Galois covers of degree p : semi-stable reduction and Galois action

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Abstract

In this paper we study the semi-stable reduction of Galois covers of degree p above semi-stable curves over a complete discrete valuation ring of inequal chracteristics $(0,p)$. We are also able to describe the Galois action on these covers in terms of some geometric and combinatorial data in characteristic p endowed with the action of the Galois group of the residue field.

0. Introduction. In this paper we study the semi-stable reduction of Galois covers of degree p above semi-stable curves over a complete discrete valuation ring of inequal chracteristics $(0,p)$, as well as the Galois action on these data.

More precisely, let R be a complete discrete valuation ring of inequal chracteristics $(0,p)$, with fraction field K, and residue field k. In the first part of the paper we consider the following local situation. Let $f:\mathcal{Y}\to\mathcal{X}$ be a Galois cover of degree p between formal germs of R-curves at the closed points y and x respectively, with $\mathcal{Y}$ normal, and $\mathcal{X}$ semi-stable i.e. $\mathcal{X}$ is the formal germ of an R-curve at a closed point x which is either a smooth point or an ordinary double point. It follows from the theorem of semi-stable reduction for curves that the germ $\mathcal{Y}$ admits potentially a semi-stable reduction i.e. there exists, after eventually a finite extension of R, a semi-stable model $\tilde{\mathcal{Y}}\to\mathcal{Y}$ which is "essencially" unique. In particular, the Galois group of the cover f acts on $\tilde{\mathcal{Y}}$ and the quotient $\tilde{\mathcal{X}}$ of $\tilde{\mathcal{Y}}$ by this action is a semi-stable blow-up of $\mathcal{X}$. We have a canonical morphism $\tilde{f}:\tilde{\mathcal{Y}}\to\tilde{\mathcal{X}}$ which is Galois of degree p. To the above cover $f:\mathcal{Y}\to\mathcal{X}$ we associate canonically some *degeneration data* which determine completely the special fibre $\tilde{\mathcal{Y}}_k:=\tilde{\mathcal{Y}}\times_R k$ of $\tilde{\mathcal{Y}}$. These degeneration data consist of the (canonically marked) tree associated to the special fibre $\tilde{\mathcal{X}}_k:=\tilde{\mathcal{X}}\times_R k$ of $\tilde{\mathcal{X}}$, plus some geometric data which consists of a torsor $f_i:V_i\to U_i$ under a finite and flat group scheme of rank p above each irreducible marked component U_i of $\tilde{\mathcal{X}}_k$ such that the "conductors" of these torsors at the marked points satisfy certain "compatibility" conditions (these are what we called in [Sa] Kummerian mixed torsors) (cf.

1.2, 1.3 for a more precise definition). For an illustration of this situation we refeer to the example 1.2.1. Let's denote by Deg_p the set of isomorphism classes of such degeneration data which are defined over an algebraic closure $\overline{k}$ of k (cf. 1.2.2, 1.3.1 for a precise definition). We show that the set Deg_p is a G_k-set, where G_k is the Galois group of the separable closure of k contained in $\overline{k}$ (cf. 1.2.3). Let $\overline{K}$ be an algebraic closure of K, and let $\operatorname{Spec} L$ be the geometric generic point of $\mathcal{X}$. The étale cohomology group $H^1_{\text{et}}(\operatorname{Spec} L, \mu_p)$ classifies (pointed)-geometric Galois covers of $\mathcal{X}$ of degree p. Moreover our result which exhibit the above degeneration data can be translated as the existence of a canonical specialisation map $\mathrm{Sp} : H^1_{\text{et}}(\operatorname{Spec} L, \mu_p) \to \mathrm{Deg}_p$. Note that both $H^1_{\text{et}}(\operatorname{Spec} L, \mu_p)$ and Deg_p are G_K-sets, where G_K is the Galois group of $\overline{K}$ over K, and G_K acts on Deg_p via its canonical quotient G_k. Our main result in this part of the paper is the following:

Theorem (1.2.7, 1.3.2). *The above specialisation map* $\mathrm{Sp} : H^1_{\text{et}}(\operatorname{Spec} L, \mu_p) \to \mathrm{Deg}_p$ *is surjective and* G_K*-equivariant.*

In other words the association of degeneration data to a cover $f : \mathcal{Y} \to \mathcal{X}$ as above is compatible with the action of the Galois group on covers and on degeneration data, Moreover one has a realisation result for such degeneration data (cf. example 1.2.5 which illustrates the realistaion of degeneration data). The proof of the above result relies heavily on the results in [Sa] and [Sa-1]. We also use formal patching techniques à la Harbater. As an application of the above result we construct in 2.1.8 an example of a covers $f : \mathcal{Y} \to \mathcal{X}$ as above where $\mathcal{X}$ is smooth at the closed point x, and where the special fibre $\mathcal{Y}_k := \mathcal{Y} \times_R k$ of $\mathcal{Y}$ is singular and unibranche at the closed point y, and such that the configuration of the special fibre of the semi-stable model $\tilde{\mathcal{Y}}$ of $\mathcal{Y}$ is not a tree-like. This indeed answers a question raised by Lorenzini whether such a situation can occur in the inequal characteristic case.

In the second part of the paper we study the global situation of a Galois cover $f : Y \to X$ of degree p above a proper and semi-stable R-curve X with Y normal. As above, and using the theorem of semi-stable reduction for curves (cf. [De-Mu]), one obtains after eventually a finite extension of R a semi-stable model $\tilde{Y} \to Y$ of Y and a Galois cover $f : \tilde{Y} \to \tilde{X}$ of degree p, where $\tilde{X}$ is a semi-stable blow-up of X. We associate then canonically to the cover $f : Y \to X$ some "degeneration data" which determine completely the special fibre $\tilde{Y}_k := \tilde{Y} \times_R k$ of the semi-stable model $\tilde{Y}$ of Y (cf. 2.2). This consists of the graph associated to the semi-stable k-curve X_k, and a given "mixed torsor" above X_k, plus given local degeneration data at the critical points of this mixed torsor (cf. 2.2.1 for a more precise definition). For an illustration of this we refeer to the example 2.2.3. Let's denote by $\mathrm{Deg}_p(X_k)$ the set of isomorphism classes of such degeneration data which are defined over $\overline{k}$. We show that this set is a G_k-set in a canonical way. Let $\overline{\eta}$ be the geometric generic point of X. The étale cohomology group $H^1_{\text{et}}(\overline{\eta}, \mu_p)$ classifies (pointed)-geometric Galois covers of X of degree p. Moreover our result which exhibit the above degeneration data can be translated as the existence of a canonical specialisation map $\text{Sp} : H^1_{\text{et}}(\overline{\eta}, \mu_p) \to \text{Deg}_p(X_k)$. Note that both $H^1_{\text{et}}(\overline{\eta}, \mu_p)$ and $\text{Deg}_p(X_k)$ are G_K-sets, where G_K is the Galois group of $\overline{K}$ over K, and G_K acts on Deg_p via its canonical quotient G_k. Our main result in the second part of the paper is the following:

Theorem (2.3.2). *The above specialisation map* $\text{Sp} : H^1_{\text{et}}(\overline{\eta}, \mu_p) \to \text{Deg}_p(X_k)$ *is* G_K-*equivariant.*

The above specialisation map Sp can not be surjective in this global case. However we prove the following result of realisation of degeneration data:

Theorem (2.3.1). *Let X_k be a proper and semi-stable k-curve. Let R be a complete discrete valuation ring of inequal characteristics with residue field k, and fractions field K. Suppose given a $\overline{k}$-degeneration data $\deg(X_k)$ of rank p associated to X_k (in other words suppose given an element of $\text{Deg}_p(X_k)$). Then there exists, after eventually a finite extension of R, a proper and semi-stable R-curve $\tilde{X}$ with smooth generic fibre and a special fibre $\tilde{X}_k := \tilde{X} \times_R k$ isomorphic to X_k, and a Galois cover $\tilde{f} : \tilde{Y} \to \tilde{X}$, such that the degeneration data associated to f is isomorphic to the given data $\deg(X_k)$.*

We refeer to the example 2.3.4 which illustrate the realisation of global degeneration data. Finally, I believe the ideas presented in this paper provide the framework to construct Hurwitz spaces over $\mathbb{Z}$ whose fibre in characteristic zero classifies Galois covers of degree p plus extra data. Also in the case of Galois covers of degree p of the projective line over a p-adic field these results should lead to an algorithm which compute the semi-stable reduction of these covers at least in the case where the number of branched points is smaller than p.

I. Semi-stable reduction of Galois covers of degree p above formal germs of curves with Galois action.

1.0. In what follows we use the following notations: R is a complete discrete valuation ring of inequal characteristics, with residue characteristic $p > 0$, and which contains a primitive p-th root of unity ζ. We denote by K the fraction field of R, π a uniformising parameter, and k the residue field of R. Let $\overline{K}$ be a fixed algebraic closure of K, and let G_K be the Galois group of $\overline{K}$ of K. Let $\overline{R}$ be the integral closure of R in $\overline{K}$ which is a valuation ring, and let $\overline{k}$ be the residue field of $\overline{R}$ which is an algebraic closure of k. Let G_k be the Galois group of a separable closure of k contained in $\overline{k}$. We have the following canonical exact sequence, where I_K denotes the inertia subgroup:

$$0 \to I_K \to G_K \to G_k \to 0$$

In this section we will consider a formal germ $\mathcal{X}$ of a semi-stable R-curve at a closed point, a Galois cover $f : \mathcal{Y} \to \mathcal{X}$ with group $G = \mathbb{Z}/p\mathbb{Z}$, and we will study the semi-stable reduction of $\mathcal{Y}$, as well as the action of G_K on these data.

1.1. Let $\mathcal{X} := \operatorname{Spec} \hat{\mathcal{O}}_{X,x}$ be the formal germ of an R-curve X at a closed point x, and let $f : \mathcal{Y} \to \mathcal{X}$ be a Galois cover with group $G \simeq \mathbb{Z}/p\mathbb{Z}$, such that $\mathcal{Y}$ is normal and local. It follows then easily from the theorem of semi-stable reduction for curves (cf. [De-Mu], as well as the compactification process in [Sa-1] 2.3), that after eventually a finite extension R' of R with residue field k', and fractions field K', the formal germ $\mathcal{Y}$ has a semi-stable reduction. More precisely there exists a birational and proper morphism $\tilde{f} : \tilde{\mathcal{Y}} \to \mathcal{Y}'$, where $\mathcal{Y}'$ is the normalisation of $\mathcal{Y} \times_R R'$, such that $\tilde{\mathcal{Y}}_{K'} \simeq \mathcal{Y}'_{K'}$, and the following conditions hold:

(i) The special fibre $\tilde{\mathcal{Y}}_{k'} := \tilde{\mathcal{Y}} \times_{\operatorname{Spec} R'} \operatorname{Spec} k'$ of $\tilde{\mathcal{Y}}$ is reduced.
(ii) $\tilde{\mathcal{Y}}_k$ has only ordinary double points as singularities.

