INVERSIONS OF INTEGRAL OPERATORS AND ELLIPTIC BETA INTEGRALS ON ROOT SYSTEMS

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Abstract. We prove a novel type of inversion formula for elliptic hypergeometric integrals associated to a pair of root systems. Using the (A,C) inversion formula to invert one of the known C-type elliptic beta integrals, we obtain a new elliptic beta integral for the root system of type A. Validity of this integral is established by a different method as well.

Contents

1. Introduction

Beta-type integrals are fundamental objects of applied analysis, with numerous applications in pure mathematics and mathematical physics. The classical Euler beta integral

$$\int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \qquad \min\{\mathrm{Re}(x), \mathrm{Re}(y)\} > 0,$$

determines the measure for the Jacobi family of orthogonal polynomials expressed as certain ${}_2F_1$ hypergeometric functions [3]. Its multi-dimensional extension due

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to Selberg [21] plays an important role in harmonic analysis on root systems, the theory of special functions of many variables, the theory of random matrices, and so forth.

Important generalizations of beta integrals arise in the theory of basic or q-hypergeometric functions. The Askey–Wilson q-beta integral depends on four independent parameters and a base q, and fixes the orthogonality measure for the Askey–Wilson polynomials, the most general family of classical single-variable orthogonal polynomials [4]. Closely related to the Askey–Wilson integral is the integral representation for a very-well-poised ${}_8\phi_7$ basic hypergeometric series found by Nassrallah and Rahman [15]. Through specialization this led Rahman to the discovery of a one-parameter extension of the Askey–Wilson integral [16]. Finally, several multi-dimensional generalizations of the Askey–Wilson and Rahman integrals, including a q-Selberg integral, were found by Gustafson [10, 11, 12]. For some time, these multi-dimensional q-beta integrals were believed to be the most general integrals of beta type.

A new development in the field was initiated by the first author with the discovery of an elliptic generalization of Rahman's q-beta integral [22]. This elliptic beta integral depends on five free parameters and two basic variables — or elliptic moduli — p and q. As a further development two n-dimensional elliptic beta integrals associated to the C_n root system were proposed by van Diejen and the first author [6, 7]. In the $p \to 0$ limit these integrals reduce to Gustafson's C_n q-beta integrals. More elliptic beta integrals, all related to either the A_n or C_n root systems and all but one generalizing integrals of Gustafson [11, 12] and Gustafson and Rakha [13], were subsequently given in [26].

Roughly, n-dimensional elliptic beta integrals come in three different types. Most fundamental are the type-I integrals. These contain $2n + 3$ free parameters (as well as the bases p and q), and one of the A_n integrals of [24] and one of the C_n integrals of [7] are of type I. The first complete proofs were found by Rains [17] who derived them as a consequence of a symmetry transformation for more general elliptic hypergeometric integrals. More elementary proofs using difference equations were subsequently given in [26]. The elliptic beta integrals of type II contain less than $2n + 3$ parameters and can be deduced from type I integrals via the composition of higher-dimensional integrals [7, 10, 12, 13, 24]. The second C_n elliptic beta integral of [7] (see also [6]), depending on six parameters (only 5 when $n = 1$), provides an example of a type II integral. Finally, type III elliptic beta integrals arise through the computation of n-dimensional determinants with entries composed of one-dimensional integrals [24]. Originally, all of the above beta integrals were defined for bases p and q inside the unit circle (due to the use of the standard elliptic gamma function described in the next section). Another class of elliptic hypergeometric integrals, which are well defined in the larger region $|p| < 1$, $|q| \le 1$ (by employing a different elliptic gamma function), has been introduced in [24]. We shall not discuss here the corresponding elliptic beta integrals, and refer the reader to [8, 26] for more details.

Further progress on the subject is associated with symmetry transformations of elliptic hypergeometric integrals. Certain hypergeometric identities are well-known to be related to the notion of matrix inversions and Bailey pairs. At the level of hypergeometric series — ordinary, basic or elliptic — the Bailey pair machinery allows for the derivation of infinite sequences of symmetry transformations [1, 2, 23, 27, 29]. A formulation of the notion of Bailey pairs for integrals was proposed in [25] (on the basis of a transformation for univariate elliptic hypergeometric integrals proved in [24]). Using the univariate elliptic beta integral, this led to a binary tree of identities for multiple elliptic hypergeometric integrals. A generalization of these results to elliptic hypergeometric integrals labelled by root systems has been one of the motivations for the present paper. Indeed, the integral analogues of the matrix inversions underlying the Bailey transform for series are provided by the integral inversions of this paper. A powerful set of transformations for elliptic hypergeometric integrals was introduced by Rains [17]. Although some of them can be reproduced with the help of the Bailey technique, the complete correspondence is not established yet.

