STABILITY OF HODGE BUNDLES AND A NUMERICAL CHARACTERIZATION OF SHIMURA VARIETIES

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Abstract. Let U be a connected quasi-projective manifold and $f : A \to U$ a family of abelian varieties of dimension g. Suppose that the induced map $U \to \mathcal{A}_g$ is generically finite and there is a compactification Y with complement $S = Y \setminus U$ a normal crossing divisor such that $\Omega^1_Y(\log S)$ is nef and $\omega_Y(S)$ is ample with respect to U.

We characterize whether U is a Shimura variety only by numerical data attached to the variation of Hodge structures, rather than by properties of the map $U \to \mathcal{A}_g$ or by the existence of CM points.

More precisely, we show that U is a Shimura variety, if and only if two conditions hold. First, each irreducible local subsystem $\mathbb{V}$ of $R^1 f_* \mathbb{C}_A$ is either unitary or satisfies the Arakelov equality. Second, for each factor M in the universal cover of U whose tangent bundle behaves like the one of a complex ball, an iterated Kodaira-Spencer map associated with $\mathbb{V}$ has minimal possible length in the direction of M.

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Let Y be a complex projective manifold of dimension n, and let U be the complement of a normal crossing divisor S. We are interested in families $f : A \to U$ of abelian varieties, up to isogeny, and we are looking for numerical invariants which take the minimal possible value if and only if U is a Shimura variety of certain type, or to be more precise, if $f : A \to U$ is a Kuga fibre space as recalled in Section 1.1. Those invariants will be attached to $\mathbb{C}$-subvariations of Hodge structures $\mathbb{V}$ of $R^1 f_* \mathbb{C}_A$. We will always assume that the family has semistable reduction in

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codimension one, hence that the local system $R^1 f_* \mathbb{C}_A$ has unipotent monodromy in the general points of the components of S.

In [VZ04] we restricted ourselves to curves Y, and we gave a characterization of Shimura curves in terms of the degree of $\Omega^1_Y(\log S)$ and the degree of the Hodge bundle $f_*\Omega^1_{X/Y}(\log f^{-1}(S))$ for a semistable model $f : X \to Y$. For infinitesimal rigid families this description was an easy consequence of Simpson's correspondence, whereas in the non-rigid case we had to use the classification of certain discrete subgroups of $\mathbb{P}\mathrm{Sl}_2(\mathbb{R})$. In [VZ07] we started to study families over a higher dimensional base U, restricting ourselves to the rigid case. There it became evident that one has to consider numerical invariants of all the irreducible subvariations $\mathbb{V}$ of $R^1 f_* \mathbb{C}_A$, and that for ball quotients one needed some condition on the second Chern classes, or equivalently on the length of the Higgs field of certain wedge products of $\mathbb{V}$. In [VZ07] we have chosen the condition that the discriminant of one of the Hodge bundles is zero. This was needed to obtain the purity of the Higgs bundles (see Definition 0.4) for the special variations of Hodge structures considered there, but it excluded several standard representations.

In this article we give a numerical characterization of an arbitrary Shimura variety or Kuga fibre space, including rigid and non-rigid ones. In order to state and to motivate the results, we need some notations.

Consider a complex polarized variation of Hodge structures $\mathbb{V}$ on U of weight k and with unipotent local monodromy around the components of S. The $\mathcal{F}$-filtration on $\mathcal{H}_0 = \mathbb{V} \otimes_{\mathbb{C}} \mathcal{O}_U$ extends to a filtration of the Deligne extension $\mathcal{H}$ of $\mathcal{H}_0$ to Y, again denoted by $\mathcal{F}$ (see [Sch73]). By Griffiths' Transversality Theorem (see [Gr70], for example) the Gauss-Manin connection $\mathcal{H} \to \mathcal{H} \otimes \Omega^1_Y(\log S)$ induces an $\mathcal{O}_Y$-linear map

$$\mathfrak{g}r_{\mathcal{F}}(\mathcal{H}) = \bigoplus_{p+q=k} E^{p,q} \xrightarrow{\bigoplus \theta_{p,q}} \bigoplus_{p+q=k} E^{p,q} \otimes \Omega^1_Y(\log S) = \mathfrak{g}r_{\mathcal{F}}(\mathcal{H}) \otimes \Omega^1_Y(\log S),$$

with $\theta_{p,q} : E^{p,q} \to E^{p-1,q+1} \otimes \Omega^1_Y(\log S)$. So by [Si92] $\big(E = \mathfrak{g}r_{\mathcal{F}}(\mathcal{H}),\ \theta = \bigoplus \theta_{p,q}\big)$ is the (logarithmic) Higgs bundle induced by $\mathbb{V}$. We will write $\theta^{(m)}$ for the iterated Higgs field

$$(0.1) \quad E^{k,0} \xrightarrow{\theta_{k,0}} E^{k-1,1} \otimes \Omega^1_Y(\log S) \xrightarrow{\theta_{k-1,1}} E^{k-2,2} \otimes S^2(\Omega^1_Y(\log S)) \xrightarrow{\theta_{k-2,2}} \cdots \xrightarrow{\theta_{k-m+1,m-1}} E^{k-m,m} \otimes S^m(\Omega^1_Y(\log S)).$$

For families of abelian varieties we are considering subvariations $\mathbb{V}$ of the complex polarized variation of Hodge structures $R^1 f_* \mathbb{C}_A$, so we will assume that $\mathbb{V}$ has weight one. Then its Higgs field is of the form

$$(E = E^{1,0} \oplus E^{0,1}, \theta) \quad \text{with} \quad \theta : E^{1,0} \to E^{0,1} \otimes \Omega^1_Y(\log S).$$

The most important numerical invariant will be the slope of $\mathbb{V}$ or of the Higgs bundle (E, θ). Recall that the slope $\mu(\mathcal{F})$ of a torsion free coherent sheaf $\mathcal{F}$ on Y, is defined by

$$\Upsilon(\mathcal{F}) = \frac{\mathrm{c}_1(\mathcal{F})}{\mathrm{rk}(\mathcal{F})} \in H^2(Y, \mathbb{Q}) \quad \text{and} \quad \mu(\mathcal{F}) = \Upsilon(\mathcal{F}).\mathrm{c}_1(\omega_Y(S))^{\dim(Y)-1}.$$

We write

$$\mu(\mathbb{V}) := \mu(E^{1,0}) - \mu(E^{0,1}).$$

Variations of Hodge structures of weight $k>1$ will only occur as tensor representations of $\mathbb{W}_{\mathbb{Q}}=R^1f_*\mathbb{Q}_A$ or of irreducible direct factors $\mathbb{V}$ of $R^1f_*\mathbb{C}_A$, in particular in the definition of the second numerical invariant:

Given any (logarithmic) Higgs bundle

$$\big(E=E^{1,0}\oplus E^{0,1},\ \theta:E^{1,0}\to E^{0,1}\otimes\Omega^1_Y(\log S)\big)$$

and some $\ell>0$ one has the induced Higgs bundle

$$\bigwedge^{\ell}(E,\theta)=\big(\bigoplus_{i=0}^{\ell}E^{\ell-i,i},\ \bigoplus_{i=0}^{\ell-1}\theta_{\ell-i,i}\big)\quad\text{with}$$

$$E^{\ell-m,m}=\bigwedge^{\ell-m}(E^{1,0})\otimes\bigwedge^{m}E^{0,1}\quad\text{and with}\tag{0.2}$$

$$\theta_{\ell-m,m}:\bigwedge^{\ell-m}(E^{1,0})\otimes\bigwedge^{m}(E^{0,1})\longrightarrow\bigwedge^{\ell-m-1}(E^{1,0})\otimes\bigwedge^{m+1}(E^{0,1})\otimes\Omega^1_Y(\log S)$$

induced by θ.

If $\ell=\mathrm{rk}(E^{1,0})$, then $E^{\ell,0}=\det(E^{1,0})$. In this case $\langle\det(E^{1,0})\rangle$ denotes the Higgs subbundle of $\bigwedge^{\ell}(E,\theta)$ generated by $\det(E^{1,0})$. Writing as in (0.1)

$$\theta^{(m)}=\theta_{\ell-m+1,m-1}\circ\cdots\circ\theta_{\ell,0},$$

we define as a measure for the complexity of the Higgs field

$$\varsigma((E,\theta)):=\mathrm{Max}\{\ m\in\mathbb{N};\ \theta^{(m)}(\det(E^{1,0}))\neq 0\}=\mathrm{Max}\{\ m\in\mathbb{N};\ \langle\det(E^{1,0})\rangle^{\ell-m,m}\neq 0\}.$$

If (E,θ) is the Higgs bundle of a variation of Hodge structures $\mathbb{V}$ we will usually write $\varsigma(\mathbb{V})=\varsigma((E,\theta))$.

We require some positivity properties of the sheaf of differential forms on the compactification Y of U:

Assumptions 0.1. *Y is a connected projective manifold and U is the complement of a normal crossing divisor S such that:*

- *$\Omega^1_Y(\log S)$ is nef and $\omega_Y(S)$ is ample with respect to U.*

By definition a locally free sheaf $\mathcal{F}$ is *numerically effective (nef)* if for all curves $\tau:C\to Y$ and for all invertible quotients $\mathcal{N}$ of $\tau^*\mathcal{F}$ one has $\deg(\mathcal{N})\geq 0$. An invertible sheaf $\mathcal{L}$ is *ample with respect to* U if for some $\nu\geq 1$ the sections in $H^0(Y,\mathcal{L}^\nu)$ generate the sheaf $\mathcal{L}^\nu$ over U and if the induced morphism $U\to\mathbb{P}(H^0(Y,\mathcal{L}^\nu))$ is an embedding.

If the universal covering $\pi:\tilde{U}\to U$ is a bounded symmetric domain, hence isomorphic to $M_1\times\cdots\times M_s$ for irreducible bounded symmetric domains M_i of dimension n_i, Mumford constructed in [Mu77, Section 4] a non-singular compactification satisfying the Assumption 0.1. We will call it the *Mumford compactification* in the sequel.

In addition, as recalled in Section 2, it is easy to verify for the Mumford compactification the following property:

Condition 0.2.

- $\Omega^1_Y(\log S)$ is μ-polystable. If $\Omega^1_Y(\log S) = \Omega_1 \oplus \cdots \oplus \Omega_{s'}$ is the decomposition as a direct sum of stable direct factors, then $s = s'$ and for a suitable choice of the indices the pullback of $\Omega_i|_U$ to $\tilde{U}$ coincides with $\mathrm{pr}_i^* \Omega^1_{M_i}$.

In particular the Mumford compactification exists for a Shimura variety of Hodge type or for the base of a Kuga fibre space.

Proposition 0.3. *Let $f : A \to U$ be a Kuga fibre space, such that $\mathbb{W} = R^1 f_* \mathbb{C}_A$ has unipotent local monodromies at infinity. Then there exists a compactification Y satisfying the Assumption 0.1 and the Condition 0.2 such that for all irreducible non-unitary $\mathbb{C}$ subvariation of Hodge structures $\mathbb{V}$ of $\mathbb{W}$ with Higgs bundle (E, θ) one has:*

i. *There exists some $i = i(\mathbb{V})$ such that the Higgs field θ factorizes through*
$$\theta : E^{1,0} \longrightarrow E^{0,1} \otimes \Omega_i \xrightarrow{\subset} E^{0,1} \otimes \Omega^1_Y(\log S).$$
ii. *The "Arakelov equality" $\mu(\mathbb{V}) = \mu(\Omega^1_Y(\log S))$ holds.*
iii. *The sheaves $E^{1,0}$ and $E^{0,1}$ are μ-stable.*
iv. *The sheaf $E^{1,0} \otimes E^{0,1\vee}$ is μ-polystable.*
v. *Assume for $i = i(\mathbb{V})$ that M_i is a complex ball of dimension $n_i \geq 1$. Then*
$$\varsigma(\mathbb{V}) = \frac{\mathrm{rk}(E^{1,0}) \cdot \mathrm{rk}(E^{0,1}) \cdot (n_i + 1)}{\mathrm{rk}(E) \cdot n_i}.$$

We will verify the first four of those properties, presumably well known to experts, at the end of Section 2. The fifth one will be shown in Section 6.

The "Arakelov Equality" in ii) will be our main condition. By Lemma 2.5 its validity is independent of the compactification, as long as the Assumptions 0.1 on positivity hold. As we will see in Section 6, assuming the Arakelov equality and assuming that Y is the compactification constructed by Mumford, the properties iii) and iv) are equivalent, and in case that $M_{i(\mathbb{V})}$ is a complex ball they are equivalent to v), as well.

The Proposition 0.3 can serve as a "Leitfaden" for this article. For simplicity let us assume for a moment that $Y = U$ is a compact submanifold of the moduli stack $\mathcal{A}_g$ of polarized abelian varieties of dimension g. Then the Assumptions 0.1 hold. Let us assume moreover, that the universal covering $\tilde{U}$ does not decompose as a product containing a bounded symmetric domain of rank strictly larger than 1 as a factor. Then the Arakelov equality for all irreducible non-unitary $\mathbb{C}$ subvariation of Hodge structures $\mathbb{V} \subset R^1 f_* \mathbb{C}_A$, together with the equality for $\varsigma(\mathbb{V})$ in v), will hold, if and only if $f : A \to Y$ is a Kuga fibre space.

In general we have to assume that U has a compactification Y satisfying the Assumptions 0.1. Those allow to apply Yau's Uniformization Theorem ([Ya93] recalled in [VZ07, Theorem 1.4]). In particular the sheaf $\Omega^1_Y(\log S)$ is μ-polystable and the Condition 0.2 holds true. So one has again a direct sum decomposition

$$\Omega^1_Y(\log S) = \Omega_1 \oplus \cdots \oplus \Omega_s. \qquad (0.3)$$

in stable sheaves of rank $n_i = \mathrm{rk}(\Omega_i)$. We say that Ω_i *is of type A*, if it is invertible, and *of type B*, if $n_i > 1$ and if for all $m > 0$ the sheaf $S^m(\Omega_i)$ is stable. Finally it is *of type C* in the remaining cases, i.e. if for some $m > 1$ the sheaf $S^m(\Omega_i)$ is unstable.

Let again $\pi : \tilde{U} \to U$ denote the universal covering with covering group Γ. The decomposition (0.3) of $\Omega^1_Y(\log S)$ gives rise to a product structure

$$\tilde{U} = M_1 \times \cdots \times M_s, \tag{0.4}$$

where $n_i = \dim(M_i)$. If $\tilde{U}$ is a bounded symmetric domain, the M_i in (0.4) are irreducible bounded symmetric domains, and on a Mumford compactification the decomposition (0.3) coincides with the one in Property 0.2.

Yau's Uniformization Theorem gives a criterion for the M_i to be bounded symmetric domains. In fact, if Ω_i is of type A, M_i is a one-dimensional complex ball, and it is a bounded symmetric domain of rank > 1, if Ω_i is of type C.

If Ω_i is of type B, then M_i is a n_i-dimensional complex ball if and only if

$$\big[2 \cdot (n_i + 1) \cdot \mathrm{c}_2(\Omega_i) - n_i \cdot \mathrm{c}_1(\Omega_i)^2\big].\mathrm{c}(\omega_Y(S))^{\dim(Y)-2} = 0. \tag{0.5}$$

Fix an irreducible polarized $\mathbb{C}$-variation of Hodge structures $\mathbb{V}$ on U of weight one and with Higgs bundle (E, θ). By [VZ07, Theorem 1] one has the Arakelov type inequality

$$\mu(\mathbb{V}) = \mu(E^{1,0}) - \mu(E^{0,1}) \leq \mu(\Omega^1_Y(\log S)). \tag{0.6}$$

The Arakelov equality

$$\mu(\mathbb{V}) = \mu(\Omega^1_Y(\log S)) \tag{0.7}$$

implies that $E^{1,0}$ and $E^{0,1}$ are both μ-semistable.

Before being able to give the numerical characterization of Kuga fibre spaces, hence a converse of Proposition 0.3, ii), and v), we will have to state some result on the splitting of variations of Hodge structures, similar to Proposition 0.3, i).

Definition 0.4.

1. A subsheaf $\mathcal{F} \subset E^{1,0}$ is *pure of type* i if the composition
$$\mathcal{F} \xrightarrow{\subset} E^{1,0} \xrightarrow{\theta} E^{0,1} \otimes \Omega^1_Y(\log S) \xrightarrow{\mathrm{pr}} E^{0,1} \otimes \Omega_j$$
is zero for $j \neq i$ and non-zero for $j = i$.
2. A Higgs Bundle $(E^{1,0} \oplus E^{0,1}, \theta)$ (or the corresponding variation of Hodge structures $\mathbb{V}$) is *pure of type* i, if $E^{1,0}$ is pure of type i.
3. If (E, θ) or $\mathbb{V}$ is of type i and if Ω_i is of type A, B, or C, we sometimes just say that (E, θ) or $\mathbb{V}$ *is pure of type A, B, or C.*

Obviously $(E^{1,0} \oplus E^{0,1}, \theta)$ is pure of type i if the Higgs field θ is non zero and if it factors like

$$E^{1,0} \longrightarrow E^{0,1} \otimes \Omega_i \xrightarrow{\subset} E^{0,1} \otimes \Omega^1_Y(\log S).$$

The Property 0.3, i), states that for Kuga fibre spaces uniformizing variations of Hodge structures decompose as a direct sum of pure and of unitary subvariations.

If in the decomposition (0.3) all the stable direct factors Ω_i are of type C, hence if $\tilde{U}$ is the product of bounded symmetric domains $M_i = G_i/K_i$ of rank > 1, the Margulis Superrigidity Theorem and a simple induction argument (see the proof of 5.8) imply that each representation of Γ is the tensor product of a unitary representation and a second one, ρ, coming from a representation of the group $G = G_1 \times \cdots \times G_s$.

By Schur's lemma the irreducible representation ρ of Γ is the tensor product of representations ρ_j of the G_j. Correspondingly $\mathbb{V}$ is the tensor product of a unitary bundle and of a polarized $\mathbb{C}$ variations of Hodge structures $\mathbb{V}_j$ induced by ρ_j. Since the weight of $\mathbb{V}$ is one, all the $\mathbb{V}_j$, except one, have to be variations of Hodge structures of weight zero, hence they are unitary and the induced Higgs field is zero. So an irreducible $\mathbb{C}$ variation of Hodge structures $\mathbb{V}$ of weight one is pure of type $i = i(\mathbb{V})$. As we will see in Proposition 5.8 one can extend this result to all U with $\tilde{U}$ a bounded symmetric domain.

The next Theorem extends this property in another way, replacing the condition that $\tilde{U}$ a bounded symmetric domain by the Arakelov equality.

Theorem 0.5. *Under the Assumptions 0.1 consider an irreducible polarized $\mathbb{C}$-variation of Hodge structures $\mathbb{V}$ of weight one with unipotent monodromy at infinity. If $\mathbb{V}$ satisfies the Arakelov equality (0.7) then $\mathbb{V}$ is pure for some $i = i(\mathbb{V})$.*

The proof of Theorem 0.5 will cover most of the Sections 3, 4 and 5. In particular we will have to consider small twists of the slopes $\mu(\mathcal{F})$.

Applying Simpson's Correspondence [Si92] to the Higgs subbundle $\langle \det(E^{1,0}) \rangle$ of $\bigwedge^{\mathrm{rk}(E^{1,0})}(E, \theta)$ we will obtain in Lemma 6.1:

Corollary 0.6. *Assume in the situation of Theorem 0.5 that $\mathbb{V}$ is non-unitary and that for $i = i(\mathbb{V})$ the sheaf Ω_i is of type A or B. Then*

$$\varsigma(\mathbb{V}) \geq \frac{\mathrm{rk}(E^{1,0}) \cdot \mathrm{rk}(E^{0,1}) \cdot (n_i + 1)}{\mathrm{rk}(E) \cdot n_i}. \tag{0.8}$$

In fact, if Ω_i is invertible, hence of type A, we will see in Lemma 6.2 that both, the left hand side and the right hand side of (0.8) are equal to $\mathrm{rk}(E^{1,0})$.

If one does not want to use the invariant $i(\mathbb{V})$ at this stage, one can consider the Higgs bundles (E, θ_j) with the pure Higgs field θ_j, given by the composite

$$\theta_j : E^{1,0} \xrightarrow{\theta} E^{0,1} \otimes \Omega^1_Y(\log S) \xrightarrow{\mathrm{pr}_j} E^{0,1} \otimes \Omega_j \xrightarrow{\subset} E^{0,1} \otimes \Omega^1_Y(\log S).$$

If (E, θ) is pure of type i, then θ_j is zero for $j \neq i$ and $\theta_i = \theta$. So Theorem 0.5 allows to restate the Corollary 0.6 as follows.

Variant 0.7. *Under the Assumptions 0.1 consider an irreducible polarized $\mathbb{C}$ variation of Hodge structures $\mathbb{V}$ of weight one with unipotent monodromy at infinity. If $\mathbb{V}$ satisfies the Arakelov equality (0.7) then for each stable direct factor Ω_j of $\Omega^1_Y(\log S)$ of type B (or A) either $\theta_j = 0$ or*

$$\varsigma((E, \theta_j)) \geq \frac{\mathrm{rk}(E^{1,0}) \cdot \mathrm{rk}(E^{0,1}) \cdot (n_j + 1)}{\mathrm{rk}(E) \cdot n_j}.$$

Finally we will obtain the numerical characterization of Kuga fibre spaces in the following form.

Theorem 0.8. *Let $f : A \to U$ be a smooth family of abelian varieties, such that the induced morphism $U \to \mathcal{A}_g$ is generically finite. Assume that U has a projective compactification Y satisfying the Assumptions 0.1.*

Then $f : A \to U$ is a Kuga fibre space if and only if for each irreducible subvariation of Hodge structures $\mathbb{V}$ of $R^1 f_* \mathbb{C}_A$ with Higgs bundle (E, θ) one has:

1. *If $\mathbb{V}$ is non-unitary, the Arakelov equality $\mu(\mathbb{V}) = \mu(\Omega^1_Y(\log S))$ holds.*

2. *For each stable direct factor* Ω_j *of* $\Omega^1_Y(\log S)$ *of type B either the composition*

$$\theta_j : E^{1,0} \xrightarrow{\theta} E^{0,1} \otimes \Omega^1_Y(\log S) \xrightarrow{\mathrm{pr}} E^{0,1} \otimes \Omega_j$$

is zero, or

$$\varsigma((E,\theta_j)) = \frac{\mathrm{rk}(E^{1,0}) \cdot \mathrm{rk}(E^{0,1}) \cdot (n_j+1)}{\mathrm{rk}(E) \cdot n_j}.$$

If in addition $f : A \to U$ *is infinitesimally rigid* U *is a Shimura variety of Hodge type.*

Remark that the condition 2) in Theorem 0.8 automatically holds true if $\mathbb{V}$ is unitary, or if it is pure of type A or C. Moreover the condition 1) allows to apply Theorem 0.5. So we know that each non-unitary irreducible subvariation of Hodge structures $\mathbb{V}$ of $R^1 f_* \mathbb{C}_A$ is pure. Since θ_j is zero for $j \neq i(\mathbb{V})$ and equal to θ for $j = i(\mathbb{V})$, we can replace the condition 2) by

2'. *If* $\mathbb{V}$ *is non unitary and pure of type* $i = i(\mathbb{V})$ *with* Ω_i *of type B, then*

$$\varsigma(\mathbb{V}) = \frac{\mathrm{rk}(E^{1,0}) \cdot \mathrm{rk}(E^{0,1}) \cdot (n_i+1)}{\mathrm{rk}(E) \cdot n_i}.$$

Obviously $\varsigma(\mathbb{V})$ is determined by the Higgs bundle on any open dense subset. By Lemma 2.5 the slopes $\mu(\mathbb{V})$ and $\mu(\Omega^1_Y(\log S))$ are independent of the compactification Y, as long as the Assumption 0.1 holds. So both conditions, 1) and 2), in Theorem 0.8 are independent of the compactification.

Using the Proposition 0.3 one obtains:

Corollary 0.9. *Let* $f : A \to U$ *be a smooth family of abelian varieties, such that the induced morphism* $U \to \mathcal{A}_g$ *is generically finite. Assume that* U *has one projective compactification, satisfying the Assumptions 0.1, such that on this compactification the conditions 1) and 2) hold for the Higgs bundles of all irreducible subvariation of Hodge structures* $\mathbb{V}$ *of* $R^1 f_* \mathbb{C}_A$. *Then there exists a compactification* Y *of* U *satisfying the assumptions made in 0.1, with*

3. $E^{1,0}$ *and* $E^{0,1}$ *are* μ*-stable.*
4. $E^{1,0} \otimes E^{0,1\vee}$ *is* μ*-polystable.*

The proof of Theorem 0.8 will be given in Section 6. As indicated by the formulation of the condition 2) or 2') the arguments will depend on the type of $\Omega_{i(\mathbb{V})}$. The subvariations of Hodge structures which are pure of type B will play a special role. In Section 6 we will obtain a slightly more precise information.

Addendum 0.10. *Consider the following conditions for the Higgs bundle* (E,θ) *of an irreducible complex polarized variation of Hodge structures* $\mathbb{V}$ *of weight one and pure of type* $i = i(\mathbb{V})$*:*

α. $E^{1,0}$ *and* $E^{0,1}$ *are* μ*-stable.*

β. *The kernel of the natural map* $\mathcal{H}om(E^{0,1}, E^{1,0}) \to \Omega_i$ *is a direct factor of* $\mathcal{H}om(E^{0,1}, E^{1,0})$.

γ.
$$\varsigma(\mathbb{V}) = \frac{\mathrm{rk}(E^{1,0}) \cdot \mathrm{rk}(E^{0,1}) \cdot (n_i+1)}{\mathrm{rk}(E) \cdot n_i}.$$

δ. *M_i is the complex ball $\mathrm{SU}(1,n_i)/K$, and $\mathbb{V}$ is the tensor product of a unitary representation with a wedge product of the standard representation of $\mathrm{SU}(1,n_i)$.*

η. *Let M' denote the period domain for $\mathbb{V}$. Then the period map factors as the projection $\tilde{U} \to M_i$ and a totally geodesic embedding $M_i \to M'$.*

Then, the Arakelov equality for $\mathbb{V}$ implies:

I. *If Ω_i is of type A, then α), β), γ), δ) and η) hold true.*
II. *If Ω_i is of type C, then η) holds true.*
III. *If Ω_i is of type B, then the conditions β), γ), δ) and η) are equivalent. Moreover η) implies that α) holds on a Mumford compactification of U.*
IV. *If U is projective, or if $\dim(U) = 1$ then α) implies β)*

In IV the additional assumption "U projective, or $\dim(U) = 1$" is needed since it forces $\omega_Y(S)$ to be ample. This in turn implies the following property of μ-stable sheaves:

Property 0.11.

- If $\mathcal{F}$ and $\mathcal{G}$ are two μ-stable locally free sheaves, then $\mathcal{F} \otimes \mathcal{G}$ is μ-polystable.

S.T. Yau informed us, that Property 0.11 holds true for $\omega_Y(S)$ nef and big, and that a proof will be in a forthcoming article by Sun and Yau. So hopefully the equivalence of all the conditions in Addendum 0.10, III, will follow from the Assumptions 0.1 and from the Arakelov equality. In Addendum 0.10, δ), the Higgs field of the standard representation (or of its dual) is given by

$$E^{1,0} = \omega_i^{-\frac{1}{n_i+1}} \otimes \Omega_i, \quad E^{0,1} = \omega_i^{-\frac{1}{n_i+1}} \quad \text{and} \quad \theta = \mathrm{id} : \omega_i^{-\frac{1}{n_i+1}} \otimes \Omega_i \longrightarrow \omega_i^{-\frac{1}{n_i+1}} \otimes \Omega_i,$$

where $\omega_1^{-\frac{1}{n_i+1}}$ stands for an invertible sheaf, whose (n_i+1)-st power is $\det(\Omega_i)$.

In [VZ07] we had to exclude direct factors of $\Omega^1_Y(\log S)$ of type C, and we used a different numerical condition for $\mathbb{V}$ of type B.

Recall that the discriminant of a torsion free coherent sheaf $\mathcal{F}$ on Y is given by

$$\delta(\mathcal{F}) = \big[2 \cdot \mathrm{rk}(\mathcal{F}) \cdot \mathrm{c}_2(\mathcal{F}) - (\mathrm{rk}(\mathcal{F}) - 1) \cdot \mathrm{c}_1(\mathcal{F})^2\big].\mathrm{c}_1(\omega_Y(S))^{\dim(Y)-2},$$

and that the μ-semistability of $E^{1-q,q}$ implies that $\delta(E^{1-q,q}) \geq 0$. So the Arakelov equality implies that

$$\delta(\mathbb{V}) = \mathrm{Min}\{\delta(E^{1,0}), \delta(E^{0,1})\} \geq 0.$$

In [VZ07], using the Property 0.11, we gave two criteria, forcing $f : A \to U$ to be a Kuga fiber space. The first one, saying that all the direct factors of $\Omega^1_Y(\log S)$ are of type A, is now a special case of Theorem 0.8. In the second criterion we allowed the direct factors of $\Omega^1_Y(\log S)$ to be of type A and B, but excluded factors of type C. There, for all irreducible subvariations $\mathbb{V}$ of Hodge structures we required $\delta(\mathbb{V}) = 0$. This additional condition, needed in [VZ07] to prove the purity of irreducible subvariations of Hodge structures, forced at the same time the representations in Addendum 0.10, δ) to be the tensor product of the standard representations of $\mathrm{SU}(1,n_i)$ with a unitary local system. So the condition $\delta(\mathbb{V}) = 0$ did not allow wedge products of those representations.

The bridge between the criterion [VZ07] and Theorem 0.8 is already contained in [VZ07, Proposition 3.4]:

Proposition 0.12. *Let $f : A \to U$ be a smooth family of abelian varieties, such that the induced morphism $U \to \mathcal{A}_g$ is generically finite. Assume that U has a projective compactification Y satisfying the Assumptions 0.1 and assume that Property 0.11 holds.*

Let $\mathbb{V}$ be an irreducible subvariation of Hodge structures of $R^1 f_* \mathbb{C}_A$ with Higgs bundle (E, θ), pure of type $i = i(\mathbb{V})$ and with Ω_i of type B. If $\mathbb{V}$ satisfies the Arakelov equality and if $\delta(\mathbb{V}) = 0$, then either

$$\mathrm{rk}(E^{1,0}) = \mathrm{rk}(E^{0,1}) \cdot n_i \quad \textit{or} \quad \mathrm{rk}(E^{1,0}) \cdot n_i = \mathrm{rk}(E^{0,1}).$$

In particular the condition γ) in Addendum 0.10 holds.

In fact, assuming that $\mathrm{rk}(E^{1,0}) \leq \mathrm{rk}(E^{0,1})$, Lemma 0.6 implies that $\varsigma(\mathbb{V}) \geq \mathrm{rk}(E^{1,0})$, and by definition one knows that $\varsigma(\mathbb{V}) \leq \mathrm{rk}(E^{1,0})$.

