STABLE HOMOTOPY GROUPS OF SPHERES AND HOMOLOGY OF CO-*B*- CONSTRUCTIONS

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Here we will consider algebraic methods of the problem of describing stable homotopy groups of spheres, wich were discussed on the first report 09. 02. 1994. Recall the notions of B-construction and co-B- construction introduced by Adams [1] and generalized on the case of A_∞-algebras and A_∞-coalgebras by Stasheff [2].

Let A be a graded A_∞-algebra. It means there are products $\pi_i : A^{\otimes(i+2)} \to A$ and for any $n \geq 0$ the following relations are satisfied

$$\sum_{i=0}^{n} (-1)^\epsilon \pi_i(1 \otimes ... \otimes \pi_{n-i} \otimes ... \otimes 1) = 0$$

where the sum is taken also over all places of π_{n-i}. Then B- construction $\tilde{B}A$ is a differential coalgebra, wich as a graded coalgebra coincides with the tensor coalgebra TSA over the suspension SA. The elements in TSA denote $[x_1, ..., x_n], x_i \in A$, and have dimensions $\sum_{i=1}^{n} dim(x_i) + n$. A coproduct in TSA is given by the formular

$$\nabla[x_1, ..., x_n] = \sum_i [x_1, ..., x_i] \otimes [x_{i+1}, ..., x_n]$$

A differential on the elements $[x_1, ..., x_n]$ is defined by the formular

$$d[x_1, ..., x_n] = \sum_i (-1)^\epsilon [x_1, ..., \pi_k(x_i \otimes ... \otimes x_{i+k+1}), ..., x_n]$$

In the case when A is usual algebra without higher products $\pi_i, i \geq 1$ the corresponding B- construction denoted BA.

By dual manner, let K be a graded A_∞- coalgebra. It means there are coproducts $\nabla_i : K \to K^{\otimes(i+2)}$ and for any $n \geq 0$ the following relations are satisfied

$$\sum_{i=0}^{n} (-1)^\epsilon (1 \otimes ... \otimes \nabla_{n-i} \otimes ... \otimes 1)\nabla_i = 0$$

where the sum is taken also over all places of ∇_{n-i}. Then co-B-construction $\tilde{F}K$ is a differential algebra, wich as a graded algebra coincides with a tensor algebra $TS^{-1}K$

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over a desuspension $S^{-1}K$. The elements in $TS^{-1}K$denote $[x_1, ..., x_n], x_i \in K$ and have dimensions $\sum_{i=1}^{n} dim(x_i) - n$. A product in $TS^{-1}K$ is given by the formula

$$\pi([x_1, ..., x_n] \otimes [y_1, ..., y_m]) = [x_1, ..., x_n, y_1, ..., y_m]$$

A differential on the elements $[x]$ is defined by the formula $d[x] = \sum_i \nabla_i(x)$. On the others elements the differential defined as on the products of the elements $[x]$.

In the case when K is usual coalgebra without higher coproducts $\nabla_i, i \geq 1$ the corresponding co-B-construction denoted FK.

B-constructions and co-B-constructions prove to be very useful in algebraic topology. In particular Adams has used the co-B- construction FK overe Milnor coalgebra K (dual to Steenrod algebra) to describe the second term of his spectral sequence for stable homotopy groups of spheres. He has proved that this second term is isomorphic to the homology of the co-B-construction over Milnor coalgebra. We have proved the next

Theorem 1. *On Milnor coalgebra K there is A_∞- coalgebra structure and the homology of corresponding co-B-construction $\tilde{F}K$ are isomorphic to E_∞- term of the Adams spectral sequence of stable homotopy groups of spheres.*

The meaning of this theorem is in the possibility to choose higher differentials of the Adams spectral sequence in such manner, that they will form A_∞-coalgebra structure on Milnor coalgebra. To prove this theorem we use Bousfield-Kan spectral sequence [3], functional homology operations [4] and operad methods [5],[6].

