# Weight structures vs. $t$-structures; weight filtrations, spectral sequences, and complexes (for motives and in general)

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**Abstract**

In this paper we introduce a new notion of a *weight structure* ($w$) for a triangulated category $\underline{C}$; this notion is an important natural counterpart to the notion of $t$-structures. This allows to generalize certain results of the previous paper [11] to a large class of triangulated categories and functors.

The *heart* of $w$ is an additive $Hw \subset \underline{C}$; there are no non-trivial $\underline{C}$-distinguished triangles in $Hw$. We prove that a weight structure defines Postnikov towers for any $X \in Obj\underline{C}$ (whose "factors" are $X^i \in ObjHw$); these towers are canonical and functorial "up to zero cohomology". For any (co)homological functor $H : \underline{C} \to A$ ($A$ is abelian) we construct a *weight spectral sequence* $T : H(X^i[j]) \implies H(X[i+j])$; it is canonical and functorial starting from $E_2$. This spectral sequences specializes to the usual weight spectral sequences for "classical" realizations of (Voevodsky's) motives and to the Atiyah-Hirzebruch sequence in topology. We prove that $K_0(\underline{C}) \cong K_0(Hw)$ in the bounded case if $Hw$ is idempotent complete. Under certain restrictions, we prove a similar equality for $K_0(\operatorname{End}\underline{C})$. We define a canonical conservative *weakly exact* functor $t$ from $\underline{C}$ to a certain *weak category of complexes* $K_{\mathfrak{w}}(Hw)$.

We also define *adjacent* weight and $t$-structures; their hearts are dual in a very interesting sense.

These results give us a better understanding of Voevodsky's motives $DM^{eff}_{gm} \subset DM^{eff}_{-}$ (that were studied in [11]) and also of the

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stable homotopy category $SH$ (and cellular towers for spectra). We define a new (*Chow*) $t$-structure for $DM_-^{eff}$ which is adjacent to the Chow weight structure. The philosophy of adjacent structures also allows to express torsion motivic cohomology of certain motives in terms of étale cohomology of their "submotives". The latter fact is an extension of the calculation of $E_2$ of the coniveau spectral sequence (by Bloch and Ogus). We also calculate very explicitly the groups $K_0(SH_{fin})$ and $K_0(\operatorname{End} SH_{fin})$ (and also certain $K_0(\operatorname{End}^n SH_{fin})$ for $n \in \mathbb{N}$).

# Contents

# Introduction

The purpose of this paper is the introduction of a new formalism of *weight structures* for triangulated categories; this notion is an important natural counterpart to the notion of $t$-structures.

In [11] for a triangulated category $\underline{C}$ with a (negative) differential graded enhancement a conservative exact *weight complex* functor $t_0 : \underline{C} \to K(Hw)$ for a certain additive $Hw$ was constructed. For any 'enhanceble' realization functor $G : \underline{C} \to K(A)$ for an abelian $A$ a spectral sequence starting from the cohomology of $t_0(X)$ and converging to the cohomology of $G(X)$ was constructed. It was proved that $K_0(\underline{C}) \cong K_0(Hw)$ if $Hw$ is idempotent complete.

The first goal of the current paper is to generalize these results to categories and functors that do not (necessarily) have a differential graded enhancement. Our main tool is the (new) formalism of a *weight structure* ($w$ for $\underline{C}$ is defined via $\underline{C}^{w\le 0}, \underline{C}^{w\ge 0} \subset \underline{C}$). We obtain the generalizations wanted (up to certain modifications) as well as several new results. The main idea is that a weight structure defines Postnikov towers of objects; these towers are canonical and functorial "up to cohomology zero". These results give us a better understanding of Voevodsky's motives (that were studied in [11]) and of the stable homotopy category $SH$ (that doesn't possess any enhancements of this sort!). In particular, we calculate very explicitly the groups $K_0(SH_{fin})$ and $K_0(\operatorname{End} SH_{fin})$ (and also certain $K_0(\operatorname{End}^n SH_{fin})$ for $n \in \mathbb{N}$). Besides we obtain a certain "weight filtration" on homotopy groups of spectra; the corresponding "weight" spectral sequence is usually called the Atiyah-Hirzebruch spectral sequence. The author doesn't think that all of these results are new; however they illustrate our methods very well. Note also that in the topological case the Postnikov tower corresponding to the weight structure constructed is called a *cellular tower*; its most basic properties (parallel to those for the general case studied in the current paper) are described in §6.3 of [26].

Another goal of the current paper is the exposition of the yoga of *adjacent structures* and related spectral sequences results. This concept seems to be completely new. Under certain conditions (see §4) for a weight structure $w$ one can construct a certain $t$-structure which is *adjacent* to $w$. Vice versa, for a $t$-structure one can often construct adjacent weight structures (such that either $\underline{C}^{w\le 0} = \underline{C}^{t\le 0}$ or $\underline{C}^{w\ge 0} = \underline{C}^{t\ge 0}$). In particular, this is the case for the Voevodsky's category $DM_-^{eff}$ (and the Chow weight structure essentially constructed in [11]) and for the stable homotopy category (and the Postnikov $t$-structure). The hearts of *adjacent* structures are dual in a very interesting sense (see Theorem 4.4.2). Moreover, even in the case when an adjacent $w$ does not actually exist, under certain conditions (that $w$ exists "in the limit") one could relate the cohomology of $X$ with coefficients in the $t$-structure truncations of a "theory" $H$ with the limit of $H$-cohomology of certain objects related to $X$. These calculations are closely related with the well-known calculations of the $E_2$-terms of the coniveau spectral sequence; see [9] and

[12]. In particular, one can express torsion motivic cohomology of some motives in terms of étale cohomology of their "submotives" (in a certain sense). In topology, our spectral sequence results corresponds to the fact that the Atiyah-Hirzebruch spectral sequence for the cohomology of a space $X$ with coefficient in a spectrum $S$ could be obtained either by considering the cellular filtration of $X$ or the Postnikov $t$-truncations of $S$. An analogue of our formulas with étale cohomology replaced by the singular one should be related to a certain (non-existent yet) "theory of mixed motives up to algebraic equivalence".

The definition of a weight structure for a triangulated category $\underline{C}$ is almost dual to those of a $t$-structure; yet some properties of these definitions are surprisingly distinct. The heart $Hw$ of a weight structure is defined in the same way as the heart of a $t$-structure; yet if $A \to B \to C \to A[1]$ is a distinguished triangle in $\underline{C}$ whose terms belong to $Hw$ then it necessarily splits. Besides, any weight structure defines a canonical conservative *weakly exact* functor $t$ (the *weight complex* functor) from $\underline{C}$ to a certain *weak category of complexes* $K_{\mathfrak{w}}(Hw)$ (which is a factor of $K(Hw)$). For $\underline{C} = SH$ we have $K_{\mathfrak{w}}(Hw) = K(Hw) \cong K(Ab_{fr})$ (the homotopy category of complexes of free abelian groups); $t$ is exact in the usual sense.

For any (co)homological functor $H$ and $X \in Obj\underline{C}$ one has a spectral sequence $T : H(X^i[j]) \implies H(X[i+j])$ where $X^i$ are the terms of $t(X)$. This spectral sequence is canonical and functorial starting from $E_2$; it specializes to the usual weight spectral sequences for "classical" realizations of motives and to the Atiyah-Hirzebruch sequence for cohomology of spectra (in topology).

Predecessors of the definition of a weight structure were the classical notions of "filtration bete" (see §3.1.7 of [5]) and of connective spectra (see §7 of [22]). Still, our axiomatics and most of main results are completely new. The only exception known to the author is that a part of Theorem 4.5.2 (those that concerns $t$-structures) is a slight generalization of Theorem 1.3 of [19]. Besides, recently weight structures were also (independently) introduced by D. Pauksztello (he called them co-$t$-structures), see [28]; some of easier results of the current paper were also proved there. In [11] a weight structure was (essentially) constructed for Voevodsky's motives. A similar construction for Hanamura's motives was described in §1 of [18] (see property (6) in the end of the §1 loc. cit.); note that in §4 of [11] it was proved that Hanamura's category of motives is anti-equivalent to (the rational hull of) those of Voevodsky. Possibly, the so-called cofiltrations considered in [4] are also related to the subject.

A general example of a category which has a natural weight structure is the category of twisted complexes over a negative differential graded category; all these notions are defined in section 6. In [11] these concepts were studied in detail; it was shown that the Voevodsky's category of motives $DM^s \subset DM_{gm}^{eff}$ is an example of our situation (without mentioning weight structures explicitly); we also recall those results here. The relevant definitions and constructions are described in subsection 6.4 independently from [11]; yet an interested reader could certainly compare the differential graded versions of proofs (presented in [11]; see also [6]) with those here.

Another important example of a category with adjacent weight and $t$-structures is the stable homotopy category $SH$. The weight complex functor in this case computes singular homology and cohomology of spectra, see §4.6. Note that $SH$ certainly cannot have a differential graded description! The *spherical* weight structure is "generated" by the sphere spectrum; it is *left adjacent* to the (usual) Postnikov $t$-structure on $SH$. A Postnikov tower corresponding to this weight structure is called a *cellular tower*; its most basic properties (parallel to those described here for the general situation) are described in §6.3 of [26]. Note also that the spherical weight structure is also defined on the category of finite spectra, whereas the Postnikov $t$-structure is not. The reason for this is that the sphere spectrum is finite whereas Eilenberg-Maclane spectra aren't.

Our results in [11] easily yield a *Chow* weight structure for $DM_{gm}^{eff}$ (and hence also for $DM_{gm}$) such that $DM_{gm}^{eff\,w=0} = Chow^{eff}$; we describe it in §6. We also prove in §7.1 that there is a *Chow* $t$-structure on $DM_{-}^{eff}$ that is *right adjacent* to the Chow weight structure. Moreover, an easy application of the weight spectral sequence yields that if the "mixed motivic" cohomology of motives exists, then its images have certain canonical weight filtration, which behaves well under regulators.

Now we list the contents of the paper.

In section 1 we give the definition of a weight structure $w$ in a triangulated category $\underline{C}$. We describe some other basic definitions and prove their (relatively) simple properties. Our central objects of study are *weight decompositions* of objects and morphisms. We also describe certain Postnikov towers for object of $\underline{C}$ that come from weight structures.

In section 2 we describe the *weight spectral sequence* $T(H,X)$ (for $X \in Obj\underline{C}$ and a (co)homological functor $H : \underline{C} \to A$) that comes from the Postnikov towers described. It is canonical and functorial starting from $E_2$. It specializes to the "usual" weight spectral sequences for "classical realizations" of varieties (or motives; at least with rational coefficients). Moreover, in this case the spectral sequence degenerates at $E_2$ and its $E_2$-terms are exactly the graded pieces of the weight filtration.

In section 3 we define the weight complex functor $t$. Its target is a certain "weak category of complexes" $K_{\mathfrak{w}}(Hw)$. $K_{\mathfrak{w}}(Hw)$ is a factor of $K(Hw)$

which is no longer triangulated; yet the kernel of the projection $K(Hw) \to K_{\mathfrak{w}}(Hw)$ is an ideal whose square is zero so our ("weak") weight complex functor is not much worse than the "strong" one (as constructed in [11] in the differential graded case). In particular, $t$ is conservative, *weakly exact*, and preserves the filtration given by the weight structure (in the bounded case). We conjecture that the "strong" weight complex functor exists also; see Remark 3.3.4 and §8.3. Besides, in some cases (for example, for all subcategories of $SH$ mentioned in this paper) we have $K_{\mathfrak{w}}(Hw) = K(Hw)$. Our main tool of study is the *weight decomposition functor* $WD : \underline{C} \to K_{\mathfrak{w}}^{[0,1]}(\underline{C})$; see Theorem 3.2.2.

In section 4 we prove that weight structures are closely related to $t$-structures. In particular, in several cases a triangulated category possesses simultaneously a $t$-structure and a weight structure which are "dual" in a very interesting sense. In the case when structures are *adjacent* we have a certain duality of their hearts, whereas spectral sequences coming from these structures are closely related (by Deligne's decalage). We also prove that a weight structure could often be described in terms of some "negative" additive subcategory of $\underline{C}$.

In §4.6 we apply our results to the study of the stable homotopy category. It turns out that the weight complex for it calculates the singular (co)homology of spectra. Besides our results immediately yield a certain "weight filtration" on homotopy groups of spectra (and the corresponding "weight" spectral sequence).

In section 5 we prove that a bounded $\underline{C}$ is idempotent complete iff $Hw$ is; the idempotent completion of a general bounded $\underline{C}$ has a weight structure whose heart is the idempotent completion of $Hw$. If $\underline{C}$ is bounded and idempotent complete then $K_0(\underline{C}) \cong K_0(Hw)$. In §5.4 we study a certain Grothendieck group of endomorphisms in $\underline{C}$. Though it is not always isomorphic to $K_0(\operatorname{End} Hw)$, it is if $Hw$ is *regular* in a certain sense. Besides, we can still say something about $K_0(\operatorname{End} \underline{C})$ in the general case. In particular, this allows us to generalize Theorem 3.3 of [8] (on independence of $l$ for traces of "open correspondences"); see also §8.4 of [11]. As an application of our results, we also calculate explicitly the groups $K_0(SH_{fin})$ and $K_0(\operatorname{End} SH_{fin})$ (along with their ring structure). We also extend these results to the calculation of certain $K_0(\operatorname{End}^n SH_{fin})$ for $n \in \mathbb{N}$.

In section 6 we translate the results of [11] into the language of weight structures. In particular, we show that Voevodsky's $DM_{gm}^{eff}$ ($\subset DM_{gm}$) admits the *Chow weight structure* whose heart it $Chow^{eff}$ (resp. $Chow$). This allows us to prove that weight spectral sequences for realizations ("almost the same as" those described in §7 of [11]; see §6.4) exists for any realizations (not necessarily admitting a differential graded enhancement) and do not depend on the choice of enhancements.

In section 7 we show that the Chow weight structure of $DM_{gm}^{eff}$ extends to $DM_-^{eff}$ and admits a right adjacent *Chow* $t$-structure (whose heart is the category $Chow_*^{eff} = AddFun(Chow^{eff}, Ab) \supset Chow^{eff}$). We prove that any possible (conjectural) "mixed motivic" $t$-structure induces a canonical "weight filtration" on the values of the corresponding homological functor $DM_{gm}^{eff} \to MM$. We prove that (a certain version of) the weight complex functor can be defined on $DM_{gm}^{eff}$ without using the resolution of singularities (so one can define it for motives over any perfect field).

Next, we apply the philosophy of adjacent structures to express the cohomology of a motif $X$ with coefficients in homotopy ($t$-structure) truncations of any $H \in ObjDM_-^{eff}$ in terms of the limit of $H$-cohomology of certain "submotives" of $X$. These calculations are closely related with the well-known calculations of the $E_2$-terms of the coniveau spectral sequence; see [9], [12], and also [27]. In particular, one can express torsion motivic cohomology of certain motives in terms of étale cohomology of their "submotives" (this requires Beilinson-Lichtenbaum conjecture). As a partial case, we obtain a formula for the (torsion) motivic cohomology with compact support of a smooth quasi-projective variety.

In section 8 we show that a weight structure $w$ on $\underline{C}$ which induces a weight structure on a triangulated $\underline{D} \subset \underline{C}$ yields also a weight structure on the localization $\underline{C}/\underline{D}$. Next we prove a funny result: functors represented by compositions of $t$-truncations with respect to distinct $t$-structures could be expressed in terms of the corresponding adjacent weight structures (as certain images). We prove (by an argument due to A. Beilinson) that any $f$-category enhancement of $\underline{C}$ yields a "strong" weight complex functor $\underline{C} \to K(Hw)$. We also describe other possible sources of conservative "weight complex-like" functors (they are "usually" conservative) and related spectral sequences.

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**Notation.** For a category $C$, $A, B \in ObjC$, we denote by $C(A, B)$ the set of $A$-morphisms from $A$ into $B$.

For categories $C, D$ we write $C \subset D$ if $C$ is a full strict subcategory of $D$. Recall that $D$ is called *strict* if it contains all objects in $ObjC$ isomorphic to those from $ObjD$.

For a category $C$, $X, Y \in ObjC$ we say that $X$ is a *retract* of $Y$ if $id_X$ could be factorized through $Y$. Note that if $C$ is triangulated or abelian then

$X$ is a retract of $Y$ iff $X$ is its direct summand. For an additive $D \subset C$ the subcategory $D$ is called *Karoubi-closed* in $C$ if it contains all retracts of its objects in $C$.

$X \in Obj\underline{C}$ will be called compact if the functor $X^* = \underline{C}(X, -)$ commutes with arbitrary direct sums.

For a category $\underline{C}$ we denote by $\underline{C}^{op}$ the opposite category.

$\underline{C}$ will usually denote a triangulated category; usually it will be endowed with a weight structure $w$ (see Definition 1.1.1 below). We will use the term "exact functor" for a functor of triangulated categories (i.e. for a for a functor that preserves the structures of triangulated categories). We will call a covariant additive functor $\underline{C} \to A$ for an abelian $A$ *homological* if it converts distinguished triangles into long exact sequences; homological functors $\underline{C}^{op} \to A$ will be called *cohomological* when considered as contravariant functors $\underline{C} \to A$.

For $f \in \underline{C}(X, Y)$, $X, Y \in Obj\underline{C}$ we will call the third vertex of (any) distinguished triangle $X \xrightarrow{f} Y \to Z$ a cone of $f$. Recall that different choices of cones are connected by non-unique isomorphisms (easy, see IV.1.7 of [15]). Besides, in $C(A)$ we have canonical cones of morphisms, see section §III.3 of [15].

We will often specify a distinguished triangle by two of its morphisms. The author apologizes for this as well as for absence of certain diagrams (mostly "octahedron diagrams") that could help to understand this text.

When dealing with triangulated categories we (mostly) use conventions and auxiliary statements of [15]. For a set of objects $C_i \in Obj\underline{C}$, $i \in I$, we will denote by $\langle C_i \rangle$ the smallest strictly full subcategory containing all $C_i$; for $D \subset \underline{C}$ we will write $\langle D \rangle$ instead of $\langle Obj D \rangle$.

We will say that $C_i$ generate $\underline{C}$ if $\underline{C}$ equals $\langle C_i \rangle$. We will say that $C_i$ *weakly generate* $\underline{C}$ if for $X \in Obj\underline{C}$ we have $\underline{C}(C_i[j], X) = \{0\}\ \forall i \in I,\ j \in \mathbb{Z} \implies X = 0$. We will say that a set of $C_i$ *weakly cogenerates* $\underline{C}$ if for $X \in Obj\underline{C}$ we have $\underline{C}(X, C_i[j]) = \{0\}\ \forall i \in I,\ j \in \mathbb{Z} \implies X = 0$.

In this paper all complexes will be cohomological i.e. the degree of all differentials is $+1$; respectively, we will use cohomological notation for their terms.

For an additive category $A$ we denote by $C(A)$ the unbounded category of complexes over $A$; $K(A)$ is the homotopy category of $C(A)$ i.e. the morphisms of complexes are considered up to homotopy equivalence; $C^-(A)$ denotes the category of complexes over $A$ bounded above; $C^b(A) \subset C^-(A)$ is the subcategory of bounded complexes; $K^b$ denotes the homotopy category of bounded complexes. We will denote by $C(A)^{\leq i}$ (resp. $C(A)^{\geq i}$) the unbounded category of complexes concentrated in degrees $\leq i$ (resp. $\geq i$).

For an abelian $A$ we will denote by $D(A)$, $D^-(A)$, $D^b(A)$ the corresponding versions of the derived category of $A$.

$Ab$ is the category of abelian groups; $Ab_{fr}$ is the category of free abelian groups; $Ab_{fin.fr}$ is the category of finitely generated free abelian groups.

For additive $C, D$ we denote by $AddFun(C, D)$ the category of additive functors from $C$ to $D$ (we will be always able to assume that $C, D$ are small). For an additive $A$ we will denote by $A^*$ the category $AddFun(A, Ab)$ and by $A_*$ the category $AddFun(A, Ab^{op})$. Note that both of these are abelian. Moreover, Yoneda's lemma gives full embeddings of $A$ into $A_*$ and of $A^{op}$ into $A^*$ (these send $X \in ObjA$ to $X_* = A(-, X)$ and to $X^* = A(X, -)$, respectively). $A'_*$ will denote the full abelian subcategory of $A_*$ generated by $A$.

It is easily seen that any object of $A$ becomes projective in $A_*$. Besides, any object of $A_*$ has a resolution by (infinite) direct sums of objects of $A$. These fact are rather easy; the proofs can be found in the beginning of §8 of [25].

The definition of a cocompact object is dual to those of a compact one: $X \in Obj\underline{C}$ is cocompact if $\underline{C}(\prod_{i\in I} Y_i, X) = \bigoplus \prod_{i\in I} \underline{C}(Y_i, X)$ for any set $I$ and any $Y_i \in Obj\underline{C}$ such that the product exists.

We list the main definitions of this paper. Weight structures, $\underline{C}^{w\ge 0}$, $\underline{C}^{w\le 0}$, and weight decompositions of objects are defined in Definition 1.1.1; $Hw$ (the *heart* of $w$), $\underline{C}^{w=0}$, $\underline{C}^{w\ge l}$, $\underline{C}^{w\le l}$, $\underline{C}^{[j,i]}$, non-degenerate, and bounded (above, below or both) weight structures are defined in Definition 1.2.1; $X^{w\le i}$ and $X^{w\ge i+1}$ are defined in Remark 1.2.2; $\underline{C}^-$, $\underline{C}^+$, and $\underline{C}^b$ are defined in Definition 1.3.3; several notation and definitions for weight decomposition of morphisms, (infinite) weight decomposition of objects, and Postnikov towers for objects are introduced in §1.5; the weight filtration of functors is introduced in Definition 2.1.1; the weight complex of objects is defined in Definition 2.2.1; weight spectral sequences (denoted by $T(H, X)$) are introduced in §2.3 and §2.4; $T \lhd MorA$ ($T$ is an ideal of morphisms for $A$) is defined in Definition 3.1.1; $A/T$ is defined in Remark 3.1.2; the weak category of complexes $K_{\mathfrak{w}}(A)$, distinguished triangles in it, and weakly exact functors are defined in Definition 3.1.6; the weight decomposition functor $WD$ and the weight complex functor $t$ are described in Theorem 3.2.2; $t$-structures are recalled in §4.1; countable homotopy colimits and their properties are described in §4.2; negative subcategories, Karoubi-closures, and small envelopes are introduced in Definition 4.3.1; the categories $SH$ and $SH_{fin}$ of spectra are mentioned in Corollary 4.3.3; adjacent (weight and $t$-structures) are defined in Definition 4.4.1; negatively well-generating sets of objects are defined in Definition 4.5.1; more categories of spectra, singular cohomology, and singular homology $H^{sing}$ of spectra are considered in §4.6; we discuss idempotent completions in §5.1; $K_0$-groups of $Hw$, $\underline{C}$, $\operatorname{End} Hw$, $\operatorname{End} \underline{C}$, $\operatorname{End}^n Hw$, and $\operatorname{End}^n \underline{C}$ are defined in §5.3 and §5.4; regular additive categories are defined in Definition 5.4.2; differential graded categories and twisted complexes over them are defined in §6.1; truncation functors $t_N$ are constructed in §6.3; the spectral sequence $S(H,X)$ is considered in §6.4; we recall $SmCor$, $J$, $\mathfrak{H}$, $DM_-^{eff}$, $DM^s$, $DM_{gm}^{eff}$, and $DM_{gm}$ in §6.5.

# 1 Weight structures in triangulated categories: basic definitions and properties; auxiliary statements

In this section we give the definition of a weight structure $w$ in a triangulated category $\underline{C}$ (in §1.1) (this includes the notion of a *weight decomposition* of an object). We give other basic definitions and prove their certain simple properties in §1.2 and §1.3. We recall certain auxiliary statements that will help us to prove that the weight decomposition is functorial (in a certain sense) in §1.4. We study weight decompositions of morphisms, *infinite weight decompositions* and Postnikov towers for objects in §1.5.

## 1.1 Weight structures: definition and simple examples

**Definition 1.1.1** (Definition of a weight structure)**.** A pair of subclasses $\underline{C}^{w\le 0}, \underline{C}^{w\ge 0} \subset Obj\underline{C}$ for a triangulated category $\underline{C}$ will be said to define a weight structure $w$ for $\underline{C}$ if they satisfy the following conditions:

(i) $\underline{C}^{w\ge 0}, \underline{C}^{w\le 0}$ are additive and Karoubi-closed (i.e. contain all retracts of their objects that belong to $Obj\underline{C}$).

(ii) **"Semi-invariance" with respect to translations.**
$\underline{C}^{w\ge 0} \subset \underline{C}^{w\ge 0}[1]$, $\underline{C}^{w\le 0}[1] \subset \underline{C}^{w\le 0}$.

(iii) **Orthogonality.**
For any $X \in \underline{C}^{w\ge 0}$, $Y \in \underline{C}^{w\le 0}[1]$ we have $\underline{C}(X,Y) = \{0\}$.

(iv) **Weight decomposition**.
For any $X \in Obj\underline{C}$ there exists a distinguished triangle

$$B[-1] \to X \to A \xrightarrow{f} B \qquad (1)$$

such that $A \in \underline{C}^{w\le 0}, B \in \underline{C}^{w\ge 0}$.

The triangle (1) will be called a *weight decomposition* of $X$.

The basic example of a weight structure is given by the stupid filtration on the homotopy category of complexes over an arbitrary additive category

$A$. We will omit $w$ in this case and denote by $K(A)^{\leq 0}$ (resp. $K(A)^{\geq 0}$) the set of complexes that are (up to homotopy) retracts of complexes concentrated in degrees $\leq 0$ (resp. $\geq 0$). Its heart (see Definition 1.2.1 below) lies in the idempotent completion of $A$ (it is its *small envelope*; see part 3 of Definition 4.3.1 below). Moreover, we will see below (cf. Theorems 3.2.2, 3.3.1, and Remark 3.3.4) that this example is "almost universal" if one fixes the heart.

Note that in the case when $A$ is an abelian category with enough projectives and injectives then often the appropriate version (i.e. we impose some boundedness conditions) of $D(A)$ is equivalent to $K(I)$ and $K(P)$ where $P$ and $I$ denote the categories of projective and injective objects of $A$. Hence we see that some triangulated categories can support more then one weight structure; note that their hearts are usually not isomorphic.

*Remark* 1.1.2. Obviously, the axioms of weight structures are self-dual (recall that the same is true for axioms of triangulated categories). This means that $(C_1, C_2)$ define a weight structure for $\underline{C}$ iff $(C_2^{op}, C_1^{op})$ define a weight structure for $\underline{C}^{op}$. Recall also the same is true for $t$-structures (see Definition 4.1.1). We will apply these observations several times.

## 1.2 Other definitions

We will also need the following definitions.

**Definition 1.2.1.** [Other basic definitions]

1. The category $Hw \subset \underline{C}$ whose objects are $\underline{C}^{w=0} = \underline{C}^{w\geq 0} \cap \underline{C}^{w\leq 0}$, $Hw(X,Y) = \underline{C}(X,Y)$ for $X, Y \in \underline{C}^{w=0}$, will be called the *heart* of the weight structure $w$. We will see below that $Hw$ is additive.

2. $\underline{C}^{w\geq l}$ (resp. $\underline{C}^{w\leq l}$) will denote $\underline{C}^{w\geq 0}[-l]$ (resp. $\underline{C}^{w\leq 0}[-l]$).

3. For all $i, j \in \mathbb{Z}$, $i \geq j$ we define $\underline{C}^{[j,i]} = \underline{C}^{w\geq j} \cap \underline{C}^{w\leq i}$. B abuse of notation, we will sometimes identify $\underline{C}^{[j,i]}$ with the corresponding full additive subcategory of $\underline{C}$.

4. $w$ will be called *non-degenerate* if

$$\cap_l \underline{C}^{w\geq l} = \cap_l \underline{C}^{w\leq l} = \{0\}.$$

5. $w$ will be called *bounded above* (resp. bounded below) if $\cup_l \underline{C}^{w\leq l} = Obj\underline{C}$ (resp. $\cup_l \underline{C}^{w\geq l} = Obj\underline{C}$).

6. $w$ will be called *bounded* if it is bounded both above and below.

Next we observe an important difference between 'decompositions of objects' with respect to $t$-structures and weight structures.

*Remark* 1.2.2. In contrast to the $t$-structure situation, the presentation of $X$ in the form (1) is (almost) never unique. The only exception is the following *totally degenerate* situation: for any $X \in Obj\underline{C}$ there exist $Y \in \cap_l \underline{C}^{w\le l}$ and $Z \in \cap_l \underline{C}^{w\ge l}$ such that for some $f \in \underline{C}(Y,Z)$ we have $X \approx \mathrm{Cone}(f)$. Indeed, otherwise we can replace $(A,B,f)$ by $(A\oplus D,\ B\oplus D,\ f\oplus id_D)$ for any $D \in \underline{C}^{w=0}$. It could be easily seen that $\underline{C}^{w=0}$ is zero only in the totally degenerate case (we don't give the proof since degenerate cases are not very interesting).

Yet we will need to choose some $(A,B,f)$ several times. We will write that $A = X^{w\le 0}$, $B = X^{w\ge 1}$ if there exists a distinguished triangle (1). In Theorem 3.2.2 below we will verify that $X \to (A,B,f)$ is a functor 'up to zero maps on cohomology'.

We will also often denote $X[-i]^{w\le 0}$ by $X^{w\le i}$ and $X[i]^{w\ge 1}$ by $X^{w\ge i+1}$ for all $i \in \mathbb{Z}$. Note that we have $X^{w\le i} \in \underline{C}^{w\le 0}$ and $X^{w\ge i} \in \underline{C}^{w\le 0}$.

Below we will introduce a similar convention for the *weight complex* of $X$.

Besides, we will sometimes denote $X^{w\le i}[-i]$ by $w_{\le i}(X)$ and $X^{w\ge i}[-i]$ by $w_{\ge i}(X)$. So, for any $i \in \mathbb{Z}$ we have a distinguished triangle

$$w_{\ge i+1}(X) \to X \to w_{\le i}(X).$$

Yet if $X$ *does not have weight* 0 (a term proposed by J. Wildeshaus) then there exists a unique "nice" choice of a weight decomposition for $X$; see part 2 of Remark 1.5.2 below.

## 1.3 Simple basic properties of weight structures

For any $\underline{C}, w$ also the following fundamental properties are fulfilled. These properties (except part 7) are parallel to those of $t$-structures; part 7 illustrates the distinction between these notions.

**Proposition 1.3.1.** *1. If $\underline{C}(Y,X) = \{0\}$ for some $X \in Obj\underline{C}$ and all $Y \in \underline{C}^{w\ge 1}$ then $X \in \underline{C}^{w\le 0}$.*

*2. Vice versa, if $\underline{C}(X,Y) = \{0\}$ for some $X \in Obj\underline{C}$ and any $Y \in \underline{C}^{w\le -1}$ then $X \in \underline{C}^{w\ge 0}$.*

*3. If $A \to B \to C \to A[1]$ is a distinguished triangle and $A, C \in \underline{C}^{w\ge 0}$ (resp. $A, C \in \underline{C}^{w\le 0}$, resp. $A, C \in \underline{C}^{w=0}$) then $B \in \underline{C}^{w\ge 0}$ (resp. $B \in \underline{C}^{w\le 0}$, resp. $B \in \underline{C}^{w=0}$).*

*4. All $\underline{C}^{w\le i}$ are closed with respect to arbitrary (small) direct products (those, which exist in $\underline{C}$).*

*5. All $\underline{C}^{w\ge i}$ are closed with respect to arbitrary (small) direct sums (those, which exist in $\underline{C}$).*

*6. For any weight decomposition of $X \in \underline{C}^{w\geq 0}$ (see (1)) we have $A \in \underline{C}^{w=0}$, $B \in \underline{C}^{w\geq 0}$.*

*7. If $A \to B \to C \to A[1]$ is a distinguished triangle and $A, C \in \underline{C}^{w=0}$ then $B = A \oplus C$.*

*Proof.* 1. Let $B[-1] \to X \to A \to B$ be a weight decomposition of $X$. Since $\underline{C}(B[-1], X) = \{0\}$ we obtain that $X$ is a retract of $A$; hence $X \in \underline{C}^{w\leq 0}$.

2. The proof is similar to those of part 1 and could be obtained by dualization (see Remark 1.1.2). If $B[-1] \to X[-1] \to A \to B$ is a weight decomposition of $X[-1]$ then $\underline{C}(X[-1], A) = \{0\}$. Hence $X$ is a retract of $B$.

3. Let $A, C \in \underline{C}^{w\geq 0}$. For any $Y \in Obj\underline{C}$ we have a (long) exact sequence $\dots \to \underline{C}(C, Y) \to \underline{C}(B, Y) \to \underline{C}(A, Y) \to \dots$; hence by part (ii) of Definition 1.1.1 we obtain that $\underline{C}(B, Y) = \{0\}$ for any $Y \in \underline{C}^{w\leq -1}$. Now assertion 2 implies that $B \in \underline{C}^{w\geq 0}$.

The proof for the case $A, C \in \underline{C}^{w\geq 0}$ could be obtained by dualization.

The statement for the case $A, C \in \underline{C}^{w=0}$ now follows immediately from the definition of $\underline{C}^{w=0}$.

4. Obviously, assertion 1 implies that $\underline{C}^{w\leq i} = \{Y \in Obj\underline{C} : \underline{C}(X, Y) = \{0\} \forall X \in C^{w\geq i+1}\}$. This yields the result immediately.

5. Similarly, by assertion 2 we have $\underline{C}^{w\geq i} = \{X \in Obj\underline{C} : \underline{C}(X, Y) = \{0\}\ \forall Y \in C^{w\leq i-1}\}$; this yields the result.

6. $A \in \underline{C}^{w\leq 0}$ by definition. Since we have a distinguished triangle $X \to A \to B \to X[1]$, assertion 3 implies that $A \in \underline{C}^{w\geq 0}$.

7. Since $C \in \underline{C}^{w\geq 0}$ and $A[1] \in \underline{C}^{w\leq -1}$, the morphism $C \to A[1]$ in the the distinguished triangle is zero; so the triangle splits. □

*Remark* 1.3.2. 1. We try to answer the questions when a morphism $b[-1] \in \underline{C}(B[-1], X)$ for $B \in \underline{C}^{w\geq 0}$ extends to a weight decomposition of $X$ and $a \in \underline{C}(X, A)$ for $A \in \underline{C}^{w\leq 0}$ extends to a weight decomposition of $X$ (i.e. $\text{Cone}(f) \in \underline{C}^{w\geq 0}$) using parts 1 and 2 of Proposition 1.3.1.

We apply the long exact sequence corresponding to the functor $C^*$ for $C \in \underline{C}^{w\geq 0}$ (resp. to $C_*$ for $C \in \underline{C}^{w\leq 0}$). In the first case we obtain that $b[-1]$ extends to a weight decomposition iff the map $\underline{C}(C[i], B[-1]) \to \underline{C}(C[i], X)$ induced by $b$ is bijective for $i = -2$ and is surjective for $i = -1$ for all $C \in \underline{C}^{w\geq 0}$. Dually, $a$ extends to a weight decomposition iff for any $C \in \underline{C}^{w\leq 0}$ the map $\underline{C}(A, C) \to \underline{C}(X, C)$ induced by $a$ is bijective for $i = 1$ and is injective for $i = 0$.

Moreover, in many important cases (cf. section 4 below) it suffices to check the conditions of part 1 (resp. part 2) of Proposition 1.3.1 only for $Y = C[i]$ for $C \in \underline{C}^{w=0}$, $i < 0$ (resp. for $i > 0$). Then these conditions are equivalent to the bijectivity of all maps $\underline{C}(C[i], B[-1]) \to \underline{C}(C[i], X)$ induced by $b$ for $i < -1$ and their surjectivity for $i = -1$ for all $C \in \underline{C}^{w=0}$ (resp. to the bijectivity of all maps $\underline{C}(A, C) \to \underline{C}(X, C)$ induced by $a$ for $i > 0$ and their injectivity for $i = 0$).

We will use this observation below.

2. Certainly, parts 4 and 5 of the Proposition imply that all $\underline{C}^{w\geq i}, \underline{C}^{w\leq i}, \underline{C}^{w=i}$ are additive (i.e. closed with respect to direct sums of two objects) for any $i \in \mathbb{Z}$.

3. Since all (co)representable functors are additive, for any class of $C \subset Obj\underline{C}$ the classes of $X \in Obj\underline{C}$ satisfying $\underline{C}(X, Y) = \{0\}$ for all $Y \in C$ and $\underline{C}(Y, X) = \{0\}$ for all $Y \in C$ are Karoubi-closed in $\underline{C}$. We will use this fact below.

**Definition 1.3.3.** We consider $\underline{C}^- = \cup \underline{C}^{w\leq i}$ and $\underline{C}^+ = \cup \underline{C}^{w\geq i}$

We call $\underline{C}^b = \underline{C}^+ \cap \underline{C}^-$ the set of *bounded* objects of $\underline{C}$.

**Proposition 1.3.4.** *1. $\underline{C}^-$, $\underline{C}^+$, $\underline{C}^b$ are Karoubi-closed triangulated subcategories of $\underline{C}$.*

*2. $w$ induces weight structures for $\underline{C}^-$, $\underline{C}^+$, $\underline{C}^b$ whose hearts equal $Hw$.*

*3. $w$ is non-degenerate when restricted to $\underline{C}^b$.*

*Proof.* 1. From part 3 of Proposition 1.3.1 we easily deduce that $\underline{C}^-$, $\underline{C}^+$, $\underline{C}^b$ are closed with respect to direct sums, cones of morphisms, and retracts.

2. It suffices to verify that for any object $X$ of $\underline{C}^-$, $\underline{C}^+$, $\underline{C}^b$, respectively, the components of all of its possible weight decompositions belong to the corresponding category.

Let a distinguished triangle $B[-1] \to X \to A \to B \to X[1]$ be a weight decomposition of $X$, i.e $A \in \underline{C}^{w\leq 0}, B \in \underline{C}^{w\geq 0}$.

If $X$ in $\underline{C}^{w\leq i}$ for some $i > 0$ then part 3 of Proposition 1.3.1 implies $B \in \underline{C}^{w\leq i-1}$. Similarly, if $X$ in $\underline{C}^{w\geq i}$ for some $i \leq 0$ then $A \in \underline{C}^{w\geq i}$. We obtain the claim.

3. Let $X \in Obj\underline{C}^b \bigcap (\cap \underline{C}^{w\geq i})$; in particular, $X \in \underline{C}^{w\leq j}$ for some $j \in \mathbb{Z}$. Then, by the orthogonality property for $w$, we have $\underline{C}(X, X) = \{0\}$, hence $X = 0$.

A similar argument proves that $Obj\underline{C}^b \bigcap (\cap \underline{C}^{w\leq i}) = \{0\}$.