We do not know whether the condition 2) in Theorem 0.8 is really needed, or whether in Addendum 0.10, for $\mathbb{V}$ of type B, the condition γ) follows from the Arakelov equality. The necessity of the equality (0.5) in the characterization of ball quotients might indicate that a condition on the first Chern class, as it given by the Arakelov equality, can not be sufficient to characterize complex balls. As we will show in Section 8 however, the ampleness of $\omega_Y(S)$ or, more generally, the Property 0.11 allows to obtain the condition 2) as a consequence of the Arakelov equality, for $\mathrm{rk}(\mathbb{V}) \leq 7$.

Up to now we did not mention any condition guaranteeing the existence of fibres with complex multiplication or the equality between the monodromy group and the derived Mumford-Tate group $\mathrm{MT}(f)^{\mathrm{der}}$, usually needed in the construction of Kuga fibre spaces or of Shimura varieties of Hodge type. In fact, as in [Mo98], we will rather concentrate on the condition that $U \to \mathcal{A}_g$ is totally geodesic. This will allow in the proof of Theorem 0.8 to identify $f : A \to U$ with a Kuga fibre space $\mathcal{X}(G, \tau, \varphi_0)$. For rigid families we will refer to [Abd94] and [Mo98] for the proof that they are of Hodge type (see Section 1 for more details).

This implies that for a rigid family $f : A \to U$ the group $\mathrm{MT}(f)^{\mathrm{der}}$ is the smallest the $\mathbb{Q}$-algebraic subgroup containing the monodromy group and that U is up to étale coverings equal to $\mathcal{X}(\mathrm{MT}(f)^{\mathrm{der}}, \mathrm{id}, \varphi_0)$. In [VZ07] we used for the last step an explicit identification of possible Hodge cycles. Although not really needed, we will sketch a similar calculation in Section 7. There it will be sufficient to assume that the irreducible direct factors of $R^1 f_* \mathbb{C}_A$ satisfy the Arakelov equality, and we will explicitly identify the monodromy group $\mathrm{Mon}^0(f)$ and the Mumford-Tate group $\mathrm{MT}(f)^{\mathrm{der}}$. We obtain morphisms

$$U \xrightarrow{\delta_1} \mathcal{X}(\mathrm{Mon}^0(f), \mathrm{id}, \varphi_0) \xrightarrow{\delta_2} \mathcal{X}(\mathrm{MT}(f)^{\mathrm{der}}, \mathrm{id}, \varphi_0).$$

Again we will see that the rigidity implies that $\mathrm{Mon}^0(f) = \mathrm{MT}(f)^{\mathrm{der}}$, hence that δ_2 is an isomorphism (or étale). As in Theorem 0.8 one needs a second condition to guarantee that δ_1 is generically finite, for example the one that the dimensions of U and $\mathcal{X}(\mathrm{Mon}^0(f), \mathrm{id}, \varphi_0)$ coincide. Then it is easy to show that δ_1 is étale.

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1. Kuga fibre spaces and Shimura varieties

1.1. **Kuga fibre spaces and totally geodesic subvarieties.** A *Kuga fibre space* (see [Mu69] and the references therein) consists of

i. a rational vector space V of dimension $2g$ with a lattice L,
ii. a symmetric bilinear form $Q : V \times V \to \mathbb{Q}$, integral on $L \times L$,
iii. a $\mathbb{Q}$-algebraic group G and an injective map $\tau : G \to \mathrm{Sp}(V, Q)$,
iv. an arithmetic subgroup $\Gamma \subset G$ such that $\tau(\Gamma)$ preserves L,
v. a complex structure
$$\varphi_0 : S^1 = \{z \in \mathbb{C}^* ; |z| = 1\} \to \mathrm{Sp}(V, Q)$$
such that $\tau(G)$ is normalized by $\varphi_0(S^1)$.

For Γ sufficiently small, a Kuga fibre space $(L, Q, G, \tau, \varphi_0, \Gamma)$ defines a family of abelian varieties by the following procedure. Let $K^0_{\mathbb{R}}$ be the connected component of the centralizer of $\varphi(S^1)$ in $G_{\mathbb{R}}$. Then there is a map
$$M := G^0_{\mathbb{R}}/K^0_{\mathbb{R}} \longrightarrow \mathrm{Sp}(V, Q)_{\mathbb{R}}/(\text{centralizer of}\, \varphi_0) \cong \mathbb{H}_g$$
and the pullback of the universal family over $\mathbb{H}_g$ descends to the the desired family over
$$\mathcal{X} := \mathcal{X}(G, \tau, \varphi_0) := \Gamma\backslash G^0_{\mathbb{R}}/K^0_{\mathbb{R}}.$$
In the sequel we will usually suppress V and Q from the notation and write just $\mathrm{Sp}(Q)$ or Sp, if no ambiguity arises.

We say that two Kuga fibre spaces $(L, Q, G, \tau, \varphi_0, \Gamma)$ and $(L', Q', G', \tau', \varphi'_0, \Gamma')$ are isomorphic, if the families of abelian varieties over $\mathcal{X}(G, \tau, \varphi_0)$ and over $\mathcal{X}(G', \tau', \varphi'_0)$ are isomorphic. Note that different groups G and G' might lead to the same Kuga fibre space and that $K^0_{\mathbb{R}}$ is not necessarily compact but the extension of a central torus in $G_{\mathbb{R}}$ by a compact group. Note moreover that replacing φ_0 by $\tau(g)\varphi_0\tau(g)^{-1}$ for any $g \in G$ gives an isomorphic Kuga fibre space - this just changes the reference point.

Kuga fibre spaces are the objects that naturally arise when studying variations of Hodge structures satisfying the Arakelov equality. We restrict the translation procedure into the language of Shimura varieties to the case of 'Hodge type', see Section 1.3.

We provide symmetric domains throughout with the Bergman metric (e.g. [Sa80, §II.6]). By condition v) in Mumford's definition of a Kuga fibre space, $M \to \mathbb{H}_g$ is a strongly equivariant map in the sense of [Sa80]. By [Sa80, Theorem II.2.4], it is a *totally geodesic embedding*, i.e. each geodesic curve in $\mathbb{H}_g$ which is tangent to M at some point of M is a curve in M. The converse will be dealt with in Section 1.3.

1.2. **The Hodge group, the Mumford-Tate group and the monodromy group.** We start be recalling the definitions of the Hodge- and Mumford-Tate group. The *Hodge group* $\mathrm{Hg}(A_0)$ of an abelian variety A_0 with polarization Q is defined in [Mu66] as the smallest $\mathbb{Q}$-algebraic subgroup of $\mathrm{Sp}(H^1(A_0, \mathbb{Q}), Q)$, whose extension to $\mathbb{R}$ contains the complex structure
$$\varphi_0 : S^1 \longrightarrow \mathrm{Sp}(H^1(A_0, \mathbb{R}), Q).$$
(see also [Mu69]), where z acts on (p, q) cycles by multiplication with $z^p \cdot \bar{z}^q$.

As in [An92], for a totally real subfield $K \subset \mathbb{R}$ one can extend this definition. This will be used in Section 7 only.

As usual, let $f : A \to U$ be a family of polarized abelian varieties. Let $\mathbb{W}_K$ be a K-subvariation of Hodge structures of $R^1 f_* K_A$, and let W_K be the corresponding subspace of $H^1(A_0, K)$, where A_0 is a fibre of A. There is a decomposition $H^1(A_0, K) = W_K \oplus W'_K$, orthogonal with respect to Q. We still write Q for the restriction of the polarization. Since K is real, the image of the complex structure

$$\varphi_0 : S^1 \longrightarrow \mathrm{Sp}(H^1(A_0, \mathbb{R}), Q)$$

lies in

$$\mathrm{Sp}(W_K \otimes_K \mathbb{R}, Q) \times \mathrm{Sp}(W'_K \otimes_K \mathbb{R}, Q),$$

and the complex structure on W_K is obtained by the projection to the first factor. We denote this composition by $\varphi_0^{W_k}$. Define the *Hodge group of* W_K to be the smallest K-algebraic subgroup of $\mathrm{Sp}(W_K, Q)$ such that $\mathrm{Sp}(W_K, Q) \otimes_K \mathbb{R}$ contains the image of the complex structure $\varphi_0^{W_k}$.

In a similar way, one defines the *Mumford-Tate group* $\mathrm{MT}(W_K)$. The complex structure$\varphi_0^{W_k}$ extends to a morphism of real algebraic groups

$$h^{W_K} : \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \longrightarrow \mathrm{Gl}(W_K \otimes_K \mathbb{R}),$$

and $\mathrm{MT}(W_K)$ is the smallest K-algebraic subgroup of $\mathrm{Gl}(W_K)$, whose extension to $\mathbb{R}$ contains the image of h^{W_K} (see [De82], [An92], and [Mo98]).

By [An92], and [De82] the group $\mathrm{MT}(W_K)$ is reductive, and it can also be defined as the largest K-algebraic subgroup of the linear group $\mathrm{Gl}(W_K)$, which leaves all K-Hodge tensors invariant, hence all elements

$$\eta \in \left[W_K^{\otimes m} \otimes W_K^{\vee \otimes m'} \right]^{0,0}.$$

Here $W_K^\vee$ is regarded as a Hodge structures concentrated in the bidegrees $(0, -1)$ and $(-1, 0)$, and hence $W_K^{\otimes m} \otimes W_K^{\vee \otimes m'}$ is of weight $m - m'$. So the existence of some η forces m and m' to be equal.

For the smooth family of abelian varieties $f : A \to U$ there exist a union Σ of countably many proper closed subvarieties of Y such that for $A_0 = f^{-1}(y_0)$ the group $\mathrm{MT}(H^0(A_0, \mathbb{Q}))$ is independent of y_0 for $y_0 \in U \setminus \Sigma$ (see [De82], [Mo98] or [Sc96]). The same holds true for the sub variation $\mathbb{W}_K$ of Hodge structures of $R^1 f_* K_A$. So we may assume that $\mathrm{MT}(\mathbb{W}_K|_y)$ is independent of y on the complement of Σ. Let us fix such a very general point $y \in U \setminus \Sigma$ in the sequel, and write $A_0 = f^{-1}(y)$. If a local system is denoted by a boldface letter, the corresponding non-boldface letter will stand for the fibre at y, for example $W_k = \mathbb{W}_K|_y$. The Mumford-Tate group $\mathrm{MT}(\mathbb{W}_\mathbb{Q})$ of $\mathbb{W}_\mathbb{Q} = R^1 f_* \mathbb{Q}_A$ is defined as the Mumford-Tate group $\mathrm{MT}(H^0(A_0, \mathbb{Q}))$ and the Mumford-Tate group $\mathrm{MT}(\mathbb{W}_K)$ is $\mathrm{MT}(W_K)$.

Note that the derived subgroup of Mumford's Hodge group coincides with the derived Mumford-Tate group $\mathrm{MT}(W_\mathbb{Q})^{\mathrm{der}}$.

1.3. **Shimura varieties of Hodge type and totally geodesic subvarieties.** A Kuga fibre space $\mathcal{X}(G, \tau, \varphi_0)$ is *of Hodge type*, if it is isomorphic to a Kuga fibre space $\mathcal{X}(G', \tau', \varphi'_0)$ such that G' is the Hodge group of the abelian variety defined by φ'_0. Let us next compare this notion with the one of a Shimura varieties of Hodge type.

In [De79], the notion of a *connected Shimura datum* (G, M) consists of a reductive $\mathbb{Q}$-algebraic group G and a $G(\mathbb{R})^+$-conjugacy class of homomorphisms $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to G_\mathbb{R}$ with the following properties:

(SV1) for $h \in M$, only the characters $z/\overline{z}$, 1, $\overline{z}/z$ occur in the representation of $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ on $\mathrm{Lie}(G)$.
(SV2) $\mathrm{ad}(h(i))$ is a Cartan involution of G.
(SV3) G^{ad} has no $\mathbb{Q}$-factor on which the projection of h is trivial.

A *Shimura variety* is defined to be the pro-system $(\Gamma\backslash M)_\Gamma$, running over all arithmetic subgroups Γ of $G(\mathbb{Q})$ whose image in G^{ad} is Zariski-dense. Since we do not bother about canonical models etc. and since we allow to replace the base variety U by an étale cover any time, we say that U *is* a Shimura variety of Hodge type, if U is a connected component of $\Gamma\backslash M$ for some Γ. Usually Γ is required moreover to be a congruence subgroup, but we drop this condition to simplify matters of passing to étale covers at some places.

We let $\mathrm{CSp}(Q)$ (or CSp for short) be the group of symplectic similitudes with respect to a symplectic form Q. The Shimura datum $(\mathrm{CSp}(Q), M(Q))$ attached to the symplectic space consists of all maps $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to \mathrm{CSp}(Q)$ defined on $\mathbb{C}$-points by

$$h(x+iy) = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \cdot \begin{pmatrix} a_i \\ b_i \end{pmatrix} \qquad (1.1)$$

for a symplectic basis $\{a_i, b_i\}$, $i = 1, \ldots, g$ of the underlying vector space V.

A Shimura datum (G, M) is *of Hodge type*, if there is a map $\tau : G \to \mathrm{CSp}(Q)$ such that composition with τ maps M to $M(Q)$.

There is a bijection between isomorphism classes of Kuga fibre spaces of Hodge type and Shimura varieties of Hodge type:

Given $(L, Q, G, \tau, \varphi_0, \Gamma)$, let $Z \cong \mathbb{G}_m$ be the center of CSp, define $G' := G \cdot Z \subset \mathrm{CSp}$ and define $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G'_{\mathbb{R}}$ by on $\mathbb{C}$-points by $h(z) = \varphi_0(z/\overline{z})|z|$. Finally, let M' be the $G'_{\mathbb{R}}$ conjugacy class of h. One checks that (G', M') is a Shimura datum of Hodge type. Conversely given (G', M') of Hodge type, let $G := G' \cap \mathrm{Sp}$ and let φ_0 be the restriction of a generic $h \in M'$ to $S^1 \subset \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m(\mathbb{C})$. Together with τ being the inclusion map, this defines a Kuga fibre space of Hodge type.

We start with a subvariety U of the moduli stack $\mathcal{A}_g$, not necessarily a Shimura variety or satisfying numerical conditions on the variation of Hodge structure. We follow Moonen ([Mo98]) and recall the construction of the smallest Shimura subvariety $\mathcal{X}^{\mathrm{MT}}$ of Hodge type in $\mathcal{A}_g$ that contains U.

Theorem 1.1 ([Mo98])**.** *There exists a Shimura datum (G, M) such that a Shimura variety $\mathcal{X}^{\mathrm{MT}} \cong \Gamma\backslash M$ attached to this Shimura datum is the unique smallest Shimura subvariety of Hodge type in $\mathcal{A}_g$ that contains U.*

G may be chosen to be the Mumford-Tate group at a very general point y of U.

Although the Shimura variety $\mathcal{X}^{\mathrm{MT}}$ is unique, the Shimura datum is unique only up to the centralizer of G in CSp, see [Mo98, Remark 2.9].

Proof. Let G be the Mumford-Tate group at a very general point y of U. In the topological space of all maps $h \in M(Q)$ that factor though $G_{\mathbb{R}}$, chose M to be the connected component containing the complex structure at y. By definition of the Mumford-Tate group M is not empty and by the argument of [De79, Lemma 1.2.4], M is an $G(\mathbb{R})$-conjugacy class. Hence (G, M) is a Shimura datum of Hodge type. Since y was very general, $U \to \mathcal{A}_g$ factors through $\mathcal{X}^{\mathrm{MT}}$. The minimality of $\mathcal{X}^{\mathrm{MT}}$

follows from the minimality condition in the definition of the Mumford-Tate group. □

We now suppose that U is totally geodesic in a $\mathcal{A}_g$, hence totally geodesic in the Shimura variety $\mathcal{X}^{\mathrm{MT}}$.

Theorem 1.2 ([Mo98] Corollary 4.4)**.** *If $U \subset \mathcal{X}^{\mathrm{MT}}$ is totally geodesic, then U is a Kuga fibre space. It is a Shimura variety of Hodge type up to some translation in the following sense:*

After replacing U by a finite étale cover, there are Kuga fibre spaces $\mathcal{X}_1$ and $\mathcal{X}_2$ and an isomorphism $\mathcal{X}_1 \times \mathcal{X}_2 \to S^{\mathrm{MT}}$, such that U is the image of $\mathcal{X}_1 \times \{b\}$ for some point $b \in \mathcal{X}_2(\mathbb{C})$.

For some $a \in \mathcal{X}_2(\mathbb{C})$, the subvariety $\mathcal{X}_1 \times \{a\}$ in $\mathcal{X}^{\mathrm{MT}}$ is a Shimura variety of Hodge type.

Proof. In loc. cit. the author deals with Shimura subvarieties of arbitrary period domains and shows that there totally geodesic subvarieties $\mathcal{X}_i$ such that U is the image of $\mathcal{X}_1 \times \{b\}$.

We repeat part of his arguments to justify that $\mathcal{X}_1$ is a Kuga fibre space.

More precisely, let (G, M) be the Shimura datum underlying $\mathcal{X}^{\mathrm{MT}}$. By [An92], Mon^0 is a normal subgroup of $\mathrm{MT}(R^1 f_* \mathbb{Q}_A)^{\mathrm{der}}$, where $f : A \to U$ is the universal family of abelian varieties. We have a decomposition of the adjoint Shimura datum

$$(G^{\mathrm{ad}}, M) \cong ((\mathrm{Mon}^0)^{\mathrm{ad}}, M_1) \times (G_2^{\mathrm{ad}}, M_2)$$

into connected Shimura data given as follows. Since G is reductive, there is a complement G_2 of Mon^0, i.e. such that $\mathrm{Mon}^0 \times G_2 \to G$ is surjective with finite kernel. Write $G_1 := \mathrm{Mon}^0$ and let M_i be the set of maps

$$\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \longrightarrow G \longrightarrow G^{\mathrm{ad}} \longrightarrow (G_i)^{\mathrm{ad}}.$$

For suitable arithmetic subgroups Γ_i a components of the quotients $\mathcal{X}_i := \Gamma_i \backslash M_i$ have the claimed property by [Mo98] Corollary 4.4.

It suffices to take $\tau : \mathrm{Mon}^0 \to \mathrm{Sp}$ the natural inclusion and φ_0 the restriction of any $h \in M$ to $S^1 \subset \mathbb{C}^*$. Then φ_0 normalizes Mon^0 and for a suitable choice of Γ, U is the Kuga fibre space given by $(L, Q, \mathrm{Mon}^0, \tau, \varphi_0, \Gamma)$. □

Corollary 1.3 (See also [Abd94])**.** *If U is totally geodesic and rigid, then U is a Shimura variety of Hodge type.*

2. Stability for homogeneous bundles and the Arakelov equality for Shimura varieties

To prove a first part of the properties of Kuga fibre space stated in Proposition 0.3 we recall from [Mu77] and [Mk02] some facts on homogeneous vector bundles on Hermitian symmetric domains and deduce stability results.

Let M be a Hermitian symmetric domain and let $G = \mathrm{Aut}(M)$ be the holomorphic isometries of M. $\mathrm{Aut}(M)$ is the connected component of one in the isometry group of M and $M \cong G/K$ for a maximal compact subgroup $K \subset G$. Let V_0 be a vector space with a representation $\rho : K \to \mathrm{Gl}(V_0)$ and a ρ-invariant metric s_0. Then

$$V = G \times_K V_0 := G \times V_0 / \sim, \quad \text{where} \quad (g, v) \sim (gk, \rho(k^{-1})v) \quad \text{for} \quad k \in K$$

with the metric h inherited from h_0 is a vector bundle on G/K, homogeneous under the action of G.

In this section we suppose that the universal covering of U is a symmetric domain M. We call a bundle E_U on U *homogeneous*, if its pullback to M is homogeneous. We call a homogeneous bundle *irreducible*, if the representation ρ is irreducible.

For the rest of this section, we work over the smooth compactification Y of U as in [Mu77]. In fact, if Y^* denotes the Baily-Borel compactification of U, there exists a morphism $\delta : Y \to Y^*$ whose restriction to U is the identity.

Recall that the cotangent bundle of a symmetric domain $M = G/K$ is the homogeneous bundle associated with $(\mathrm{Lie}(G)/\mathrm{Lie}(K))^\vee$.

Theorem 2.1 ([Mu77] Theorem 3.1 and Proposition 3.4)**.**

a. *Suppose that E_U is a homogeneous bundle with Hermitian metric h induced by h_0 as above. Then there exists a unique locally free sheaf E on Y with $E|_U = E_U$, such that h is a singular Hermitian metric good on Y.*
b. *For $E_U = \Omega^1_U$ one obtains the extension $E = \Omega^1_Y(\log S)$.*
c. *For $E_U = \omega_U$ one obtains the extension $E = \omega_Y(S)$ and this sheaf is the pullback of an invertible ample sheaf on Y^*.*

We will not need the exact definition of a singular Hermitian metric, "good on Y". Let us just recall that this implies that the curvature of the Chern connection ∇_h of h represents the first chern class of E.

Corollary 2.2. *Assume that U maps to the moduli stack $\mathcal{A}_g$ of polarized abelian varieties, and that this morphism is induced by a homomorphism $G \to \mathrm{Sp}$. Then the Mumford compactification Y satisfies the Assumptions 0.1 and Conditions 0.2.*

Proof. If the bounded symmetric domain M decomposes as $M_1 \times \cdots \times M_s$, hence if $\mathrm{Aut}(M) =: G = G_1 \times \cdots \times G_s$, the sheaves $\Omega^1_{M_i}$ are homogeneous bundles associated with $(\mathrm{Lie}(G_i)/\mathrm{Lie}(K_i))^\vee$. They descend to sheaves $\Omega_{i\,U}$ on U which extend to Ω_i on Y. The uniqueness of the extensions implies that $\Omega^1_Y(\log S) = \Omega_1 \oplus \cdots \oplus \Omega_s$.

Let $f : A \to U$ denote the universal family over U, and let $F^{1,0}_U = f_*\Omega^1_{A/U}$ denote the Hodge bundle. Since $U \to \mathcal{A}_g$ is induced by a homomorphism $G \to \mathrm{Sp}$, and since the bundle $\Omega^1_{\mathcal{A}_g}$ is homogeneous on $\mathcal{A}_g$, its pullback to U is homogeneous under G. The latter is isomorphic to $S^2(F^{1,0}_U)$.

The sheaf Ω^1_U is a homogeneous direct factor, hence the uniqueness of the extension in Theorem 2.1 implies that $\Omega^1_Y(\log S)$ is a direct factor of the extension of $S^2(F^{1,0}_U)$ to Y. We may assume that the local monodromies of $R^1 f_*\mathbb{C}_A$ around the components of $S = Y \setminus U$ are unipotent. Then the Mumford extension is $S^2(F^{1,0})$, where $F = F^{1,0} \oplus F^{0,1}$ is the logarithmic Higgs bundle of $R^1 f_*\mathbb{C}_A$. Moreover, as shown by Kawamata (see [Vi95, Theorem 6.12], for example), the sheaf $F^{1,0}$ is nef. So $S^2(F^{1,0})$ and the direct factor $\Omega^1_Y(\log S)$ are both nef.

The ampleness of $\omega_Y(S)$ follows directly from the second part of [Mu77, Proposition 3.4]. In fact, as remarked in the proof of [Mu77, Proposition 4.2], this sheaf is just the pullback of the ample sheaf on the Baily-Borel compactification of U.

It remains to verify that $\Omega^1_Y(\log S)$ is μ-polystable and that for all i Ω_i is μ-stable.

Using standard calculation of Chern characters on products, as in Section 4, it is easy to show that the slopes $\mu(\Omega_i)$ coincide with $\mu(\Omega^1_Y(\log S))$. The μ-stability of Ω_i follows from the next lemma and a case by case verification that for M_i irreducible the representation attached to the homogeneous bundle Ω_{M_i} is irreducible.

Alternatively, since we have verified the Assumptions 0.1, we may use Yau's Uniformization Theorem, stated in [VZ07, Theorem 1.4]. It implies that $\Omega^1_Y(\log S)$ is μ-polystable. Then the sheaves Ω_i, constructed above, are μ-polystable as well. Moreover, if Ω_i decomposes as a direct sum of two μ-polystable subsheaves the corresponding M_i will be the product of two subspaces. So if we choose the decomposition $M = M_1 \times \cdots \times M_s$ with M_i irreducible, the sheaves Ω_i will be stable. □

Lemma 2.3. *Suppose that the vector bundle E on Y is Mumford's extension of an irreducible homogeneous vector bundle $E|_U$. Then E is stable with respect to the polarization $\omega_Y(S)$.*

Proof. By definition of Mumford's extension ([Mu77, Theorem 3.1]), E carries a metric h coming from the G-invariant metric, again denoted by h, on the pull back $\tilde{E}$ of E to M. As mentioned already, for a singular metric, good in the sense of Mumford, the curvature of the Chern connection ∇_h of h represents the first chern class of E.

We claim that the restriction of ∇_h to U is a Hermitian Yang-Mills connection with respect to the Kähler-Einstein metric g on Ω^1_U. In fact, the pull back vector bundle $\tilde{E}$ on M is an irreducible homogeneous vector bundle.

So our claim says that this G-invariant metric h on $\tilde{E}$ is Hermitian-Yang-Mills with respect to the G-invariant (Kähler-Einstein) metric g on Ω^1_M with the argument adapted from the proof of [Ko86, Theorem 3.3 (1)]. The $g-$trace of the curvature $\wedge_g(\Theta_h)$ of h is a $G-$invariant endomorphism on the vector bundle $\tilde{E}$, and

$$\wedge_g(\Theta_h)_0 := \wedge_g(\Theta_h)|_{\tilde{E}_0}$$

is an $K-$invariant endomorphism on the vector space $\tilde{E}_0$. Since the maximal compact subgroup K acts on $\tilde{E}_0$ irreducibly, $\wedge_g(\Theta_h)_0$ must be a scalar multiple of the identity on $\tilde{E}_0$. The facts that G operates on M transitively and that the induced action of G on $\tilde{E}$ commutes with $\wedge_g(\Theta_h)$ imply that $\wedge_g(\Theta_h)$ is a constant scalar multiple of the identity endomorphism. So, h is a Hermitian-Yang-Mills metric with respect to the G-invariant (Kähler-Einstein) metric g on Ω^1_M. Here we regard Ω^1_M as an irreducible homogeneous vector bundle. Going downstairs we obtain the Hermitian-Yang-Mills metric h on $E|_U$ with respect to the Kähler-Einstein metric g on Ω^1_U.

Suppose that $F \subset E$ is a subbundle and let s_U be the C^∞ orthogonal splitting over U. By Theorem 5.20 in [Kol85] the curvature of the Chern connection to $h|_F$ represents the $c_1(F)$. The Chern-Weil formula implies

$$R(\nabla_{(h|_F)}) = R(\nabla_h)|_F + s_u \wedge s_u^*.$$

The Hermitian Yang-Mills property of h yields $\mu(F) \leq \mu(E)$ and equality holds if and only if s_U is holomorphic.

If the equality holds, the pull back of s_U to M gives an orthogonal splitting of Hermitian vector bundles

$$\pi^* E|_U \cong \pi^* F|_U \oplus \pi^* F^\perp|_U.$$

By Proposition 2 on p. 198 of [Mk02] this contradicts the irreducibility of $E|_U$. Thus E is stable. □

Lemma 2.4. *Suppose that E_i are vector bundles on Y, that are Mumford's extensions of irreducible homogeneous vector bundles $E_i|_U$. Then $E_1 \otimes E_2$ is polystable.*

Proof. Let ρ_i be the representation corresponding to E_i. Since the E_i are stable, $E_1 \otimes E_2$ is semistable. Repeating the calculation of the curvature of the Chern connection from the previous Lemma, the existence of a subbundle of $E_1 \otimes E_2$ of the same slope as $E_1 \otimes E_2$ implies that the respresentation $\rho_1 \otimes \rho_2$ corresponding to $E_1 \otimes E_2$ is not irreducible. Since K is reductive, $\rho_1 \otimes \rho_2$ decomposes as a direct sum of irreducible representations. Each of them defines a stable bundle, again by the previous Lemma, and equality of slopes follows from semistability. □

Before proving the first part of Proposition 0.3 for the Mumford compactification Y, let us show that the Arakelov equality is independent of the compactification Y, as long as the Assumptions 0.1 on positivity hold.

Lemma 2.5. *Let Y_1, S_1 and Y_2, S_2 be two compactifications of U, both satisfying the Assumptions 0.1. Let μ_i denote the slope on Y_i with respect to $\omega_{Y_i}(S_i)$. Given a complex polarized variation of Hodge structures $\mathbb{V}$ on U with Higgs bundle (E_i, θ_i) on Y_i one has:*

i. $\mu_1(E_1^{1-q,q}) = \mu_2(E_2^{1-q,q})$, *for* $q = 0, 1$.
ii. $\mu_1(\Omega^1_{Y_1}(\log S_1)) = \mu_2(\Omega^1_{Y_2}(\log S_2))$.
iii. *In particular the Arakelov equality on Y_1 implies the one on Y_2.*

Proof. Choose a compactification $\bar{Y}$ of U which allows birational morphisms

$$\sigma_1 : \bar{Y} \longrightarrow Y_1 \quad \text{and} \quad \sigma_2 : \bar{Y} \longrightarrow Y_2.$$

The Assumptions 0.1 implies that the sheaf $\mathcal{L}_i = \sigma_i^* \omega_{Y_i}(S_i)$ is nef and big, for $i = 1, 2$. Moreover, writing $\bar{S} = \bar{Y} \setminus U$, for some effective divisors E_i one has

$$\omega_{\bar{Y}}(\bar{S}) = \mathcal{L}_1 \otimes \mathcal{O}_{\bar{Y}}(E_1) = \mathcal{L}_2 \otimes \mathcal{O}_{\bar{Y}}(E_2).$$

Choose effective divisors F_i and some ν sufficiently large, such that the sheaves $\mathcal{L}_i^\nu \otimes \mathcal{O}_{\bar{Y}}(-F_i)$ are very ample, as well as $\mathcal{L}_i^\nu \otimes \mathcal{O}_{\bar{Y}}(-F_i) \otimes \omega_{\bar{Y}_i}(S_i)^{-1}$. So replacing F_i and ν by some multiple, one may even assume that the sheaves $\mathcal{L}_i^\beta \otimes \mathcal{O}_{\bar{Y}}(-F_i)$ are both generated by global sections for $\beta \geq \nu$ (see [Vi95, Corollary 2.36], for example).

On the other hand, for all $\beta \geq 0$

$$H^0(\bar{Y}, \mathcal{L}_1^\beta) = H^0(\bar{Y}, \omega_{\bar{Y}}(\bar{S})^\beta) = H^0(\bar{Y}, \mathcal{L}_2^\beta).$$

This is only possible if $\mathcal{L}_1 = \mathcal{L}_2$. Let us write $\bar{\mu}$ for the slope with respect to the invertible sheaf $\mathcal{L}_1 = \mathcal{L}_2$.