In two words the proof is the next. It is known that higher differentials in the Adams spectral sequence are determined by Massey-Peterson functional cohomology operations [7], wich are partial defined and multivalued mappings. We have defined such functional homology operations [4], wich are usual operations, everywhere defined and unique valued. They determine Massey-Peterson operations and hence the higher differentials in the Adams spectral sequence. These functional homology operations give us the desirable A_∞-coalgebra structure on Milnor coalgebra K.

So to describe the second term and E_∞ term of the Adams spectral sequence we must describe the homology of the corresponding co-B-constructions FK overe Milnor coalgebra in the case of the second term, and $\tilde{F}K$ over A_∞-Milnor coalgebra in the case of E_∞-term. Indeed our methods are working in general situation of arbitrary A_∞-coalgebra. In the case of Milnor coalgebra we will obtain the second term, in the case of A_∞- Milnor coalgebra we will obtain E_∞ term of the Adams spectral sequence.

Let now K be a graded A_∞- coalgebra. Consider a question of describing homology $A_* = H_*(\tilde{F}K)$ of co-B- construction $A = \tilde{F}K$. First question we must answer - what structure is on $A_* = H_*(\tilde{F}K)$. Of course there is an algebra structure $\pi_* : A_* \otimes A_* \to A_*$ induced by an algebra structure $\pi : A \otimes A \to A$ in co-B-construction $A = \tilde{F}K$. But besides that there are, for example, Massey products $\mu : A_* \otimes ... \otimes A_* \to A_*$, wich are partial defined and multivalued operations. Much more convenient language for our purpose is Stasheff language of A_∞- structures. A general theorem states that on the homology of any differential algebra A there is A_∞- algebra structure, wich defines all Massey products and there is an isomorphism of homology of B-constructions $H_*(BA) \cong H_*(\tilde{B}A_*)$. It was proved by

Kadeishvili [8] and also follows from the fact that A_∞- algebra structure is a homotopy invariant one in the sense that if it is on a chain complex A and a chain complex A' is homotopy equivalent to A, then on A' also is A_∞- algebra structure and there is A_∞- chain equivalence between A and A'. In the case when a ground ring is a field the homology A_* of a differential algebra A may be considered as a chain complex with zero differential, wich is chain equivalent to A. Then on A_* there is the required A_∞- algebra structure and an isomorphism $H_*(\tilde{B}A_*) \cong H_*(BA)$.

So we will describe $A_* = H_*(\tilde{F}K)$ as A_∞- algebra. It means to find indecomposable elements and relations between its products. In the first report we have defined what it means decomposable and indecomposable elements for A_∞- algebra. Now we reformulate these definitions on the language of B- constructions.

Let A be a graded A_∞- algebra. Consider B- construction $\tilde{B}A$. There is an injection $A \to \tilde{B}A, x \longmapsto [x]$ and a short exact sequence

$$0 \to A \xrightarrow{i} \tilde{B}A \xrightarrow{p} \tilde{B}^1A = \tilde{B}A/A \to 0$$

wich induces a long exact sequence of homology

$$... \to A \xrightarrow{i_*} H_*(\tilde{B}A) \xrightarrow{p_*} H_*(\tilde{B}^1A) \xrightarrow{\mu_*} A \xrightarrow{i_*} H_*(\tilde{B}A) \to ...$$

From the definition of Massey sequence it follows that the sequence $(x^2, ..., x^n)$ of the elements $x^i \in A^{\otimes i}$ is Massey sequence if and only if the element $x^2 + ... + x^n$ is a cycle in $\tilde{B}^1A$. Moreover, a map $\mu_* : H_*(\tilde{B}^1A) \to A$ is induced by Massey products. The element $x \in A$ will be decomposable if it belong to the image of μ_* . The module of indecomposable elements of A will be denoted QA. It is isomorphic to $A/Im\mu_* \cong Imi_*$.