Moreover there exists such a semi-stable model $\tilde{f} : \tilde{\mathcal{Y}} \to \mathcal{Y}'$ which is **minimal** for the above properties. In particular the action of G on $\mathcal{Y}'$ extends to an action on $\tilde{\mathcal{Y}}$. Let $\tilde{\mathcal{X}}$ be the quotient of $\tilde{\mathcal{Y}}$ by G, which is a semi-stable model of $\mathcal{X}$. One has the following commutative diagram:

$$\begin{array}{ccc} \tilde{\mathcal{Y}} & \xrightarrow{\tilde{f}} & \mathcal{Y}' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f'} \\ \tilde{\mathcal{X}} & \xrightarrow{\tilde{g}} & \mathcal{X}' \end{array}$$

With the same notations as above, one can moreover choose the semi-stable models $\tilde{\mathcal{Y}}$ and $\tilde{\mathcal{X}}$ such that the set of points $B_{K'} := \{x_{i,K'}\}_{1 \le i \le r}$, consisting of the branch locus in the morphism $\mathcal{Y}'_{K'} \to \mathcal{X}'_{K'}$, which we may assume to be **rational**, specialise in **smooth** (resp. **smooth distincts**) points of $\mathcal{X}'_{k'}$. We may also, after eventually a finite extension of K, suppose that the double points of $\tilde{\mathcal{X}}_k$ are rational. Moreover one can choose such $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{Y}}$ which are minimal for these properties. We will denote by $\tilde{f}^{\mathrm{nsp}} : \tilde{\mathcal{Y}}^{\mathrm{nsp}} \to \mathcal{Y}'$ (resp. $\tilde{f}^{\mathrm{sp}} : \tilde{\mathcal{Y}}^{\mathrm{sp}} \to \mathcal{Y}'$) the minimal semi-stable model of f such that the points of B_K specialise in smooth (resp. smooth distincts) points of $\tilde{\mathcal{X}}^{\mathrm{nsp}} := \tilde{\mathcal{Y}}^{\mathrm{nsp}}/G$ (resp. of $\tilde{\mathcal{X}}^{\mathrm{sp}} := \tilde{\mathcal{Y}}^{\mathrm{sp}}/G$). We call $\tilde{\mathcal{Y}}^{\mathrm{nsp}}$ (resp. $\tilde{\mathcal{Y}}^{\mathrm{sp}}$) the minimal **non split** (resp. **split**) semi-stable model of $\mathcal{Y}$. We have the following commutative diagrams:

$$\begin{array}{ccc} \tilde{\mathcal{Y}}^{\mathrm{nsp}} & \xrightarrow{\tilde{f}^{\mathrm{nsp}}} & \mathcal{Y}' \\ {\scriptstyle g^{\mathrm{nsp}}}\downarrow & & {\scriptstyle f'}\downarrow \\ \tilde{\mathcal{X}}^{\mathrm{nsp}} & \xrightarrow{\tilde{g}^{\mathrm{nsp}}} & \mathcal{X}' \end{array}$$

and

$$\begin{array}{ccc} \tilde{\mathcal{Y}}^{\mathrm{sp}} & \xrightarrow{\tilde{f}^{\mathrm{sp}}} & \mathcal{Y}' \\ {\scriptstyle g^{\mathrm{sp}}}\downarrow & & {\scriptstyle f'}\downarrow \\ \tilde{\mathcal{X}}^{\mathrm{sp}} & \xrightarrow{\tilde{g}^{\mathrm{sp}}} & \mathcal{X}' \end{array}$$

Moreover the morphism $\tilde{f}^{\mathrm{sp}} : \tilde{\mathcal{Y}}^{\mathrm{sp}} \to \mathcal{Y}'$ factors throught $\tilde{f}^{\mathrm{nsp}}$ as follows: $\tilde{\mathcal{Y}}^{\mathrm{sp}} \to \tilde{\mathcal{Y}}^{\mathrm{nsp}} \to \mathcal{Y}'$, and the later commutative diagram factors through the first one. In the case where $\mathcal{X}$ is the formal germ of a semi-stable R-curve at a closed point x, the fibre $(\tilde{g}^{\mathrm{nsp}})^{-1}(x)$ (resp. $(\tilde{g}^{\mathrm{sp}})^{-1}(x)$) of the closed point x in $\tilde{\mathcal{X}}^{\mathrm{nsp}}$ (resp. in $\tilde{\mathcal{X}}^{\mathrm{sp}}$) is a tree Γ^{nsp} (resp. Γ^{sp}) of projective lines. This tree is canonically endowed with some "degeneration data" that we will exhibit below, and which follow mainly from the results in [Sa] and [Sa-1].

1.2. We will use the same notations as in 1.1. We first consider the case where $\mathcal{X} \simeq \operatorname{Spf} A$ is the formal germ of a semi-stable R-curve at a **smooth** point x. Let R' be a finite extension of R as in 1.1, and let π' be a uniformiser of R'. We assume that $\mathcal{X}$ is geometrically connected, in which case $\mathcal{Y}$ is also geometrically connected. Below we exhibit the **degeneration data** associated to the non split (resp. split) semi-stable reduction of $\mathcal{Y}$, and which are consequences of the results in [Sa] and [Sa-1].

Deg.1. Let $\wp := (\pi')$ be the ideal of $A' := A \otimes_R R'$ generated by π', and let $\hat{A}'_\wp$ be the completion of the localisation of A' at $\wp$. Let $\mathcal{X}'_\eta := \operatorname{Spf} \hat{A}'_\wp$ be the boundary of $\mathcal{X}'$, and let $\mathcal{X}'_\eta \to \mathcal{X}'$ be the canonical morphism. Consider the following cartesian diagram:

$$\begin{array}{ccc} \mathcal{Y}_\eta & \xrightarrow{f_\eta} & \mathcal{X}'_\eta \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{f'} & \mathcal{X}' \end{array}$$

Then $f_\eta : \mathcal{Y}_\eta \to \mathcal{X}'_\eta$ is a torsor under a commutative finite and flat R'-group scheme $G_{R'}$ of rank p. Let δ be the degree of the different associated to the torsor f_η. One has the degeneration type $(G_{k'}, m, h)$ of the torsor f_η (cf. [Sa] 3.2) which is canonically associated to f. The geometric genus g_y of the point y equals $(r - m - 1)(p - 1)/2$ (cf. [Sa-1] 3.1.1), where r is the cardinality of $B_{K'}$.

Deg.2. The fibre $(\tilde{g}^{\text{nsp}})^{-1}(x)$ (resp. $(\tilde{g}^{\text{sp}})^{-1}(x)$) of the closed point x of $\mathcal{X}'$ in $\tilde{\mathcal{X}}^{\text{nsp}}$ (resp. in $\tilde{\mathcal{X}}^{\text{sp}}$) is a tree Γ^{nsp} (resp. Γ^{sp}) of projective lines, Γ^{nsp} is a sub-tree of Γ^{sp}. Let $\text{Vert}(\Gamma^{\text{nsp}}) := \{X_i\}_{i\in I^{nsp}}$ (resp. $\text{Vert}(\Gamma^{\text{sp}}) := \{X_i\}_{i\in I^{sp}}$) be the set of irreducible components of $(\tilde{g}^{\text{nsp}})^{-1}(x)$ (resp. of $(\tilde{g}^{\text{sp}})^{-1}(x)$), which are also the vertices of the tree Γ^{nsp} (resp. Γ^{sp}). The tree Γ^{sp} (hence also Γ^{nsp}) is canonically endowed with an origin vertex X_{i_0}, which is the unique irreducible component of $(\tilde{g}^{\text{sp}})^{-1}(x)$ which meets the point x. We fix an orientation of the tree Γ^{sp} starting from X_{i_0} in the direction of the ends. Such an orientation induces of course an orientation of the subtree Γ^{nsp}.

Deg.3. For each $i \in I^{\text{nsp}}$ (resp. I^{sp}), let $\{x_{i,j}\}_{j\in S_i}$ be the set of points of X_i in which specialise some points of $B_{K'}$ (S_i may be empty), say in each point $x_{i,j}$ specialise $r_{i,j}$ points of B_K. If $i \in I^{\text{sp}}$, and S_i is non empty, we have $r_{i,j} = 1$. Also let $\{z_{i,j}\}_{j\in D_i}$ be the set of points of the irreducible component X_i where X_i meets the rest of the components of $\tilde{\mathcal{X}}_{k'}^{\text{nsp}}$ (resp. of $\tilde{\mathcal{X}}_{k'}^{\text{sp}}$). These are the double points of $\tilde{\mathcal{X}}_{k'}^{\text{nsp}}$ (resp. $\tilde{\mathcal{X}}_{k'}^{\text{sp}}$) supported by X_i. We denote by $B_{k'}^{\text{nsp}}$ (resp. $B_{k'}^{\text{sp}}$) the set of all points $\cup_{i\in I^{\text{nsp}}}\{x_{i,j}\}_{j\in S_i}$ (resp. $\cup_{i\in I^{\text{sp}}}\{x_{i,j}\}_{j\in S_i}$), which is the set of specialisation of the branch locus $B_{K'}$, and by $D_{k'}^{\text{nsp}}$ (resp. $D_{k'}^{\text{sp}}$) the set of double points of $\tilde{\mathcal{X}}_{k'}^{\text{nsp}}$ (resp. $\tilde{\mathcal{X}}_{k'}^{\text{sp}}$). In particular $x_{i_0,j_0} := x$ is a double point of $\tilde{\mathcal{X}}_{k'}^{\text{nsp}}$ (resp. of $\tilde{\mathcal{X}}_{k'}^{\text{sp}}$).

Deg.4. Let $\mathcal{U}^{\text{nsp}} := \mathcal{X}^{\text{nsp}} - \{B_{k'}^{\text{nsp}} \cup D_{k'}^{\text{nsp}}\}$ (resp. $\mathcal{U}^{\text{sp}} := \mathcal{X}^{\text{sp}} - \{B_{k'}^{\text{sp}} \cup D_{k'}^{\text{sp}}\}$). The cover f induces a *mixed torsor* $f^{\text{nsp}} : \mathcal{V} \to \mathcal{U}^{\text{nsp}}$ above $\mathcal{U}^{\text{nsp}}$ (resp. $f^{\text{sp}} : \mathcal{V}' \to \mathcal{U}^{\text{sp}}$ above $\mathcal{U}^{\text{sp}}$), whose special fibre $f_{k'}^{\text{nsp}} : \mathcal{V}_{k'} \to \mathcal{U}_{k'}^{\text{nsp}}$ (resp. $f_{k'}^{\text{sp}} : \mathcal{V}'_{k'} \to \mathcal{U}_{k'}^{\text{sp}}$) is an element of $H^1_{\text{fppf}}(\mathcal{U}_{k'}^{\text{nsp}})_p$ (resp. an element of $H^1_{\text{fppf}}(\mathcal{U}_{k'}^{\text{sp}})_p$) (cf. [Sa] 1.6 for the definition of $H^1_{\text{fppf}}(\ )_p$). In particular the restriction of f^{nsp} (resp. f^{sp}) to each connected component $\mathcal{U}_i$ of $\mathcal{U}^{\text{nsp}}$ (resp. $\mathcal{U}^{\text{sp}}$) is a torsor $f_i : \mathcal{V}_i \to \mathcal{U}_i$ under a commutative finite and flat R'-group scheme $G_{R',i}$ of rank p, and $f_{i,k'} : \mathcal{V}_{i,k'} \to \mathcal{U}_{i,k'} := \mathcal{U}_i \times_{R'} k'$ is a torsor under the k'-group scheme $G_{k',i} := G_{R',i} \times_{R'} k'$. Moreover if $G_{k',i}$ is radicial, and if ω_i is the associated differential form, then the set of zeros and poles of w_i is necessarily contained in $\{x_{i,j}\}_{j\in S_i} \cup \{z_{i,j}\}_{j\in D_i}$, as $\mathcal{V}_{i,k'}$ is smooth (cf. [Sa], I).