$\square$

$\underline{C}^b$ is especially important; note that it equals $\underline{C}$ if $(\underline{C}, w)$ is bounded.

Now we prove a simple lemma that will help us several times below to verify that a pair of subcategories satisfy axioms of weight structures.

**Lemma 1.3.5.** *1. Let $C \subset Obj\underline{C}$. Then the classes $C_1 = \{T \in Obj\underline{C} : \underline{C}(Y, T[1]) = \{0\}\ \forall Y \in C\}$ and $C_2 = \{T \in Obj\underline{C} :\ \underline{C}(T, Y[1]) = \{0\}\ \forall Y \in$*

$C\}$ *are strict, additive, Karoubi-closed; for a distinguished triangle* $U \to V \to W$ *if* $U, W \in C_i$ *then* $V \in C_i$ *(for* $i = 1, 2$*).*

*2. Let* $C$ *be additive, Karoubi-closed, and satisfy* $C \subset C[1]$*. Suppose also that for any* $X \in Obj\underline{C}$ *there exist* $A \in C_1,\ B \in C$ *and a distinguished triangle* $B[-1] \to X \to A \to B$*. Then the pair* $(C_1, C)$ *defines a weight structure for* $\underline{C}$*.*

*3. If for* $D_1, D_2 \subset Obj\underline{C}$ *we have* $\underline{C}(X, Y[1]) = \{0\}$ *for any* $X \in D_1$*,* $Y \in D_2$*, then the same is true for Karoubi-closures of* $D_1, D_2$*.*

*4. Let* $C$ *be additive, Karoubi-closed, and satisfy* $C[1] \subset C$*. Suppose also that for any* $X \in Obj\underline{C}$ *there exist* $A \in C,\ B \in C_2$ *and a distinguished triangle* $B[-1] \to X \to A \to B$*. Then the pair* $(C, C_2)$ *defines a weight structure for* $\underline{C}$*.*

*Proof.* 1. The assertion follows immediately form the fact that (co)representable functors are additive and cohomological (resp. homological).

2. Applying assertion 1, we obtain that it suffices to check that $C_1[1] \subset C_1$. Now, for any $X \in C_1$, $Y \in C$ we have $\underline{C}(Y, X[2]) = \underline{C}(Y[-1], X[1]) = \{0\}$ (from the definition of $C_1$ and $C[-1] \subset C \iff C \subset C[1]$).

3. Immediate from the biadditivity of $\underline{C}(-,-)$.

4. This is exactly the dual of assertion 2 (see Remark 1.1.2). $\square$

Lastly we prove a simple statement on comparison of weight structures.

**Lemma 1.3.6.** *Suppose that* $v, w$ *are weight structures for* $\underline{C}$*; let* $\underline{C}^{v\le 0} \subset \underline{C}^{w\le 0}$ *and* $\underline{C}^{v\ge 0} \subset \underline{C}^{w\ge 0}$*. Then* $v = w$ *(i.e. the inclusions are equalities).*

*Proof.* Let $X \in Obj\underline{C}^{w\le 0}$; let $B[-1] \xrightarrow{h} X \to A \to B$ be a weight decomposition of $X$ with respect to $v$. Since $B[-1] \in \underline{C}^{w\ge 1}$, the orthogonality property for $w$ implies $h = 0$. Hence $X$ is a retract of $A$. Since $\underline{C}^{v\le 0}$ is Karoubi-closed, we have $X \in \underline{C}^{v\le 0}$.

We obtain that $\underline{C}^{v\le 0} = \underline{C}^{w\le 0}$. The equality $\underline{C}^{v\ge 0} = \underline{C}^{w\ge 0}$ is proved similarly. $\square$

## 1.4 Some auxiliary statements: "almost functoriality" of distinguished triangles

We will prove below that the weight decomposition is functorial in a certain sense ("up to zero cohomology"). We will need some (general) statements on "almost functoriality" of distinguished triangles for this. This means that a morphism of between (two) vertices of two distinguished triangles can often be completed to a large commutative diagram.

**Lemma 1.4.1.** *Let $T: X \to A \to B \to X[1]$ and $T': X' \to A' \to B' \to X'[1]$ be distinguished triangles.*

*1. Let $\underline{C}(B, A'[1]) = \{0\}$. Then for any morphism $g: X \to X'$ there exist $h: A \to A'$ and $i: B \to B'$ completing $g$ to a morphism of triangles $T \to T'$.*

*2. Let moreover $\underline{C}(B, A') = \{0\}$. Then $g$ and $h$ are unique.*

*Proof.* This fact can be easily deduced from Proposition 1.1.9 of [5] (or Corollary IV.1.4 of [15]); we use the same argument here.

1. Since the sequence $\underline{C}(B, A') \to \underline{C}(B, B') \to \underline{C}(B, X'[1]) \to \underline{C}(B, A'[1])$ is exact, there exists $i: B \to B'$ such that $b' \circ i = g[1] \circ b_0$. By axiom Tr3 (see §IV.1 of [15]) there also exist a morphism $h: A \to A'$ that completes $(g, i)$ to a morphism of triangles.

2. Now we also have $\underline{C}(B, A') = \{0\}$. Hence the exact sequence mentioned in the proof of part I now also yields the uniqueness of $i$.

The condition on $h$ is that $h \circ a_0 = a' \circ f$. We have an exact sequence $\underline{C}(B, A') \to \underline{C}(A, A') \to \underline{C}(X, A')$. Since $\underline{C}(B, A') = \{0\}$, we obtain that $h$ is unique also. □

**Proposition 1.4.2.** *[$3 \times 3$-Lemma]*

*Any commutative square*

$$\begin{array}{ccc} X & \xrightarrow{a} & A \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ X' & \xrightarrow{a'} & A' \end{array}$$

*could be completed to a $4 \times 4$ diagram (we will mainly need its upper left $3 \times 3$ part) of the following sort:*

$$\begin{array}{ccccccc} X & \xrightarrow{a} & A & \xrightarrow{f} & B & \longrightarrow & X[1] \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle g[1]} \\ X' & \xrightarrow{a'} & A' & \xrightarrow{f'} & B' & \longrightarrow & X'[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X'' & \xrightarrow{a''} & A'' & \xrightarrow{f''} & B'' & \longrightarrow & X''[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X[1] & \xrightarrow{a[1]} & A[1] & \xrightarrow{f[1]} & B[1] & \longrightarrow & X[2] \end{array} \qquad (2)$$

*such that all rows and columns are distinguished triangles and all squares are commutative, except the right lowest square which anticommutes.*

*Proof.* The proof is mostly a repetitive use of the octahedron axiom. However it requires certain unpleasant diagrams. It is written in [5], Proposition 1.1.11. $\square$

We will also apply the octahedron axiom (see §IV.1.1 of [15]) directly. We recall that it states that any diagram $X \xrightarrow{f} Y \xrightarrow{g} Z$ can be completed to an octahedron diagram. In particular, there exists a distinguished triangle $\mathrm{Cone}(g \circ f) \to \mathrm{Cone}(g) \to \mathrm{Cone}(f)[1]$, whereas the map $\mathrm{Cone}(g) \to \mathrm{Cone}(f)[1]$ is obtained by composing of two of the maps in the distinguished triangles that define $\mathrm{Cone}(f)$ and $\mathrm{Cone}(g)$ (see §IV.1.8 of [15]).

## 1.5 Weight decomposition of morphisms; multiple weight decomposition of objects

Starting from this moment the triangle

$$T_k[k] : X[k] \xrightarrow{a_k} X^{w \le k} \xrightarrow{f_k} X^{w \ge k+1} \xrightarrow{b_k} X[k+1] \tag{3}$$

will be a weight decomposition of $X[k]$ for some $X \in Obj\underline{C}$, $k \in \mathbb{Z}$; $T'_k[k]$ : $X'[k] \xrightarrow{a'_k} X'^{w \le k} \xrightarrow{f'_k} X'^{w \ge k+1} \xrightarrow{b'_k} X'[k+1]$ will be a weight decomposition of $X'[k]$. Sometimes we will drop the index $k$ in the case $k = 0$.

**Lemma 1.5.1.** *1. Let $l \le m$. Then for any morphism $g : X \to X'$ there exist $h : w_{\le m}(X) \to w_{\le l}(X')$ and $i : X^{w \ge m+1}[-m] \to X'^{w \ge l+1}[-l]$ completing $g$ to a morphism of triangles $T_m \to T'_l$.*

*2. Let $l < m$. Then $h$ and $i$ are unique.*

*3. For $l = m$ any two choices $(h, i)$ and $(h', i')$ we have $h - h' = (s \circ f_m)[-m]$ and $i - i' = (f'_m \circ s')[-m]$ for some $s, s' \in \underline{C}(X^{w \ge m+1} \to X'^{w \ge m})$.*

*Proof.* 1,2: Immediate from Proposition 1.4.1.

3. If suffices to consider the case $g, h, i = 0$. Since $a_k[-k] \circ h = 0$, $T_k$ is a distinguished triangle, we obtain that $h'$ can be presented as $(s \circ f_m)[-m]$. Dually, $i'$ can be presented as $(f'_m \circ s')[-m]$

$\square$

*Remark* 1.5.2. 1. For $l < m$ we will denote $i, h$ constructed by $g_{X^{w \le m}, X'^{w \le l}}$ and $g_{X^{w \ge m+1}, X'^{w \ge l+1}}$, respectively.

For $l = m = 0$ we will call any pair $(h, i)$ a *weight decomposition* of $g$.

2. Suppose that $X$ admits a weight decomposition such that $X^{w \le 0} \in \underline{C}^{w \le -1}$ (in the terminology of J. Wildeshaus, $X$ *does not have weight* 0). Then such a decomposition is unique up to a unique isomorphism. Indeed,

in this case we can take $X[-1]^{w\le 0}=X^{w\le 0}$. Therefore, we can apply part 2 of the previous lemma for $l=-1$, $m=0$, $X'=X$, $g=id_X$.

This statement was communicated to the author by prof. J. Wildeshaus.

3. The statement of part 3 of Lemma 1.5.1 is the best possible in a certain sense. It is not possible (in general) to choose $s=s'$. In particular, one can take

$$X=X'=\mathbb{Z}/8\mathbb{Z}\stackrel{\times 2}{\to}\mathbb{Z}/8\mathbb{Z}\in ObjC^{[0,1]}((\mathbb{Z}/8\mathbb{Z})-mod)\subset C(Ab).$$

Then for $g=0$ there exists a pair $(h,i)=(\times 4,0)$ that is not homotopic to 0. Certainly, this example could be generalized to $X=X'=R/r^3R$ for any commutative ring $R$, $r\in R$, such that $r^2\nmid r^3$. In particular, this problem is not "torsion".

Note that the example of the weight decomposition described is obviously not a "nice" one. In particular, it cannot be extended to a $3\times 3$ diagram. Yet adding this example to the obvious weight decomposition of $id_X$ one obtains another weight decomposition of $id_X$ that is not homotopy equivalent to the first one; yet it does not seem to be "bad" in any sense.

Still one can check that extending morphisms $X\to X'$ to $(X^{w\le i},X^{w\ge i})\to (X'^{w\le i},X'^{w\ge i})$ using part 1 is sufficient to prove the functoriality of the cohomology of the weight complex of $X$ as defined in §2.2 below (the cohomology objects belong to $Hw'_*$, see the Notation and part 2 of Remark 3.1.7 below).

We check that $g_{X^{w\le m},X'^{w\le l}}$ and $g_{X^{w\ge m+1},X'^{w\ge l+1}}$ are "functorial".

**Lemma 1.5.3.** *Let $T''[j]:X''[j]\stackrel{a''_j}{\to}X''^{w\le j}\stackrel{f''_j}{\to}X''^{w\ge j+1}\stackrel{b''_j}{\to}X''[j+1]$ be a weight decomposition of $X''[j]$ for $X''\in Obj\underline{C}$ for some $j\le i\le 0$; let $p\in\underline{C}(X,X')$ and $q\in\underline{C}(X',X'')$.*

*If $j<0$ then for any choice of $(h',h'')$ satisfying*

$$h'\circ a_0=a'_i\circ p \text{ and } h''\circ a'_i=a''_j\circ q \qquad (4)$$

*we have $(q\circ p)_{X^{w\le 0},X''^{w\le j}}=h'\circ h$, while for any choice of $(i',i'')$ satisfying $b'_i\circ i'=p[1]\circ b_0$ and $b''_j\circ i''=q[1]\circ b'_i$ we have $(q\circ p)_{X^{w\ge 1},X''^{w\ge j+1}}=i'\circ i$.*

*Proof.* We apply the uniqueness proved in the previous lemma.

Both sides of the first equality calculate the only map $h$ that satisfies $h\circ a_0=a''_j\circ(q\circ p)$, while both sides of the second equality calculate the only map $i$ that satisfies $b''_j\circ i=(q\circ p)[1]\circ b_0$.

□

To prove the (weak) exactness of the weight complex functor (below), we will need the following lemma.

**Lemma 1.5.4.** *Let $DT : C \to X \xrightarrow{g} X'$ be a distinguished triangle. Then $DT$ could be completed to a diagram*

$$\begin{array}{ccccc}
X & \xrightarrow{a_i[-i]} & X^{w\le i}[-i] & \xrightarrow{f_i[-i]} & X^{w\ge i+1}[-i] \\
\downarrow g & & \downarrow g_{X^{w\le i},X'^{w\le i-1}} & & \downarrow g_{X^{w\ge i+1},X'^{w\ge i}} \\
X' & \xrightarrow{a'_{i-1}[1-i]} & X'^{w\le i-1}[1-i] & \xrightarrow{f'_{i-1}[1-i]} & X'^{w\ge i}[1-i] \\
\downarrow & & \downarrow & & \downarrow \\
C[1] & \longrightarrow & C_i[1-i] & \longrightarrow & C'_i[1-i]
\end{array} \qquad (5)$$

*whose rows and columns are distinguished triangles, all squares commute, $C_i, C'_i \in Obj\underline{C}$. Moreover, the last row (shifted by $[i-1]$) gives a weight decomposition of $C[i]$.*

*Besides, the choice of the part of (5) consisting of six upper objects and arrows connecting them is unique (even if we don't demand that this part could be completed to the whole (5)).*

*Proof.* By part 2 of Lemma 1.5.1, $g$ could be uniquely completed to a morphism of triangles that are the first two rows of (5). Since the left upper square of (5) is commutative, it could be completed to a $3 \times 3$-diagram (see Proposition 1.4.2). Hence the first two rows of this diagram will be as in (5). It remains to study the third row.

By part 3 of Proposition 1.3.1, the second column yields $C_i \in \underline{C}^{w\le 0}$, whereas the third column yields $C'_i \in \underline{C}^{w\ge 0}$. Hence $C[i] \to C_i \to C'_i$ is a weight decomposition of $C[i]$. $\square$

*Remark* 1.5.5. 1. In fact, the lemma is valid in a more general situation. Suppose that we have a pair of full subcategories $D, E \subset \underline{C}$ satisfying the orthogonality condition of weight structures (i.e. $\underline{C}(X,Y) = \{0\}$ for all $X \in ObjE$ and $Y \in ObjD[1]$) and "closed with respect respect to taking middle terms of distinguished triangles" (as $C^{w\le 0}$ and $C^{w\ge 0}$ in part 3 of Proposition 1.3.1). Then any "weight decompositions" of $X[i]$ and $X'[i-1]$ (defined similarly to the case when $D, E$ form a weight structure) can be completed to a diagram 5 with $C_i \in ObjD$, $C'_i \in ObjE$.

Indeed, it suffices to use the orthogonality to construct the diagram required (for some $C_i, C'_i \in Obj\underline{C}$). Next, the second column yields yields $C_i \in ObjD$, whereas the third column yields $C'_i \in ObjE$.

We will use this statement below for constructing weight decompositions for certain "candidate weight structures".

2. Lemma 1.5.4 and its expansion show that it suffices to know weight decompositions for some "basic" objects of $\underline{C}$ in order to obtain weight decompositions for all objects; see Theorems 4.3.2 and 4.5.2, and §7.1 below. The situation is quite different for $t$-structures; for this reason weight structures are more likely to exist (than $t$-structures), especially in "small" triangulated categories; cf. part 4 of Remark 4.3.4.

Now we study what happens if one combines more than one weight decompositions $T_k$.

**Proposition 1.5.6.** *[Multiple weight decomposition]*

*1. [Double weight decomposition]*

*Let $T_k$ be fixed for some $X \in Obj\underline{C}$ for $k$ being equal to some $i, j \in \mathbb{Z}$, $i > j$.*

*Then there exist unique morphisms $s_{ij} : X^{w\le i}[j-i] \to X^{w\le j}$, $q_{ij} : X^{w\ge i+1}[j-i] \to X^{w\ge j+1}$ making the corresponding triangles commutative. There also exists $X^{[i,j]} \in \underline{C}^{[0,i-j-1]}$, and distinguished triangles*

$$X^{w\le i}[j-i] \stackrel{s_{ij}}{\to} X^{w\le j} \stackrel{c_{ij}}{\to} X^{[i,j]} \stackrel{d_{ij}}{\to} X^{w\le i}[j-i+1] \tag{6}$$

*and*

$$X^{[i,j]}[-1] \stackrel{x_{ij}}{\to} X^{w\ge i+1}[j-i] \stackrel{q_{ij}}{\to} X^{w\ge j+1} \stackrel{y_{ij}}{\to} X^{[i,j]} \tag{7}$$

*for some $\underline{C}$-morphisms $c_{ij}, d_{ij}, x_{ij}, y_{ij}$.*

*2. [Infinite weight decomposition]*

*Let $T_k$ be fixed for all $k \in \mathbb{Z}$. Then for all $k \in \mathbb{Z}$ there exist unique morphisms $s_k : X^{w\le k}[-1] \to X^{w\le k-1}$, $q_k : X^{w\ge k+1}[-1] \to X^{w\ge k}$ making the corresponding triangles commutative. There also exists $X^k \in \underline{C}^{w=0}$, and distinguished triangles*

$$X^{w\le k}[-1] \stackrel{s_k}{\to} X^{w\le k-1} \stackrel{c_k}{\to} X^k \stackrel{d_k}{\to} X^{w\le k} \tag{8}$$

*and*

$$X^k[-1] \stackrel{x_k}{\to} X^{w\ge k+1}[-1] \stackrel{q_k}{\to} X^{w\ge k} \stackrel{y_k}{\to} X^k \tag{9}$$

*for some $\underline{C}$-morphisms $c_k, d_k, x_k, y_k$.*

*Moreover, $c_k$ and $x_k$ can be chosen equal to $y_k \circ f_{k-1}$ and $(f_k \circ d_k)[-1]$, respectively.*

*Proof.* 1. Applying Lemma 1.5.1 for $X = X'$ and $g = id_X$ we obtain the existence and uniqueness of $s_{ij}, q_{ij}$. It remains to study cones of these morphisms.

The $3 \times 3$-Lemma (i.e. Proposition 1.4.2) implies that the map of triangles $T_i[i] \to T_j[j]$ could be completed to a $3 \times 3$ diagram whose rows and columns are distinguished triangles. Hence there exists an distinguished triangle $\operatorname{Cone}(id_X) \to \operatorname{Cone}(s_{ij}) \to \operatorname{Cone}(q_{ij})$; hence $\operatorname{Cone}(s_{ij}) \cong \operatorname{Cone}_{q_{ij}}$. Part 3 of Proposition 1.3.1 applied to the distinguished triangle (6) implies $X^{[i,j]} \in \underline{C}^{w \le i-j-1}$; the same statement applied to the distinguished triangle (7) implies $X^{[i,j]} \in \underline{C}^{w \ge 0}$.

2. The first part of the assertion is immediate from part 1 applied for $(i,j) = (k+1,k)$ for all $k \in \mathbb{Z}$.

To prove the second part it suffices to complete the commutative triangle $X[k] \stackrel{a_k}{\to} X^{w \le k} \stackrel{s_k[1]}{\to} X^{w \le k-1}[1]$ to an octahedron diagram.

□

**Corollary 1.5.7.** *$\underline{C}^{w=0}$ generates $\underline{C}^b$ (as a triangulated category).*

*Proof.* Since $\underline{C}^b$ is a triangulated category that contains $Hw$, it suffices to prove that any object of $\underline{C}^b$ could be obtained from objects of $Hw$ by a finite number of taking cones of morphisms.

Let $X \in \underline{C}^{w \ge j} \cap \underline{C}^{w \le i}$. Then we can take $X^{w \le k} = 0$ for $k < j$ and $X^{w \ge k} = 0$ for $k > i$ in (3). Then $X = X^{w \le i}$ and the formula (8) gives a sequence of distinguished triangles implying that $X \in \langle \underline{C}^{w=0} \rangle$.

□

We will need the following definition several times.

**Definition 1.5.8.** We will denote by $Po(X)$ (a Postnikov tower for $X$) the following data: all $a_k$ (in (3)) and all triangles (8).

*Remark* 1.5.9. 1. A Postnikov tower (of $X$) is certainly not unique; yet we will prove below that it is "unique and functorial up to cohomology zero" in a certain sense.

The corresponding functoriality facts for the Postnikov towers of spectra are described in §6.3 of [26].

2. $Po(X)$ could be recovered from $a_k$ and $X^{w \le k}$ uniquely up to a non-canonical isomorphism.

# 2 The weight spectral sequence

The goal of this section is to describe the *weight spectral sequence* $T(H,X)$ (for $X \in Obj\underline{C}$ and a functor $H : \underline{C} \to A$) generalizing the classical one as one as (in a certain sense) those of §7 of [11]; cf. Remark 7.4.4 loc. cit. and §6.4 below. It will specialize to the "usual" weight spectral sequence for "classical realizations" of varieties (or motives); see part 2 of Remark 2.4.2.

Moreover, in this case the spectral sequence degenerates at $E_2$ (rationally) and its $E_2$ terms are exactly the graded pieces of the weight filtration.

In §2.1 we define the "weight filtration" for any functor from $\underline{C}$ with an abelian target. This will allow us to prove that the $D_2$-term of the derived exact couple for $T(H, X)$ is functorial in $X$.

In §2.2 we define the *weight complex* of $X$ in the terms of $Po(X)$. Its "cohomology" will yield the $E_2$ terms of the weight spectral sequence.

For simplicity we only construct in detail only the spectral sequence for homological functors (in §2.3); dualization immediately extends the result to the cohomological functor case (see §2.4). We conclude the section by noting that our spectral sequence induces the standard weight filtration for the rational étale and Hodge realizations of varieties (and motives); see Remark 2.4.2.

## 2.1 The weight filtration for (co)homological functors

We fix some choice of $T_k$ (in the notation of subsection 1.5). Let $A$ be an abelian category.

**Definition 2.1.1.** 1. If $H : \underline{C} \to A$, is any covariant functor then for any $i \in \mathbb{Z}$ we define $W_i(H)(X) = \operatorname{Im}(H(w_{\geq i}(X)) \to H(X))$.

2. If $H$ is contravariant then we define $W^i(H)(X) = \operatorname{Im}(w_{\leq i}(X)) \to H(X))$.

**Proposition 2.1.2.** *1. Let $H$ be covariant. Then the correspondence $X \to W_i(H)(X)$ gives a canonical subfunctor of $H(X)$. This means that $W_i(H)(X)$ does not depend on the choice of the weight decomposition of $X[i]$ and for any $f : X \to Y$ we have $f_*(W_i(H)(X)) \subset W_i(H)(Y)$ (for $X, Y \in Obj\underline{C}$).*

*2. The same is true for contravariant $H$ and $W^i(H)(X)$.*

*Proof.* 1. Part 1 of Lemma 1.5.1 implies that for any choice of weight decompositions of any $X[i], Y[i]$ we have $f_*(W_i(H)(X)) \subset W_i(H)(Y)$. In particular, taking $Y = X$, $f = id_X$ we obtain that $W_i(H)(X)$ does not depend on the choice of the weight decomposition of $X[i]$

2. The statement is exactly the dual of assertion 1 (see Remark 1.1.2).

□

*Remark* 2.1.3. 1. In the case when $H$ is (co)homological, we can replace the images in Definition 2.1.1 by certain kernels (since they coincide).

Besides, Proposition 2.1.2 is true if $A$ is **any** category with well-defined images of morphisms.

2. A partial case of this method for defining weight filtration for cohomology was (essentially) considered in Proposition 3.5 of [18].

3. [Universal functor; semi-motives]

Recall that we have a natural embedding $\underline{C} \to \underline{C}_*$. Then for any $i \in \mathbb{Z}$ and $X \in Obj\underline{C}$ we can define a functor $A \to W_i(X_*(A)) : \underline{C} \to Ab$. This yields an object of $\underline{C}_*$; it equals $W_i(i)(X)$ for $i$ being the embedding functor that sends $X \in Obj\underline{C}$ to $X_* = \underline{C}(-, X)$ . We obtain a sequence of functors $W_i : \underline{C} \to \underline{C}_*$. The usual Yoneda's isomorphism $F(X) \cong Mor_{Fun(\underline{C},-)}((X \to X_*), F)$ for a covariant functor $F : \underline{C} \to A$ can be easily generalized to

$$W_i(F)(X) \cong Mor_{Fun(\underline{C},-)}((X \to W_i(X)), F) \tag{10}$$

Hence the sequence of functors $_*, (W_i, -)$ is universal in the category of (covariant functors from $\underline{C}$ + their weight filtrations). Moreover, for any cohomological $F : \underline{C} \to A$ one could easily check that

$$W^i(F)(X) \cong Mor_{Fun(\underline{C},-)}((X \to X_*/W_{i+1}(X)), F) \tag{11}$$

In particular, one can apply this construction to the Chow weight filtration of Voevodsky's motives (see §6.5 below). For any motif $X$ (an so, for any variety) one obtains a sequence of objects of $DM^{eff}_{gm}{}_*$ which could be called *semi-motives*. These objects contain important information on $X$. In particular, (10) and (11) show that they have both homological and cohomological realizations!

## 2.2 The definition of the weight complex

Now we describe the weight complex of $X \in Obj\underline{C}$. We will prove that it is canonical and functorial (in a certain sense) in §3.2 below.

We adopt the notation of subsection 1.5.

**Definition 2.2.1.** We define the morphisms $h_i : X^i \to X^{i+1}$ as $c_{i+1} \circ d_i$. We will call $(X^i, h_i)$ the *weight complex* of $X$.

Note that all information on $t(X)$ is contained in $Po(X)$ (including the 'connection' of $t(X)$ with $X$).

**Proposition 2.2.2.** *1. Weight complex is a complex indeed i.e. an object of $C(Hw)$.*

*2. If for some $i \in \mathbb{Z}$ and $X \in \underline{C}^{w \le i}$ (resp. $X \in \underline{C}^{w \ge i}$) then there exists a choice of the weight complex of $X$ belonging to $C(Hw)^{\le i}$ (resp. to $C(Hw)^{\ge i}$).*

*Proof.* 1. We have

$$h_{i+1} \circ h_i = c_{i+2} \circ (d_{i+1} \circ c_{i+1}) \circ d_i = c_{i+2} \circ 0 \circ d_i = 0$$

for all $i$.

2. Similarly to the proof of Corollary 1.3.4, we can take $X^{w\geq k} = 0$ for $k > i$ (resp. $X^{w\leq k} = 0$ for $k < i$). Then we would have $X^{w\leq k} = X$ for $k \geq i$ (resp. $X^{w\geq k} = X$ for $k \leq i$). Therefore the corresponding choice of the weight complex of $X$ belongs to $C(Hw)^{\leq i}$ (resp. to $C(Hw)^{\geq i}$) by definition. □

We will study the functoriality of the weight complex in §3 below.

## 2.3 The weight spectral sequence for homological functors

Let $A$ be an abelian category; let $H : \underline{C} \to A$ be a homological functor (i.e. a covariant additive functor that transfers distinguished triangles into long exact sequences). The cohomological functor case will be obtained from the homological one by dualization.

Let $X \in Obj\underline{C}$, $(X^i, h_i) = t(X)$. We construct a spectral sequence whose $E_1$-terms are $H(X^i[j])$ which converges to $H(X[i+j])$ in many important cases.

**Definition 2.3.1.** We denote $H(Y[p])$ by $H^p(Y)$ for any $Y \in Obj\underline{C}$.

For a cohomological $H$ we will denote by $H^p(Y)$ the object $H(Y[-p])$.

First we describe the exact couple. It is obtained by applying $H$ to a Postnikov tower for $X$ (see Definition 1.5.8). In the first three parts of Theorem 2.3.2 we will fix the choice of a Postnikov tower.

Our exact couple is almost the same as the exact couple in §IV2, Exercise 2, of [15]. We take $E_1^{pq} = H^q(X^p)$, $D_1^{pq} = H^q(X^{\leq p})$. Then the distinguished triangles (8) give $E_1, D_1$ the structure of an exact couple.

**Theorem 2.3.2.** *[The homological weight spectral sequence]*

*I There exists a spectral sequence $T = T(H, X)$ with $E_1^{pq} = H^q(X^p)$ which weakly converges to $H^{p+q}(X)$ such that the map $E_1^{pq} \to E_1^{p+1q}$ equals $H_*^q(h_p)$; the corresponding filtration on $H(X)$ coincides with those of Definition 2.1.1.*

*II $T(H, X) \implies H^{p+q}(X)$ in either of the following cases:*

*(i) $X \in \underline{C}^b$.*

*(ii) $H$ vanishes on $\underline{C}^{w\geq q}$ for $q$ large enough and on $\underline{C}^{w\leq q}$ for $q$ small enough.*

*(iii) $X \in \underline{C}^-$ (resp $\underline{C}^+$) and $H$ vanishes on $\underline{C}^{w\leq q}$ for $q$ small enough (resp. on $\underline{C}^{w\geq q}$ for $q$ large enough).*

*III $T$ is functorial with respect to $H$ i.e. for any transformation of functors $H \to H'$ we have a canonical morphism of spectral sequences $T(H, X) \to$*

*$T(H, X')$; these morphisms respect sums and compositions of transformations.*

*IV $T$ is canonical and functorial with respect to $X$ starting from $E_2$.*

*Proof.* I The standard construction of the spectral sequence for an exact couple shows that the boundary maps for $E_1$ equal $H_*^q(h_p)$ indeed. The induced filtration on $H(X)$ is the weight filtration (of Definition 2.1.1) by definition.

II In case (ii) the exact couple is obviously bounded (this is already true at level 1).

In case (i) this will be also true if we choose the weight complex of $X$ to be bounded (we can do this by the definition of $\underline{C}^b$). Now, for an arbitrary choice of the weight complex we can connect it with some bounded choice by a quasi-isomorphism; see Remark 3.2.3 below. Applying the statement of part IV we obtain that the spectral sequence will be bounded starting from $E_2$ (moreover, the derived exact couple is bounded).

The proof in case (iii) is similar.

III This is obvious since all components of the exact couple are functorial with respect to $H$.

IV It suffices to check that the correspondence $X \to (E_2, D_2)$ (the derived exact couple) defines a functor. $D_2^{pq}(T) = \operatorname{Im} H^q(X^{\leq p}) \to H^{q+1}(X^{\leq p-1})$ is canonical and functorial by Lemma 1.5.3. Indeed, by this Lemma these terms and morphisms between them do not depend on the choices of the corresponding $h$-components of weight decompositions of morphisms.

Next, $E_2$ factorizes through the weight complex functor $t$ defined in §3.2 below; see part 3 of Remark 3.1.7. It can be also easily seen that the morphisms that define the derived couple are also functorial in $X$ (and hence, canonical).

$\square$

## 2.4 The weight spectral sequence for cohomological functors

Inverting the arrows in the proof of the previous theorem we obtain the following cohomological analogue.

**Theorem 2.4.1.** *[The cohomological weight spectral sequence]*

*I There exists a spectral sequence $T = T(H, X)$ with $E_1^{pq} = H^q(X^{-p})$ which weakly converges to $H^{p+q}(X)$ such that the map $E_1^{pq} \to E_1^{p+1q}$ equals $H^{q*}(h_{-1-p})$. The corresponding filtration on $H(X)$ coincides with those of Definition 2.1.1.*

*II* $T(H,X) \implies H^{p+q}(X)$ *in either of the following cases:*

*(i)* $X \in \underline{C}^b$.

*(ii)* $H$ *vanishes on* $\underline{C}^{w\geq q}$ *for* $q$ *large enough and on* $\underline{C}^{w\leq q}$ *for* $q$ *small enough.*

*(iii)* $X \in \underline{C}^-$ *(resp* $\underline{C}^+$*) and* $H$ *vanishes on* $\underline{C}^{w\leq q}$ *for* $q$ *small enough (resp. on* $\underline{C}^{w\geq q}$ *for* $q$ *large enough)*

*III* $T$ *is functorial with respect to* $H$ *i.e. for any transformation of functors* $H \to H'$ *we have a morphism of spectral sequences* $T(H,X) \to T(H,X')$.

*IV* $T$ *is canonical and (contravariantly) functorial with respect to* $X$ *starting from* $E_2$.

*Proof.* It suffices to apply Theorem 2.3.2 to the functor $H^{op}: \underline{C} \to A^{op}$. □

*Remark* 2.4.2. 1. Let $w$ be bounded. Suppose that there are no maps between different weights for $H$ i.e. there exists a family of full abelian subcategories $A^i \subset A$ such that $H^i(P) \in Obj A^i$ for all $i \in \mathbb{Z},\ P \in \underline{C}^{w=0}$, and there are no non-zero $A$-maps between different $A_i$. Then we easily obtain that $T(H,X)$ degenerates at $E_2$. Besides for any $a \in Obj A$ there cannot exist more than one finite filtration $W^j$ on $a$ such that $W^j(a)/W^{j+1}(a) \in Obj A^j$; whereas our definition (2.1.1) of $W(H)(-)$ provides us with such a filtration for all $H(X)$, $X \in Obj \underline{C}$.

2. We will see in §6 below that Voevodsky's $DM_{gm} \supset DM_{gm}^{eff}$ admits a *Chow weight structure* whose heart is $Chow$. Hence we obtain the weight spectral sequence and weight spectral sequence for any realization of motives. In particular, this is the case for étale and Hodge realizations of motives, and motivic cohomology.

Now, it is well known that for the rational étale and Hodge realizations there are no non-zero maps between different weights (in the corresponding categories of mixed structures) in the sense described above. Therefore for the rational étale and Hodge realizations of motives our weight filtration coincides with the usual one up to a change of indices; see also §7.4 of [11]. Still recall that "classically" the weight filtration is well-defined only for cohomology with rational coefficients. Yet our method allows to define canonical weight filtrations integrally; this generalizes the construction of Theorem 3 of [16].

Now we consider the case of motivic cohomology. A simple example of the spectral sequence obtained comes from the Bloch long exact localization sequence for motivic cohomology; see part 1 of Remark 7.3.1 in [11]. Since the latter is not trivial, the "weight" filtration obtained is non-trivial in this case either; it appears not to be mentioned in the literature (in the general case). This filtration is compatible with the regulator maps (whose targets are "classical" cohomology theories).

# 3 The weight complex functor

In §5 of [11] for a triangulated category $\underline{C}$ with a *negative differential graded enhancement* a conservative exact *weight complex* functor $t_0 : \underline{C} \to K(Hw)$ (in our notation) was constructed; see §6.3. The goal of this section is to extend this result to the case of arbitrary $(\underline{C}, w)$.

The results of §1.5 easily imply that any $g : X \to X'$ where $X, X' \in Obj\underline{C}$ could be extended to a morphism of (any possible) Postnikov towers for $X, X'$ (the "topological case" of this statement is Lemma 14 of §6.3 of [26]). Moreover, for compositions of morphisms the corresponding morphisms of Postnikov towers could be composed. Yet, as the example of part 3 of Remark 1.5.2 shows, this construction cannot give a canonic morphism of weight complexes in $K(Hw)$. We have to consider a certain factor $K_{\mathfrak{w}}(Hw)$ of this category. This factor is no longer triangulated (in the general case; yet cf. Remark 3.3.4). Still the kernel of the projection $K(Hw) \to K_{\mathfrak{w}}(Hw)$ is an ideal (of morphisms) whose square is zero; so our ("weak") weight complex functor is not much worse than the "strong" one of [11].

We define and study $K_{\mathfrak{w}}(Hw)$ in §3.1. We construct the weight complex functor $t$ in §3.2 and prove its main properties in §3.3. One of our main tools is the *weight decomposition functor* $WD : \underline{C} \to K_{\mathfrak{w}}^{[0,1]}(\underline{C})$; see Theorem 3.2.2.

One of the main properties of the functor $t$ is that it calculates the $E_2$-terms of the weight spectral sequence $T$, see part 3 of Remark 3.1.7. In fact, this is why $t$ it called the weight complex; this term was used for the first time in [16] (see §2 and §3.1 of [16]).

## 3.1 The weak category of complexes

Let $A$ be an additive category. We will need the following, very natural definition.

**Definition 3.1.1.** A class $T$ of morphisms in $A$ will be called a (two-sided) ideal if it is closed with respect to sums and differences (of two morphisms of $T$ lying in the same morphism group), direct sums and compositions with any morphism in $A$.

We will abbreviate these properties as $T \triangleleft MorA$.

*Remark* 3.1.2. Obviously, for any $T \triangleleft MorA$ we can consider an additive category $A/T$ whose object are the same as for $A$, and $A/T(X, Y) = A(X, Y)/T(X, Y)$ for all $X, Y \in ObjA$.

Besides, it is easily seen that one can naturally "multiply" ideals of $MorA$ via the composition operation.

Now, we will denote by $Z(X,Y)$ for $X,Y \in ObjK(A)$ the subgroup of $K(A)(X,Y)$ consisting of morphisms that could be presented as $(s_{i+1} \circ d_X^i + d_Y^{i-1} \circ t_i)$ for some set of $s_i, t_i \in A(X_i, Y_{i-1})$ (here $X = (X_i),\ Y = (Y_i)$).

*Remark* 3.1.3. We will often use the fact that $sd + dt = (s-t)d + (dt + td)$ is homotopy equivalent to $(s-t)d$; hence we may assume that $t = 0$ in the definition of $Z$.

Now we check that for $Z = \cup_{X,Y \in ObjK(A)} Z(X,Y)$ we have $Z \triangleleft MorK(A)$ and $Z^2 = 0$. A easy standard argument also shows that for any $\underline{C}$ all ideals $Z \triangleleft Mor\underline{C}$ that satisfy $Z^2 = 0$ satisfy a collection of nice properties.