The Deligne extension of $\mathbb{V} \otimes_C \mathcal{O}_U$ is compatible with pullbacks to blowing-ups of the boundary divisor. This implies that $\sigma_1^* E_1^{1-q,q} = \sigma_2^* E_2^{1-q,q}$, and by the projection formula

$$\mu_1(E_1^{1-q,q}) = \bar{\mu}(\sigma_1^* E_1^{1-q,q}) = \bar{\mu}(\sigma_2^* E_2^{1-q,q}) = \mu_2(E_2^{1-q,q}).$$

Finally for ii) remark that

$$\dim(U) \cdot \mu_1(\Omega^1_{Y_1}(\log S_1)) = \mu_1(\omega_{Y_1}(S_1)) = \bar{\mu}(\mathcal{L}_1) = \bar{\mu}(\mathcal{L}_2) = \mu_2(\omega_{Y_2}(S_2)) = \dim(U) \cdot \mu_2(\Omega^1_{Y_2}(\log S_2)).$$

Of course, iii) follows from i) and ii). □

We now prove Proposition 0.3 except for the statement v). The latter will be shown at the end of Section 6, by applying Addendum 0.10, III.

Proof of Proposition 0.3, part i)–iv) for Mumford's compactification.
By Theorem 1.2 we may may replace $U = \mathcal{X}_1 \times \{b\}$ by $\mathcal{X}_1 \times \{a\}$. This will affect neither stability nor slopes, hence we may suppose without loss of generality that U is a Shimura variety of Hodge type given by the datum (G, M).

Our first aim is to exhibit $E^{1,0}$ and $E^{0,1}$ as homogeneous vector bundles. Let $\tau : G \to \mathrm{CSp}$ be the map given by the property 'of Hodge type'. Choose a base point on the symmetric domain M and its image on $M' := M(Q)$. There are maximal compact subgroups K of G^{der} and $K' \cong U(g)$ of Sp such that $U \to \mathcal{A}_g$ is uniformized by the map $M = G^{\mathrm{der}}/K \to \mathrm{Sp}/K' =: M'$. Let $\pi_U : D \to U$ and $\pi_{\mathcal{A}_g} : D' \to \mathcal{A}_g$ be the natural quotients modulo arithmetic subgroups. The choice of the base point in M' is equivalent to the choice of a Q-symplectic basis $\{a_i, b_i\}$ of V such that we have $h(i)(a_i) = b_i$ and $h(b_i) = -a_i$ by (1.1).

Since the $(1,0)$- and $(0,1)$-parts of $\pi^*_{\mathcal{A}_g}(R^1 f_* \mathbb{C}_A)$ are the i resp. $-i$-eigenspaces of $h(i)$, they are homogeneous bundles. Moreover, they are given by the representations ρ_{can} and $\overline{\rho_{\mathrm{can}}}$, where $\rho_{\mathrm{can}} : U(g) \to \mathrm{GL}(g)$ is the standard representation. The $(1,0)$- and $(0,1)$-parts of $\pi^*_U(R^1 f_* \mathbb{C}_A)$ are consequently homogeneous bundles too, given by the representation $\tau|_K \circ \rho_{\mathrm{can}}$ and $\tau|_K \circ \overline{\rho_{\mathrm{can}}}$.

Next, we link two notions of irreducibility. Since π_U is the quotient map by an arithmetic group $\Gamma \subset G(\mathbb{Q})$, whose image in G^{ad} is Zariski-dense, $\mathbb{C}$-irreducible summands of $R^1 f_* \mathbb{C}_A$ are in bijection with $\mathbb{C}$ irreducible summands of the representation

$$\widetilde{\tau} : \widetilde{G^{\mathrm{ad}}} \longrightarrow G \longrightarrow \mathrm{CSp}.$$

Here $\widetilde{G^{\mathrm{ad}}} \to G^{\mathrm{ad}}$ is the universal covering and the map to $\widetilde{G^{\mathrm{ad}}} \to G$ is induced by the canonical splitting of $\mathrm{Lie}(G)$ into its abelian and its semisimple part. We determine these $\mathbb{C}$ irreducible summands, following [De79, §2.3.7 (a)], see also [Sa65] or [Sa80].

By [De79, §2.3.4], it suffices to consider irreducible summands over $\mathbb{R}$. Write

$$G^{\mathrm{ad}}_{\mathbb{R}} = \prod_{i \in I} G_i$$

and partition the index set $I = I_c \cup I_{\mathrm{nc}}$ according to whether G_i is compact or not. By [De79, §1.3.8 (a) and §2.3.7] the irreducible representations of $V_{\mathbb{C}}$ are of the form $\otimes_{t \in T} W_t$ for some $T \subset I$, where W_t is an irreducible representation of $\widetilde{G_i}$. Moreover, the condition (SV1) forces $T \cap I_{\mathrm{nc}}$ to contain at most one element, see [De79, Lemma 1.3.7] This shows i).

If $T \cap I_{\mathrm{nc}} = \emptyset$, then $\mathbb{V}$ is unitary. We thus restrict to the other case from now on. Then the condition 'Shimura variety' imposes the restrictions to the representation of the non-compact group as in the hypothesis of Lemma 2.6, stated below. From this lemma we deduce that in each case the representation of $K \subset G^{\mathrm{ad}}$ is irreducible.

Now we know by Lemma 2.3 that for each irreducible summand $\mathbb{V}$ of $R^1 f_* \mathbb{C}_A$, both $E^{1,0}$ and $E^{0,1}$ are stable. By Lemma 2.4, the bundle $\mathrm{Hom}(E^{1,0}, E^{0,1})$ is polystable with the stable summands given as homogeneous bundles by the irreducible summands of the representation $\rho \otimes \rho^\vee$, where $\rho = \tau \circ \rho_{\mathrm{can}}$. This proves iii) and iv). Since $M \to M'$ is induced by a group homomorphism and hence totally geodesic, the tangent map

$$T_M \longrightarrow T'_M|_M = \mathrm{Hom}(E^{1,0}, E^{0,1})$$

is onto a direct summand. Since it is a map between homogeneous bundles, the direct summand corresponds to an irreducible summand of the representation $\rho \otimes \rho^\vee$. Consequently, the map

$$\overline{(T_U)} \longrightarrow \overline{\mathrm{Hom}(E^{1,0}, E^{0,1})|_U}$$

between the Mumford extensions is an injection onto a stable summand. Since the Mumford extension of T_U is $T_Y(-\log S)$ and the Mumford extension of $E^{p,q}$ is the Deligne extension, we obtain

$$\mu(T_Y(-\log S)) = \mu(E^{1,0}) - \mu(E^{0,1}),$$

i.e. the Arakelov equality, stated as ii). □

We keep the notations of the preceding proof, that will be completed with the following lemma. We follow [De79] and define a cocharacter $\chi : \mathbb{G}_m \to (G_i)_{\mathbb{C}}$ induced by $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_{\mathbb{R}}$ in the following way. Fix an isomorphism

$$(\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m)_{\mathbb{C}} \cong \mathbb{G}_m \times \mathbb{G}_m$$

such that the inclusion

$$(\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m)(\mathbb{R}) \to (\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m)(\mathbb{C})$$

is given by $z \mapsto (z, \overline{z})$. Let $i : \mathbb{G}_m \to \mathbb{G}_m \times \mathbb{G}_m$ be the inclusion given by the identity in the second argument. Then $\chi := h_{\mathbb{C}} \circ i$.

Given χ, we let $\tilde{\chi}$ the inductive system of fractional lifts of χ to $\widetilde{G_i}$ ([De79, §1.3.4]).

Lemma 2.6. *Let $\tau_{i,t} : \widetilde{G_i} \to \mathrm{GL}(W_t)$ be an irreducible representation whose highest weight α is a fundamental weight and such that*

$$\langle \widetilde{\chi}, \alpha + \iota(\alpha) \rangle = 1, \tag{2.1}$$

where ι is the opposition involution. Then W_t is the sum of two non-empty weight spaces, denoted by $W_t^{1,0}$ and $W_t^{0,1}$. Both weight spaces are irreducible representations of the maximal compact subgroup K_i of G_i.

Proof. The equivalence of the condition (2.1) and the decomposition into two weight spaces is in ([De79, §1.3.8]). The possible solutions to (2.1) are listed on [Sa65, p. 461]. We distinguish the cases according to the Dynkin diagram of G_i. We use that the cocharacter $\widetilde{\chi}$ satisfying (2.1) determines a special node in the Dynkin diagram ([De79, §1.2.5]).

Type a_n: In this case $G_i = \mathrm{SU}(p, q)$ with $p + q = n - 1$, depending on the signature of the bilinear form induced by the Cartan involution $\mathrm{ad}(h(i))$. We may assume $p \geq q$. The maximal compact subgroup is

$$K_i = S(U(p) \times U(q)).$$

If $q > 1$ only the standard representation satisfies (2.1). The weight spaces $W_t^{1,0}$ and $W_t^{0,1}$ carry the standard representation of $\mathrm{SU}(p)$ and $\mathrm{SU}(q)$ respectively and are hence irreducible.

If $q = 1$ all j-th wedge product representations for $j = 1, \dots, n-1$ satisfy (2.1). The weight spaces $W_t^{1,0}$ (resp. $W_t^{0,1}$) carry the j-th (resp. $j-1$-st) exterior power representation of $\mathrm{SU}(p)$, which is also irreducible.

Type b_n: In this case is $G_i = \mathrm{SO}(2, 2n-1)$ (type IV_{2n-1} in [Sa80])) and the only representation that satisfies (2.1) is the spin representation of the double cover $\mathrm{Spin}(2, 2n-1) \to G_i$. The maximal compact subgroup is

$$K_i \cong \mathrm{SO}(2n-1, \mathbb{R}) \times \mathrm{SO}(2, \mathbb{R}).$$

We claim that one weight space carries the tensor product of the spin representation of $\mathrm{SO}(2n-1)$ and one of the natural representations $\mathrm{SO}(2,\mathbb{R}) \to U(1)$ while the other weight space carries the tensor product of the spin representation and the complex conjugate representation of $\mathrm{SO}(2,\mathbb{R})$. In both cases the representations are well known to be irreducible.

In order to prove the claim we write down the spin representation explicitly and exhibit its weight spaces. We follow the notations of [Sa65, §3.5]. Let G_i be the group of transformations of $V_{\mathbb{R}}$ preserving a bilinear form S of signature $(2n-1, 2)$. Let $\{e_1, \dots, e_{2n-1}\}$ (resp. $\{e_{2n}, e_{2n+1}\}$) be an orthonormal bases of V^+ (resp. V^-), the subspaces where the form is positive (resp. negative) definite. We let $f_j = (e_{2j-1} + ie_{2j})/2$ for $j = 1, \dots, n-1$ and $f_n = (e_{2n} + ie_{2n+1})$. Denote by W the complex vector space generated by the f_j. The exterior algebra $E = \Lambda(W)$ embeds into the Clifford algebra of $C(V, S)$. For an ordered subset $\mathcal{J} = \{i_1, \dots, i_a\} \subset N := \{1, \dots, n\}$ we consider the elements $f_{\mathcal{J}} = f_{i_1} \cdots f_{i_a}$ and their complex conjugates in the Clifford algebra. We identify E with the left ideal $E \cdot \overline{f_N}$ and obtain a representation of $\mathrm{Spin}(2, 2n-1)$ on E.

We may choose in

$$\mathrm{Lie}(G_i) = \left\{ \begin{pmatrix} X_1 & X_{12} \\ X_{12}^T & X_2 \end{pmatrix} ;\, X_1, X_{12}, X_2 \,\text{real}, X_1, X_2 \,\text{skew symmetric} \right\}.$$

a maximal abelian subalgebra,

$$\mathfrak{h} = \left\{ \mathrm{diag}\left(\begin{pmatrix} 0 & -\xi_1 \\ \xi_1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & -\xi_{n-1} \\ \xi_{n-1} & 0 \end{pmatrix}, 0, \begin{pmatrix} 0 & -\xi_n \\ \xi_n & 0 \end{pmatrix}, \xi_i \in \mathbb{R} \right) \right\}.$$

Then by the calculation in [Sa65, p. 455]) the $f_{\mathcal{J}}$ are eigenvectors with corresponding weight $\frac{i}{2}(\sum_{i \notin \mathcal{J}} \xi_i - \sum_{i \in \mathcal{J}} \xi_i)$. The map χ corresponding to the special node is generated by the element $H_0 \in \mathrm{Lie}(G_i)$ with $X_1 = 0$, $X_{12} = 0$ and $X_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We deduce that the weight spaces $W_i^{1,0}$ (resp. $W_i^{0,1}$) are generated by the $f_{\mathcal{J}}$ with $n \notin \mathcal{J}$ (resp. by the $f_{\mathcal{J}}$ with $n \in \mathcal{J}$).

From this we first read off that $\mathrm{SO}(2,\mathbb{R})$ acts on the weight spaces as claimed. Fix the root system

$$\{i(\xi_1 - \xi_2), \dots, i(\xi_{2n-2} - \xi_{2n-1}), i\xi_{2n-1}\}$$

of $\mathrm{so}(2n-1)$. Consider $W_i^{1,0}$ as a representation of $\widetilde{\mathrm{SO}(2n-1)}$ of dimension 2^{n-1}. A vector of highest weight is $f_{N \setminus \{n\}}$ with weight $i/2 \sum_{i=1}^{n-1} \xi_i$. Consequently, the representation contains a spin representation of $\mathrm{Spin}(2n-1) \to \mathrm{SO}(2n-1)$. For dimension reasons the representation is irreducible. The same argument applies to $W_i^{0,1}$.

Type c_n: In this case $G_i = \mathrm{Sp}(n)$, and as in the beginning of the proof of Proposition 0.3 above, the weight spaces carry the standard representation of $U(n)$ and its complex conjugate. Thus, they are irreducible.

Type d_n: This case splits into two subcases according to the χ or equivalently according to the position of the corresponding special node in the Dynkin diagram.

Special node at the 'fork' end. In this case

$$G_i = \mathrm{SU}^-(n,\mathbb{H}) \cong \mathrm{SU}(n,n) \cap \mathrm{SO}(2n,\mathbb{C}) \subset \mathrm{Sl}(2n,\mathbb{C})$$

where $\mathbb{H}$ denotes the Hamiltonians. In this matrix representation the weight spaces are given by the n first (resp. last) column vectors. The maximal compact subgroup $K_i \cong U(n)$ sits in G_i via

$$A + iB \mapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

Consequently, both weight spaces are n-dimensional and carry the irreducible standard representation of $U(n)$.

Special node at the opposite end. This is completely similar to the case b_n replacing 'spin' by 'half spin' representations throughout.

Exceptional Lie algebras do not admit any solution to (2.1). □

3. Slopes and filtrations of coherent sheaves

We will need small twists of the slope $\mu(\mathcal{F})$ defined by the nef and big invertible sheaf $\omega_Y(S)$, for coherent sheaves $\mathcal{F}$. So we will decompose the slope in a linear combination of different slopes and we will deform the coefficients a little bit. In particular, as in [La04], we will compare the Harder-Narasimhan filtrations for small twists of slopes.

On the projective manifold Y of dimension n consider $n-1$-tuples of $\mathbb{R}$-divisors

$$\underline{D}^{(\iota)} = (D_1^{(\iota)}, \ldots, D_{n-1}^{(\iota)}),$$

for $\iota = 1, \ldots, m$. Given two such tuples we will write

$$\underline{D}^{(\iota)} + \underline{D}^{(j)} = (D_1^{(\iota)} + D_1^{(j)}, \ldots, D_{n-1}^{(\iota)} + D_{n-1}^{(j)}).$$

Definition 3.1. We call $\underline{D}^{(\bullet)}$ a semi-polarization if the $\mathbb{R}$-divisors $D_j^{(\iota)}$ are nef for $\iota = 1, \ldots, m$ and for $j = 1, \ldots, n-1$ and if $(\underline{D}^{(\iota)})^{n-1} := D_1^{(\iota)}. \cdots .D_{n-1}^{(\iota)}$ is not numerically trivial for $\iota = 1, \ldots, m$.

For a coherent torsion free sheaf $\mathcal{F}$ on Y the slope is given by

$$\mu_{\underline{D}^{(\bullet)}}(\mathcal{F}) = \mu_{\underline{D}^{(1)},\ldots,\underline{D}^{(m)}}(\mathcal{F}) = \sum_{\iota=1}^{m} \frac{\mathrm{c}_1(\mathcal{F}).(\underline{D}^{(\iota)})^{n-1}}{\mathrm{rk}(\mathcal{F})}.$$

In the sequel we will assume that $\underline{D}^{(\bullet)}$ a semi-polarization, and we fix a torsion free coherent sheaf $\mathcal{F}$ on Y. If there is no ambiguity, we write μ' in this Section instead of $\mu_{\underline{D}^{(1)},\ldots,\underline{D}^{(m)}}$, and we reserve the notion μ for the slope with respect to $\omega_Y(S)$.

Given an exact sequence of torsion free coherent sheaves

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

an easy calculation shows that

$$\mu'(\mathcal{F}) = \frac{\mathrm{rk}(\mathcal{F}')}{\mathrm{rk}(\mathcal{F})}\mu'(\mathcal{F}') + \frac{\mathrm{rk}(\mathcal{F}'')}{\mathrm{rk}(\mathcal{F})}\mu'(\mathcal{F}''). \tag{3.1}$$

Definition 3.2.

a. A subsheaf $\mathcal{G}$ of $\mathcal{F}$ is μ'-equivalent to $\mathcal{F}$, if $\mathcal{F}/\mathcal{G}$ is a torsion sheaf and if $\mathrm{c}_1(\mathcal{F}) - \mathrm{c}_1(\mathcal{G})$ is the class of an effective divisor D with $\mu'(\mathcal{O}_Y(D)) = 0$, or equivalently with $D.(\underline{D}^{(\iota)})^{n-1} = 0$, for $\iota = 1, \ldots, m$.
b. $\mathcal{G} \subset \mathcal{F}$ is saturated, if $\mathcal{F}/\mathcal{G}$ is torsion free.
c. $\mathcal{F}$ is μ'-stable, if $\mu'(\mathcal{G}) < \mu'(\mathcal{F})$ for all subsheaves $\mathcal{G}$ of $\mathcal{F}$ with $\mathrm{rk}(\mathcal{G}) < \mathrm{rk}(\mathcal{F})$.
d. $\mathcal{F}$ is μ'-semistable, if $\mu'(\mathcal{G}) \leq \mu'(\mathcal{F})$ for all subsheaves $\mathcal{G}$ of $\mathcal{F}$.
e. $\mathcal{F}$ is μ'-polystable if it is the direct sum of μ'-stable sheaves of the same slope.
f. $\mathcal{F}$ is weakly μ'-polystable, if it is equivalent to a μ'-polystable subsheaf.
g. A saturated subsheaf $\mathcal{G}$ of $\mathcal{F}$ is called a maximal destabilizing subsheaf if for all subsheaves $\mathcal{E}$ of $\mathcal{F}$ one has $\mu'(\mathcal{E}) \leq \mu'(\mathcal{G})$ and if the equality implies that $\mathcal{E} \subset \mathcal{G}$.

Lemma 3.3.

1. *If $\mathcal{F}$ is μ'-stable and if $\mathcal{G} \subset \mathcal{F}$ is a subsheaf with $\mu'(\mathcal{G}) = \mu'(\mathcal{F})$ then $\mathcal{F}$ and $\mathcal{G}$ are μ'-equivalent.*
2. *A μ'-polystable sheaf $\mathcal{F}$ is μ'-semistable.*
3. *In particular, if $\mathcal{H}$ is invertible, then $\bigoplus \mathcal{H}$ is μ'-semistable.*

Proof. If $\mathcal{G}$ is a subsheaf of $\mathcal{F}$ with $\mathrm{rk}(\mathcal{G}) = \mathrm{rk}(\mathcal{F})$ then $\mathrm{c}_1(\mathcal{F}) - \mathrm{c}_1(\mathcal{G})$ is an effective divisor D. Since all the $D_j^{(\iota)}$ are nef, one finds $D'.(\underline{D}^{(\iota)})^{n-1} \geq 0$ and hence $\mu'(\mathcal{G}) \leq \mu'(\mathcal{F})$. This implies 2) in case that $\mathcal{F}$ is μ'-stable.

For μ'-polystable sheaves 2) follows by induction on the number of direct factors, and 3) is an example for the statement in 2).

If $\mathcal{F}$ is stable and $\mu'(\mathcal{G}) = \mu'(\mathcal{F})$, then by definition $\mathrm{rk}(\mathcal{F}) = \mathrm{rk}(\mathcal{G})$, hence $D.(\underline{D}^{(\iota)})^{n-1} = 0$ as claimed in 1). □

Consider a second tuple

$$\underline{H}^{(\iota)} = (H_1^{(\iota)}, \ldots, H_{n-1}^{(\iota)})$$

of nef $\mathbb{R}$-divisors, for $\iota = 1, \ldots, m$, and for $t \in \mathbb{R}$ the slope

$$\mu'_t(\mathcal{F}) = \mu_{\underline{D}^{(\bullet)} + t \cdot \underline{H}^{(\bullet)}}(\mathcal{F}) = \sum_{\iota=1}^{m} \frac{\mathrm{c}_1(\mathcal{F}).(\underline{D}^{(\iota)} + t \cdot \underline{H}^{(\iota)})^{n-1}}{\mathrm{rk}(\mathcal{F})},$$

regarded as a polynomial in t. Of course one has $\mu'_0(\mathcal{F}) = \mu'(\mathcal{F})$. The cycle $(\underline{D}^{(\iota)} + t \cdot \underline{H}^{(\iota)})^{n-1}$ can be written as

$$D_1^{(\iota)}. \cdots .D_{n-1}^{(\iota)} + \sum_{I \in \mathcal{I}} t^{n-|I|-1} \cdot D_{i_1}^{(\iota)}. \cdots .D_{i_{|I|}}^{(\iota)}.H_{j_1}^{(\iota)}. \cdots .H_{j_{n-1-|I|}}^{(\iota)}$$

where the sum is taken over the set $\mathcal{I}$ of ordered subsets

$$I = \{i_1, \ldots, i_{|I|}\} \quad \text{of} \quad \{1, \ldots, n-1\}$$

of cardinality $|I| < n-1$, and where $\{j_1, \ldots, j_{j_{n-1-|I|}}\}$ is the complement of I in $\{1, \ldots, n-1\}$, again as an ordered set. For a coherent sheaf $\mathcal{G}$ one has

$$\mu'_t(\mathcal{F}) - \mu'_t(\mathcal{G}) = \mu'(\mathcal{F}) - \mu'(\mathcal{G}) + \sum_{\mathcal{I}} t^{n-|I|-1} \cdot (\mu'^I(\mathcal{F}) - \mu'^I(\mathcal{G})), \tag{3.2}$$

$$\text{with} \quad \mu'^I(\mathcal{G}) = \sum_{\iota=1}^{m} \frac{\mathrm{c}_1(\mathcal{G}).D^{(\iota)}_{i_1}. \cdots .D^{(\iota)}_{i_{|I|}}.H^{(\iota)}_{j_1}. \cdots .H^{(\iota)}_{j_{n-1-|I|}}}{\mathrm{rk}(\mathcal{G})}. \tag{3.3}$$

Lemma 3.4. *Fix a coherent sheaf $\mathcal{F}$ of rank r.*

i. *The set $S = \{\mu'(\mathcal{G});\ \mathcal{G} \subset \mathcal{F}\} \subset \mathbb{R}$ is discrete and bounded from above.*
ii. *There exists some $\epsilon_0 > 0$ and some "maximal" element*

$$G(t) \in \mathcal{S} = \left\{\mu'_t(\mathcal{G}) = \sum_{\nu=0}^{n-1} a_\nu \cdot t^\nu;\ \mathcal{G} \subset \mathcal{F}\right\}$$

such that for all $F(t) \in \mathcal{S}$ with $F(t) \neq G(t)$ one has $G(\epsilon) > F(\epsilon)$ for $0 < \epsilon \leq \epsilon_0$.

Proof. Let S' be the set of all coefficients occurring in $F(t) \in \mathcal{S}$. We will first show, that the set S' is discrete and bounded from above. Since $S \subset S'$, this implies i).

Embed $\mathcal{F}$ in $\bigoplus \mathcal{H}$, for some $\mathcal{H}$ sufficiently ample and invertible. Then under the projection to suitable factors, any subsheaf $\mathcal{G} \subset \mathcal{F}$ of rank r' is isomorphic to a subsheaf of $\bigoplus^{r'} \mathcal{H}$ and $\mathrm{c}_1(\mathcal{G}) = r \cdot \mathrm{c}_1(\mathcal{H}) - D$ for some effective divisor D.

Since the divisors $D^{(\iota)}_j$ and $H^{(\iota)}_j$ are all nef, the intersection of the 1-dimensional cycles

$$D^{(\iota)}_{i_1}. \cdots .D^{(\iota)}_{i_{|I|}}.H^{(\iota)}_{j_1}. \cdots .H^{(\iota)}_{j_{n-1-|I|}}$$

in (3.3) with any divisor is a natural number, so one may write

$$\sum_{i=1}^{m} (\underline{D}^{(\iota)} + t \cdot \underline{H}^{(\iota)})^{n-1} = \sum_{\nu=0}^{n-1} t^\nu \sum_{\mu} \alpha_{\nu,\mu} C_{\mu,\nu}$$

for $\alpha_{\mu,\nu} \in \mathbb{R}$ and for linear combinations $C_{\mu,\nu}$ of curves with $D.C_{\mu,\nu} \geq 0$ for all effective divisors D. Then $-S'$ is discrete, as a subset of the union of translates of finite many copies of $\bigcup_{\mu,\nu} \alpha_{\mu,\nu} \cdot \mathbb{N}$. Moreover S' it is bounded above by the maximal coefficient c of $\mu'_t(\mathcal{H})$.

On the set $\mathcal{S}$ consider the lexicographical order. So $\sum_{\nu=0}^{n-1} a_\nu \cdot t^\nu < \sum_{\nu=0}^{n-1} b_\nu \cdot t^\nu$ if $a_\nu = b_\nu$ for $\nu < j$ and if $a_j < b_j$. Obviously $\mathcal{S}$ contains a maximal element $G(t) = \sum_{\nu=0}^{n-1} b_\nu \cdot t^\nu$.

Choose $\epsilon_0 < 1$ to be a real number with

$$\frac{1}{\sqrt{\epsilon_0}} \geq \sup_{c \in S} \left\{ \sum_{\nu=j+1}^{n-1} (c - b_\nu) t^{\nu-j-1};\ t \in [0,1],\ j = 1, \ldots, r-1 \right\},$$

and such that an $\sqrt{\epsilon_0}$-interval around b_ν contains no element of S', except b_ν, for $\nu = 0, \dots, r$. Since $G(t) > F(t)$, for some j and for $0 < \epsilon \leq \epsilon_0$ one finds

$$G(\epsilon) - F(\epsilon) = \sum_{\nu=j}^{n-1} (b_\nu - a_\nu) \cdot \epsilon^\nu \geq$$

$$\epsilon^j \cdot \big((b_j - a_j) + \epsilon \cdot \sum_{\nu=j+1}^{n-1} (b_\nu - c) \cdot \epsilon^{\nu-j-1} \big) \geq \epsilon^j \cdot (\sqrt{\epsilon_0} - \epsilon \cdot \frac{1}{\sqrt{\epsilon_0}}) \geq 0.$$

□

Definition 3.5. For $\epsilon \in \mathbb{R}_{\geq 0}$ consider a filtration $0 = \mathcal{G}_0 \subset \mathcal{G}_1 \subset \cdots \subset \mathcal{G}_\ell = \mathcal{F}$ with $\mathcal{G}_\alpha / \mathcal{G}_{\alpha-1}$ torsion free and μ'_ϵ-semistable, for $\alpha = 1, \dots, \ell$, and with

(3.4) $$\mu'_{\epsilon,\max}(\mathcal{F}) = \mu'_\epsilon(\mathcal{G}_1) \geq \mu'_\epsilon(\mathcal{G}_2/\mathcal{G}_1) \geq \cdots \geq \mu'_\epsilon(\mathcal{G}_\ell/\mathcal{G}_{\ell-1}) = \mu'_{\epsilon,\min}(\mathcal{F}).$$

The filtration is called a *μ'_ϵ-Harder-Narasimhan filtration* if the inequalities in (3.4) are all strict, and it is called a *weak μ'_ϵ-Jordan-Hölder filtration* if $\mu'_{\epsilon,\max}(\mathcal{F}) = \mu'_{\epsilon,\min}(\mathcal{F})$.

Lemma 3.6. *Let $\mathcal{F}$ be a coherent torsion free sheaf on Y.*

a. *For all $\epsilon \geq 0$ there exists the Harder-Narasimhan filtration*
$$\mathcal{G}_0 = 0 \subset \mathcal{G}_1 \subset \cdots \subset \mathcal{G}_\ell = \mathcal{F}$$
of $\mathcal{F}$ with respect to μ'_ϵ.

b. *There exists some $\epsilon_0 > 0$ such that the filtration in a) is independent of ϵ for $\epsilon_0 \geq \epsilon > 0$.*

c. *If $\mathcal{F}$ is μ' stable, then for some $\epsilon_0 > 0$ and for all $\epsilon_0 \geq \epsilon \geq 0$ the sheaf $\mathcal{F}$ is μ'_ϵ-semistable.*

Proof. For $\epsilon > 0$ we apply Lemma 3.4, ii). For the polynomial $G(t)$, given there, choose a subsheaf $\mathcal{G} \subset \mathcal{F}$ with $G(t) = \mu'_t(\mathcal{G})$, for all $t \in \mathbb{R}$. Moreover for $0 < \epsilon \leq \epsilon_0$ the slope $\mu_\epsilon(\mathcal{G}) = G(\epsilon)$ is maximal among the possible slopes of subsheaves of $\mathcal{F}$. This allows to assume that $\mathcal{G}$ is saturated. If there are several subsheaves of $\mathcal{F}$ with the same slope, we choose a saturated one of maximal rank.

If for $\mathcal{E} \subset \mathcal{F}$ one has $\mu'_\epsilon(\mathcal{E}) = \mu'_\epsilon(\mathcal{G})$, then by (3.1) the slope of $\mathcal{E} \oplus \mathcal{G}$ is $\mu'_\epsilon(\mathcal{G})$. The maximality of the slope of $\mathcal{G}$ implies $\mu'_\epsilon(\mathcal{E} \cap \mathcal{G}) \leq \mu'_\epsilon(\mathcal{G})$ and $\mu'_\epsilon(\mathcal{E} + \mathcal{G}) \leq \mu'_\epsilon(\mathcal{G})$. By (3.1) this is only possible if $\mu'_\epsilon(\mathcal{E} + \mathcal{G}) = \mu'_\epsilon(\mathcal{G})$. Then the maximality of the rank of $\mathcal{G}$ implies that $\mathrm{rk}(\mathcal{E} + \mathcal{G}) = \mathrm{rk}(\mathcal{G})$, and $\mathcal{E} \subset \mathcal{G}$.