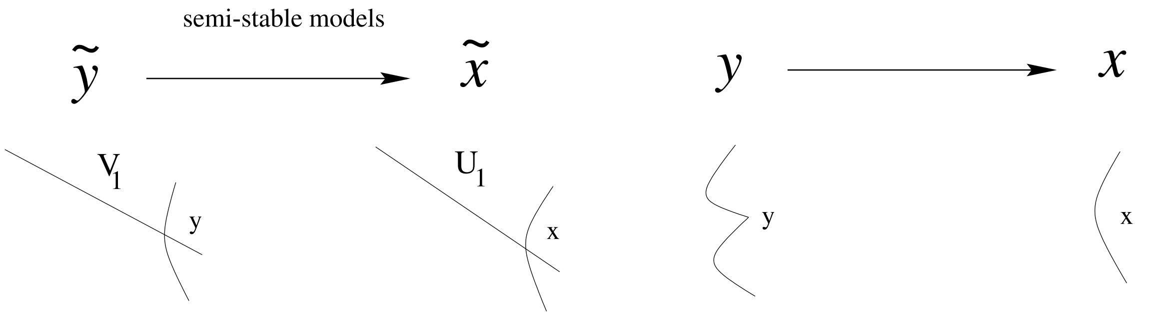
Deg.5. Each smooth point $x_{i,j} \in B_{k'}^{\text{nsp}}$ (resp. $\in B_{k'}^{\text{sp}}$) is endowed via f with degeneration data on the boundary of the formal fibre at $x_{i,j}$, as in Deg.1 above, and which satisfy certain *compatibility conditions*. More precisely for each smooth point $x_{i,j}$ we have the reduction type $(G_{k',i}, m_{i,j}, h_{i,j})$ on the boundary of the formal fibre at this point, induced by g^{nsp} (resp. g^{sp}), and such that $r_{i,j} = m_{i,j} + 1$, as a consequence of [Sa-1] 3.1.1. Also if $x_{i,j} \in B_{k'}^{\text{sp}}$ (in other words if $i \in I^{\text{sp}}$, and S_i is non empty), then necessarily $G_{k',i} = \mu_p$, $m_{i,j} = -1$ and $h_{i,j} = 0$ (cf. [Sa-1, 3.1.2).

Deg.6. Each double point $z_{i,j} = z_{i',j'} \in X_i \cap X_{i'}$ of $\tilde{X}^{\text{nsp}}$ (resp. $\tilde{X}^{\text{sp}}$) is endowed with degeneration data $(G_{i,k'}, m_{i,j}, h_{i,j})$ and $(G_{i',k'}, m_{i',j'}, h_{i',j'})$, induced by g^{nsp} (resp. g^{sp}), on the two boundaries of the formal fibre at this point as in Deg.1, such that $m_{i,j}+m_{i',j'}=0$, and $h_{i,j}+h_{i',j'}=0$, as a consequence of [Sa-1] 3.2.3. In particular $m_{i_0,j_0}+m=0$, and $h_{i_0,j_0}+h=0$. In other words the above induced by f element of $H^1_{\text{fppf}}(\mathcal{U}^{\text{nsp}}_{k'})_p$ (resp. of $H^1_{\text{fppf}}(\mathcal{U}^{\text{sp}}_{k'})_p$) is indeed an element of $H^1_{\text{fppf}}(\mathcal{U}^{\text{nsp}}_{k'})^{\text{kum}}_p$ (resp. an element of $H^1_{\text{fppf}}(\mathcal{U}^{\text{sp}}_{k'})^{\text{kum}}_p$) (cf. [Sa] 1.7 for the definition of $H^1_{\text{fppf}}(\ )^{\text{kum}}$).

Deg.7. For each double point $z_{i,j}=z_{i',j'}\in X_i\cap X_{i'}$ of $\tilde{X}^{\text{nsp}}$ (resp. $\tilde{X}^{\text{sp}}$), let $e_{i,j}$ be the thikeness of $z_{i,j}$. Then $e_{i,j}=pt_{i,j}$ is necessarily divisible by p. For each irreducible component X_i, $i\in I^{nsp}$ (resp. $i\in I^{sp}$), let η_i be the generic point of X_i. Let δ_i be the degree of different above η_i in the cover g^{nsp} (resp. g^{sp}). Then we have $\delta_i-\delta_{i'}=t_{i,j}m_{i,j}(p-1)$ as follows from [Sa-1] 3.2.5. In particular $\delta-\delta_{i_0}=t_{i_0,j_0}m(p-1)$, and $|D_i|\delta_i=\sum_{j\in D_i}(\delta_j+t_{i,j}m_{i,j}(p-1))$, where $|D_i|$ denotes the cardinality of D_i.

Deg.8. The contribution to the arithmetic genus g_y of the point y was computed in [Sa-1] 3.1.1, in terms of the degeneration data $(G_{k'},m,h)$ on the boundary of the formal fibre at x, and the cardinality r of B_K. It follows also from the above considerations and after easy calculation that: $g_y=\sum_{i\in I_{\text{et}}}(-2+\sum_{j\in S_i}(m_{i,j}+1)+\sum_{j\in D_i}(m_{i,j}+1))(p-1)/2$, where I_{et} denote the subset of I^{nsp}, or I^{sp}, which among to the same, consisting of those i for which $G_{k',i}$ is étale.

1.2.1. Example. In the following we give an example where one can exhibit the degeneration data associated to a Galois cover $f:\mathcal{Y}\to\mathcal{X}$ of degree p where $\mathcal{X}\simeq\operatorname{Spf}R[[T]]$ is the formal germ of a smooth point. More precisely, for $m>0$ an integer prime to p consider the cover given generically by the equation $X^p=1+\lambda^p(T^{-m}+\pi T^{-m+1})$ (this is the example 1 in [Sa-1], 3.2.4, with $m'=m+1$). Hier $r=m+2$ and this cover has a reduction of type $(\mathbb{Z}/p\mathbb{Z},m,0)$ on the boundary. In particular the geometric genus g_y of the closed point y of $\mathcal{Y}$ equals $(p-1)/2$. The non split degeneration data associated to the above cover consists necessarily of a treee with only one vertex and no edges, i.e. a unique projective line X_1 with a marked $\overline{k}$-point x_1, and an étale torsor $f_1:V_1\to U_1:=X_1-\{x_1\}$ above U_1 with conductor 2 at the point x_1.



The above considerations leads naturally to the following abstract geometric and combinatorial definition of degeneration data.

1.2.2. Definition. Let k' be an algebraic extension of k. A k'**-simple non split (resp. split) degeneration data** $\mathrm{Deg}(\mathrm{x})$ of type $(r, (G_{k'}, m, h))$ and rank p consists of the following data:

Deg.1. $G_{k'}$ is a commutative finite and flat k'-group scheme of rank p, $r \geq 0$ is an integer, m is an integer prime to p such that $r - m - 1 \geq 0$, and $h \in \mathbb{F}_p$ equals 0 unless $G_{k'} = \mu_p$ and $m = 0$.

Deg.2. $\Gamma := X_{k'}$ is an oriented tree of projective lines with vertices $\mathrm{Vert}(\Gamma) := \{X_i\}_{i \in I}$ (resp. $\mathrm{Vert}(\Gamma) := \{X_i\}_{i \in I^{\mathrm{sp}}}$), endowed with an origin vertex X_{i_0}, and a marked smooth point $x := x_{i_0,j_0}$ on X_{i_0} which is a k'-rational point. We denote by $\{z_{i,j}\}_{j \in D_i}$ the set of double points, or (non oriented) edges, of Γ which are supported by X_i, and which we assume to be k'-rational points.

Deg.3. For each vertex X_i of Γ is given a set, may be empty, of smooth marked k'-rational points $\{x_{i,j}\}_{j \in S_i}$.

We will summarise the data Deg.2 and Deg.3 by saying that we are given a k'**-rational marked tree** of projective lines, or a marked semi-stable k'-curve with arithmetic genus 0.

Deg.4. For each $i \in I$ (resp. $i \in I^{\mathrm{sp}}$), is given a torsor $f_{i,k'} : V_{i,k'} \to U_{i,k'} := X_{i,k'} - \{\{x_{i,j}\}_{j \in S_i} \cup \{z_{i,j}\}_{j \in D_i}\}$ under a commutative finite and flat k'-group scheme $G_{k',i}$ of rank p, where $X_{i,k'} := X_i \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k'$, with $V_{i,k'}$ **smooth**. Let $U_{k'}$ be the open subscheme of $X_{k'}$ obtained by deleting the double and marked points of $X_{k'}$. Then the above data determines an element of $H^1_{\mathrm{fppf}}(U_{k'})_p$ (cf. [Sa], I). Moreover if $G_{k',i}$ is radicial, and if ω_i is the associated differential form, then the set of zeros and poles of w_i is necessarily contained in $\{x_{i,j}\}_{j \in S_i} \cup \{z_{i,j}\}_{j \in D_i}$, as $V_{i,k'}$ is smooth. Also if $i \in I^{\mathrm{sp}}$, and S_i is non empty, we assume that $G_{i,k'} = \mu_p$.

Deg.5. For each $i \in I$ (resp. $i \in I^{\mathrm{sp}}$), are given for each $j \in S_i$ the integres $(m_{i,j}, h_{i,j})$, where $m_{i,j}$ is the conductor of the torsor f_i at the point $x_{i,j}$, and $h_{i,j}$ its residue at this point. Let $r_{i,j} := m_{i,j} + 1$. If $i \in I^{\mathrm{sp}}$ then we assume that $r_{i,j} = 1$. Also for the marked point x_{i_0,j_0} is given the data $(m_{i_0,j_0}, h_{i_0,j_0})$, such that $m_{i_0,j_0} + m = 0$, and $h_{i_0,j_0} + h = 0$.

Deg.6. For each double point $z_{i,j} = z_{i',j'} \in X_i \cap X_{i'}$ is given the pair $(m_{i,j}, h_{i,j})$ (resp. $(m_{i',j'}, h_{i',j'})$), where $m_{i,j}$ (resp. $m_{i',j'}$) is the conductor of the torsor $f_{i,k'}$ (resp. $f_{i',k'}$) at the point $z_{i,j}$ (resp. $z_{i',j'}$), and $h_{i,j}$ (resp. $h_{i',j'}$) its residue at this point. These data must satisfy $m_{i,j} + m_{i',j'} = 0$, and $h_{i,j} + h_{i',j'} = 0$. In other words the element of $H^1_{\mathrm{fppf}}(U_{k'})_p$ determined by the data Deg.4 must belong to the subgroup $H^1_{\mathrm{fppf}}(U_k)_p^{\mathrm{kum}}$ (cf. [Sa]. I).

Deg.7. For each irreducible component X_i of Γ is given an integer $\delta_i \leq v_K(p)$ which is divisible by $p-1$. For each double point $z_{i,j} = z_{i',j'} \in X_i \cap X_{i'}$ of Γ is given an integer $e_{i,j} = pt_{i,j}$ divisible by p, such that with the same notations as above we have $\delta_i - \delta_{i'} = m_{i,j} t_{i,j}(p-1)$.

Deg.8. Let I_{et} denote the subset of I (resp. of I^{sp}) consisting of those i for which $G_{k',i}$ is étale. Then the following equality should hold: $(r-m-1)(p-1)/2 = \sum_{i\in I_{\text{et}}}(-2 + \sum_{j\in S_i}(m_{i,j}+1) + \sum_{j\in D_i}(m_{i,j}+1))(p-1)/2$. The integer $g_x := (r-m-1)(p-1)/2$ is called the **genus** of the degeneration data Deg(x).

There is a natural notion of isomorphism of simple split (resp. non split) degeneration data of a given type over a given algebraic extension of k. We will denote by SS-Deg$_{\mathbf{p}}$ (resp. SNS-Deg$_{\mathbf{p}}$) the set of isomorphism classes of $\overline{k}$-simple split (resp. k'-simple non split) degeneration data of rank p.

1.2.3. Galois action on degeneration data.

The Galois group G_k of the separable closure of k contained in $\overline{k}$ acts in a canonical way on the set SS-Deg$_p$ (resp. SNS-Deg$_p$) of isomorphism classes of $\overline{k}$-simple split (resp. non split) degeneration data of rank p, in a way that is compatible with the group law on G_k. In other words SS-Deg$_p$ and SNS-Deg$_p$ are G_k-sets. More precisely, to a $\overline{k}$-simple split (resp. non split) degeneration data Deg(x) of type $(r,(G_{\overline{k}},m,h))$, and an element $\sigma \in G_k$, we associate the simple degeneration data $\text{Deg}(x)^\sigma$ of the same type $(r, G_{\overline{k}}, m, h))$. We explain briefly the defining data for $\text{Deg}(x)^\sigma$:

Deg$^\sigma$.1. The data Deg$^\sigma$.1 are the same as in Deg.1.