**Lemma 3.1.4.** *I1. $Z \triangleleft MorK(A)$.*

*2. Let $L, M, N \in ObjK(A)$, let $g \in Z(L,M) \subset K(A)(L,M)$, $h \in Z(M,N) \subset K(A)(M,N)$. Then $h \circ g = 0$ (in $K(A)$).*

*II Let $T \triangleleft Mor\underline{C}$ for some additive category $\underline{C}$, suppose also that $T^2 = 0$; let $D$ be an additive category. Let $p : \underline{C} \to D$ be an additive functor such that for any $X, Y \in Obj\underline{C}$ we have $\operatorname{Ker}(\underline{C}(X,Y) \to D(p(X), p(Y))) = T(X,Y)$.*

*Then the following statements are valid.*

*1. Let $p$ be a full functor. Then it is conservative i.e. $p(g)$ is an isomorphism iff $g$ is (for any morphism $g$ in $\underline{C}$).*

*2. For any $X \in Obj\underline{C}$ and $r \in \underline{C}(X,X)$ if $p(r)$ is an idempotent then it could be lifted to an idempotent $r' \in \underline{C}(X,X)$ (i.e. $p(r') = p(r)$).*

*3. If $\underline{C}$ is idempotent complete then its categorical image in $D$ also is. Here we consider a not necessarily full subcategory of $D$ such that all its objects and morphisms are exactly those that come from $\underline{C}$.*

*Proof.* I1. Obviously, $Z$ is closed with respect to sums and direct sums.

Lastly, let $d$ denote the differential, let $f, g$, and $h$ be composable morphisms; let $g = s \circ d$, for $s$ being a collection of arrows shifting the degree by $-1$. Then we have $f \circ g = (f \circ s) \circ d$ and $g \circ h = -(s \circ h) \circ d$; note that $h$ "anticommutes with the differential".

2. Let $L = (L_i),\ M = (M_i),\ N = N_i$. Suppose that for all $i \in \mathbb{Z}$ we have $g_i = s_{i+1} \circ d_L^i$ for some set of $s_i \in A(L_i, M_{i-1})$, whereas $h_i = u_{i+1} \circ d_M^i$ for some set of $u_i \in A(M_i, N_{i-1})$

Then $h_i \circ g_i = u_{i+1} \circ d_M^i \circ s_{i+1} \circ d_L^i$. Recall now that $g$ is a morphism of complexes; hence for all $i \in \mathbb{Z}$ we have $d_M^i \circ s_{i+1} \circ d_L^i = d_M^i \circ d_M^{i-1} \circ s_i = 0$. We obtain that $h \circ g$ is homotopic to 0.

II1. Since $p$ is a functor, it sends isomorphisms to isomorphisms.

Now we prove the converse statement. Let $g \in \underline{C}(X, X')$ for $X, X' \in Obj\underline{C}$, let $p(h)$ for some $h \in \underline{C}(X', X)$ be the inverse to $p(g)$. We have $h \circ g - id_X \in T(X,X)$ and $g \circ h - id_{X'} \in T(X', X')$. It suffices to check that $h \circ g$ and $g \circ h$ are invertible in $\underline{C}$. The last assertion follows from equalities

$(h\circ g - id_X)^2 = 0$ and $(g\circ h - id_{X'})^2 = 0$ in $\underline{C}$, that yield $(h\circ g)(2id_X - h\circ g) = id_X$ and $(g\circ h)(2id_{X'} - g\circ h) = id_{X'}$.

2. This is just the standard statement that idempotents could be lifted (in rings).

We consider $r' = -2r^3 + 3r^2$. Since $p(r)^2 = p(r)$ in $D$ and $r' = r + (r^2 - r)\circ(id_X - 2r)$, we have $p(r') = p(r)$. Since $r_i'^2 - r' = (r^2-r)^2 \circ (4r^2 - 4r - 3id_X)$, we obtain that $r'$ is an idempotent.

3. The assertion follows immediately from II2. Indeed, any idempotent $d$ in the image could be lifted to an idempotent $c$ in $\underline{C}$. Since $c$ splits in $\underline{C}$, $p(c) = d$ splits in the image.

$\square$

*Remark* 3.1.5. The assertions of part II remain valid for any nilpotent $T$. For $l$ that satisfies $l^n = 0$, $n > 0$, the inverse to $id_X - l$ is given by $id_X + l + l^2 + \cdots + l^{n-1}$. If $l^n = 0$, $r^2 - r = l$, then the equality $(x - (x-1))^{2n-1} = 0$ allows to construct explicitly a polynomial $P(x)$ such that $P \equiv 0 \mod x^n\mathbb{Z}[x]$ and $P \equiv 1 \mod (x-1)^n\mathbb{Z}[x]$. Then $P(r)^2 = P(r)$; $P(r) - r$ could be factorized through $l$.

**Definition 3.1.6.** [The definition of $K_{\mathfrak{w}}(A)$]

We define $K_{\mathfrak{w}}(A)$ as $K(A)/Z$ (in the sense of Remark 3.1.2) with isomorphic (i.e. homotopy equivalent) objects identified.

We have the obvious shift functor $[1] : K_{\mathfrak{w}}(A) \to K_{\mathfrak{w}}(A)$.

A triangle $X \to Y \to Z \to X[1]$ in $K_{\mathfrak{w}}(A)$ will be called distinguished if any of its two sides could be lifted to two sides of some distinguished triangle in $K(A)$.

An additive functor $F : \underline{C} \to K_{\mathfrak{w}}(A)$ for a triangulated $\underline{C}$ will be called *weakly exact* if it commutes with shifts and sends distinguished triangles to distinguished triangles.

The bounded subcategories of $K_{\mathfrak{w}}(A)$ are defined in the obvious way.

*Remark* 3.1.7. [Why $K_{\mathfrak{w}}(A)$ is a category; cohomology ]

1. $K_{\mathfrak{w}}(A)$ is a category since we just factorize the class of objects of $K(A)/Z$ with respect to a class of invertible morphisms; see Remark 3.1.2.

2. Let $B$ be an abelian category; let $F : A \to B$ be an additive functor. Then any $g \in Z(X,Y)$ gives a zero morphism on cohomology of $F_*(X)$. It follows that the cohomologies of $F_*(X)$ give well-defined functors $K_{\mathfrak{w}}(A) \to B$. Besides, these functors are easily seen to be "cohomological" i.e. they translate distinguished triangles in $K_{\mathfrak{w}}(A)$ into long exact sequences.

In particular, this is true for the "universal" functor $A \to A'_*$ (recall that $A'_*$ is the full abelian subcategory of $A_*$ generated by $A$). Hence there are well defined cohomology functors $H_i : K_{\mathfrak{w}}(A) \to A'_*$.

3. Now suppose that for a triangulated $\underline{C}$ we have a weakly exact functor $u : \underline{C} \to K_{\mathfrak{w}}(A)$. Then the cohomology of $F_*(u(X))$ gives well-defined functors $\underline{C} \to B$. Again, distinguished triangles in $\underline{C}$ are translated into long exact sequences.

In particular, this statement can be applied to the weight complex functor $t : \underline{C} \to K_{\mathfrak{w}}(Hw)$ described in part II of Theorem 3.2.2 below. This concludes the proof of (part IV of) Theorem 2.3.2.

Lemma 3.1.4 immediately yields the following statement.

**Proposition 3.1.8.** *1. The projection $p : K(A) \to K_{\mathfrak{w}}(A)$ is conservative.*

*2. Let $A$ be idempotent complete. Then $K^b_{\mathfrak{w}}(A)$ is idempotent complete also.*

*Proof.* 1. Immediate from part II1 of Lemma 3.1.4.

2. It is well known that $K^b(A)$ is idempotent complete; see, for example, [3]. Hence part II3 of Lemma 3.1.4 yields the result.

□

## 3.2 The functoriality of the weight complex

We will use the following simple fact.

**Lemma 3.2.1.** *If $X \in \underline{C}^{w\ge 0}$, $Y \in \underline{C}^{w\le 0}$ then any $f \in \underline{C}(X,Y)$ could be factorized through some morphism $X^0 \to Y^0$ (of the zeroth terms of weight complexes).*

*Proof.* Easy from the equality $\underline{C}(X^{w\ge 1}[-1], Y) = \underline{C}(X^0, Y^{w\le -1}[1]) = \{0\}$. □

Now we prove that the "single" and the "infinite" weight decompositions define functors. Let $X, X'$ denote arbitrary objects of $\underline{C}$.

**Theorem 3.2.2.** *I1. The (single) weight decomposition of objects and morphisms gives a functor $WD : \underline{C} \to K^{[0,1]}_{\mathfrak{w}}(\underline{C})$ (i.e. the image is concentrated in degrees $0, 1$).*

*2. Morphisms $g \in \underline{C}(X, X')$, $h \in \underline{C}(X^{w\le 0}, X'^{w\le 0})$ and $i : \underline{C}(X^{w\ge 1}, X'^{w\ge 1})$ give a morphism of weight decompositions (of $X$ and $X'$) iff $(h, i) = WD(g)$ in $K_{\mathfrak{w}}(\underline{C})$.*

*3. The homomorphism $\underline{C}(X, X') \to K^{[0,1]}_{\mathfrak{w}}(\underline{C})(WD(X), WD(X'))$ is surjective.*

*4. For all $X, X' \in Obj\underline{C}$ we make the notation*

$$T(X, X') = \operatorname{Ker}(\underline{C}(X, Y) \to K_{\mathfrak{w}}(Hw)(WD(X), WD(X')).$$

*Then $T \triangleleft Mor\underline{C}$; $T^2 = 0$.*

*5. If $WD(X) \cong WD(X')$ in $K_{\mathfrak{w}}(\underline{C})$ then $X \cong X'$ in $\underline{C}$.*

*6. For any $X \in Obj\underline{C}$, $p \in \underline{C}(X, X)$, if $WD(p)$ is idempotent then $WD(p)$ could be lifted to an idempotent $p'$ in $\underline{C}(X, X)$.*

*II The infinite weight decomposition of objects and morphisms (cf. the construction described in the proof) gives a functor $\underline{C} \to K_{\mathfrak{w}}(Hw)$.*

*Proof.* 1. By part 1 of Lemma 1.5.1, any morphism $X \to X'$ could be extended to a morphism of their (fixed) weight decompositions. This extension is uniquely defined in $K_{\mathfrak{w}}^{[0,1]}(\underline{C})$ by part 3 of loc. cit. One can compose such homomorphisms in $K_{\mathfrak{w}}(\underline{C})$ since one of the possible extensions of the composition of morphisms $X \to X' \to X''$ (in $C(\underline{C})$) is the composition of (arbitrary) extensions for the morphisms $X \to X'$ and $X' \to X''$.

It remains to check that the image of $X$ in $Obj K_{\mathfrak{w}}^{[0,1]}(\underline{C})$ does not depend on the choice of the weight decomposition. Let $K, K' \in Obj K(\underline{C})$ be given by two weight decompositions of $X$; $id_X$ induces $g \in K(\underline{C})(K, K')$ and $h \in K(\underline{C})(K', K)$. By part 3 of Lemma 1.5.1, $h \circ g - id_K \in Z(K, K)$ and $g \circ h - id_{K'} \in Z(K', K')$. It suffices to check that $h \circ g$ and $g \circ h$ are invertible in $K(\underline{C})$; this follows from part 1 of Proposition 3.1.8.

2. By definition of $WD$, the triple $(g, WD(g))$ gives a morphism of weight decompositions.

Now suppose that $(h, i) = WD(g)$ i.e. $(h, i) \in C(\underline{C})(WD(X), WD(X'))$ and $(h, i) \equiv WD(g) \mod T(WD(X), WD(X'))$. It follows that $i \circ f = f' \circ h$ (in the notation of (3)). Besides, there exist $(h', i')$ that give a morphism of weight decompositions; $h - h' = s \circ f$ and $i - i' = f' \circ t$ for some $s, t \in \underline{C}(X^{w \geq 1}, X'^{w \leq 0})$. We obtain that $h \circ a = h' \circ a = a' \circ g$ and $b' \circ i = b' \circ i' = g[1] \circ b$.

Hence $(g, h, i)$ give a morphism $T_0 \to T'_0$.

3. By definition, any $h \in K_{\mathfrak{w}}^{[0,1]}(\underline{C})(WD(X), WD(X'))$ comes from some commutative square

$$\begin{array}{ccc} X^{w \leq 0} & \xrightarrow{f_0} & X^{w \geq 1} \\ \downarrow & & \downarrow \\ X'^{w \leq 0} & \xrightarrow{f'_0} & X'^{w \geq 1} \end{array}$$

Extending this square to a morphisms of triangles $T_0 \to T'_0$ (i.e. of weight decompositions of $X$ and $X'$) immediately yields the result.

4. Since $WD$ is a functor, $T$ is an ideal.

We prove that $T^2 = 0$ similarly to the proof of I2 of Lemma 3.1.4.

Let $X, X', X'' \in Obj\underline{C}$, let $g \in T(X', X'') \subset \underline{C}(X, X')$, $h \in T(X', X'') \subset \underline{C}(X', X'')$.

We should check that $h \circ g = 0$ (in $\underline{C}$). We can choose any weight decompositions of $X, X', X'$; denote them by $T, T, T''$ (similarly to (3)).

Since $WD(g) = WD(h) = 0$, by assertion I2 we obtain that $(g, 0, 0)$ and $(h, 0, 0)$ give morphisms of weight decompositions. This means that $a' \circ g = a'' \circ h = g[1] \circ b = h[1] \circ b' = 0$. Hence $g$ could be presented as $b'[-1] \circ c$ for some $c \in \underline{C}(X, X'^{w\geq 1}[-1])$. Then $h \circ g = (h[1] \circ b')[-1] \circ c = 0$.

5. By assertion I3 any isomorphism $WD(X) \to WD(X')$ is induced by some morphism $X \to X'$. Now by part II1 of Lemma 3.1.4, $t$ is conservative (we apply assertion I4); this yields the result.

6. Immediate from part II2 of Lemma 3.1.4.

II Exactly the same reasoning as in part I1 will prove the assertion after we verify that morphisms in $\underline{C}$ give well-defined morphisms of weight complexes (in $K_{\mathfrak{w}}(Hw)$).

Applying part I1 to weight decompositions of $X[k], X'[k]$ for $X, X' \in Obj\underline{C}$ and all $k \in \mathbb{Z}$, we obtain that any $g \in \underline{C}(X, X')$ gives a (non-unique) family of $g_k : X^{w\leq k} \to X'^{w\leq k}$. Besides, for all $k \in \mathbb{Z}$ we have $g_k \circ s_{k+1} = s'_{k+1} \circ g_{k+1}[-1]$ (see Lemma 1.5.3); here we extend the notation of part 2 of Proposition 1.5.6 to $X'$. These morphisms can be extended to a morphism $Po(X) \to Po(X')$; hence we obtain some morphism $t(g) : t(X) \to t(X')$. It remains to verify that for $g = 0$ we have $t(g) \in Z(t(X), t(X'))$.

We study the possibilities for $g_i : X^i \to X'^i$. Note that $X_i$ depends on the maps $r_k : X^{w\leq k} \to X'^{w\leq k}$ only for $k = i, i-1$. This dependence is linear. Moreover, any pair of $(r_i, r_{i-1})$ could be presented as $(0, r_{i-1}) + (r_i, 0)$. Hence it suffices to prove that $g_i$ could be presented as $(s_{i+1} \circ h_{iX} + h_{i-1,X'} \circ t_i)$ for some $s_{i+1} \in Hw(X^{i+1}, X'^i)$, $t_i \in Hw(X^i, X'^{i-1})$ in two cases: either $r_i$ or $r_{i-1}$ equals 0. (Recall that $h$ denotes the boundary of a weight complex).

In the case $r_i = 0$ we can present $g_i[-1]$ as the second component of $WD(0 : X^{w\leq i}[-1] \to X'^{w\leq i}[-1])$. Hence $g_i$ equals $c_{i-1,X'} \circ u_i$ for some $u_i \in \underline{C}(X^i,\ X'^{w\leq i-1})$. Note now that $u_i$ could be factorized through $X'^{i-1}$ (see Lemma 3.2.1).

In the case $r_{i-1} = 0$ we can present $g_i$ as the first component of $WD(0 : X^{w\geq i} \to X'^{w\geq i})$. Hence $g_i$ equals $v_{i+1} \circ x_{iX}[1]$ for some $v_{i+1} \in Hw(X^{w\geq i+1}, X'^i)$. It remains to note that $v_i$ could be factorized through $X^{i+1}$.

Combining the two cases, we obtain our claim.

□

*Remark* 3.2.3. The functoriality of $t$ implies that for any $X \in Obj\underline{C}$ any two choices for $t(X)$ are connected by a (canonical) isomorphism in $K_{\mathfrak{w}}(Hw)$. Then part II1 of Lemma 3.1.4 (combined with part I2 of the Lemma) implies that they are isomorphic (not necessarily canonically) in $K(Hw)$ i.e. they are homotopy equivalent (in $C(Hw)$).

$WD$ and $t$ "commute" in the following sense.

**Proposition 3.2.4.** *Let $X, X' \in Obj\underline{C}$, $g \in \underline{C}(X, X')$.*

*1. Any choice of $(t(i), t(l))$ for $(i, l) = WD(g)$ comes from a truncation of $t(g)$ (here we fix some weight decompositions of $X$ and $X'$ and consider all compatible lifts of $t(g)$ to $MorC(Hw)$).*

*2. Let some $(r', s') = (t(i'), t(l'))$ for some weight decomposition $(i', l')$ of $g$, let $r + s : t(X) \to t(X')$ be homotopic to $r' + s'$ (here we consider sums of collections of arrows). Then $(r, s) = (t(i), t(l))$ for some (other) weight decomposition $(i, l)$ of $g$.*

*Proof.* 1. By the definition of $t(g)$ (see part II of Theorem 3.2.2) any choice of $(t(i), t(l))$ is a possible truncation of $t(h)$ over $C(Hw)$.

2. It suffices to prove the statement for $g = 0$. Suppose that $(r, s)$ could be obtained from some $WD(0)$ via $t$. Note that (replacing $r, s$ by equivalent morphisms if needed) we can assume that $r = r_0$, $s = s_1$ (i.e. they are concentrated in degrees $0, 1$). Hence there exists some $l \in Hw(X^1, X'^0)$ such that $r_0 = l \circ h_0$, $s_1 = h'_0 \circ l$.

Now it remains to note that the triple $(0, d'_0 \circ l \circ c_1, x'_0[1] \circ l \circ y_1)$ gives a weight decomposition of $0 : X \to X'$. This fact follows from the equalities $d'_0 \circ l \circ c_1 \circ a_0 = 0 = b'_0 \circ x'_0[1] \circ l \circ y_1$ (see (8) and (9)), whereas

$$f'_0 \circ d'_0 \circ l \circ c_1 = x'_0[1] \circ l \circ c_1 = x'_0[1] \circ l \circ y_1 \circ f_0.$$

□

## 3.3 Main properties of the weight complex

Now we prove the main properties of the weight complex functor.

**Theorem 3.3.1.** *[The weight complex theorem]*

*I Exactness.*

*$t$ is a weakly exact functor.*

*II Nilpotence.*

*$I(-,-) = \operatorname{Ker} \underline{C}(-,-) \to K_{\mathfrak{w}}(t(-), t(-))$ defines an ideal in $Mor\underline{C}$. For any $i \le j \in \mathbb{Z}$ the restriction $I^{[i,j]}$ of $I$ to $\underline{C}^{[i,j]}$ satisfies $(I^{[i,j]})^{j-i+1} = 0$.*

*III Idempotents.*

*If $X \in \underline{C}^b$, $g \in \underline{C}(X, X)$, $t(g) = t(g \circ g)$, then $t(g)$ could be lifted to an idempotent $g' \in \underline{C}(X, X)$.*

*IV Filtration.*

*If $X \in \underline{C}^{w \le i}$ (resp. $\underline{C}^{w \ge i}$) for some $i \in \mathbb{Z}$ then $t(X) \in K_{\mathfrak{w}}(Hw)^{w \le i}$ (resp. $K_{\mathfrak{w}}(Hw)^{w \ge i}$) i.e. it is homotopy equivalent to a complex concentrated in degrees $\le i$ (resp. $\ge i$).*

*If $X$ is bounded from above (resp. from below) then the converse implications are valid also.*

*V Conservativity.*

*If $w$ is non-degenerate then the functor $t$ is conservative on $\underline{C}^+$ and $\underline{C}^-$.*

*VI If $X, Y \in \underline{C}^{[0,1]}$ then $t(X) \cong t(Y) \implies X \cong Y$.*

*VII Let $X \in \underline{C}^{w \geq a}$ for some $a \in \mathbb{Z}$; consider the homomorphism $t_* : \underline{C}(X, X') \to K_{\mathfrak{w}}(Hw)(t(X), t(X'))$. Then the following statements are valid.*

*1. $t$ is bijective if $X' \in \underline{C}^{w \leq a}$.*

*2. $t$ is bijective if $X' \in \underline{C}^{w \leq a+1}$.*

*Proof.* I Let $C \xrightarrow{a} X \xrightarrow{f} X' \xrightarrow{b} C[1]$ be a distinguished triangle. We should prove that the triangle $t(C) \xrightarrow{t(a)} t(X) \xrightarrow{t(f)} t(X') \xrightarrow{t(b)} t(C)[1]$ is distinguished. It suffices to construct a triangle of morphisms

$$V : t(X'[-1]) \xrightarrow{m} t(C) \xrightarrow{n} t(X) \tag{12}$$

that splits componentwisely (in $C(Hw)$) such that $m$ is some choice for $t(b)[-1]$ and $n$ is some choice for $t(a)$. Indeed, it is a well known fact that any such $V$ gives a distinguished triangle in $K(Hw)$. Hence any two sides of $t(V)$ could be lifted to two sides of a distinguished triangle in $K(Hw)$; so $t(V)$ is distinguished (see Definition 3.1.6).

In order to prove our claim we apply Lemma 1.5.4 for all $i \in \mathbb{Z}$. By the Lemma, the triangles $C[i] \to C_i \to C'_i$ obtained from (5) by shifting the last row are weight decompositions of $C[i]$ for all $i \in \mathbb{Z}$. Hence first two columns could be completed to morphisms $Po(X') \to Po(C)[1] \to Po(X)[1]$.

Now we check that the corresponding map of weight complexes splits componentwisely.

We apply Lemma 1.5.4 to the morphism $g_{X^{w \leq i}, X'^{w \leq i-1}}[i]$ and the weight decompositions $X^i \xrightarrow{d_i} X^{w \leq i} \to X^{w \leq i-1}[1]$ and $X'^{i-1}[1] \xrightarrow{d'_{i-1}[1]} X'^{w \leq i-1}[1] \to X'^{w \leq i-2}[2]$ of the corresponding objects. We obtain a diagram

$$\begin{CD}
X^i @>{d_i}>> X^{w\leq i} @>>> X^{w\leq i-1}[1]\\
@VV{0}V @VV{g_{X^{w\leq i},X'^{w\leq i-1}}[-i]}V @VVV\\
X'^{i-1}[1] @>{d'_{i-1}[1]}>> X'^{w\leq i-1}[1] @>>> X'^{w\leq i-2}[2]\\
@VVV @VVV @VVV\\
D_i[1] @>>> C_i[1] @>{t_i[2]}>> C_{i-1}[2]
\end{CD} \tag{13}$$

for some $D_i \in Obj\underline{C}$ and some $t_i$. The first column gives $D_i \cong X^i \oplus X'^{i-1}$. Hence $C_i[-1] \xrightarrow{t_i} C_{i-1} \to D_i$ is a weight decomposition of $C_i[-1]$. Applying the fact that the morphisms $C_i[-1] \to C_{i-1}$ that correspond to $id_{C_i[-1]}$ is unique, we obtain that $t_i$ equals the corresponding morphism coming from the infinite weight decomposition of $C$ (see Proposition 1.5.6). Hence we obtain our claim.

II $I$ is an ideal since $t$ is an additive functor.

Obviously, it suffices to check that for $X \in \underline{C}^{[0,n]}$ the ideal $J = \{g \in \underline{C}(X,X) : \ t(g) = 0\}$ of the ring $\underline{C}(X,X))$ satisfies $J^{n+1} = 0$. We will prove this fact by induction in $n$. In the case $n = 0$ we have $\underline{C}^{[0,n]} = Hw$, hence $J = \{0\}$.

To make the inductive step we consider $g_0 \circ g_1 \circ \dots g_n$, $g_i \in J$, let $r = (g_0 \circ g_1 \circ \dots g_{n-1})[n-1]$, $s = g_n[n-1] \circ r$. By Proposition 3.2.4, we can choose a representative $(h_i, l_i)$ of $WD(g_i[n-1])$ such that $t(h_i) = 0$. Then by the inductive assumption we have $WD(r) = (0, m)$ for some $m : X^n \to X^n$. Considering the morphism of triangles corresponding to $WD(r)$ we obtain that $r = b_{n-1}[-1] \circ q$ for some $q : X[n-1] \to X^n[-1]$. Next, since $t(g_n) = 0$, we can assume that $t(g_n[n-1]) = (u, 0)$ for some $u$ (by Proposition 3.2.4). Hence $g_n[n-1] = v \circ a_{n-1}$ for some $v \in \underline{C}(X^{w\le n-1}, X[n-1])$ and we obtain $s = v \circ (a_{n-1} \circ b_{n-1}[-1]) \circ q = 0$. The assertion is proved.

III Follows from assertion II by a standard reasoning, see Remark 3.1.5.

IV By part 2 of Proposition 2.2.2 if $X \in \underline{C}^{w\le i}$ (resp. $\underline{C}^{w\ge i}$) then choosing $X^{w\ge i+1} = 0$ (resp. $X^{w\le i-1} = 0$) we obtain that the corresponding choice of $t(X)$ is concentrated in degrees $\le i$ (resp. $\ge i$). Now note that all choices of $t(X)$ are homotopy equivalent by part 1 of Proposition 3.1.8.

Conversely, let $w$ be non-degenerate, let $t(X) \in K_{\mathfrak{w}}(Hw)^{w\le i}$. We can assume that $i = 0$; let $X \in \underline{C}^{w\le n}$. Then $t(id_X)$ is homotopy equivalent to a morphism concentrated in degrees $\le 0$. Hence Proposition 3.2.4 implies that for $WD(id_X) = (l, m)$ we can assume that $t(m) = 0$.

Then by assertion II we have $WD(id_X^n) = (l^n, 0)$. Considering the distinguished triangle corresponding to $WD(id_X^n)$ we obtain that $id_X = id_X^n$ could be factorized through $X^{w\le 0}$. Hence $X$ is a retract of $X^{w\le 0}$; since $\underline{C}^{w\le 0}$ is Karoubi-closed in $\underline{C}$ we obtain that $X \in \underline{C}^{w\le 0}$.

The case $t(X) \in K_{\mathfrak{w}}(Hw)^{w\ge i}$ is considered similarly.

V Since $t$ is weakly exact (see Definition 3.1.6), it suffices to check that $t(X) = 0$ implies $X = 0$. This is immediate from assertion IV.

VI Immediate from part I5 of Theorem 3.2.2.

VII We can assume that $a = 0$.

1. The proof is just a repetitive application of axioms (of weight structures).

Note first that $t_*$ is bijective for $X, X' \in \underline{C}^{w=0}$. Next, for $X \in \underline{C}^{w=0}$ and any $X'$ we consider the distinguished triangle $X'^{w\le -1} \to X'^0 \to X' \to X'^{w\le -1}[1]$. Then orthogonality yields that any $h : \underline{C}(X, X'^0)$ gives a morphism $X \to X'$; hence $t$ is surjective in this case. We also can apply this statement for $X'' = X'^{w\le -1}$ Hence considering the diagram

$$\begin{array}{ccccccc}
(\underline{C}(X, X'^{w\le -1}) & \longrightarrow & \underline{C}(X, X'^0) & \longrightarrow & \underline{C}(X, X') & \longrightarrow & 0\\
\downarrow & & \downarrow & & \downarrow & & \\
K_{\mathfrak{w}}(Hw)(t(X), t(X'^{w\le -1})) & \longrightarrow & K_{\mathfrak{w}}(Hw)(t(X), t(X'^0)) & \longrightarrow & K_{\mathfrak{w}}(Hw)(t(X), t(X')) & \longrightarrow & 0
\end{array}$$

induced by $t$ we obtain that $t_*$ is bijective in this case.

Now considering the distinguished triangle $X^{w\ge 1}[1] \to X \to X^0 \to X^{w\ge 1}$ and applying the dual argument one can easily obtain the claim.

2. Let $h \in K_{\mathfrak{w}}(Hw)(t(X), t(X'))$. By definition, we can "cut" $h$ to obtain a commutative diagram

$$\begin{array}{ccc}
t(X^0) & \xrightarrow{t(f_0)} & t(X^{w\ge 1})\\
\downarrow & & \downarrow\\
t(X')^{w\le 0} & \xrightarrow{t(f_0')} & t(X'^1)
\end{array}$$

By assertion VII1, this diagram corresponds to some homomorphism $WD(X) \to WD(X')$. It remains to apply part I3 of Theorem 3.2.2. □

*Remark* 3.3.2. 1. Parts IV and V imply that $t$ it is always conservative and respect the filtrations on $\underline{C}^b$; see part 3 of Proposition 1.3.4.

2. In fact, our notion of a distinguished triangle in $K_{\mathfrak{w}}(A)$ (see Definition 3.1.6) is rather weak. It is similar to the notion of an *exact triangle* in Definition 0.3 of [31]. Since exact triangles are not distinguished in general (see loc.cit.), part I of our theorem does not imply that for a distinguished triangle $C \to X \to X'$ the triangle $t(C) \to t(X) \to t(X')$ comes from some distinguished triangle in $K(Hw)$.

We will not prove the latter fact in detail, since we will only need it in Remark 3.3.4 below. Yet the proof is rather easy. In the proof of part I of our theorem it suffices to check that some choice of $t(f)$ in $K(Hw)$ yields the third side of the distinguished triangle in question. Using (obvious) functoriality properties of the construction in the proof, one can reduce the latter claim to the case $X, X' \in \underline{C}^{w=0}$. Certainly, the statement is obvious in this case.

It seems probable that $t$ could de lifted to a certain "strong weight complex" functor.

**Conjecture 3.3.3.** *$t$ could be lifted to an exact functor $t^{st} : \underline{C} \to K(Hw)$.*

*Remark* 3.3.4. 1. Let $Hw$ be (fully) embedded into the subcategory $B = Proj\ A$ of projective objects of an abelian category $A$ (probably, the most reasonable choice for $A$ is $Hw'_*$; cf. Lemma 5.4.3 below). Then we have a full embedding $K(Hw) \subset K(B) \subset D(A)$. Suppose now that $A$ is of projective dimension 1. Then any complex over $A$ is quasi-isomorphic to a complex with zero differentials; hence it could be presented in $D(A)$ as a direct sum of some monomorphisms in $B$ (i.e. of complexes of the form $\dots 0 \to X \xrightarrow{i} Y \to 0 \to \dots$ placed in pairwise distinct dimensions). We check that $K_{\mathfrak{w}}(B) = K(B)$. Note that it suffices to prove the corresponding fact for $K^b$. Therefore it suffices to check that $K_{\mathfrak{w}}(B)(X,Y) = K(B)(X,Y)$ for $X, Y$ being monomorphisms (as two-term complexes); let $X = X^{-1} \to X^0$. If $Y \in C^{[-1,0]}(B)$ then $K(B)(X,Y) = A(H^0(X), H^0(Y)) = K_{\mathfrak{w}}(B)(X,Y)$ (see part 2 of Remark 3.1.7). If $Y \in C^{[-2,-1]}(B)$ then the equality $K(B)(X,Y) = K_{\mathfrak{w}}(B)(X,Y)$ is obvious (cf. part VII of Theorem 3.3.1). For $Y$ placed in all other positions we have $K(B)(X,Y) = \{0\} = K_{\mathfrak{w}}(B)(X,Y)$.

We conclude that $K_{\mathfrak{w}}(Hw) = K(Hw)$. Therefore part 2 of Remark 3.3.2 implies that $t$ is exact (as a functor of triangulated categories).

In particular, this reasoning can be applied if $Hw = Ab_{fin.fr}$ or $Hw = Ab_{fr}$. Hence this is the case for all categories of spectra considered in §4.6 below.

2. In §6.3 below we will also verify the conjecture in the case when $\underline{C}$ has a *differential graded enhancement*.

3. Prof. A. Beilinson has kindly communicated to the author a proof of the conjecture in the case when $\underline{C}$ has a *filtered triangulated enhancement*; see §8.3 below. Probably, a filtered triangulated enhancement exists for any "reasonable" triangulated category.

# 4 Connection between weight structures and $t$-structures; duality of hearts

In this section we prove that weight structures are closely related to $t$-structures.

In §4.1 we recall the definition of a $t$-structure in a triangulated $\underline{C}$.

In §4.2 we recall the (standard) construction of countable homotopy colimits in triangulated categories and study its properties.

In §4.3 we show that in many cases a weight structure could be described by specifying a *negative* $H \subset \underline{C}$. In particular, this is the case for the category of finite spectra ($\subset SH$).

In §4.4 we define the notion of *adjacent* weight and $t$-structure for $\underline{C}$; their hearts are "dual" in a very interesting sense (see Theorem 4.4.2). We also compare spectral sequences arising from adjacent weight and $t$-structures.

In §4.5 we study the conditions for adjacent weight and $t$-structures to exist. We only consider in detail the cases which are relevant for our main examples (motives and $SH$); other possibilities are described in Remark 4.5.3. Note also that the *weight resolution* construction used in the proof of Theorem 4.5.2 allows to construct Eilenberg-Maclane spectra in $SH$.

In 4.6 we apply the results of this section to the study of $SH$. In particular, we obtain a certain "weight filtration" on homotopy groups of spectra. In §7.1 below we will apply our results to $DM_-^{eff}$ (the category of motivic complexes of Voevodsky).

## 4.1 $t$-structures: reminder

To fix the notation we recall the definition of a $t$-structure.

**Definition 4.1.1.** A pair of subclasses $\underline{C}^{t\ge 0}, \underline{C}^{t\le 0} \subset Obj\underline{C}$ for a triangulated category $\underline{C}$ will be said to define a $t$-structure $t$ if $\underline{C}^{t\ge 0}, \underline{C}^{t\le 0}$ satisfy the following conditions:

(i) $\underline{C}^{t\ge 0}, \underline{C}^{t\le 0}$ are strict i.e. contain all objects of $\underline{C}$ isomorphic to their elements.

(ii) $\underline{C}^{t\ge 0} \subset \underline{C}^{t\ge 0}[1]$, $\underline{C}^{t\le 0}[1] \subset \underline{C}^{t\le 0}$.

(iii) **Orthogonality**. For any $X \in \underline{C}^{t\le 0}[1]$, $Y \in \underline{C}^{t\ge 0}$, we have $\underline{C}(X, Y) = \{0\}$.

(iv) $t$**-decomposition**. For any $X \in Obj\underline{C}$ there exists a distinguished triangle

$$A \to X \to B \to A[1] \tag{14}$$

such that $A \in \underline{C}^{t\le 0}, B \in \underline{C}^{t\ge 0}[-1]$.

Non-degenerate and bounded (above, below, or both) $t$-structures could be defined similarly to Definition 1.2.1.

We will need some more notation for $t$-structures.

**Definition 4.1.2.** 1. A category $Ht$ whose objects are $\underline{C}^{t=0} = \underline{C}^{t\ge 0} \cap \underline{C}^{t\le 0}$, $Ht(X, Y) = \underline{C}(X, Y)$ for $X, Y \in \underline{C}^{t=0}$, will be called the *heart* of $t$. Recall (cf. Theorem 1.3.6 of [5]) that $Ht$ is abelian (short exact sequences come from distinguished triangles in $\underline{C}$).

2. $\underline{C}^{t\ge l}$ (resp. $\underline{C}^{t\le l}$) will denote $\underline{C}^{t\ge 0}[-l]$ (resp. $\underline{C}^{t\le 0}[-l]$).

*Remark* 4.1.3. Recall (cf. Lemma IV.4.5 in [15]) that (14) defines additive functors $\underline{C} \to \underline{C}^{t\le 0} : X \to A$ and $C \to \underline{C}^{t\ge 1} : X \to B$. We will denote $A, B$ by $X^{t\le 0}$ and $X^{t\ge 1}[-1]$, respectively.

Moreover, the functor $X \to X^{t\le 0}$ is right adjoint to the inclusion $\underline{C}^{t\le 0} \to \underline{C}$. It follows that this functor commutes with all those direct sums that exist in $\underline{C}$. Besides, if $\oplus X_i$, $\oplus X_i^{t\le 0}$, and $\oplus X_i^{t\ge 1}$ exist in $\underline{C}$ then the distinguished triangle $\oplus X_i^{t\le 0} \to \oplus X_i \to \oplus X_i^{t\ge 1}[-1]$ yields that $(\oplus X_i)^{t\ge 1} = \oplus(X_i^{t\ge 1})$.

The $t$-components of $X[i]$ will be denoted by $X^{t\le i} \in \underline{C}^{t\le 0}$ and $X^{t\ge i+1} \in \underline{C}^{t\ge 0}$, respectively. (14) will be called the *t-decomposition* of $X$.

We denote by $H^{0t}$ the zeroth cohomology functor corresponding to $t$ (cf. part 10 of §IV.4 of [15]); i.e. $H^{0t}(X)$ is defined similarly to $X^{[0,1]}$ in part 1 of Proposition 1.5.6. Shifting the $t$-decomposition of $X^{t\le 0}[-1]$ by [1] we obtain a canonical and functorial (with respect to $X$) distinguished triangle $X^{t\le -1}[1] \to X^{t\le 0} \to H^{0t}(X)$ with $X^{t\le -1} \in \underline{C}^{t\le 0}$.

Lastly, $\tau_{\le i}X$ will denote $X^{t\le i}[-i]$; $\tau_{\ge i}X = X^{t\ge i}[-i]$.

## 4.2 Countable homotopy colimits in triangulated categories: the construction and properties

The triangulated construction of countable (filtered) homotopy colimits is fairly standard, cf. Definition 1.6.4 of [23].

**Definition 4.2.1.** Suppose that we have a sequence of objects $Y_i$ (starting from some $j \in \mathbb{Z}$) and maps $\phi_i : Y_i \to Y_{i+1}$. Let there exist $D = \oplus Y_i$ in $\underline{C}$. We consider the map $d : \oplus id_{Y_i} \bigoplus \oplus(-\phi_i) : D \to D$ (we can define it since its $i$-th component is could be easily factorized as a composition $Y_i \to Y_i \oplus Y_{i+1} \to D$). Denote the cone of $a$ as $Y$. We will write $Y = \varinjlim Y_i$ and call $Y$ the *homotopy colimit* of $Y_i$.

We will say that the colimit exists (in $\underline{C}$) if the direct sum $D$ exists.

*Remark* 4.2.2. 1. By Lemma 1.7.1 of [23] the homotopy colimit of $Y_{i_j}$ is the same for any subsequence of $Y_i$. In particular, we can discard any (finite) number of first terms in $Y_i$.

2. By Lemma 1.6.6 of [23] the homotopy colimit of $X \overset{id_X}{\to} X \overset{id_X}{\to} X \overset{id_X}{\to} X \overset{id_X}{\to} \dots$ is $X$. Hence we obtain that $\varinjlim X_i \cong X$ if for $i \gg 0$ all $\phi_i$ are isomorphisms and $X_i \cong X$.

