3.5.2 holds provided $E = y_1 E_1 \oplus y_2 E_2$, where E_1 is an exceptional bundle and $E_2 = \mathcal{O}_e(-1)$;
3. the procedure described in 3.5.1 is called the transfer of the collection $(G_{k+1}, \dots, G_s)$ to the right, and the similar procedure from 3.5.2 is the transfer of the collection $(E_{s+1}, \dots, E_k)$ to the left.

Now let us prove a proposition concluding the proof of the main theorem.

3.5.4 Proposition. *Suppose a superrigid bundle $F = \mathcal{B}(F_0, F_1, F_2, \dots, F_k)$ on the surface S with $K_S^2 > 0$ is included in the exact sequence*

$$0 \longrightarrow G \longrightarrow F \longrightarrow E \longrightarrow 0, \tag{20}$$

where G is a superrigid bundle with an exceptional filtration

$$Gr(G) = (y_n G_n, y_{n-1} G_{n-1}, \dots, y_s G_s),$$

and E is a superrigid sheaf. In addition we assume that E is either locally free and

$$Gr(E) = (y_{s-1} G_{s-1}, y_{s-2} G_{s-2}, \dots, y_0 G_0)$$

or $E = y_0 G_0 = y_0 \mathcal{O}_e(-1)$; but the collection $(G_0, G_1, G_2, \dots, G_n)$ is exceptional in all cases. Then

1. $k \leq n$;

2. *the collection* $(F_0, \dots, F_k)$ *is included in an exceptional collection obtained from* $(G_0, G_1, G_2, \dots, G_n)$ *by mutations;*

3. $\sum_{i=0}^{k} r(F_i) \geq \sum_{j=0}^{n} r(G_j)$;

4. *If* E *is locally free then the equality* $\sum_{i=0}^{k} r(F_i) = \sum_{j=0}^{n} r(G_j)$ *yields the equality* $k = n$. *Moreover, in this case we have* $F_i = G_i$ *after some mutations of the neighboring zero pairs.*

PROOF. The proof is by induction on the number of sheaves in the collection

$$(G_0, G_1, G_2, \dots, G_n).$$

The case $n = 1$ has been checked in the previous section.

STATEMENT. *It can be assumed without loss of generality that* E *and* G *is locally free.*

Proof. Suppose $E = y_0 G_0 = y_0 \mathcal{O}_e(-1)$. Following remark 3.5.3 , let us apply the transfer of G_1 to the right. Namely, let us denote by G' the bundle $\mathcal{B}(G_2, G_3, \dots, G_n)$ and let us consider the exact sequences

$$0 \longrightarrow G' \longrightarrow F \longrightarrow E' \longrightarrow 0,$$

$$0 \longrightarrow y_1 G_1 \longrightarrow E' \longrightarrow y_0 G_0 \longrightarrow 0.$$

Taking into account lemma 3.5.1 and proposition 3.4.1 we obtain that E' is a superrigid bundle such that $E' = x_0 E'_0 \oplus x_1 E'_1$ (or $E' = x_0 E'_0$), where the exceptional pair (E'_0, E'_1) (or E'_0) is obtained by mutations of the pair (G_0, G_1). Moreover, E'_0 is locally free and

$$r(E'_0) + r(E'_1) \geq r(G_0) + r(G_1) \qquad (r(E'_0) \geq r(G_0) + r(G_1)).$$

Let us show that the collection $(E'_0, E'_1, G_2, \dots, G_n)$ is exceptional. From lemma 3.5.1 it follows that $\mathrm{Ext}^k(G', E') = 0 \quad \forall k$. But, $E' = x_0 E'_0 \oplus x_1 E'_1$ and $G' = \mathcal{B}(G_2, G_3, \dots, G_n)$.

Provided E'_i is locally free, the triviality of the groups $\mathrm{Ext}^k(G_j, E'_i)$ for $j = 2, \dots, n$ follows from lemma 2.5.7 . Let us check this property for the case $E'_1 = \mathcal{O}_e(-1)$. Since $\mathrm{Ext}^k(G', \mathcal{O}_e(-1)) = 0 \quad \forall k$, we see that the restriction of G' to the curve e is trivial. Therefore there exists a superrigid bundle L on the surface S' obtained from S by blowing down the curve e ($\sigma : S \longrightarrow S'$) such that $\sigma^*(L) = G'$.

Since L is superrigid, we see that it has the exceptional filtration

$$Gr(L) = (z_m L_m, z_{m-1} L_{m-1}, \dots, z_2 L_2).$$

Besides, $Gr(G') = (z_m \sigma^*(L_m), z_{m-1}\sigma^*(L_{m-1}), \dots, z_2 \sigma^*(L_2))$ is the exceptional filtration of the bundle G'. Now by theorem 2.5.1 $m = n$ and $G_i = \sigma^*(L_i)$. Thus the collection $(E'_0, E'_1, G_2, \dots, G_n)$ is exceptional.