More specifically, we provide the following new framework for viewing elliptic beta integrals on root systems. First, we introduce certain multi-dimensional integral transformations with integration kernels determined by the structure of the type I elliptic beta integrals on the A_n and C_n root systems. The A_n and C_n elliptic beta integrals then acquire the new interpretation as examples for which these integral transformations can be performed explicitly. Second, we prove two theorems describing inversions of the corresponding integral operators on a certain class of functions and conjecture a third inversion formula. These inversion formulas naturally carry two root system labels, our three results corresponding to the pairs $(\mathrm{A}_n,\mathrm{A}_n)$, $(\mathrm{A}_n,\mathrm{C}_n)$ and $(\mathrm{C}_n,\mathrm{A}_n)$. Third, using the $(\mathrm{A}_n,\mathrm{C}_n)$ inversion formula we 'invert' the type I C_n beta integral to prove a new type I A_n elliptic beta integral. It appears that this exact integration formula is new even at the q-hypergeometric and plain hypergeometric levels. Finally, for completeness, we give an alternative proof of this integral using the method for proving type I integrals developed in [26].

In the univariate case, all three integral inversions coincide and the resulting formula establishes the inversion of the integral Bailey transform of [25]. Also our multi-variable integral transformations on root systems can be put into the framework of integral Bailey pairs. This will be the topic of a subsequent publication together with a consideration of integral operators associated with the type II elliptic beta integrals.

2. Notation and preliminaries

Throughout this paper $p,q\in\mathbb{C}$ such that

$$M := \max\{|p|,|q|\} < 1. \tag{2.1}$$

For fixed p and q, and $z\in\mathbb{C}\backslash\{0\}$ the elliptic gamma function is defined as [19]

$$\Gamma(z;p,q)=\prod_{\mu,\nu=0}^{\infty}\frac{1-z^{-1}p^{\mu+1}q^{\nu+1}}{1-zp^{\mu}q^{\nu}}, \tag{2.2}$$

and satisfies

$$\Gamma(z;p,q)=\Gamma(z;q,p), \tag{2.3a}$$

$$\Gamma(z;p,q)=\frac{1}{\Gamma(pq/z;p,q)}. \tag{2.3b}$$

Defining the theta function

$$\theta(z;p)=(z,p/z;p)_\infty,$$

where

$$(a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k) \quad \text{and} \quad (a_1,\dots,a_k;q)_n=(a_1;q)_n\cdots(a_k;q)_n,$$

it follows that

$$\Gamma(qz;p,q)=\theta(z;p)\Gamma(z;p,q), \quad \Gamma(pz;p,q)=\theta(z;q)\Gamma(z;p,q) \tag{2.4}$$

and

$$\Gamma(z;p,q)\Gamma(z^{-1};p,q)=\frac{1}{\theta(z;p)\theta(z^{-1};q)}. \tag{2.5}$$

A useful formula needed repeatedly for calculating residues is

$$\lim_{z\to a}(1-z/a)\Gamma(z/a;p,q)=\frac{1}{(p;p)_\infty(q;q)_\infty}. \tag{2.6}$$

For n an integer the elliptic shifted factorial is defined by [28]

$$(a;q,p)_n=\frac{\Gamma(aq^n;p,q)}{\Gamma(a;p,q)}$$

(this was denoted as $\theta(a;p;q)_n$ in [6, 7, 24]). When n is non-negative it may also by written as

$$(a;q,p)_n=\prod_{j=0}^{n-1}\theta(aq^j;p).$$

For both the elliptic gamma function and the elliptic shifted factorial we employ standard condensed notation, i.e.,

$$\begin{aligned}\Gamma(z_1,\dots,z_k;p,q)&=\Gamma(z_1;p,q)\cdots\Gamma(z_k;p,q)\\(a_1,\dots,a_k;q,p)_n&=(a_1;q,p)_n\cdots(a_k;q,p)_n.\end{aligned}$$

Also, we will often suppress the p and q dependence and write $\Gamma(z)=\Gamma(z;p,q)$, $\theta(z)=\theta(z;p)$ and $(a)_n=(a;q,p)_n$.

As a final notational point we write the sets $\{1,\dots,n\}$ and $\{0,\dots,n-1\}$ as $[n]$ and $\mathbb{Z}_n$, and adopt the convention that μ and ν are non-negative integers.