3. The construction of $\varinjlim Y_i$ easily yields: if countable direct sums exist in $\underline{C}^{w\le 0}$ then $\underline{C}^{w\le 0}$ is closed (in $\underline{C}$) with respect to homotopy colimits. Indeed, we have $D \in \underline{C}^{w\le 0}$; hence it suffices to apply part 3 of Proposition 1.3.1. On the other hand, it is easy to construct a counterexample to the similar statement on $\underline{C}^{w\ge 0}$ (though colimits of object of $\underline{C}^{w\ge 0}$ always belong to $\underline{C}^{w\ge -1}$). To settle this problem we will describe a "clever" method for passing to the colimit in $\underline{C}^{w\ge 0}$ below.

We study the behaviour of colimits under (co)representable functors.

**Lemma 4.2.3.** *1. For any $C \in Obj\underline{C}$ we have a natural epimorphism $\underline{C}(Y,C) \to \varprojlim \underline{C}(Y_i,C)$.*

*2. This epimorphism is bijective if all $\phi_i[1]^* : \underline{C}(Y_{i+1}[1],C) \to \underline{C}(Y_i[1],C)$ are surjective for all $i \gg 0$.*

*3. If $C$ is compact then $\underline{C}(C,Y) = \varinjlim \underline{C}(C,Y_i)$.*

*Proof.* 1. For any $C$ we have $\underline{C}(D,C) = \prod \underline{C}(Y_i,C)$.

This yields a long exact sequence

$$\dots \to \underline{C}(D[1],C) \stackrel{a[1]^*}{\to} \underline{C}(D[1],C) \to \underline{C}(Y,C) \to \underline{C}(D,C) \stackrel{a^*}{\to} \underline{C}(D,C) \to \dots.$$

It is easily seen that the kernel of $a^*$ equals

$$\{(s_i) : s_i \in \underline{C}(Y_i,C),\ s_{i+1} = s_i \circ \phi_i\} = \varprojlim \underline{C}(Y_i,C);$$

this yields the result.

2. By part 1 of Remark 4.2.2, we can assume that the homomorphisms $\phi[1]^*$ are surjective for all $i$. In this case $a[1]^*$ is easily seen to be surjective; this yields the result.

3. Similarly to the proof of part 1, we consider the long exact sequence

$$\dots \to \underline{C}(C,D) \stackrel{a_*}{\to} \underline{C}(C,D) \to \underline{C}(C,Y) \to \underline{C}(C,D[1]) \stackrel{a[1]_*}{\to} \underline{C}(C,D[1]) \to \dots.$$

Since $C$ is compact, we have $\underline{C}(C,D) = \oplus \underline{C}(C,Y_i)$. Then it is easily seen that map $a[1]_*$ is surjective, whereas the cokernel of $a_*$ is $\varinjlim \underline{C}(C,Y_i)$. See also Lemma 2.8 of [24].

$\square$

Now we describe a "clever" method for passing to the colimit in $\underline{C}^{w\geq 0}$. Since we will use it to prove that a certain "candidate" for being a weight structure is a weight structure indeed, we will describe it in a (somewhat) more general setting than those of weight structures.

Suppose that we have a full subcategory $D \subset \underline{C}$ that is "closed with respect to taking middle terms of distinguished triangles". Define a full subcategory $E \subset \underline{C}$ by

$$X \in ObjE \iff \underline{C}(X,Y) = \{0\}\ \forall Y \in ObjD[1].$$

Note that $E$ is also "closed with respect to taking middle terms of distinguished triangles", wheareas the pair $(D,E)$ satisfies the conditions of Remark 1.5.5.

**Lemma 4.2.4.** *Let $\phi_i : Y_i \to Y_{i+1}$ be a sequence of $\underline{C}$-morphisms; denote* $\operatorname{Cone}\phi_i$ *by $C_i$; let the first of $Y_i$ be $Y_l$. Suppose that $Y_l$ and all $C_i$ have "weight decompositions with respect to $D, E$" i.e. that there exist distinguished triangles $Y_l \to D_l \to E_l$ and $C_i \to F_i \to G_i$ with $D_l, F_i \in ObjD$ and $E_l, G_i \in ObjE$. Suppose also that for any possible choice of weight decompositions of $Y_i$ ($Y_i \to D_i \to E_i$ with $D_i \in ObjD$ and $E_i \in ObjE$) the sum $\oplus E_i$ exists. Then there exists a choice of $E_i$ and of the morphisms $\phi'_i : E_i \to E_{i+1}$ coming from $\phi_i$ (defined similarly to part 1 of Lemma 1.5.1) such that $\varinjlim E_i \in ObjE$ (note that the colimit exists!).*

*Proof.* We fix weight decompositions for all $C_i$ and for $Y_l$.

Now we fix $\phi'_i$ and the weight decompositions of $Y_{i+1}$ starting from $i = l$ inductively.

Suppose that we have fixed some weight decomposition of $Y_i$. By Remark 1.5.5 we can construct $E_{i+1}$ and $\phi'_i$ that fit into a distinguished triangle $E_i \xrightarrow{\phi'_i} E_{i+1} \to G_i$, whereas $E_{i+1}[-1] \to Y_{i+1}$ yields a weight decomposition of $Y_i$.

Now we check that passing to the limit of $E_i$ this way we obtain an object of $E$. Let $Z$ be the limit of $E_i$. We should check that $\underline{C}(Z, C) = \{0\}$ for any $C \in ObjD[1]$. By part 2 of Lemma 4.2.3 to this end it suffices to check that all $\phi'_i[1]^* : \underline{C}(E_{i+1}, C) \to \underline{C}(E_i^{w\geq 1}[1], C)$ are surjective. Indeed, then we will have $\underline{C}(Z, C) = \varprojlim \underline{C}(E_i, C) = \{0\}$. Lastly, the surjectivity is immediate from the long exact sequences (for all $i$)

$$\dots \to \underline{C}(E_{i+1}[1], C) \to \underline{C}(E_i[1], C) \to \underline{C}(G_i, C)(= \{0\}) \to \dots.$$

□

Lastly we prove that $t$-truncations "approximate" objects. We will prove this statement in the form that is relevant for §7.1; certainly, some other versions of it are valid for similar reasons.

**Lemma 4.2.5.** *Let $t$ be a non-degenerate $t$-structure; suppose that all countable direct sums exist for objects of $\underline{C}^{t\leq 0}$. Let $Y_i \in \underline{C}^{t\leq l}$ for some $l \in \mathbb{Z}$, let $\phi_i : Y_i \to Y_{i+1}$ be a sequence of $\underline{C}$-morphisms.*

*Suppose that there exists such an $Y$ that for any $i$ and all $j \geq i$ we have $Y_j^{t\geq i} \cong Y_j^{t\geq i}$ and these isomorphisms commute with $\phi_{j*} :\ Y_j^{t\geq i} \to Y_j^{t\geq i+1}$. Then $\varinjlim Y_i$ exists and $\cong Y$.*

*Proof.* Since countable direct sums exist in $\underline{C}^{t\leq 0}$, they also exist in $\underline{C}^{t\leq 0}$. This implies the existence of $\varinjlim Y_i$. We denote $\varinjlim Y_i$ by $Z$.

We obviously have $Y \in \underline{C}^{t\le l}$. Then the definition of $\varinjlim$ easily yields (at least, one) morphism $\varinjlim(Y \stackrel{id_Y}{\to} Y \stackrel{id_Y}{\to} Y \stackrel{id_Y}{\to} \dots) = Y \to \varinjlim Y_i = Z$; to this end one should apply Proposition 1.4.2.

Since $t$ is non-degenerate, it suffices to prove that $Y^{t\le j} \cong Z^{t\le j}$ for any $j$. We fix $j$.

By part 1 of Remark 4.2.2 we can assume that $\phi_{i*}$ are isomorphisms for all $i$. We apply the functor $X \to X^{t\le j}$ to the definition of $\varinjlim Y_i$. Since countable direct sums exist in $\underline{C}^{t\le l}$ and (by Remark 4.1.3) the functor $X \to X^{t\ge i}$ commutes with them we obtain: $Z^{t\ge j} \cong \varinjlim(Y^{t\ge j} \stackrel{id_{Y^{t\ge j}}}{\to} Y^{t\ge j} \stackrel{id_{Y^{t\ge j}}}{\to} Y^{t\ge j} \to \dots)$. By part 2 of Remark 4.2.2 we conclude that $Z^{t\ge i} \cong Y^{t\ge i}$.

$\square$

## 4.3 Recovering $w$ from $Hw$

In many cases instead of describing $\underline{C}^{w\le 0}$ and $\underline{C}^{w\ge 0}$ it is easier to specify only $\underline{C}^{w=0}$. We describe some conditions that ensure that $w$ could be recovered from $Hw$.

**Definition 4.3.1.** Let $H$ be a strict full additive subcategory of $\underline{C}$.

1. We will say that $H$ is *negative* if for any $X, Y \in ObjH$ and $i > 0$ we have $\underline{C}(X, Y[i]) = \{0\}$.

2. We will say that $H' \subset \underline{C}$ is the *Karoubi-closure* of $H$ if the objects of $H'$ are exactly all retracts of objects of $H$ (in $\underline{C}$).

3. We define the *small envelope* of an additive category $A$ as a category $A'$ whose objects are $(X, p)$ for $X \in ObjA$ and $p \in A(X, X):\ p^2 = p$ such that there exist $Y \in ObjA$ and $q \in A(X, Y)$, $s \in A(Y, X)$ satisfying $sq = 1 - p$, $qs = id_Y$. We define

$$A'((X,p),(X',p')) = \{f \in A(X,X'):\ p'f = fp = f\}. \qquad (15)$$

The small envelope of $A$ is (naturally) a full subcategory of the idempotent completion of $A$ (cf. subsection 5.1 below). One should think of $A'$ as of the category of $X \ominus Y$ for $X, Y \in ObjA$, $Y$ is a retract $X$. Here $X \ominus Y$ is a certain "complement" of $Y$ to $X$.

It can be easily checked that the small envelope of an additive category is additive; $X \to (X, id_X)$ gives and full embedding $A \to A'$.

**Theorem 4.3.2.** *I Let $A$ be a full additive subcategory of some triangulated $\underline{C}$. Then the embedding $A \to \underline{C}$ could be extended to a full embedding of the small envelope of $A$ into $\underline{C}$.*

*II Let $H$ be negative and generate $\underline{C}$.*

*1. There exists a unique weight structure $w$ for $\underline{C}$ such that $H \subset Hw$. Moreover, it is bounded.*

*2. $Hw$ equals the small envelope of $H$.*

*III Let $H$ be negative and weakly generate $\underline{C}$, suppose that for any $X \in Obj\underline{C}$ there exists a $j \in \mathbb{Z}$ such that*

$$\forall\ Y \in ObjH \text{ we have } \underline{C}(Y, X[i]) = \{0\}\ \forall i > j. \tag{16}$$

*Let $H' \subset H$ be additive. Suppose that either*

*(i) There exists a cardinality $c$ such that for any direct of sum of $< c$ objects of $H$ exists and belongs to $H$, whereas $Card\ H' < c$. For any $X \in Obj\underline{C}$ and any $Y \in ObjH'$ the group $\underline{C}(Y, X)$ considered as a $\underline{C}(Y, Y)$-module can be generated by $< c$ elements. Any object of $H$ can be presented as $\oplus_{i\in I} C_i$ for $C_i \in ObjH'$, $Card\ I < c$. For any $I:\ Card(I) < c$, $Y \in ObjH'$, $j \in \mathbb{Z}$, and $X_i \in ObjH,\ i \in I$, we have*

$$\underline{C}(Y, \oplus_{j\in I} X_j) = \oplus \underline{C}(Y, X_j) \tag{17}$$

*or*

*(ii) Arbitrary direct sums exist in $H$; all objects of $H'$ are compact; $ObjH'$ is a set; any object of $H$ can be presented as $\oplus_{i\in I} C_i$ for $C_i \in ObjH'$ and some set $I$.*

*Then there exists a weight structure $w$ for $\underline{C}$ such that $H \subset Hw$. Moreover, it is non-degenerate and bounded above. In case (ii) it admits negative direct sums, in case (i) it admits negative direct sums of $< c$ objects.*

*IV Suppose that all conditions of part III ((i) or (ii)) except (16) are fulfilled. Denote the set of objects of $\underline{C}$ satisfying (16) for some $j \in \mathbb{Z}$ by $\underline{C}^-$; denote the class of objects of $\underline{C}$ satisfying (16) for a fixed $j \in \mathbb{Z}$ by $\underline{C}^{w\le j}$. Then the category $\underline{C}^-$ is triangulated and satisfies all conditions of part III (we will identify the class $\underline{C}^-$ with the corresponding full subcategory of $\underline{C}$).*

*Proof.* I We map $(X, p)$ to (any choice of) $\mathrm{Cone}(q)$; we denote this object by $Z$.

Now we define the embedding on morphisms. We note that in $A$ the map $q$ is a projection of $X$ onto $Y$. Hence $A'$ we have $X \cong (X, p) \oplus Y$, the isomorphism is given by $(p, q)$. Since $q$ has a section in $\underline{C}$, we have a distinguished triangle $Z \to X \xrightarrow{q} Y \xrightarrow{0} Z[1]$ i.e. we also have a similar decomposition of $X$ in $\underline{C}$. It is easily seen that $\underline{C}(Z, Z')$ is given exactly by the formula (15) if we assume that $Z$ is a subobject of $X$ i.e. if we fix the splitting of the projection $X \to Z$. Hence if we fix the embedding $Z \to X$ for each $(X, p)$ then (al possible choices) of objects $\mathrm{Cone}(q)$ would give a subcategory that is equivalent to the small envelope of $A$; it is obviously additive.

II 1. We define $\underline{C}'^{w\geq 0}$ as the smallest subset of $Obj\underline{C}$ that contains $ObjH[i]$ for $i \leq 0$ and satisfies the property 3 of Proposition 1.3.1; for $\underline{C}'^{w\leq 0}$ we take a similar 'closure' of the set $\cup ObjH[i]$ for $i \geq 0$.

Obviously, $\underline{C}'^{w\geq 0}$ and $\underline{C}'^{w\leq 0}$ satisfy property (ii) of Definition 1.1.1; we define $\underline{C}'^{w\geq i}$ and $\underline{C}'^{w\leq i}$ for $i \in \mathbb{Z}$ in the usual way.

If we have a distinguished triangle $X \to Y \to Z \to X[1]$ with $\underline{C}(X, A) = \underline{C}(Z, A) = \{0\}$ for some $X, Y, Z, A \in Obj\underline{C}$ then $\underline{C}(Y, A) = \{0\}$; the same statement is valid for a functor of the type $\underline{C}(B, -)$. Hence the equality $\underline{C}(X[i], Y[j]) = \{0\}$ for $i < 0 \leq j$, $X, Y \in H$ easily implies (by induction) that $\underline{C}(Z, T) = \{0\}$ for all $Z \in C^{w\geq 1}$, $T \in C^{w\leq 0}$.

Now we verify that any $X \in Obj\underline{C}$ has a "weight decomposition" (with respect to $\underline{C}'^{w\geq 0}$ and $\underline{C}'^{w\leq 0}$). We prove this by induction on the "complexity" of $X$ i.e. on the number of distinguished triangles that we have to consider to "generate" $X$ from objects of $H[i]$, $i \in \mathbb{Z}$.

For $X$ of "complexity zero" (i.e. for $X \in ObjH[i]$) we can take a "trivial" weight decomposition i.e. define $X^{w\leq 0}$ as $X$ for $i \geq 0$ and 0 otherwise; $X^{w\geq 0}$ will be 0 and $X$, respectively.

Suppose now that $X \cong \operatorname{Cone}(Y \xrightarrow{d} Z)$ for $Y, Z$ of "complexity less than that of $X$". By the inductive assumption there exist "weight decompositions" of $Y$ and $Z[-1]$ i.e. distinguished triangles $Y \xrightarrow{a} A \to B$ and $Z[-1] \xrightarrow{a'[-1]} A'[-1] \to B'[-1]$ for $A \in \underline{C}'^{w\leq 0}$, $A' \in \underline{C}'^{w\leq -1}$, $B \in \underline{C}'^{w\geq 1}$, $B' \in \underline{C}'^{w\geq 0}$. We apply Remark 1.5.5 for $D = \underline{C}'^{w\leq 0}$ and $E = \underline{C}'^{w\leq 0}$. It yields a "weight decomposition" of $X$.

Now we take for $\underline{C}^{w\geq 0}$ and $\underline{C}^{w\leq 0}$ the Karoubi-closures of $\underline{C}'^{w\geq 0}$ and $\underline{C}'^{w\leq 0}$, respectively. By part 3 of Lemma 1.3.5, they satisfy the orthogonality axiom of weight structures. Hence they define a weight structure $w$ for $\underline{C}$.

Now, since any object of $\underline{C}$ could be obtained by a finite sequence of considerations of cones of morphisms of objects of $Hw$, we obtain that $w$ is bounded.

It remains to check that $w$ is the only weight structure such that $H \subset Hw$. By part 3 of Proposition 1.3.1 for any weight structure $u$ satisfying $H \subset Hu$ we have $\underline{C}'^{w\geq 0} \subset \underline{C}^{u\geq 0}$ and $\underline{C}'^{w\leq 0} \subset \underline{C}^{u\leq 0}$. Since $\underline{C}^{u\geq 0}$ and $\underline{C}^{u\leq 0}$ are Karoubi-closed, we also have $\underline{C}^{w\geq 0} \subset \underline{C}^{u\geq 0}$ and $\underline{C}^{w\leq 0} \subset \underline{C}^{u\leq 0}$. Now Lemma 1.3.6 implies our claim immediately.

2. By assertion I, $\underline{C}$ contains the small envelope of $H$. To check that this envelope is actually contained in $Hw$ it suffices to note that the object $X \ominus Y$ could be presented both as a cone of the "embedding" $Y \to X$ and of the "projection" $X \to Y$.

To check the inverse inclusion we can assume that $H$ equals its small envelope. Let $X \in \underline{C}^{w=0}$.

We apply the weight complex functor $t$. We obtain that $t(X) = X$ is a retract of two objects $A, B \in ObjK^b_{\mathfrak{w}}(Hw)$; $A \in ObjK^{b,\geq 0}_{\mathfrak{w}}(Hw)$ and $B \in ObjK^{b,\leq 0}_{\mathfrak{w}}(Hw)$. Next, applying Lemma 3.1.4 we obtain that the same is true in $K^b(Hw)$. This easily implies that $t(X) = t(Z)$ for some $Z$ that could be presented as an object of a small envelope of $H$ in $\underline{C}$. Hence the assertion follows from part VI of Theorem 3.3.1.

III Again, for $\underline{C}^{w\geq 0}$ we take the smallest Karoubi-closed subset of $Obj\underline{C}$ that contains $H[i]$ for $i \leq 0$ and satisfies the property 3 of Proposition 1.3.1.

For $\underline{C}^{w\leq 0}$ we take

$$\{X \in Obj\underline{C} : \ \underline{C}(X, Y) = \{0\} \ \forall Y \in ObjH[i], \ i < 0\}.$$

The proof of orthogonality is by induction on "complexity" of $Y \in \underline{C}^{w\geq 0}$ as in the proof of part II1. We have $\underline{C}(X, Y) = \{0\}$ for any $X \in \underline{C}^{w\leq 0}$ and any $Y \in \underline{C}^{w\geq 0}$ of "complexity zero". Now using the fact that all (co)representable functors are homological on $\underline{C}$, we obtain that the same is true for objects of $\underline{C}^{w\geq 0}$ of arbitrary complexity. Obviously, the same is true for their direct sums i.e. for the whole $\underline{C}^{w\geq 0}$.

$\underline{C}^{w\leq 0}$ is Karoubi-closed, additive, and strict by part 1 of Lemma 1.3.5. Besides, for any distinguished triangle $X \to Y \to Z$ we have $X, Z \in \underline{C}^{w\leq 0} \implies Y \in \underline{C}^{w\leq 0}$. $\underline{C}^{w\geq 0}$ has these properties also. Moreover, part 3 of Proposition 1.3.1 shows that for any $w'$ satisfying the conditions we have $\underline{C}^{w\geq 0} \subset \underline{C}^{w'\geq 0}$, hence $\underline{C}^{w\leq 0} \supset \underline{C}^{w'\leq 0}$.

To prove that $w$ is a weight structure, it remains to prove the existence of weight decompositions (see part 2 of Lemma 1.3.5). We will construct $X^{w\leq 0}$ and $X^{w\geq 1}$ for a fixed $X \in Obj\underline{C}$ explicitly. The construction could be called the *weight resolution*, cf. the proof of Proposition 4.5.2 below and Proposition 7.1.2 of [22].

First we treat case (i). For each object $Y$ of $H'$ any $Z \in Obj\underline{C}$ we choose some set of $f_i(Y, Z) \in \underline{C}(Y, Z)$ of cardinality $< c$ that $f_i(Y, Z)$ are $\underline{C}(Y, Y)$-generators of $\underline{C}(Y, Z)$. Let $j \in \mathbb{Z}$ satisfy (16).

Now we construct a certain sequence of $X_k$ for $k \leq j$ starting from $X_j = j$. For $k = j$ we take $P_j = \bigoplus_{Y \in ObjH', f_i(Y, X_j[j])} Y$. Note that the number of summands is $< c$, hence the sum exists and belongs to $ObjH$. Then we have a morphism $f_j : P_j \to X_j[j]$ given by $\prod f_i(Y, X[j])$. Let $X_{j-1}[j]$ denote a cone of $f_j$. Repeating the construction for $X_{j-1}$ instead of $X_j$ and with $k = j - 1$ we get an object $P_{j-1} \in ObjH'$, $f_{j-1} : P_{j-1} \to X_{j-1}[j-1]$; we denote a cone of $f_1$ by $X_{j-2}[j-1]$. Proceeding, we get an infinite sequence of $(P_i, f_i, X_i)$. Note that we have $P_i \in \underline{C}^{w\geq 0}$.

We denote the maps $X_i \to X_{i-1}$ given by the construction by $g_i$, $h_i = g_j \circ \cdots \circ g_{i+2} \circ g_{i+1} : X \to X_i$. We denote a cone of $h_i$ by $Y_i[-1]$; the map $Y \to X_i$ given by the corresponding distinguished triangle by $r_i$.

The octahedron axiom implies that the commutative triangle $X \xrightarrow{h_i} X_i \xrightarrow{g_i} X_{i-1}$ can be completed to an octahedron diagram (cf. §IV.1 of [15], or the last paragraph of 1.4). Hence we obtain a distinguished triangle $P_i[-i] \to Y_i \to Y_{i-1} \to P_i[1-i]$. By induction on $i$ we obtain that $Y_i[i] \in \underline{C}^{w \geq 0}$ for all $i \leq j$ (using the definition of $\underline{C}^{w \geq 0}$).

Now we denote $Y_0$ by $Y$ and $X_0$ by $Z$. $Y, Z$ will be our candidates for $X^{w \geq 0}$ and $X^{w \leq 0}$.

It remains to prove that $Z \in \underline{C}^{w \leq 0}$. We should check that $\underline{C}(C, Z[k]) = \{0\}$ for all $k > 0$, $C \in Obj H$. Since $\underline{C}(-, Z)$ transforms arbitrary sums into products, it suffices to consider $C \in Obj H'$.

First we prove that $\underline{C}(C, X_{k-1}[k]) = \{0\}$ for all $k \leq j$.

We apply the distinguished triangle

$$V_k : P_k \to X_k[k] \to X_{k-1}[k] \to P_k[1]. \tag{18}$$

Using (17), we obtain $\underline{C}(C, P_k[k]) = \oplus_{Y \in Obj H', f_i(Y, X_k[k])} \underline{C}(C, Y)$. By the definition of $f_i(C, Y)$ we obtain that this group surjects onto $\underline{C}(C, X_k[k])$. Moreover, $\underline{C}(C, P_k[1] = \oplus_{Y \in Obj H', f_i(Y, X_k[k])} \underline{C}(C, Y[1]) = \{0\}$. We obtain $\underline{C}(C, X_{k-1}[k]) = \{0\}$.

Now we use distinguished triangles $V_l$ for all $l < k$. Again (17) yields $\underline{C}(C, P_l[1]) = \underline{C}(C, P_l[2]) = \{0\}$. Hence $\underline{C}(C, X_{l-1}[k]) = \underline{C}(C, X_l[k]) = \{0\}$ for all $l < k$.

Hence $\underline{C}(C, Z[k]) = \{0\}$ for all $j \leq k > 0$.

Lastly, the distinguished triangles $V_k$ easily yield by induction that $\underline{C}(C, X_l[k]) = \{0\}$ for all $l \leq j$ and $k > j$.

The proof in case (ii) is almost the same; one should only always replace some choice of generators $f_i(Y, Z) \in \underline{C}(Y, Z)$ by all elements of $\underline{C}(Y, Z)$.

$(\underline{C}, w)$ is obviously bounded above by (16).

Now we check that $(\underline{C}, w)$ is non-degenerate. The condition (16) implies that $\cap \underline{C}^{w \geq i} = \{0\}$. Next, for any $X \in Obj\underline{C} \setminus \{0\}$ there exists an $f \in \underline{C}(Y[i], X)$ for some $Y \in H$ and $i \in \mathbb{Z}$ such that $f \neq 0$. Hence such an $X$ does not belong to $\underline{C}^{w \leq -1-i}$ (see the definition of $\underline{C}^{w \leq i}$ in the proof of part III).

IV Everything is obvious except that a cone of a morphism of objects of $\underline{C}^-$ belongs to $Obj\underline{C}^-$. This fact is easy also since the functors $\underline{C}(Y, -)$ are homological.

$\square$

Note that in the proofs it was specified explicitly how to recover $w$ from $\underline{C}^{w=0}$.

**Corollary 4.3.3.** *It is well known that there are no morphisms of positive degrees between (copies) of the sphere spectrum $S^0$ in the stable homotopy category $SH$; cf. §4.6 below. Hence part II of Theorem 4.3.2 immediately implies that the category of finite spectra $SH_{fin}$ (i.e. the full subcategory of $SH$ generated by $S^0$) has a bounded weight structure $w$. Its heart can be described as a category $H$ of finite sums of (copies of) $S^0$ (since any retract of $S^0$ is trivial, no new objects appear in the small envelope of $H$). Since $SH(S^0, S^0) = \mathbb{Z}$, $Hw$ is equivalent to $Ab_{fin.fr}$ (the category of finitely generated free abelian groups).*

*This weight structure obtained is a certain "dual" of the Postnikov $t$-structure for $SH$; cf. Theorem 4.4.2 and §4.6 below.*

*We will prove that the whole $SH$ satisfies the conditions of part IV2, while a certain category of quasi-finite spectra satisfies the conditions of part III for $c = \omega$, in §4.6.*

*Remark* 4.3.4. 1. Recall that in the proof of Theorem 4.3.2 it was specified explicitly how to recover $w$ from $\underline{C}^{w=0}$.

2. The conditions of parts III and IV could seem to be rather exotic. Yet they can be easily checked for a natural subcategory of *quasi-finite* objects in $SH$, see §4.6.

3. Obviously, if for any $X \in Obj\underline{C}$ and any $Y \in Obj H'$ the group $\underline{C}(Y, X)$ is generated by $< c$ elements as a group, then it is also generated by $< c$ elements as a $\underline{C}(Y, Y)$-module. In particular, this is the condition which we will actually check for the category $SH_{fin}$.

4. $\underline{C}^{w\le 0}$ described in the proof of part III of Theorem 4.3.2 is often a $\underline{C}^{t\le 0}$-part of a certain $t$-structure; then this $t$-structure is *left adjacent* to $w$ (see Definition 4.4.1 below). Yet in order for the $t$-decompositions to exist when we take the only possible candidate for $\underline{C}^{t\ge 0}$ (cf. Proposition 4.4.4 below) the homotopy colimit of all $Y_i$ (defined as in the proof of part III) should exist for all $X \in Obj\underline{C}$ (cf. the proof of Proposition 4.5.2). Note that this is not true for the category $SH_{fin}$ (see Corollary 4.3.3). For example, one can note that Eilenberg-MacLane spectra do not belong to $SH_{fin}$. Another example: one could define the Chow weight structure on $DM_{gm}^{eff}$ whearas the corresponding $t$-structure is only defined on $DM_{-}^{eff}$; see §6.5 and §7.1.

This shows that weight structures "exist more often than $t$-structures", while Theorem 4.4.2 below shows that they "contain almost the same information" as the corresponding $t$-structures. This evidence (along with the corresponding results for Voevodsky's motives below, see Proposition 6.5.3) supports author's opinion that weight structures are more relevant for 'general' triangulated categories than $t$-structures.

Moreover, it could be easily seen that the natural "opposite" to the statement of part II (i.e. we take a positive generating subcategory $H$ and ask whether a $t$-structure with $H \subset Ht$ exists) is false.

## 4.4 Adjacent weight and $t$-structures

**Definition 4.4.1.** We say that a weight structure $w$ is left (resp. right) adjacent to a $t$-structure $t$ if $\underline{C}^{w\le 0} = \underline{C}^{t\le 0}$ (resp. $\underline{C}^{w\ge 0} = \underline{C}^{t\ge 0}$).

In this situation we will also say that $t$ is right (resp. left) adjacent to $w$.

A simple example is obtained if one takes the canonical $t$-structure of (some version of) $D(A)$ for an abelian $A$. Then we have adjacent weight structures given by projective and injective resolutions in degrees $\ge 0$ and $\le 0$ if such resolutions exist.

The following result shows that adjacent structures could be uniquely recovered from each other. It also shows that $Ht$ and $Hw$ are connected by a natural generalization of the relation between the categories $A$ and $ProjA$ for an abelian $A$. Still note that in the latter case we also have $Hw \subset Ht$; this is a rather non-typical situation.

**Theorem 4.4.2** (Duality theorem)**.** *Let $w$ be left adjacent to $t$. Then the following statements are fulfilled.*

*1. $\underline{C}^{w\ge 0} = \{X \in Obj\underline{C} : \underline{C}(X,Y) = \{0\}\ \forall Y \in \underline{C}^{w\le -1} = \underline{C}^{t\le -1}\}$.*

*2. $\underline{C}^{t\ge 0} = \{X \in Obj\underline{C} : \underline{C}(Y,X) = \{0\}\ \forall Y \in \underline{C}^{w\le -1} = \underline{C}^{t\le -1}\}$.*

*3. The functor $\underline{C}(-,Ht) : Hw \to Ht^*$ (see the Notation) that sends $X \in \underline{C}^{w=0}$ to $Y \to \underline{C}(X,Y)$, $Y \in \underline{C}^{t=0}$ is a full embedding of $Hw$ into the full subcategory $Ex(Ht,Ab) \subset Ht^*$ which consists of exact functors.*

*4. The functor $\underline{C}(-,Ht) : Ht \to Hw_*$ that sends $X \in \underline{C}^{t=0}$ to $Y \to \underline{C}(Y,X)$, $Y \in \underline{C}^{w=0}$ is a full exact embedding of $Ht$ into the abelian category $Hw_*$.*

*5. Let $t$ be non-degenerate. Then $\underline{C}^{t=0}$ equals the class*

$$S = \{X \in Obj\underline{C} :\ \underline{C}(Y,X[i]) = \{0\}\ \forall Y \in \underline{C}^{w=0},\ i \ne 0\}.$$

*6. Let $i \in \mathbb{Z}$, let $Y \in Obj\underline{C}$ be fixed. Then for the functor $F(X) = \underline{C}(X,Y)$ we have $W^i(F)(X) = \operatorname{Im}(\underline{C}(X,Y^{t\le -i}[i]) \to \underline{C}(X,Y))$ for any $X \in Obj\underline{C}$.*

*7. For any $i,j$ we have a functorial isomorphism*

$$\underline{C}(X,Y^{t\le i}[j]) \cong \operatorname{Im}(\underline{C}(X^{w\le -j},Y[i]) \to \underline{C}(X^{w\le 1-j},Y[i+1])).$$

*8. For any $i, j$ we have a functorial isomorphism*

$$\underline{C}(X, Y^{t\geq i}[j]) \cong \operatorname{Im}(\underline{C}(X^{w\geq -j}, Y[i]) \to \underline{C}(X^{w\geq -1-j}, Y[i-1])).$$

*9. For any $X, Y \in Obj\underline{C}$ let $\dots \to Y^{-1} \to Y^0 \to Y^1 \to \dots$ denote an arbitrary choice of the weight complex for $Y$. Then we have*

$$\underline{C}(Y, X^{t=0}) = (\operatorname{Ker}(\underline{C}(Y^0, X) \to \underline{C}(Y^{-1}, X)) / \operatorname{Im}(\underline{C}(Y^1, X) \to \underline{C}(Y^0, X)). \tag{19}$$

*10. For $X \in \underline{C}^b$ we have*

$$X \in \underline{C}^{b,w\geq 0} \iff \underline{C}(X, Y) = \{0\}\ \forall i > 0,\ Y \in \underline{C}^{t=0}$$

*and*

$$X \in \underline{C}^{b,w\leq 0} \iff \underline{C}(X, Y[i]) = \{0\}\ \forall Y \in \underline{C}^{t=0},\ i < 0.$$

*Proof.* 1. Immediate from part 2 of Proposition 1.3.1 applied to $w$.

2. A well-known property of $t$-structures (note that one doesn't have to consider $w$).

3. First we note that for any $X \in \underline{C}^{w=0}$ orthogonality for $w$ implies $\underline{C}(X, Y) = \{0\}$ for all $Y \in \underline{C}^{w\leq -1} = \underline{C}^{t\leq -1}$, while orthogonality for $t$ gives $\underline{C}(X, Y) = \{0\}$ for all $Y \in \underline{C}^{t\geq 1}$. In particular,

$$\underline{C}(X, Y[i]) = \{0\}\ \forall\ X \in \underline{C}^{w=0},\ Y \in \underline{C}^{t=0},\ i \neq 0. \tag{20}$$

Now, short exact sequences in $Ht$ give distinguished triangles in $\underline{C}$. Hence for any homological functor $F : \underline{C} \to Ab$ and for $0 \to A \to B \to C \to 0$ being a short exact in $Ht$ we have a long exact sequence $\dots \to F(C[-1]) \to F(A) \to F(B) \to F(C) \to F(A[1]) \to \dots$. If $F = \underline{C}(X, -)$ then $F(C[-1]) = F(A[1]) = \{0\}$ (as was just noted). Hence objects of $Hw$ induce exact functors on $Ht$.

To prove that the restriction $Hw \to Ht^*$ is a fully faithful functor it suffices to prove that the restriction of the functor $\underline{C}(X, -)$ to $Ht$ for $X \in Hw$ determines $X$ in a functorial way. Using Yoneda's lemma, we see that it suffices to recover $\underline{C}(-, X)$ from its restriction.

We prove that

$$\underline{C}(X, Y) = \underline{C}(X, H^{0t}(Y))\ \forall X \in \underline{C}^{w=0},\ Y \in Obj\underline{C}. \tag{21}$$

We apply the $t$-decomposition (i.e. (14)) twice.

We have a distinguished triangle $Y^{t\geq 1}[-1] \to Y^{t\leq 0} \to Y \to Y^{t\geq 1}$. Since $\underline{C}(X, Y^{t\geq 1}[-1]) = \underline{C}(X, Y^{t\geq 1})$, we obtain $\underline{C}(X, Y) = \underline{C}(X, Y^{t\leq 0})$.

Next, we have a distinguished triangle $Y^{t\leq -1} \to Y^{t\leq 0} \to H^{0t}(Y) \to Y^{t\leq -1}[1]$. Since $\underline{C}(X, Y^{t\leq -1}) = \underline{C}(X, Y^{t\leq -1}[1]) = \{0\}$, we obtain (21).

4. Again, it suffices to prove that the restriction of the functor $\underline{C}(-, X)$ to $Hw$ for $X \in \underline{C}^{t=0}$ determines $X$ functorially.

We note that for any $X \in \underline{C}^{t=0}$ the orthogonality axiom for $w$ implies $\underline{C}(Y, X) = \{0\}$ for all $Y \in \underline{C}^{t\le -1} = \underline{C}^{w\le -1}$, while orthogonality for $t$ gives $\underline{C}(Y, X) = \{0\}$ for all $Y \in \underline{C}^{w\ge 1}$.

Now we prove that

$$\underline{C}(Y, X) = (\operatorname{Ker}(\underline{C}(Y^0, X) \to \underline{C}(Y^{-1}, X)) / \operatorname{Im}(\underline{C}(Y^1, X) \to \underline{C}(Y^0, X)). \tag{22}$$

Indeed, consider the (infinite) weight decomposition of $Y$ that gives our choice of the weight complex and apply Theorem 2.4.1 to the functor $\underline{C}(-, X)$. The spectral sequence obtained converges since it satisfies condition II(ii) of Theorem 2.4.1 (it has only one non-zero column!). It remains to note that this only non-zero column of $(E_1^{pq}(T(\underline{C}(-, X), Y))) = (\underline{C}(Y^p, X[q]))$ is exactly $(E_1^{p0}(T)) = \dots \to \underline{C}(Y^1, X) \to \underline{C}(Y^0, X) \to \underline{C}(Y^{-1}, X) \to \dots$.

We obtain (22).

5. By assertion 4, an object of $Ht$ is non-zero iff it represents a non-zero functor on $Hw$. Hence applying (21) we obtain that $S$ is exactly the class of objects that satisfy $H^{it}(X) = 0$ for all $i \neq 0$. It remains to note that for a non-degenerate $t$ this set is exactly $\underline{C}^{t=0}$.

6. We can assume that $i = 0$. We should check that $g \in \underline{C}(X, Y)$ lifts to some $h \in \underline{C}(X^{w\le 0}, Y)$ iff it lifts to some $l \in \underline{C}(X, Y^{t\le 0})$. Now note that the equality

$$\underline{C}(w_{\ge 0}X, \tau_{\ge 1}Y) = \underline{C}(w_{\ge 1}X, \tau_{\ge 0}Y) = \{0\}$$

yields that any morphism of these two morphism groups could be lifted to some $m \in \underline{C}(w_{\ge 0}X, \tau_{\ge 0}Y)$. Hence if one of $(h, l)$ exists then the other one could be constructed from the corresponding $m$ in the obvious way.

7. Shifting $X, Y$ we can easily reduce the statement to the case $i = j = 0$.

The $t$-decomposition of $Y$ yields exact sequences $\{0\} = \underline{C}(X^{w\le 0}, Y^{t\ge 1}[-2]) \to \underline{C}(X^{w\le 0}, Y^{t\le 0}) \to \underline{C}(X^{w\le 0}, Y) \to \underline{C}(X^{w\le 0}, \tau_{\ge 1}Y) = \{0\}$ and

$$\{0\} = \underline{C}(X^{w\le 1}, \tau_{\ge 1}Y) \to \underline{C}(X^{w\le 1}, Y^{t\le 0}[1]) \to \underline{C}(X^{w\le 1}, Y[1]) \to \underline{C}(X^{w\le 1}, Y^{t\ge 1}) \to \dots$$

Next, weight decompositions of $X$ and $X[1]$ similarly yield that the obvious morphism $\underline{C}(w_{\le 0}X, Y^{t\le 0}) \to \underline{C}(w_{\le 1}X, Y^{t\le 0})$ is surjective whereas $\underline{C}(w_{\le 1}X, Y^{t\le 0}) \cong \underline{C}(X, Y^{t\le 0})$.