Deg$^\sigma$.2. and Deg$^\sigma$.3. The $\overline{k}$-rational marked tree Γ^σ is the marked tree Γ.

Deg$^\sigma$.4. For each $i \in I$ (resp. $i \in I^{\text{sp}}$), is given the torsor $f_i^\sigma : \overline{V}_i^\sigma \to \overline{U}_i := \overline{X}_i - \{\{x_{i,j}\}_{j\in S_i} \cup \{z_{i,j}\}_{j\in D_i}\}$ under the commutative finite and flat $\overline{k}$-group scheme $G_{\overline{k},i}$ of rank p, where $\overline{X}_i := X_i \times_k \overline{k}$, and f_i^σ is the image of the torsor f_i above $\overline{U}_i$ given by the data Deg.4, via the action of σ on torsors (cf. [Sa], I). In other words the element $(f_t')_t$ of $H^1_{\text{fppf}}(U_{\overline{k}})_p$ determined by the data Deg$^\sigma$.4 is the transform $(f_t)_t{}^\sigma$ of the element $(f_t)_t$ of $H^1_{\text{fppf}}(U_{\overline{k}})_p$ determined by the data Deg.4, under the canonical action of σ on $H^1_{\text{fppf}}(U_{\overline{k}})_p$ (cf. [Sa], I).

Deg$^\sigma$.5. For each $i \in I$ (resp. $i \in I^{\text{sp}}$), are given for each $j \in S_i$ the integers $(m_{i,j}^\sigma, h_{i,j}^\sigma)$, where $m_{i,j}^\sigma$ is the conductor of the torsor f_i^σ at the point $x_{i,j}^\sigma$ of S_i^σ, and $h_{i,j}^\sigma$ its residue at this point. We have $m_{i,j}^\sigma = m_{i,j}$ and $h_{i,j}^\sigma = h_{i,j}$.

Deg$^\sigma$.6. For each double point $z_{i,j}^\sigma = z_{i',j'}^\sigma \in X_i \cap X_{i'}$ is given the pair $(m_{i,j}^\sigma, h_{i,j}^\sigma)$ (resp. $(m_{i',j'}^\sigma, h_{i',j'}^\sigma)$), where $m_{i,j}^\sigma$ (resp. $m_{i',j'}^\sigma$) is the conductor of the torsor f_i^σ (resp. $f_{i'}^\sigma$) at the point $z^\sigma_{i,j}$ (resp. $z^\sigma_{i',j'}$), and $h^\sigma_{i,j}$ (resp. $h^\sigma_{i',j'}$) its residue at this point. We have $m^\sigma_{i,j} = m_{i,j}$ and $h^\sigma_{i,j} = h_{i,j}$.

Deg$^\sigma$.7. For each double point $z^\sigma_{i,j} = z^\sigma_{i',j'} \in X^\sigma_i \cap X^\sigma_{i'}$ of Γ^σ is given the integer $e^\sigma_{i,j} = pt^\sigma_{i,j}$ divisible by p. For each irreducible component X_i of Γ is given the integer $\delta^\sigma_i \le v_K(p)$ which is divisible by $p-1$, such that with the same notations as above $\delta^\sigma_i - \delta^\sigma_{i'} = m^\sigma_{i,j} t^\sigma_{i,j}(p-1)$. We have $e^\sigma_{i,j} = e_{i,j}$, and $\delta^\sigma_i = \delta_i$.

Deg$^\sigma$.8. The integer $g_x := (r-m-1)(p-1)/2$ is the **genus** of the degeneration data D^σ_x.

In conclusion, the action of the group G_k on the set of isomorphism classes of simple degeneration data is given, and completely determined, through the canonical action of G_k on the data Deg.4, which is essencially given by the canonical action of G_k on the group $H^1_{\text{fppf}}(U_{\overline{k}})_p$ and which was studied in [Sa].

Let $\mathcal{X} := \operatorname{Spf} A$ be the formal germ of an R-curve at a smooth closed point x, and let $\overline{\mathcal{X}} := \operatorname{Spf} \overline{A}$ where $\overline{A} := A \otimes_R \overline{R}$. Let $L := \operatorname{Fr} \overline{A}$ be the fractions field of $\overline{A}$. In 1.2 above we associated to a Galois cover $f : \mathcal{Y} \to \mathcal{X}$ of degree p, above a formal germ of an R-curve at a smooth point x, a simple non split (resp. split) degeneration data D_x. This indeed can be interpreted as the existence of a canonical specialisation map:

$$\operatorname{Sp} : H^1_{\text{et}}(\operatorname{Spec} L, \mu_p) \to \text{SS-Deg}_p$$

resp.

$$\operatorname{Sp} : H^1_{\text{et}}(\operatorname{Spec} L, \mu_p) \to \text{SNS-Deg}_p$$

Reciproqually, we have the following result of realisation for degeneration data for such covers:

1.2.4. Theorem. *Let k' be a finite extention of k. Let $\operatorname{Deg}(x)$ be a k'-simple non split (resp. split) degeneration data of type $(r, (G_k, m, h))$ and rank p. Then there exists, after eventually a finite ramified extension of R, a non-uniquely determined Galois cover $f : \mathcal{Y} \to \mathcal{X}$ of degree p, where $\mathcal{X}$ is a formal germ of an R-curve at a smooth point x, and such that the degeneration data associated to f via 1.2. is isomorphic to $\operatorname{Deg}(x)$. In other words the above specialisation maps $\operatorname{Sp} : H^1_{\text{et}}(\operatorname{Spec} L, \mu_p) \to \text{SS-Deg}_p$ and $\operatorname{Sp} : H^1_{\text{et}}(\operatorname{Spec} L, \mu_p) \to \text{SNS-Deg}_p$ are surjective.*

Proof. One uses the formal patching results as explained in [Sa-1], plus the examples given in [Sa-1] 3.1.3, 3.1.4 and 3.2.4. We may also assume that $k = k'$. First, for each $i \in I^{\text{nsp}}$ (resp. $i \in I^{\text{sp}}$), let $\mathcal{U}_i$ be a formal affine scheme with special fibre $U_i := U_{k,i}$. The given torsor $f_i : V_i \to U_i$ is admissible (cf. [Sa], IV), hence can be lifted, after eventually a ramified extension of R, to a torsor $f_i' : \mathcal{V}_i \to \mathcal{U}_i$ under a finite and flat R-group scheme of rank p, which is either μ_p or $\mathcal{H}_{R,n}$ for $0 < n \leq v_K(\lambda)$ where $n := (v_K(p) - \delta_i)/(p-1)$. Such a lifting is non-unique in the radicial case (cf. loc. cit). Moreover, for each marked point $x_{i,j}$ (resp. double point $z_{i,j}$) one can find a Galois cover $f_{i,j} : \mathcal{Y}_{i,j} \to \mathcal{X}_{i,j}$ of degree p where $\mathcal{X}_{i,j}$ is a formal germ of a smooth point $x_{i,j}$, and $\mathcal{Y}_{i,j}$ is smooth (resp. $f_{i,j} : \mathcal{Y}_{i,j} \to \mathcal{Z}_{i,j}$ of degree p, where $\mathcal{Z}_{i,j}$ is a formal germ of a double point $z_{i,j} = z_{i'j'}$ of thikeness $e_{i,j}$, and $\mathcal{Y}_{i,j}$ is semi-stable) and with reduction type $(G_{i,k}, m_{i,j}, h_{i,j})$ (resp. $((G_{k,i}, m_{i,j}, h_{i,j})$, $(G_{k,i'}, m_{i',j}, h_{i',j}))$ on the boundaries) (cf. [Sa-1] 3.1.3, 3.1.4, 3.2.6). Also one can find a Galois cover $f_x : \mathcal{Y}_x \to \mathcal{X}_x$ of degree p, above a formal germ of a double point x, and with reduction type (G_k, m, h) and $(G_{i_0}, -m, -h)$ on the boundaries of the formal fibre at x, as well as a Galois cover $f_x' : \mathcal{Y}_x' \to \mathcal{X}_x'$ above a formal closed disc $\mathcal{X}_x' := \operatorname{Spf} R < T >$, with degeneration data (G_k, m, h) on the boundary of the formal fibre at the closed point $T = 0$ (cf. [Sa-1] 2.3.1). Now using the formal patching result as in [Sa-1] I, as well as the result 3.1 in [Sa] , and after carefully adjusting the Galois action on the $f_{i,j}$ and the f_i', in order to obtain Galois patching data, one can patch the covers f_i', $f_{i,j}$, f_x and f_x' along the points $x_{i,j}$, $z_{i,j}$ and x, in order to obtain a Galois cover $f' : \tilde{\mathcal{Y}} \to \tilde{\mathcal{X}}$ of degree p, where $\tilde{\mathcal{X}}$ is a formal proper semi-stable R-curve, whose special fibre consists of the tree Γ plus a projective line linked to Γ via the double point x. A G-equivariant contraction of the vertices $(X_i)_i$ of the tree Γ in $\tilde{\mathcal{X}}_k$, will yield to a Galois cover $f : \mathcal{Y}' \to \mathcal{X}'$ of degree p, where $\mathcal{X}'$ is a formal proper and smooth projective line, with a marked point x on $\mathcal{X}_k'$. A formal localisation now at the point x will give the desired cover $f : \mathcal{Y}_x \to \mathcal{X}_x := \operatorname{Spf} \hat{\mathcal{O}}_{\mathcal{X}',x}$, and by construction the simple degeneration data associated to f via 1.2 is isomorphic to D_x.

1.2.5. Example. This in particular is the realisation given by the above theorem 1.2.4 of the degeneration data which arises in the example 1.2.1. Consider the non split simple degeneration data of rank p which consists of a tree with one vertex and no edges. Hence a projective line X_1 with one $\overline{k}$-marked point $x = x_1$, and a given étale torsor $f_1 : V_1 \to U_1 := X_1 - \{x_1\}$ with conductor $m' - m + 1$ at the point x_1 where $m' > m$ are positif integers. Then by 1.2.4 one can construct (after eventually a finite extension of R) a Galois cover $f : \mathcal{Y} \to \mathcal{X}$ of degree p above a formal germ $\mathcal{X}$ at a smooth point x, such that the singularity of $\mathcal{Y}$ at the closed point y is unibranche and the geometric genus of the point y equals $(m' - m)(p-1)/2$, moreover the semi-stable reduction of $\mathcal{Y}$ consists of one irreducible component Y_1 of genus $(m' - m)(p-1)/2$, which is the projective completion of the above affine curve V_1, and which is linked to the point y by a double point.