So $\mathcal{G}$ is a maximal destabilizing subsheaf of $\mathcal{F}$, and it is independent of $\epsilon \in (0, \epsilon_0]$. The existence and uniqueness of a μ'_ϵ-Harder-Narasimhan filtration follows by induction on the rank. Here of course we have to lower ϵ_0 in each step.

For $\epsilon = 0$ the existence and uniqueness of the Harder-Narasimhan filtration follows by the same argument, replacing the reference to part ii) of Lemma 3.4 by the one to part i).

Assume now that $\mathcal{F}$ is μ'-stable and consider the Harder-Narasimhan filtration in a). Then
$$\mu'(\mathcal{G}_1) = \lim_{\epsilon \to 0} \mu'_\epsilon(\mathcal{G}_1) \geq \lim_{\epsilon \to 0} \mu'_\epsilon(\mathcal{F}) = \mu'(\mathcal{F}).$$
By assumption, $\mathcal{F}$ is stable, with respect to μ', hence $\mathcal{G}_1 = \mathcal{F}$, and $\ell = 1$. □

Although this will not be used in the sequel, let us state a strengthening of the last part of Lemma 3.6.

Addendum 3.7. *For ϵ_0 sufficiently small, the sheaf $\mathcal{F}$ in part c) is μ'_ϵ-stable for all $\epsilon_0 \geq \epsilon \geq 0$*

Proof. Part i) of Lemma 3.4 and the μ'-stability of $\mathcal{F}$ imply that

$$\gamma = \mathrm{Inf}\{\mu'(\mathcal{F}) - \mu'(\mathcal{G});\ \mathrm{rk}(\mathcal{G}) < \mathrm{rk}(\mathcal{F})\} > 0.$$

Let us return to the slopes μ'^I introduced in (3.2) and (3.3). By part a) of Lemma 3.6 there exists a Harder-Narasimhan filtration

$$\mathcal{G}_0^I = 0 \subset \mathcal{G}_1^I \subset \cdots \subset \mathcal{G}_{\ell_I}^I = \mathcal{F}$$

with respect to μ'^I. In particular for $\mathcal{G} \subset \mathcal{F}$ one has $\mu'^I(\mathcal{G}) \leq \mu'^I(\mathcal{G}_1^I)$.

Choose $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$, and for all $I \in \mathcal{I}$ with $|I| < n-1$ one has

$$\frac{1}{|\mathcal{I}|+1} \cdot \gamma \geq \epsilon^{n-|I|-1} \cdot \big(\mu'^I(\mathcal{G}_1^I) - \mu'^I(\mathcal{F})\big)$$

For a subsheaf $\mathcal{G} \subset \mathcal{F}$ of strictly smaller rank one finds

$$\mu'^I(\mathcal{F}) - \mu'^I(\mathcal{G}) \geq \mu'^I(\mathcal{F}) - \mu'^I(\mathcal{G}_1^I),$$

and thereby

$$\mu'_\epsilon(\mathcal{F}) - \mu'_\epsilon(\mathcal{G}) \geq \gamma + \sum_{I\in\mathcal{I}} \epsilon^{n-1-|I|} \cdot (\mu'^I(\mathcal{F}) - \mu'^I(\mathcal{G})) \geq$$

$$\gamma + \sum_{I\in\mathcal{I}} \epsilon^{n-1-|I|} \cdot (\mu'^I(\mathcal{F}) - \mu'^I(\mathcal{G}_1^I)) \geq \gamma - \frac{|\mathcal{I}|}{|\mathcal{I}|+1} \cdot \gamma > 0.$$

□

Corollary 3.8. *Assume in Lemma 3.6 that $\mathcal{F}$ is μ'-semistable. Then there exists a weak μ'-Jordan-Hölder filtration*

$$\mathcal{G}_0 = 0 \subset \mathcal{G}_1 \subset \cdots \subset \mathcal{G}_\ell = \mathcal{F}$$

and some $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0]$ the filtration $\mathcal{G}_\bullet$ is a μ'_ϵ-Harder-Narasimhan-filtration.

Proof. The filtration $\mathcal{G}_\bullet$ constructed Lemma 3.6, b), is a μ'_ϵ-Harder-Narasimhan filtration for all $0 < \epsilon \leq \epsilon_0$. Taking the limit of the slopes for μ'_ϵ one obtains

$$\mu'_{\max}(\mathcal{F}) = \mu'(\mathcal{G}_1) \geq \mu'(\mathcal{G}_2/\mathcal{G}_1) \geq \cdots \geq \mu'(\mathcal{G}_\ell/\mathcal{G}_{\ell-1}) = \mu'_{\min}(\mathcal{F}),$$

and since $\mathcal{F}$ is μ'-semistable, those are all equalities. □

4. Splittings of Higgs bundles

Assume again that Y is non-singular, that $U \subset Y$ the complement of a normal crossing divisor, and that the positivity conditions stated as Assumptions 0.1 hold true. In particular one has the decomposition (see (0.3))

$$\Omega_Y^1(\log S) = \Omega_1 \oplus \cdots \oplus \Omega_s$$

as a direct sum of μ-stable subsheaves Ω_i. Let us recall some of the numerical properties shown in [VZ07, Lemmata 1.6 and 1.9].

Lemma and Definition 4.1.

i. *The stable direct factors* Ω_i *and their determinants* $\det(\Omega_i)$ *are nef. The cycle* $c_1(\Omega_j)^{n_j+1}$ *is numerically trivial.*
ii. *For* $\nu_1, \ldots, \nu_s$ *with* $\nu_1 + \cdots + \nu_s = n$ *the product* $c_1(\Omega_1)^{\nu_1}. \cdots .c_1(\Omega_s)^{\nu_s}$ *is a positive multiple of* $c_1(\omega_Y(S))^n$*, if* $\nu_\iota = n_\iota$ *for* $\iota = 1, \ldots, s$*. Otherwise it is zero.*
iii. *In particular* $c_1(\Omega_1)^{n_1}. \cdots .c_1(\Omega_s)^{n_s} > 0$.
iv. *Let* D *be an effective* $\mathbb{Q}$ *divisor. Then* $D.c_1(\omega_Y(S))^{n-1} = 0$ *if and only if*
$$D.c_1(\Omega_1)^{\nu_1}. \cdots .c_1(\Omega_s)^{\nu_s} = 0$$
for all $\nu_1, \ldots, \nu_s$ *with* $\nu_1 + \cdots + \nu_s = n - 1$.
v. *Let* NS_0 *denote the subspace of the Neron-Severi group* $\mathrm{NS}(Y)_{\mathbb{Q}}$ *of* Y *which is generated by all effective divisors* D *satisfying the equivalent conditions in iv). If for some* $\alpha \in \mathbb{Q}$ *one has* $c_1(\Omega_i) - \alpha \cdot c_1(\Omega_j) \in \mathrm{NS}_0$ *then* $i = j$.

Using the notations from Section 3 consider $m = 1$ and the tuple $\underline{D}^{(1)}$ where all divisors are $D_j^{(1)} = K_Y + S$ for some canonical divisor K_Y. Then the slope $\mu_{\underline{D}^{(1)}}(\mathcal{F})$, considered there, is equal to $\mu(\mathcal{F})$. Using Lemma 4.1, the μ-equivalence, stated in Definition 3.2, can be made more precise:

Addendum 4.2.

vi. *Let* $\mathcal{G} \hookrightarrow \mathcal{F}$ *be an inclusion of* μ*-semistable sheaves of the same slope and rank. Then* $c_1(\mathcal{F}) - c_1(\mathcal{G})$ *lies in the subspace* NS_0 *defined in Lemma 4.1 v).*
vii. *Let* $\theta : \mathcal{G} \to \mathcal{F}$ *be a morphism of* μ*-semistable sheaves of the same slope, and let* $\mathrm{Im}'(\theta)$ *denote the saturated image, i.e. the kernel of*
$$\mathcal{F} \longrightarrow (\mathcal{F}/\mathrm{Im}(\theta))/_{\mathrm{torsion}}.$$
Then $\mathrm{Im}'(\theta)$ *is a semistable subsheaf of* $\mathcal{F}$ *of slope* μ_0.

In the sequel we consider again an irreducible polarized $\mathbb{C}$ variation of Hodge structures $\mathbb{V}$ of weight one with Higgs bundle
$$\big(E = E^{1,0} \oplus E^{0,1},\ \theta : E^{1,0} \to E^{0,1} \otimes \Omega^1_Y(\log S)\big).$$
We assume that $\mathbb{V}$ is non-unitary, hence that $\theta \neq 0$. By [VZ07, Proposition 2.4] one obtains as a corollary of Simpson's correspondence:

Lemma 4.3. *Let* $\underline{D}^{(\iota)}$ *be a finite system of* $n-1$*-tuples of nef* $\mathbb{R}$*-divisors. Let* (E, θ) *be the Higgs bundle of a complex polarized variation of Hodge structures. Let* $\mathcal{G}$ *be a Higgs subsheaf, i.e. a subsheaf with* $\theta(\mathcal{G}) \subset \mathcal{G} \otimes \Omega^1_Y(\log S)$*, and let* $\mathcal{Q}$ *be a torsion free quotient Higgs sheaf. Then:*

i. $\mu_{\underline{D}^{(\bullet)}}(\mathcal{G}) \leq 0$.
ii. $\mu_{\underline{D}^{(\bullet)}}(\mathcal{Q}) \geq 0$.
iii. *If for one* ι *and for all* j *the divisors* $\underline{D}_j^{(\iota))}$ *are ample with respect to* U*, then* $\mu_{\underline{D}^{(\bullet)}}(\mathcal{G}) = 0$ *if and only if the saturated hull of* $\mathcal{G}$ *is a direct factor of the Higgs bundle* E.

Let us write D_i for a divisor with $\mathcal{O}_Y(D_i) = \det(\Omega_i)$. For $\iota = 1, \ldots, s$ we choose $\underline{D}^{(\iota)}$ to be the tuple

$$(\overbrace{D_j, \ldots, D_j}^{n_j - 1}, \overbrace{D_1, \ldots, D_1}^{n_1}, \ldots \overbrace{D_{j-1}, \ldots, D_{j-1}}^{n_{j-1}}, \overbrace{D_{j+1}, \ldots, D_{j+1}}^{n_{j+1}}, \ldots, \overbrace{D_s, \ldots, D_s}^{n_s}).$$

For some binomial coefficients one can write

$$\mu(\mathcal{F}) = \sum_{\iota=1}^{s} \alpha_\iota \cdot \mu_{\underline{D}^{(\iota)}}(\mathcal{F}) = \sum_{\iota=1}^{s} \mu^{(\iota)}(\mathcal{F}),$$

where $\mu^{(\iota)} = \alpha_\iota \cdot \mu_{\underline{D}^{(\iota)}}$. Remark that $\mu^{(\iota)}(\Omega_i) \neq 0$ if and only if $\iota = i$.

Properties 4.4.

1. If $\mathbb{V}$ is irreducible and non-unitary, there exists some ι with $\mu^{(\iota)}(E^{1,0}) > 0$.
2. If $\mathcal{L}$ is an invertible sheaf, nef and big, then for all j one has $\mu^{(j)}(\mathcal{L}) > 0$.

Proof. For part 1) just remark that Lemma 4.3, iii) implies that $\mu(E^{1,0}) > 0$.

For 2) recall that for $\nu \gg 1$ the sheaf $\mathcal{L}^\nu \otimes \Omega_j^{-1}$ has a section with divisor B. Since the D_j are all nef, $\mu^{(j)}(\mathcal{O}_Y(B)) \geq 0$ and hence

$$\nu \cdot \mu^{(j)}(\mathcal{L}) \geq \mathrm{c}_1(\Omega_1)^{n_1} . \cdots . \mathrm{c}_1(\Omega_s)^{n_s} > 0.$$

□

Choose the tuples of $\mathbb{Q}$-divisors $\underline{H}^{(\bullet)}$ with $H_j^{(i)} = 0$ for $i = 1, \ldots, s$ and for $j = 1, \ldots, n-1$, except for $H_1^{(\iota)}$ which is chosen to be $\alpha_\iota \cdot D_\iota$. As in Section 3 we define for $\epsilon > 0$

$$\mu_\epsilon^{(\iota)}(\mathcal{F}) = \mu_{\underline{D}^{(\bullet)} + \epsilon \cdot \underline{H}^{(\bullet)}} = (1 + \epsilon) \cdot \mu^{(\iota)}(\mathcal{F}) + \sum_{j \neq \iota} \mu^{(j)}(\mathcal{F}) = \epsilon \cdot \mu^{(\iota)}(\mathcal{F}) + \mu(\mathcal{F}).$$

Remark that the limit of $\mu_\epsilon^{(\iota)}(\mathcal{F})$ for $\epsilon \to 0$ is $\mu(\mathcal{F})$.

For $s > 1$ none of the divisors $\underline{D}_j^{(\iota)}$ will be ample. So we are not allowed to apply part iii) of Lemma 4.3 to the slope $\mu^{(\iota)}$.

For a for a Higgs subbundle $\mathcal{G}$ of E the first part of Lemma 4.3 just implies that $\mu^{(\iota)}(\mathcal{G}) \leq 0$. For $\epsilon > 0$ the equality $\mu_\epsilon^{(\iota)}(\mathcal{G}) = 0$ can only hold if $\mu(\mathcal{G}) = \mu^{(\iota)}(\mathcal{G}) = 0$. This would imply that the saturated hull of $\mathcal{G}$ in E is a direct factor, contradicting the irreducibility of $\mathbb{V}$. So $\mathrm{rk}(G) < \mathrm{rk}(E)$ implies that $\mu_\epsilon^{(\iota)}(G) < 0$.

As we will show in Section 5.1 this remains true for the slopes $\mu^{(\iota)}$ if the universal covering $\tilde{U}$ is a bounded symmetric domain. Without this information, one just has the following criterion.

Proposition 4.5. *Let*

$$0 \longrightarrow \mathcal{K} \longrightarrow E^{1,0} \longrightarrow \mathcal{Q} \longrightarrow 0$$

be an exact sequence, and let $s : \mathcal{Q} \to E^{1,0}$ be the orthogonal complement of $\mathcal{K}$. Assume that for some ι the slope $\mu^{(\iota)}(\mathcal{Q}) = 0$. Then

a. *The composition*

$$\mathcal{Q} \xrightarrow{s} E^{1,0} \xrightarrow{\theta} E^{0,1} \otimes \Omega^1_Y(\log S) \xrightarrow{\mathrm{pr}_\iota} E^{0,1} \otimes \Omega_\iota$$

is zero.

b. *$s : \mathcal{Q} \to E^{1,0}$ is holomorphic in the direction Ω_ι.*

Remark that a priori s will be a C^∞ map. So part b) of the Proposition needs some explanation. Recall that we have the decomposition

$$\tilde{U} = M_1 \times \cdots \times M_s,$$

corresponding to the decomposition of $\Omega^1_Y(\log S)$ in stable direct factors. Write $n_0 = 0$ and again $n_i = \mathrm{rk}(\Omega_i) = \dim(M_i)$.

Given a point $y \in U$ let us choose a local coordinate system $z_1, \ldots, z_n$ in a neighborhood of y such that $\pi^*(z_{n_{i-1}+1}), \ldots, \pi^*(z_{n_i})$ are coordinates on M_i.

Definition 4.6. The inclusion $s : \mathcal{Q} \to E^{1,0}$ is holomorphic in the direction Ω_ι if its image is invariant under the action of $\partial/\partial\bar{z}_k$ on $E^{1,0}$ for $k = n_{\iota-1}+1, \ldots, n_\iota$.

Proof of Proposition 4.5. We assume $\iota = 1$. Locally, in some open set $W \subset U$ choose complex coordinates $z_1, \ldots, z_n$ as above and unitary frames of $E^{1,0}$ and $E^{0,1}$. That is, choose C^∞-sections $e_1, \ldots, e_\ell$ of $E^{1,0}$ and $f_1, \ldots, f_{\ell'}$ of $E^{0,1}$ orthogonal with respect to the scalar product $h(\cdot,\cdot)$ coming from the Hodge metric, and such that $e_1, \ldots, e_k$ generate $\mathcal{K}$ while $e_{k+1}, \ldots, e_\ell$ generate $s(Q)$. Write the Higgs field θ in these coordinates as

$$\theta(e_\alpha) = \sum_{i=1}^{n} \sum_{\beta=1}^{\ell'} \theta^i_{\alpha,\beta} f_\beta dz_i.$$

By [Gr70, Theorem 5.2] the curvature R of the metric connection ∇_h on $E^{1,0}$ is given by

$$R_{E^{1,0}} = \theta \wedge \theta^* = \sum_{i,j=1}^{n} (R_{E^{1,0}})^{i,j} dz_i \wedge \bar{z}_j,$$

$$\text{where} \quad (R_{E^{1,0}})^{i,j}_{\alpha,\beta} = \sum_{\gamma=1}^{\ell'} \theta^i_{\alpha,\gamma} \overline{\theta^i_{\beta,\gamma}}. \tag{4.1}$$

For the subbundle $\mathcal{K} \subset E^{1,0}$ the composition

$$b : \mathcal{K} \longrightarrow E^{1,0} \xrightarrow{\nabla_h} E^{1,0} \otimes \Omega^1_U \longrightarrow \mathcal{Q} \otimes \Omega^1_U$$

of the metric connection and the quotient is called second fundamental form. Taking complex conjugates we obtain a map

$$c : \bar{\mathcal{K}} \cong \mathcal{K}^\vee \longrightarrow \bar{\mathcal{Q}} \otimes \Omega^{0,1}_U \cong \mathcal{Q}^\vee \otimes \Omega^{0,1}_U.$$

Both maps are only C^∞. We write the map c in coordinates

$$c(e_\alpha) = \sum_{i=1}^{n} \sum_{\beta=1}^{k} c^i_{\alpha,\beta} e_\beta dz_i.$$

By [Gr70, Theorem 5.2] the curvature of the metric connection on $\mathcal{Q}$ is given by

$$R_{\mathcal{Q}} = (q\theta s) \wedge (q\theta s)^* + c \wedge c^* = \sum_{i,j=1}^{n} (R_{\mathcal{Q}})^{i,j} dz_i \wedge \bar{z}_j,$$

where

$$(R_{\mathcal{Q}})^{i,j}_{\alpha,\beta} = \sum_{\gamma=1}^{\ell'} \theta^i_{\alpha,\gamma} \overline{\theta^i_{\beta,\gamma}} + \sum_{\gamma=1}^{k} c^i_{\alpha,\gamma} \overline{c^i_{\beta,\gamma}}, \quad \text{for} \quad \alpha, \beta \in \{k+1, \ldots, \ell\}. \tag{4.2}$$

We conclude that for all i, the matrices $(R_{\mathcal{Q}})^{i,i}$ are positive semi-definite. Moreover their traces are zero if and only if $\theta^i_{\alpha,\beta} = 0$ and $c^i_{\alpha,\beta} = 0$ for all $\alpha, \beta \in \{k+1, \ldots, \ell\}$.

We write $R(\Omega_i)$ for the curvature of $\det(\Omega_i)$. By Lemma 4.1 ii) and after rescaling z_i by suitable constants we may assume that over W

$$R(\Omega_i) = dz_{n_{j-1}+1} \wedge d\bar{z}_{n_{j-1}+1} + \cdots + dz_{n_j} \wedge d\bar{z}_{n_j},$$

keeping the convention $n_0 = 0$. Then

$$R(\Omega_1)^{n_1-1} \wedge R(\Omega_2)^{n_2} \wedge \cdots \wedge R(\Omega_s)^{n_s} = \sum_{i=1}^{n_1} C_i \cdot \bigwedge_{j\neq i} dz_j \wedge d\bar{z}_j,$$

for some binomial coefficients $C_i > 0$. The hypothesis $\mu^{(1)}(\mathcal{Q}) = 0$ is equivalent to

$$0 = \Big(\frac{\sqrt{-1}}{2\pi}\Big) \cdot \int_U \mathrm{tr}(R_{\mathcal{Q}}) \wedge R(\Omega_1)^{n_1-1} \wedge R(\Omega_2)^{n_2} \wedge \cdots \wedge R(\Omega_s)^{n_s}.$$

Since $\mathrm{tr}(R_{\mathcal{Q}})$ and all the $R(\Omega_i)$ are positive semidefinite, the integral has to be zero on all open sets, in particular on W. We deduce

$$\begin{aligned} 0 &= \int_W \Big(\sum_{i,j=1}^{n} \mathrm{tr}(R_Q)^{i,j} dz_i \wedge d\bar{z}_j\Big) \wedge \Big(\sum_{i=1}^{n_1} C_i \cdot \bigwedge_{j\neq i} dz_j \wedge d\bar{z}_j\Big) \\ &= \int_W C_i \,\mathrm{tr}(R_{\mathcal{Q}})^{i,i} \bigwedge_{j=1}^{n} dz_j \wedge d\bar{z}_j. \end{aligned} \tag{4.3}$$

Hence $\mathrm{tr}(R_{\mathcal{Q}})^{i,i} = 0$ for all i and we obtain the vanishing on U of the composition $s \circ \theta \circ \mathrm{pr}_\iota$ as claimed in a) and of all $c^i_{\alpha,\beta}$. Since the $(0,1)$-part of the metric connection ∇_h is $\bar{\partial}$, the vanishing of $c^i_{\alpha,\beta}$ is what is claimed in b). Both vanishing statements extend to the whole of Y by continuity. □

5. Purity of Higgs bundles with Arakelov equality

In this section we will prove Theorem 0.5. Keeping the assumptions from Section 4 we will assume in addition that $\mathbb{V}$ satisfies the Arakelov equality

$$\mu(\mathbb{V}) = \mu(E^{1,0}) - \mu(E^{0,1}) = \mu(\Omega^1_Y(\log S)).$$

In particular we will assume that $\mathbb{V}$ is non-unitary. By [VZ07, Theorem 1] we know that $E^{1,0}$ and $E^{0,1}$ are both μ-semistable. By Corollary 3.8 one finds a weak μ-Jordan-Hölder filtration $\mathcal{G}^{(\iota)}_\bullet$ of $E^{1,0}$ and some ϵ_0 such that $\mathcal{G}^{(\iota)}_\bullet$ is a $\mu^{(\iota)}_\epsilon$-Harder-Narasimhan filtration of $E^{1,0}$ for all $\epsilon \in (0, \epsilon_0]$. Of course we may choose ϵ_0 to be independent of ι. Recall that the sheaves $\mathcal{G}^{(\iota)}_i/\mathcal{G}^{(\iota)}_{i-1}$ are μ-semistable and $\mu^{(\iota)}_\epsilon$-polystable. However the sheaves $\mathcal{G}^{(\iota)}_i/\mathcal{G}^{(\iota)}_{i-1}$ are not necessarily $\mu^{(\iota)}$-semistable.

Lemma 5.1. *Let $\mathcal{F}$ be a μ-stable subsheaf of $E^{1,0}$. Then $\mathcal{F}$ is pure of type ι for some $\iota \in \{1, \ldots, s\}$. Moreover, each subsheaf $\mathcal{F}'$ of $E^{1,0}$ which is isomorphic to $\mathcal{F}$ is pure of the same type ι.*

Recall from Definition 0.4 that $\mathcal{F}$ is pure of type ι if the restriction $\theta|_{\mathcal{F}}$ of the Higgs field factors like

$$\mathcal{F} \xrightarrow{\theta_\iota} \mathcal{F} \otimes \Omega_\iota \xrightarrow{\subset} \mathcal{F} \otimes \Omega^1_Y(\log S).$$

Equivalently, writing T_i for the dual of Ω_i and $\theta_i^\vee$ for the the composite

$$E^{1,0} \otimes T_i \xrightarrow{\theta\otimes \mathrm{id}_{T_i}} E^{0,1} \otimes \Omega_i \otimes T_i \xrightarrow{\text{contraction}} E^{0,1},$$

one requires $\theta_i^\vee(\mathcal{F} \otimes T_i)$ to be zero for $i \neq \iota$. Since $\mathbb{V}$ is non-unitary this is only possible if $\theta_\iota^\vee(\mathcal{F} \otimes T_\iota) \neq 0$.

Proof of Lemma 5.1. Assume that $\mathcal{F}' \cong \mathcal{F}$ and that for some $i \neq i'$ one has

$$\theta_i^\vee(\mathcal{F} \otimes T_i) \neq 0 \quad \text{and} \quad \theta_{i'}^\vee(\mathcal{F}' \otimes T_{i'}) \neq 0.$$

We will write $\mathcal{B}_i$ and $\mathcal{B}_{i'}$ for the saturated hull of those images. The Arakelov equality implies that $\theta_i^\vee$ and $\theta_{i'}^\vee$ are morphisms between μ-semistable sheaves of the same slope, hence $\mu(\mathcal{B}_\iota) = \mu(\mathcal{F}) + \mu(T_\iota)$ for $\iota = i,\ i'$.

The sheaves $\mathcal{F}$ and T_ι are μ-stable. By Lemma 3.6 for $\epsilon > 0$, sufficiently small, and for all j the sheaves $\mathcal{F}$ and T_ι are $\mu_\epsilon^{(j)}$-semistable. Hence $\mathcal{F} \otimes T_\iota$ is $\mu_\epsilon^{(j)}$-semistable, and consequently,

$$\mu_\epsilon^{(j)}(\mathcal{B}_\iota) \geq \mu_\epsilon^{(j)}(\mathcal{F}) + \mu_\epsilon^{(j)}(T_\iota) \quad \text{and} \quad \mu^{(j)}(\mathcal{B}_\iota) \geq \mu^{(j)}(\mathcal{F}) + \mu^{(j)}(T_\iota).$$

For $\iota = i$ and $j \neq i$ one obtains

$$0 \geq \mu^{(j)}(\mathcal{B}_i) \geq \mu^{(j)}(\mathcal{F}) + \mu^{(j)}(T_1) = \mu^{(j)}(\mathcal{F}),$$

and for $\iota = i' \neq k$

$$0 \geq \mu^{(k)}(\mathcal{B}_{i'}) \geq \mu^{(k)}(\mathcal{F}) + \mu^{(k)}(T_2) = \mu^{(k)}(\mathcal{F}') = \mu^{(k)}(\mathcal{F}).$$

Then $i \neq i'$ implies that $\mu^{(j)}(\mathcal{F}) \leq 0$ for all j, hence $\mu(\mathcal{F}) \leq 0$. Since $\mathbb{V}$ is non-unitary and since $\mu(\mathcal{F}) = \mu(E^{1,0})$ this contradicts part iii) of Lemma 4.3. □

Let us define

$$\mathcal{K}^{(\iota)} = \mathrm{Ker}\big(E^{1,0} \longrightarrow E^{0,1} \otimes \Omega^1_Y(\log S) \longrightarrow E^{0,1} \otimes \bigoplus_{j\neq\iota} \Omega_j\big).$$

Corollary 5.2. *There exists some ι with $\mathcal{K}^{(\iota)} \neq 0$.*

Proof. Choose any μ-stable subsheaf $\mathcal{F}$. Then by Lemma 5.1 the bundle $\mathcal{F}$ is contained in $\mathcal{K}^{(\iota)}$ for some ι. □

Lemma 5.3. *Assume that $E^{1,0} = \mathcal{K}^{(\iota)}$ for some ι. Then for all $j \neq \iota$ one has $\mu^{(j)}(E^{1,0}) = 0$.*

Proof. If $E^{1,0} = \mathcal{K}^{(\iota)}$, the saturated image $\mathcal{B}_\iota$ of

$$\theta_\iota^\vee : E^{1,0} \otimes T_\iota \longrightarrow E^{0,1}$$

has to be non-zero. $\theta_\iota^\vee$ is a map of μ-semistable sheaves of the same slope, hence $\mu(\mathcal{B}_\iota) = \mu_\epsilon(E^{1,0}) - \mu_\epsilon(\Omega_\iota)$. For ϵ sufficiently small $E^{1,0} \otimes T_\iota$ is $\mu_\epsilon^{(j)]}$-semistable, and

$$\mu_\epsilon^{(j)}(\mathcal{B}_\iota) \geq \mu_\epsilon^{(j)]}(E^{1,0}) - \mu_\epsilon^{(j)]}(\Omega_\iota).$$

Then for $j \neq \iota$ one finds

$$\mu^{(j)}(\mathcal{B}_\iota) \geq \mu^{(j)]}(E^{1,0}) - \mu^{(j)]}(\Omega_\iota) = \mu^{(j)]}(E^{1,0}),$$

which by Lemma 4.3 can neither be positive, nor negative, hence it must be zero. □

A similar argument will be used to obtain a stronger statement, which finally will lead to a contradiction, except if $E^{1,0} = \mathcal{K}^{(\iota)}$ for some ι.

Lemma 5.4. *Let ℓ be the length of the filtration $\mathcal{G}_\bullet^{(\iota)}$.*

a. *Then $\mathcal{G}_{\ell-1}^{(\iota)} \subset \mathcal{K}^{(\iota)}$.*

b. *If $\mathcal{K}^{(\iota)} \neq E^{1,0}$ then $\mu^{(\iota)}(\mathcal{G}_\ell^{(\iota)}/\mathcal{G}_{\ell-1}^{(\iota)}) = 0$.*

Proof. Assume we know for some ν that $\mathcal{G}_{\nu-1}^{(\iota)} \subset \mathcal{K}^{(\iota)}$, hence that $\theta_j^\vee$ factors like

$$\theta_j^\vee : E^{1,0} \otimes T_j \longrightarrow \mathcal{G}_\nu^{(\iota)}/\mathcal{G}_{\nu-1}^{(\iota)} \otimes T_j \longrightarrow E^{0,1}, \tag{5.1}$$

for $j \neq \iota$. We write $\mathcal{B}_j$ for the saturated image of $\theta_j^\vee$. Since $\mathcal{G}_\bullet^{(\iota)}$ is a weak μ-Jordan Hölder filtration, the second morphism in (5.1) is a morphism between μ-semistable sheaves of the same slope. So $\mathcal{B}_j \neq 0$ implies that

$$\mu(\mathcal{G}_\nu^{(\iota)}/\mathcal{G}_{\nu-1}^{(\iota)} \otimes T_j) = \mu(E^{1,0}) + \mu(T_j) = \mu(E^{0,1}) = \mu(\mathcal{B}_j),$$

and by Lemma 4.3 that $\mu^{(\iota)}(\mathcal{B}_j) \leq 0$. Since $j \neq \iota$ we know that

$$\mu^{(\iota)}(\mathcal{G}_\nu^{(\iota)}/\mathcal{G}_{\nu-1}^{(\iota)} \otimes T_j) = \mu^{(\iota)}(\mathcal{G}_\nu^{(\iota)}/\mathcal{G}_{\nu-1}^{(\iota)}), \quad \text{and hence}$$
$$\mu_\epsilon^{(\iota)}(\mathcal{G}_\nu^{(\iota)}/\mathcal{G}_{\nu-1}^{(\iota)} \otimes T_j)) = \epsilon \cdot \mu^{(\iota)}(\mathcal{G}_\nu^{(\iota)}/\mathcal{G}_{\nu-1}^{(\iota)}) + \mu(\mathcal{B}_j).$$

For $0 < \epsilon \leq \epsilon_0$ the sheaf $\mathcal{G}_\nu^{(\iota)}/\mathcal{G}_{\nu-1}^{(\iota)} \otimes T_j$ is $\mu_\epsilon^{(\iota)}$-semistable. Then

$$\epsilon \cdot \mu^{(\iota)}(\mathcal{B}_j) + \mu(\mathcal{B}_j) = \mu_\epsilon^{(\iota)}(\mathcal{B}_j) \geq \epsilon \cdot \mu^{(\iota)}(\mathcal{G}_\nu^{(\iota)}/\mathcal{G}_{\nu-1}^{(\iota)}) + \mu(\mathcal{B}_j), \quad \text{and hence}$$
$$\mu^{(\iota)}(\mathcal{G}_\nu^{(\iota)}/\mathcal{G}_{\nu-1}^{(\iota)}) \leq 0. \tag{5.2}$$

The slope $\mu(\mathcal{G}_\nu^{(\iota)}/\mathcal{G}_{\nu-1}^{(\iota)})$ is independent of ν. If $\nu < \ell$ the choice of $\mathcal{G}_\bullet^{(\iota)}$ implies that

$$\mu_\epsilon^{(\iota)}(\mathcal{G}_\ell^{(\iota)}/\mathcal{G}_{\ell-1}^{(\iota)}) < \mu_\epsilon^{(\iota)}(\mathcal{G}_\nu^{(\iota)}/\mathcal{G}_{\nu-1}^{(\iota)}),$$

and hence that $\mu^{(\iota)}(\mathcal{G}_\ell^{(\iota)}/\mathcal{G}_{\ell-1}^{(\iota)}) < \mu^{(\iota)}(\mathcal{G}_\nu^{(\iota)}/\mathcal{G}_{\nu-1}^{(\iota)})$. So $\mu^{(\iota)}(\mathcal{G}_\ell^{(\iota)}/\mathcal{G}_{\ell-1}^{(\iota)}) < 0$ contradicting Lemma 4.3, ii). So $\theta_j^\vee$ must be zero for all $j \neq \iota$ and $\mathcal{G}_\nu^{(\iota)} \subset \mathcal{K}^{(\iota)}$. By induction one obtains part a) of 5.4.