We obtain a commutative diagram

$$\begin{array}{ccc}
\underline{C}(w_{\le 0}X, Y^{t\le 0}) & \xrightarrow{f} & \underline{C}(w_{\le 0}X, Y) \\
\Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle h} \\
\underline{C}(w_{\le 1}X, Y^{t\le 0}) & \xrightarrow{p} & \underline{C}(w_{\le 1}X, Y)
\end{array}$$

with $f$ being bijective, $g$ being surjective, and $p$ being injective. Hence $\underline{C}(X, Y^{t\le 0}) \cong \underline{C}(w_{\le 1}X, Y^{t\le 0}) \cong \operatorname{Im} g \cong \operatorname{Im} h$.

Note that the isomorphism constructed is obviously natural in $Y$ whereas it is natural in $X$ by part 2 of Lemma 1.5.1.

8. This assertion is exactly the dual of the previous one (see Remark 1.1.2).

9. Immediate from (22) and (21).

10. For $X \in \underline{C}^{w\ge 0}$ (resp. $X \in \underline{C}^{w\ge 0}$) the orthogonality statements desired are valid by assertion 1.

Now we prove the converse implication. Let $\underline{C}(X, Y[i]) = \{0\}$ $\forall Y \in \underline{C}^{t=0}$, $i > 0$. We should check that $\underline{C}(X, Z) = \{0\}$ $\forall Z \in \underline{C}^{t\le -1}$. We have $\underline{C}(X, H^{it}(Z)[-i]) = \{0\}$ for all $i$. Besides, since $X$ is bounded, we have $\underline{C}(X, \tau_{\le j}Z) = \{0\}$ for some $j$ (that is small enough). Hence considering the $t$-decompositions of $Z^{t\le k}[-1]$ for all $k > j$ one can easily obtain the orthogonality statement required.

The case $X:\ \underline{C}(X, Y[i]) = \{0\}$ $\forall Y \in \underline{C}^{t=0},\ i < 0$ is considered similarly.

$\square$

*Remark* 4.4.3. 1. Usually one can describe the images of embeddings in parts 3 and 4 more explicitly. If $\underline{C}$ is 'large enough' then these embedding are equivalences; see Theorem 4.5.2 below.

2. Dually to assertion 6: for $w$ right adjacent to $t$ and $F(X) = \underline{C}(Y, X)$ we obtain $W_i(F)(X) = \operatorname{Im}(\underline{C}(\tau_{\le -i}Y, X) \to \underline{C}(Y, X))$.

3. Assertion 6 and its dual show that the weight truncations are "almost adjoint" to the corresponding $t$-truncations. These statements are counterparts to the fact that (for an arbitrary $t$-structure) any morphism $X \to Y$ for $Y \in \underline{C}^{t\ge 0}$ could be uniquely factorized through $X^{t\ge 0}$ (and to its dual).

4. Assertions 7 and 9 imply that the derived exact couple for the spectral sequence $\underline{C}(X^{-p}[q], Y) \implies \underline{C}(X[p+q], Y)$ (as in Theorem 2.4.1) could also be described in terms of $\underline{C}(X[p], Y^{t\le q})$; see the proof of part IV of Theorem 2.3.2. It follows that the spectral sequence $S$ converging to $\underline{C}(X, Y)$ corresponding to the $t$-truncations of $Y$ could be "embedded into" our $T$ (i.e. for all $i > 0$ any $E_i^{pq}(S) \cong E_{i+1}^{p'q'}(T)$ for $p' = q + 2p, q' = -p$; these isomorphisms respect the structure of spectral sequences).

In algebraic topology, this result corresponds to the fact (and implies it) that the Atiyah-Hirzebruch spectral sequence for the cohomology of a space $X$ with coefficient in a spectrum $S$ could be obtained either by considering the cellular filtration of $X$ or the Postnikov $t$-truncations of $S$.

Note that in our method we describe all terms of exact couples (in contrast to [27], for example). The advantage of this is that the $D$-terms could be very interesting; see Proposition 7.4.1 and Corollary 7.5.2 below.

5. In fact, one could extend the notion of an adjacent structures to the case when there are two distinct triangulated categories $\underline{C}$ and $\underline{D}$ equipped with a *duality* $\Phi : \underline{C}^{op} \times \underline{D} \to Ab$ (that generalizes $\underline{C}(-,-) : \underline{C}^{op} \times \underline{C} \to Ab$. $\underline{D}^{t\geq 1}$ should annihilate of $\underline{C}^{w\leq 0}$ with respect to $\Phi$, while $\underline{D}^{t\leq -1}$ should annihilate $\underline{C}^{w\geq 0}$. In this situation one can easily prove the natural analogue of parts 6, 7, 8, and 9 of theorem 4.4.2 (the proofs are the same as above). In particular, this would yield an alternative proof of the comparison of spectral sequence statement of 6.4 (in the general case; see Remark 6.4.1). If $\Phi$ also satisfies certain 'perfectness' conditions (similar to parts 1,2 of the theorem) then one would probably obtain a natural analogue of part 5 of the theorem.

See also §8.2 below for further ideas in this direction.

6. More generally, for parts 7,8 of the theorem it suffices for $X, Y$ (lying either in the same category or in different ones) to have Postnikov towers whose terms satisfy the orthogonality conditions the same as those provided by the definition of the weight structure. Indeed, the same proofs work!

Furthemore, it is sufficient for the orthogonality conditions to be satisfied "in the limit" for a directed set of Postnikov towers (for $X$). Again, it is no problem to generalize the proof to this case. Still, in order to make the statement easier to understand, the author chose to formulate it in §7.4 below only for a partial (yet very important!) case corresponding to the coniveau spectral sequence.

Theorem 4.4.2 yields a simple description of adjacent structures (of any type) when they exist.

**Proposition 4.4.4.** *1. Let $w$ be a weight structure for $\underline{C}$. Then there exists a t-structure which is left (resp. right) adjacent to $w$ iff for $\underline{C}^{t\leq 0} = \underline{C}^{w\leq 0}$ and $\underline{C}^{t\geq 0} = \{X \in Obj\underline{C} : \underline{C}(Y,X) = \{0\}\ \forall Y \in \underline{C}^{w\leq -1}\}$ (resp. $\underline{C}^{t\geq 0} = \underline{C}^{w\geq 0}$ and $\underline{C}^{t\leq 0} = \{X \in Obj\underline{C} : \underline{C}(X,Y) = \{0\}\ \forall Y \in \underline{C}^{w\geq 1}$), and any $X \in Obj\underline{C}$ there exists a t-decomposition (14) of $X$. In this case our choice of $\underline{C}^{t\leq 0}$ and $\underline{C}^{t\geq 0}$ is the only one possible.*

*2. Let $t$ be a t-structure for $\underline{C}$. Then there exists a weight structure which is left (resp. right) adjacent to $w$ iff for $\underline{C}^{w\leq 0} = \underline{C}^{t\leq 0}$ and $\underline{C}^{w\geq 0} = \{X \in Obj\underline{C} : \underline{C}(X,Y) = \{0\}\ \forall Y \in \underline{C}^{t\leq -1}\}$ (resp. $\underline{C}^{w\geq 0} = \underline{C}^{t\geq 0}$ and $\underline{C}^{w\leq 0} = \{X \in Obj\underline{C} : \underline{C}(Y,X) = \{0\}\ \forall Y \in \underline{C}^{t\geq 1}$), and any $X \in Obj\underline{C}$ there exists a weight decomposition (1) of $X$. In this case our choice of $\underline{C}^{t\leq 0}$ and $\underline{C}^{t\geq 0}$ is the only one possible.*

*Proof.* First we note that by Theorem 4.4.2 our choices of the structures are the only one possible. Hence it suffices to check when these choices indeed give the corresponding structures.

1. We only consider the left adjacent structure case; the "right" case is similar (and, in fact, dual; see Remark 1.1.2).

The set $\underline{C}^{t\geq 0}$ is automatically strict and since $\underline{C}^{w\leq 0}[1]\subset \underline{C}^{w\leq 0}$, we have $\underline{C}^{t\geq 0}\subset \underline{C}^{t\geq 0}[1]$.

Hence we obtain a $t$-structure if and only if there always exists a $t$-decompositions.

2. As in part 1, we consider only the "left" case (for the same reason).

It is well known that $\underline{C}^{t\leq 0}$ is Karoubi-closed; hence both $\underline{C}^{w\leq 0}$ and $\underline{C}^{w\geq 0}$ are Karoubi-closed also. Again $\underline{C}^{t\leq 0}[1]\subset \underline{C}^{t\leq 0}$ implies $\underline{C}^{t\geq 0}\subset \underline{C}^{t\geq 0}[1]$.

Hence we obtain a weight structure if and only if there always exist weight decompositions. □

## 4.5 Existence of adjacent structures

Now we study certain sufficient conditions for adjacent weight and $t$-structures to exist.

First we prove a statement that is relevant for Voevodsky's $DM_-^{eff}$ and for $SH$. We describe a certain version of the compactly generated category notion; $DM_-^{eff}$ and $SH$ will satisfy our conditions.

**Definition 4.5.1.** We will say that a set of objects $C_i\in Obj\underline{C}$, $i\in I$ ($I$ is a set) *negatively well-generate* $\underline{C}$ if

(i) $C_i$ are compact; they weakly generate $\underline{C}$ (cf. the Notation).

(ii) For all $j>0$, $i,i'\in I$ and $j>0$ we have $\underline{C}(C_i,C_{i'}[j])=\{0\}$ (i.e. the set $\{C_i\}$ is negative).

(iii) $\underline{C}$ contains the category $H$ whose objects are arbitrary (small) direct sums of $C_i$; $\underline{C}$ also contains all homotopy colimits of $X_i\in Obj\underline{C}$ (see Definition 4.2.1) such that $X_{-1}=0$ and the cone of $X_i\to X_{i+1}\in H[i]$.

**Theorem 4.5.2.** *I1. Suppose that $C_i\in Obj\underline{C}$, $i\in I$, negatively well-generate $\underline{C}$. For $H$ described in (iii) of Definition 4.5.1 we consider a full subcategory $\underline{C}^-\subset\underline{C}$ whose objects are*

$$X\in Obj\underline{C}:\ \forall\, Y\in ObjH \text{ there exists } j\in\mathbb{Z} \text{ such that } \underline{C}(Y,X[i])=\{0\}\ \forall i>j. \tag{23}$$

*Then there exist a weight structure $w$ on $\underline{C}^-$ and a $t$-structure $t$ on $\underline{C}$ such that $H\subset Hw$, $t$ restricts to a $t$-structure on $\underline{C}^-$, and $\underline{C}^{t\leq 0}=\underline{C}^{-,w\leq 0}$.*

*(Note that $w$ and $t$ restricted to $\underline{C}^-$ are adjacent by definition.)*

*2. If $\underline{C}$ also admits arbitrary countable direct sums, then $w$ could be extended to the whole $\underline{C}$.*

*II Let $C_i,\underline{C},w,t$ be either as in part I2 or as in I1 with the additional condition $\underline{C}=\underline{C}^-$ (i.e. $w$ is defined on $\underline{C}$) fulfilled. Then the following statements are valid.*

*1. $Hw$ is the small envelope of the category $H$ (whose objects are direct sums of $C_i$) in $\underline{C}$.*

*2. Restrict the functors from $Ht$ (considered as a subset of $Hw_*$ by part 4 of Theorem 4.4.2) to the full additive subcategory $C \subset Hw$ consisting of finite direct sums of $C_i$. Then this restriction functor gives an equivalence of $Ht$ with $C_*$.*

*3. For any object of $Y \in \underline{C}^{t=0}$ and any $X \in Obj\underline{C}$ we have $\underline{C}(X,Y) = (\mathrm{Ker}(\underline{C}(X^0,Y) \to \underline{C}(X^{-1},Y))/\mathrm{Im}(\underline{C}(X^1,Y) \to \underline{C}(X^0,Y))$ where $\dots \to X^{-1} \to X^0 \to X^1 \to \dots$ is an arbitrary choice of the weight complex for $X$.*

*Proof.* I1. The existence of $w$ on $\underline{C}^-$ is immediate from part III (version (ii)) of Theorem 4.3.2.

We define $\underline{C}^{t\ge 0} = \{X \in Obj\underline{C} : \underline{C}(Y,X) = \{0\}\ \forall Y \in \underline{C}^{w\le -1}\}$. Then to prove that $t$ is a $t$-structure it suffices (cf. Proposition 4.4.4) to check that for any $X \in Obj\underline{C}$ there exists a $t$-decomposition (14).

We will construct $X^{t\le 0}$ and $X^{t\ge 1}$ explicitly. Our construction is uses almost the same argument as in the proof of part III version (ii) of Theorem 4.3.2. It could also be thought about as of a triangulated version of the construction of Eilenberg-MacLane spaces (this construction really allows to construct Eilenberg-MacLane spectra from $S^0$ in $SH$, see §4.6 below!).

We take $P_0 = \bigoplus_{i\in I, s\in \underline{C}(C_i,X)} C_i$. Then we have a morphism $f_0 : P_0 \to X$ whose component that corresponds to $(C_i, s)$ is given by $s$. Let $X_0$ denote a cone of $f_0$. Repeating the construction for $X_0[-1]$ instead of $X$ we get an object $P_1$ being a direct sum of certain $C_i$, $f_1 : P_1 \to X_0[-1]$; we denote a cone of $f_1$ by $X_1[-1]$. Proceeding (with $X_i[-1-i]$), we get an infinite sequence of $(P_i, f_i, X_i)$. We denote the map $X \to X_0$ given by the construction by $g_0$, $g_i : X_{i-1} \to X_i$, $h_i = g_i \circ \dots \circ g_1 \circ g_0 : X \to X_i$. We denote a cone of $h_i$ by $Y_i[1]$; the map $Y_i \to X[1]$ given by the corresponding distinguished triangle by $r_i$. We have $P_i \in \underline{C}^{w=0}$ by the definition.

We have $Y_0 = P_0$. The octahedron axiom implies that the commutative triangle $X \xrightarrow{h_{i-1}} X_{i-1} \xrightarrow{g_i} X_i$ could be completed to an octahedron diagram. This yields a distinguished triangle $Y_i \to P_i[i] \to Y_{i-1}[1] \to Y_i[1]$, we denote the map $Y_{i-1} \to Y_i$ by $\phi_{i-1}$. The octahedron diagram (cf. §IV.1 of [15]) also gives $r_{i-1} = r_i \circ \phi_{i-1}$. Hence $Y_i \in \underline{C}^{w\le 0}$ by definition.

Now we consider the homotopy colimit of $Y_i$; cf. Definition 4.2.1. By part 1 of Lemma 4.2.3, the sequence $r_i$ could be lifted to some morphism $f : Y \to X$. We denote its cone as $Z$.

Since $(\underline{C}, w)$ admit negative direct sums, by part 3 of Remark 4.2.2 we have $Y \in \underline{C}^{w\le 0}$ $(= \underline{C}^{t\le 0})$. $Y, Z$ will be our candidates for $X^{t\le 0}$ and $X^{t\ge 1}$.

We verify that $Z \in \underline{C}^{t\geq 1}$. First we check that $\underline{C}(C_i[j], Z) = \{0\}$ for all $i \in I,\ j \geq 0$. This is equivalent to the fact that the map $f_* : \underline{C}(C_i[j], Y) \to \underline{C}(C_i[j], X)$ is an isomorphism for all $i \in I,\ j \geq 0$ and is injective for $j = -1$ (cf. part 1 of Remark 1.3.2).

By part 3 of Lemma 4.2.3, for any compact $C$ we have $\underline{C}(C, Y) = \varinjlim \underline{C}(C, Y_i)$. Moreover, we have $\underline{C}(C_i[-1], Y) = \{0\}$ since $Y \in \underline{C}^{w\leq 0}$ and $C_i[-1] \in \underline{C}^{w=1}$. Hence it suffices to verify that $\underline{C}(C_i[j], X_l) = \{0\}$ for $l > j \geq 0$ (this gives $\underline{C}(C_i[j], Y_l) \cong \underline{C}(C_i[j], X)$).

We apply the distinguished triangle $P_j[j] \to X_{j-1} \to X_j \to P_j[j+1]$. Since $C_i[j]$ is compact, we easily obtain

$$\underline{C}(C_i[j], P_j[j]) = \bigoplus_{m\in I, s\in \underline{C}(C_m[j], X_{j-1})} \underline{C}(C_i, C_m).$$

Hence this group has an element for each morphism $C_m[j] \to X_{j-1}$; it follows that the map $\underline{C}(C_i[j], P_j[j]) \to \underline{C}(C_i[j], X_{j-1})$ is surjective. Next, since $C_i[j]$ is compact and $C_i[j] \in \underline{C}^{w=j}$, $\underline{C}(C_i[j], P_j[j+1]) = \underline{C}(C_i, P_j[1])$ equals the direct sum of corresponding $\underline{C}(C_i, C_m[1])$; hence it is zero by the orthogonality property for $w$ (cf. Definition 1.1.1). We obtain $\underline{C}(C_i[j], X_j) = \{0\}$.

Now we use distinguished triangle $P_l[l] \to X_{l-1} \to X_l \to P_j[l+1]$ for $l > j$. Again compactness of $C_i[j]$ yields $\underline{C}(C_i[j], P_l[l+1]) = \underline{C}(C_i[j], P_l[l]) = \{0\}$. Hence $\underline{C}(C_i[j], X_l) = \underline{C}(C_i[j], X_{l-1}) = \{0\}$ for all $l > j$.

It remains to check that for any $T \in Obj\underline{C}$ the condition $\underline{C}(C_i[j], T) = \{0\}$ for all $i \in I,\ j \geq 0$ implies that $\underline{C}(C, T) = \{0\}$ for all $C \in \underline{C}^{w\leq 0}$. This follows immediately from part III version (ii) of Theorem 4.3.2.

Moreover, part III version (ii) of Theorem 4.3.2 implies that if on $\underline{C}^-$ (defined as in part IV of Theorem 4.3.2) there exists a weight structure such that $H \subset Hw$. Note that $\underline{C}^-$ also satisfies the conditions of the theorem. The description of $\underline{C}^{w\leq 0}$ in the proof of Theorem 4.3.2 shows that $w$ is left adjacent to $t$ on $\underline{C}^-$.

2. We should check that $w$ could be extended to the whole $\underline{C}$. We define $X^{w\geq 0}$ using the orthogonality axiom (of weight structures).

For any $X \in Obj\underline{C}$ we denote $X^{t\leq i}[-i]$ by $X_i$ for $i > 1$ and take $Y$ being the homotopy colimit of $Y_i$ for $Y_i = X_i^{w\geq 1}$ (see Definition 4.2.1). Here the morphism $Y_i \to Y_{i+1}$ are obtained by applying part 1 of Lemma 1.5.1 to the natural morphisms $X_i \to X_{i+1}$. By Lemma 4.2.4 we can assume that $Y \in \underline{C}^{w\geq 0}$.

$Y$ will be our candidate for $X^{w\geq 1}$ (cf. the proof of part I1). By part 1 of Lemma 4.2.3 the system of composition maps $Y_i[-1] \to X_i \to X$ could be lifted to some $f \in \underline{C}(Y[-1], X)$.

Now we show that $f$ extends to a weight decomposition of $X$ using part 1 of Remark 1.3.2. We should check that $\underline{C}(C_k[j], \mathrm{Cone}(f)) = \{0\}$ for all $k \in I$

and $j < 0$ (see the description of $\underline{C}^{w\le 0} = \underline{C}^{-,w\le 0}$ in the proof of part III of Theorem 4.3.2). Since all $C_k$ are compact, as in the proof of part I1 we obtain that $\underline{C}(C_k[j], Y) = \varinjlim \underline{C}(C_k[j], Y_i)$. Moreover, $\underline{C}(C_k[j], X_i) = \underline{C}(C_k[j], X)$ for $i > -j$. Hence it suffices to note that the direct limit of isomorphisms is an isomorphism, while a direct limit of surjections is surjective if the targets stabilize (obvious!).

Now it remains to apply part 4 of Lemma 1.3.5.

II1. Obviously, $Hw$ contains $ObjH$. Since $Hw$ is Karoubi-closed in $\underline{C}$, it also contains all retracts of objects of $H$. Hence it suffices that any object of $Hw$ is such a retract.

We consider the "weight resolution" of $X \in \underline{C}^{w=0}$ (in fact, it suffices to consider first few terms). We obtain that the weight complex of $X$ can be presented by $\cdots \to P_1 \to P_0$. Since it is homotopy equivalent to $X$, we obtain that $X$ is a retract of $P_0$. The assertion is proved.

2. By assertion II1, the restriction of representable functors to the category of all direct sums of $C_i$ is fully faithful on $Ht$ (see part 4 of Theorem 4.4.2). Since $C_i$ are compact, we can fully faithfully restrict these functors further to $C$. So it remains to compute the categorical image of this restriction.

Since $(\underline{C}, w)$ admits negative direct sums, $\underline{C}$ contains all direct sums of $C(-, C_i)$. Since $C_i$ are compact, these sums represent functors $\oplus C(-, C_i)$ on $C$. Since $Ht$ is abelian, its image also contains all cokernels of morphisms of objects that could be presented as $\oplus C(-, C_i)$.

It remains to note that cokernels of morphisms of objects of the type $\oplus C(-, C_i)$ give the whole $C_*$. This fact was mentioned in the Notation, see also Lemma 8.1. of [25]. In fact, this is very easy: every $F : C \to Ab^{op}$ can be presented as a factor of the natural

$$h : \sum_{i\in I, x\in F(C_i)} C_i \to F,$$

and the same could be said about the kernel of $h$.

3. This is just the formula (22).

□

*Remark* 4.5.3. 1. Dualizing part I2, one obtains certain sufficient conditions for the right adjoint weight and $t$-structures to exist. Unfortunately, this requires 'positive products' and cocompact weak cogenerators which do not usually exist.

2. If $\underline{C}$ is endowed with a $t$-structure then the question of existence of an adjacent weight structure seems to be difficult in general; cf. Remark 7.1.1

below. Yet see Theorem 4.1 of [28] for an interesting result in this direction (though in rather restrictive conditions).

## 4.6 The spherical weight structure for the stable homotopy category

We consider the stable homotopy category $SH$. Recall some of its basic properties.

The objects of $SH$ are called *spectra*. $SH$ contains the *sphere spectrum* $S^0$ that weakly generates it.

The groups $A_i = SH(S^0[i], S^0)$ are called the stable homotopy groups of spheres. We have $A_i = 0$ for $i < 0$, $A_i = \mathbb{Z}$ for $i = 0$; $A_i$ are finite for $i > 0$. For an arbitrary $A \in ObjSH$ the groups $SH(S^0[i], A)$ are called the homotopy groups of $A$ (they are denoted by $\pi_i(A)$)).

The category $SH_{fin} \subset SH$ of finite spectra was defined in Corollary 4.3.3. We will also consider the category $SH_{qfin} \subset SH$ of *quasi-finite spectra*. Its objects are described by the following conditions: all $\pi_i(A)$ are finitely generated and $\pi_i(A) = 0$ for all $i > j$ for some $j \in \mathbb{Z}$. Lastly, we will also mention the full subcategory $SH^- \subset SH$ whose objects are spectra with homotopy groups that are zero for $i > j$ (for some $j$ that depends on the spectrum chosen). Obviously, all categories mentioned are triangulated subcategories of $SH$.

We see that $SH_{fin}$ and $SH_{qfin}$ satisfy the conditions of part III (version (i)) of Theorem 4.3.2 if we take $H = H'$ equal to the category of finite direct sums of $S^0$ and $c = \omega$. Indeed, in this case we only need finite sums and their properties which are valid for arbitrary $\underline{C}$.

Hence we obtain a certain non-degenerate weight structure $w$ on $SH_{fin} \subset SH_{qfin}$. It is bounded above for $SH_{qfin}$, whereas $SH_{fin}$ is bounded since it is generated by $S^0$. Recall that $S^0$ are compact hence all objects of $H'$ also are. Hence using part III version (ii) of Theorem 4.3.2 we can extend $w$ to $SH^-$.

Now we describe the heart of $w$ for different categories of spectra. Since $SH(S^0, S^0) = \mathbb{Z}$, we obtain that $H' \cong Ab_{fin.fr}$ (the category of finitely generated free abelian groups); note that $H = H'$ in this case. Since $H$ it is idempotent complete, part III (i) of Theorem 4.3.2 implies that $Hw_{SH_{fin}} = Hw_{SH_{qfin}} \cong Ab_{fin.fr}$.

In $SH^-$ we have $H \cong Ab_{fr}$ (the category of all free abelian groups). Since $Ab_{fr}$ is idempotent complete, we obtain $Hw_{SH^-} \cong Ab_{fr}$.

Now recall that $SH$ admits countable (and also, in fact, arbitrary) direct sums. Hence by part I2 of Theorem 4.5.2 we can extend $w$ to the whole $SH$.

This certainly means that $Hw_{SH} \cong Ab_{fr}$. Hence the functor $t$ is "strong" for all categories of spectra mentioned, see part 1 of Remark 3.3.4. Besides we obtain a certain "weight filtration" on homotopy groups of spectra. By definition, it is trivial (i.e. "canonical") on the homotopy of $S^0$.

Note that any object of $SH^{w=0}$ is isomorphic to a direct sum of spherical spectra. Hence Postnikov towers (coming form $w$) in this case become *cellular towers* for spectra in topology. Its construction and the functoriality properties (in the topological case) are described in §6.3 of [26]; certainly, the results of loc. cit. are parallel to ours.

Now we describe the connection of the weight complex functor for this weight structure with singular homology and cohomology of spectra.

To this end we recall that $SH$ supports a non-degenerate *Postnikov* $t$-*structure* $t_{Post}$; the corresponding cohomology functor is given by $SH(S^0,-)$. We obtain that $SH^{-,w\le 0} = SH^{t_{Post}\le 0}$. Hence $t_{Post}$ is exactly the $t$-structure described in part I1 of Theorem 4.5.2. Besides by part 5 of Theorem 4.4.2, any Eilenberg-MacLane spectrum belongs to $SH^{t_{Post}=0}$. Recall that the singular cohomology theory $H^i_{sing}$ for spectra is represented by the Eilenberg-Maclane spectrum $H\mathbb{Z}$ that corresponds to $\mathbb{Z}$, while the singular homology of $X$ can be calculated as $SH(S^0, H\mathbb{Z}\wedge X)$; we will denote it by $H_i^{sing}(X)$ (see the notation of Definition 2.3.1).

We identify $Hw = H$ with $Ab_{fr}$ using the functor $H(S^0,-)$.

**Proposition 4.6.1.** *Let $X$ be a spectrum.*

*1. $H^i(t(X)) \cong H_i^{sing}(X)$.*

*2. $H^0(Ab(X^{-i},\mathbb{Z})) \cong H^0_{sing}(X)$.*

*3. For $X \in ObjSH_{fin}$ we have $X \in SH_{fin}^{w\ge 0} \iff H^i_{sing}(X) = 0\ \forall i > 0$ and $X \in SH_{fin}^{w\le 0} \iff H^i_{sing}(X) = 0\ \forall i < 0$.*

*Proof.* 1. We apply Theorem 2.3.2 to the functor $H_0^{sing}$. We have $E_1^{pq} = H_q^{sing}(X^p)$ while each $X^p$ is a (possibly, infinite) direct sum of copies of $S^0$. Now, the only non-zero homology group of $S^0$ is $\mathbb{Z}$ placed in dimension 0; the functor $Y \to H\mathbb{Z}\wedge Y$ commutes with (arbitrary) homotopy colimits and sums.

Hence the spectral sequence $T(H^{sing}, X)$ degenerates to the weight complex of $X$. By the convergence condition II(ii) of loc. cit. we have $T(H^{sing}, X) \implies H^{sing}(X)$.

2. Part II3 of Theorem 4.5.2 calculates $SH(X,Y)$ for any Eilenberg-MacLane spectrum $Y$. In particular, taking $Y = H\mathbb{Z}$ we obtain the claim.

3. As in part 10 of Theorem 4.4.2, if $X \in SH^{w\ge 0}$ (resp. $X \in SH^{w\le 0}$) then the corresponding conditions on the cohomology of $X$ are fulfilled.

Conversely, let $H^i_{sing}(X) = 0\ \forall i > 0$. Then by (22) the complex $t(X)$ is acyclic in negative degrees. Since it is a complex of free abelian groups, it is homotopy equivalent to a complex concentrated in non-negative degrees. Hence part IV of Theorem 3.3.1 yields the assertion desired.

The case of $X:\ H^i_{sing}(X) = 0\ \forall i < 0$ is considered similarly.

$\square$

*Remark* 4.6.2. 1. Alternatively, if we take an Eilenberg-MacLane spectrum $HI$ corresponding to some injective group $I$ instead, we will get $SH(X, HI) = Ab(H^0(t(X)), I)$.

2. Note also that $S^0[-1]^{t\geq 1}$ is exactly $H\mathbb{Z}$. Hence $H\mathbb{Z}$ could be obtained by applying the construction described in the proof of Theorem 4.5.2 to $S^0$.

3. The proof of part 1 of Proposition 4.6.1 shows that the weight filtration given by the spherical weight structure on singular homology coincides with the canonical filtration. This is not the case for homotopy groups of spectra.

# 5 Idempotent completions; $K_0$ of categories with bounded weight structures

In §5.1 we recall that an idempotent completion of a triangulated category is triangulated. In §5.2 we prove that a bounded $\underline{C}$ is idempotent complete iff $Hw$ is; in general, the idempotent completion of a bounded $\underline{C}$ has a weight structure whose heart is the idempotent completion of $Hw$.

In §5.3 we prove that if $\underline{C}$ is bounded and idempotent complete then the embedding $Hw \to \underline{C}$ induces an isomorphism $K_0(\underline{C}) \cong K_0(Hw)$. It is a ring isomorphism if $Hw \subset \underline{C}$ are endowed with compatible tensor structures. In §5.4 we study a certain Grothendieck group of endomorphisms in $\underline{C}$. Unfortunately, it is not always isomorphic to $K_0(\operatorname{End} Hw)$; yet it is if $Hw$ is *regular*; see Definition 5.4.2. Besides, we can still say something about it in other cases. In particular, this allows us to generalize Theorem 3.3 of [8] to arbitrary endomorphisms of motives (in Corollary 5.4.6); see also §8.4 of [11].

In §5.5 we calculate explicitly the groups $K_0(SH_{fin})$ and $K_0(\operatorname{End} SH_{fin})$. It turns out that the classes of $[X]$ and $[g : X \to X]$ are easily recovered from the rational singular homology of $X$; see Proposition 5.5.1. More generally, one could calculate certain groups $K_0(\operatorname{End}^n SH_{fin})$ for $n \in \mathbb{N}$ in a similar way, see Remarks 5.5.2 and 5.4.7.

## 5.1 Idempotent completions: reminder

We recall that an additive category $A$ is said to be *idempotent complete* if for any $X \in ObjA$ and any idempotent $p \in A(X,X)$ there exists an image of $p$ in $A$.

Any additive $A$ can be canonically idempotent completed. Its idempotent completion is (by definition) the category $A'$ whose objects are $(X,p)$ for $X \in ObjA$ and $p \in A(X,X): \ p^2 = p$; we define

$$A'((X,p),(X',p')) = \{f \in A(X,X'): \ p'f = fp = f\}.$$

It can be easily checked that this category is additive and idempotent complete, and for any idempotent complete $B \supset A$ we have a natural unique embedding $A' \to B$.

The main result of [3] (Theorem 1.5) states that an idempotent completion of a triangulated category $\underline{C}$ has a natural triangulation (with distinguished triangles being all direct summands of distinguished triangles of $\underline{C}$).

In this section $\underline{C}'$ will denote the idempotent completion of $\underline{C}$, $Hw'$ will denote the idempotent completion of $Hw$.

Note that if $\underline{C}$ is idempotent complete then $Hw$ is also, since $Hw \subset \underline{C}$ and $Hw$ is Karoubi-closed.

## 5.2 Idempotent completion of a triangulated category with a weight structure

We prove that $\underline{C}^b$ is idempotent complete if $Hw$ is.

**Lemma 5.2.1.** *If $w$ is bounded, $Hw$ is idempotent complete, then $\underline{C}$ also is.*

*Proof.* We prove that all $\underline{C}^{[i,j]}$ are idempotent complete by induction on $j-i$. The base is: $\underline{C}^{[i,i]} = \underline{C}^{w=0}[-i]$ is idempotent complete.

To make the inductive step it suffices to prove that $\underline{C}^{[-i,1]}$ if idempotent complete if $\underline{C}^{[-i,0]}$ is (for $i > 0$). For $X \in \underline{C}^{[-i,1]}$ and an idempotent $p \in \underline{C}(X,X)$ we consider the functor $WD$ (see part I of Theorem 3.2.2). We obtain an idempotent $q = WD(p) \in K_{\mathfrak{w}}^{[0,1]}(\underline{C})(WD(X),WD(X))$ whereas $Y = WD(X)$ has the form $(Z \to T)$ for $Z,T \in ObjK_{\mathfrak{w}}^b(\underline{C}^{[-i,0]})$. Since $\underline{C}^{[-i,0]}$ is idempotent complete, $K_{\mathfrak{w}}^b(\underline{C}^{[-i,0]})$ also is by part 2 of Proposition 3.1.8. Hence there exists a $Z' \to T'$ and idempotent endomorphisms $r,s$ of $Z'$ and $T'$, respectively, such that $(Y,q)$ could be presented by the diagram

$$\begin{array}{ccc} Z' & \longrightarrow & T' \\ \Big\downarrow{\scriptstyle r} & & \Big\downarrow{\scriptstyle s} \\ Z' & \longrightarrow & T' \end{array}$$

(in $K_{\mathfrak{w}}^{[0,1]}(\underline{C}^{[-i,0]})$).

By Part I5 of Theorem 3.2.2, $Z', T'$ come from a certain weight decomposition of $X$. Then any corresponding weight decomposition of $p$ is homotopy equivalent to $(r,s)$. Then part I2 of Theorem 3.2.2 yields that $(r,s)$ also give a weight decomposition of $p$. Hence the object $(X,p) \in Obj\underline{C}'$ (see §5.1) could be presented as a cone of a certain map $(Z',r) \to (T',s)$ in $\underline{C}'$; whereas $(Z',r),(T',s) \in Obj\underline{C}$ by the inductive assumption.

□

Now we prove that in the general (bounded) case a weight structure could be extended from $\underline{C}$ to its idempotent completion $\underline{C}'$.

**Proposition 5.2.2.** *Let $w$ be bounded. Then the following statements are valid.*

*1. $w$ extends to a bounded weight structure $w'$ for $\underline{C}'$.*

*2. The heart of $\underline{C}'$ equals $Hw'$ (the idempotent completion of $Hw$).*

*Proof.* 1. By Part II1 of Theorem 4.3.2, we have a bounded weight structure that extends $w$ on the subcategory $D \subset \underline{C}'$ generated by $Hw'$. Hence it suffices to recall that $D$ is idempotent complete; see Lemma 5.2.1.

2. Since $Hw'$ is idempotent complete, the assertion follows from part II2 of Theorem 4.3.2.

□

**Corollary 5.2.3.** *If $(\underline{C}, w)$ is bounded and non-degenerate then $Hw'$ generates $\underline{C}'$.*

*Proof.* By Proposition 5.2.2, $Hw'$ is the heart of a bounded non-degenerate weight structure for $\underline{C}'$. Now Corollary 1.5.7 yields the result.

□

*Remark* 5.2.4. It seems possible that the boundedness condition for $w$ is in Proposition 5.2.2 could be weakened. However this does not seem to be actual since in all "natural" cases either $(\underline{C}, w)$ is bounded or $\underline{C}$ admits countable direct sums. In the latter case $\underline{C}$ is idempotent complete, see Proposition 1.6.8 of [23].

## 5.3 $K_0$ of a triangulated category with a bounded weight structure

We recall some standard definitions (cf. 3.2.1 of [16]). We define the Grothendieck group of an additive category $A$ as a group whose generators are of the form $[X]$, $X \in ObjA$; the relations are $[X \oplus Y] = [X] + [Y]$ for $X, Y \in ObjA$. The

$K_0$-group of a triangulated category $T$ is defined as the group whose generators are $[t]$, $t \in ObjT$; if $A \to B \to C \to A[1]$ is a distinguished triangle then $[B] = [A] + [C]$. Note that $X \oplus 0 \cong X$ implies that $[X] = [Y]$ if $X \cong Y$ (in $A$ or in $T$).

For an additive $A$ we define $K_0(K^b_{\mathfrak{w}}(A))$ similarly to $K_0(K^b(A))$; hence it equals $K_0(K^b(A))$ (see Definition 3.1.6).

The existence of a bounded $w$ allows to calculate $K_0(\underline{C})$ easily.

**Theorem 5.3.1.** *Let $(\underline{C}, w)$ be bounded, let $Hw$ be idempotent complete. Then the inclusion $i : Hw \to \underline{C}$ induces an isomorphism $K_0(Hw) \to K_0(\underline{C}^b)$.*

*Proof.* Since $t$ is an weakly exact functor (see Definition 3.1.6), it gives an abelian group homomorphism $a : K_0(\underline{C}) \to K_0(K^b_{\mathfrak{w}}(Hw)) = K_0(K^b(Hw))$. By Lemma 3 of 3.2.1 of [16], there is a natural isomorphism $b : K_0(K^b(Hw)) \to K_0(Hw)$. The embedding $Hw \to \underline{C}$ gives a homomorphism $c : K_0(Hw) \to K_0(\underline{C})$. The definitions of $a, b, c$ imply immediately that $b \circ a \circ c = id_{K_0(Hw)}$. Hence $a$ is surjective, $c$ is injective.

It remains to verify that $c$ is surjective. It follows immediately from the fact that $Hw$ generates $\underline{C}$, see Corollary 1.5.7.

$\square$

*Remark* 5.3.2. Obviously, if $\underline{C}$ is a tensor triangulated category then $K_0(\underline{C})$ is a ring. If the tensor structure on $\underline{C}$ induces a tensor structure on $Hw$ then $K_0(Hw)$ is a ring also and $c$ is a ring isomorphism.

For the convenience of citing we concentrate certain assertions relevant for motives in a single statements.