Notice that in the case of usual algebras we obtain usual definition of indecomposable elements.

So to find indecomposable elements in A_∞- algebra A we must find the image of the homomorphism $i_* : A \to H_*(\tilde{B}A)$, and its elements will be the generator elements.

By dual manner for A_∞- coalgebra K it may be defined a notion of primitive elements. Namely, consider co-B-construction $\tilde{F}K$. There is a projection $p : \tilde{F}K \to K$ and short exact sequence

$$0 \to \tilde{F}^1K = Ker(p) \xrightarrow{i} \tilde{F}K \xrightarrow{p} K \to 0$$

wich induces a long exact sequence of homology

$$... \to H_*(\tilde{F}^1K) \xrightarrow{i_*} H_*(\tilde{F}K) \xrightarrow{p_*} K \xrightarrow{\tau_*} H_*(\tilde{F}^1K) \to ...$$

An element $x \in K$ will be called primitive if it belongs to the kernal of the mapping $\tau_* : K \to H_*(\tilde{F}^1K)$, or that is the same, to the image of the mapping $p_* : H_*(\tilde{F}K) \to K$. The module of primitive elements of K will be denoted PK.

Notice that if K be usual coalgebra with coproduct $\nabla : K \to K \otimes K$ then we obtain usual definition of primitive elements $PK = \{x \in K : \nabla(x) = x \otimes 1 + 1 \otimes x\}$.

Next theorem gives us the opportunity to find indecomposable elements of the homology of co-B-construction over arbitrary A_∞-coalgebra.

Theorem 2. *For any graded A_∞-coalgebra K there is an isomorphism*

$$QH_*(\tilde{F}K) \cong PK$$

, where $QH_*(\tilde{F}K)$ -a module of indecomposable elements of $A_* = H_*(\tilde{F}K)$, PK - a module of primitive elements in K.

The proof follows from the above definitions and commutative diagram

$$\begin{array}{ccc} H_*(\tilde{F}K) & \xrightarrow{p_*} & K \\ \downarrow = & & \downarrow \cong \\ A_* & \xrightarrow{i_*} & H_*(\tilde{B}A_*) \end{array}$$

Similary, for the primitive elements of the homology of B- construction over A_∞-algebra there is a dual theorem.

Now we consider the question about relations for A_∞- algebra. Let A be a graded A_∞- algebra. Any relation in A may be written in a form $\mu(x^2, ..., x^n) = \pi_0(x^2) + ... + \pi_{n-2}(x^n) = 0$, where $(x^2, ..., x^n)$ - Massey sequence and $\mu(x^2, ..., x^n)$- Massey product in A_∞ - agebra A. To find the relations it means to find those Massey sequences, for wich Massey products are zero. Of course, some of such relations follows from A_∞- algebra structure. So we need to find only such relations, wich don't follows simply from A_∞ - algebra structure.

As we have seen it earlier Massey sequences $(x^2, ..., x^n)$ represents cycles in $\tilde{B}^1 A$. Show that homological zero cycles give the relations $\mu(x^2, ..., x^n) = 0$, wich follows from A_∞- algebra structure in A.

Indeed let $y \in A^{\otimes(n+1)}, d(y) = x^2 + ... + x^n$. Then $x^2 = (\pi_{n-2} \otimes 1 + 1 \otimes \pi_{n-2})(y), ..., x^n = (\pi_0 \otimes ... \otimes 1 + ... + 1 \otimes ... \otimes \pi_0)(y)$. Massey sequence $(x^2, ..., x^n)$ in this case determines the relation $\mu(x^2, ..., x^n) = \pi_0(x^2) + ... + \pi_{n-2}(x^n) = \pi_0(\pi_{n-2} \otimes 1 + 1 \otimes \pi_{n-2})(y) + ... + \pi_{n-2}(\pi_0 \otimes ... \otimes 1 + ... + 1 \otimes ... \pi_0)(y) =$

$$\sum_{i=0}^{n-2} \pi_i(1 \otimes ... \otimes \pi_{n-2-i} \otimes ... \otimes 1)(y) = 0$$

wich follows from A_∞- algebra structure.