Our statement is correct in the case $E' = y_0 E'_0$.

Assume that $E' = x_0 E'_0 \oplus x_1 E'_1$ for some positive x_0, x_1. Let us apply the transfer of E'_1 to the left:

$$0 \longrightarrow \tilde{G} \longrightarrow F \longrightarrow x_0 E'_0 \longrightarrow 0,$$

$$0 \longrightarrow G' \longrightarrow \tilde{G} \longrightarrow x_1 E_1' \longrightarrow 0.$$

Using lemma 3.5.2 and the inductive hypothesis we obtain that $\tilde{G}$ is a superrigid sheaf with the exceptional filtration $Gr(\tilde{G}) = (x_m' G_m'', x_{m-1}' G_{m-1}'', \dots, x_1' G_1'')$. In addition, the collection $(G_1'', \dots, G_m'')$ is included in an exceptional collection obtained from $(E_1', G_2, \dots, G_n)$ by mutations and

$$\sum r(G_i'') \geq \sum r(G_i) + r(E_1').$$

Note that the sheaf $\tilde{G}$ has no torsion, as a subsheaf of a bundle. Since $\tilde{G}$ is rigid we see that it is locally free. It can be checked as above that the collection $(E_0', G_1'', \dots, G_m'')$ is exceptional. This completes the proof.

We shall name *bounding* the collection $(G_0, G_1, G_2, \dots, G_n)$ from the formulation of our proposition and all collections obtained from it by mutations.

Now, consider exact sequence (20). We shall use the transfer of the bundle G_s to the right and to the left. Recall that in this procedure the sum of ranks of the bounding collections do not decrease.

Since the sum of ranks of the bounding collections is less than or equals to the rank of the bundle F we see that this process cannot continue ad infinitum. Hence starting with some moment the sum of ranks is a constant. We study this moment in the following statement.

STATEMENT. *Assume that under the assumptions of our proposition the sum of ranks of bundles from the bounding collection does not change after the transfers of the bundle G_s to the right and to the left. Then $k = n$ and*

$$(F_0, F_1, F_2, \dots, F_k) = (G_0, G_1, G_2, \dots, G_n).$$

up to mutations of neighboring zero-pairs.

Proof. After the transfer of the bundle G_s to the right one gets two exact sequence:

$$0 \longrightarrow \mathcal{B}(G_{s+1}, \dots, G_n) \longrightarrow F \longrightarrow \mathcal{B}(G''_0, G''_1, G''_2, \dots, G''_l) \longrightarrow 0,$$

$$0 \longrightarrow y_s G_s \longrightarrow \mathcal{B}(G''_0, G''_1, G''_2, \dots, G''_l) \longrightarrow \mathcal{B}(G_0, G_1, G_2, \dots, G_{s-1}) \longrightarrow 0.$$

Since $G_0, \dots, G_{s-1}, G_s$ are locally free we obtain that the inductive hypothesis and the relation

$$\sum_{i=0}^{l} r(G_i'') = \sum_{i=0}^{s} r(G_i)$$

imply that $l = s$ and $G_i = G_i''$ (up to mutations of neighboring zero-pairs). Therefore there exists an exact sequence

$$0 \longrightarrow \mathcal{B}(G_{s+1}, \dots, G_n) \longrightarrow F \longrightarrow \mathcal{B}(G_0, G_1, G_2, \dots, G_s) \longrightarrow 0.$$

Moreover, $(G_0, G_1, G_2, \dots, G_s)$ is the *hom*-collection. Whereby, $\mu_H(G_i) \geq \mu_H(G_j)$ for $s \geq i > j$.

Now let us do the transfer of the bundle G_s to the left (by assumption, the sum of ranks does not change as well):

$$0 \longrightarrow \mathcal{B}(G_s''', \dots, G_m''') \longrightarrow F \longrightarrow \mathcal{B}(G_0, G_1, G_2, \dots, G_{s-1}) \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{B}(G_{s+1}, \dots, G_n) \longrightarrow \mathcal{B}(G''_s, \dots, G''_m) \longrightarrow y_s G_s \longrightarrow 0.$$

As before, by inductive hypothesis, we obtain that the collection $(G''_s, \dots, G''_m)$ coincides with the collection $(G_s, \dots, G_n)$ up to mutations of neighboring zero-pairs. Hence $(G_s, \dots, G_n)$ is the *hom*-collection and $\mu_H(G_j) \leq \mu_H(G_i)$. for $s \leq j < i$.

As a result we obtain that the all bounding collections $(G_0, G_1, G_2, \dots, G_n)$ are the *hom*-collections. Thus we can construct the exceptional filtrations of the bundle

$$F = \mathcal{B}(F_0, F_1, F_2, \dots, F_k)$$

from the exceptional filtrations of the bundles E and G in sequence (20).

Now the proof follows from the uniqueness of the exceptional filtration.