**Proposition 5.3.3.** *Suppose that $\underline{C}$ contains an additive negative (see Definition 4.3.1) subcategory $H$ such that $H$ is idempotent complete and $\underline{C}$ is the idempotent completion of $\langle H \rangle$. Then the following statements are valid.*

*1. $\langle H \rangle = \underline{C}$.*

*2. There exists a conservative weight complex functor $\underline{C} \to K^b_{\mathfrak{w}}(H)$ which sends $h \in ObjH$ to $h[0] \in ObjK^b_{\mathfrak{w}}(H)$. It could be extended to an exact functor $t' : \underline{C} \to K^b(H)$ in the case when $\underline{C}$ has a differential graded enhancement (see part 4 of Definition 6.1.2 below).*

*3. $K_0(\underline{C}) \cong K_0(H)$.*

*Proof.* 1. By part II of Theorem 4.3.2 there exists a bounded weight structure $w'$ on $\langle H \rangle$ whose heart equals the small envelope of $H$ i.e. to $H$. Next, by Proposition 5.2.2 $w$ extends to some bounded $w$ on $\underline{C}$ whose heart equals the idempotent completion of $H$ i.e. $H$. Hence Corollary 5.2.3 immediately yields assertion 1.

2. The weight complex functor $t : \underline{C} \to K_{\mathfrak{w}}(H)$ could be factorized through $K^b_{\mathfrak{w}}(H)$ since $w$ is bounded. $t$ is conservative by part V of Theorem 3.3.1. If $\underline{C}$ has a differential graded enhancement then $t$ could be lifted to $t'$ by part 2 of Remark 6.2.2 below.

3. Immediate from Theorem 5.3.1.

$\square$

## 5.4 $K_0$ for categories of endomorphisms

Now we define various Grothendieck groups of endomorphisms in an additive category $A$. Our definitions are similar to those of [1].

**Definition 5.4.1.** 1. The generators of $K_0^{add}(\operatorname{End} A)$ are endomorphism of objects of $A$; the relations are of the form $[g] = [f] + [h]$ if $(f, g, h)$ give an endomorphism of a split short exact sequence.

2. If $A$ is also abelian then we also consider the group $K_0^{ab}(\operatorname{End} A)$. Its generators again are endomorphism of objects of $A$; the relations are of the form $[g] = [f] + [h]$ if $(f, g, h)$ give an endomorphism of an arbitrary short exact sequence.

3. If $A$ is triangulated then we consider the group $K_0^{tr}(\operatorname{End} A)$.

The generators of $K_0^{tr}(\operatorname{End} A)$ are endomorphism of objects of $A$ again; the relations are $[g] = [f] + [h]$ if $(f, g, h)$ give an endomorphism of a distinguished triangle in $A$.

Note that $K_0^{ab}(\operatorname{End} A)$ and $K_0^{tr}(\operatorname{End} A)$ are natural factors of $K_0^{add}(\operatorname{End} A)$ when these groups are defined. Indeed, $K_0^{ab}(\operatorname{End} A)$ and $K_0^{tr}(\operatorname{End} A)$ have the same generators as $K_0^{add}(\operatorname{End} A)$ and more relations.

Let $\underline{C}$ be bounded. We provide some sufficient conditions for $K_0(\operatorname{End} \underline{C})$ to be isomorphic to $K_0(\operatorname{End} Hw)$. We need a notion of a *regular* additive category $A$. Recall that $A'_*$ is the full abelian subcategory of $A_*$ generated by $A$.

**Definition 5.4.2.** An additive category $A$ will be called *regular* if it satisfies the following conditions.

1. $A$ equals its small envelope (see part 3 of Definition 4.3.1) i.e. if $X, Y \in Obj A$, $X$ is a retract of $Y$, is then $X$ has a complement to $Y$ (in $A$).

2. Every object of $A'_*$ has a finite resolution by objects of $A$.

The most simple examples of regular categories are abelian semisimple categories and the category of finitely generated projective modules over a noetherian (commutative) local ring all of whose localizations are regular local; cf. the end of §1 of [1].

We will need the following technical statement. Let $R$ be an associative ring with a unit.

**Lemma 5.4.3.** *1. If $A$ is regular then $K_0^{add}(\operatorname{End} A) \cong K_0^{ab}(\operatorname{End} A'_*)$.*

*2. If $A$ is the category of finitely generated projective modules over $R$ then $A_*$ is the category of all (left) modules over $R$.*

*Proof.* 1. We apply the method of the proof of Proposition 5.2 of [2]. First we consider the obvious category $\operatorname{End} Hw'_*$ and note that it is abelian. Next, the objects of $Hw$ become projective in $Hw'_*$. Hence all 3-term complexes in $Hw$ that become exact in $Hw'_*$ do split in $Hw$. Therefore we can define $K_0^{add}(\operatorname{End} Hw_*)$ as the Grothendieck group of an exact subcategory of $\operatorname{End} Hw'_*$.

Condition 1 of Definition 5.4.2 ensures that for any short exact sequence $0 \to G' \to G \to G'' \to 0$ in $\operatorname{End} Hw'_*$ if $G, G'' \in \operatorname{End} Hw$ then $G' \in \operatorname{End} Hw$ (i.e. $G'$ is an endomorphism of an object of $Hw$). Lastly, condition 2 of Definition 5.4.2 easily implies that any $G \in \operatorname{End} Hw'_*$ has a finite resolution by objects of $\operatorname{End} Hw$ (again note that objects of $Hw$ become projective in $Hw'_*$!). Hence applying Theorem 16.12 of [30] (page 235) we obtain the result.

2. The equivalence is given by sending a functor $F$ to $F(R)$ and a module $Q/R$ to $P \to \operatorname{Hom}_R(P, Q)$ (here $R$ is also considered as right $R$-module). Note that all $F(P)$ could be uniquely recovered from $F(R)$ since all finitely generated projective modules are direct summands of $R^m$, $m \in \mathbb{N}$.

□

**Proposition 5.4.4.** *1. There exist natural homomorphisms $K_0(\operatorname{End} Hw) \xrightarrow{c} K_0^{tr}(\operatorname{End} \underline{C}) \xrightarrow{d} K_0^{ab}(\operatorname{End} Hw'_*)$; $c$ is a surjection.*

*2. $c$ is an isomorphism if $Hw$ is regular.*

*Proof.* 1. $c$ is induced by $i: Hw \to \underline{C}$. For $g: X \to X$ we define

$$d(g) = \sum (-1)^i [g_{i*} : H^i(t(X)) \to H^i(t(X))]. \tag{24}$$

Here $H^i(t(X)) \in Obj Hw'_*$ are the cohomology of the weight complex; see part 2 of Remark 3.1.7. We obtain a well-defined homomorphism since $t$ is a weakly exact functor (see Definition 3.1.6); see part 3 of Remark 3.1.7.

$c$ is surjective since for $g: X \to X$ we have the equality $[g] = \sum (-1)^i [g^i : X^i \to X^i]$. This equality follows easily from the fact that repetitive application of the (single, shifted) weight decomposition functor to a morphism yields its infinite weight decomposition (see Theorem 3.2.2; note that $X$ is bounded).

2. In the case when $Hw$ is abelian semi-simple we have $Hw = Hw'_*$. Hence the equality $d \circ c = id_{K_0(\operatorname{End} \underline{C})}$ yields the assertion (in this case).

Now, in the general (regular) case it suffices to apply the equality $K_0(\operatorname{End} \underline{C}) = K_0(\operatorname{End} Hw'_*)$ (this is part 1 of Lemma 5.4.3).

$\square$

*Remark* 5.4.5. 1. Unfortunately, $c$ is not an isomorphism in the general case. To see this it suffice to consider the example described in part 3 of Remark 1.5.2 for $\underline{C} = K^b(Z)$ where $Z$ is the category of free $\mathbb{Z}/8\mathbb{Z}$-modules. This fact is also related to the observation in the end of §1 of [1]. Certainly, $Z$ is not regular.

2. Certainly, if $i : Hw \to \underline{C}$ is a tensor functor then $c, d$ are ring homomorphisms, cf. Remark 5.3.2.

The surjectivity of $c$ immediately implies the following fact.

**Corollary 5.4.6.** *Let $r : \underline{C} \to D^b(R)$ and $s : \underline{C} \to D^b(S)$ be exact functors for an abelian $R, S$; let $r_* : K_0(\underline{C}) \to K_0(D^b(R))$ and $s_* : K_0(\underline{C}) \to K_0(D^b(S))$ be the induced homomorphisms. Let $u : K_0(\operatorname{End} D^b(R)) \to K_0(\operatorname{End} R)$ and $v : K_0(\operatorname{End} D^b(S)) \to K_0(\operatorname{End} S)$ be defined as $(g : X \to X) \to [g_{i*} : H^i(X) \to H^i(X)]$. Let $T$ be an abelian group; $x : K_0(\operatorname{End} R) \to T$ and $y : K_0(\operatorname{End} S) \to T$ be group homomorphisms. Then the equality $x \circ u \circ r_* \circ c = y \circ v \circ s_* \circ c$ implies $x \circ u \circ r_* = y \circ v \circ s_*$.*

*In particular, one could take $\underline{C} = DM^{eff}_{gm}$, $Hw = Chow^{eff}$ (see §6 below), $r, s$ be given by $l$-adic cohomology realizations (for two different $l$), $x, y$ be given by traces of endomorphisms. It follows that the alternated sum of traces of maps induced by $g \in DM^{eff}_{gm}(X, X))$ on the cohomology of $X$ does not depend on $l$. We also obtain the independence from $l$ of $n_\lambda(H) = (-1)^i n_\lambda g^*_{H^i(X)}$; here $n_\lambda g^*_{(H^i(X))}$ for a fixed algebraic $\lambda$ denotes the algebraic multiplicity of the eigenvalue $\lambda$ for the operator $g^*_{H^i(X)}$.*

*This generalizes Theorem 3.3 of [8] to arbitrary correspondences of motives; see §8.4 of [11] for more details.*

Lastly, we consider some more general $K_0$-groups.

*Remark* 5.4.7. 1. For an additive $A$ instead of $\operatorname{End} A$ one could for any $n \geq 0$ consider the category $\operatorname{End}^n A$ whose objects are the following $n+1$-tuples: $(X \in Obj A; g_1, \dots, g_n \in A(X, X))$. We have $\operatorname{End}^0(A) = A$, $\operatorname{End}^1(A) = \operatorname{End} A$. Generalizing Definition 5.4.1 in an obvious way one defines $K_0^{add}(\operatorname{End}^n A)$, $K_0^{ab}(\operatorname{End}^n A)$, and $K_0^{tr}(\operatorname{End}^n A)$ (for $A$ additive, abelian or triangulated, respectively). Next, one can define $c, d$ as in Proposition 5.4.4; exactly the same argument as in the proof of the Proposition shows that $c$ is always surjective and it is also injective if $Hw$ is regular. In particular, this is true for $\underline{C} = SH_{fin}$; see Proposition 5.5.1 below.

2. Even more generally, for any ring $R$ one could consider the category $\mathrm{End}(R, A)$ of *$R$-representations* in $A$ i.e. of pairs $(X, H : R \to A(X, X))$; here $X \in Obj A$, $H$ is a unital homomorphism of rings. In particular, we have $\mathrm{End}(R, A) = A$ for $R = \mathbb{Z}$, $= \mathrm{End}\, A$ for $R = \mathbb{Z}[t]$, and $= \mathrm{End}^n A$ for $R = \mathbb{Z}\langle t_1, \dots, t_n\rangle$ (the algebra of non-commutative polynomials). Again one defines $K_0^{add}(\mathrm{End}(R, A))$, $K_0^{ab}(\mathrm{End}(R, A))$, and $K_0^{tr}(\mathrm{End}(R, A))$, $c$ and $d$. Yet the method of the proof of Proposition 5.4.4 fails for a general $R$; one can only note that $d \circ c$ is an isomorphism if $Hw$ is abelian semi-simple.

## 5.5 An application: calculation of $K_0(SH_{fin})$ and $K_0(\mathrm{End}\, SH_{fin})$

Now we calculate explicitly the groups $K_0(SH_{fin})$ and $K_0^{tr}(\mathrm{End}\, SH_{fin})$. The author doesn't think that (all of) these results are new; yet they illustrate our methods very well.

We will need the following simple observation: $K_0(A)$ is naturally a direct summand of $K_0(\mathrm{End}\, A)$ (both in the "triangulated" and in the "additive" case). The splitting is induced by $[f : X \to X] \to [X] \to [0 : X \to X]$; see §1 of [1].

We define the group $\Lambda$ as a subgroup of the multiplicative group $\Lambda(\mathbb{Z}) = \{1 + t\mathbb{Z}[[t]]\}$ that is generated by polynomials (with constant term 1). $\Lambda$ and $\Lambda(\mathbb{Z})$ are also rings; see Proposition 3.4 of [2] for $\Lambda$ and [17] for $\Lambda(\mathbb{Z})$.

**Proposition 5.5.1.** *1. $K_0(SH_{fin}) \cong \mathbb{Z}$ with the isomorphism sending $X \in Obj SH_{fin}$ to $[X] = \sum (-1)^i \dim_{\mathbb{Q}}(H_i^{sing}(X) \otimes \mathbb{Q})$ (the rational singular homology of $X$).*

*2. $K_0^{tr}(\mathrm{End}\, SH_{fin}) \cong \mathbb{Z} \oplus \Lambda$ with the isomorphism sending $g : X \to X$ to $[X] \oplus \prod_i (\det_{\mathbb{Q}[t]}(id - g_i t \otimes \mathbb{Q}))^{(-1)^i}$; here $g_i t \otimes \mathbb{Q}$ is the map induced by $g \otimes t$ on $H_i^{sing}(X) \otimes_{\mathbb{Z}} \mathbb{Q}[t]$.*

*Proof.* 1. We have $Hw = Ab_{fin.fr}$ for the spherical weight structure $w$ on $SH_{fin}$; see §4.6. Hence $K_0(SH_{fin}) \cong K_0(Ab_{fin.fr}) = K_0(\mathbb{Z}) = \mathbb{Z}$.

The second assertion could easily be deduced from part 1 of Proposition 4.6.1. Note that $K_0(SH_{fin})$ is a direct summand of $K_0^{tr}(\mathrm{End}\, SH_{fin})$; hence

$$[X] = \sum (-1)^i [H^i(t(X))] = \sum (-1)^i [H_i^{sing}(X)]$$

by (24). We also use the fact that $K_0(\mathbb{Z})$ injects into $K_0(\mathbb{Q})$, so $[H_i^{sing}(X)]$ could be computed rationally.

2. By part 2 of Lemma 5.4.3 we have $Hw_* \cong Ab_{fin.fr}$ (the category of finitely generated abelian groups). Hence $Hw$ is regular (see Definition 5.4.2). Therefore by part 2 of Proposition 5.4.1 we have $K_0^{tr}(\mathrm{End}\, SH_{fin}) \cong$

$K_0^{add}(\operatorname{End} Ab_{fin.fr})$. Then the Main Theorem in §1 of [1] implies that $K_0^{tr}(\operatorname{End} SH_{fin}) \cong \mathbb{Z} \oplus \Lambda$.

Next, (24) implies $[g] = (-1)^i[g_{i*}]$. Now note that $\Lambda(\mathbb{Q}) \to \Lambda(\mathbb{Z})$ is injective; so it suffices to calculate $[g_{i*}]$ rationally. Lastly, the equality

$$[g_{i*} \otimes \mathbb{Q}] = \dim_{\mathbb{Q}}(H_i^{sing}(X) \otimes \mathbb{Q}) \oplus \det{}_{\mathbb{Q}[t]}(id - g_i t \otimes \mathbb{Q})$$

follows from the formula at the bottom of p. 376 of [1].

□

*Remark* 5.5.2. 1. Note that the isomorphisms described are compatible with the natural ring structures of $K_0$-groups involved.

2. Assertion 1 doesn't seem to be new; yet the author doesn't know of any paper that contains assertion 2 in its current form.

3. One also has $K_0^{tr}(\operatorname{End}^n SH_{fin}) \cong K_0^{add}(\operatorname{End}^n Ab_{fin.fr})$; see Remark 5.4.7.

# 6 Twisted complexes over a negative differential graded category; Voevodsky's motives

The goal of this section is to apply our theory to triangulated categories that have differential graded enhancements of a certain sort (as considered in [11]); this will allow to use it for motives.

In §6.1 we recall the definitions of differential graded categories and twisted complexes over them. In 6.2 we consider negative differential graded categories; we obtain a weight structure on the category of twisted complexes (over them). In §6.3 we construct the so-called *truncation functors* $t_N$; $t_0$ is the *strong weight complex functor* for this case, see Conjecture 3.3.3.

In §6.4 we recall the spectral sequence $S(H, X)$ constructed in §7 of [11] for $H$ having a differential graded enhancement and prove that it could be obtained from $T(H, X)$ by decalage. In particular, this shows that $S$ does not depend on the choice of enhancements. We prove that *truncated realizations* for representable realizations are represented by the adjacent $t$-truncations of representing objects (see also §7.1).

In §6.5 we apply our theory to Voevodsky's motivic categories $DM_{gm}^{eff}$ and $DM_{gm}$; we calculate the heart of the corresponding *Chow* weight structures obtained in §6.6.

## 6.1 Basic definitions

We recall the basic definitions and results of the theory as they were presented in [11]; cf. also [6] and [10]

Categories of *twisted complexes* were first considered in [10]. However our notation differs slightly from those of [10]; some of the signs are also different.

An additive category $C$ is called graded if for any $P, Q \in ObjC$ there is a canonical decomposition $C(P,Q) \cong \oplus_i C_i(P,Q)$ defined; this decomposition satisfies $C_i(*,*) \circ C_j(*,*) \subset C_{i+j}(*,*)$. A differential graded category (cf. [10] or [6]) is a graded category endowed with an additive operator $\delta : C_i(P,Q) \to C_{i+1}(P,Q)$ for all $i \in \mathbb{Z}, P, Q \in ObjC$. $\delta$ should satisfy the equalities $\delta^2 = 0$ (so $C(P,Q)$ is a complex of abelian groups); $\delta(f \circ g) = \delta f \circ g + (-1)^i f \circ \delta g$ for any $P, Q, R \in ObjC$, $f \in C_i(P,Q)$, $g \in C(Q,R)$. In particular, $\delta(id_P) = 0$.

We denote $\delta$ restricted to morphisms of degree $i$ by $\delta^i$.

For an additive category $A$ one can construct the following differential graded categories.

We denote the first one by $S(A)$. We set $ObjS(A) = ObjA$; $S(A)_i(P,Q) = A(P,Q)$ for $i = 0$; $S(A)_i(P,Q) = 0$ for $i \neq 0$. We take $\delta = 0$.

We also consider the category $B^b(A)$ whose objects are the same as for $C^b(A)$ whereas for $P = (P_i)$, $Q = (Q_i)$ we define $B^-(A)(P,Q)_i = \oplus_{j \in \mathbb{Z}} A(P_j, Q_{i+j})$. Obviously $B^b(A)$ is a graded category. $B(A)$ will denote the unbounded analogue of $B^b(A)$.

We set $\delta f = d_Q \circ f - (-1)^i f \circ d_P$, where $f \in B_i(P,Q)$, $d_P$ and $d_Q$ are the differentials in $P$ and $Q$. Note that the kernel of $\delta^0(P,Q)$ coincides with $C(A)(P,Q)$ (the morphisms of complexes); the image of $\delta^{-1}$ are the morphisms homotopic to 0.

$B^b(A)$ can be obtained from $S(A)$ by means of the category functor Pre-Tr described below.

For any differential graded $C$ we define a category $H(C)$; its objects are the same as for $C$; its morphisms are defined as

$$H(C)(P,Q) = \operatorname{Ker} \delta^0_C(P,Q) / \operatorname{Im} \delta^{-1}_C(P,Q).$$

Having a differential graded category $C$ one can construct another differential graded category Pre-Tr$(C)$ as well as a triangulated category $Tr(C)$. The simplest example of these constructions is Pre-Tr$(S(A)) = B^b(A)$.

**Definition 6.1.1.** The objects of Pre-Tr$(C)$ are

$$\{(P_i),\ P_i \in ObjC, i \in \mathbb{Z}, q_{ij} \in C_{i-j+1}(P_i, P_j)\};$$

here almost all $P_i$ are 0; for any $i, j \in \mathbb{Z}$ we have

$$\delta q_{ij} + \sum_l q_{lj} \circ q_{il} = 0 \tag{25}$$

We call $q_{ij}$ *arrows* of degree $i-j+1$. For $P = \{(P_i), q_{ij}\}$, $P' = \{(P'_i), q'_{ij}\}$ we set

$$\text{Pre-Tr}_l(P, P') = \bigoplus_{i,j\in\mathbb{Z}} C_{l+i-j}(P_i, P'_j).$$

For $f \in C_{l+i-j}(P_i, P'_j)$ (an arrow of degree $l+i-j$) we define the differential of the corresponding morphism in $\text{Pre-Tr}(C)$ as

$$\delta_{\text{Pre-Tr}(C)} f = \delta_C f + \sum_m (q'_{jm} \circ f - (-1)^{(i-m)l} f \circ q_{mi}).$$

It can be easily seen that $\text{Pre-Tr}(C)$ is a differential graded category (see [10]). There is also an obvious translation functor on $\text{Pre-Tr}(C)$. Note also that the terms of the complex $\text{Pre-Tr}(C)(P, P')$ do not depend on $q_{ij}$ and $q'_{ij}$ whereas the differentials certainly do.

We denote by $Q[j]$ the object of $\text{Pre-Tr}(C)$ that is obtained by putting $P_i = Q$ for $i = -j$, all other $P_j = 0$, all $q_{ij} = 0$. We will write $[Q]$ instead of $Q[0]$.

Immediately from the definition we have $\text{Pre-Tr}(S(A)) \cong B^b(A)$.

A morphism $h \in \operatorname{Ker} \delta^0$ (a closed morphism of degree 0) is called a *twisted morphism*. For a twisted morphism $h = (h_{ij}) \in \text{Pre-Tr}((P_i, q_{ij}), (P'_i, q'_{ij}))$, $h_{ij} \in C(P_i, P'_j)$ we define $\text{Cone}(h) = P''_i, q''_{ij}$, where $P''_i = P_{i+1} \oplus P'_i$,

$$q''_{ij} = \begin{pmatrix} q_{i+1,j+1} & 0 \\ h_{i+1,j} & q'_{ij} \end{pmatrix}$$

We have a natural triangle of twisted morphisms

$$P \xrightarrow{f} P' \to \text{Cone}(f) \to P[1], \tag{26}$$

the components of the second map are $(0, id_{P'_i})$ for $i = j$ and 0 otherwise. This triangle induces a triangle in the category $H(\text{Pre-Tr}(C))$.

Now we define a certain distinguished triangles in $Tr(C)$ and a certain differential graded subcategory $\text{Pre-Tr}^+(C) \subset \text{Pre-Tr}(C)$.

**Definition 6.1.2.** 1. For distinguished triangles in $Tr(C)$ we take the triangles isomorphic to those that come from the diagram (26) for $P, P' \in \text{Pre-Tr}(C)$.

2. $\text{Pre-Tr}^+(C)$ is defined as a full subcategory of $\text{Pre-Tr}(C)$. $A = \{(P_i), q_{ij}\} \in Obj\,\text{Pre-Tr}^+(C)$ if there exist $m_i \in \mathbb{Z}$ such that for all $i \in \mathbb{Z}$ we have $q_{ij} = 0$ for $i + m_i \geq j + m_j$.

3. $Tr^+(C)$ is defined as $H(\text{Pre-Tr}^+(C))$; the definition of distinguished triangles is the same as for $Tr(C)$.

4. We will say that $\underline{C}$ admits a *differential graded enhancement* if it is equivalent to $Tr(C)$ for some differntial graded $C$.

We summarize the properties of of categories defined that are most relevant for the current paper. See [11] and [10] for the proofs.

**Proposition 6.1.3.** *I $Tr^+(C) \subset Tr(C)$ (the embedding is full) are triangulated categories.*

*II For any additive category $A$ there are natural isomorphisms*

*1.* $\text{Pre-Tr}(B(A)) \cong B(A)$.

*2.* $Tr(B(A)) \cong K(A)$.

*3.* $Tr(S(A)) \cong B^b(A)$

*III 1. There are natural embeddings of categories $i : C \to \text{Pre-Tr}^+(C)$ and $H(C) \to Tr^+(C)$ sending $P$ to $[P]$.*

*2.* Pre-Tr*, $Tr$,* Pre-Tr$^+$*, are $Tr^+$ are functors on the category of differential graded categories i.e. any differential category functor $F : C \to C'$ naturally induces functors* Pre-Tr$F$*, $TrF$,* Pre-Tr$^+ F$*, and $Tr^+F$.*

*4. Let $F : \text{Pre-Tr}^+(C) \to D$ be a differential graded functor. Then the restriction of $F$ to $C \subset \text{Pre-Tr}(C)$ gives a differential graded functor $FC : C \to D$. Moreover, since $FC = F \circ i$, we have* $\text{Pre-Tr}^+(FC) = \text{Pre-Tr}^+(F) \circ \text{Pre-Tr}^+(i)$*; therefore* $\text{Pre-Tr}^+(FC) \cong \text{Pre-Tr}^+(F)$*.*

*IV $Tr^+(C)$ as a triangulated category is generated by the image of the natural map $ObjC \to ObjTr^+(C) : P \to [P]$.*

For example, for $X = (P_i, q_{ij}) \in Obj\,\text{Pre-Tr}(C)$ we have $\text{Pre-Tr}F(X) = (F(P_i), F(q_{ij}))$; for a morphism $h = (h_{ij})$ of $\text{Pre-Tr}(C)$ we have $\text{Pre-Tr}F(h) = (F(h_{ij}))$. Note that the definition of Pre-Tr$F$ on morphisms does not involve $q_{ij}$; yet Pre-Tr$F$ certainly respects differentials for morphisms.

*Remark* 6.1.4. By definition, any morphism $g : A = (P_i, f_{ij}) \to B = (P'_i, f'_{ij})$ can be described as sets $(g_{ij}) \in C_{i-j}(P'_j)(P_i),\ i, j \in \mathbb{Z}$, where $g_{ij}$ satisfy

$$\delta_C g_{ij-1} + \sum_m (f'_{mj} \circ g_{im} - g_{mj} \circ f_{im}) = 0\ \forall i, j \in \mathbb{Z}. \tag{27}$$

Moreover, two sets of $g_{ij}$ give the same morphism whenever they are "homotopy equivalent" i.e. there exist $h_{ij} \in C_{i-j-1}(P'_j)(P_i), i \leq j$, such that

$$\delta_C h_{ij-1} + \sum_m (f'_{mj} \circ h_{im} - (-1)^{i-m} h_{mj} \circ f_{im}) = g_{ij}\ \forall i, j \in \mathbb{Z}.$$

## 6.2 Negative differential graded categories; a weight structure for $Tr(C)$

Suppose now that a differential graded category $C$ is *negative* i.e. for any $X, Y \in ObjC$ we have $i > 0 \implies C^i(X, Y) = \{0\}$ (cf. Definition 4.3.1).

For $\underline{C} = Tr(C)$ we define $\underline{C}^{w\le 0}$ as a set that contains all objects that are isomorphic to those that satisfy $P_i = 0$ for $i > 0$. $\underline{C}^{w\ge 0}$ is defined similarly by the condition $P_i = 0$ for $i < 0$.

**Proposition 6.2.1.** *1. $\underline{C}^{w\le 0}$ and $\underline{C}^{w\ge 0}$ give a non-degenerate weight structure for $\underline{C}$.*

*2. $Hw$ is isomorphic to the small envelope of $HC$ (cf. Definition 4.3.1).*

*Proof.* The definition of morphisms in $\underline{C}$ immediately yields that $\underline{C}(\underline{C}^{w\ge 0}, \underline{C}^{w\le 0}) = 0$. We obviously have $\underline{C}^{w\le 0}[1] \subset \underline{C}^{w\le 0}$; $\underline{C}^{w\ge 0} \subset \underline{C}^{w\ge 0}[1]$. The verification of the fact that $\underline{C}^{w\le 0}$ and $\underline{C}^{w\ge 0}$ are Karoubi-closed in $\underline{C}$ is straightforward. However we will never actually use this statement below (so we can replace $\underline{C}^{w\le 0}$ and $\underline{C}^{w\ge 0}$ described by their Karoubizations in the definition of $w$).

It remains to check that any object $X$ of $\underline{C}$ admits a weight decomposition. We follow the proof of Proposition 2.6.1 of [11].

We take $(P_i, f_{ij},\ i,j \le 0)$ as $X^{w\le 0}$ and $(P_i, f_{ij},\ i,j \ge 1)[1]$ as $X^{w\ge 1}$. We should verify that $X^{w\le 0}$ and $X^{w\ge 1}$ are objects of $\underline{C}$.

We have to check that the equality (25) is valid for $X^{w\le 0}$ (resp. $X^{w\ge 1}$). All terms of (25) are zero unless $i \le j \le 0$ (resp. $1 \le i \le j$). Moreover, in the case $i \le j \le 0$ (resp. $1 \le i \le j$) the terms of (25) are the same as for $X$. Both of these facts follow immediately from the negativity of $C$.

Now we verify that $(id_{P_i},\ i \le 0)$ gives a morphism $X \to X^{w\le 0}$ and $(id_{P_i},\ i \le 1)$ gives a morphism $X^{w\ge 1}[-1] \to X$. The condition (27) for these cases is obvious by the negativity of $C$.

Next we should check that $X \to X^{w\le 0}$ is the second morphism of the triangle corresponding to $X^{w\le 1}[1] \to X$; this easily follows from (26).

Lastly, $w$ is non-degenerate since any object of $\underline{C}$ is bounded from both sides (in the obvious sense).

2. Obviously, the objects of $HC$ belong to $\underline{C}^{w=0}$. Next, the definition of $\underline{C}$ easily yields that $Hw(X,Y) \cong HC(X,Y)$ for $X, Y \in HC$.

Moreover, part 1 implies that any object of $\underline{C}$ has a "filtration" by subobjects whose "successive factors" come from $HC$. By part II2 of Theorem 4.3.2 we obtain that $Hw$ is isomorphic to the small envelope of $HC$. $\square$

Obviously, the same construction also gives weight structures for all unbounded versions of $Tr(C)$.

*Remark* 6.2.2. 1. Alternatively, Proposition 6.2.1 could be deduced from part II of Theorem 4.3.2. In particular, this method easily deduces the fact that $\underline{C}^{w\le 0}$ and $\underline{C}^{w\ge 0}$ are Karoubi-closed from the assertion that the small envelope of $HC$ lies in both of them (cf. the beginning of the proof of Part II2 loc. cit.).

2. Let $\underline{C} = Tr^+(C)$ and $H^iC(X,Y) = 0$ for all $i > 0$, $X, Y \in ObjC$ (the cohomology of $C(-,-)$ is concentrated in non-positive degrees). Let $C_-$ be a (non-full!) subcategory of $C$ with the same objects and $C_-(X,Y) = C(X,Y)^{t\leq 0}$ (morphisms are the zeroth canonical truncation of those of $C$). Then by part 2 of Remark 2.7.4 of [11] the embedding $C_- \to C$ induces an equivalence of triangulated categories $Tr^+(C_-) \to Tr^+(C)$.

It follows: if $\underline{C} \cong Tr^+(C)$ and $C(-,-)$ is acyclic in positive degrees then we can assume $C$ to be negative. In particular, the strong (i.e. exact) weight complex functor $\underline{C} \to K^b(Hw)$ exists in this case.

## 6.3 Truncation functors; comparison of weight complexes

For $N \geq 0$, $P, Q \in ObjC$ we denote the $-N$-th canonical filtration of $C(P,Q)$ (i.e. $C_{-N}(P,Q)/d_P C_{-N-1}(P,Q) \to C_{-N+1}(P,Q) \to \dots \to C_0(P,Q) \to 0$) by $C^N(P,Q)$.

We denote by $C_N$ the following differential graded category. Its objects are the same as for $C$ whereas $C_N(P,Q)_i = C_i^N(P,Q)$. The composition of morphisms is induced by those in $C$. For morphisms in $C_N$ presented by $g \in C_i(P,Q)$, $h \in C_j(Q,R)$, we define their composition as the morphism represented by $h \circ g$ for $i + j \geq -N$ and zero for $i + j < -N$. Certainly, all $C_N$ are negative (i.e. all morphisms of positive degrees are zero).

We have an obvious functor $C \to C_N$. As noted in Proposition 6.1.3,this gives canonically a functor $t_N : \underline{C} \to Tr(C_N)$. We denote $Tr(C_N)$ by $\underline{C}_N$.

Obviously, objects of $\underline{C}_N$ could be represented as certain $(P_i, f_{ij} \in C^N_{i-j+1}(P_i, P_j), i < j \leq i + N + 1)$, the morphisms between $(P_i, f_{ij})$ and $(P'_i, f'_{ij})$ are represented by certain $g_{ij} \in C^N_{i-j}(P_i, P_j)$, $i \leq j \leq i + N$, etc. The functor $t_N$ "forgets" all elements of $C_m([P],[Q])$ for $P, Q \in ObjC$, $m < -N$, and factorizes $C_{-N}([P],[Q])$ modulo coboundaries. In particular, for $N = 0$ we get ordinary complexes over $HC$ i.e. $\underline{C}_0 = K^b(HC)$.

$t_0$ will be called the *strong weight complex* functor.

One could easily verify that the strong weight complex functor constructed is a lift of the weight complex functor $t$ corresponding to the weight structure $w$ to an exact functor $t^{st}$ (as in Conjecture 3.3.3). This follows immediately from the explicit description of $X^{w\leq 0}$ and $X^{w\geq 1}$ for any $X \in Obj\underline{C}$ (in the proof of Proposition 6.2.1).

**Conjecture 6.3.1.** *1. For a general $(\underline{C}, w)$ there also exist exact "higher truncation functors" $t_N$ such that $t_0$ is the "strong" weight complex functor; cf. Conjecture 3.3.3. Their targets $\underline{C}_N$ should satisfy: if $X, Y \in \underline{C}^{w=0}$ then $\underline{C}_N(t_N(X), t_N(Y)[-i]) = \underline{C}(X,Y)$ for $0 \leq i \leq N$ and $= \{0\}$ otherwise. These categories should admit full embeddings $i_N : \underline{C}^{[0,N]} \to \underline{C}_N$; distin-*

*guished triangles of $\underline{C}$ consisting of elements of $\underline{C}^{[0,N]}$ should be mapped to distinguished triangles by $i_N$.*

*2. Let $I : \underline{C} \to D(A)$ be an exact functor, where $\underline{C}, w$ is a triangulated category with a weight structure, $A$ is an abelian category. If $I(\underline{C}^{w=0}) \subset D_{[0,N]}(A)$ (i.e. acyclic for degrees outside $[0,N]$) then $I$ could be canonically factorized through $t_N$.*

## 6.4 The weight spectral sequence for enhanced realizations

The method of construction of the weight spectral sequences in [11] was somewhat distinct from the method we use here. In [11] we used a certain filtration on the complex that computes cohomology; that filtration could be obtained from the filtration corresponding to our current method by Deligne's decalage (see §1.3 of [13] or [27]). So the spectral sequence there was "shifted one level down" (in particular, it was functorial starting from $E_1$). We compare the methods here.

Let $J$ be some negative differential graded category, let $\mathfrak{H} = Tr(J)$, $J' =$ Pre-Tr$(J)$. Below we will use the same notation for Voevodsky's motives (which are the most important example of this situation).

In [11] weights were constructed only for (co)homological functors that admit an enhancement i.e. those that could be factorized through $Tr(F)$ for a differential graded functor $F : J \to C$. Here we consider only $C = B(A)$ for an abelian $A$ and homological functors of the form $H_{K_A} \circ Tr(F)$ (here $H_{K_A}$ denotes the zeroth cohomology functor for $C(A)$). The cohomological functor case was considered in §7.3 of [11] (certainly, reversing the arrows is no problem). Note that for those realizations for which $C \neq B(A)$ one can sometimes reduce the situation to the case $C = B(A')$ for a large $A'$ (for example, this seems to be the case for the Hodge realization). Alternatively, one could apply decalage to the spectral sequence coming from our current method.

Now we recall the formalism of [11] (modified for the homological functor case).

We denote the functor Pre-Tr$(F) : J' \to B(A)$ by $G$, denote $Tr(F) : \mathfrak{H} \to K(A)$ by $E$.

We recall that for a complex $Z$ over $A$, $b \in \mathbb{Z}$, its $b$-th canonical truncation from above is the complex $\dots Z_{b-1} \to \mathrm{Ker}(Z_b \to Z_{b+1})$, here $\mathrm{Ker}(Z_b \to Z_{b+1})$ is put in degree $b$.

For any $b \geq a \in \mathbb{Z}$ we consider the following functors. By $F_{\tau_{\leq b}}$ we denote the functor that sends $[P]$ to $\tau_{\leq b}(F([P]))$. These functors are differential graded; hence they extend to $G_i = \text{Pre-Tr}(F_{\tau_{\le -i}}) : J' \to B(A)$. Note that we consider the $-i$-th filtration here in order to make the filtration decreasing (which is usual when the decalage is applied); this is another minor distinction of the current exposition from those of [11]. The functors $Tr(F_{\tau_{\le -i}})$ were called truncated realizations in loc.cit.

Let $X = (P_i, q_{ij}) \in Obj J'$. The complexes $G_b(X)$ give a filtration of $G(X)$; one could also consider $G_{a,b}(X) = G_b(X)/G_{a-1}(X)$. We obtain the spectral sequence of a filtered complex (see §III.7.5 of[15])

$$S : E_1^{ij}(S) \implies H^{i+j}(G(X)) \tag{28}$$

we call it the *spectral sequence of motivic descent*. Here $E_1^{ij}(S) = H^{i+j}(G_{1-j}(X)/G_{-j}(X))$.

All $G_b(X)$ are $J'$-contravariantly functorial with respect to $X$. Besides, starting from $E_1$ the terms of $S$ depend only on the homotopy classes of $G_b(X)$. Hence starting from $E_1$ the terms of $S$ are functorial with respect to $X$ (considered as an object of $\mathfrak{H}$).

Now we compare the spectral sequences obtained using this method with the ones provided by Theorem 2.4.1. The comparison statement could be obtained as a certain extension of Theorem 4.4.2; see parts 5 and 4 of Remark 4.4.3.

To this end we compare the filtrations of $G(X)$ corresponding to $T$ and $S$. Fortunately, we don't have to write down the differential in $G$; it suffices to recall that $G_j(X) = \bigoplus_{k+l=j} F_k(P_l)$.

The method of Theorem 2.4.1 gives the following filtration on $G(X)$: $Q_i G_j(X) = \bigoplus_{k+l=j, l \ge i} F_k(P_l)$.

Now we apply decalage to this filtration. It is easily seen that we obtain the filtration given by $G_i$ i.e.

$$(DecQ)_i(G_j(X)) = \bigoplus_{k+l=j, l \ge j+i+1} F_k(P_l) \oplus \operatorname{Ker}(F_{-b}(P_{l+i}) \to F_{-b+1}(P_{l+i})).$$

Hence $T_{n+1}^{pq} = S_n^{-q,p+2q}$ for all integral $i, j$ and $n > 0$; the corresponding filtrations on the limit (i.e. on $H^{i+j}(E(X))$) coincide up to a certain shift of indices.