Thus the relations what we need are determined by the elements in homology $H_*(\tilde{B}^1 A)$ such that $\mu_*(x) = 0$, where $\mu_* : H_*(\tilde{B}^1 A) \to A$, and to find relations in A_∞-algebra A we need to find the kernal of μ_*, or that is the same, the image of$p_* : H_*(\tilde{B}A) \to H_*(\tilde{B}^1 A)$.

From the other side let $A_\infty A$ be a free A_∞- algebra generated by A and $A^1_\infty A = A_\infty A/A$. Then to find relations in A it means to find such elements $x \in A^1_\infty A$ for wich $\mu(x) = 0$, where $\mu : A^1_\infty A \to A$ is a mapping induced A_∞- algebra structure in A. Of course, some of such relations follows from A_∞ -algebra structure in A. To find these relations consider a map $\gamma \times 1 - 1 \times \mu : A_\infty A^1_\infty A \to A^1_\infty A$, where γ is a mapping induced by a monad structure in A_∞, considered as a monad in the category of graded modules. If x belongs to the image of this mapping $\gamma \times 1 - 1 \times \mu$ then it generates a relation $\mu(x) = 0$, wich follows from A_∞- structure. Denote

$A^1_\infty * A$ a factor of $A^1_\infty A$ under the image of $\gamma \times 1 - 1 \times \mu$. Note that elements in $A^1_\infty * A$ have a form $\pi_n(x_1 \otimes ... \otimes x_{n+2})$, where $x_i \in A, 1 \leq i \leq n$. Then to find relations in A, wich not follows simply from A_∞- algebra structure we need to find such elements $x \in A^1_\infty * A$ wich maps into zero under the mapping $\mu : A^1_\infty * A \to A$.

Define a mapping $i : \tilde{B}^1 A \to A^1_\infty * A$ by putting $i(x^2 + .. + x^n) = \pi_0(x^2) + ... + \pi_{n-2}(x^n)$. It is easy to see, that if x be homological zero in $\tilde{B}^1 A$ then $i(x) = 0$, and conversly if $i(x) = 0$ then x is homological zero. Therefore i induces a monomorphism $i_* : H_*(\tilde{B}^1 A) \to A^1_\infty * A$.

Consider a composition $\psi : H_*(\tilde{B}A) \to A^1_\infty * A$ of the mapping $p_* : H_*(\tilde{B}A) \to H_*(\tilde{B}^1 A)$ and this monomorphism $H_*(\tilde{B}^1 A) \to A^1_\infty * A$. Then to find relations we need to find the image of ψ. All required relations will have a form $\psi(x) = 0, x \in H_*(\tilde{B}A)$.

Let now K - A_∞- coalgebra. Consider A_∞- algebra $A_* = H_*(\tilde{F}K)$. Taking into account an isomorphism $H_*(\tilde{B}A_*) \cong K$, from previous considerations it follows that all required relations have a form $\psi(x) = 0, x \in K$ and to find them we need a formula for ψ. To write such formula consider Stasheff operad A_∞, wich is generated by the operations π_i. Remark, that we can denote it's elements as ∇_i also. When we are saying about A_∞-algebras we use the notations π_i, when about A_∞-coalgebras then we use the notations ∇_i. There is a coproduct $\Delta : A_\infty \to A_\infty \otimes A_\infty$, wich turn A_∞ into a Hopf operad. We will denote values of this coproduct as $\Delta(\pi_i) = \sum \pi'_i \otimes \nabla''_i$. Denote also $p : K \to PK \cong QA_*$ a projection. Then we will have