3. Elliptic beta integrals

The single-variable elliptic beta integral — due to the first author [22] — corresponds to the following generalization of the celebrated Rahman integral [16] (obtained as an important special case of the Nassrallah–Rahman integral [15]). Let $t_1,\dots,t_6\in\mathbb{C}$ such that $t_1\cdots t_6=pq$ and

$$\max\{|t_1|,\dots,|t_6|\}<1, \tag{3.1}$$

and let $\mathbb{T}$ denote the positively oriented unit circle. Then

$$\frac{1}{2\pi \mathrm{i}}\int_{\mathbb{T}}\frac{\prod_{i=1}^{6}\Gamma(t_iz,t_iz^{-1})}{\Gamma(z^2,z^{-2})}\frac{\mathrm{d}z}{z}=\frac{2\prod_{1\le i<j\le 6}\Gamma(t_it_j)}{(p;p)_\infty(q;q)_\infty}. \tag{3.2}$$

Defining $T=t_1\cdots t_5$, eliminating t_6 and using the symmetry (2.3b), this may also be put in the form

$$\frac{1}{2\pi \mathrm{i}}\int_{\mathbb{T}}\frac{\prod_{i=1}^{5}\Gamma(t_iz,t_iz^{-1})}{\Gamma(z^2,z^{-2},Tz,Tz^{-1})}\frac{\mathrm{d}z}{z}=\frac{2\prod_{1\le i<j\le 5}\Gamma(t_it_j)}{(p;p)_\infty(q;q)_\infty\prod_{i=1}^{5}\Gamma(Tt_i^{-1})} \tag{3.3}$$

from which the Rahman integral follows by letting p (or, equivalently, q) tend to 0. For (3.3) to be valid we must of course replace (3.1) by

$$\max\{|t_1|,\dots,|t_5|,|pq/T|\}<1.$$

Two multivariable generalizations of (3.2) associated with the root systems of type A and C will be needed. In order to state these we require some further notation. Throughout, n will be a fixed positive integer, $z=(z_1,\dots,z_n)$ and

$$\frac{\mathrm{d}z}{z}=\frac{\mathrm{d}z_1}{z_1}\cdots\frac{\mathrm{d}z_n}{z_n}.$$

Whenever the variable z_{n+1} occurs it will be fixed by $z_1\cdots z_{n+1}=1$ unless stated otherwise. For reasons of printing economy we also employ the notation $f(z_i^{\pm})$ for $f(z_i,z_i^{-1})$, $f(z_i^{\pm}z_j^{\pm})$ for $f(z_1z_j,z_iz_j^{-1},z_i^{-1}z_j,z_i^{-1}z_j^{-1})$ and so on.

The A_n generalization of (3.2) depends on $2n+4$ complex parameters $t_1,\dots,t_{n+2}$ and $s_1,\dots,s_{n+2}$ such that $\max\{|t_1|,\dots,|t_{n+2}|,|s_1|,\dots,|s_{n+2}|\}<1$ and $ST=pq$ for $T=t_1\cdots t_{n+2}$ and $S=s_1\cdots s_{n+2}$. Hence we effectively have only $2n+3$ free parameters, making it an elliptic beta integral of type I;

$$\begin{aligned}(3.4)\quad \frac{1}{(2\pi\mathrm{i})^n}\int_{\mathbb{T}^n}\prod_{i=1}^{n+2}\prod_{j=1}^{n+1}\Gamma(s_iz_j,t_iz_j^{-1})\prod_{1\le i<j\le n+1}\frac{1}{\Gamma(z_iz_j^{-1},z_i^{-1}z_j)}\frac{\mathrm{d}z}{z}\\ =\frac{(n+1)!}{(p;p)_\infty^n(q;q)_\infty^n}\prod_{i=1}^{n+2}\Gamma(Ss_i^{-1},Tt_i^{-1})\prod_{i,j=1}^{n+2}\Gamma(s_it_j).\end{aligned}$$

As already mentioned in the introduction this integral was conjectured by the first author [24] and subsequently proven by Rains [17, Corollary 4.2] and by the first author [26, Theorem 3].