In §7.3 of [11] so-called *truncated realizations* were considered. They were defined as $Tr(F_{\tau \le b})$ and $Tr(F_{\tau \le b}/F_{\tau \le a-1}) : \mathfrak{H} \to K(A)$ (for $a \le b \in \mathbb{Z}$). The formula (15) of [11] computes all $E_n^{ij}(S)$ for $n \ge 1$ in terms of the weight filtration of truncated realizations of $X$; this description is $\mathfrak{H}$-functorial.

*Remark* 6.4.1. 1. Suppose now that there exists a differential graded functor $F_1 : J \to B(A)$ and a differential graded transformation $F_1 \to F$ such that the induced cohomology functor maps are isomorphisms in degrees $\le$

$b$ and are zero in degree $> b$. Let $F_2$ denote $F_{1\tau\leq b}$. We have a natural transformation $F_2 \to F_1$ which is an isomorphism on cohomology. Hence by part 2 of Corollary 2.7.2 of [11], the transformation of functors $Tr(F_2 \to F_1)$ induces quasi-isomorphisms of their values. Next, the transformation $F_1 \to F$ induces a transformation $F_2 \to F_{\tau\leq b}$. Applying part 2 of Corollary 2.7.2 of [11] again we obtain that $Tr(F_{\tau\leq b}) \approx Tr(F_2)$; hence both of them are quasi-isomorphic to $Tr(F_1)$.

In particular, let $A = Ab$; let $F$ be (contravariant) representable by some $Y$ in some differential graded $K \supset J$ such that $TrK$ possesses a weight structure extending $w$ and its adjacent $t$-structure $t$. Then our reasoning shows that the objects $Y^{t\leq -i}[i]$) represent the truncated realizations for $Tr(K(-,Y))$ (in $Tr(K) \supset \mathfrak{H}$; up to quasi-isomorphism i.e. they give the cohomology groups required). This is a differential graded version of part 6 of Theorem 4.4.2. Besides, in this case the fact that the filtrations induced by the morphisms $X^{w\leq i}[-i] \to X$ and by $Y^{t\leq -i}[i] \to Y$ coincide also follows from part 6 of Theorem 4.4.2.

So, the results of §7.1 below yield that the truncated realizations both for the "classical" (Weil) realizations of motives and for motivic cohomology are representable. This fact seems to be far from obvious.

2. As was mentioned in part 4 of Remark 4.4.3, one could deduce the comparison of spectral sequence statement in the representable case from the remark above and Theorem 4.4.2. Moreover, one could construct a "duality" of $\underline{C}, w$ with the $t$-structure on the category $Tr(DG-Fun(J, B(A)))$ (differential graded functors) that corresponds to the canonical truncation of $A$-complexes; see part 5 of Remark 4.4.3. The realizations of the type considered above correspond to some objects of this category; truncations of a realization with respect to this $t$-structure would be exactly its truncated realizations. So, one could compare spectral sequences by this method in the general case also.

## 6.5 $SmCor$, $DM_{gm}^{eff}$ and $DM_{gm}$; the "Chow" weight structure

We recall some definitions of [32].

$k$ will denote our perfect ground field; we will mostly assume that the characteristic of $k$ is zero. $pt$ is a point, $\mathbb{A}^n$ is the $n$-dimensional affine space (over $k$), $x_1, \dots, x_n$ are the coordinates, $\mathbb{P}^1$ is the projective line.

$Var \supset SmVar \supset SmPrVar$ will denote the class of all varieties over $k$, resp. of smooth varieties, resp. of smooth projective varieties.

We define the category of smooth correspondences: $ObjSmCor = SmVar$,

$SmCor(X,Y)=\sum_U \mathbb{Z}$ for all $U \subset X \times Y$ that are integral closed finite subschemes which are dominant (over a connected component of) $X$. The elements of $SmCor(X,Y)$ are called *finite correspondences* from $X$ to $Y$.

*Remark* 6.5.1. The composition of $U_1 \subset X \times Y$ and $U_2 \subset Y \times Z$ as in the definition of finite correspondences is defined as always in the categories of motives i.e. one considers the obvious scheme-theoretic analogue of $\{(x \in X, z \in Z) : \exists y \in Y : \ (x,y) \in U_1, \ (y,z) \in U_2\}$. Note that the composition is well-defined without any factorization by equivalence relations needed. Next one extends composition to all $SmCor(-,-)$ by linearity.

Note that this definition is compatible with the intuitive notion of composition of multivalued functions. Now, to $\sum c_i U_i \in SmCor(X,Y)$, $c_i \neq 0$ one could associate a multi-valued function whose graph is $\cup U_i$. Applying this method, one could define images and preimages of finite correspondences (and their restrictions).

$SmCor$ is additive: the addition of objects is given by the disjoint union operation for varieties.

$Shv(SmCor)$ is the abelian category of additive cofunctors $SmCor \to Ab$ that are sheaves in the Nisnevich topology.

$DM_-^{eff} \subset D^-(Shv(SmCor))$ is defined as the subcategory defined by the condition that the cohomology sheaves are homotopy invariant (i.e. $S(X) \cong S(X \times \mathbb{A}^1)$ for any $S \in SmVar$).

There is a natural functor $RC \circ L : K^b(SmCor) \to DM_-^{eff}$ (cf. Theorem 3.2.6 of [32]) given by Suslin complexes (see below); that could be factorized as a composition of the "localization by homotopy invariance and Mayer-Viertoris" and a full embedding; it categorical image will be denoted by $DM^s$. One can restrict $RC \circ L$ to obtain a functor $M_{gm} : SmVar \to DM^s$. Moreover, one could extend $M_{gm}$ to $Var$ (see §4.1 of [32]); unfortunately, in the case $\operatorname{char} k > 0$ one would have to take $DM_-^{eff}$ as the target of this (extended) $M_{gm}$. Therefore, cohomology of varieties could be expressed in terms of cohomology of motives.

$DM_-^{eff}$ is idempotent complete; hence it contains the idempotent completion of $DM^s$ which is Voevodsky's $DM_{gm}^{eff}$ (by definition; see [32]).

Now we define a differential category $J$ with $Obj J = SmPrVar$ (the addition of objects is the same as for $SmCor$. The morphisms of $J$ are given by *cubical Suslin complexes* $J_i(Y,P) \subset SmCor(\mathbb{A}^{-i} \times Y, P)$ consisting of correspondences that "are zero if one of the coordinates is zero". Being precise: consider $C_i'(P,Y) = SmCor(\mathbb{A}^{-i} \times Y, P)$ for all $P, Y \in SmVar$; note that $C_i'$ are zero for positive $i$. For all $1 \le j \le -i$, $x \in k$, we define $d_{ijx} = d_{jx} : C_i' \to C_{i+1}'$ as $d_{jx}(f) = f \circ g_{jx}$, where $g_{jx} : \mathbb{A}^{-i-1} \times Y \to \mathbb{A}^{-i} \times Y$ is induced by the map $(x_1, \dots, x_{-1-i}) \to (x_1, \dots, x_{j-1}, x, x_j, \dots, x_{-1-i})$. We define $J_i(Y,P)$ as $\cap_{1\le j\le -i}\operatorname{Ker} d_{j0}$. The boundary maps $\delta^i : J_i(-,-) \to J_{i+1}(-,-)$ are defined as $\sum_{1\le j\le -i}(-1)^j d_{j1}$.

The composition of morphisms in $J$ is induced by the obvious composition $C'_i(Y\times\mathbb{A}^{-j}, X\times\mathbb{A}^{-j})\times C'_j(Z,Y) \to C'_{i+j}(Z,X)$ combined with the embedding of $C'_i(Y,X)$ into $C'_i(Y\times\mathbb{A}^{-j}, X\times\mathbb{A}^{-j})$ via "tensoring" its elements by $id_{\mathbb{A}^{-j}}$; here $X,Y,Z\in SmPrVar,\ i,j\le 0$.

It was checked in §2 of [11] that $J$ is a differential graded category. It is negative by definition.

We denote $Tr(J)$ by $\mathfrak{H}$. $\mathfrak{H}$ is equivalent to $DM^s$ (if $\operatorname{char} k = 0$) by Theorem 3.1.1 of [11].

By Proposition 6.2.1 we obtain that there exists a weight structure $w$ in $\mathfrak{H}$; hence it also gives a weight structure for $DM^s$. We have $Hw = J'_0$ where $J'_0$ is the small envelope of $J_0 = HJ$ (cf. Definition 4.3.1 and part II2 of Theorem 4.3.2).

*Remark* 6.5.2. 1. In §4.1 of [32] the *motif with compact support* for any $X\in Var$ was defined as the Suslin complex of a certain sheaf $L^c(X)$. For a proper $X$ we have $M^c_{gm}(X) = M_{gm}(X)$. However, in order to increase the chances to obtain a geometric motif (with compact support) one could define $M^c_{gm}(X)$ using Poincare duality; see Appendix B of [20]. In the case $\operatorname{char} k = 0$ these definitions coincide and yield an object of $DM^s$ for any $X\in Var$.

In Theorem 6.2.1 of [11] it was proved that for a smooth $X$ we have $M_{gm}(X)\in DM^{sw\ge 0}$, $M^c_{gm}(X)\in DM^{sw\le 0}$. Using the blow-up distinguished triangle (see Proposition 4.1.3 of [32]) one could also show that for a proper $X$ we have $M_{gm}(X) = M^c_{gm}(X)\in DM^{sw\le 0}$.

2. As in part 3 of Remark 2.1.3 one could consider semi-motives $W_i(M_{gm}(X))$ and $W_i(M^c_{gm}(X))$ for all $i\in\mathbb{Z}$, $X\in Var$; they lie in $DM^{eff}_{gm}{}_*$. We obtain that $W_0(M_{gm}(X)) = M_{gm}(X)_*$ for proper $X$, whereas $W_{-1}(M_{gm}(X)) = 0$ for $X\in SmVar$. Recall that (11) allows to express the weight filtration on the cohomology of (the motif of) $X$ in terms of $W_i(M_{gm}(X))$.

In $DM^{eff}_{gm}$ we have a decomposition $[P^1] = [pt]\oplus\mathbb{Z}(1)[2]$ for $\mathbb{Z}(1)$ being the *Tate motif*. Moreover, $DM^{eff}_{gm}$ is a tensor category with $\otimes\mathbb{Z}(1)$ being a full embedding of $DM^{eff}_{gm}$ into itself (the *Cancellation Theorem*, see [32] and [36]). Hence one could define Voevodsky's $DM_{gm}$ as the direct limit of $DM^{eff}_{gm}$ with respect to tensoring by $\mathbb{Z}(1)$; it also could be described as the "union" of $DM^{eff}_{gm}(-i)$ (whereas each $DM^{eff}_{gm}(-i)$ is isomorphic to $DM^{eff}_{gm}$).

**Proposition 6.5.3.** *$w$ extends to a weight structure for $DM^{eff}_{gm}$ and $DM_{gm}$.*

*Proof.* I Extending $w$ to $DM^{eff}_{gm}$.

We define $DM_{gm}^{eff\,w\leq 0}$ as the set of retracts of $DM^{s,w\leq 0}$ in $DM_{gm}^{eff}$; the same for $DM_{gm}^{eff\,w\geq 0}$. By Proposition 5.2.2, this gives a weight structure on $DM_{gm}^{eff}$.

II Extending $w$ to $DM_{gm}$.

We note that tensoring by $\mathbb{Z}(1)[2]$ sends $[P]$ to a retract of $[P\times\mathbb{P}^1]$. Hence $\otimes\mathbb{Z}(1)[2]$ maps $DM_{gm}^{eff\,w\leq 0}$ and $DM_{gm}^{eff\,w\geq 0}$ into themselves. It follows that one can define $DM_{gm}^{w\leq 0}$ and $DM_{gm}^{w\geq 0}$ as $\cup DM_{gm}^{eff\,w\leq 0}(-i)[-2i]$ and $DM_{gm}^{eff\,w\geq 0}(-i)[-2i]$ respectively. Indeed, the Cancellation Theorem gives us orthogonality; since each object of $DM_{gm}$ belongs to $DM_{gm}^{eff}(-i)=DM_{gm}^{eff}(-i)[2i]$ for some $i\in\mathbb{Z}$, we also have the weight decomposition property.

$\square$

*Remark* 6.5.4. Note that (for any $\underline{C}$) if $w$ is bounded then $\underline{C}^{w\leq 0}$ consists exactly of objects that could be "decomposed" into a Postnikov tower as in (8) with $X_k=0$ for $k>0$; for $X\in\underline{C}^{w\geq 0}$ we can assume that $X_k=0$ for $k<0$.

Besides (see Proposition 6.2.1) for $\underline{C}=DM^s$ we can assume that all $X_k$ could be presented as $M_{gm}(P_k)$ for $P_k\in SmPrVar$. For $\underline{C}=DM_{gm}^{eff}$ or $\underline{C}=DM_{gm}$ we have $P_k\in ObjChow\subset ObjDM_{gm}$ (see §6.6 below).

We call the weight structure constructed the *Chow* weight structure (for any of $\mathfrak{H}$, $DM^s$, $DM_{gm}^{eff}$, $DM_{gm}$, and also for $DM_{-}^{eff}$ considered below).

Note that the same arguments prove the existence of weight structures on rational hulls of $DM^s$, $DM_{gm}^{eff}$ and $DM_{gm}$ (i.e. we tensor the groups of morphisms by $\mathbb{Q}$) as well as on their idempotent completions (which do not coincide with $DM_{gm}^{eff}\otimes\mathbb{Q}$ and $DM_{gm}\otimes\mathbb{Q}$).

## 6.6 The heart of the Chow weight structure

Now we calculate the hearts of $w$ in each of the categories constructed.

For our choice of $J$ we have $ObjJ=[P],\ P\in SmPrVar$, while $J_0([P],[Q])=DM_{gm}^{eff}([P],[Q])=Chow([P],[Q])$ (cf. section §4.2 of [32]). Hence the heart of $DM^s$ is the small envelope of the category $Corr_{rat}$ of effective rational correspondences (see Definition 4.3.1 and part II2 of Theorem 4.3.2). Note that the small envelope of $Corr_{rat}$ contains $\mathbb{Z}(1)[2]$ whereas $Corr_{rat}$ does not. Now Proposition 5.2.2 implies that the heart of $DM_{gm}^{eff}$ is the idempotent completion of $Corr_{rat}$ i.e. the whole category $Chow^{eff}$. Lastly, this easily implies that the heart of $DM_{gm}$ equals $Chow$.

We obtain that for **any** (co)homological functor from $DM_{gm}^{eff}$ (or $DM_{gm}$) there exist weight spectral sequences and weight filtrations. Note that we don't need any enhancements here (in contrast to [11])! Moreover, the weight spectral sequences are functorial with respect to natural transformations of (co)homological functors (we don't need transformations for enhancements).

Lastly we recall (from [11]) that the results obtained also concern motivic cohomology.

# 7 New facts on motives

The first subsection is dedicated to the study of $DM_-^{eff}$. We prove that the Chow weight structure extends to it; moreover $DM_-^{eff}$ supports a (right) adjacent *Chow* $t$-structure. It follows the Chow $t$-truncations of the objects that respresent the classical realizations of motives and motivic cohomology represent their *truncated realizations*; see Remark 6.4.1.

In §7.2 we note that (any) "mixed motivic" $t$-structure induces a canonical "weight filtration" on the values of the corresponding homological functor $DM_{gm}^{eff} \to MM$.

In §7.3 we prove that a certain (possibly, "infinite") weight complex functor could be defined for motives over any perfect field (without any resolution of singularities assumptions).

In §7.4 we apply the philosophy of adjacent structures to express the cohomology of a motif $X$ with coefficients in homotopy ($t$-structure) truncations of any $H \in ObjDM_-^{eff}$ in terms of the limit of $H$-cohomology of certain "submotives" of $X$. Luckily, to this end (instead of the "Gersten weight structure") it suffices to have the Gersten resolution for homotopy invariant pretheories (constructed in [33]). These calculations are closely related with the well-known calculations of the $E_2$-terms of the coniveau spectral sequence; see [9], [12], and [27].

In §7.5 we recall that (by the Beilinson-Lichtenbaum conjecture which was recently proved) torsion motivic cohomology is the truncation of the (torsion) étale one. Hence one can express torsion motivic cohomology of certain motives in terms of étale cohomology of their "submotives". In particular, we obtain a formula for the (torsion) motivic cohomology with compact support of a smooth quasi-projective variety.

## 7.1 "Chow" weight and $t$-structures for $DM_-^{eff}$

We recall (see §3 of [32]) that for any $S \in DM_-^{eff}$ and $X \in SmVar$ we have $DM_-^{eff}(M_{gm}(X), S) = \mathbb{H}^0(S)(X)$ (here $S$ is considered as a complex of sheaves). It follows (cf. 6.5) that $M_{gm}(X)$ for $X \in SmPrVar$ weakly generate $DM_-^{eff}$.

Now we take $\{C_i\} = ObjChow \subset ObjDM_-^{eff}$ (we can assume that $ObjChow$ is a set). We obtain that $(DM_-^{eff}, \{C_i\})$ satisfy the conditions of part I1 of Theorem 4.5.2. Hence it has is $t$-structure whose heart is $Chow_*$. Unfortunately, it seems that this $t$-structure cannot be restricted to $DM_{gm}^{eff}$ (i.e. it is not "geometric").

Now we check that the Chow weight structure of $DM_{gm}^{eff}$ could be extended to $DM_-^{eff}$. Till the end of this subsection $\underline{C} = DM_-^{eff}$, $t$ will denote the homotopy $t$-structure of $DM_-^{eff}$ (defined as in [32]). This is the $t$-structure corresponding to Nisnevich hypercohomology i.e. $X \in \underline{C}^{t\le 0}$ (resp. $X \in \underline{C}^{t\ge 0}$) whenever its Nisnevich hypercohomology is concentrated in non-positive (resp. non-negative) degrees. Note that in all $\underline{C}^{t\le i}$ arbitrary direct sums exist.

We define $\underline{C}^{w\le 0}$ as the Karoubi-closure in $\underline{C}$ of the "closure" of $DM_{gm}^{eff\,w\le 0}$ in $\underline{C}$ with respect to arbitrary direct sums and to "taking middle terms of distinguished triangles" (as in the proof of part II1 of Theorem 4.3.2). Note that $\underline{C}^{w\le 0} \subset \underline{C}^{t\le 0}$. We recover $\underline{C}^{w\ge 0}$ from $\underline{C}^{w\le 0}$ from the orthogonality condition in the usual way (cf. part 4 of Lemma 1.3.5). $\underline{C}^{w\ge 0}$ satisfies the "middle terms of distinguished triangles property" of part 1 of Lemma 1.3.5. Besides, it contains arbitrary direct sums of objects of $DM_{gm}^{eff\,w=0}$ (here we apply the compactness of objects of $DM_{gm}^{eff}$ in $\underline{C}$).

As usual, the only non-trivial axiom check here is the verification of the existence of weight decompositions. Recall that any object of $Shv(SmCor)$ has a "canonical resolution" by direct sums of $L(X) = SmCor(-, Y)$ for $Y \in SmVar$ (placed in degrees $\le 0$; see §3.2 of [32]). Hence any object $X$ of $DM_-^{eff}$ is a homotopy colimit of certain $X_i$ for the cone of $X_i \to X_{i+1}$ being a direct sum of some $M_{gm}(Y_{ij})[i]$; $X_l = 0$ for some $l \in \mathbb{Z}$. The limit of $X_i$ equals $X$ indeed by Lemma 4.2.5.

We construct $Z = X^{w\ge 1}$ as a homotopy colimit of $X_l^{w\ge 1}$ (see Definition 4.2.1). Note that the weight decomposition of $X_l^{w\ge 1}$ could be "constructed from" the weight decompositions of $\oplus M_{gm}(Y_{ij})[i]$ by Remark 1.5.5.

We should check that the colimit exists (i.e. is bounded from above). For any $Y \in SmVar$, $i > 0$, we have $(M_{gm}(Y)[i])^{w\ \ge 1} \in \underline{C}^{t\le 0}$ (for any choice of $(M_{gm}(Y)[i])^{w\ \ge 1}$). This is easy since $(M_{gm}(Y)[i]) \in \underline{C}^{t\le -i}$ and $(M_{gm}(Y)[i])^{w\le 0} \in \underline{C}^{w\le 0} \subset \underline{C}^{t\le 0}$. Combining these statements for all $Y_{ij}$ and $i$ yields the boundedness required.

We have the composition morphisms $X_l^{w\ge 1} \to X_l[1] \to X[1]$; by part 1 of Lemma 4.2.3 this system of morphisms could be lifted to some morphism $Z \to X[1]$. We should check that it yields a weight decomposition (if we make the choices in the construction in a "clever" way). By Lemma 4.2.4 we can assume that $Z \in \underline{C}^{w\ge 0}$. We denote a cone of $Z[-1] \to X$ by $Y$.

Now, it suffices (see part 1 of Remark 1.3.2) to check that the induced map $\underline{C}(C, Z) \to \underline{C}(Z, X[1])$ is an isomorphism for any $C \in \underline{C}^{w \geq 1}$ and is surjective for $C \in \underline{C}^{w \geq 0}$.

First suppose that for some $i \in \mathbb{Z}$ we have $\underline{C}(C, R) = \{0\}$ for any $R \in \underline{C}^{t \leq i}$. Then the sequence $\underline{C}(C[1], X_i)$ stablizes; this yields the result required by part 2 of Lemma 4.2.3. Hence for any such $C$ we have $\underline{C}(C, Y[1]) = \{0\}$.

We denote $C \in Obj\underline{C} : \{\underline{C}(C, Y[1] = \{0\})\} \cap \underline{C}^{w \geq 0}$ by $S$. Certainly, $S$ is closed with respect to arbitrary direct sums (in $\underline{C}$) and to "taking middle terms of distinguished triangles".

We have $DM_{gm}^{eff}{}^{w \geq 0} \subset S$. Indeed, any $C \in ObjDM_{gm}^{eff}$ is a direct summand of an object that could be obtained from (a finite number of) motives of smooth varieties by considering cones of morphisms; whereas for $X \in SmVar$ we have $DM_{-}^{eff}(M_{gm}(X), R) = \{0\}$ for any $R \in \underline{C}^{t \leq -\dim X - 1}$ (since the Nisnevich cohomological dimension of a scheme is not greater than its dimension). Next, all direct sums of objects of $DM_{gm}^{eff}{}^{w \geq 0}$ (belonging to $DM_{-}^{eff}$) also belongs to $S$. Therefore, it suffices to prove that any object of $\underline{C}^{w \geq 0}$ could be "approximated" by such direct sums.

By the same method as above, we present $C \in \underline{C}^{w \geq 0}$ as a homotopy colimit of certain $C_i$ for the cone of $C_i \to C_{i+1}$ being a direct sum of some $M_{gm}(E_{ij})[i]$; $C_l = 0$ for some $l \in \mathbb{Z}$.

Since any direct sum of distinguished triangles is a distinguished triangle, we can construct distinguished triangles $(\oplus M_{gm}(E_{ij})[i])^{\geq 0} \to A_i \to B_i$ for $A_i \in \underline{C}^{w \leq 0}$ and $B_i \in S$ (they will be direct sums of objects of $DM_{gm}^{eff}$). Next, applying Remark 1.5.5 for $D = \underline{C}^{w \leq 0}$, $E = S$, we can (starting from $C_l$) inductively construct distinguished triangles $C_i \to F_i \to G_i$ for $F_i \in \underline{C}^{w \leq 0}$, $G_i \in S$. We also construct distinguished triangles $C_i[-1] \to L_i \to M_i$ for $L_i \in \underline{C}^{w \leq 0}$, $M_i \in S$.

By Definition 4.2.1, we have a distinguished triangle $\oplus C_i \to \oplus C_i \to C$. Now note that $\oplus F_i$, $\oplus G_i$, $\oplus L_i$ and $\oplus M_i$ exist in $\underline{C}$ (since by the same arguments as the ones used above all of the summands belong to $\underline{C}^{t \leq l}$ for some $l \in \mathbb{Z}$). Since any direct sum of distinguished triangles in $\underline{C}$ is a distinguished triangle and $\underline{C}^{w \leq 0}$ and $S$ are closed with respect to all direct sums, we obtain distinguished triangles $\oplus C_i \to \oplus F_i \to \oplus G_i$ and $\oplus C_i[-1] \to \oplus L_i \to \oplus M_i$ with $\oplus F_i, \oplus L_i \in \underline{C}^{w \leq 0}$, $\oplus G_i, \oplus M_i \in S$.

Applying Remark 1.5.5 again we obtain a distinguished triangle $C[-1] \xrightarrow{f} U \to V$. for some $U \in \underline{C}^{w \leq 0}$ and $V \in S$. Hence $f = 0$; therefore $C$ is a direct summand of $V$. Thus $C \in S$.

*Remark* 7.1.1. Unfortunately, one cannot define a weight structure that would be left adjacent to the homotopy $t$-structure. Indeed, its heart should contain "motives of points" i.e. motives of (possibly, infinite) extensions of

$k$. However one could try to define the corresponding weight structure in a certain "closure" of $DM^{eff}_{gm}$ with respect to (certain) homotopy limits. Note that this weight structure would be closely connected with the Gersten resolutions of homotopy invariant pretheories and to the conievau filtration on cohomology; cf. Remark 7.4.2 below.

Still, the results of [33] allow as to prove certain analogues of parts 7,8 of Theorem 4 with cohomology of weight decompositions replaced by certain limits of cohomology. See part 6 of Remark 4.4.3 and §7.4 below.

## 7.2 Weight filtration for (conjectural) mixed motives

Suppose now that there exists so-called "mixed motivic" $t$-structure on $DM^{eff}_{gm}$ or $DM^{eff}_{gm}\mathbb{Q}$ (then one could extend it to $DM_{gm}$ and $DM_{gm}\mathbb{Q}$, respectively). We will not discuss any of its properties here; however it would automatically induce a homological functor $H_{MM}: DM^{eff}_{gm} \to MM$ for some abelian category $MM$ (of so-called mixed motives) that is the heart of the $t$-structure. Hence for any $X \in DM^{eff}_{gm}$ there will be a certain (weight) filtration on $H^i_{MM}(X)$ (cf. Remark 2.4.2). This filtration should be trivial (i.e. "canonical") when $X$ is smooth projective. It can be easily checked that there could exist only one filtration on $H_{MM,i}(X)$ which is $DM^{eff}_{gm}$-functorial and satisfies this property.

Moreover, any transformation $H_{MM} \to H$ for $H$ being a realization (of $DM^{eff}_{gm}$) with values in an abelian category would induce the transformation of the weight filtration for $H_{MM}$ to the weight filtration of $H$. Here the weight filtration of $H$ is defined by the weight structure method, yet it coincides with the "classical one" (cf. part 2 of Remark 2.4.2).

Therefore we obtain that our results will give the weight filtration for $H^i_{MM}(X)$ (an the corresponding weight spectral sequence) automatically when $H_{MM}$ will be defined. Note we don't need any information on $H_{MM}$ for this! However it seems difficult to prove that the filtration on $H^i_{MM}(X)$ depends only on the object $H^i_{MM}(X)$ and does not depend on the choice of possible $X$. To this end one possibly needs the degeneration of the weight spectral sequence for $H_{MM}$ and all $X$; this is probably one of the most important properties of the motivic $t$-structure.

## 7.3 Motives over perfect fields of finite characteristic

In our study of motives (here and in [11]) we applied several results of [32] that use resolution of singularities. So we had assume that the characteristic of the ground field $k$ is 0. In §8.3 of [11] it was shown that using de Jong's alterations one could extend most of our results to motives with rational coefficients over an arbitrary perfect $k$.

In this subsection (and also in all remaining parts of this section) we consider motives with integral coefficients over a perfect field $k$ of characteristic 0. Our goal is to justify a certain claim made in §8.3.1 of [11].

In [6] it was proved unconditionally that $DM^s$ has a differential graded enhancement. In fact, this fact could be easily obtained by applying Drinfeld's description of localizations of enhanced triangulated categories. Moreover, Proposition 5.6 of [6] extends the Poincare duality for Voevodsky motives to our case. Therefore for $P, Q \in SmPrVar$ we obtain

$$DM^s(M_{gm}(P), M_{gm}(Q)[i]) = Corr_{rat}([P], [Q]) \text{ for } i = 0;\ 0 \text{ for } i > 0.$$

Hence the triangulated subcategory $DM_{pr}$ of $DM^s$ generated by $[P]$, $P \in SmPrVar$ could be described as $Tr(I)$ for a certain negative differential graded $I$. In particular, we obtain the existence of a conservative weight complex functor $t_0 : DM_{pr} \to K^b(Corr_{rat})$. Moreover, for any realization of $DM_{pr}$ and any $X \in Obj DM_{pr}$ one has the weight spectral sequence $T$.

The problem is that (to the knowledge of the author) at this moment there is no way to prove that $DM_{pr}$ contains the motives of all smooth varieties (though it contains the motives of varieties that admit "smooth projective stratifications").

Instead we will prove that the weight structure on $DM_{pr}$ could be extended to a weight structure on a larger category containing all $M_{gm}(X)$.

Recall that $M_{gm}$ is a full embedding of $DM_{gm}^{eff} \supset DM_{pr}$ into $DM_{-}^{eff}$, whereas $DM_{-}^{eff} \subset D(Shv(SmCor))$ ($M_{gm}$ is denoted by $i$ in Theorem 2.3.6 of [32]). We denote by $D \subset D(Shv(SmCor))$ the full category of complexes with homotopy invariant hypercohomology. We have a full embedding $DM_{-}^{eff} \to D$.

We can extend to $D \subset D(Shv(SmCor))$ the assertion of Proposition 3.2.3 of [32] (i.e. construct a projection $D(Shv(SmCor)) \to D$ which is left adjoint to the embedding) using the fact that

$$D(Shv(SmCor))(D(Shv(SmCor))^{t\le 0}, D(Shv(SmCor))^{t\ge 1}) = 0.$$

Here $t$ denotes the usual $t$-structure of $D(Shv(SmCor))$ (corresponding to the homotopy $t$-structure for $DM_{-}^{eff}$). It follows that all objects of $M_{gm}(DM_{gm}^{eff})$ are compact. Indeed, it is sufficient to prove this for $DM_{gm}^{eff}([X])$ where $X \in SmVar$; Proposition 3.2.3 of [32] implies that $D(DM_{gm}^{eff}([X], -)$ is the corresponding hypercohomology functor which commutes with arbitrary direct sums.

Consider $S = \{X \in ObjD : \ D(Y,X) = \{0\} \ \forall Y \in ObjDM_{pr}\}$. Note that in the definition of $S$ it suffices to consider $Y = M_{gm}(P)[i]$, $P \in SmPrVar$, $i \in \mathbb{Z}$, since $[P]$ generate $DM_{pr}$. Obviously, $S$ is the class of objects for a certain full triangulated subcategory of $D(Shv(SmCor))$. We denote the localization of $D$ by $S$ by $D_S$. By definition of $S$, the set $H = \{[P],\ P \in SmPrVar\}$ weakly generates $D_S$. Since objects of $DM_{pr}$ are compact, $S$ is closed with respect to arbitrary direct sums. It follows that $D_S$ admits arbitrary direct sums. Note that $DM_{pr} \subset D_S$ by Proposition III.2.10 of [15]; hence we have a full embedding $Chow^{eff} \to D$.

By part I2 of Theorem 4.5.2 we obtain that $D_S$ supports adjacent weight and $t$-structures which we will call Chow ones. By part II of Theorem 4.5.2 we have $Ht_{Chow} = Chow^{eff}_*$. Moreover, $Hw$ is the category $Chow^{eff}_\oplus$ of arbitrary direct sums of effective Chow motives since $Chow^{eff}$ is idempotent complete.

Note that the definition of $w_{Chow}$ is compatible with the definition of the Chow weight structure on $DM_{pr}$. In particular, this reasoning extends the weight complex functor to a functor $D \to K_{\mathfrak{w}}(Chow^{eff}_\oplus)$. This would give a (possibly, infinite) weight complex for any $X \in ObjDM^{eff}_{gm}$. Recall that (by the results of §8.3.2 of [11]) $t(X)$ becomes (homotopy equivalent to) a finite complex after tensoring the coefficients by $\mathbb{Q}$. This weight complex functor could be "strengthened" (see part 2 of Remark 6.2.2) since $D(Shv(SmCor))$ has a differential graded enhancement.

## 7.4 Coniveau and truncated cohomology

Let $k$ be an arbitrary perfect field, $H \in ObjDM^{eff}_-$. We denote $\tau_{\le i}H$ by $H'$ ($i$ is fixed, $\tau$ is the $t$-truncation with respect to the homotopy $t$-structure). We denote by $H''$ the "complement of $H'$ to $H$" i.e. $H'' = \mathrm{Cone}(H' \to H)$. Note that the cohomology of $H''$ is concentrated in degrees $> i$. $j \in \mathbb{Z}$ will be a fixed integral number up to the end of the section.

Let

$$M = U_u \xrightarrow{d_u} U_{u-1} \xrightarrow{d_{u-1}} \dots U_1 \xrightarrow{d_1} U_0 \in ObjDM^{eff}_{gm} \tag{29}$$

be a complex in $SmCor$; $U_l$ is in degree $-l$. We demand that for all $r$, any (closed) point $u \in U_{r-1}$ the codimension of the preimage (in the sense of Remark 6.5.1) $\mathrm{codim}_{U_r} d_r^{-1}(u) \ge \mathrm{codim}_{U_{r-1}} u - 1$; here we define the codimension of a subvariety as the minimum of codimensions of its parts in the corresponding connected components.

We fix some $j \in \mathbb{Z}$, $DM^{eff}_-$ will be denoted by $\underline{C}$.

**Theorem 7.4.1.** *I Let $(Y_l^0, Y_l^1)$ run through open subschemes of $U_l$ such that: $U_l \setminus Y_k$ is everywhere of codimension $\ge j-i-k+1-l$ in $U_l$ ($k = 0, 1$,*

*$0 \le l \le u$), the images (in the sense of Remark 6.5.1) $d_l(Y_l^k) \subset d_l(Y_{l-1}^k)$ for all $k, l$. We define the motives $L^k = Y_u^k \to \dots \to Y_1^k \to Y_0^k$ for $k = 0, 1$ using the corresponding restrictions of $d_l$ (so if $Y_l^1 \subset Y_l^0$ for all $l$ then we have natural morphisms $L^1 \to L^0 \to M$).*

*Then we have an isomorphism*

$$\underline{C}(M, H''[j]) \cong \operatorname{Im}(\varinjlim \underline{C}(L^0, H[j]) \to \varinjlim \underline{C}(L^1, H[j])). \tag{30}$$

*Here connections between the cohomology of $L^k$ for different sets $(Y_l^k)$ are induced by open embeddings of varieties.*

*II Let $N^k = \operatorname{Cone}(L^k \to M) \in Obj DM_{gm}^{eff}$. For any $j$ we have:*

$$\underline{C}(M, H'[j]) \cong \operatorname{Im}(\varinjlim \underline{C}(N^0, H[j]) \to \varinjlim \underline{C}(N^1, H[j])), \tag{31}$$

*the limit is defined as in assertion I.*

*III 1. The isomorphisms described above are functorial in the obvious way with respect to "nice" morphisms of complexes of correspondences $(f_l) : M' \to M$. Here $M_l$ is a complex of $U_l'$, $(f_l)$ is nice if for any $l$, for any (closed) point $u \in U_l$ we have $\operatorname{codim}_{U_l'} f_l^{-1}(u) \ge \operatorname{codim}_{U_l} u$ (in the sense of Remark 6.5.1).*

*2. Furthermore, suppose that for some $(f_l)$ and fixed set of $Y_l^0 \subset U_l$ (satisfying the above conditions) we have $\operatorname{codim}_{U_l'} f_l^{-1}(U_l \setminus Y_l^0) \ge j - i - l$. Then the morphism $f_{H''}^* : H''(M) \to H''(M')$ (resp. $f_{H'}^* : H'(M) \to H'(M')$) is compatible with the natural morphism $\underline{C}(L^0, H[j]) \to \underline{C}(L^{1\prime}, H[j]))$ (resp. $\underline{C}(N^0, H[j]) \to \underline{C}(N^{1\prime}, H[j]))$) via the isomorphism of assertion I (resp. assertion II).*

*Proof.* I Shifting $H$ we easily reduce the statement to the case $i = 0$.

We claim that $\underline{C}(N^k, H''[r]) = 0$ for any $r \le j - k + 1$ and any $Y_l^k$.

It easily seen (by "slicing" $H[i]$ into its $t$-cohomology) that it suffices to prove a similar statement is valid for $H''$ replaced by any homotopy invariant $S \in Shv(SmCor)$ shifted by $v \le j - k$.

First let all $U_l$ except $U_t$ be empty (and all $d_l = 0$). Then our (last) assertion could be easily deduced from Lemma 4.36 of [33] (and some cohomological comparison results of Voevodsky) using the standard coniveau spectral sequence argument. We write down a (short) proof here. The classical coniveu spectral sequence could be found, for example, in [12]: for any $U \in SmVar$ by formula (1.2) ibid. there exists a spectral sequence

$$E_1^{p,q} = \coprod_{x \in U^{(p)}} H_{x,Zar}^{p+q}(U, S) \implies H_{Zar}^{p+q}(U, S);$$

here $U^{(p)}$ denotes the set of points of $U$ of codimension $p$, $H^{p+q}_{x,Zar}(U,S)$ is the local Zarisky cohomology group (see §4.6 of [33] and Lemma 1.2.1 of [12]). Note that this spectral sequence is functorial with respect to open embeddings. Now, Lemma 4.36 of [33] yields cohomological purity in this case; in particular, $H^{p+q}_{x,Zar}(U,S)=0$ for any $x\in U^{(p)}$ unless $q=0$. It follows that the map $H^v_{Zar}(U_t,S)\to H^v_{Zar}(Y^k_t,S)$ is bijective for $v<j-k-t$ and injective for $v=j-k-t$. Next, the Zarisky cohomology of $S$ coincides with its Nisnevich cohomology by Theorem 5.3 of [33], whereas the latter equals $\underline{C}(-,S[v])$ by Proposition 3.2.3 of [32]. Hence the long exact sequence of relative cohomology of $(Y^k_t,Y_t)$ yields our (last) claim in this case.

Now let $u=1$. We have an exact sequence

$$\{0\}=\underline{C}(Y^k_1\to U_1,S[v-1])\to\underline{C}(N^k,S[v])\to\underline{C}(Y^k_0\to U_0,S[v])=\{0\}$$

for $k=0,1$, $v\le j-k$; this yields the claim in this case. The case of $u>1$ could be easily obtained from similar exact sequences by induction.