References

[1] M. F. Atiyah: Vector Bundles Over an Elliptic Curve.// Proc. Lond. Math. Soc., VII (1957), 414-452.

[2] A. I. Bondal: Helices, Representations of Quivers and Koszul Algebras.// Helixes and Vector Bundles, London Math. Soc., Lecture Noute Series 148. Cambridge Univ. Press, p.75-96.

[3] R. Bott: Homogeneus vector bundles.// Ann. of Math., v.66, p.203-248.

[4] A. L. Gorodentsev: Exceptional bundles on a surface with a moving anticanonical class.// Math. USSR Izv.**33** (1989) 740-755.

[5] A. L. Gorodentsev: Exceptional Objects and Mutations in Derived Categories.// Helixes and Vector Bundles, London Math. Soc., Lecture Noute Series 148. Cambridge Univ. Press, p.57-74.

[6] J.-M. Drezet and J.Le Potier: Fibres stables et fibres exceptionnels sur $\mathbb{P}_2$.// Ann. Sci. ENS(4)18(1985),193-243.

[7] J.-M. Drezet: Fibres exceptionnels et suite spectrale de Beilinson generalisee sur $\mathbb{P}_2(\mathbb{C})$.// Math. Ann. 275,(1) (1986), 25-48.

[8] S. K. Zube, D. Yu. Nogin: Computing Invariants of Exceptional Bundles on a Quadric.// Helixes and Vector Bundles, London Math. Soc., Lecture Noute Series 148. Cambridge Univ. Press, p.23-32.

[9] S. Yu. Zyuzina. Constructibility of exceptional pairs of vector bundles on a quadric.// (Russian) Akad.Nauk SSSR Ser. Mat. **57** (1993), no 1, 183-191.

[10] S. A. Kuleshov: Construction of Bundles on an Elliptic Curve.// Helixes and Vector Bundles, London Math. Soc., Lecture Noute Series 148. Cambridge Univ. Press, p.7-22.

[11] S. A. Kuleshov, D. O. Orlov: Exceptional sheaves on Del Pezzo surfaces. // (Russian) Izv. Akad. Nauk Russia Ser. Mat. **58** (1994), no. 3, 59–93.

[12] Yu. I. Manin: Cubic forms.// M.: Nauka. 1972.

[13] A. A. Markov: About binary quadratic forms of positive definition.// SPb., 1880, p.44.

[14] S.Mukai: On the Moduli Spaces of Bundles on K3 Surfaces,I// in Vector Bundles ed. Atiyah et al, Oxford Univ. Press, Bombey, (1986) P. 67-83.

[15] D. Yu. Nogin: Helices of period four and Markov-type equations.// Math. USSR - Izv. **37** (1991), no. 1, 209–226.

[16] C. Okonek, M. Scneider, H. Spindler: Vector Bundles on Complex Projective Spaces.// Birkhauser, Boston, 1980.

[17] D. O. Orlov: Projective bundles, monoidal transformations and derived categories of coherent sheaves.// (Russian) Isv. Akad. Nauk USSR Ser. Mat. **56** (1992), no 4, 852-862.

[18] A. N. Rudakov: Markov numbers and exceptional bundles on $\mathbb{P}^2$.// Math. USSR - Izv. **32** (1989), no. 1, 99–112.

[19] A. N. Rudakov: Exceptional Collections, Mutations and Helixes.// Helixes and Vector Bundles, London Math. Soc., Lecture Noute Series 148. Cambridge Univ. Press, p.1-6.

[20] A. N. Rudakov: Exceptional vector bundles on a quadric.// Math. USSR - Izv. **33** (1989), no. 1, 115–138.

[21] A. N. Rudakov: A description of Chern classes of semistable sheaves on a quadric surface.// Schriftenriehe. Hett Nr.88. Forschungsschwerpunkt Komplexe Mannigfattigkeiten. Erlangen(1990).

[22] R. Hartshorne: Algabraic Geometry.// Springer – Verlag New York Heidelberg Berlin, 1977.

Contents

Introduction. **1**

Notations. **2**

1 Axioms of Stability. **2**
1.1 Definitions and Simple Properties. 2
1.2 The Harder-Narasimhan Filtration. 4
1.3 Examples of Slopes and Types of Stability. 8

2 Rigid Sheaves. **11**
2.1 Preliminary Information. 11
2.2 Exceptional Sheaves. 13
2.3 Exceptional Collections. 15
2.4 Structure of Rigid Sheaves. 18
2.5 Structure of Superrigid Sheaves. 25

3 Constructibility of Exceptional bundles. 31

3.1 Introduction to the Helix Theory. 31
3.2 Restriction of Superrigid Bundles to an Exceptional Curve. 37
3.3 Equivalence of Collections and the Key Exact Sequence. 41
3.4 Category Generated by a Pair. 45
3.5 Proof of the Main Theorem. 53