The type I elliptic beta integral for the root system C_n depends on the parameters $t_1,\dots,t_{2n+4}\in\mathbb{C}$ such that $t_1\cdots t_{2n+4}=pq$ and $\max\{|t_1|,\dots,|t_{2n+4}|\}<1$, and can be stated as

$$\begin{aligned}(3.5)\quad \frac{1}{(2\pi\mathrm{i})^n}\int_{\mathbb{T}^n}\prod_{j=1}^{n}\frac{\prod_{i=1}^{2n+4}\Gamma(t_iz_j^{\pm})}{\Gamma(z_j^{\pm 2})}\prod_{1\le i<j\le n}\frac{1}{\Gamma(z_i^{\pm}z_j^{\pm})}\frac{\mathrm{d}z}{z}\\ =\frac{2^nn!}{(p;p)_\infty^n(q;q)_\infty^n}\prod_{1\le i<j\le 2n+4}\Gamma(t_it_j).\end{aligned}$$

This was conjectured by van Diejen and Spiridonov [7, Theorem 4.1] who gave a proof based on a certain vanishing hypothesis and proven in full by Rains [17, Corollary 3.2] and the first author [26, Theorem 2].

In the limit when p tends to 0 the type I A_n and C_n elliptic beta integrals reduce to multiple integrals of Gustafson [11, Theorems 2.1 and 4.1].

The identification with the A_n and C_n root systems in the above two integrals is simple. In the case of A_n the set of roots $\Delta^{\mathscr{A}}$ is given by $\Delta^{\mathscr{A}}=\{\epsilon_i-\epsilon_j|\ i,j\in[n+1],i\ne j\}$ with ϵ_i the ith standard unit vector in $\mathbb{R}^{n+1}$. Setting $\phi_i=\epsilon_i-(\epsilon_1+\dots+\epsilon_{n+1})/(n+1)$, we formally put $z_i=\exp(\phi_i)$. Hence $z_1\cdots z_{n+1}=1$, and the permutation symmetry in the z_i, $i\in[n+1]$ of the integrand in (3.4) is in accordance with the A_n Weyl group, which acts on $\Delta^{\mathscr{A}}$ by permuting the indices of the ϵ_i. The factor $\prod_{1\le i<j\le n+1}\Gamma(z_iz_j^{-1},z_i^{-1}z_j)$ in (3.4) is identified with $\prod_{\alpha\in\Delta^{\mathscr{A}}}\Gamma(\mathrm{e}^{\alpha})$.

In the case of C_n the set of roots is given by $\Delta^{\mathscr{C}} = \{\pm 2\varepsilon_i | i \in [n]\} \cup \{\pm\varepsilon_i \pm \varepsilon_j |\ 1 \leq i < j \leq n\}$ with ε_i the ith standard unit vector in $\mathbb{R}^n$ and the two $\pm$'s in $\pm\varepsilon_i \pm \varepsilon_j$ taken independently. Furthermore, $z_i = \exp(\varepsilon_i)$, and the hyperoctahedral (i.e., signed permutation) symmetry of the integrand in (3.5) reflects the C_n Weyl group symmetry of $\Delta^{\mathscr{C}}$. The factor $\prod_{i=1}^n \Gamma(z_i^{\pm 2}) \prod_{1\leq i<j\leq n} \Gamma(z_i^{\pm} z_j^{\pm})$ in (3.5) is now identified with $\prod_{\alpha\in\Delta^{\mathscr{C}}} \Gamma(\mathrm{e}^{\alpha})$.

4. Inversion formulas. I. The single variable case

4.1. **Motivation.** To explain the origin of the inversion formula given in Theorem 4.1 below let us take the elliptic beta integral (3.2) and remove the restrictions (3.1). The price to be paid is that the contour $\mathbb{T}$ has to be replaced by C, where C is a contour[1] such that the sequences of poles of the integrand converging to zero (i.e., the poles at $z = t_i p^{\mu} q^{\nu}$ for $i \in [6]$) lie in the interior of C. Defining

$$\kappa = \frac{(p;p)_\infty (q;q)_\infty}{4\pi \mathrm{i}} \tag{4.1}$$

we thus have

$$\kappa \int_C \frac{\prod_{i=1}^6 \Gamma(t_i z^{\pm})}{\Gamma(z^{\pm 2})} \frac{\mathrm{d}z}{z} = \prod_{1\leq i<j\leq 6} \Gamma(t_i t_j) \tag{4.2}$$

for $t_i \in \mathbb{C}$ such that $t_1 \cdots t_6 = pq$.