1.2.6. Now we will study the action of G_K on Galois covers of degree p above formal germs of smooth curve at closed points. This action as we will see extends in a functorial way to an action of G_K on the corresponding semi-stable models. More precisely, let $\mathcal{X} := \operatorname{Spf} A$

be the formal germ of an R-curve at a smooth closed point x, and let $\overline{\mathcal{X}} := \operatorname{Spf} \overline{A}$ where $\overline{A} := A \otimes_R \overline{R}$. Let $L := \operatorname{Fr} \overline{A}$ be the fractions field of $\overline{A}$. Then the Galois group G_K acts in a natural way on cyclic extentions of degree p of L, and we have a canonical homomorphism:

$$(1)\ G_K \to \operatorname{Aut} H^1_{\text{et}}(\operatorname{Spec} L, \mu_p)$$

The above action (1) is equivalent to the action of G_K on normal Galois covers of degree p above $\overline{\mathcal{X}}$. More precisely to each such a cover $f : \mathcal{Y} \to \overline{\mathcal{X}}$ with $\mathcal{Y}$ local and normal, which corresponds to the extension of functions fields: $\operatorname{Spec} L' \to \operatorname{Spec} L$, and an element $\sigma \in G_K$, one associates the Galois cover $f^\sigma : \mathcal{Y}^\sigma \to \overline{\mathcal{X}}$ of degree p, with $\mathcal{Y}^\sigma$ local and normal, which corresponds to the extension of functions fields: $\operatorname{Spec} {L'}^\sigma \to \operatorname{Spec} L$. This action of G_K on covers extends to an action of G_K on corresponding minimal semi-stable models (because of the minimality condition). Hier by a minimal semi-stable model we mean either a split or a non split one. More precisely, let $\tilde{f} : \tilde{\mathcal{Y}} \to \mathcal{Y}$ (resp. $\tilde{f}' : \tilde{\mathcal{Y}}' \to \mathcal{Y}^\sigma$) be a minimal semi-stable model of $\mathcal{Y}$ (resp. of $\mathcal{Y}^\sigma$), then the element $\sigma \in G_K$ maps the semi-stable model $\tilde{f}$ to the semi-stable model $\tilde{f}'$, in the sens that we have a commutative diagram:

$$\begin{array}{ccccc}
\tilde{\mathcal{Y}} & \xrightarrow{\tilde{f}} & \mathcal{Y} & \xrightarrow{f} & \overline{\mathcal{X}} \\
{\scriptstyle\sigma}\downarrow & & {\scriptstyle\sigma}\downarrow & & \downarrow \\
\tilde{\mathcal{Y}}' & \xrightarrow{\tilde{f}^\sigma} & \mathcal{Y}^\sigma & \xrightarrow{f^\sigma} & \overline{\mathcal{X}}
\end{array}$$

where the horizontal arrows are $\overline{\mathcal{X}}$-automorphisms. We first show how the above action of G_K induces an action of G_K on simple degeneration data via its canonical quotient G_k. Our main result then is that this action coincides with the canonical action of G_k on degeneration data as defined in 1.2.2, and that the specialisation maps $\operatorname{Sp} : H^1_{\text{et}}(\operatorname{Spec} L, \mu_p) \to \text{SS-Deg}_p$ and $\operatorname{Sp} : H^1_{\text{et}}(\operatorname{Spec} L, \mu_p) \to \text{SNS-Deg}_p$ are G_K-equivariant.

First we may assume that the Galois cover $f : \mathcal{Y} \to \mathcal{X}' := \mathcal{X} \times_R R'$, as well as its Galois conjugate via σ, are defined over a finite extension R' of R. Let π' be a uniformiser of R', and let $A' := A \otimes_R R'$. Let $\wp := (\pi')$ be the ideal of A' generated by π', and let $\hat{A}'_\wp$ be the completion of the localisation of A' at $\wp$. Let $\mathcal{X}'_\eta := \operatorname{Spf} \hat{A}'_\wp$ be the boundary of $\mathcal{X}'$, and let $\mathcal{X}'_\eta \to \mathcal{X}'$ be the canonical morphism. Consider the following cartesian diagram:

$$\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{f} & \mathcal{X}' \\
\downarrow & & \downarrow \\
\mathcal{Y}_\eta & \xrightarrow{f_\eta} & \mathcal{X}'_\eta
\end{array}$$

Then $f_\eta : \mathcal{Y}_\eta \to \mathcal{X}'_\eta$ is a torsor under a commutative finite and flat R'-group scheme $G_{R'}$ of rank p. Let $(G_{k'}, m, h)$ be the degeneration type of f_η where k' is the residue field of R'. We have also the following cartesian diagram:

$$\begin{array}{ccc} \mathcal{Y}^\sigma & \xrightarrow{f^\sigma} & \mathcal{X}' \\ \downarrow & & \downarrow \\ \mathcal{Y}^\sigma_\eta & \xrightarrow{f^\sigma_\eta} & \mathcal{X}'_\eta \end{array}$$

Moreover the torsor f^σ_η is the Galois conjugate of the torsor f_η by σ as one sees easily. In particular both of these torsors have the same reduction type.

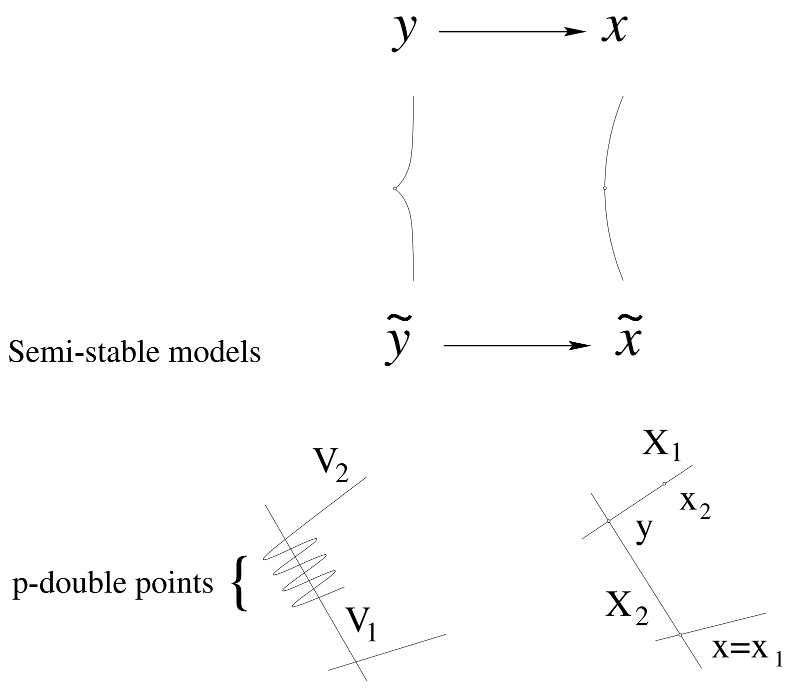
The Galois group $G \simeq \mathbb{Z}/p\mathbb{Z}$ of the cover f (resp. f^σ) acts in a canonical way on $\tilde{\mathcal{Y}}$ (resp. $\tilde{\mathcal{Y}}' := \tilde{\mathcal{Y}}^\sigma$). Let $\tilde{\mathcal{X}} := \tilde{\mathcal{Y}}/G$ which also equals $\tilde{\mathcal{Y}}'/G$. We have a commutative diagram:

$$\begin{array}{ccc} \tilde{\mathcal{Y}} & \longrightarrow & \tilde{\mathcal{X}} \\ {\scriptstyle\sigma}\downarrow & & \downarrow \\ \tilde{\mathcal{Y}}^\sigma & \longrightarrow & \tilde{\mathcal{X}} \end{array}$$

Where the horizontal arrows are Galois covers with group G, and σ is an $\tilde{\mathcal{X}}$-automorphism. Let $\mathrm{Deg}(x_1)$ (resp. $\mathrm{Deg}(x_2)$) be the simple degeneration data associated to the cover f (resp. f^σ). Let $\overline{\sigma}$ be the image of σ in G_k via the canonical surjective homomorphism $G_K \to G_k$. We will prove that $\mathrm{Deg}(x_2)$ is the transform $\mathrm{Deg}(x_1)^{\overline{\sigma}}$ of $\mathrm{Deg}(x,1)$ by $\overline{\sigma}$, via the canonical action of G_k on degeneration data as explained in [Sa], I. The only think we have to check to prove this is that this is true when considering the degenertaion data Deg.4. Let X_i be an irreducible component of $\tilde{\mathcal{X}}_{k'} := \tilde{\mathcal{X}} \times_R k'$. Let D_i be the set of double points of $\tilde{\mathcal{X}}_{k'}$ supported by X_i, and let S_i be the set of smooth points of X_i in which specialise some branched points of f. Let $U_i := X_i - \{S_i \cup D_i\}$, and let $\mathcal{U}_i$ be a formal open subset of $\tilde{\mathcal{X}}$ with special fibre U_i. Let $f_i : \mathcal{V}_i \to \mathcal{U}_i$ (resp. $f'_i : \mathcal{V}'_i \to \mathcal{U}_i$) be the torsor above $\mathcal{U}_i$ induced by f (resp. by f^σ). Then f'_i is the transform f^σ_i by σ of the torsor f_i, via the canonical action of G_K on these torsors. Let $f_{i,k'} : \mathcal{V}_{i,k'} \to \mathcal{U}_{i,k'}$ (resp. $f'_{i,k'} : \mathcal{V}'_{i,k'} \to \mathcal{U}_{i,k'}$) be the special fibre of the torsor f_i (resp. $f'_i := f^\sigma_i$). Let $\overline{\sigma}$ be the image of σ in G_k via the canonical homomorphism $G_K \to G_k$. Then $f'_{i,k'}$ is nothing else but the transform ${f_{i,k'}}^{\overline{\sigma}}$ of $f_{i,k'}$ via $\overline{\sigma}$ (cf. [Sa], I). So indeed we have proven the following:

1.2.7. Theorem. *Let G_K acts on the sets* SS-Deg$_p$ *(resp.* SNS-Deg$_p$*) of isomorphism classes of simple split (resp. simple non split) degeneration data of rank p, via its canonical quotient G_k, and through the natural action of G_k on these sets as defined in 1.2.3. Then the surjective specialisation maps* $\mathrm{Sp} : H^1_{\mathrm{et}}(\mathrm{Spec}\, L, \mu_p) \to$ SS-Deg$_p$ *and* $\mathrm{Sp} : H^1_{\mathrm{et}}(\mathrm{Spec}\, L, \mu_p) \to$ SNS-Deg$_p$ *are G_K-equivariant.*

1.2.8. Remark. It is easy to construct examples of covers $f : \mathcal{Y} \to \mathcal{X}$ as in [Sa-1] 3.1.4, where the special fibre $\mathcal{Y}_k$ is singular and unibranche at the closed point y of $\mathcal{Y}$, and such that the configuration of the special fibre of a semi-stable model $\tilde{\mathcal{Y}}$ of $\mathcal{Y}$ is not a tree-like. This indeed answers a question raised by Lorenzini whether such a situation can occur in the inequal characteristic case. More precisely, consider the non split simple degeneration data $\operatorname{Deg}(x)$ of type $(2, (\mu_p, -1, 0))$ which consists of a graph Γ with two vertices X_1 and X_2 linked by a unique edge y, with given $\overline{k}$-marked points $x = x_1$ on X_1 and $\overline{k}$-marked point x_2 on X_2, and given étale torsors of rank p : $f_1 : V_1 \to U_1 := X_1 - \{x_1\}$ with conductor $m_1 = 1$ at $x = x_1$ and $f_2 : V_2 \to U_2 := X_2 - \{x_2\}$ with conductor $m_2 = 1$ at x_2. Then it follows from 1.2.4 that there exists, after eventually a finite extension of R, a Galois cover $f : \mathcal{Y} \to \mathcal{X}$ of degree p above the formal germ $\mathcal{X} \simeq \operatorname{Spf} R[[T]]$ at the smooth R-point x, such that the $\overline{k}$-simple non split degeneration data associated to the above cover f is the above given one. Moreover by construction the singularity of the closed point y of $\mathcal{Y}$ is unibranche, and the configuration of the (non-split) semi-stable reduction of $\mathcal{Y}$ consists of two projective lines which meet at p-double points (the above cover will be étale in reduction above the double point y), in particular one has $p - 1$ cycles in this configuration.