Therefore we have exact sequences

$$\begin{aligned}\{0\}=\varinjlim\underline{C}(N^0,H''[j])\to\underline{C}(M,H''[j])\\ \to\varinjlim\underline{C}(L^0,H''[j])\to\varinjlim\underline{C}(N^0,H''[j+1])=\{0\}\end{aligned}\tag{32}$$

and

$$\{0\}=\varinjlim\underline{C}(N^1,H''[j])\to\underline{C}(M,H''[j])\to\varinjlim\underline{C}(L^1,H''[j])\to\dots\tag{33}$$

An argument similar to those above proves that $\varinjlim\underline{C}(L^1,H'[j+1])=\varinjlim\underline{C}(L^1,H'[j])=\varinjlim\underline{C}(L^0,H'[j+1])=\{0\}$. Here one should "slice" $H'$ into $t$-pieces and apply the coniveau spectral sequence arguments. To this end one should recall that the inductive limit of (long) exact sequences is exact. Besides, the codimension condition (on $U_l$) implies that for sets of $Y^k_l$ as in the assertion all (single) $Y^k_l$ could be "as small as possible". This means that for any open $Y\subset U_l$ such that $U_l\setminus Y$ is everywhere of codimension $\ge j-i-k+1-l$ in $U_l$ ($k=0,1$) could be completed to some set of $Y^k_l$; besides, we can intersect such sets (componentwisely).

Now we can argue similarly to the proof of assertion 7 of Theorem 4.4.2; note that our assertion is an analogue of part 8 of loc.cit.

We proved that the obvious morphism $\varinjlim\underline{C}(L^0,H[j])\to\varinjlim\underline{C}(L^0,H''[j])$ is surjective whereas $\varinjlim\underline{C}(L^1,H[j])\cong\varinjlim\underline{C}(L^1,H''[j])$.

Now we consider the commutative diagram (rows are not exact!):

$$\begin{array}{ccccc}0 & \longrightarrow & \varinjlim\underline{C}(L^0,H[j]) & \longrightarrow & \varinjlim\underline{C}(L^1,H[j])\\ \downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h}\\ \underline{C}(M,H''[j]) & \xrightarrow{t} & \varinjlim\underline{C}(L^0,H''[j]) & \xrightarrow{p} & \varinjlim\underline{C}(L^1,H''[j])\end{array}\tag{34}$$

We have proved that $t$ and $h$ are bijective, $g$ is surjective, and $p$ is injective. This immediately yields the assertion required.

II Having (32) and (33), we can argue exactly as in the proof of assertion 7 of Theorem 4.4.2 (and dually to the reasoning above).

III1. We describe the functoriality in question for "nice" $(f_l)$.

For $r = 0$ or 1 let $Y_l^r$ be fixed for all $l$. Then we can take $Y_l^{r\prime} = U_l' \setminus f_l^{-1}(U_l \setminus Y_l^r)$; $(f_i)$ induces morphisms $L'^r \to L^r$ and $N'^r \to N^r$. It remains to note that the proofs of assertions 1 and 2 are compatible with these morphisms.

III2. We check the compatibility desired for $H''$; the statement for $H'$ could be proved similarly (and dually in the categorical sense).

We should check the following. Let $v \in \underline{C}(M, H''[j])$ come from some $w \in \underline{C}(L^0, H[j])$ (for our fixed $Y_l^0$) via (30). Denote by (30') the isomorphism (30) with $M$ replaced by $M'$, $L^r$ replaced by $L^{r\prime}$. Then $f_{H''}^*(v)$ should be mapped via (30') to the image of $f_H^*(w)$ in $\varinjlim \underline{C}(M^{1\prime}, H[j])$. Here $f_H^*$ is the map $\underline{C}(L^0, H[j]) \to \underline{C}(L', H[j])$ induced by $(f_l)$, and $L'$ is the complex of $U_l' \setminus f_l^{-1}(U_l \setminus Y_l^0)$.

The latter fact follows easily from the commutativity of the diagrams

$$\begin{array}{ccccc}
\underline{C}(M, H''[j]) & \longrightarrow & \underline{C}(L^0, H''[j]) & \longrightarrow & \varinjlim \underline{C}(L^0, H''[j]) \\
\downarrow & & \downarrow & & \\
\underline{C}(L', H''[j]) & \longrightarrow & \underline{C}(L', H''[j]) & \longrightarrow & \varinjlim \underline{C}(L'^1, H''[j])
\end{array}$$

and

$$\begin{array}{ccc}
\underline{C}(L^0, H[j]) & \longrightarrow & \underline{C}(L^0, H''[j]) \\
\downarrow & & \downarrow \\
\underline{C}(L', H[j]) & \longrightarrow & \underline{C}(L', H''[j])
\end{array}$$

□

*Remark* 7.4.2. 1. The main differences of this result from the usual comparison of spectral sequences (as in [27]) is: we calculate the $D$-terms of the corresponding exact couple instead of the $E$-ones; we compute cohomology of certain motives (instead of varieties as in [9] and [12]).

2. Instead of applying assertion III2 to a single "nice" set of $Y_l^0$ one could consider a (directed) system of those. This is especially actual if the right parts of (30) or (31) could be calculated using such a "nice" directed subset of the set of all possible $(Y_l^0)$ (which is often the case). In this case part III2 allows to "calculate" $f_{H''}^*$ (resp. $f_{H'}^*$) completely.

3. One could generalize (31) in the following way. Let $r \geq i$, denote $\tau_{\leq r} H$ by $G$. Then for the corresponding map of cohomology theories $H' \to G$ we

have

$$\mathrm{Im}(G^j(M)) \to H'[j](M)) \cong \varinjlim \mathrm{Im}(H^j(N^0)) \to H^j(N^{r-i+1})). \qquad (35)$$

Here $N^{r-i+1}$ is defined similarly to $N^0, N^1$ in the theorem. This statement could be easily obtained by calculation of the $D$-terms of the higher derived couples for the coniveau spectral sequence; cf. Theorem 4.4.2.

4. Instead of considering limits of cohomology of motives we could have conisdered the cohomology of the corresponding pro-motives as it was done in §4 of [13]; this wouldn't have affected the proof substantially. Unfortunately, the category of pro-motives is not triangulated (if we define it in the obvious way).

5. In order to apply the weight structure formalism directly one would have to define a certain Gersten weight structure on a certain (more complicated) category of pro-motives which is triangulated (see Remark 7.1.1). This would probably allow (somehow) to get rid off the codimension condition for $d_l$.

6. As was noted in (part 6 of) Remark 4.4.3, all these statements could be vastly generalized.

## 7.5 Expressing torsion motivic cohomology (with compact support) in terms of étale one

For fixed $n > 0$, $(l, p) = 1$, $r \geq 0$ we denote by $\mathcal{H}^{et}_{l^n}(r) \in Obj D^-(Shv(SmCor))$ some étale resolution of $\mu_{l^n}^{\otimes r}$ by injective étale sheaves with transfers. This object does not depend on the choice of resolution by obvious reasons (it is the total derived image of $\mu_{l^n}^{\otimes r}$ with respect to the corresponding change of topologies functor). $\mathcal{H}^{et}_{l^n}(r)$ is homotopy invariant, so we can substitute it for $H$ in the statements above (since for any fixed $M$ it suffices to consider some homotopy $t$-truncation of $\mathcal{H}^{et}_{l^n}(r)$, whereas the latter belong to $Obj DM^{eff}_-$).

**Proposition 7.5.1** (The Beilinson-Lichtenbaum Conjecture)**.** *The (well-known) cycle class map $\mathbb{Z}/l^n\mathbb{Z}(r) \to \mathcal{H}^{et}_{l^n}(r)$ identifies the former object with $\tau_{\leq r}\mathcal{H}^{et}_{l^n}(r)$.*

We recall that this statement is equivalent to the Bloch-Kato Conjecture (see [29] and [14]). The latter is known for $l = 2$ (see [35]); so reader may assume that the next result is stated for $l = 2$. For odd $l$ it was was recently announced (by Rost and Voevodsky, and (independently) by Weibel) that the missing details for the famous Voevodsky's plan of the proof of the conjecture are completed.

For a motif $X$ we denote $DM_-^{eff}(X,\mathbb{Z}/l^n\mathbb{Z}(s)[i])$ by $H^i(X,\mathbb{Z}/l^n\mathbb{Z}(s))$; $H^i_{et}(X,\mathbb{Z}/l^n\mathbb{Z}(s))=D^-(Shv(SmCor))(X,\mathcal{H}^{et}_{l^n}(s)[i])$.

Now, Theorem 7.4.1 easily yields the following statement (in the notation of loc.cit.).

**Corollary 7.5.2.** *1. For $M$ as in (29), we have*

$$H^j(M,\mathbb{Z}/l^n\mathbb{Z}(s))\cong \operatorname{Im}(\varinjlim H^j_{et}(N^0,\mathbb{Z}/l^n\mathbb{Z}(s))\to \varinjlim H^j_{et}(N^1,\mathbb{Z}/l^n\mathbb{Z}(s))). \tag{36}$$

*The corresponding functor $H''_s$ could be calculated as follows:*

$$H''^j_s(M)\cong \operatorname{Im}(\varinjlim H^j_{et}(L^0,\mathbb{Z}/l^n\mathbb{Z}(s))\to \varinjlim H^j_{et}(L^1,\mathbb{Z}/l^n\mathbb{Z}(s))).$$

*These isomorphisms satisfy those functoriality properties that were described in part III of Theorem 7.4.1.*

*2. Let $Y_h$, $1\le h\le u$, $u>0$, be smooth of the same dimension; let $Y=\cup Y_h$ be a normal crossing scheme i.e. all intersections (in some large basic scheme) are normal and smooth. Consider the motif $M$ corresponding to the complex $(U_l)$; here $U_l=\sqcup_{(i_j)}Y_{i_1}\cap Y_{i_2}\cap\dots\cap Y_{i_{l+1}}$ for all $1\le i_1\le\dots\le i_{r+1}\le u$, $d_l$ is the alternated sum of $l+1$ natural maps $U_l\to U_{l-1}$.*

*Let $(Y^0,Y^1)$ run through open subschemes of $Y$ such that $Y\setminus Y_r$ is (everywhere) of codimension $\ge j-r-s+1$ in $Y$ $(r=0,1)$. Then we have*

$$H''^j_s(M)\cong \operatorname{Im}(\varinjlim H^j_{et}(Y^0,\mathbb{Z}/l^n\mathbb{Z}(s))\to \varinjlim H^j_{et}(Y^1,\mathbb{Z}/l^n\mathbb{Z}(s))). \tag{37}$$

*For $N^r=M_{gm}(Y^r\to Y)$ $(Y$ is in degree $0)$ we have*

$$H^j(M,\mathbb{Z}/l^n\mathbb{Z}(s))\cong \operatorname{Im}(\varinjlim H^j_{et}(N^0,\mathbb{Z}/l^n\mathbb{Z}(s))\to \varinjlim (H^j_{et}(N^1,\mathbb{Z}/l^n\mathbb{Z}(s))). \tag{38}$$

*3. Let $Y'=\cup Y'_i$, $M'$ is defined similarly to $M$, let $f:Y'\to Y$ be a morphism of schemes, suppose that for any (closed) point $u\in Y$ we have* $\operatorname{codim}_{Y'}f^{-1}(u)\ge \operatorname{codim}_Y u-1$. *Then the morphisms $f^*_{H'_s}$ and $f^*_{H''_s}$ could be computed by the way described in part III2 of Theorem 7.4.1 (see also its proof and part 2 of Remark 7.4.2).*

*4. Let $U\in SmVar$ equal $P\setminus Y=\cup Y_i, 1\le i\le m$, where $P\in SmPrVar$, $Y$ is a smooth normal crossing divisor. Then for any cohomological functor $G$ defined on $DM_-^{eff}$ we have a long exact sequence*

$$\dots G^j(M^c_{gm}(U))\to G^j(X)\to G^j(M)\to\dots \tag{39}$$

*Proof.* 1. This is immediate from part 2 of Theorem 7.4.1 applied for $H=\mathcal{H}^{et}_{l^n}(s)$, $i=s$.

2. We should prove that the formulas (37) and (38) compute the limits described in parts 1,2 of Theorem 7.4.1.

First, we note that if $Y^r$ $(r = 0, 1)$ satisfies the condition of the assertion then the set

$$Y_l^r = \sqcup_{(i_j)} Y^r \cap Y_{i_1} \cap Y_{i_2} \cap \cdots \cap Y_{i_{l+1}} \tag{40}$$

satisfies the conditions of part 1 of Theorem 7.4.1. Next, using proper descent we easily obtain that the étale cohomology of $Y^r$ is isomorphic to those of $L^r = (Y_l^r)$. It suffices to note that any set of $Y_l^r$ as in part 1 of Theorem 7.4.1 could be shrunk to a one coming from some $Y^r$ as in (40).

3. It suffices to note that the functoriality provided by part III2 of Theorem 7.4.1 is compatible with those of the formulas (37) and (38).

4. By definition, $H_c^j(U, \mathbb{Z}/l^n\mathbb{Z}(s)) = H^j(M_{gm}^c(U), \mathbb{Z}/l^n\mathbb{Z}(s))$. Hence it suffices to recall that $M_{gm}^c(U) \cong \mathrm{Cone}(M \to M_{gm}(X))$. The latter fact in the characteristic 0 case is Proposition 6.5.1 of [11]. In the characteristic $p$ case one could deduce the statement from the results of [13] (and the Poincare duality properties).

$\square$

*Remark* 7.5.3. 1. Note that for $G = H'$ or $G = H''$ one could compute the map $G^*(M) \to G^*(Y)$ in (39) using part 3 of the Corollary.

Besides, one could write down the formula for the (motivic and $H''$-) cohomology of $M_{gm}^c(U)$ by substituting the "complex" $Y \to X$ for $M$ into part 1 of the Corollary; here one should ignore the fact that $Y$ could be singular.

2. Let $K$ contain a primitive $l^n$-th root of unity. Then using it one can identify all $\mathcal{H}_{l^n}^{et}(r)$, and so obtain certain maps $H^i(-, \mathbb{Z}/l^n\mathbb{Z}(s)) \to H^{i+j}(-, \mathbb{Z}/l^n\mathbb{Z}(s + j))$ (induced by the multiplication on the corresponding motivic Bott elements, as in [21]). Then (35) allows to calculate the image of these maps.

One could prove natural analogues of part 2 of the Corollary and part 1 of this Remark.

3. It seems very interesting to replace étale cohomology in the right part of (38) by singular cohomology (in the case $\mathrm{char}\, k = 0$). One could easily deduce from Corollary 7.4 of [9] that in the case $u = 0$, $j = 2s$, the formula would calculate the group of algebraic cycles in $U_0$ of codimension $s$ modulo algebraic equivalence. So it seems that homotopy $t$-structure truncations of singular cohomology should be related to a certain (non-existent yet) "theory of mixed motives up to algebraic equivalence"; see the end of [37]. The author plans to study this subject in a forthcoming paper.

4. Certainly, the cohomology of $M$ is a very natural candidate for the cohomology of $Y$; note that $\cup Y_j \to Y$ is a *cdh*-covering (see [34]). Yet

this does not automatically imply the isomorphism of cohomologies for all "reasonable" cohomology theories.

5. Recall that if $k$ admits resolution of singularities any smooth quasi-projective $U$ could be presented as $X \setminus \cup Y_i$.

# 8 Supplements

In §8.1 we show that a weight structure $w$ on $\underline{C}$ which induces a weight structure on a triangulated $\underline{D} \subset \underline{C}$ yields also a weight structures on the localization $\underline{C}/\underline{D}$.

In §8.2 we show that functors represented by compositions of $t$-truncations with respect to (possibly) distinct $t$-structures could be expressed in terms of the corresponding adjacent weight structures; the formulas are similar to those of Theorem 4.4.2 (and are deduced from it). We also extend this result to the case of several triangulated categories connected by exact functors.

In §8.3 we prove (using an argument due to A. Beilinson) that any $f$-category enhancement of $\underline{C}$ yields a lift of $t$ to a "strong" weight complex functor $\underline{C} \to K(Hw)$; cf. Remark 3.3.4.

In §8.4 we discuss other possible sources of conservative "weight complex-like" functors and related spectral sequences.

## 8.1 Weight structures in localizations

We call a category a localization of an additive category $A$ by its full additive subcategory $B$ (we denote it by $\frac{A}{B}$) if $Obj\big(\frac{A}{B}\big)= ObjA$ and $(\frac{A}{B})(X,Y) = A(X,Y)/(\sum_{Z\in ObjB} A(Z,Y) \circ A(X,Z))$.

**Proposition 8.1.1.** *1. Let $\underline{D} \subset \underline{C}$ be a strict triangulated subcategory of $\underline{C}$; suppose that $w$ induces a $t$-structure on $\underline{D}$ i.e. $Obj\underline{D} \cap \underline{C}^{w\le 0}$ and $Obj\underline{D} \cap \underline{C}^{w\le 0}$ give a weight structure for $\underline{D}$. We denote the heart of the latter weight structure by $HD$.*

*Then $w$ induces a weight structure on $\underline{C}/\underline{D}$ (the localization of $\underline{C}$ with respect to $\underline{D}$). This means that the Karoubi-closures of $\underline{C}^{w\le 0}$ and $\underline{C}^{w\le 0}$ (in $\underline{C}/\underline{D}$) give a weight structure for $\underline{C}/\underline{D}$ (note that $Obj\underline{C} = Obj\underline{C}/\underline{D}$).*

*2. $H(\underline{C}/\underline{D})$ is the Karoubi-closure of $\frac{Hw}{HD}$ in $\underline{C}/\underline{D}$.*

*3. If $\underline{C}, w$ is bounded (above, below, or both), then $\underline{C}/\underline{D}$ also is.*

*Proof.* 1. It clearly suffices to prove that for any $X \in (\underline{C}/\underline{D})^{w\ge 0}$ and $Y \in (\underline{C}/\underline{D})^{w\le -1}$ we have $(\underline{C}/\underline{D})(X,Y) = \{0\}$; all other axioms of Definition 1.1.1 are fulfilled automatically since $\underline{C}/\underline{D}$ is a localization of $\underline{C}$.

Recall now (see Lemma III.2.8 of [15]) that any morphism in $(\underline{C}/\underline{D})(X,Y)$ can be presented as $fs^{-1}$ where $f \in \underline{C}(T,Y)$ for some $T \in Obj\underline{C}$, $s \in \underline{C}(T,X)$, $\mathrm{Cone}(s) = Z \in Obj\underline{D}$.

By our assertion, there exists a choice of $Z^{w\geq 0}$ that belongs to $Obj\underline{D}$. Since $\underline{C}(X, w_{\leq -1}(Z)) = \{0\}$ we can factorize the morphism $X \to Z$ (induced by $s$) through $Z^{w\geq 0}$.

Hence (applying the octahedron axiom) we obtain that there exist $T' \in Obj\underline{C}$, a morphism $d : T' \to T$, such that $\mathrm{Cone}\, d = w_{\leq -1}(Z) \in Obj\underline{D}$, whereas a cone of the composite morphism $s' : T' \to X$ equals $Z^{w\geq 0}$. It follows that $fs^{-1} = (fd)s'^{-1}$ in $\underline{C}/\underline{D}$. Now note that $T' \in \underline{C}^{w\geq 0}$ by part 3 of Proposition 1.3.1). Hence $\underline{C}(T,Y) = \{0\}$, which yields $fd = 0$.

2. By construction, $\underline{C}^{w=0} \subset (\underline{C}/\underline{D})^{w=0}$.

Now we prove that any object of $H(\underline{C}/\underline{D})$ is a retract of an object of $Hw$ (in $\underline{C}/\underline{D}$).

Let $Z \in \underline{C}/\underline{D}^{w=0} \subset Obj\underline{C}$. We consider a weight decomposition $w_{\geq 1}(Z) \to Z \to w_{\leq 0}(Z)$ of $Z$ in $\underline{C}$. In $\underline{C}/\underline{D}$ we have $w_{\geq 1}(Z) \in \underline{C}/\underline{D}^{w\geq 1}$, hence $\underline{C}/\underline{D}(w_{\geq 1}(Z), Z) = \{0\}$. Therefore $Z$ in $\underline{C}/\underline{D}$ is a retract of $Z^{w\leq 0}$. Moreover, $Z^{w\geq 1} \in \underline{C}/\underline{D}^{w=0}$ since it is a retract of $Z^{w\leq 0} \in \underline{C}/\underline{D}^{w\leq 0}$; therefore $Z^{w\leq 0} \in \underline{C}/\underline{D}^{w=0}$. Now applying the dual argument to $Z^{w\leq 0}$ (see Remark 1.1.2), we obtain that $Z$ in $\underline{C}/\underline{D}$ is a retract of some $Z^0 \in Obj\underline{C}^{w=0}$.

To conclude the proof it suffice to check that the natural functor $i : Hw/HD \to H(\underline{C}/\underline{D})$ is a full embedding. We consider the composition $\underline{C} \xrightarrow{t} K_{\mathfrak{w}}(Hw) \to K_{\mathfrak{w}}(Hw/HD)$. Obviously, it maps all objects of $\underline{D}$ to 0. Hence $i$ is injective on morphisms.

It remains to prove that any morphism $g : X \to Y$ in $\underline{C}/\underline{D}$ comes from $\underline{C}(X,Y)$. Applying the same argument as in the proof of assertion 1 we obtain that $g$ could be presented as $fs^{-1}$ where $f \in \underline{C}(T,Y)$ for some $T \in Obj\underline{C}$, $s \in \underline{C}(T,X)$, $\mathrm{Cone}\, s = Z \in \underline{D}^{w\geq 0}$. Then $\underline{C}(X,Y)$ surjects onto $\underline{C}(T,Y)$. Now the "calculus of fractions" yields the result.

3. Since $Obj\underline{C}/\underline{D} = Obj\underline{C}$, we obtain the claim.

$\square$

**Corollary 8.1.2.** *Let $E \subset Hw$ be an additive subcategory. If $X$ belongs to the Karoubi-closure $Obj\langle E\rangle$ then $t(X)$ is a retract of some object of $K^b_{\mathfrak{w}}(E)$ (here we mean that $K^b_{\mathfrak{w}}(E) \subset K_{\mathfrak{w}}(HC)$).*

*If $(\underline{C}, w)$ is bounded then the converse implication also holds.*

*Proof.* We can assume that $X \in Obj\langle E\rangle$. Then $X$ could be obtained from objects of $E$ by repetitive consideration of cones of morphisms. Since $t(ObjE) \subset ObjK_{\mathfrak{w}}(E)$ and $t$ is a weakly exact functor in the sense of Definition 3.1.6 we obtain that $t(X) \in ObjK^b_{\mathfrak{w}}(E)$.

Conversely, let $t(X)$ be a retract of $Y \in Obj K^b_{\mathfrak{w}}(E) \subset Obj K^b_{\mathfrak{w}}(Hw)$. By Proposition 8.1.1 we obtain that $\underline{C}/\langle E\rangle$ possesses a bounded weight structure whose heart contains $\frac{Hw}{E}$ as a full subcategory. Hence, by part V of Theorem 3.3.1 we obtain that $t_{\underline{C}/\langle E\rangle}$ is conservative. $Y \in Obj K^b_{\mathfrak{w}}(E)$ gives $t_{\underline{C}/\langle E\rangle}(Y) = 0$, hence $X$ and $Y$ belong to the Karoubi-closure of $\langle E\rangle$.

□

*Remark* 8.1.3. 1. Note that (in general) one cannot be sure in general that the "factor weight structure" on $\underline{C}/\underline{D}$ is non-degenerate.

2. Corollary 8.1.2 is parallel to part 3 of Proposition 8.2.1 of [11]. In particular, it could be used to prove that a motif of a smooth variety is mixed Tate whenever its weight complex (defined in [16]) is (that is Corollary 8.2.2 of [11]).

3. Adding certain additional restrictions, one could also formulate a criterion for $t(X)$ to belong to the Karoubi-closure of $Obj K_{\mathfrak{w}}(E)$ (instead of $Obj K^b_{\mathfrak{w}}(E)$).

## 8.2 Composing t- and weight truncations

Now suppose that $\underline{C}$ is endowed with some weight structures $w_i$, $1 \le i \le m$, such that there exist right adjacent $t$-structures $t_i$.

Then applying parts 7, 8 of Theorem 4.4.2 one can easily and naturally express the functors represented by all possible compositions of $t_i$-truncations as certain images (as in loc.cit).

For example, applying part 7 of loc.cit. twice we obtain

$$\underline{C}(X, \tau_{1,\le i}(\tau_{2,\le j}Y)) \cong \operatorname{Im}(\underline{C}(w_{1,\le i}(w_{2,\le j}X), Y) \to \underline{C}(w_{1,\le i+1}(w_{2,\le j+1}X, Y)); \tag{41}$$

for all $i, j \in \mathbb{Z}$; this isomorphism is functorial in both $X$ and $Y$. Here (and in the proof of more complicated formulas) we recall that morphisms of objects can be (non-uniquely) extended to morphisms of their weight decompositions. In these formulas one can also shift $t$-truncations by $[l]$, $l \in \mathbb{Z}$, and compose truncations from different sides.

Now we somewhat extend these result. Note that a duality as in part 5 of Remark 4.4.3 could be given by $\Phi(X, Y) = \underline{C}(X, F(Y))$, where $F : \underline{D} \to \underline{C}$ is an exact functor; in this case $A = Ab$. The case of adjacent structures then corresponds to $F = id_{\underline{C}}$.

So, suppose that $\underline{C}$, $\underline{D}$, and $\underline{E}$ are triangulated categories, let $F : \underline{D} \to \underline{C}$, $G : \underline{E} \to \underline{D}$ be exact functors; let $W_1$, $w_2$ be weight structures for $\underline{C}$, $t_1$ be a $t$-structure for $\underline{D}$ and $t_2$ be a $t$-structure for $\underline{E}$, suppose that they satisfy the orthogonality conditions: $\underline{C}(\underline{C}^{w_1\le 0}, F(\underline{D}^{t_1\ge 1})) = \underline{C}(\underline{C}^{w_1\ge 1}, F(\underline{D}^{t_1\le 0})) =$

$\underline{C}(\underline{C}^{w_2\le 0}, F\circ G(\underline{E}^{t_2\ge 1})) = \underline{E}(G\circ F(\underline{C}^{w_2\ge 1}), \underline{E}^{t_2\le 0}) = 0$. Then one can express $\underline{C}(X, F(\tau_{1,\le i}G(\tau_{2,\le j}Y))$ for $Y\in Obj\underline{E}$ as a certain image similar to (41).

Certainly, one can also consider compositions of more than two exact functors.

## 8.3 A strong weight complex functor for triangulated categories that admit an $f$-triangulated enhancement

Now we check that the strong weight complex functor $t$ exists if there exists an $f$-category enhancement of our category (we will define this notion very soon); see part 3 of Remark 3.3.4. The argument below was kindly communicated to the author by prof. A. Beilinson. To make our notation compatible with those of [7] we will denote our basic triangulated category (which is usually $\underline{C}$) by $D$. As usual, $D$ is endowed with a weight structure $w$.

The plan of the construction is the following one. Suppose that there exists an $f$-category $DF$ over $D$. In particular, this yields the existence of the "forgetting of filtration" functor $\omega:\ DF\to D$. We describe a class of objects $DF^s\subset ObjDF$ such that:

(i) any object of $X\in ObjD$ could be "lifted" to an element of $X^*\in DF^s$;

(ii) For every $M,N\in DF^s$ the map $DF(N,M)\to D(\omega(N),\omega(M))$ is surjective;

(iii) There exists a functor $e: DF\to C^b(D)$ such that $e(DF^s\subset Obj(C^b(Hw))$ and for any $M,N\in DF^s$ the functor $e$ maps $\operatorname{Ker} DF(N,M)\to D(\omega(N),\omega(M))$ to morphisms that are homotopic to 0.

We will denote the induced functor $DF\to K^b(D)$ by $e'$.

Then $X\to e(X^*)$ yields an additive functor $T:D\to K$ where $K$ is a certain triangulated category isomorphic to $K^b(Hw)$. Indeed, by (ii) any two choices of $X'$ are connected by (possibly, non-unique) morphisms. By (iii) these morphisms become canonical isomorphisms after the application of $e'$. Hence it suffices to take $K$ being the category obtained from $K^b(Hw)$ by factorizing by these isomorphisms. Indeed, this family respects direct sums since $\omega$ and $e$ do.

Now we recall the relevant definitions of the Appendix of [7].

**Definition 8.3.1.** I A triangulated category $DF$ will be called a filtered triangulated one if it is endowed with strict triangulated subcategories $DF(\le 0)$ and $DF(\ge 0)$; an exact autoequivalence $s: DF\to DF$; and a morphism of functors $\alpha: id_{DF}\to s$, such that the following axioms hold (for $DF(\le n)=s^n(DF(\le 0))$ and $DF(\ge n)=s^n(DF(\ge 0))$).

(i) $DF(\geq 1) \subset DF(\geq 0)$; $DF(\leq 1) \supset DF(\leq 0)$; $\cup_{n\in\mathbb{Z}} DF(\geq n) = \cup_{n\in\mathbb{Z}} DF(\leq n) = DF$.

(ii) For any $X \in ObjDF$ we have $\alpha_X = s(\alpha_{s^{-1}X})$.

(iii) For any $X \in ObjDF(\geq 1)$ and $Y \in ObjDF(\leq 0)$ we have $DF(X,Y) = \{0\}$; whereas $\alpha$ induces an isomorphism $DF(Y, s^{-1}X) \cong DF(sY, X) \to DF(Y,X)$.

(iv) Any $X \in ObjDF$ could be completed to a distinguished triangle $A \to X \to B$ with $A \in ObjDF(\geq 1)$ and $B \in ObjDF(\leq 0)$.

II $DF$ is called an f-category over $D$ if $D \subset DF$; $ObjD = ObjDF(\leq 0) \cap ObjDF(\geq 0)$.

III We will denote by $\omega$ (see Proposition A3 of [7]) the only exact functor $DF \to D$ such that:

(i) Its restrictions are right adjoint to the inclusion $D \to DF(\leq 0)$ and left adjoint to the inclusion $D \to DF(\geq 0)$ respectively.

(ii) $\omega(\alpha_X)$ is an isomorphism.

(iii) $DF(X,Y) = D(\omega X, \omega Y)$ for any $X \in ObjDF(\leq 0)$, $Y \in ObjDF(\geq 0)$.

A simple example of this axiomatics is described in Example A2 loc. cit.

By Proposition A3 loc. cit. there also exist exact functors $\sigma_{\geq n}: DF \to DF(\geq n)$, and $\sigma_{\leq n}: DF \to DF(\leq n)$ that are respectively right and left adjoint to the corresponding inclusions. We denote $gr_F^{[a,b]} := \sigma_{\leq b}\sigma_{\geq a}$, $gr_F^a = gr_F^{[a,a]}$. Note that there exist canonical and functorial (in $X$) morphisms $d: \sigma_{\leq 0}X \to \sigma_{\geq 1}X[1]$ that could be completed to a distinguished triangle in I(iv) of Definition 8.3.1.

Now we define $e$. For $M \in ObjDF$ the complex $e(M)$ has components equal to $s^{-a} gr_F^a M[a]$ (this lies in $ObjD \subset ObjDF$), the differential will be equal to $s^{-a-1}(s(d') \circ \alpha_{gr_F^a})[a]$; here $d'$ is the boundary map of the canonical triangle $gr_F^{a+1} \to gr_F^{[a,a+1]} \to gr_F^a \xrightarrow{d'} gr_F^{a+1}[1]$. $M$ is a complex indeed by the axiom I(ii). We have $e(s(X)) \cong e(X)$.

Now for a weight structure $w$ on $D$ we define $DB^s = e^{-1}(C^b(Hw))$ i.e. we demand $gr_F^a(X) \in s^a D^{w=a}$.

We will use the following statement.

**Lemma 8.3.2.** *For every $M, N \in DF^s$ the map $\alpha_* DF(N,M) \to DF(N, s(M))$ is surjective; all $DF(N, s^a(M)) \to DF(N, s^{a+1}(M))$ for $a > 0$ are bijective.*

*Proof.* Set $P = \operatorname{Cone}(\alpha_M : M \to s(M))$. By the long exact sequence for $DF(N, -)$, it suffices to show that $DF(N, s^a(P)[b]) = \{0\}$ for $a + b \geq 0$. Since $s^a M[-a] \in DF^s$, it suffices to show that $DF(N, P[b]) = \{0\}$ for $b \geq 0$.

By devissage, we can assume that $gr_F^a M$ and $gr_F^b N$ vanish for $a \neq m$, $b \neq n$, $m, n \in \mathbb{Z}$. In other words, $M = s^m(K)[-m]$, $N = s^n(L)[-n]$ for some $K, L \in \underline{C}^{w=0} \subset Obj D \subset Obj DF$.

One has $DF(N, P[b]) = D(L[-n], \omega(\sigma_{\geq n} P)[b])$. To see that this group vanishes, consider 3 cases.

(a) Suppose that $n > m+1$. Then $\sigma_{\geq n} P = 0$.

(b) Suppose $n \leq m$. Then $\sigma_{\geq n} P = P$, so $\omega(\sigma_{\geq n} P) = \omega(P) = 0$.

(c) Suppose $n = m+1$. Then $\sigma_{\geq n} P = s(M)$, so $\omega(\sigma_{\geq n} P) = K[-m]$ and $D(N, P[b]) = D(L[-n], K[-m+b]) = D(L, K[b+1]) = \{0\}$ since $w$ is a weight structure. $\square$

Now (ii) follows from Lemma 8.3.2 immediately since for any $X, Y \in Obj DF$ we have $DF(X, s(Y)) \cong D(\omega(X), \omega(s^n Y)) \cong D(\omega(X), \omega(Y))$ for $n$ large enough by parts III(ii), III(ii) of Definition 8.3.1.

(ii) easily yields (i). Indeed, we can prove the statement for $X \in D^{[i,j]}$ by the induction on $j - i$. We have obvious inclusions $D^{w=i} \to DF^s$ (that split $\omega$).

To make the inductive step it suffices to consider $X \in D^{[0,m]}$ for $m > 0$. Then $X^{w \leq 0}$ and $X^{\geq 1}$ could be lifted to $DF^s$ by the inductive assumption. The map $X^{w \leq 0} \to X^{\geq 1}$ lifts to $DF^s$ by Lemma 8.3.2; its cone will belong to $DF^s$ and so will be a lift of $X$.

Now we verify (iii). By Lemma 8.3.2, for $M, N \in DF^s$ we have

$$\operatorname{Ker} DF(N, M) \to D(\omega(N), \omega(M)) = \operatorname{Ker}(\alpha_* DF(N, M) \to DF(N, s(M))).$$

Since $\omega(M) \to \omega(s(M))$ is an isomorphism, we obtain (iii). Hence $T$ is a well-defined functor.

Now we note that $T$ is an "enhancement" of our "weak weight complex functor" $t$. Indeed, $T$ and $t$ coincide on $Hw$; both of them respect weight decompositions of objects and morphisms in a compatible way.

Lastly we check that $T$ is an exact functor. As in the proof of part I of Theorem 3.3.1, it suffices to lift any distinguished triangle $C \to X \to X'$ so that the sequence $e(C^*) \to e(X^*) \to e(X'^*)$ splits termwisely (in $C^b(Hw)$). Indeed, this would yield that any distinguished triangle is mapped by $T$ to a triangle $Tr$ any of whose two sides are two sides of a of a distinguished triangle in $K^b(Hw)$; hence $Tr$ is distinguished in $K^b(Hw)$.

Now, to find such lifting it suffices to choose the weight decompositions of $X$ and $X'$ arbitrarily; choose a weight decomposition of $C$ as in the proof of part I of Theorem 3.3.1; and lift to $DB^s$ the map $t(C) \to t(X)$ as in the proof of (i). Then the map $a^* : C^* \to X^*$ will become split surjective after the application of each $gr_F^a$; hence we can choose $\operatorname{Cone} a^* \in DF^s$ as a lift for $X'$. This yields the lift desired.

Hence $T$ is a strong weight complex functor for $D, w$. This argument is a certain weight structure counterpart of Proposition A5 of [7].

It also seems possible that an $f$-category enhancement of $D$ would allow to define certain *higher truncation* functors; see remark 6.3.1.

## 8.4 Possible variations of the weight complex functor

Now we try to answer the question: could the main results of this paper be generalized to a more general setting. We cannot prove any if and only if conditions; however we try to clarify the picture. Since we include this subsection only to explain our choice of definitions, it is rather sketchy.

First we study the question where do exact conservative functors come from.

Suppose that $f : \underline{C} \to \underline{C}'$ is an exact functor (here $\underline{C}, \underline{C}'$ are triangulated categories). We denote by $\operatorname{Ker} f$ the set of morphisms that are mapped to 0 by $f_*$. $\operatorname{Ker} f$ satisfies a certain set of obvious properties; hence it could be called a *triangulated ideal* of $Mor\underline{C}$.

It is easily seen that if $f$ is conservative if and only if $id_X \notin \operatorname{Ker} f$ for any $X \in Obj\underline{C}$. Note that in this case $\operatorname{Ker} f$ could be called a radical ideal since for any $X \in Obj\underline{C}$, $s \in \underline{C}(X, X) \cap \operatorname{Ker} f$, $id_X + s$ will be an automorphism.

Now we study an inverse problem: which triangulated ideals can correspond to conservative exact functors. Unfortunately, it seems that there does not exist a nice way to kill morphisms in an arbitrary $I$ unless $\underline{C}$ has a differential graded enhancement. So we suppose that $\underline{C} = Tr^+(D)$ for a dg-category $D$; a triangulated ideal $I \subset Mor\underline{C}$ comes from a differential graded nilpotent (or formally nilpotent in an appropriate sense) ideal $I'$ of $\text{Pre-Tr}^+ D$. Then one can form a category $\underline{C}' = Tr^+(D/I')$; using a certain spectral sequence argument for representable functors $X_*$ for $X \in Obj\underline{C}$ similar to those described below (for realizations) one can verify that the natural differential graded functor $\underline{C} \to \underline{C}'$ is conservative. However one cannot hope for a spectral sequence for a realization $H$ unless $H(I)$ belongs to some nice radical ideal (probably more conditions are needed). Note that this is obviously the case for representable functors.

We describe one of the cases when it makes sense to construct such a theory (and which does not come from a weight structure). Let $\underline{C}, D$ be pro-$p$-categories (i.e. the set of morphisms is an abelian profinite $p$-group for any pair of objects), $\underline{C} = Tr^+D$, and $I' = pMor(\underline{C})$. Let $H = Tr^+(E)$ for a dg-functor $E : D \to B(pro - p - Ab)$, where $B(pro - p - Ab)$ is the 'big' category of complexes of abelian profinite $p$-groups (see subsection 6.4). Then the complex that computes $H(X)$ for $X \in Obj\underline{C}$ has a natural filtration by subcomplexes given by $p^i E$. These subcomplexes correspond to the functors $Tr^+(p^iE)$ and the factors of the filtration are quasi-isomorphic to those calculating the functors $F_i = Tr^+(p^iE/p^{i+1}E)$. It remains to note that $F_i$ can be factorized through the natural functor $\underline{C} \to Tr^+(D/p)$. Hence in this case the spectral sequence of a filtered complex has properties similar to those of the spectral sequence $S$ in 6.4; $F_i$ are similar to truncated realizations (see §6.4 and §7.3 of [11]).

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