For b) remark that $\mathcal{K}^{(\iota)} \neq E^{1,0}$ implies that for some $j \neq \iota$ the morphism $\theta_j^\vee$ is non-zero. By part a) one has the factorization (5.1) for $\nu = \ell$. Since $\mathcal{B}_j \neq 0$ the inequality (5.2) together with Lemma 4.3 imply that $\mu^{(\iota)}(\mathcal{G}_\ell^{(\iota)}/\mathcal{G}_{\ell-1}^{(\iota)}) = 0$. □

Corollary 5.5. *If in Lemma 5.4 the sheaf $\mathcal{Q} = E^{1,0}/\mathcal{K}^{(\iota)}$ is non-zero, it is μ and $\mu_\epsilon^{(\iota)}$-semistable. One has*

$$\mu_\epsilon^{(\iota)}(\mathcal{Q}) = \mu(\mathcal{Q}) = \mu(E^{1,0}) = \mu(E^{0,1}) + \mu(\Omega_Y^1(\log S)),$$

and hence $\mu^{(\iota)}(\mathcal{Q}) = 0$.

Proof. Since $\mathcal{K}^{(\iota)}$ as the kernel of a morphism between μ-semistable sheaves of the same slope is μ-semistable, $\mathcal{Q}$ has the same property.

By Lemma 5.4, b) the slope $\mu^{(\iota)}(\mathcal{G}_\ell^{(\iota)}/\mathcal{G}_{\ell-1}^{(\iota)}) = 0$. Since $\mathcal{G}_\bullet^{(\iota)}$ is a weak μ-Jordan-Hölder filtration and a $\mu_\epsilon^{(\iota)}$-Harder-Narasimhan filtration, for all $0 \leq \epsilon \leq \epsilon_0$ the quotient $\mathcal{G}_\ell^{(\iota)}/\mathcal{G}_{\ell-1}^{(\iota)}$ is $\mu_\epsilon^{(\iota)}$-semistable and has slope $\mu(\mathcal{G}_\ell^{(\iota)}/\mathcal{G}_{\ell-1}^{(\iota)}) = \mu(E^{1,0})$. For $j \neq \iota$ the sheaf Ω_j is $\mu_\epsilon^{(\iota)}$-stable and has slope $\mu(\Omega_j) = \mu(\Omega_Y^1(\log S))$, hence

$$\mathcal{G}_\ell^{(\iota)}/\mathcal{G}_{\ell-1}^{(\iota)} \otimes \bigoplus_{j \neq \iota} T_j$$

is again $\mu_\epsilon^{(\iota)}$-semistable of slope $\mu(E^{0,1})$.

Let $\mathcal{B}$ be the saturated image of $\mathcal{G}_\ell^{(\iota)}/\mathcal{G}_{\ell-1}^{(\iota)}\otimes\bigoplus_{j\neq\iota}T_j$ in $E^{0,1}$. Then $\mu(\mathcal{B})=\mu(E^{0,1})$ and $\mu_\epsilon^{(\iota)}(\mathcal{B})\geq\mu(E^{0,1})$.

On the other hand Lemma 4.3 implies that $\mu^{(\iota)}(\mathcal{B})\leq 0$, hence $\mu_\epsilon^{(\iota)}(\mathcal{B})=\mu(E^{0,1})$. So $\mathcal{B}$ as a quotient of a semistable sheaf of the same slope has to be $\mu_\epsilon^{(\iota)}$-semistable of slope

$$\mu(E^{0,1})=\mu_\epsilon^{(\iota)}\big(\mathcal{G}_\ell^{(\iota)}/\mathcal{G}_{\ell-1}^{(\iota)}\otimes\bigoplus_{j\neq\iota}T_j\big).$$

Since $\mathcal{Q}=E^{1,0}/\mathcal{K}^{(\iota)}$ is a subsheaf of $\mathcal{B}\otimes\bigoplus_{j\neq\iota}\Omega_j$ one finds

$$\mu_\epsilon^{(\iota)}(\mathcal{Q})\leq\mu_\epsilon^{(\iota)}\big(\mathcal{B}\otimes\bigoplus_{j\neq\iota}\Omega_j\big)=\mu(E^{0,1}),$$

and since it is a quotient of $\mathcal{G}_\ell^{(\iota)}/\mathcal{G}_{\ell-1}^{(\iota)}$ one has $\mu_\epsilon^{(\iota)}(\mathcal{Q})\geq\mu(E^{0,1})$. One obtains the equality of slopes in Corollary 5.5. Finally $\mathcal{Q}$ as a subsheaf of a $\mu_\epsilon^{(\iota)}$-semistable sheaf of the same slope is itself $\mu_\epsilon^{(\iota)}$-semistable. □

Proof of Theorem 0.5. Renumbering the factors we will assume that $\mathcal{K}^{(1)}\neq 0$, and we will write

$$\Omega=\bigoplus_{j=2}^{s}\Omega_j \quad\text{and}\quad T=\Omega^\vee.$$

So $\mathcal{K}^{(1)}$ is the kernel of the composition

$$E^{1,0}\longrightarrow E^{0,1}\otimes\Omega^1_Y(\log S)\xrightarrow{\text{pr}}E^{0,1}\otimes\Omega.$$

Let $\mathcal{K}_1$ be the kernel of

$$E^{1,0}\longrightarrow E^{0,1}\otimes\Omega^1_Y(\log S)\xrightarrow{\text{pr}_1}E^{0,1}\otimes\Omega_1.$$

Claim 5.6. $E^{1,0}$ is the direct sum $\mathcal{K}^{(1)}\oplus\mathcal{K}_1$.

Proof. By Corollary 5.5 the sheaf $\mathcal{Q}=E^{1,0}/\mathcal{K}^{(1)}$ satisfies $\mu^{(1)}(\mathcal{Q})=0$. So Proposition 4.5, a), tells us that the orthogonal complement $s(\mathcal{Q})$ is contained in $\mathcal{K}_1$.

The intersection of $\mathcal{K}^{(1)}$ and $\mathcal{K}_1$ lies in the kernel of θ. Hence it is zero and the induced map $\mathcal{K}_1\to\mathcal{Q}$ is injective. On the other hand

$$\mathcal{Q}\xrightarrow{s}E^{1,0}\longrightarrow\mathcal{Q}$$

factors through $\mathcal{K}_1\to\mathcal{Q}$, and the latter must be surjective. This implies that

$$E^{1,0}=\mathcal{K}^{(1)}\oplus\mathcal{K}_1.$$

□

Let $\mathcal{B}^{(1)}$ and $\mathcal{B}_1$ be the saturated images of

$$E^{1,0}\otimes T\longrightarrow E^{0,1}\quad\text{and}\quad E^{1,0}\otimes T_1\longrightarrow E^{0,1},$$

respectively.

Claim 5.7. $\mathcal{B}^{(1)}\cap\mathcal{B}_1=0$.

Proof. Dualizing the exact sequences

$$0 \longrightarrow \mathcal{B}^{(1)} \longrightarrow E^{0,1} \longrightarrow \mathcal{C}^{(1)} = E^{0,1}/\mathcal{B}^{(1)} \longrightarrow 0$$
$$\text{and} \quad 0 \longrightarrow \mathcal{B}_1 \longrightarrow E^{0,1} \longrightarrow \mathcal{C}_1 = E^{0,1}/\mathcal{B}_1 \longrightarrow 0$$

one obtains that

$$\mathcal{C}^{(1)\vee} = \mathrm{Ker}(E^{0,1\vee} \xrightarrow{\tau} \mathcal{B}^{(1)\vee}) \quad \text{and} \quad \mathcal{C}_1^\vee = \mathrm{Ker}(E^{0,1\vee} \xrightarrow{\tau_1} \mathcal{B}_1^\vee).$$

The dual Higgs bundle $E^\vee$ has $E^{0,1\vee}$ as subsheaf of bidegree $(1,0)$ and $E^{0,1\vee}$ is of bidegree $(0,1)$. The composite

$$E^{0,1\vee} \xrightarrow{\tau} \mathcal{B}^{(1)\vee} \xrightarrow{\subset} E^{1,0\vee} \otimes \Omega \quad \text{and} \quad E^{0,1\vee} \xrightarrow{\tau_1} \mathcal{B}_1^\vee \xrightarrow{\subset} E^{1,0\vee} \otimes \Omega_1$$

are the components of the dual Higgs field.

Applying Claim 5.6 to $E^\vee$ one obtains a decomposition $E^{0,1\vee} = \mathcal{C}^{(1)\vee} \oplus \mathcal{C}_1^\vee$, hence $E^{0,1} \cong \mathcal{C}^{(1)} \oplus \mathcal{C}_1$ and $\mathcal{B}^{(1)} \cap \mathcal{B}_1 = 0$. □

So one obtains a direct sum decomposition of Higgs bundles

$$(E,\theta) = \big(\mathcal{K}^{(1)} \oplus \mathcal{B}^{(1)}, \theta^{(1)} = \theta|_{\mathcal{K}^{(1)}}\big) \oplus \big(\mathcal{K}_1 \oplus \mathcal{B}_1, \theta_1 = \theta|_{\mathcal{K}_1}\big)$$

corresponding to a decomposition $\mathbb{V} = \mathbb{V}^{(1)} \oplus \mathbb{V}_1$. The irreducibility of $\mathbb{V}$ and the assumption $\mathcal{K}^{(1)} \neq 0$ imply $\mathbb{V}_1 = 0$, hence $\mathcal{K}_1 = 0$. □

5.1. **Using superrigidity.** As mentioned in the introduction, the purity of the Higgs fields in Theorem 0.5 follows from the Margulis Superrigidity Theorem, without using the Arakelov equality, provided all the direct stable factors of $\Omega^1_Y(\log S)$ are of type C. We will show below, that for variations of Hodge structures of weight one it is sufficient to assume that the universal covering $\tilde{U}$ of U is a bounded symmetric domain. In different terms, if Ω_i is of type B we suppose that it satisfies the Yau-equality

$$2(n_i+1)\cdot \mathrm{c}_2(\Omega_i).\mathrm{c}_1(\Omega_i)^{n_i-2}.\mathrm{c}_1(\omega_Y(S))^{n-n_i}$$
$$= n_i \cdot \mathrm{c}_1(\Omega_i)^{n_i}.\mathrm{c}_1(\omega_Y(S))^{n-n_i}$$

([Ya93], see also [VZ07, Theorem 1.4]).

Proposition 5.8. *Suppose that $\tilde{U}$ is a bounded symmetric domain and that $\mathbb{V}$ is an irreducible complex polarized variation of Hodge structures of weight 1. Then the associated Higgs bundle $(E^{1,0} \oplus E^{0,1}, \theta)$ is pure of type ι for some $\iota \in \{1, \ldots, s\}$.*

Sketch of the proof. By [Ya93] the assumption implies that $U = \Gamma\backslash\tilde{U}$ is the quotient of a bounded symmetric domain by a lattice Γ. We can write $M_i = G_i/K_i$ as quotient of a real, non-compact, simple Lie group by a maximal compact subgroup.

Assume first that $U = U_1 \times U_2$. By [VZ05, Proposition 3.3] an irreducible local system on $\mathbb{V}$ is of the form $\mathrm{pr}_1^*\mathbb{V}_1 \otimes \mathrm{pr}_2^*\mathbb{V}_2$, for irreducible local systems $\mathbb{V}_i$ on U_i with Higgs bundles (E_i, θ_i). Since $\mathbb{V}$ is a variation of Hodge structures of weight one, one of those, say $\mathbb{V}_2$ has to have weight zero, hence it must be unitary.

Then the Higgs field on U factors through $E^{0,1} \otimes \Omega^1_{U_1}$. By induction on the dimension we may assume that $\mathbb{V}_1$ is pure of type ι for some ι with M_ι a factor of $\tilde{U}_1$. So the same holds true for $\mathbb{V}$.

Hence we may assume that U is irreducible, or even that

(5.3) no finite étale covering of U is a product of proper subvarieties.

By [Zi84] § 2.2, passing to a finite unramified cover of U there is a partition of $\{1,\ldots,s\}$ into subsets I_k such that $\Gamma=\prod_k \Gamma_k$ and Γ_k is an irreducible lattice in $\prod_{i\in I_k} G_i$. Here irreducible means that for any $N\subset \prod_{i\in I_k} G_i$ a normal subgroup, Γ_k is dense in $\prod_{i\in I_k} G_i/N$. The condition (5.3) is equivalent to the irreducibility of Γ, so $I_1=\{1,\ldots,s\}$.

If $s=1$ or if $\mathbb{V}$ is unitary, the statement of the proposition is trivial. Otherwise, $G:=\prod_{i=1}^s G_i$ is of real rank ≥ 2 and the conditions of Margulis' superrigidity theorem (e.g. [Zi84, Theorem 5.1.2 ii)]) are met. As consequence, the homomorphism $\Gamma\to \mathrm{Sp}(V,Q)$, where V is a fibre of $\mathbb{V}$ and is Q be the symplectic form on V, factors through a representation $\rho: G\to \mathrm{Sp}(V,Q)$. Since the G_i are simple, we can repeat the argument used in the proof of [VZ05, Proposition 3.3] in the product case: ρ is a tensor product of representations, all of which but one have weight 0. □

Corollary 5.9. *Under the assumptions made in Proposition 5.8 let $\mathcal{Q}\neq 0$ be a quotient of $E^{1,0}$ with $\mu^{(i)}(\mathcal{Q})=0$, for some $i\in\{1,\ldots,s\}$. Then $\mathcal{Q}=E^{1,0}$.*

Proof. By Proposition 5.8 $\mathbb{V}$ is pure of type ι for some ι. On the other hand Proposition 4.5 implies that the orthogonal complement of $\mathcal{Q}$ lies in the kernel of the composite

$$E^{1,0}\xrightarrow{\theta} E^{0,1}\otimes\Omega^1_Y(\log S)\xrightarrow{\mathrm{pr}_i} E^{0,1}\otimes\Omega_i.$$

Since θ is injective and factors through $E^{0,1}\otimes\Omega_\iota$ this implies that $i=\iota$ and we assume that both are 1.

Now one argues as in the proof of Proposition 4.5. The metric connection ∇_h is zero in directions not contained in M_1, hence in the equation (4.1) one finds that $(R_{E^{1,0}})^{i,j}=0$ as soon as $i>n_1$ or $j>n_1$. Similarly $c^i_{\alpha,\beta}=0$ for $i>n_1$, and hence the equation (4.2) implies that $(R_{\mathcal{Q}})^{i,j}_{\alpha,\beta}=0$ for $i>n_1$ or $j>n_1$. One has again

$$\mu^{(j)}(\mathcal{Q})=\Big(\frac{\sqrt{-1}}{2\pi}\Big)\cdot\int_U \mathrm{tr}(R_{\mathcal{Q}})\wedge R(\Omega_1)^{n_j-1}\wedge R(\Omega_1)^{n_1}\wedge\cdots \wedge R(\Omega_{j-1})^{n_{j-1}}\wedge R(\Omega_{j+1})^{n_{j+1}}\wedge\cdots\wedge R(\Omega_s)^{n_s}.$$

As in equation (4.3) this is the same as

$$\int_W\Big(\sum_{i,j=1}^{n}\mathrm{tr}(R_Q)^{i,j}dz_i\wedge d\bar z_j\Big)\wedge\Big(\sum_{i=n_{j-1}+1}^{n_j} C_i\cdot\bigwedge_{j\neq i} dz_j\wedge d\bar z_j\Big).$$

For $j>1$ this is zero, hence $\mu(\mathcal{Q})=0$ and one can apply Lemma 4.3. □

6. Stability of Higgs bundles, lengths of iterated Higgs fields and splitting of the tangent map

In this section we prove Theorem 0.8, the numerical characterization of Shimura varieties, and we prove the equivalent numerical and geometrical characterizations of ball quotients stated as Addendum 0.10. Moreover, we recall the proof of Corollary 0.6, essentially contained in [VZ07, Section 5].

So we consider again an irreducible complex polarized variation of Hodge structures $\mathbb{V}$ on U, satisfying the Arakelov equality, and of course with unipotent local monodromy operators.

By Theorem 0.5, the logarithmic Higgs bundle $(E = E^{1,0} \oplus E^{0,1}, \theta)$ of $\mathbb{V}$ is pure of type ι, i.e. the Higgs field factors through $E^{0,1} \otimes \Omega_\iota$. We write $\ell = \mathrm{rk}(E^{1,0})$ and $\ell' = \mathrm{rk}(E^{0,1})$, and n denotes $\mathrm{rk}(\Omega_\iota) = \dim(M)$. The Arakelov equality says that

$$\mu(E^{1,0}) - \mu(E^{0,1}) = \mu(\Omega^1_Y(\log S)) = \mu(\Omega_\iota).$$

Moreover $\mathrm{c}_1(E^{1,0}) + \mathrm{c}_1(E^{0,1}) = 0$ implies that

$$\ell \cdot \mu(E^{1,0}) + \ell' \cdot \mu(E^{0,1}) = 0,$$

hence the Arakelov equality can be restated as

$$\frac{\ell + \ell'}{\ell'} \cdot \mu(E^{1,0}) = \mu(\Omega_\iota). \tag{6.1}$$

Let us return to the Higgs bundle $\bigwedge^\ell(E, \theta)$ introduced in (0.2) and to the Higgs subbundle $\langle \det(E^{1,0}) \rangle$ generated by $\det(E^{1,0})$

By abuse of notations, we will write $\langle \det(E^{1,0}) \rangle$ for the saturated sub Higgs bundle of $\bigwedge^\ell(E, \theta)$. So the sheaf $\langle \det(E^{1,0}) \rangle^{\ell-m,m}$ is the saturated hull of the image of the induced map

$$\theta^{(m)^\vee} : \det(E^{1,0}) \otimes S^m(T) \longrightarrow E^{\ell-m,m} = \bigwedge^{\ell-m}(E^{1,0}) \otimes \bigwedge^{m} E^{0,1}.$$

Choosing for $\langle \det(E^{1,0}) \rangle^{\ell-m,m}$ the saturated images does not change the length

$$\varsigma(E) = \varsigma((E, \theta)) = \mathrm{Max}\{ \ m \in \mathbb{N}; \ \langle \det(E^{1,0}) \rangle^{\ell-m,m} \neq 0\}.$$

So the next Lemma implies Corollary 0.6.

Lemma 6.1. *Assume that Ω_ι is of type A or B, hence that $S^m(\Omega_\iota)$ is stable for all m. Then the Arakelov equality implies that*

$$\varsigma(E) \geq \frac{\ell \cdot \ell' \cdot (n+1)}{(\ell + \ell') \cdot n}. \tag{6.2}$$

If (6.2) is an equality, $\langle \det(E^{1,0}) \rangle$ is a direct factor of (F, τ).

Proof. For $0 \leq m \leq \varsigma = \varsigma(E)$ the sheaf $\langle \det(E^{1,0}) \rangle^{\ell-m,m}$ is a stable sheaf of slope

$$(\ell - m) \cdot \mu(E^{1,0}) + m \cdot \mu(E^{0,1}) = \ell \cdot \mu(E^{1,0}) - m \cdot \mu(\Omega^1_Y(\log S)) = \Big(\frac{\ell \cdot \ell'}{\ell + \ell'} - m\Big) \cdot \mu(\Omega^1_Y(\log S)),$$

and of rank $\binom{n+m-1}{m}$. The degree of this sheaf with respect to the polarization $\omega_Y(S)$ is non-positive, hence

$$0 \geq \frac{\mu(\langle \det(E^{1,0}) \rangle)}{\mu(\Omega^1_Y(\log S))} = \sum_{m=0}^{\varsigma} \binom{n+m-1}{m} \cdot \Big(\frac{\ell \cdot \ell'}{\ell + \ell'} - m\Big) = \left(\frac{\ell \cdot \ell'}{n \cdot (\ell + \ell')} - \frac{\varsigma}{n+1}\right) \cdot (\varsigma + 1) \cdot \binom{\varsigma + n}{\varsigma + 1}, \tag{6.3}$$

and one obtains the inequality stated in (6.2). If this is an equality, (6.3) is an equality, and the Higgs bundle $\langle \det(E^{1,0}) \rangle$ is a direct factor of (F, τ). □

We now distinguish according to the coarse type of the bounded symmetric domain attached to Ω_ι.

6.1. **Type A: Ω_ι is invertible.** This case is easy to understand [VZ07]. Let us recall the arguments. We have $n = 1$ and we may assume that $\ell \leq \ell'$. The Higgs field is an injective map

$$E^{1,0} \longrightarrow E^{0,1} \otimes \Omega_\iota, \tag{6.4}$$

and it must be surjective on some open dense subscheme. So $\ell = \ell'$ and the inequality (6.2) says that $\varsigma((E,\theta)) \geq \ell$. On the other hand, $\varsigma((E,\theta))$ can not be larger than the weight of $\bigwedge^\ell \mathbb{V}$, so it is equal to ℓ.

Both sides in (6.4) are semistable of the same slope, and they are equivalent. A stable subsheaf $\mathcal{F}$ of $E^{1,0}$ of slope $\mu(E^{1,0})$ generates a Higgs subbundle $\mathcal{F} \oplus \mathcal{F} \otimes \Omega_\iota$, whose first Chern class is zero. So the irreducibility implies that $\mathcal{F} = E^{1,0}$ and we can state:

Proposition 6.2. *If Ω_ι is invertible, then the Arakelov equality (6.1) implies that $E^{1,0}$ and $E^{0,1}$ are both stable of the same rank, that $\varsigma((E,\theta)) = \ell$ and that $\langle \det(E^{1,0}) \rangle$ is a direct factor of $\bigwedge^\ell (E,\theta)$.*

6.2. **Type B: $S^m(\Omega_\iota)$ stable.** Assume next that Ω_ι is of type B, i.e. that Ω_ι is not invertible but $S^m(\Omega_\iota)$ is stable for all m. Here the Arakelov equality just implies that certain numerical and stability conditions are equivalent.

Proposition 6.3. *Let $\mathbb{V}$ be an irreducible complex polarized variation of Hodge structures of weight one, pure of type A or B, and with Higgs bundle (E,θ). Consider the following conditions:*

a. *$E^{1,0}$ and $E^{0,1}$ are μ-stable.*
b. *$E^{1,0\vee} \otimes E^{0,1}$ is μ-polystable.*
c. *The saturated image of $T_\iota \to \mathcal{H}om(E^{1,0}, E^{0,1})$ is a direct factor of the sheaf $\mathcal{H}om(E^{1,0}, E^{0,1})$.*
d. *The Higgs bundle $\langle \det(E^{1,0}) \rangle$ is a direct factor of the Higgs bundle $\bigwedge^\ell (E,\theta)$.*
e. *$\mu(\langle \det(E^{1,0}) \rangle) = 0$.*
f. *$\varsigma((E,\theta)) = \frac{\ell \cdot \ell' \cdot (n+1)}{(\ell+\ell') \cdot n}$.*

Then the Arakelov equality for $\mathbb{V}$ implies:

i. *The conditions c), d), e), and f) are equivalent and they imply that M_i is a complex ball of dimension n.*
ii. *The condition b) implies c).*
iii. *Whenever Property 0.11 is satisfied, for example if U is projective or of dimension one, a) implies b).*

If $\mathbb{V}$ is pure of type A, we know that the conditions a), d), and f) automatically hold true. Nevertheless we allow this case, since we will later refer to the equivalence between c) and f).

Proof of Propostion 6.3. The stability of $E^{1,0}$ implies the one for $E^{1,0\vee}$, and hence b) follows from a) and from the Property 0.11.

For part ii) remark that the Arakelov equality says that

$$\mu(T_\iota) = \mu(\mathcal{H}om(E^{1,0}, E^{0,1})).$$

So c) is a consequence of b).

By Simpson's correspondence d) and e) are equivalent, and by Lemma 6.1 the numerical condition in f) is equivalent to d). So for i) it remains to verify the equivalence of c) and d).

Claim 6.4. The condition d) implies c).

Proof. The map $T_\iota \to \mathcal{H}om(E^{1,0}, E^{0,1})$, tensorized with $\det(E^{1,0})$ is the map

$$\theta^{(1)^\vee} : \det(E^{1,0}) \otimes T \longrightarrow E^{\ell-1,1} = \bigwedge^{\ell-1}(E^{1,0}) \otimes E^{0,1} \cong E^{1,0^\vee} \otimes E^{0,1} \otimes \det(E^{1,0}).$$

Condition d) implies that the saturated image of $\theta^{(1)^\vee}$ is a direct factor of the sheaf $E^{1,0^\vee} \otimes E^{0,1} \otimes \det(E^{1,0})$, hence c) holds. □

For the implication "c) implies d)" we start with:

Claim 6.5. c) implies that the images of the natural maps

$$S^m(T_\iota) \xrightarrow{\alpha_m} \bigwedge^{\ell-m}(E^{1,0}) \otimes \bigwedge^{m}(E^{0,1}) \otimes \det(E^{1,0})^{-1} = \bigwedge^{m}(E^{1,0^\vee}) \otimes \bigwedge^{m}(E^{0,1})$$

split.

Proof. α_m is a morphism of semistable sheaves of the same slope and by assumption $S^m(T_\iota)$ is stable for all m. So the saturated image is either the stable sheaf $S^m(T)$ or zero. So $G^{\ell-m,m} = \langle \det(E^{1,0}) \rangle^{\ell-m,m}$, as the image of

$$S^m(T_\iota) \otimes \det(E^{1,0}) \longrightarrow \bigwedge^{\ell-m}(E^{1,0}) \otimes \bigwedge^{m}(E^{0,1}) = \bigwedge^{m}(E^{1,0^\vee}) \otimes \bigwedge^{m}(E^{0,1}) \otimes \det(E^{1,0})$$

is either zero or the saturated hull of $S^m(T) \otimes \det(E^{1,0})$.

In order to obtain a splitting of $G^{\ell-m,m}$ write $E^{1,0^\vee} \otimes E^{0,1} = T_\iota \oplus R$. The sheaf $\bigwedge^m(E^{1,0^\vee}) \otimes \bigwedge^m(E^{0,1})$ is a direct factor of $S^m(E^{1,0^\vee} \otimes E^{0,1})$, and we write

$$S^m(E^{1,0^\vee} \otimes E^{0,1}) = \bigwedge^{m}(E^{1,0^\vee}) \otimes \bigwedge^{m}(E^{0,1}) \oplus R_m.$$

So $S^m(T)$ is a direct factor of $\bigwedge^m(E^{1,0^\vee}) \otimes \bigwedge^m(E^{0,1}) \oplus R_m$. Consider composite of the inclusion and the induced projection

$$S^m(T_\iota) \longrightarrow \bigwedge^{m}(E^{1,0^\vee}) \otimes \bigwedge^{m}(E^{0,1}) \oplus R_m \longrightarrow \bigwedge^{m}(E^{1,0^\vee}) \otimes \bigwedge^{m}(E^{0,1}).$$

The stability of $S^m(T_\iota)$ implies that the image of

$$S^m(T_\iota) \longrightarrow \bigwedge^{m}(E^{1,0^\vee}) \otimes \bigwedge^{m}(E^{0,1}) \oplus R_m \longrightarrow R_m$$

must be zero or $S^m(T_\iota)$. If it is zero, and we are done. Otherwise we obtain an inclusion

$$S^m(T_\iota) \oplus S^m(T_\iota) \longrightarrow \bigwedge^{m}(E^{1,0^\vee}) \otimes \bigwedge^{m}(E^{0,1}) \oplus R_m,$$

where the first factor lies in $\bigwedge^m(E^{1,0^\vee}) \otimes \bigwedge^m(E^{0,1})$ and the second in R_m. So under the projection to $\bigwedge^m(E^{1,0^\vee}) \otimes \bigwedge^m(E^{0,1})$ one finds a subsheaf, isomorphic to $S^m(T_\iota)$, which necessarily splits. □

Claim 6.6. The condition c) implies d).

Proof. To construct the splitting of Higgs bundles, one follows the arguments used in [VZ07, Section 5]. As we have just seen, for $m \leq \varsigma((E,\theta))$ the sheaves

$$\det(E^{\ell,0}) \otimes S^m(T) \subset G^{\ell-m,m} = \langle \det(E^{1,0}) \rangle^{\ell-m,m}$$

are μ-equivalent and the right hand side is a direct factor of $\bigwedge^m(E^{1,0\vee}) \otimes \bigwedge^m(E^{0,1})$. It remains to show, that the projections

$$\Phi_m : \bigwedge^{\ell-m}(E^{1,0}) \otimes \bigwedge^{m}(E^{0,1}) \longrightarrow G^{\ell-m,m},$$

to $G^{\ell-m,m}$ can be chosen such that $G = G^{\ell,0} \oplus G^{\ell-1,1} \oplus \cdots G^{\ell-r,r}$ is a quotient Higgs bundle. We will construct the splittings by descending induction in such a way that the diagram

$$\begin{array}{ccc} \bigwedge^{\ell-m}(E^{1,0}) \otimes \bigwedge^{m}(E^{0,1}) & \xrightarrow{\theta^{\ell-m,m}} & \bigwedge^{\ell-m-1}(E^{1,0}) \otimes \bigwedge^{m+1}(E^{0,1}) \otimes \Omega_\iota \\ \Phi_m \downarrow & & \downarrow \Phi_{m+1} \\ G^{\ell-m,m} & \longrightarrow & G^{\ell-m-1,m+1} \otimes \Omega_\iota \end{array}$$

commutes. As long as $G^{\ell-m-1,m+1} = 0$ there is nothing to construct, and we can choose Φ_m to be any splitting.