Theorem 3. *Let K be a graded A_∞- coalgebra, then all required relations in the homology $A_* = H_*(\tilde{F}K)$, considered as A_∞- algebra have the form $\psi(x) = 0$, where $\psi : K \to A^1_\infty * A_*$ is given by the formula $\psi(x) = \sum_i \pi'_i * (p \otimes ... \otimes p)\nabla''_i(x), x \in K$*

Note that in the case when K be a usual coalgebra the formula for ψ takes more simple form: $\psi(x) = \sum_i \pi_i * (p \otimes ... \otimes p)\nabla(i)(x)$, where $\nabla(i) : K \to K^{\otimes(i+2)}$ obtained by iterating a coproduct $\nabla : K \to K \otimes K$.

Let now K be Milnor coalgebra (dual to Steenrod algebra). Show how these methods are working for describing E_2-term of the Adams spectral sequence. Recall that Milnor coalgebra indeed is a Hopf algebra with associative product $\pi : K \otimes K \to K$, associative coproduct $\nabla : K \to K \otimes K$ and Hopf relation

$$\nabla \circ \pi = (\pi \otimes \pi) \circ (1 \otimes T \otimes 1) \circ (\nabla \otimes \nabla),$$

where $T : K \otimes K \to K \otimes K$ is a permutation mapping.

If a ground ring R is $\mathbb{Z}/2$ then K is polinomial algebra with generators ξ_i of dimensions $2^i - 1, \xi_0 = 1$ and coproduct $\nabla : K \to K \otimes K$ defined by the formula

$$\nabla(\xi_i) = \sum_{k=0}^{i} \xi_{i-k}^{2^k} \otimes \xi_k$$

If $R = \mathbb{Z}/p$ then K is commutative Hopf algebra with generators ξ_i of dimensions $2(p^i - 1)$ and τ_i of dimensions $2p^i - 1$. A coproduct $\nabla : K \to K \otimes K$ defines as follows

$$\nabla(\xi_i) = \sum_{k=0}^{i} \xi_{i-k}^{p^k} \otimes \xi_k, \nabla(\tau_i) = \tau_i \otimes 1 + \sum_{k=0}^{i} \xi_{i-k}^{p^k} \otimes \tau_k$$

Directly from the definition of coproduct in Milnor coalgebra it follows that its primitive elements are $\xi_1^{2^i}, i \geq 0$ if $R = \mathbb{Z}/2$ and $\xi_1^{p^i}, \tau_0$, if $R = \mathbb{Z}/p$. Corresponding indecomposable elements in the second term of the Adams spectral sequence denoted h_i for ξ_i and g_0 for τ_0. From Theorem 2 it follows

Theorem 4. *If $R = \mathbb{Z}/2$ then the module of indecomposable elements in $A_* = H_*(FK)$ considered as graded A_∞- algebra is generated by the elements h_i of dimensions $2^i - 1$. If $R = \mathbb{Z}/p$ then corresponding module of indecomposable elements is generated by the elements h_i of dimensions $(2p-2)^i - 1$ and by the element g_0 of dimension* 0.

Of course these results are simply reformulations of May results [9].

Now we consider the question of finding the relation in the second term of the Adams spectral sequence. Let $R = \mathbb{Z}/2$. Define a projection $p : K \to QA_*$, where $A_* = H_*(FK)$, putting $p(\xi_1^{2^i}) = h_i$, and $p(x) = 0$ otherwise. Then a mapping $\psi : K \to A_\infty * A_*$ calculates by the formula

$$\psi(x) = \sum_{i \geq 0} \pi_i(p(x_1) \otimes ... \otimes p(x_{i+2})),$$

where $\sum x_1 \otimes ... \otimes x_{i+2} = \nabla(i)(x)$.