1.3. We use the same notations as in 1.1. We consider now the case where x is an **ordinary double** point. Let $f : \mathcal{Y} \to \mathcal{X}$ be a Galois cover of degree p, where $\mathcal{X}$ is the formal germ of an R-curve at an ordinary double point x. In a similar way as in 1.2, and using the results of [Sa] and [Sa-1], one can associate a *double degeneration* data to f whose definition we summarise in the following:

1.3.1. Definition. Let k' be an algebraic extension of k. A k'-**double non split (resp. split) degeneration data** $\mathrm{Deg}(x)$ of type $(r,(G_{k',1},m_1,h_1),(G_{k',2},m_2,h_2))$ and rank p consists of the following:

Deg.1. For $i\in\{1,2\}$, $G_{k',i}$ is a finite and flat k'-group scheme of rank p, $r\geq 0$ is an integer, m_i is an integer prime to p such that $r-m_1-m_2\geq 0$, and $h_i\in\mathbb{F}_p$ equals 0 unless $G_{k',i}=\mu_p$ and $m_i=0$.

Deg.2. $\Gamma:=X_{k'}$ is an oriented tree of projective lines with vertices $\mathrm{Vert}(\Gamma):=(X_i)_{i\in I}$ (resp. $\mathrm{Vert}(\Gamma):=(X_i)_{i\in I^{\mathrm{sp}}}$), and $\tilde{\Gamma}=\cup_{i\in\tilde{I}}X_i$, where $\tilde{I}\subset I$, is a geodesic in Γ linking two given vertices X_{i_1} and X_{i_2}, with a given marked smooth k'-rational point $x_1\in X_{i_1}$ (resp. $x_2\in X_{i_2}$). We denote by $\{z_{i,j}\}_{j\in D_i}$ the set of double points, or edges, of Γ which are supported by X_i.

Deg.3. For each vertex X_i of Γ is given a set, may be empty, of smooth marked k'-rational points $\{x_{i,j}\}_{j\in S_i}$.

Deg.4. For each $i\in I$ (resp. $i\in I^{\mathrm{sp}}$), is given a torsor $f_{i,k'}:V_{i,k'}\to U_{i,k'}:=X_{i,k'}-\{\{x_{i,j}\}_{j\in S_i}\cup\{z_{i,j}\}_{j\in D_i}\}$ under a commutative finite and flat k'-group scheme $G_{k',i}$ of rank p, where $X_{i,k'}:=X_i\times_{\mathrm{Spec}\,k}\mathrm{Spec}\,k'$, with $V_{i,k'}$ **smooth**. Let $U_{k'}$ be the open subscheme of $X_{k'}$ obtained by deleting the double and marked points of $X_{k'}$. Then the above data determines an element of $H^1_{\mathrm{fppf}}(U_{k'})_p$. Moreover if $G_{k',i}$ is radicial, and if ω_i is the associated differential form, then the set of zeros and poles of w_i is necessarily contained in $\{x_{i,j}\}_{j\in S_i}\cup\{z_{i,j}\}_{j\in D_i}$, as $V_{i,k'}$ is smooth. Also if $i\in I^{\mathrm{sp}}$, and S_i is non empty, we assume that $G_{i,k'}=\mu_p$.

Deg.5. For each $i\in I$ (resp. $i\in I^{\mathrm{sp}}$), are given for each $j\in S_i$ integers $(m_{i,j},h_{i,j})$, where $m_{i,j}$ is the conductor of the torsor f_i at the point $x_{i,j}$, and $h_{i,j}$ its residue at this point. Let $r_{i,j}:=m_{i,j}+1$. If $i\in I^{\mathrm{sp}}$ then $r_{i,j}=1$.

Deg.6. For each double point $z_{i,j}=z_{i',j'}\in X_i\cap X_{i'}$ is given the pair $(m_{i,j},h_{i,j})$ (resp. $(m_{i',j'},h_{i',j'})$), where $m_{i,j}$ (resp. $m_{i',j'}$) is the conductor of the torsor $f_{i,k'}$ (resp. $f_{i',k'}$) at the point $z_{i,j}$ (resp. $z_{i',j'}$), and $h_{i,j}$ (resp. $h_{i',j'}$) its residue at this point. These data must satisfy $m_{i,j}+m_{i',j'}=0$, and $h_{i,j}+h_{i',j'}=0$. In other words the element of $H^1_{\mathrm{fppf}}(U_{k'})_p$ determined by the data Deg.4 must belong to the subgroup $H^1_{\mathrm{fppf}}(U_k)_p^{\mathrm{kum}}$. In particular for the double point $x_{i_1,j_1}:=x_1$ (resp. $x_{i_2,j_2}:=x_2$) is given the data $(m_{i_1,j_1},h_{i_1,j_1})$ (resp. $(m_{i_2,j_2},h_{i_2,j_2})$, such that $m_{i_1,j_1}+m_1=0$ (resp. $m_{i_2,j_2}+m_2=0$), and $h_{i_1,j_1}+h_1=0$ (resp. $h_{i_2,j_2}+h_2=0$).

Deg.7. For each irreducible component X_i of Γ is given an integer $\delta_i\leq v_K(p)$ which is divisible by $p-1$. For each double point $z_{i,j} = z_{i',j'} \in X_i \cap X_{i'}$ of Γ is given an integer $e_{i,j} = pt_{i,j}$ divisible by p, such that with the same notations as above we have $\delta_i - \delta_{i'} = m_{i,j} t_{i,j}(p-1)$.

Deg.8. Let I_{et} denote the subset of I (resp. of I^{sp}) consisting of those i for which $G_{k',i}$ is étale, then : $(r-m-1)(p-1)/2 = \sum_{i\in I_{\text{et}}}(-2+\sum_{j\in S_i}(m_{i,j}+1)+\sum_{j\in D_i}(m_{i,j}+1))(p-1)/2$. The integer $g_x := (r-m_1-m_2)(p-1)/2$ is called the **genus** of the degeneration data $\text{Deg}(x)$.

There is a natural notion of isomorphism of double split (resp. non split) degeneration data of a given type over a given algebraic extension of k. We will denote by $\text{DS-Deg}_{\mathbf{p}}$ (resp. $\text{DNS-Deg}_{\mathbf{p}}$) the set of isomorphism classes of $\overline{k}$-double split (resp. $\overline{k}$-double non split) degeneration data of rank p. Also as in 1.2.3 one shows that there exists a canonical action of G_k on these sets, via the canonical action of G_k on the degeneration data Deg.4, and these sets are G_k-sets, hence also canonically G_K-sets. The following theorem is proven in the same way as in 1.2.4 and 1.2.6.

1.3.2. Theorem. *Let $\mathcal{X} := \operatorname{Spf} A$ be the formal germ of an R-curve at an ordinary double point. Let $\overline{A} := A \otimes_R \overline{R}$, and let $L := \operatorname{Fr}(\overline{A})$. Then the canonical specialisation maps:* $\text{Sp} : H^1_{\text{et}}(\operatorname{Spec} L, \mu_p) \to \text{DS-Deg}_p$ *and* $\text{Sp} : H^1_{\text{et}}(\operatorname{Spec} L, \mu_p) \to \text{DNS-Deg}_p$ *are surjective and G_K-equivariant.*

1.3.3. remark. The surjectivity result in 1.2.4 and 1.2.7 was shown in [He] in the case of degeneration data of genus 0 in the framework of automorphisms of order p of open discs and annuli. Our method based on the techniques developed in [Sa] and [Sa-1] avoid the use of this language and uses only kummer theory.

II. Semi-stable reduction of Galois covers of degree p obove proper curves.

2.0. In this section we will use the same notations as in 1.0. We will consider a **proper** and **semi-stable** R-curve X, a Galois cover $f : Y \to X$ with group $\mathbb{Z}/p\mathbb{Z}$, and with Y normal. We will study the semi-stable reduction of Y as well as the Galois action on these data.

2.1. Let X be a proper and semi-stable R-curve, with smooth and connected generic fibre $X_K := X \times_R K$. We assume that the double points of X_k are k-rationals. Let $f : Y \to X$ be a Galois cover with group $G \simeq \mathbb{Z}/p\mathbb{Z}$, and with Y normal. It follows from the theorem of semi-stable reduction for curves (cf. [De-Mu]) that, after eventually a finite extension R' of R, with fractions field K' and residue field k', the curve Y has a semi-stable reduction. More precisely, there exists a birational and proper morphism $\tilde{f} : \tilde{Y} \to Y' := Y \times_R R'$

such that $\tilde{Y}_{K'} \simeq Y'_{K'}$, and the following conditions hold:
(i) The special fibre $\tilde{Y}_{k'}$ of $\tilde{Y}$ is reduced.
(ii) $\tilde{Y}_{k'}$ has only ordinary double points as singularities.

Moreover there exists such a semi-stable model $\tilde{Y}$ which is minimal. In particular the action of G on Y' extends to an action on $\tilde{Y}$. Let $\tilde{X}$ be the quotient of $\tilde{Y}$ by G which is a semi-stable model of $X' := X \times_R R'$. In the following we will choose $\tilde{Y}$ and $\tilde{X}$ as above such that the set of points $B_K := \{x_{i,K}\}_{i=1}^r$ consisting of the branch locus of the morphism $f_K : Y_K \to X_K$ are rational. Moreover we choose such $\tilde{X}$ and $\tilde{Y}$ which are minimal for these properties. One has the following commutative diagram:

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{g} & \tilde{X} \\ \tilde{f}\downarrow & & \tilde{g}\downarrow \\ Y' & \xrightarrow{f'} & X' \end{array}$$

Let $(X_i)_{i\in I}$ be the irreducible components of the special fibre $X'_{k'}$ of X'. For each $i \in I$, let $(x_{i,j})_{j\in S_i}$ be the set (may be empty) of those smooth points of X_i in which specialise some points of B_K, say in $x_{i,j}$ specialise $r_{i,j}$ points of B_K, and let $(z_t)_{t\in J}$ be the set of double points of X_k. For each $t \in J$, let r_t be the number of points of B_K which specialise in z_t, we have $r = \sum_{t\in J} r_t + \sum_{i\in I}\sum_{j\in S_j} r_{i,j}$. Let $U := X' - \{D \cup S\}$, where $S := \cup_{i\in I}\{x_{i,j}\}_{j\in S_i}$ and $D = \{z_t\}_{t\in J}$. We will denote by $\{U_i\}_{i\in I}$ the set of irreducible components of U.

2.2. Below we will use the notations of 2.0 and 2.1 and will exhibit the degeneration data associated to the semi-stable reduction of Y. Let R' be a finite extension of R such that the conditions of 2.1 hold. The following data are canonically associated to the Galois cover $f : Y \to X$:

Deg.1. For each irreducible component X_i of $X'_{k'}$, let η_i be the generic point of X_i, and let X'_{η_i} be the spectrum of the completion of the localisation of X' at η_i. We have a canonical morphism : $X'_{\eta_i} \to X'$. Consider the following cartesian diagram:

$$\begin{array}{ccc} Y_{\eta_i} & \xrightarrow{f_{\eta_i}} & X'_{\eta_i} \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{f'} & X' \end{array}$$

Then either $f_{\eta_i} : Y_{\eta_i} \to X'_{\eta_i}$ is completely split, or is a torsor under a finite and flat commutative R'-group scheme G_i of rank p.

Deg.2. For each irreducible component X_i of X'_k, let $U_i := X_i - \{\{x_{i,j}\}_{j\in S_i} \cup \{z_t\}_{t\in J_i}\}$, where $J_i \subset J$ denotes the index subset indexing those double points of $X'_{k'}$ which are supported by X_i. Then $V_i := f^{-1}(U_i) \to U_i$ is an admissible torsor under the finite and flat group scheme $G_{k',i} := G_i \times_{R'} k'$. Moreover, if $G_{k',i}$ is étale then V_i is smooth, and if $G_{k',i}$ is radicial then the only singularities of V_i lie above the zeros $\{y_l\}_{l \in Z_i}$ of the differential form ω_i associated to the above torsor (cf. [Sa], I).