If r is the largest integer with $G^{\ell-r,r} \neq 0$, assume by induction, that we found the $\Phi_{m'}$ for all $m' > m$ and that $m < r$.

So $\theta^{\ell-m,m}|_{G^{\ell-m,m}}$ is non-zero. Since $G^{\ell-m,m}$ is stable and since $\theta^{\ell-m,m}$ a morphism between polystable sheaves of the same slope, one finds

$$G^{\ell-m,m} \xrightarrow[\subset]{\theta^{\ell-m,m}} G^{\ell-m-1,m+1} \otimes \Omega_\iota \subset \bigwedge^{\ell-m-1}(E^{1,0}) \otimes \bigwedge^{m+1}(E^{0,1}) \otimes \Omega_\iota.$$

So the saturated image of $G^{\ell-m,m}$ under $\Phi_{m+1} \circ \theta^{\ell-m,m}$ is isomorphic to $G^{\ell-m,m}$, and

$$G^{\ell-m,m} \xrightarrow{\subset} \bigwedge^{\ell-m}(E^{1,0}) \otimes \bigwedge^{m}(E^{0,1}) \xrightarrow{\Phi_{m+1}\circ\theta^{\ell-m,m}} G^{\ell-m,m}.$$

defines a splitting Φ_m with the desired properties.

So G splits as a sub-Higgs bundle of $E^{(\ell)}$, hence it is itself a Higgs bundle arising from a local system. □

To finish the proof of Proposition 6.3 it remains to verify:

Claim 6.7. The splitting in d) implies that M_i is an n-dimensional complex ball.

Proof. Claim 6.6 implies that all the Chern classes $\mathrm{c}_i(G)$ are zero, where again $G = \langle \det(E^{1,0}) \rangle$.

The Higgs bundle $G = \langle \det(E^{1,0}) \rangle$. splits as a sub-Higgs bundle of $\bigwedge^\ell E$, hence it is itself a Higgs bundle arising from a local system. In particular the Chern classes $\mathrm{c}_2(G)$ are zero.

To show that, as in [VZ07, Section 5] this implies that M is a ball quotient,we use a formal calculation of Chern numbers. Hence we may replace Y by any finite covering, and assume that there exists an invertible sheaf $\mathcal{L}$ with $\det(E^{1,0}) = \mathcal{L}^\ell$. Or we may calculate with $\mathbb{Q}$-Chern classes. Consider the sheaf

$$F = F^{1,0} \oplus F^{0,1} \quad \text{with} \quad F^{1,0} = \mathcal{L}, \quad F^{1,0} = \mathcal{L} \otimes T.$$

Then $S^\ell(F)$ is a Higgs bundle with $\mathcal{L}^\ell \otimes S^m(T)$ in bidegree $(\ell - m, m)$, hence isomorphic to G. By e) the first Chern class of G is zero, hence $\mathrm{c}_1(F)$ as well. On the other hand,

$$\mathrm{c}_1(F) = \mathrm{c}_1(\mathcal{L}) + n \cdot \mathrm{c}_1(\mathcal{L}) - \mathrm{c}_1(\Omega_\iota) = \frac{n+1}{\ell}\mathrm{c}_1(E^{1,0}) - \mathrm{c}_1(\Omega_\iota),$$

and $\mathrm{c}_1(\mathcal{L}) = \frac{1}{n+1}\mathrm{c}_1(\Omega_\iota)$. For the second Chern class it is easier to calculate the discriminant

$$\Delta(\mathcal{F}) = 2 \cdot \mathrm{rk}(\mathcal{F}) \cdot \mathrm{c}_2(\mathcal{F}) - (\mathrm{rk}(\mathcal{F}) - 1) \cdot \mathrm{c}_1(\mathcal{F})^2.$$

By [VZ07, Lemma 3.3], a), the discriminant is invariant under tensor products with invertible sheaves, hence $\Delta(\mathcal{L} \oplus \mathcal{L} \otimes T) = \Delta(\mathcal{O}_Y \oplus T)$.

Since $\mathrm{c}_1(G)^2 = \mathrm{c}_2(G) = 0$ one finds $\Delta(G) = 0$, and [VZ07, Lemma 3.3] implies that $\Delta(F) = 0$. Then

$$0 = \Delta(\mathcal{O}_Y \oplus T) = 2 \cdot (n+1) \cdot \mathrm{c}_2(T) - n \cdot \mathrm{c}_1(T)^2,$$

and by Yau's Uniformization Theorem, recalled in [VZ07, Theorem 1.4] Y is a complex ball quotient □

The Proposition 6.3 gives a numerical condition on the length of the wedge product of the Higgs field, which together with the Arakelov equality implies that M_ι is a complex ball. A similar condition holds automatically for local systems which are pure of type A. This is not surprising, since in this case the corresponding factor M_ι automatically is a one dimensional ball.

Proposition 6.8. *Let $\mathbb{V}$ be an irreducible complex polarized variation of Hodge structures of weight one, pure of type ι, and with Higgs bundle (E, θ). Assume that Ω_ι is of type A or B, and that the saturated image of $T_\iota \to \mathcal{H}om(E^{1,0}, E^{0,1})$ is a direct factor of the sheaf $\mathcal{H}om(E^{1,0}, E^{0,1})$.*

Then $\mathbb{V}$ is the tensor product of a unitary representation with a wedge product of the standard representation of $\mathrm{SU}(1, n)$. In particular the period map $\tau : \tilde{U} \to M'$ factors as the projection $\tilde{U} \to M_\iota$ and a totally geodesic embedding $M_\iota \to M'$.

Proof. If Ω_ι is invertible, M_ι is a complex ball. Proposition 6.3, i), implies the same, if Ω_ι is of type B.

Before we proceed, we fix some notation. For a bounded symmetric domain M we denote by $\mathrm{Aut}(M)$ the group of holomorphic isometries of M. We write, as usual, $\tilde{U} = \prod_i M_i$ and fix origins o_i in all M_i. Let $G_i := \mathrm{Aut}(M_i)$, $K_i := \mathrm{Stab}(o_i)$, hence $M_i = G_i/K_i$.

Let $\tau : \tilde{U} \to M'$ be the period map for the bundle $\mathbb{V}$. In M' fix o', let $G' := \mathrm{Aut}(M') \cong \mathrm{SU}(\ell, \ell')$, and let $K' := \mathrm{Stab}(o')$. By the purity of the Higgs bundle, τ factors as the projection $\tilde{U} \to M_\iota$ composed with a map $\tau_1 : M_\iota \to M'$. By definition as a period map, τ is equivariant with respect to the action of $\pi_1(U)$ via

$$\rho_1 : \pi_1(U) \longrightarrow \mathrm{Aut}(\tilde{U}) \cong G_i \quad \text{and} \quad \rho_2 : \pi_1(U) \longrightarrow \mathrm{Aut}(M') \cong G'$$

on domain and range.

Our aim is to lift τ to a map $\tilde{\tau} : G \to G'$, equivariant for ρ_1 and ρ_2 up to the compact factor that accounts for the unitary representation.

The hypothesis is used in the next claim. Remember that, since the splitting $T_\iota \to \mathcal{H}om(E^{1,0}, E^{0,1})$ comes from a splitting of Higgs bundles, it is orthogonal for the Hodge metric, hence for the Kähler metric.

Claim 6.9. Let $\tau_1 : M \to M' = G'/K'$ be a holomorphic map between symmetric domains. Assume that $\tau_1^* T_{M'} = T_M \oplus R$ is a holomorphic splitting, orthogonal with respect to the Kähler metric on M'. Then $M \to M'$ is a totally geodesic embedding.

Proof. (From a letter by N. Mok.) First, the splitting condition on $\tau_1^* T_{M'}$ implies that τ_1 is locally an embedding. Second, we check that the image $\tau_1(M)$ is totally geodesic in M'. This is again a local condition. By [He62, Theorem I.14.5] it suffices to check that the splitting $T_{M'}|_{\tau_1(M)} = T_M \oplus R$ is preserved under parallel transport.

Take any two local holomorphic sections s respectively t of T_M respectively of R. Then $\langle s, t\rangle = 0$ with respect to the Hermitian inner product. The derivative of t with respect to a $(1,0)$ vector is orthogonal to s because s is holomorphic and because $\langle \cdot, \cdot \rangle$ is Hermitian bilinear. Since s and t are arbitrary, it follows that R is invariant under differentiation in the $(1,0)$ direction. But since R is a holomorphic subbundle, it is invariant under differentiation in the $(0,1)$-direction. As a consequence R is parallel, so is its orthogonal complement T_M.

Finally, since M' is a global symmetric domain, it has geodesic symmetries at each point of $\tau_1(M)$. Since $\tau_1(M)$ is totally geodesic in M', these are geodesic symmetries of $\tau_1(M)$. Consequently, $\tau_1(M)$ is a global symmetric domain and τ_1 is an embedding. □

We continue with the proof of Proposition 6.8 and let

$$B := \{\varphi \in \mathrm{Aut}(M') : \varphi(\tau_1(M)) = \tau_1(M)\} \subset G'.$$

In the next step we deduce from Claim 6.9 that $\tau_1(M) = B/K_B$, where K_B is a maximal compact subgroup. The first observation is:

Claim 6.10. The embedding $\tau_1 : M_\iota \to M'$ is induced by a homomorphism from $\tilde{\tau_1} : G_\iota \to G'$ that factors through B.

Proof. As explained in [Sa65, §1.1] or [Sa80, II §2], the geodesic holomorphic isometry $M \to M''$ is induced by a local isomorphism $G_\iota \to G'$ and hence a homomorphism of Lie algebras $\mathrm{Lie}(G_\iota) \to \mathrm{Lie}(G')$. This induces a homomorphism $\tilde{G}_\iota \to G'$ from the universal covering $\tilde{G}_\iota$ of G_ι. Our group G_ι is a real form of the complex group $(G_\iota)_{\mathbb{C}} = \mathrm{Sl}(1+n)$, and the later is simply connected. Writing $_{\mathbb{C}}$ for the extension of scalars form $\mathbb{Q}$ to $\mathbb{C}$, the induced morphism $\mathrm{Lie}((G_\iota)\mathbb{C}) \to \mathrm{Lie}(H_{\mathbb{C}})$ gives rise to $\widetilde{((G_\iota)_{\mathbb{C}})} = G_{\mathbb{C}} \to H_{\mathbb{C}}$, extending $\tilde{G}_\iota \to H$. The group $\tilde{G}_{\iota\mathbb{C}}$ maps isomorphically to $G_{\mathbb{C}}$, and one has a commutative diagram

$$\begin{array}{ccccc} \tilde{G} & \longrightarrow & G & & \\ \subset\downarrow & & \downarrow\subset & & \\ \tilde{G}_{\mathbb{C}} & \xrightarrow{\cong} & G_{\mathbb{C}} & \longrightarrow & H_{\mathbb{C}}. \end{array}$$

So the map $\tilde{G}_\iota \to H$ factors through the image of $\tilde{G}_\iota$ in $(G_\iota)_{\mathbb{C}}$, hence through G_ι. The factorization through B is obvious by definition. □

By this claim, the natural map $\mathrm{res} : B \to \mathrm{Aut}(M) \cong G$ induces a surjection $B/K_B \to M$. This map is also injective, since elements in B preserve M. Consequently, the kernel Υ of res is a compact subgroup. By Claim 6.10 again, this kernel is a direct factor. In fact the kernel is a maximal direct factor, since $G_\iota = \mathrm{Aut}(M_\iota)$ does not contain direct compact factors. We deduce that given the choice of origins, the product decomposition $B \cong G_\iota \times \Upsilon$ is canonical.

The image of ρ_2 lies in B by definition. The equivariance of τ is equivalent to $\rho_1 = \mathrm{res} \circ \rho_2$ as maps $\pi_1(U) \to \mathrm{Aut}(M) = G_\iota$. Since the above splitting of B was canonical, we know that

$$\tilde{\tau} := (\mathrm{incl}) \circ (\tilde{\tau}_1 \times \mathrm{id}_\Upsilon) : G \times \Upsilon \longrightarrow B \subset G'$$

is the lift we wanted, equivariant with respect to the action of $\pi_1(U)$ by $\rho_1 \times (\mathrm{pr}_\Upsilon \circ \rho_2)$ and ρ_2 on domain and range.

Consequently, $\mathbb{V}$ decomposes as a tensor product of the unitary representation $\mathrm{pr}_{\Upsilon_0} \circ \rho_2$ and the representation given by

$$\pi_1(U) \longrightarrow G \longrightarrow G_\iota \longrightarrow G'.$$

The last remaining step in the proof of Proposition 6.8 is:

Claim 6.11. The representation $G \to B \to G'$ is a wedge product of the standard representation.

Proof. In order to match the hypothesis of [Sa80] precisely, we should postcompose the map $\mathrm{incl} \circ \tilde{\tau}_1 : G_\iota \to G'$ by a natural inclusion of G' into the symplectic group. By the table p. 461 and Proposition 1 in [Sa80], $\mathrm{incl} \circ \tilde{\tau}_1$ is a direct sum wedge products of the standard representations. This direct sum has only one summand since $\mathbb{V}$ was reducible otherwise. □

Proposition 6.12. *Assume that Ω_ι is of type A or B, that M_ι is the complex ball $\mathrm{SU}(1,n)/K$ and that $\mathbb{V}$ is the tensor product of a unitary representation with a wedge product of the standard representation of* $\mathrm{SU}(1,n)$.

1. *Then $\mathbb{V}$ satisfies the Arakelov equality.*
2. *There exists a compactification Y' such that for the Higgs bundle (E', θ') the sheaves $E'^{1,0}$ and $E'^{0,1}$ are μ-stable and such that $E'^{1,0\vee} \otimes E'^{0,1}$ is μ-polystable.*

Proof. Let us first show that 1) and 2) hold true for the Mumford compactification Y', S' considered in Section 2. The bundles $E'^{1,0}$ and $E'^{0,1}$ are irreducible homogeneous bundles as in Lemma 2.6, case a_n and $q = 1$, given by the wedge products of the standard representation of $U(n)$. The same arguments as in the proof of Proposition 0.3 now imply 1) and 2).

By Lemma 2.5 the Arakelov equality on the Mumford compactification implies the one on any compactification, satisfying the positivity statement in Assumption 0.1. □

6.3. **Type C: $S^m(\Omega_\iota)$ is unstable for some m.** Suppose that $\mathbb{V}$ is pure of type ι and ι and $S^m(\Omega_\iota)$ is unstable for some $m > 1$. Yau's Uniformization Theorem, recalled in [VZ07, Theorem 1.4], implies that M_ι is a bounded symmetric domain of rank greater than one. Using the characteristic subvarieties, introduced by Mok presumably one can write down an explicit formula for $\varsigma(\mathbb{V})$. However we do not need this, since in this case the superrigidity theorems apply. Recall the notations introduced at the beginning of the proof of Proposition 6.8.

Proposition 6.13. *If $\mathbb{V}$ is pure of type ι, The period map factors as the projection $\tilde{U} \to M_\iota$ and a totally geodesic embedding $M_\iota \to M'$.*

Proof. Purity of $\mathbb{V}$ implies that the period map factors through the projection to M_ι. In the case we treat, M_ι has rank greater than one, hence the metric rigidity theorems of Mok and their generalizations due to To apply. More precisely, Let h be the pullback the restriction of the Bergman-metric on M' to M_ι. By purity and since M' is a bounded symmetric domain of non-compact type, h descends to a metric of seminegative curvature on the bundle $(\Omega_\iota)^\vee$ on U. Thus the hypothesis of [Mk87, Theorem 4] are met, if one takes into account the arguments of To ([To89, Corollary 2] and the subsequent remark) to extend from U compact to U of finite volume. We conclude that up, to a constant multiple, h is the Bergman-metric on M_ι and $M_\iota \to M'$ a totally geodesic embedding. □

Proof of Proposition 0.3. Parts i)–iv) have been verified at the end of Section 2. By assumption $f : A \to U$ is a Kuga fibre space, and $\mathbb{V}$ is pure of type B. In particular the assumption made in Proposition 6.12 hold, and on a suitable compactification the sheaf $E^{1,0^\vee} \otimes E^{0,1}$ is μ-polystable. So Proposition 6.3 implies that $\varsigma((E,\theta)) = \frac{\ell \cdot \ell' \cdot (n+1)}{(\ell+\ell')\cdot n}$. Of course this equality is independent of the compactification. □

Proof of Addendum 0.10. Part I) is Proposition 6.2, if one uses in addition the equivalence between f) and c) in Proposition 6.3 and the Proposition 6.8. Part II) is just repeating the conclusion of Proposition 6.13.

For Part III) remark first that the equivalence of the conditions β) and γ) is part of Proposition 6.3. By Proposition 6.8 β) implies δ) and hence η). Finally Proposition 6.12, 2), shows that on a Mumford compactification of U the condition β) holds true, as well asα).

Since we already obtained the equivalence of β) and γ) on any Y satisfying the assumptions in 0.1, we know that γ) holds on a Mumford compactification. Obviously γ) is independent of the compactification, hence γ) and β) hold on Y.

For the last part remark that a) and the Condition 0.11 imply that $E^{1,0^\vee} \otimes E^{0,1}$ is polystable, and β) follows directly from the Arakelov equality. □

Proof of Theorem 0.8. If $f : A \to U$ is a Kuga fibre space, and if $\mathbb{V}$ is a non-unitary irreducible subvariation of Hodge structures in $R^1 f_* \mathbb{C}_A$, then part ii) of Proposition 0.3 gives the Arakelov equality, and part i) implies that $\mathbb{V}$ is pure of type $i = i(\mathbb{V})$.

If Ω_i is of type A or C, there is nothing to verify in 2). If Ω_i is of type B, then $\theta_j = 0$ for $i \neq j$, and equal to θ for $j = i$. Part v) of Proposition 0.3 shows that

$$\varsigma(\mathbb{V}) = \varsigma((E,\theta)) = \varsigma((E,\theta_i)) = \frac{\mathrm{rk}(E^{1,0}) \cdot \mathrm{rk}(E^{0,1}) \cdot (n_j+1)}{\mathrm{rk}(E) \cdot n_j}.$$

Assume now that the conditions 1) and 2) in Theorem 0.8 hold. Let $F^{1,0}$ be the Hodge bundle of $R^1 f_* \mathbb{C}_A$. Since U is generically finite over $\mathcal{A}_g$ the sheaf $\det(f_*\omega_{X/Y})$ is big. Since it is nef, using the slopes introduced in Section 5, one finds by the property 2) in Lemma 4.4 that

$$\mu^{(j)}(f_*\omega_{X/Y}) = g \cdot \mu^{(j)}(F^{1,0}) > 0$$

for all j. By 5.3 this is only possible if for each stable direct factor Ω_ι of $\Omega^1_Y(\log S)$ there exists some non-unitary complex polarized subvariation of Hodge structures $\mathbb{V}$, pure of type ι.

If Ω_ι is of type C, we find that the map $M_\iota \to M'$ to the period domain M' of $\mathbb{V}$ factors as the projection $\tilde{U} \to M_\iota$ and a totally geodesic embedding $M_\iota \to M'$.

By Proposition 6.8 the same holds if ω_ι is of type A, or if it is of type B and if the condition 2) Theorem 0.8 holds.

So $\tilde{U} \to \tilde{\mathcal{A}}_g$ is a totally geodesic embedding, hence $f : A \to U$ a Kuga fibre space by Theorem 1.2.

Finally the last statement in Theorem 0.8 follows from Corollary 1.3. □

7. The Arakelov equality and the Mumford-Tate group

We will keep the assumption that U is the complement of a normal crossing divisor S in a projective manifold Y, that $\Omega^1_Y(\log S)$ is nef, that $\omega_Y(S)$ is ample with respect to U. Given a smooth family $f : A \to U$ we will assume that each irreducible subvariation of Hodge structures of $R^1 f_* \mathbb{C}_A$ is either unitary or it satisfies the Arakelov equality.

If in addition the second condition in Theorem 0.8 holds, we have shown in the last Section that the induced morphism $U \to \mathcal{A}_g$ is totally geodesic. By Moonen's Theorem, stated as Theorem 1.2, we know that U is a Kuga fibre space, and that it is the translate of a Shimura variety of Hodge type.

In particular this implies that the monodromy group Mon^0 of $R^1 f_* \mathbb{C}_A$ is normalized by the complex structure, hence by the derived Hodge group $\mathrm{MT}(R^1 f_* \mathbb{C}_A)^{\mathrm{der}}$. In this section we will verify the last property as a direct consequence of the Arakelov equality, without using the second condition in Theorem 0.8, and we will determine the invariant cycles under Mon^0 explicitly.

The methods will be similar to those used in [VZ07, Sections 9 and 10]. However, we will allow $R^1 f_* \mathbb{C}_A$ to have unitary direct factors, and we will try to avoid some of the explicit calculations used in [VZ07, Sections 9]. Let us formalize the conditions we will use.

Definition 7.1. Let $(E = \bigoplus_{m=0}^k E^{k-m,m}, \theta)$ be the Higgs bundle of a complex polarized variation of Hodge structures $\mathbb{V}$ of weight k, with unipotent local monodromy around the components of S.

i. The *support* $\mathrm{supp}(E, \theta)$ is the set of all m with $E^{k-m,m} \neq 0$.
ii. (E, θ) has a *connected support*, if there exists some $m_0 \leq m_1 \in \mathbb{Z}$ with
$$\mathrm{supp}(E, \theta) = \{m;\ m_0 \leq m \leq m_1\} \quad \text{and if}$$
$$\theta_{k-m,m} = \theta|_{E^{k-m,m}} \neq 0 \quad \text{for} \quad m_0 \leq m \leq m_1 - 1.$$
iii. (E, θ) (or $\mathbb{V}$) satisfies the *Arakelov condition* if (E, θ) has a connected support and if for all m with $m,\ m+1 \in \mathrm{supp}(E, \theta)$ the sheaves $E^{k-m,m}$ and $E^{k-m-1,m+1}$ are μ-semistable and
$$\mu(E^{k-m,m}) = \mu(E^{k-m-1,m+1}) + \mu(\Omega^1_Y(\log S)).$$

Lemma 7.2.

1. *The tensor product of Higgs bundles with connected support has a connected support.*
2. *The Higgs bundle (E, θ) of a polarized irreducible complex variation of Hodge structures $\mathbb{V}$ of weight k has a connected support.*
3. *A variation of Hodge structures $\mathbb{V}$ as in 2. satisfies the Arakelov condition if and only if one of the following conditions holds*
   a. $\mathbb{V}$ *is unitary.*
   b. $\mathrm{supp}(E, \theta) = \{m_0, \dots m_1\}$ *for $m_0 < m_1$, and for $m_0 \leq m \leq m_1$ the sheaves $E^{k-m,m}$ are μ-semistable of slope*
   $$\mu(E^{k-m,m}) = \mu(E^{k-m_0,m_0}) - (m - m_0) \cdot \mu(\Omega^1_Y(\log S)).$$

Proof. 2) follows directly from the Simpson correspondence, and 1) and 3) are obvious. □

If $k = 1$ and if $\mathbb{V}$ is irreducible and non-unitary [VZ07, Theorem 1] says that the Arakelov equality for (E, θ) implies the Arakelov condition. If $\mathbb{V}$ is unitary, the connectedness of the support in Lemma 7.2, 2), says that the Higgs bundle is concentrated in one bidegree.

Lemma 7.3. *Consider for $i = 1, \dots, s$ irreducible $\mathbb{C}$ variations of Hodge structures $\mathbb{V}_i$ with Higgs bundles $(E_i = \bigoplus_{m=0}^{k_i} E_i^{k_i-m,m}, \theta_i)$, and $\mathbb{V} = \mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_s$.*

a. *If the Arakelov condition holds for all the $\mathbb{V}_i$, it holds for $\mathbb{V}_1 \otimes \cdots \otimes \mathbb{V}_s$.*
b. *If $k_1 = \cdots = k_s = k$ the Arakelov condition for $\mathbb{V}$ implies the Arakelov condition for each of the direct factors $\mathbb{V}_i$.*

Proof. Let (E, θ) denote again the Higgs bundle of $\mathbb{V}$. By Lemma 7.2, 2), the support of each of the $\mathbb{V}_i$ is connected. Since $E^{k-m,k}$ is the direct sum of the sheaves $E_i^{k-m,m}$ one obtains b).

For a) we will replace the condition that the $\mathbb{V}_i$ are irreducible by the one saying that the support of all the $\mathbb{V}_i$ is connected. Using Lemma 7.2, 1), this allows by induction to assume that $s = 2$.

Assume that the weight of $\mathbb{V}_i$ is $k^{(i)}$ and write $\{m_0^{(i)}, \dots m_1^{(i)}\}$ for the support of (E_i, θ_i). Then $\mathbb{V}$ has weight $k = k^{(1)} + k^{(2)}$. Since the Higgs bundle

$$(E, \theta) = (E_1, \theta_1) \otimes (E_2, \theta_2)$$

is given by

$$E^{k^{(1)}+k^{(2)}-r,r} = \bigoplus_{\ell_1+\ell_2=r} E_1^{k^{(1)}-\ell_1} \otimes E_2^{k^{(2)}-\ell_2,\ell_2},$$

one easily sees that (E, θ) is supported in $\{m_0 = m_0^{(1)} + m_0^{(2)}, \dots, m_1 = m_1^{(1)} + m_1^{(2)}\}$.

As the tensor product of semistable sheaves $E_1^{k^{(1)}-\ell_1} \otimes E_2^{k^{(2)}-\ell_2,\ell_2}$ is semi-stable of slope

$$\mu(E_1^{k^{(1)}-m_0^{(1)},m_0^{(1)}}) + \mu(E_2^{k^{(2)}-m_0^{(2)},m_0^{(2)}}) - (\ell_1 + \ell_2 - m^{(1)} - m^{(2)}) \cdot \mu(\Omega^1_Y(\log S)),$$

for $m_0^{(1)} \leq \ell_1 \leq m_1^{(1)}$ and $m_0^{(2)} \leq \ell_2 \leq m_1^{(2)}$, and zero otherwise. So $E^{k-m,m}$ is semi-stable of slope $\mu(E^{k-m_0,m_0}) - (m - m_0) \cdot \mu(\Omega^1_Y(\log S))$, if non-zero. □

Lemma 7.4. *Let $\mathbb{V}$ be a $\mathbb{C}$ sub variation of Hodge structures of $R^k f_* \mathbb{C}_V$ and let $\mathbb{V}^\vee$ be its complex conjugate. If $\mathbb{V}$ satisfies the Arakelov condition, then the same holds for $\mathbb{V}^\vee$.*

Proof. This is obvious since (as indicated by the notations) the polarization allows to identify the complex conjugate with the dual local system. □

Lemma 7.5. *Let $\mathbb{V}$ be a sub variation of Hodge structures of $R^k f_* \mathbb{C}_V$ with connected support, satisfying the Arakelov condition. Let $s \in H^0(U, \mathbb{V})$ be a global section.*

a. *There is a unique m such that s is of bidegree $(k-m, m)$.*
b. *If $\mathbb{V}$ is defined over $\mathbb{R}$, then k is even and $m = \frac{k}{2}$.*

Proof. Obviously b) follows from a).

For a) consider the trivial sub local system $\mathbb{U}$ generated by $H^0(U, \mathbb{V})$. By [De87] $\mathbb{U}$ is a subvariation of Hodge structures, in particular the corresponding Higgs bundle $(F, 0)$ is a direct factor of (E, θ). So $F^{p,q}$ has to be a direct factor of $E^{p,q}$, in particular it is again semistable of slope $\mu(E^{p,q})$.

Since $F^{p,q}$ is a direct sum of copies of $\mathcal{O}_Y$, and since $\mu(\Omega^1_Y(\log S)) > 0$ this is only possible for one tuple (p, q). □

Let K be a totally real numberfield, and consider a polarized K variation of Hodge structures $\mathbb{W}_K$ of weight one. As in Section 1.2 we consider a very general point $y \in U$. So we will assume that the Mumford-Tate group of $\mathbb{W}_K$ and of all subvariation of Hodge structures coincides with the one of $W_K = \mathbb{W}_K|_y$. In general, if a variation of Hodge structures is denoted by a boldface letter, the restriction to y will be denoted by the same letter, not in boldface.

Recall that the monodromy group $\mathrm{Mon}(\mathbb{W}_K)$ is defined as the smallest K-algebraic subgroup of $\mathrm{Gl}(W_K)$ which contains the image of the monodromy representation, and that $\mathrm{Mon}^0(\mathbb{W}_K)$ is its connected component containing the zero. $\mathrm{Mon}^0(\mathbb{W}_K)$ is a normal subgroup of the derived subgroup $\mathrm{MT}(\mathbb{W}_K)^{\mathrm{der}}$ (see [An92], for example). We will compare the monodromy group and the Mumford-Tate group using the following well-known criterion.

Lemma 7.6. *Let $G \subset \mathrm{Gl}(W_K)$ and H be two reductive K algebraic subgroups. Assume that each global section*

$$\eta \in H^0\big(Y, \mathbb{W}_K^{\otimes m} \otimes \mathbb{W}_K^{\vee \otimes m'}\big)$$

which is invariant under G is also invariant under H. Then $G \supset H$. In particular this holds true for G being the product of $\mathrm{Mon}^0(\mathbb{W}_K)$ and a reductive group.

Proof. This is just [De82, Proposition 3.1 (c)]. By [Si92, Lemma 4.4] $\mathrm{Mon}^0(\mathbb{W}_K)$ is reductive. So the last part follows from the first one. □

Assumptions 7.7. Let K be a totally real numberfield. Assume that $\mathbb{W}_K$ is a polarized variation of Hodge structures of weight one, and that there exist polarized K-variations of Hodge structures $\mathbb{V}'_{1\,K}, \ldots, \mathbb{V}'_{\ell\,K}$, and polarized K-Hodge structures $H_{1\,K}, \ldots, H_{\ell\,K}$, such that:

1. $\mathbb{V}'_{i\mathbb{R}} = \mathbb{V}'_{i\,K} \otimes_K \mathbb{R}$ is irreducible for $i = 1, \ldots, \ell$.
2. One has a decomposition of variation of Hodge structures

$$\mathbb{W}_K = \mathbb{W}_{1\,K} \oplus \cdots \oplus \mathbb{W}_{\ell\,K} \quad \text{with} \quad \mathbb{W}_{i\,K} = \mathbb{V}'_{i\,K} \otimes_K H_{i\,K},$$

orthogonal with respect to the polarization.