Taking different elements $x \in K$ we will obtain different relations. For example, it is to see that $\psi(\xi_i) = \pi_{i-2}(h_{i-1} \otimes ... \otimes h_0)$ and therefore we have the relations

$$\pi_{i-2}(h_{i-1} \otimes ... \otimes h_0) = 0$$

wich we call basic relations.

To describe others relations we use Hopf structure in K. Namely, define products $\cup_1$ in QA_* putting $h_i \cup_1 h_i = h_{i+1}$ and $h_i \cup_1 h_j = 0$ otherwise. Similary $h_i \cup_1 h_i \cup_1 h_{i+1} = h_i \cup_1 h_{i+1} \cup_1 h_i = h_{i+1} \cup_1 h_i \cup_1 h_i = h_{i+2}$ and $h_i \cup_1 h_j \cup_1 h_k = 0$ otherwise, and so on.

Using these products we define products of the elements $\pi_{n-2}(x_1 \otimes ... \otimes x_n)$ in $A_\infty^1 * A_*$ by putting

$$\pi_{n-2}(x_1 \otimes ... \otimes x_n) \cup_1 \pi_{m-2}(x_{n+1} \otimes ... \otimes x_{n+m}) =$$

$$= \sum \pi_{n+m+2}(x_{i_1} \otimes ... \otimes x_{i_{n+m}}) + \sum \pi_{n+m-3}(x_{i_1} \otimes ... \otimes x_{i_k} \cup_1 x_{j_k} \otimes ... x_{i_{n+m}}) + ...$$

where all sums are taken aver all (n, m) shuffles of $(1, ..., n+m)$ and $1 \leq i_1 < i_2 < ... \leq n, n+1 \leq j_1 < j_2 < ... \leq n+m$.

Similary it defines a product

$$\pi_{n-2}(x_1 \otimes ... \otimes x_n) \cup_1 \pi_{m-2}(x_{n+1} \otimes ... \otimes x_{n+m}) \cup_1 \pi_{k-2}(x_{n+m+1} \otimes ... x_{n+m+k})$$

and so on.

From basic relations by taking its products we can obtain new relations and there is the next

Theorem 5. *The second term of the Adams spectral sequence considered as A_∞-algebra has generator elements $h_i, i \geq 0$ and relations:*

$$\pi_i(h_{i+1} \otimes ... \otimes h_0) = 0,$$

$$\pi_i(h_{i+1} \otimes ... \otimes h_0) \cup_1 \pi_j(h_{j+1} \otimes ... \otimes h_0) = 0$$

and so on.

For example, multiplying the elements h_i and h_j we obtain the relation $\pi_0(h_i \otimes h_j) + \pi_0(h_j \otimes h_i) = 0$. Multiplying $\pi_0(h_1 \otimes h_0)$ with itself, we obtain the relations $\pi_0(h_2 \otimes h_1) = 0$. Repeating this procedure we obtain the relations $\pi_0(h_{i+1} \otimes h_i) = 0$. We denote them simply $h_{i+1}h_i = 0$. Remark that there are no another relations between two times products of the elements h_i. To obtain all relations with three elements h_i, h_j, h_k we must find the products $h_i \cup_1 h_j \cup_1 h_k$ and $\pi_0(h_{i+1} \otimes h_i) \cup_1 h_j$. Doing it we obtain the following relations: $\pi_1(h_i \otimes h_j \otimes h_k + h_i \otimes h_k \otimes h_j + h_j \otimes h_i \otimes h_k + h_j \otimes h_k \otimes h_i + h_k \otimes h_i \otimes h_j + h_k \otimes h_j \otimes h_i) = 0$;

$$\pi_1(h_{i+1} \otimes h_i \otimes h_j + h_{i+1} \otimes h_j \otimes h_i + h_j \otimes h_{i+1} \otimes h_i) = 0, j \neq i, i+1;$$

$$\pi_1(h_i \otimes h_{i+1} \otimes h_i) + \pi_0(h_{i+1} \otimes h_{i+1}) = 0, j = i;$$

$$\pi_1(h_{i+1} \otimes h_i \otimes h_{i+1}) + \pi_0(h_{i+2} \otimes h_i) = 0, j = i+1.$$