Deg.3. For each double point $z_j \in X_i \cup X_{i'}$, let $\mathcal{X}_j$ be the completion of the localisation of X' at z_j, and let $\mathcal{X}_j \to X'$ be the canonical morphism. Consider the cartesian diagram:

$$\begin{array}{ccc} \mathcal{Y}_j & \xrightarrow{f_j} & \mathcal{X}_j \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{f'} & X' \end{array}$$

Then either the cover $f_j : \mathcal{Y}_j \to \mathcal{X}_j$ is split, or it is a Galois cover of degree p with $\mathcal{Y}_j$ connected, in which case is associated to f_j a double degeneration data $\operatorname{Deg}(z_j)$. Let $m_{j,1}$ and $h_{j,1}$ (resp. $m_{j,2}$ and $h_{j,2}$) be the conductor and the residue at the double point z_j associated to the torsor $V_i := f^{-1}(U_i) \to U_i$ (resp. $V_{i'} := f^{-1}(U_{i'}) \to U_{i'}$). Then the above degeneration data $\operatorname{Deg}(z_i)$ is of type $(r_j, (G_{k',i}, m_{j,1}, h_{j,1}), (G_{k',i'}, m_{j,2}, h_{j,2}))$, and genus $(r_j - m_{j,1} - m_{j,2})(p-1)/2$.

Deg.4. For each irreducible component X_i of $X'_{k'}$, and a smooth point $x_{i,j}$, $j \in S_i$, let $\mathcal{X}_{i,j}$ be the completion of the localisation of X' at $x_{i,j}$, and let $\mathcal{X}_{i,j} \to X'$ be the canonical morphism. Consider the cartesian diagram:

$$\begin{array}{ccc} \mathcal{Y}_{i,j} & \xrightarrow{f_{i,j}} & \mathcal{X}_{i,j} \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{f'} & X' \end{array}$$

The cover $f_{i,j} : \mathcal{Y}_{i,j} \to \mathcal{X}_{i,j}$ is necessarily a Galois cover of degree p with $\mathcal{Y}_{i,j}$ connected, and to $f_{i,j}$ is associated a simple degeneration data $\operatorname{Deg}(x_{i,j})$. Let $m_{i,j}$ and $h_{i,j}$ be the conductor and the residue at the smooth point $x_{i,j}$ associated to the torsor $V_i := f^{-1}(U_i) \to U_i$. Then the above degeneration data $\operatorname{Deg}(x_{i,j})$ is of type $(r_{i,j}, (G_{k',i}, m_{i,j}, h_{i,j}))$, and genus $(r_{i,j} - m_{i,j} - 1)(p-1)/2$.

Deg.5. For each $i \in I$ such that the group scheme $G_{k',i}$ is radicial, let $\{y_l\}_{l \in Z_i}$ be the zeros of the differential form ω_i associated to the above torsor $V_i := f^{-1}(U_i) \to U_i$, and which we may assume to be rational over k'. For each $l \in Z_i$, let $\mathcal{Z}_l$ be the completion of the localisation of X' at y_l, and let $\mathcal{Z}_l \to X'$ be the canonical morphism. Consider the cartesian diagram:

$$\begin{array}{ccc} \mathcal{Z}'_l & \xrightarrow{f_l} & \mathcal{Z}_l \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{f'} & X' \end{array}$$

The cover $\mathcal{Z}'_l \to \mathcal{Z}_l$ is a Galois cover of degree p with $\mathcal{Z}'_l$ connected, and to f_l is associated a simple degeneration data $\operatorname{Deg}(y_l)$. Let m_l and h_l be the conductor and the residue at the smooth point y_l associated to the torsor $V_i := f^{-1}(U_i) \to U_i$. Then the above degeneration data $\operatorname{Deg}(y_l)$ is of type $(0, (G_{k',i}, m_l, h_l))$, and genus $(-m_l - 1)(p-1)/2$.

The above considerations lead to the following definition of degeneration data associated to a Galois cover of degree p above a proper and semi-stable R-curve.

2.2.1. Definition. Let X_k be a proper and semi-stable k-curve, and let $\{X_i\}_{i\in I}$ be the irreducible components of X_k. We assume that the double points $\{z_t\}_{t\in J}$ of X_k are k-rational. Let k' be an algebraic extension k. A k'-**degeneration data of rank** p associated to X_k, consists of the following data:

Deg.1. For each irreducible component X_i of X_k is given a set of smooth k'-rational points $\{x_{i,j}\}_{j\in S_i}$ of X_k which are supported by X_i.

Deg.2. For each irreducible component X_i of X_k, let $U_i := X_i - \{\{x_{i,j}\}_{j\in S_i} \cup \{z_t\}_{t\in J_i}\}$, where $J_i \subset J$ denotes the index subset indexing those double points of X_k which are supported by X_i. Then we assume given an admissible torsor $f_i : V_i \to U'_i := U_i \times_k k'$ (cf. [Sa], 4, for the definition of admissibility) under a finite and flat k'-group scheme $G_{k',i}$, and we allow the torsor f_i to be trivial. If $G_{k',i}$ is radicial then we assume that the zeros $\{y_l\}_{l\in Z_i}$ of the differential form ω_i associated to the above torsor f_i are k'-rational.

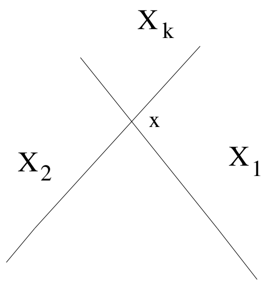
Deg.3. For each double point $z_j \in X_i \cap X_{i'}$ of X_k, let $m_{j,1}$ and $h_{j,1}$ (resp. $m_{j,2}$ and $h_{j,2}$) be the conductor and the residue at the double point z_j associated to the torsor $V_i := f^{-1}(U_i) \to U_i$ (resp. $V_{i'} := f^{-1}(U_{i'}) \to U_{i'}$). Let r_j be an integer such that $r_j - m_{j,1} - m_{j,2} \geq 0$. Then we assume given a k'-double degeneration data $\operatorname{Deg}(z_j)$ of type $(r_j, (G_{k',i}, m_{j,1}, h_{j,1}), (G_{k',i'}, m_{j,2}, h_{j,2}))$, and genus $(r_j - m_{j,1} - m_{j,2})(p-1)/2$.

Deg.4. For each irreducible component X_i of $X_{k'}$, and a smooth point $x_{i,j}$, $j \in S_i$, let $m_{i,j}$ and $h_{i,j}$ be the conductor and the residue at the smooth point $x_{i,j}$ associated to the torsor $V_i := f^{-1}(U_i) \to U_i$. Let $r_{i,j}$ be an integer such that $r_{i,j} - m_{i,j} - 1 \geq 0$. Then we assume given a k'-simple degeneration data $\operatorname{Deg}(x_{i,j})$ of type $(r_{i,j}, (G_{k',i}, m_{i,j}, h_{i,j}))$, and genus $(r_{i,j} - m_{i,j} - 1)(p-1)/2$.

Deg.5. For each $i \in I$ such that the group scheme $G_{k',i}$ is radicial, let $\{y_l\}_{l\in Z_i}$ be the zeros of the differential form ω_i associated to the above torsor $V_i := f^{-1}(U_i) \to U_i$, and which we may assume to be rational over k'. For each $l \in Z_i$, let m_l and h_l be the conductor and the residue at the smooth point y_l associated to the torsor $V_i := f^{-1}(U_i) \to U_i$. Then we assume given a degeneration data $\mathrm{Deg}(y_l)$ of type $(0, (G_{k',i}, m_l, h_l))$, and genus $(-m_l - 1)(p-1)/2$.

2.2.2. Remark. The admissibility condition on the torsors $\{f_i\}_{i\in I}$ which is part of the above definition of degeneration data is satisfied when the U_i are opens of the projective line, e.g. if X_k is a totally degenerate Mumford curve, and in the case where U_i is proper.

2.2.3. Example. In this example we consider a semi-stable and proper R-curve X with smooth and geometrically connected generic fibre X_K, and whose special fibre X_k consists of two irreducible smooth components X_1 of genus $g_1 \geq 2$ and X_2 of genus $g_2 \geq 2$ which intersect at the double point x. We suppose moreover that both X_1 and X_2 are **generic** (for this one has to choose R to be rather large).



Consider a Galois cover $f : Y \to X$ of degree p with Y normal and such that the Galois cover $f_K : Y_K \to X_K$ induced by f above the generic fibre is a μ_p-torsor (hence in particular is étale). We assume that the special fibre $Y_k := Y \times_R k$ of Y is reduced. For $i \in \{1, 2\}$, let $f_i : V_i \to U_i := X_i - \{x\}$ be the torsor above U_i induced by the cover f, and let m_i be the conductor of f_i at the point x. Then only the following two cases can ossur:

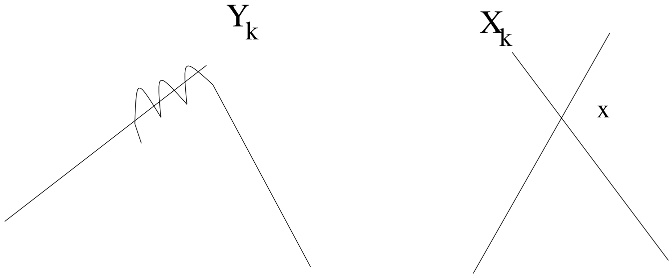
Case 1) Either f_i is an étale torsor in which case either f_i ramify above x or f_i extends to an étale torsor above X_i. In both cases V_i is smooth.

Case 2) Or f_i is a torsor under μ_p in which case is associated to f_i canonically a logarithmic differential form ω_i, and the singularities of V_i lie above the zeros of ω_i. In this case where X_i is generic it was shown in [Ra] (proposition 4) that such a differential form ω_i has $g_i - 1$ double zeros if $p = 2$ and $2g_i - 2$ simple zeros if $p \neq 2$. In the later case $m_i = -2$ or $m_i = -1$ depending on whether x is a zero of ω_i or not.

In what follows we assume for simplicity that $p \neq 2$. Supppose we are in case 2, and let y_j be a point of V_i above a zero z_j of ω_i which is different from the point x. Then V_i is singular at y_j, the singularity at y_j is unibranche, and the geometric genus of y_j equals $(p-1)/2$. Let $\mathrm{Deg}(z_j)$ be the non split $\overline{k}$-degeneration data associated to z_j via the semi-stable reduction of Y. Then $\mathrm{Deg}(z_j)$ is necessarily of type $(0, (\mu_p, -2, 0))$, and consists of one projective line P_i with a marked $\overline{k}$-point $x_i = x$ and an étale torsor $T_i \to P_i - \{x_i\}$ with conductor 2 at the point x_i.

Above the double point x three situations can occur:

Case a) Both f_1 and f_2 are étale in which case f is necessarily étale above x.



Apart from case a) Y must be local above, we will denote by y the unique point of Y above x.

Case b) Say f_1 is étale and f_2 is radicial, and assume for simplicity that Y has two branches at the point y. In this case the geometric genus of y equals $(m_1 + 2)(p - 1)/2$ if ω_2 has a zero at x, or $(m_1 + 1)(p - 1)/2$ otherwise.

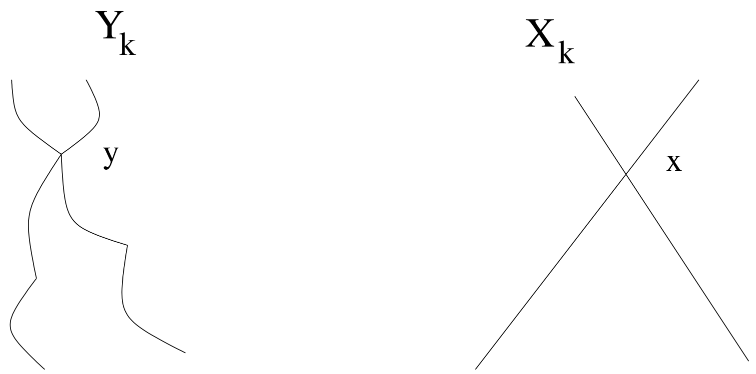
Case c) Both f_1 and f_2 are radicial in which case the geometric genus of y equals $2(p - 1)$, $3(p - 1)/2$ or $p - 1$ depending on whether both f_i have a zero at the point x, or just one of them do, or none of them.