3. $\mathrm{Hom}(\mathbb{V}'_{i\,\mathbb{R}}, \mathbb{V}'_{j\,\mathbb{R}})$ is a skew field for $i = j$ and is zero otherwise.

Lemma 7.8. *Let $\mathbb{W}_{\mathbb{Z}}$ be a polarized $\mathbb{Z}$ variation of Hodge structures of weight one. Then for some K, $\mathbb{V}'_{1\,K}, \ldots, V'_{\ell K}$, and $H_{1\,K}, \ldots, H_{\ell\,K}$, the Assumptions 7.7 hold true.*

Proof. The existence of such a decomposition for local systems over $\mathbb{R}$ is just the semi-simplicity. As shown in [VZ07, Lemma 9.3], for example, $\mathbb{W}_{\mathbb{Q}}$ decomposes as a direct sum of subvariations of Hodge structures. Such a decomposition can be defined over some totally real number field, and it can be chosen to be orthogonal with respect to the polarization.

Of course we may write the direct sum of all direct factors, isomorphic to some $\mathbb{V}_i$ in the form $\mathbb{V}'_{i\,K} \otimes_K H_{i\,K}$, for some K vector space $H_{i\,K}$. As in [De87, Proposition 1.12] or in [De71, Theorem 4.4.8] one defines a Hodge structure on $H_{i\,K}$. □

By [De87, Proposition 1.12] the variations of Hodge structures $\mathbb{V}'_i = \mathbb{V}'_{i\,K} \otimes_K \mathbb{C}$ can be written as a direct sum of polarized $\mathbb{C}$ variations o f Hodge structures. Let us distinguish different types of direct factors $\mathbb{W}_i = \mathbb{W}_{i\,K} \otimes_K \mathbb{C}$, keeping the notations from Lemma 7.8. We will write $H_i = H_{i\,K} \otimes_K \mathbb{C}$ and again W_i and V'_i for the restriction to $y \in U$.

Recall that one can identify $\bar{\mathbb{V}}_i$ and the dual $\mathbb{V}_i^\vee$ of $\mathbb{V}$ by means of the anti-symmetric intersection form Q on $H^1(f^{-1}(y), \mathbb{Z})$. It induces a non-degenerate pairing $\mathbb{V}_i \otimes \bar{\mathbb{V}}_i \to \mathbb{C}$.

In different terms, Q defines a polarization φ by

$$\varphi(\alpha, \beta) = \frac{\sqrt{-1}}{2 \cdot \pi} Q(\alpha, \bar{\beta}).$$

Remark that φ is positive definite on $H^1(f^{-1}(y), \mathbb{C})^{1,0}$ and negative definite on $H^1(f^{-1}(y), \mathbb{C})^{0,1}$. By [De87, Proposition 1.12] the factors of the tensor product decomposition are again Hodge structures or variations of Hodge structures. So there are only two possibilities.

Lemma 7.9. *Keeping the notations from Lemma 7.8, for each i the decomposition of the $\mathbb{C}$ variation of Hodge structures*

$$\mathbb{W}_i = \mathbb{W}_{i\,K} \otimes_K \mathbb{C} = \mathbb{V}'_i \otimes H_i$$

satisfies one of the following two conditions:

a. *$\mathbb{V}'_i$ is a polarized variation of Hodge structures of weight one and H_i a polarized Hodge structure of weight zero.*

b. *$\mathbb{V}'_i$ is a polarized variation of Hodge structures of weight zero and H_i a polarized Hodge structure of weight one.*

Let us assume that for some ℓ_2 and for $1 \leq i \leq \ell_2$ the condition a) holds true, whereas for $\ell_2 < i \leq \ell$ one has the condition b).

One has a decomposition $\mathrm{Gl}(W_{i\,K}) = \mathrm{Gl}(V'_{i\,K}) \times \mathrm{Gl}(H_{i\,K})$. The monodromy group $\mathrm{Mon}^0(\mathbb{W}_{i\,K})$ acts trivially on the second factor of the tensor product. Hence it is contained in $\mathrm{Gl}(V'_{i\,K}) \times \{\mathrm{id}_{H_{i\,K}}\}$.

Using the notations from Assumptions 7.7 one finds

$$\mathrm{Mon}^0(\mathbb{W}_K) \subset \bigtimes_{i=1}^{\ell} \mathrm{Mon}^0(\mathbb{V}'_{i\,K}) \times \{\mathrm{id}_{H_{i\,K}}\} \subset \bigtimes_{i=1}^{\ell} \mathrm{Gl}(W_{i\,K}). \tag{7.1}$$

For the Mumford-Tate group the situation is more complicated. On just has

$$\mathrm{MT}(\mathbb{W}_K) \subset \bigtimes_{i=0}^{\ell} \mathrm{MT}^{(}\mathbb{W}_{i\,K}) \subset \bigtimes_{i=1}^{\ell} \mathrm{Gl}(V'_{i\,K}) \times \mathrm{Gl}(H_{i\,K}) = \bigtimes_{i=1}^{\ell} \mathrm{Gl}(W_{i\,K}). \tag{7.2}$$

Corollary 7.10. *Under the convention introduced above,*

$$\mathrm{MT}(\mathbb{W}_{i\,K}) = \mathrm{MT}(\mathbb{V}_{i\,K}) \times \{\mathrm{id}_{H_{i\,K}}\} \quad \textit{for} \quad i = 1, \dots, \ell_2 \quad \textit{and}$$

$$\mathrm{MT}(\mathbb{W}_K) \subset \left(\bigtimes_{i=0}^{\ell_2} \mathrm{MT}(\mathbb{V}'_{i\,K}) \times \{\mathrm{id}_{H_{i\,K}}\} \right) \times \left(\bigtimes_{i=\ell_2+1}^{\ell} \mathrm{Gl}(\mathbb{W}_{i\,K}) \right).$$

Proof. It suffices to calculate $\mathrm{MT}(\mathbb{W}_{i\,K})$ for $1 \leq i \leq \ell_2$. A Hodge cycle, i.e. a $(0,0)$-cycle

$$\eta \in H^0\big(Y, \mathbb{W}_{i\,K}^{\otimes m} \otimes_K \mathbb{W}_{i\,K}^{\vee \otimes m'}\big) = H^0\big(Y, \mathbb{V}'^{\otimes m}_{i\,K} \otimes_K \mathbb{V}'^{\vee \otimes m'}_{i\,K}\big) \otimes_K H_{i\,K}^{\otimes m} \otimes_K H_{i\,K}^{\vee \otimes m'}$$

can only exist for $m = m'$ and it can be written as

$$\eta = \sum_{\iota} \gamma_\iota \otimes h_\iota, \quad \text{with}$$

$$\gamma_\iota \in H^0\big(Y, \mathbb{V}'^{\otimes m}_{i\,K} \otimes \mathbb{V}'^{\vee \otimes m}_{i\,K}\big) \quad \text{and} \quad h_\iota \in H_{i\,K}^{\otimes m} \otimes H_{i\,K}^{\vee \otimes m}.$$

Since all the h_ι are of bidegree $(0,0)$, each η_ι is of bidegree $(0,0)$, and hence $\gamma_\iota \otimes h_\iota$ as a Hodge cycle is invariant under $\mathrm{MT}(\mathbb{W}_K)$. Moreover it stays a Hodge cycle, if one replaces h_ι by any other element of $H_{i\,K}^{\otimes m} \otimes_K H_{i\,K}^{\vee \otimes m}$. So $\mathrm{MT}(\mathbb{W}_K)$ is contained in $\mathrm{Gl}(V_{i\,K}) \times \{\mathrm{id}_{H_{i\,K}}\}$. Since γ_ι is a Hodge cycle of $\mathbb{V}'_{i\,K}$ one finds $\mathrm{MT}(\mathbb{W}_{i\,K}) = \mathrm{MT}(\mathbb{V}_{i\,K})^{\mathrm{der}} \times \{\mathrm{id}_{H_{i\,K}}\}$ and the corollary follows from (7.2). □

Definition 7.11. We define the *restricted Mumford-Tate* groups as

$$\mathrm{MT}^{\mathrm{rest}}(\mathbb{W}_{i\,K}) = \mathrm{MT}(\mathbb{W}_{i\,K}) \cap (\mathrm{Gl}(\mathbb{V}_{i\,K}) \times \{\mathrm{id}_{H_{i\,K}}\}) \quad \text{and}$$

$$\mathrm{MT}^{\mathrm{rest}}(\mathbb{W}_K) = \mathrm{MT}(\mathbb{W}_K) \cap \left(\bigtimes_{i=1}^{\ell} \mathrm{Gl}(\mathbb{V}_{i\,K}) \times \{\mathrm{id}_{H_{i\,K}}\} \right).$$

At present the group $\mathrm{MT}^{\mathrm{rest}}(\mathbb{W}_K)$ is only defined if $\mathbb{W}_K$ decomposes in irreducible subvariation of Hodge structures, which remain irreducible over $\mathbb{R}$. Or at least, whatever one defines without this assumption, might not behave in a nice way under Galois conjugation or field extensions.

Remark 7.12. If $\mathbb{W}_\mathbb{Q}$ is a $\mathbb{Q}$-variation of Hodge structures, and if K is a totally real number field, then $\mathrm{MT}(\mathbb{W}_\mathbb{Q})$ is the smallest $\mathbb{Q}$-subgroup H of $\mathrm{Gl}(W_\mathbb{Q})$, for which one has an inclusion $\mathrm{MT}(\mathbb{W}_K) \subset H \otimes K$. In fact, since $\mathrm{MT}(\mathbb{W}_K) \otimes_K \mathbb{R}$ contains the complex structure, the same holds for $H \otimes \mathbb{R}$ and hence $\mathrm{MT}(\mathbb{W}_\mathbb{Q}) \subset H$. On the other hand, since $\mathrm{MT}(\mathbb{W}_K)$ is the smallest K subgroup containing the complex structure, $\mathrm{MT}(\mathbb{W}_K) \subset \mathrm{MT}(\mathbb{W}_\mathbb{Q}) \otimes K$, and consequently $H \subset \mathrm{MT}(\mathbb{W}_\mathbb{Q})$.

So the following definition seems reasonable and compatible with the rigid case, where $\mathrm{MT}^{\mathrm{rest}}(\mathbb{W}_K) = \mathrm{MT}(\mathbb{W}_K)$.

Definition 7.13. If $\mathbb{W}_\mathbb{Q}$ is a $\mathbb{Q}$-variation of Hodge structures, and if K is a totally real number field, such that the Assumptions 7.7 hold for $\mathbb{W}_K = \mathbb{W}_\mathbb{Q} \otimes_\mathbb{Q} K$, we define $\mathrm{MT}^{\mathrm{rest}}(\mathbb{W}_\mathbb{Q})$ to be the smallest $\mathbb{Q}$-algebraic subgroup of $\mathrm{Gl}(W_\mathbb{Q})$ whose tensor product with K contains $\mathrm{MT}^{\mathrm{rest}}(\mathbb{W}_K)$.

Properties 7.14.

i. For $1 \leq i \leq \ell_2$ one has $\mathrm{MT}^{\mathrm{rest}}(\mathbb{W}_{i\,K}) = \mathrm{MT}(\mathbb{W}_{i\,K})$.
ii. For all i one has an inclusion $\mathrm{Mon}^0(\mathbb{W}_{i\,K}) \subset \mathrm{MT}^{\mathrm{rest}}(\mathbb{W}_{i\,K})^{\mathrm{der}}$.
iii. One has an inclusion $\mathrm{Mon}^0(\mathbb{W}_K) \subset \mathrm{MT}^{\mathrm{rest}}(\mathbb{W}_K)^{\mathrm{der}}$.

Proof. i) follows from the first part of Corollary 7.10. As well known (see [An92] for example) $\mathrm{Mon}^0(\mathbb{W}_K) \subset \mathrm{MT}(\mathbb{W}_K)^{\mathrm{der}}$. Hence iii) follows from the description of $\mathrm{Mon}^0(\mathbb{W}_K)$ in (7.1), and ii) is a special case of iii). □

In Lemma 7.9 in each of the cases a) or b) one can distinguish two subcases:

Type a1. $\mathbb{V}'_i$ is an irreducible $\mathbb{C}$-variation of Hodge structures of weight one, and H_i is a $\mathbb{C}$-Hodge structure of weight zero. This implies in particular that $\mathbb{V}'_i$ is isomorphic to its complex conjugate $\mathbb{V}'^{\vee}_i$, and that $\mathbb{V}'_i$ is not unitary. In fact, if $\mathbb{V}'_i$ were unitary, it would decompose in two subsystems, one of bidegree $(1,0)$ and the other of bidegree $(0,1)$, contradicting the irreducibility.

Claim 7.15. Assume that $\mathbb{W}_{i\,K}$ is of type a1, and that it satisfies the Arakelov equality. Then all global sections

$$\eta \in H^0\big(Y, \mathbb{W}_{i\,K}^{\otimes m} \otimes_K \mathbb{W}_{i\,K}^{\vee \otimes m'}\big)$$

are of bidegree $(m-m', m-m')$.

Proof. The Arakelov equality implies that $\mathbb{V}'_i$ satisfies the Arakelov condition, and by Lemma 7.4 the same holds true for $\mathbb{W}_i = \mathbb{V}'_i \otimes H_i$. So the Claim follows from Lemma 7.5. □

Type a2. $\mathbb{V}'_i$ is a $\mathbb{C}$-variation of Hodge structures of weight one, and the direct sum of two irreducible factors $\mathbb{V}_i$ and $\mathbb{V}_i^\vee$. Again H_i is a $\mathbb{C}$-Hodge structure of weight zero.

Remark that we allow $\mathbb{V}_i$ and $\mathbb{V}_i^\vee$ to be unitary. Nevertheless, $\mathbb{V}_i$, $\mathbb{V}_i^\vee$ and their tensor product with H_i will satisfy the Arakelov condition. So Lemma 7.5 implies:

Claim 7.16. Assume that $\mathbb{W}_{i\,K}$ is of type a2, and either unitary or with Arakelov equality. Then there exist p and q such that all global sections

$$\eta \in H^0\big(Y, (\mathbb{V}_{i\,K} \otimes_K H_{i\,K})^{\otimes m} \otimes_K (\mathbb{V}_{i\,K} \otimes_K H_{i\,K})^{\vee \otimes m'}\big)$$

are of bidegree (p,q), and all global sections

$$\eta \in H^0\big(Y, (\mathbb{V}_{i\,K}^\vee \otimes_K H_{i\,K})^{\otimes m} \otimes_K (\mathbb{V}_{i\,K}^\vee \otimes_K H_{i\,K})^{\vee \otimes m'}\big)$$

are of bidegree (q,p). Moreover one has $p+q = m-m'$.

Claim 7.17. For $\mathbb{W}_{i\,K}$ of type a2 the Mumford-Tate group respects the decomposition of $\mathbb{V}'_i$, i.e. up to conjugation

$$\mathrm{MT}(\mathbb{W}_{i\,K}) \otimes_K \mathbb{C} \subset \mathrm{Gl}(V_i \otimes H_i) \times \mathrm{Gl}(V_i^\vee \otimes H_i).$$

Proof. The decomposition in a direct sum can be defined over an imaginary quadratic extension $L = K(\sqrt{b})$ of K, say with ι as a generator of the Galois group. So the Mumford-Tate group acts trivially on ι-invariant global sections of $\mathrm{End}(\mathbb{W}_i)$. Applying this to $\mathrm{id}_{V_i\otimes H_i} + \mathrm{id}_{V_i^\vee\otimes H_i}$ and to $\sqrt{b}\cdot(\mathrm{id}_{V_i\otimes H_i} - \mathrm{id}_{V_i^\vee\otimes H_i})$ one obtains the claim. □

Type b1. It remains to consider the case where $\mathbb{V}'_i$ is a $\mathbb{C}$-variation of Hodge structures of weight zero, and H_i is a $\mathbb{C}$-Hodge structure of weight one. In this case $\mathbb{V}'_{i\,K}$ and $W_{i\,K}$ are both unitary.

As in [De71, Lemma 4.4.9] the antisymmetric form $Q|_{\mathbb{W}_i}$ is the tensor product of two forms Q'_i and Q_i on $\mathbb{V}'_{i\,K}$ and $H_{i\,K}$, respectively. Here either Q'_i or $\mathbb{Q}_i$ is antisymmetric, and the other one symmetric.

We say that $W_{i\,K}$ is of type b1 if Q'_i is antisymmetric. In this case,

$$\mathrm{MT}(\mathbb{W}_{i\,K}) \subset \mathrm{CSp}(V'_{i\,K}, Q'_i) \times \mathrm{O}(H_{i\,K}, Q_i).$$

Recall that the image of the monodromy representation lies in the first factor, hence it can be written as $\mathrm{Mon}^0(\mathbb{V}'_{i\,K}) \times \{\mathrm{id}_{H_{i\,K}}\}$.

Lemma 7.18. *Assume that $\mathbb{W}_{i\,K}$ is of type b1.*

a. *One has inclusions* $\mathrm{MT}(\mathbb{W}_{i\,K}) \subset \mathrm{Mon}^0(\mathbb{V}'_{i\,K}) \times \mathrm{O}(H_{i\,K}, Q_i)$ *and*
$$\mathrm{MT}(\mathbb{W}_{i\,K})^{\mathrm{der}} \subset \mathrm{Mon}^0(\mathbb{V}'_{i\,K}) \times \mathrm{SO}(H_{i\,K}, Q_i).$$
b. *If* $\dim(H_i) = 2$, *then* $\mathrm{O}(H_{i\,K}, Q_i)$ *is commutative and hence* $\mathrm{Mon}^0(\mathbb{W}_{i\,K}) = \mathrm{MT}^{\mathrm{rest}}(\mathbb{W}_{i\,K})^{\mathrm{der}} = \mathrm{MT}(\mathbb{W}_{i\,K})^{\mathrm{der}}$.
c. *If* $\dim(H_i) > 2$ *then there exists a non-zero antisymmetric endomorphism of* W_i *of bidegree* $(-1, 1)$.

Proof. Since $\mathrm{O}(2, K)$ is commutative, b) follows from a).

For c) remark that $\dim(H_i) = \mu$ is even, and that for $\mu \geq 4$ the elements of $\mathrm{SO}(\mu, \mathbb{C})$ generate the matrix algebra $\mathrm{M}(\mu, \mathbb{C})$. So one finds an antisymmetric endomorphism of $\mathbb{V}'_i \otimes H_i$ of bidegree $(-1, 1)$.

For a) consider a section

$$\eta \in H^0\big(Y, \mathbb{V}'^{m+m'}_{i\,K}\big) \otimes_K H^{\otimes m}_{i\,K} \otimes_K H^{\vee\otimes_K m'}_{i\,K}$$

invariant under the action of $\mathrm{Mon}^0(\mathbb{V}'_{i\,K}) \times \mathrm{O}(H_{i\,K}, Q_i)$. In particular, η is invariant under the action of $\{\mathrm{id}_{V'_{i\,K}}\} \times \mathrm{O}(H_{i\,K}, Q_i)$, hence it must be the sum of cycles of the form $\eta_0 \otimes h$ for $h \in H^{\otimes m}_{i\,K} \otimes_K H^{\vee\otimes m'}_{i\,K}$ invariant under $\mathrm{O}(H_{i\,K}, Q_i)$. This implies that h is a Hodge cycle, hence that η is invariant under $\mathrm{MT}(\mathbb{W}_{i\,K})$. Since $\mathrm{Mon}^0(\mathbb{V}'_{i\,K}) \times \mathrm{O}(H_{i\,K}, Q_i)$ is again reductive, Lemma 7.6 implies that there is an inclusion

$$\mathrm{MT}(\mathbb{W}_{i\,K}) \subset \mathrm{Mon}^0(\mathbb{V}'_{i\,K}) \times \mathrm{O}(H_{i\,K}, Q_i),$$

and the second inclusion follows by considering the derived subgroups. □

Type b2. Again $\mathbb{V}'_i$ is a $\mathbb{C}$-variation of Hodge structures of weight zero, and H_i is a $\mathbb{C}$-Hodge structure of weight one, hence again $\mathbb{V}'_{i\,K}$ and $W_{i\,K}$ are both unitary. Here we assume however that the induced form Q'_i on $\mathbb{V}'_i$ is antisymmetric and that $\dim(H_i) > 0$.

Lemma 7.19. *Assume that $\mathbb{W}_{i\,K}$ is of type b2. Then there exists a non-zero antisymmetric endomorphism of W_i of bidegree* $(-1, 1)$.

Proof. Obviously there exists an antisymmetric endomorphism of H_i of bidegree $(-1,1)$, and $\mathrm{End}(\mathbb{W}_i) = \mathrm{End}(H_i)$. □

Lemma 7.20. *Assume that $\mathbb{W}_{\mathbb{Z}}$ is a rigid polarized variation of Hodge structures of weight one. Then there are no direct factors of type b2, and for each of the direct factors $\mathbb{W}_{i\,K}$ of type b1 one has* $\dim(H_{i,K}) = 2$.

Proof. By [Fa83] this follows from Lemma 7.18, c) and Lemma 7.19. □

Notations 7.21. Under the Assumptions 7.7 we choose the indices such that $\mathbb{W}_{i\,K}$ is of type a1 for $i = 1, \ldots, \ell_1$, of type a2 for $i = \ell_1 + 1, \ldots, \ell_2$, of type b1 for $i = \ell_2 + 1, \ldots, \ell_3$, and of type b2 for $i = \ell_3 + 1, \ldots, \ell$.

Proposition 7.22. *Keeping the assumptions and notations made in 7.7 and 7.21, assume that each $\mathbb{W}_{i\,K}$ is either unitary, or it satisfies the Arakelov equality. Then*

$$\mathrm{Mon}^0(\mathbb{W}_K) = \mathrm{MT}^{\mathrm{rest}}(\mathbb{W}_K)^{\mathrm{der}}.$$

If there are no direct factors of type b2, and if for each direct factor $\mathbb{W}_{i\,K}$ of type b1 one has $\dim(H_{i\,K}) = 2$ *then* $\mathrm{Mon}^0(\mathbb{W}_K) = \mathrm{MT}(\mathbb{W}_K)^{\mathrm{der}}$.

Before proving Proposition 7.22 let us state the corollary we are heading for.

Corollary 7.23. *Let Y be a projective manifold, and let $U \subset Y$ be the complement of a normal crossing divisor S. Assume that $\Omega^1_Y(\log S)$ is nef and that $\omega_Y(S)$ is ample with respect to U. Let $f : A \to U$ be a smooth family of abelian varieties such that each non-unitary irreducible subvariation of Hodge structures of $\mathbb{W}_{\mathbb{Q}} = R^1 f_* \mathbb{C}_A$ satisfies the Arakelov equality. Then* $\mathrm{Mon}^0(\mathbb{W}_{\mathbb{Q}}) = \mathrm{MT}^{\mathrm{rest}}(\mathbb{W}_{\mathbb{Q}})^{\mathrm{der}}$, *and if $f : A \to U$ is rigid, then* $\mathrm{Mon}^0(\mathbb{W}_{\mathbb{Q}}) = \mathrm{MT}(\mathbb{W}_{\mathbb{Q}})^{\mathrm{der}}$.

Proof. Choose the totally real number field K such that there exists the decomposition asked for in Assumptions 7.7. The containments

$$\mathrm{Mon}^0(\mathbb{W}_{\mathbb{Q}}) \subset \mathrm{MT}(\mathbb{W}_{\mathbb{Q}})^{\mathrm{der}} \quad \text{and} \quad \mathrm{Mon}^0(\mathbb{W}_K) \subset \mathrm{MT}^{\mathrm{rest}}(\mathbb{W}_K)^{\mathrm{der}}$$

are immediate, the second one is part iii) of Properties 7.14.

Choose y sufficiently general and let $A = f^{-1}(y)$. By Proposition 7.22

$$\mathrm{Mon}^0(\mathbb{W}_K) = \mathrm{MT}^{\mathrm{rest}}(\mathbb{W}_K)^{\mathrm{der}} \quad \text{and} \quad \mathrm{MT}^{\mathrm{rest}}(\mathbb{W}_K)^{\mathrm{der}} = \mathrm{MT}(\mathbb{W}_K)^{\mathrm{der}},$$

if f is rigid.

Recall that by definition $\mathrm{Mon}^0(\mathbb{W}_{\mathbb{Q}})$ is the smallest $\mathbb{Q}$-algebraic subgroup of $\mathrm{Gl}(H^1(A,\mathbb{Q}))$ for which $\mathrm{Mon}^0(\mathbb{W}_{\mathbb{Q}}) \otimes_{\mathbb{Q}} K$ contains $\mathrm{Mon}^0(\mathbb{W}_K)$. So the Remark 7.12 and the definition of $\mathrm{MT}^{\mathrm{rest}}(\mathbb{W}_{\mathbb{Q}})^{\mathrm{der}}$ in 7.13 imply that $\mathrm{Mon}^0(\mathbb{W}_{\mathbb{Q}}) = \mathrm{MT}(\mathbb{W}_{\mathbb{Q}})^{\mathrm{der}}$ in the rigid case, and that $\mathrm{Mon}^0(\mathbb{W}_{\mathbb{Q}}) = \mathrm{MT}^{\mathrm{rest}}(\mathbb{W}_{\mathbb{Q}})^{\mathrm{der}}$ in general. □

Remark 7.24. In Theorem 1.2 we have seen that for a suitable choice of the point $a \in \mathcal{X}_2$ the Kuga fibre space $\mathcal{X}_1 \times \{a\}$ is of Hodge type. In Corollary 7.23 this corresponds to the situation that there are direct factors of type b1 with $\dim(H_{i\,K}) \geq 4$ or of type b2, but that nevertheless $\mathrm{MT}(\mathbb{W}_{i\,K})/_{\mathrm{MT}^{\mathrm{rest}}(\mathbb{W}_{i\,K})}$ is commutative. This will happen in case b2 for example, if $H_{i\,K}$ has complex multiplication.

Proof of Proposition 7.22. By the definition of $\mathrm{MT}^{\mathrm{rest}}(\mathbb{W}_K)$, the last part follows from the first one, from Properties 7.14 and from Lemma 7.18 b).

In order to proof the first part of Proposition 7.22 we will apply arguments, similar to the ones used in the proof of [VZ07, Proposition 10.3].

Property 7.14 states that $\mathrm{Mon}^0(\mathbb{W}_K)$ is a subgroup of $\mathrm{MT}^{\mathrm{rest}}(\mathbb{W}_K)$ and we know that $\mathrm{Mon}^0(\mathbb{W}_K)$ is a subgroup of $\mathrm{MT}(\mathbb{W}_K)^{\mathrm{der}}$. By Lemma 7.6 we just have to show that all Hodge tensors which are invariant under $\mathrm{Mon}^0(\mathbb{W}_K)$ are invariant under $\mathrm{MT}^{\mathrm{rest}}(\mathbb{W}_K)^{\mathrm{der}}$. Of course, a Hodge tensor invariant under $\mathrm{Mon}^0(\mathbb{W}_K)$ is invariant under the monodromy representation, hence it is just a global section

$$\eta \in H^0\big(Y, \mathbb{W}_K^{\otimes m} \otimes_K \mathbb{W}_K^{\vee \otimes m'}\big).$$

Up to a shift of the bigrading, $\mathbb{W}_K^\vee$ can be identified with

$$\bigwedge^{\mathrm{rk}(W_K)-1} \mathbb{W}_K \otimes_K \det(\mathbb{W}_K)^{-1} = \bigwedge^{\mathrm{rk}(W_K)-1} \mathbb{W}_K \otimes_K \det(\mathbb{W}_K),$$

so we may as well consider sections of

$$\eta \in H^0\big(Y, \mathbb{W}_K^{\otimes k}\big) = \bigoplus_{\mathcal{I}'} H^0\Big(Y, \bigotimes_{i=1}^{\ell} \mathbb{W}_{i\,K}^{\otimes \kappa_i}\Big) = \bigoplus_{\mathcal{I}'} H^0\Big(Y, \bigotimes_{i=1}^{\ell} \mathbb{V}_{i\,K}'^{\otimes \kappa_i}\Big) \otimes_K \bigotimes_{i=1}^{\ell} H_{i\,K}^{\otimes \kappa_i},$$

where $\mathcal{I}'$ is the set of tuples $\underline{\kappa} = (\kappa_1, \ldots, \kappa_\ell)$ with $\sum_{i=1}^{\ell} \kappa_i = k$. So

$$\eta = \sum_{\mathcal{I}'} \eta_{\underline{\kappa}} \quad \text{and} \quad \eta_{\underline{\kappa}} = \sum_{\iota=1}^{\mu} \gamma_{\underline{\kappa},\iota} \otimes h_{\underline{\kappa},\iota}.$$

In order to show that η is invariant under $\mathrm{MT}^{\mathrm{rest}}(\mathbb{W}_K)$ we may assume that $\eta = \eta_{\underline{\kappa}^0}$ for a fixed tuple $\underline{\kappa}^0 = (\kappa_1^0, \ldots, \kappa_\ell^0)$ and by abuse of notations, that

$$\eta_{\underline{\kappa}^0} = \gamma_{\underline{\kappa}^0} \otimes h_{\underline{\kappa}^0} \quad \text{with} \quad \gamma_{\underline{\kappa}} \in H^0\Big(Y, \bigotimes_{i=1}^{\ell} \mathbb{V}_{i\,K}'^{\otimes \kappa_i}\Big) \quad \text{and} \quad h_{\underline{\kappa}} \in \bigotimes_{i=1}^{\ell} H_{i\,K}^{\otimes \kappa_i}.$$

Choose any Galois extension L of K with Galois group Γ, such that the local systems $\mathbb{V}'_{i\,L}$ decompose as a direct sum of two subsystems $\mathbb{V}_{i\,L}$ and $\mathbb{V}_{i\,L}^\vee$ for $i = \ell_1 + 1, \ldots, \ell_2$. By abuse of notation we will drop the L, hence ${}_i$ stands for ${}_{i\,L}$.