Of course when we say all relations we mean all generator relations. There are also the relations from A_∞-algebra structure and corollaries relations. For example, consider the relation $\pi_0(\pi_1 \otimes 1 + 1 \otimes \pi_1) + \pi_1(\pi_0 \otimes 1 \otimes 1 + 1 \otimes \pi_0 \otimes 1 + 1 \otimes 1 \otimes \pi_0) = 0$ and apply it to the element $h_i \otimes h_{i+1} \otimes h_i \otimes h_{i+1}$, we will obtain the relation $h_{i+1}^3 + h_i^2 h_{i+2} = 0$. If we apply the same relation to the element $h_{i+2} \otimes h_{i+1} \otimes h_i \otimes h_{i+1}$ we will obtain the relation $h_i h_{i+2}^2 = 0$, and so on.

Indeed all these relations are some kind of commutativity relations. There is another way to describe commutativity relations with the help of E_∞-algebra structure. Note that Milnor coalgebra K is a commutative algebra. In this case on its co-B-construction FK is a structure of E_∞-algebra [10]. But E_∞- structure is a homotopy invariant structure and hence E_∞-algebra structure is also on the homology $H_*(FK)$. For E_∞-algebras we also can define a notion of indecomposable elements and to look for the relations between its products.

Namely, for graded E_∞-algebra A also as for A_∞ -algebra it may be defined a notion of B-construction $B(E_\infty, A)$. There is an injection $i : A \to B(E_\infty, A)$, wich induces a short exact sequence

$$0 \to A \xrightarrow{i} B(E_\infty, A) \xrightarrow{p} B^1(E_\infty, A) \to 0$$

and a long exact sequence of homologies

$$...A \xrightarrow{i_*} H_*B(E_\infty, A) \xrightarrow{p_*} H_*B^1(E_\infty, A) \xrightarrow{\mu_*} A \to ...$$

An element $x \in A$ will be called decomposable if it belongs to the image of μ_*. The module of indecomposable elements will be isomorphic to Imi_*.

Also as in the case of A_∞- algebras the required relations are determined by the homomorphism $\psi : H_*B(E_\infty, A) \to E^1_\infty * A$ and have the form $\psi(x) = 0$. So to find indecomposable elements and relations for E_∞- algebra A we need to calculate the homology of $B(E_\infty, A)$. The answer in the case of Milnor coalgebra gives the next theorem

Theorem 6. *Let K - Milnor coalgebra and $A_* = H_*(FK)$ be considered as E_∞-algebra. Then the homology $H_*B(E_\infty, A)$ in the case $R = \mathbb{Z}/2$ be generated by the elements ξ_i of dimenson $2^i - 1$; in the case $R = \mathbb{Z}/p$ - by the elements ξ_i of dimension $2(p^i - 1)$ and τ_i of dimension $2p^i - 1$.*

After that it is easy to find the image of $i_* : A_* \to H_*B(E_\infty, A)$. It contains only one element h_0 of dimension zero. The homomorfism $\psi : H_*B(E_\infty, A_*) \to E^1_\infty * A_*$ be expressed by the formula $\psi(\xi_i) = \pi_{i-1}(h_i \otimes ... \otimes h_0)$. So we obtain the next theorem

Theorem 7. *The second term of the Adams spectral sequence considered as E_∞-algebra has only one generator element h_0 and relations $\pi_{i-1}(h_i \otimes ... \otimes h_0) = 0$.*

Indeed we don't need all E_∞-algebra structure for describing commutativity relations. For example we don't need $\cup_i$-products, since we know that the elements h_i are generator elements and all others elements may be obtained by applying to them only the operations from A_∞-algebra structure without $\cup_i$-products or any othes operations from E_∞-algebra structure.