Let $\mathrm{Deg}(x)$ be the double non split $\overline{k}$-degeneration data associated to x via the semi-stable reduction of Y. assume for example that we are in case c) and that the genus of y equals $p - 1$. Then the following two cases can occur:

First case: either $\mathrm{Deg}(x)$ consists of one projective line P with two marked $\overline{k}$-points $x_1 = x$ and $x_2 = x$, and an étale torsor $T \to P - \{x_1, x_2\}$ with conductor 2 at each of x_1 and x_2. In this case the special fibre of the semi-stable reduction of Y is a tree like which consist of the components V_1 and V_2 which are linked by the projective completion of V (which has genus $p-1$) at the points above x, and at each of the points of V_i above a zero of ω_i is linked a curve of genus $(p - 1)/2$.

Second case: $\mathrm{Deg}(x)$ consists of two projective line P_1 and P_2 with a marked $\overline{k}$-points $x_1 = x$ and $x_2 = x$ on each one, and an étale torsor $T_i \to P_i - \{x_i\}$ with conductor 2 at x_i. In this case the special fibre of the semi-stable reduction of Y contains $(p - 1)$ cycles and it consists of the components V_1 and V_2 which are linked by the projective completion of T_1 and T_2 (each has genus $(p - 1)/2$) at the points above x, moreover both T_1 and T_2 meet at p double points, and at each of the points of V_i above a zero of ω_i is linked a curve of genus $(p - 1)/2$ as in the preceeding case.

case c)



2.3. There is a natural notion of isomorphism of degeneration data of rank p associated to a proper and semi-stable k-curve X_k, over a given algebraic extension of k. We will denote by $\mathrm{Deg}_p(X_k)$ the set of isomorphism classes of $\overline{k}$-degeneration data of rank p associated to X_k. The set $\mathrm{Deg}_p(X_k)$ is equipped in a natural way with a G_k-action, via the natural action of G_k on the above data which was explained in [Sa] and I, and hence is a G_k-set. The following result is analogous to the result of realisation for degeneration data obtained in paragraph I in the local situation.

2.3.1. Theorem. *Let X_k be a proper and semi-stable k-curve. Let R be a complete discrete valuation ring of inequal characteristics with residue field k, and fractions field K. Suppose given a $\overline{k}$-degeneration data $\deg(X_k)$ of rank p associated to X_k (in other words suppose given an element of $\mathrm{Deg}_p(X_k)$). Then there exists, after eventually a finite extension of R, a proper and semi-stable R-curve $\tilde{X}$ with smooth generic fibre and a special fibre isomorphic to X_k, and a Galois cover $f : \tilde{Y} \to \tilde{X}$, such that the degeneration data associated to f as in 2.2 is isomorphic to the given data $\deg(X_k)$. The above cover f is not unique in general, moreover it can be constructed such that the cover $f_K : \tilde{Y}_K \to \tilde{X}_K$ induced by f on the generic fibres is ramified above $r = \sum_{t\in J} r_t + \sum_{i\in I}\sum_{j\in S_j} r_{i,j}$ geometric points of X_K (hier r_t and $r_{i,j}$ are the one given in the definition of the degeneration data in 2.2.1).*

The next result expreses the compatibility of degeneration with the Galois action.

2.3.2. Theorem. *Let R be a complete discrete valuation ring of inequal characteristics with residue field k, and fractions field K, and let X be a proper and semi-stable R-curve with smooth and geometrically connected generic fibre X_K and with special fibre X_k. Let $\overline{K}$ be an algebraic closure of K, and let L be the function field of the geometric generic fibre $X_{\overline{K}} := X \times_R \overline{K}$ of X. Then the canonical specialisation map $\mathrm{Sp} : H^1_{\mathrm{et}}(\mathrm{Spec}\, L, \mu_p) \to \mathrm{Deg}_p(X_k)$ is G_K-equivariant, where hier we consider the natural action of G_K on $\mathrm{Deg}_p(X_k)$ via its canonical quotient G_k.*

Note that the above specialisation map can not be surjective in this case since the set $\mathrm{Deg}_p(X_k)$ with fixed ramification data on the "generic fibre" is infinite in general. The proof of the above two results is very similar to the one obtained in paragraph I in the local case and are left to the reader. They are based on formal patching techniques and the results in [Sa] and [Sa-1]. However in order to prove 2.3.1 one will also need the following result on lifting of admissible torsors with given degeneration data at the "critical points". More precisely let R be as in 1.0. Let X be a formal smooth R-curve, whose special fibre X_k is irreducible. Let $f_k : Y_k \to X_k$ be an admissible torsor, under a finite and flat k-group scheme G_k of rank p, which is radicial. Let ω be the associated differential form, and let $Z := \{x_j\}_{j\in J}$ be the set of zeros of ω which are contained in X_k, which we call the **critical** points of f_k, in particular the singularities of Y_k lie above the points of Z. Assume that $f : Y \to X$ is a torsor under a finite and flat R-group scheme G which lifts f_k. To each point $x_j \in Z$ is associated via f, a simple degeneration data $\mathrm{Deg}(x_j)$ of type $(0, (G_k, m_j, h_j))$, where m_j is the conductor of f_k at x_j, and h_j its residue at this point. The following result shows that we can choose such a lifting f of f_k which gives rise to a given set of degeneration data at the critical points.

2.3.3. Proposition / Definition. *Let $f_k : Y_k \to X_k$ be an admissible torsor under a finite and flat k-group scheme G_k of rank p, which is radicial, and let $Z := \{x_j\}_{j\in J}$ be the set of critical point of f_k which we assume to be k-rational. For each point $x_j \in Z$ consider the pair (m_j, h_j), where m_j is the conductor of f_k at x_j, and h_j its residu at this point. Suppose given for each critical point x_j, a simple degeneration data $\mathrm{Deg}(x_j)$ of type $(0, (G_k, m_j, h_j))$. Then there exists, after eventually a finite extension of R, a smooth formal R-curve $\tilde{X}$ with a special fibre $\tilde{X}_k := \tilde{X} \times_R k$ which is isomorphic to X_k, and a torsor $\tilde{f} : \tilde{Y} \to \tilde{X}$ under a finite and flat R-group scheme G of rank p, such that the restriction of $\tilde{f}$ to $U_k := X_k - Z$ is isomorphic to the restriction of f_k to U_k, and such that the simple degeneration data assosiated to each point $x_j \in Z$, via $\tilde{f}$, is isomorphic to $\mathrm{Deg}(x_j)$. We call such a torsor $\tilde{f}$ a* **lifting** *of f_k, with the* **given degeneration data** *$\{\mathrm{Deg}(x_j)\}_{j\in Z}$ at the critical points $\{x_j\}_{j\in Z}$.*

Proof. The torsor $f_k : Y_k \to X_k$ is admissible hence by definition can be lifted, after eventually a ramified extension of R, to a torsor $f : Y' \to X$ under a finite and flat R-group scheme of rank p, which is either μ_p or $\mathcal{H}_n$ for $0 < n < v_K(\lambda)$ where v_K is the normalised valuation of the fractions field of R (such a lifting is not unique). Also for each point $x_j \in Z$ one can find a Galois cover $f_j : \mathcal{Y}_j \to \mathcal{X}_j$ of degree p, where $\mathcal{X}_j$ is the formal germ of X at x_j, and such that the degeneration data associated to x_j via f_j is isomorphic to $\mathrm{Deg}(x_j)$ (cf. 1.2.3). Let $U_k := X_k - Z$, and let g_k be the restriction of f_k to U_k. Then f induces a lifting $g : V \to U$ of g_k, which is a torsor under a finite and flat R-group scheme of rank p, and where U is obtained from X by deleting the formal fibres at the points $\{x_j\}_{j\in Z}$. Now a patching of the covers g and f_j along the formal fibres at the points $\{x_j\}_j$ will produce the desired torsor $\tilde{f} : \tilde{Y} \to \tilde{X}$.

2.3.4. Example. Hier we use the same notations as in the example 2.2.3. First we will explain the Galois action on a μ_p-torsor $f_{\overline{K}}: Y_{\overline{K}} \to X_{\overline{K}}$ above the geometric generic fibre $X_{\overline{K}}$ of X. We assume that the above torsor induces in reduction the situation in case c) where $g_y = p-1$ and where the semi-stable reduction of $Y_{\overline{K}}$ is a tree like. Let $\sigma \in G_K$, and let $\overline{\sigma}$ be the image of σ in G_k. Consider the μ_p-torsor ${f_{\overline{K}}}^{\sigma} : {Y_{\overline{K}}}^{\sigma} \to X_{\overline{K}}$ which is the conjugate of $f_{\overline{K}}$ via σ. Then using the result in 2.3.2 one can describe the semi-stable reduction of ${Y_{\overline{K}}}^{\sigma}$. More precisely the graph of the semi-stable reduction of ${Y_{\overline{K}}}^{\sigma}$ is isomorphic to the one of $Y_{\overline{K}}$, in particular it is also a tree like, and it consists of the components $V_i^{\overline{\sigma}}$, for $i = 1, 2$, where $V_i^{\overline{\sigma}}$ is the conjugate of V_i via $\overline{\sigma}$, which are linked by an irreducible curve $T^{\overline{\sigma}}$ of genus $p-1$ which is the conjugate of T by $\overline{\sigma}$. Moreover at each point of $V_i^{\overline{\sigma}}$ above a zero of ω_i is linked a curve of genus $(p-1)/2$ which is the conjugate by $\overline{\sigma}$ of the curve that is linked to the corresponding point (via the action of $\overline{\sigma}$) of V_i in the graph of semi-stable reduction of $Y_{\overline{K}}$.

Also one can establish the converse of the situation in 2.2.3, i.e. reconstruct a μ_p-torsor f_K as above from the degeneration data, using 2.3.1. More precisely, let X_k be the semi-stable curve considered in 2.2.3 and consider the non split degeneration data $\mathrm{Deg}(X_k)$ of rank p which consists in the following:

For $i = 1, 2$ is given a μ_p-torsor $Y_i \to X_i$ associated to the logarithmic differential form ω_i, and assume that the zeros $\{z_{j,i}\}_j$ of ω_i lie outside the point x. For each i, j suppose given a non split simple degeneration data $\mathrm{Deg}(z_{j,i})$ of type $(0, (\mu_p, -2, 0))$ and which consists of one projective line $P_{j,i}$ with one $\overline{k}$-marked point $z_{j,i}$ and an étale torsor above $P_{j,i} - \{z_{j,i}\}$ with conductor 2 at the point $z_{j,i}$. At the double point x is given a non split double degeneration data of type $(0, (\mu_p, -2, 0), (\mu_p, -2, 0))$ and which consists of one projective line P and two $\overline{k}$-marked points $x_1 = x$ and $x_2 = x$ and an étale torsor above $P - \{x_1, x_2\}$ with conductor 2 at both points x_1 and x_2. Then by 2.3.1 one can find a proper and semi-stable curve $\tilde{X}$ with smooth generic fibre and a special fibre $\tilde{X}_k := \tilde{X} \times R_k$ isomorphic to X_k, and (eventually after a finite extension of R) a Galois cover $\tilde{f} : \tilde{Y} \to \tilde{X}$ of degree p, such that the degeneration data of rank p associated to X_k via the semi-stable reduction of $\tilde{Y}$ is isomorphic to the above given data $\mathrm{Deg}(X_k)$.

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