Consider the set $\mathcal{I}$ of tuples of natural numbers

$$\underline{k} = (k_1, \ldots, k_{\ell_1}, k_{\ell_1+1}, k'_{\ell_1+1}, \ldots, k_{\ell_2}, k'_{\ell_2}, k_{\ell_2+1}, \ldots, k_\ell), \quad \text{with}$$
$$k_i = \kappa_i^0 \quad \text{for} \quad i \in \{1, \ldots, \ell_1\} \cup \{\ell_2 + 1, \ldots, \ell\} \quad \text{and}$$
$$k_i + k'_i = \kappa_i^0 \quad \text{for} \quad i \in \{\ell_1, \ldots, \ell_2\}.$$

Then $H^0\Big(Y, \bigotimes_{i=1}^{\ell} \mathbb{V}_{i\,K}'^{\otimes \kappa_i^0}\Big)$ decomposes as

$$\bigoplus_{\mathcal{I}} H^0\Big(Y, \bigotimes_{i=1}^{\ell_1} \mathbb{V}_i'^{\otimes k_i} \otimes \bigotimes_{i=\ell_1+1}^{\ell_2} \big(\mathbb{V}_i^{\otimes k_i} \oplus \mathbb{V}_i^{\vee \otimes k'_i}\big) \bigotimes_{i=\ell_2+1}^{\ell} \mathbb{V}_i'^{\otimes k_i}\Big).$$

Remark that the local systems $\mathbb{V}'_i$ and $\mathbb{V}_i$ occurring in this decomposition all satisfy the Arakelov condition. Hence $\gamma = \gamma_{\underline{\kappa}^0}$ and $\eta = \eta_{\underline{\kappa}^0}$ decompose as

$$\gamma = \sum_{\mathcal{I}} \gamma_{\underline{k}} \quad \text{and} \quad \eta = \sum_{\mathcal{I}} \gamma_{\underline{k}} \otimes h_{\underline{\kappa}^0}$$

where

$$\gamma_{\underline{k}} \in \bigoplus_{\mathcal{I}} H^0\Big(Y, \bigotimes_{i=1}^{\ell_1} \mathbb{V}_i'^{\otimes k_i} \otimes \bigotimes_{i=\ell_1+1}^{\ell_2} \big(\mathbb{V}_i^{\otimes k_i} \oplus \mathbb{V}_i^{\vee \otimes k'_i}\big) \otimes \bigotimes_{i=\ell_2+1}^{\ell} \mathbb{V}_i'^{\otimes k_i}\Big)$$

is pure of some bidegree $(p_{\underline{k}}, q_{\underline{k}})$.

The Galois group Γ acts on the decomposition, and since η and $h = h_{\underline{\kappa}^0}$ are defined over K the group Γ permutes the components $\gamma_{\underline{k}}$. The sum over the conjugates of a fixed $\gamma_{\underline{k}}$ will again be defined over K, and by abuse of notations, replacing $\mathcal{I}$ by a subset, we can assume that $\mathcal{I}$ consists of one Γ-orbit.

If for some $\underline{k} \in \mathcal{I}$ one has $p_{\underline{k}} \neq q_{\underline{k}}$ then $\gamma_{\underline{k}}$ is not defined over $\mathbb{R}$, and its complex conjugate is of the form $\gamma_{\underline{k'}}$ for some $\underline{k'} \in \mathcal{I}$. In particular

$$p = \sum_{\mathcal{I}} p_{\underline{k}} = \sum_{\mathcal{I}} q_{\underline{k}},$$

and hence the wedge product

$$\rho = \bigwedge_{\mathcal{I}} \gamma_{\underline{k}}$$

is pure of bidegree (p, p). Remark that the wedge product is a direct factor of some tensor product, so we may consider ρ as a section in

$$H^0\Big(Y, \bigotimes^{\nu} \big(\bigotimes_{i=1}^{\ell_1} \mathbb{V}_i'^{\otimes k_i} \otimes \bigotimes_{i=\ell_1+1}^{\ell_2} \big(\mathbb{V}_i^{\otimes k_i} \oplus \mathbb{V}_i^{\vee \otimes k_i'}\big) \otimes \bigotimes_{i=\ell_2+1}^{\ell} \mathbb{V}_i'^{\otimes k_i}\big)\Big).$$

The Galois group Γ of L over K permutes the different components $\gamma_{\underline{k}}$ and for $\sigma \in \Gamma$ one has $\rho^\sigma = \pm\rho$. This defines a homomorphism $\chi : \Gamma \to \{\pm 1\}$. Choose a generator β of the Galois extension of K, defined by this homomorphism, such that Γ acts on β by multiplication with χ. Then $\beta \cdot \rho$ is invariant under Γ, hence defined over K. For a suitable choice of some

$$h' \in \bigotimes^{\nu} \bigotimes_{i=1}^{\ell} H_{i\,K}^{\otimes \kappa_i}$$

one finds

$$\beta \cdot \rho \otimes h' \in H^0\big(Y, \mathbb{W}_K^{\otimes k \cdot \nu}\big).$$

We may assume that h' is chosen to be pure of bidegree (p', p'), hence $\beta \cdot \rho \otimes h'$ is a Hodge cycle, invariant under $\mathrm{MT}(\mathbb{W}_K)$ and therefore invariant under the subgroup $\mathrm{MT}^{\mathrm{rest}}(\mathbb{W}_K)$. The latter acts trivially on $\bigotimes^{\nu} \bigotimes_{i=1}^{\ell} H_{i\,K}^{\otimes \kappa_i}$. Hence $\beta \cdot \rho$ has to be invariant under $\mathrm{MT}^{\mathrm{rest}}(\mathbb{W}_K)$. Here we regard $\mathrm{MT}^{\mathrm{rest}}(\mathbb{W}_K)$ as a subgroup of

$$\bigtimes_{i=0}^{\ell} \mathrm{Gl}(V'_{i\,K}) \cong \bigtimes_{i=0}^{\ell} \mathrm{Gl}(V'_{i\,K}) \times \{\mathrm{id}_{H_{i\,K}}\}.$$

The group $\mathrm{MT}^{\mathrm{rest}}(\mathbb{W}_K)$ acts on the subspace $< \beta \cdot \rho >_K$ by a character. Consequently, $\mathrm{MT}^{\mathrm{rest}}(\mathbb{W}_K)^{\mathrm{der}}$ leaves $\beta \cdot \rho$ invariant. This implies that the subspace

$$J =< \gamma_{\underline{k}};\ \underline{k} \in \mathcal{I} >_L \subset \bigoplus_{\mathcal{I}} H^0\Big(Y, \bigotimes_{i=1}^{\ell_1} \mathbb{V}_i'^{\otimes k_i} \otimes \bigotimes_{i=\ell_1+1}^{\ell_2} \big(\mathbb{V}_i^{\otimes k_i} \oplus \mathbb{V}_i^{\vee \otimes k_i'}\big) \otimes \bigotimes_{i=\ell_2+1}^{\ell} \mathbb{V}_i'^{\otimes k_i}\Big)$$

is invariant under the action of $\mathrm{MT}^{\mathrm{rest}}(\mathbb{W}_K)^{\mathrm{der}} \otimes L$ and acted on by $\mathrm{MT}^{\mathrm{rest}}(\mathbb{W}_K) \otimes L$ through a character. Since

$$\mathrm{MT}^{\mathrm{rest}}(\mathbb{W}_K) \subset \big(\bigtimes_{i=0}^{\ell} \mathrm{Gl}(V'_{i\,K}) \times \{\mathrm{id}_{H_{i\,K}}\}\big)$$

and since we have seen in Claim 7.17 that $\mathrm{MT}^{\mathrm{rest}}(\mathbb{V}'_{i\,K}) \otimes_K \mathbb{C}$ respects the decomposition $\mathbb{V}'_{i\,K} \otimes_K \mathbb{C} = \mathbb{V}_i \oplus \mathbb{V}_i^\vee$, the action of $\mathrm{MT}^{\mathrm{rest}}(\mathbb{W}_K) \otimes_K L$ leaves for each $\underline{k} \in \mathcal{I}$ the subspaces

$$< \gamma_{\underline{k}} >_L = J \cap H^0\Big(Y, \bigotimes_{i=1}^{\ell_1} \mathbb{V}_i'^{\otimes k_i} \otimes \bigotimes_{i=\ell_1+1}^{\ell_2} \big(\mathbb{V}_i^{\otimes k_i} \oplus \mathbb{V}_i^{\vee \otimes k_i'}\big) \otimes \bigotimes_{i=\ell_2+1}^{\ell} \mathbb{V}_i'^{\otimes k_i}\Big)$$

invariant. So one obtains a homomorphism

$$\mathrm{MT}^{\mathrm{rest}}(\mathbb{W}_K) \otimes_K L \longrightarrow \mathrm{Gl}(< \gamma_{\underline{k}} >_L) = L^*.$$

Its restriction to $\mathrm{MT}^{\mathrm{rest}}(\mathbb{W}_K)^{\mathrm{der}} \otimes_K L$ has to be trivial, i.e. $\gamma_{\underline{k}}$ is invariant under $\mathrm{MT}^{\mathrm{rest}}(\mathbb{W}_K) \otimes_K L$.

Since both $\sum_{\underline{k}} \gamma_{\underline{k}}$ and $\eta = \sum_{\mathcal{I}} \gamma_{\underline{k}} \otimes h_{\underline{\kappa}^0}$ are defined over K, they are invariant under $\mathrm{MT}^{\mathrm{rest}}(\mathbb{W}_K)$, as claimed. □

Let us return to Corollary 7.23. Using the notations introduced in Section 1.1, we choose $V = H^1(f^{-1}(y), \mathbb{Q})$ for a very general point $y \in U$ with the induced symmetric bilinear form Q, and $G = \mathrm{MT}^{\mathrm{rest}}(R^1 f_* V_{\mathbb{Q}}))^{\mathrm{der}}$.

Obviously G is normalized by $\mathrm{MT}(R^1 f_* V_{\mathbb{Q}})^{\mathrm{der}}$, hence by the complex structure φ_0 as well. So one obtains a Kuga fibre space $\mathcal{X} = \mathcal{X}(G, \mathrm{id}, \varphi_0)$. We will assume for simplicity that the universal family over $\mathcal{X}$ carries a level N structure for some $N \geq 3$.

Corollary 7.25. *In Corollary 7.23 assume that $f : V \to U$ has a level N structure. Then the induced morphism $\varphi : U \to \mathcal{A}_g$ factors through $\mathcal{X}$. If the dimension of the quotient of $(\mathrm{MT}^{\mathrm{rest}}(\mathbb{W}_{\mathbb{Q}})) \otimes \mathbb{R})$ by a maximal compact subgroup is equal to $\dim(U)$, then the induced map $U \to \mathcal{X}$ is surjective and étale.*

If $f : A \to U$ is rigid, then $\mathcal{X}$ is a Shimura variety of Hodge type.

Proof. The natural morphism $\varphi : U \to \mathcal{A}_g^{[N]}$ factors through

$$\mathrm{Mon}^0(\mathbb{W}_{\mathbb{Q}}) = \mathrm{MT}^{\mathrm{rest}}(\mathbb{W}_{\mathbb{Q}})^{\mathrm{der}} \longrightarrow \mathrm{Sp}(H^1(A, \mathbb{Q}_A), Q) \longrightarrow \tilde{\mathcal{A}}_g \longrightarrow \mathcal{A}_g^{[N]}.$$

and one obtains a morphism $U \to \mathcal{X}$.

Again, all maximal compact subgroups lie in the kernel. By assumption φ is generically finite over its image. So the condition on the dimensions in Corollary 7.25 implies that the induced morphism $\varphi : U \to \mathcal{X}$ is surjective.

$\mathcal{X}$ is non-singular and we may choose a nonsingular compactification Z and a normal crossing divisor Σ such that $\mathcal{X} = Z \setminus \Sigma$. Let us choose a blowing up $\delta : Y' \to Y$ with centers in S such that $S' = \delta^*(S)$ is again a normal crossing divisor, and such that φ extends to $\varphi : Y' \to Y$. The Higgs field of the Higgs bundle (E, θ) of $\mathbb{W}_{\mathbb{Q}}$ factors through

$$E^{1,0} \longrightarrow E^{0,1} \otimes \varphi^*(\Omega^1_Z(\log \Sigma)) \xrightarrow{\tau} E^{0,1} \otimes \Omega^1_{Y'}(\log S').$$

The induced morphism $\varphi^*(T^1_Z(-\log \Sigma)) \to \mathcal{H}om(E^{0,1}, E^{1,0})$ splits locally on U, and the Arakelov equality implies that $T^1_{Y'}(-\log S') \to \mathcal{H}om(E^{0,1}, E^{1,0})$ has the same property. So the inclusion $T^1_{Y'}(-\log S') \to T^1_Z(-\log \Sigma)$ splits locally on U and $U \to \mathcal{X}$ is étale.

The rigidity implies by Corollary 7.23 that $\mathrm{MT}^{\mathrm{rest}}(\mathbb{W}_{\mathbb{Q}})^{\mathrm{der}} = \mathrm{MT}(\mathbb{W}_{\mathbb{Q}})^{\mathrm{der}}$, and hence that $\mathcal{X}$ is a Shimura variety of Hodge type. □

8. Variations of Hodge structures of low rank

Assume that $\mathbb{V}$ is a complex polarized variation of Hodge structures of weight one and pure for some i. We drop the subscript i, i.e. we write Ω, T, and n for Ω_i, its dual, and its rank. We denote the corresponding factor of the universal covering $\tilde{U}$ by M.

In some cases the second condition in Theorem 0.8 is automatically satisfied. Of course this is the case if Ω is of type A or C. So we will assume in the sequel:

Assumptions 8.1. The sheaf Ω is of type B. $\mathbb{V}$ is non unitary and it satisfies the Arakelov equality. Writing again $(E = E^{1,0} \oplus E^{0,1}, \theta)$ for the Higgs bundle of $\mathbb{V}$, the Hodge numbers are $\ell = \mathrm{rk}(E^{1,0})$ and $\ell' = \mathrm{rk}(E^{0,1})$, hence the period map is given by a morphism $M \to \mathrm{SU}(\ell, \ell')$.

We will assume moreover, that $\omega_Y(S)$ is ample, or that the slight generalization Property 8.2 of Property 0.11 holds.

Property 8.2.

i. If $\mathcal{F}$ and $\mathcal{G}$ are two μ-stable torsion free coherent sheaves, then $\mathcal{F} \otimes \mathcal{G}$ is μ-polystable.
ii. If $\mathcal{F}$ is a μ-stable torsion free coherent sheaf, then $\mathcal{F}$ admits an admissible Hermite-Einstein metric, as defined in [BS94].

Lemma 8.3. *If $\omega_Y(S)$ is ample, then the Properties 8.2 hold true.*

Proof. In [BS94] it is shown, that a reflexive sheaf on a compact Kähler manifold admits an admissible Hermite-Einstein metric if and only if it is polystable. Part follows from the fact, that a tensor product of two admissible Hermite-Einstein metrics is again admissible Hermite-Einstein. In fact, in [BS94] admissibility of metrics h_i on bundles $\mathcal{V}_i$ asks for two conditions. First, the curvatures F_i should be square integrable and second their traces ΛF_i should be uniformly bounded. The curvature of $h_1 \otimes h_2$ is $F_1 \otimes \mathrm{Id}_2 + \mathrm{Id}_1 \otimes F_2$. Thus, if h_i are admissible, so is $h_1 \otimes h_2$, and the claim follows. □

The Property 8.2 will allow to apply [VZ07, Lemma 2.7], saying that the Higgs field θ respects the socle filtration. In particular, the μ-polystability of $E^{1,0}$ will imply the μ-polystability of $E^{1,0} \otimes T$, hence by the Arakelov inequality the μ-polystability of $E^{0,1}$. Replacing $\mathbb{V}$ by its dual, we are allowed to assume that $\ell \leq \ell'$.

Recall that $\varsigma(\mathbb{V}) = \varsigma((E,\theta))$ is the length of the Higgs subbundle $\langle \det(E^{1,0}) \rangle$ of $\bigwedge^{\ell}(E,\theta)$, and that by Corollary 0.6

$$\varsigma(\mathbb{V}) \geq \frac{\ell \cdot \ell' \cdot (n+1)}{(\ell + \ell') \cdot n}. \tag{8.1}$$

By Addendum 0.10, III, the bundle $E^{1,0}$ is stable, if and only if (8.1) is an equality. Obviously $\varsigma(\mathbb{V})$ can not exceed the weight of $\bigwedge^{\ell}(E,\theta)$, hence $\ell \geq \varsigma(\mathbb{V})$. One obtains:

Property 8.4. The irreducibility of $\mathbb{V}$ implies that $n \cdot \ell \geq \ell' \geq \ell$. If $\ell' = n \cdot \ell$ the numerical condition 2) in Theorem 0.8 holds, i.e. (8.1) is an equality.

In particular (8.1) is an equality for $n = 1$. So for the rest of the section we will assume that $n > 1$.

A second easy case is the one $\ell = 1$. Since $E^{1,0}$ is invertible, $E^{0,1}$ is the saturated hull of the stable sheaf $E^{1,0} \otimes T$, hence of rank $\ell' = n$. One obtains:

Example 8.5. If $\ell = 1$ then (8.1) is an equality.

Lemma 8.6.

i. *The Hodge bundle $E^{1,0}$ can not have a torsion free μ-stable quotient sheaf $\mathcal{V}$ such that $\mu(\mathcal{V}) = \mu(E^{1,0})$ and such that $\mathcal{V} \otimes T$ is μ-stable.*
ii. *In particular $E^{1,0}$ can not have a torsionfree rank one quotient sheaf $\mathcal{N}$ with $\mu(\mathcal{N}) = \mu(E^{1,0})$.*

Proof. Obviously ii) is a special case of i). Assume there exists a torsion free μ-stable quotient sheaf $\mathcal{V}$ with $\mu(\mathcal{V}) = \mu(E^{1,0})$, such that $\mathcal{V} \otimes T$ is μ-stable. We may replace $\mathcal{V}$ by its reflexive hull. So we will still require that $\mu(\mathcal{V}) = \mu(E^{1,0})$, but just that we have a morphism $E^{1,0} \to \mathcal{V}$ which is surjective on some open dense subscheme.

In order to keep notations consistent with [VZ07, Section 2], we will first study the dual situation, hence a subbundle $\mathcal{V}'$ of $E^{0,1}$. Recall that the socle $\mathcal{S}_1(\mathcal{F})$ of a coherent sheaf $\mathcal{F}$ is the smallest saturated subsheaf containing all μ-polystable subsheaves of $\mathcal{F}$. By definition it is weakly μ-polystable. By [VZ07, Lemma 2.7] the Property 8.2, i), implies that the Higgs field θ respects the socle, in particular for $\mathcal{V}' \subset \mathcal{S}_1(E^{0,1})$ the preimage $\theta^{-1}(\mathcal{V}' \otimes \Omega)$ is contained in $\mathcal{S}(E^{1,0})$. Since (E, θ) is the Higgs bundle of an irreducible variation of Hodge structures, $\theta^{-1}(\mathcal{V}' \otimes \Omega) \neq 0$. In fact, $\theta^{\vee} : E^{1,0} \otimes T \to E^{0,1}$ is surjective, since the cokernel would be a Higgs subbundle of degree zero.

So $\theta^{-1}(\mathcal{V}' \otimes \Omega)$ is a non-trivial subsheaf of the socle, hence μ-polystable. The μ-stability of $\mathcal{V}' \otimes \Omega$ implies that $\theta^{-1}(\mathcal{V}' \otimes \Omega)$ contains a direct factor which is μ-equivalent to $\mathcal{V}' \otimes \Omega$.

Applying this to the cosocle $\mathcal{S}'(E^{1,0})$, i.e. to the dual of $\mathcal{S}(E^{1,0^{\vee}})$ one finds a quotient sheaf of $E^{0,1}$ which is μ-equivalent to $\mathcal{V} \otimes T$. So (E, θ) has a quotient Higgs bundle whose reflexive hull is isomorphic to $\mathcal{Q} = \mathcal{V} \oplus \mathcal{V} \otimes T$. Lemma 4.3, ii), applied to $\mathcal{Q} = \mathcal{V} \oplus \mathcal{V} \otimes T$ and the Arakelov equality imply that

$$\begin{aligned} 0 \leq \mu(\mathcal{Q})\mathrm{rk}(\mathcal{Q}) &= \mathrm{rk}(\mathcal{V}) \cdot \mu(\mathcal{V}) + \mathrm{rk}(\mathcal{V}) \cdot n \cdot (\mu(\mathcal{V}) - \mu(\Omega)) = \\ &\mathrm{rk}(\mathcal{V}) \cdot (\mu(E^{1,0}) + n \cdot (\mu(E^{1,0}) - \mu(\Omega))) = \mathrm{rk}(\mathcal{V}) \cdot (\mu(E^{1,0}) + n \cdot \mu(E^{0,1})). \end{aligned}$$

On the other hand, the property 8.4 implies that

$$0 = \ell \cdot \mu(E^{1,0}) + \ell' \cdot \mu(E^{1,0}) \geq \ell \cdot (\mu(E^{1,0}) + n \cdot \mu(E^{1,0})),$$

hence that $\mu(\mathcal{Q}) = 0$. The irreducibility of $\mathbb{V}$ implies that (E, θ) can not have a Higgs subbundle of degree zero, a contradiction. □

If $\ell = 2$ and if the μ-semistable sheaf $E^{1,0}$ was not μ-stable, one would find an invertible quotient, as in Lemma 8.6. So we can state:

Example 8.7. Assume that $\ell = 2$. (8.1) is an equality, hence

$$2 \geq \varsigma(\mathbb{V}) = \frac{2 \cdot \ell' \cdot (n+1)}{(2 + \ell') \cdot n}.$$

The only solution is $\ell' = 2 \cdot n$ and $\varsigma(\mathbb{V}) = 2$.

Next we will consider the case of a rank two quotient of $E^{1,0}$. To this aim, we will need a careful analysis of the holonomy group:

Lemma 8.8. *Let $\mathcal{V}$ be a μ-stable torsion free quotient sheaf of $E^{1,0}$ of rank two and with $\mu(\mathcal{V}) = \mu(E^{1,0})$. Then $n = 2$ and for some invertible sheaf $\mathcal{N}$ one has an isomorphism $\mathcal{V}^{\vee\vee} \cong T \otimes \mathcal{N}$.*

Proof. By Lemma 8.6, ii), $\mathcal{V}$ has to be stable. Moreover, since the assumptions are compatible with replacing U by an étale covering, $\mathcal{V}$ remains stable under pullback to such a covering. By Lemma 8.6, i) the sheaf $\mathcal{V} \otimes T$ can not be μ-stable. So in order to finish the proof of the Lemma 8.8 it just remains to verify:

Claim 8.9. Let $\mathcal{V}$ be a rank 2 torsion free sheaf on Y, whose pullback to any étale covering remains stable. If $\mathcal{V} \otimes T$ is not μ-stable, then $n = 2$ and $\mathcal{V}^{\vee\vee} \cong T \otimes \mathcal{N}$.

Proof. For a sheaf $\mathcal{V}$ of rank two, the only irreducible Schur functors are of the form $\{k-a, a\}$, for $a \leq \frac{k}{2}$. By [FH91], 6.9 on p. 79, one has

$$\mathbb{S}_{\{k-a,a\}}(\mathcal{V}) = \begin{cases} \mathbb{S}_{\{k-2a\}}(\mathcal{V}) = S^{k-2a}(\mathcal{V}) \otimes \det(\mathcal{V})^a & \text{if} \quad 2a < k \\ \mathbb{S}_{\{a,a\}}(\mathcal{V}) = \det(\mathcal{V})^a & \text{if} \quad 2a = k \end{cases}.$$

Claim 8.10. The sheaves $S^m(\mathcal{V})$ and $S^m(T)$ are μ-stable, for all m. Moreover, the holonomy group of $S^m(T)$ with respect to the Hermite-Einstein metric is the full group $\mathrm{U}(n)$.

Proof. Otherwise, the holonomy group with respect to the Hermitian-Einstein metric on $S^m(\mathcal{V})$ or on $S^m(T)$ is not irreducible. Note that the holonomy group of the tensor product of Hermitian vector bundles is just the tensor product of the holonomy groups of the different factors.

Consequently, a non-trivial splitting of $S^m(\mathcal{V})$ (resp. of $S^m(T)$) forces the holonomy groups of $\mathcal{V}$ (resp. of T) with respect to the Hermite-Einstein metric to be strictly smaller than $\mathrm{U}(2)$ (resp. smaller than $\mathrm{U}(n)$).

It is known that a proper subgroup of $\mathrm{U}(2)$ is a semi-product of the torus with $\mathbb{Z}_2$. So one obtains a splitting of $\mathcal{V}$ on some étale double cover.

For T we use instead that by [Ya93] (see also [VZ07, Section 1]) the holonomy group of T is $\mathrm{U}(n)$. □

Assume that $\mathcal{V} \otimes T$ contains a subsheaf $\mathcal{N}$ of the same slope and of rank $r < 2 \cdot \mathrm{rk} T = 2 \cdot n$. Since $\mathcal{V} \otimes T$ is μ-polystable, $\mathcal{N}$ is a direct factor. Replacing $\mathcal{N}$ by its complement in $\mathcal{V} \otimes T$, if necessary, we may assume that $r \leq n$.

By taking the r-th wedge product one obtains an inclusion of $\mathcal{L} = \bigwedge^r \mathcal{N}$ into $\bigwedge^r (\mathcal{V} \otimes T)$, and both sheaves have the same slope. Here and later on, the wedge products of a torsion free sheaf will be the reflexive hull of the corresponding wedge product on the open set, where the sheaf is locally free.

By [FH91, p. 80], for example, one has a decomposition

$$\bigwedge^r (\mathcal{V} \otimes T) = \bigoplus \mathbb{S}_\lambda(\mathcal{V}) \otimes \mathbb{S}_{\lambda'}(T)$$

where the sum is taken over all partitions λ of r with at most 2 rows and n columns and where λ' is the partition complementary to λ. The rank one subsheaf $\mathcal{L}$ of $\bigwedge^r (\mathcal{V} \otimes T)$ must inject to $\mathbb{S}_\lambda(\mathcal{V}) \otimes \mathbb{S}_{\lambda'}(T)$ for some λ. Again both sheaves are μ-semistable of slope $\mu(\mathcal{L})$. Moreover, for $\lambda = \{a, a\}$ the rank of $\mathbb{S}_{\lambda'}(T)$ is strictly larger than one, and the Claim 8.10 implies that neither $\mathbb{S}_\lambda(\mathcal{V})$ nor $\mathbb{S}_{\lambda'}(T)$ can be invertible.

Let us assume that $n = 2$. If $r = 2$, the only possibilities for λ are $\{2, 0\}$ or $\{1, 1\}$. In the first case $\mathbb{S}_\lambda(\mathcal{V}) = \det(\mathcal{V})$, and in the second case $\mathbb{S}_{\lambda'}(T) = \det(T)$. So both are excluded.

If $\mathcal{N}$ is a subbundle of rank one, we obtain a non-trivial map

$$\mathcal{N} \otimes \Omega \longrightarrow \mathcal{V}.$$

Since both sheaves are μ-stable of the same slope this must be an isomorphism on some dense open subset, and since $\Omega = T \otimes \det(\Omega)$ we are done.

So assume from now on that $n \geq 3$. A non-zero projection of $\mathcal{L}$ to some Schur functor

$$\mathcal{L} \longrightarrow \mathbb{S}_\lambda(\mathcal{V}) \otimes \mathbb{S}_{\lambda'}(T)$$

gives again rise to a non-zero map

$$\mathbb{S}_\lambda(\mathcal{V})^\vee \otimes \mathcal{L} \longrightarrow \mathbb{S}_{\lambda'}(T)$$

between polystable bundles of rank strictly larger than 1 and of the same slope. Claim 8.10 implies that this is an isomorphism.

Hence the holonomy group of $\mathbb{S}_{\lambda'}(T)$ with respect to the Hermitian-Yang-Mills connection is isomorphic to the holonomy group of $\mathbb{S}_\lambda(\mathcal{V})^\vee$, up to twisting by scalars. Holonomy groups are compatible with Schur functors, so the $\mathbb{S}_\lambda$-representation of the holonomy group of $\mathcal{V}$ is isomorphic to $\mathbb{S}_{\lambda'}$ applied to the holonomy group of T_Y, which by Claim 8.10 is $\mathrm{U}(n)$.

Since $\mathbb{S}'_\lambda$ is not the determinant representation, this representation is almost faithful (with the kernel contained in the subgroup of scalar matrices). Since the holonomy group of $\mathcal{V}$ is $\mathrm{U}(2)$, it is too small to contain an almost faithful representation of $\mathrm{U}(n)$ for $n \geq 3$ one obtains a contradiction. So n must be two, and we handled this case already. □

This also completes the proof of Lemma 8.8. □

Example 8.11. If $\ell = 3$ and if $n \geq 3$, then (8.1) is an equality, hence

$$3 \geq \varsigma(\mathbb{V}) = \frac{3 \cdot \ell' \cdot (n+1)}{(3 + \ell') \cdot n} > 1.$$

For $\varsigma(\mathbb{V}) = 3$ one finds $\ell' = n \cdot \ell$. For $\varsigma(\mathbb{V}) = 2$ the only possibility is $n = \ell' = 3$.

Proof. If $E^{1,0}$ is not μ-stable, it has a torsion free quotient sheaf $\mathcal{V}$ of slope $\mu(E^{1,0})$, either of rank one or of rank two. Both cases have been excluded, by the Lemmata 8.6 and 8.8.

For $\varsigma = \varsigma(\mathbb{V})$ the equality implies that $\ell' = \frac{\varsigma \cdot 3 \cdot n}{(3-\varsigma) \cdot n + 3}$. For $\varsigma = 1$ there is no solution in $\mathbb{Z}_{\geq 3}$, and for $\varsigma = 2$ the only solutions are $(\ell', n) = (3, 3)$, $(4, 6)$ or $(5, 15)$. To exclude the last two cases, consider the non-trivial map

$$S^2(T) \otimes \det(E^{1,0}) \xrightarrow{\tau^{(2)}} E^{1,0} \otimes \bigwedge^2 (E^{0,1}).$$

Since both sides have the same slope, $\tau^{(2)}$ must be injective. However the inequality

$$\frac{(n+1) \cdot n}{2} \leq \ell \cdot \frac{\ell' \cdot (\ell' - 1)}{2}.$$

is violated for $(\ell', n) = (4, 6)$ or $(5, 15)$. □

Example 8.12. For $n = 2$ (8.1) is an equality, except possibly for $\ell' = 5$.

Proof. The inequality (8.1) says that

$$3 \geq \varsigma(\mathbb{V}) \geq \frac{3 \cdot \ell' \cdot 3}{(3 + \ell') \cdot 2}.$$

Since $\ell' \geq 3$ the right hand side is strictly larger than 2, hence $\varsigma(\mathbb{V}) = 3$, and the morphism

$$\det(E^{1,0}) \otimes S^3(T) \xrightarrow{\tau^{(3)}} \bigwedge^3 (E^{0,1})$$

is non-zero. Since both sides have the same slope, for $\ell' = 3$ this contradicts the stability of $S^3(T)$. For $\ell' = 4$ the saturated image of $\tau^{(3)}$ is $\bigwedge^3(E^{0,1})$. Hence the latter and $E^{0,1}$ are both μ-stable. The compatibility of the Higgs field with the socle filtration implies that $E^{1,0}$ is stable, and hence (8.1) must be an equality. Obviously this is a contradiction. □

Altogether we verified:

Proposition 8.13. *Under the Assumptions 8.1 the equality (8.1) holds in the following cases:*

1. $n = 1$.
2. $n = 2$, $\ell \leq 3$, $\ell \leq \ell'$ *and* $\ell' \neq 5$.
3. $n \geq 3$, $\ell \leq 3$, *and* $\ell \leq \ell'$.

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