So we need such structure, what will give commutativity relations and contains not so many operations as E_∞- algebra structure. To obtain such structure we consider the monads A_∞, E_∞ in a category of graded modules wich correspond to graded module A a free A_∞-algebra $A_\infty A$ and a free E_∞- algebra $E_\infty A$. If A be a A_∞ or E_∞- algebra on the language of monads it means that A be an algebra over correspondent monad.

There is a mapping of monads $A_\infty \to E_\infty$. For any graded module A denote $S_\infty A$ the image of the mapping $A_\infty A \to E_\infty A$. Then a correspondence $A \longmapsto S_\infty A$ generates a monad S_∞ in the category of graded modules. Algebras over this monad will be called S_∞-algebras. It is clear that if A be E_∞-algebra then it will be S_∞-algebra, and we have the next

Therem 8. *The second term of the Adams spectral sequence, considered as S_∞-algebra, generated by the elements $h_i, i \geq 0$ and relations $\pi_i(h_{i+1} \otimes ... \otimes h_0) = 0$.*

Pass now to E_∞-term of the Adams spectral sequence of stable homotopy groups of spheres. To describe it in such manner we need to study a structure on Milnor coalgebra more carefully. There is not only A_∞-coalgebra structure. On Milnor coalgebra there is E_∞-algebra structure and relations similar to Hopf relations between E_∞-algebra structure and A_∞-coalgebra structure. Such a structure we call (E_∞, A_∞)- Hopf algebra structure. This structure on Milnor coalgebra K makes it possible to calculate higher coproducts $\nabla_i : K \to K^{\otimes(i+2)}$ on the products or $\cup_i$- products of the elements $x, y \in K$.

Such calculations were produced for the operation $\nabla_1 : K \to K \otimes K \otimes K$ in [11]. In particular from these calculations follows that $\nabla_1(\xi_1^{2^i}) = \xi_1^{2^{i-1}} \otimes \xi_1^{2^{i-1}} \otimes \xi_1, i \geq 2$ and hence primitive elements in K are only the elements $\xi_1, \xi_1^2, \xi_1^4, \xi_1^8$. The corresponding elements h_0, h_1, h_2, h_3 are indecomposable and therefore we have

Theorem 9. *The module of indecomposable elements in E_∞-term of the Adams spectral sequence considered as A_∞algebra is generated by the elements h_0, h_1, h_2, h_3*

Of course this result is simply a reformulation of Cohen result [12].

And what about the relations. There are two ways of obtaining relations. First - using Hopf relations between E_∞- algebra structure and A_∞- coalgebra structure to calculate the operations ∇_i. The results of [11] show that it is possible but formulas are very complicated. Second - use the next general theorem

Theorem 10. *Let K be a graded (E_∞, A_∞) - Hopf algebra. Then on it's co-B-construction $A = \tilde{F}K$ there is E_∞- algebra structure, and hence on it's homology $A_* = H_*(\tilde{F}K)$ there is also E_∞- algebra structure*

Recall that to find indecomposable elements and relations for graded E_∞- algebra A we need to calculate the homology of $B(E_\infty, A)$. The answer in the case of Milnor coalgebra gives the next theorem

Theorem 11. *Let K - Milnor coalgebra considered as (E_∞, A_∞)- Hopf algebra and $A_* = H_*(\tilde{F}K) - E_\infty$- term of the Adams spectral sequence considered as E_∞-algebra. Then the homology $H_*B(E_\infty, A_*)$ is generated by one element ξ_1.*

After that it is easy to find the indecomposable elements and relations.

Theorem 12. *E_∞- term of the Adams spectral sequence considered as E_∞- algebra generates by one element h_0 and all relations follows from E_∞- algebra structure.*

Of course the last part of my report need to be more precisely. But to do it we must use developed operad methods and there is no time for it. I hope that I will tell about it in the next time.

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