Making the substitutions

$$(t_1, \dots, t_6) \to (t^{-1}w, t^{-1}w^{-1}, ts_1, \dots, ts_4) \tag{4.3}$$

we obtain

$$\kappa \int_C \frac{\Gamma(t^{-1}w^{\pm}z^{\pm}) \prod_{i=1}^4 \Gamma(ts_i z^{\pm})}{\Gamma(z^{\pm 2})} \frac{\mathrm{d}z}{z} = \Gamma(t^{-2}) \prod_{i=1}^4 \Gamma(s_i w^{\pm}) \prod_{1\leq i<j\leq 4} \Gamma(t^2 s_i s_j)$$

with $t^2 s_1 \cdots s_4 = pq$ and C a contour that has the poles of the integrand at

$$z = t^{-1}w^{\pm} p^{\mu} q^{\nu} \text{ and } z = ts_i p^{\mu} q^{\nu}, \quad i \in [4] \tag{4.4}$$

in its interior. Multiplying both sides by $\kappa\,\Gamma(tw^{\pm}x^{\pm})/\Gamma(w^{\pm 2})$ and integrating w along a contour $\hat{C}$ around

$$w = tx^{\pm} p^{\mu} q^{\nu} \text{ and } w = s_i p^{\mu} q^{\nu}, \quad i \in [4] \tag{4.5}$$

we get

$$\begin{aligned}
\kappa^2 &\int_{\hat{C}} \int_C \frac{\Gamma(tw^{\pm}x^{\pm}, t^{-1}w^{\pm}z^{\pm}) \prod_{i=1}^4 \Gamma(ts_i z^{\pm})}{\Gamma(z^{\pm 2}, w^{\pm 2})} \frac{\mathrm{d}z}{z} \frac{\mathrm{d}w}{w} \\
&= \kappa\, \Gamma(t^{-2}) \prod_{1\leq i<j\leq 4} \Gamma(t^2 s_i s_j) \int_{\hat{C}} \frac{\Gamma(tw^{\pm}x^{\pm}) \prod_{i=1}^4 \Gamma(s_i w^{\pm})}{\Gamma(w^{\pm 2})} \frac{\mathrm{d}w}{w} \\
&= \Gamma(t^{\pm 2}) \prod_{i=1}^4 \Gamma(ts_i x^{\pm}).
\end{aligned}$$

Here the second equality follows by application of the elliptic beta integral (4.2).

[1] When dealing with one-dimensional contour integrals we always assume the contour C to be a positively oriented Jordan curve such that $C = C^{-1}$, i.e., such that if $z \in C$ then also $z^{-1} \in C$. Consequently, if a point z lies in the interior of C then its reciprocal z^{-1} lies in the exterior of C.

Inspection of the left and right-hand sides of the above result reveals that for

$$f(z)=\prod_{i=1}^{4}\Gamma(ts_iz^{\pm}),\qquad t^2s_1\cdots s_4=pq \tag{4.6}$$

the following reproducing double integral holds

$$\frac{\kappa^2}{\Gamma(t^{\pm 2})}\int_{\hat{C}}\int_{C}\frac{\Gamma(tw^{\pm}x^{\pm},t^{-1}w^{\pm}z^{\pm})}{\Gamma(z^{\pm 2},w^{\pm 2})}f(z)\frac{\mathrm{d}z}{z}\frac{\mathrm{d}w}{w}=f(x) \tag{4.7}$$

provided the contours C and $\hat{C}$ are chosen in accordance with (4.4) and (4.5).

4.2. **The $n=1$ integral inversion.** If we choose $|t|<1$ and $\max\{|s_1|,\dots,|s_4|\}<1$ then the function f in (4.6) is free of poles for $|t|\le|z|\le|t|^{-1}$. Moreover, if we also take $|t|<|x|<|t|^{-1}$ then all the points listed in (4.5) have absolute value less than one, so that we may choose $\hat{C}$ to be the unit circle $\mathbb{T}$. But assuming $w\in\mathbb{T}$ in (4.4) and further demanding that $M<|t|^2$ with M defined in (2.1), it follows that for the above choice of parameters all the points listed in (4.4) have absolute value less than $|t|$ with the exception of $z=t^{-1}w^{\pm}$.

These considerations suggest the following generalization of (4.7) to a larger class of functions.

Theorem 4.1. *Let $p,q,t\in\mathbb{C}$ such that $M<|t|^2<1$. For fixed $w\in\mathbb{T}$ let C_w denote a contour inside the annulus $\mathbb{A}=\{z\in\mathbb{C}|\ |t|-\epsilon<|z|<|t|^{-1}+\epsilon\}$ for infinitesimally small but positive ϵ, such that C_w has the points $t^{-1}w^{\pm}$ in its interior. Let $f(z)=f(z;t)$ be a function such that $f(z)=f(z^{-1})$ and such that $f(z)$ is holomorphic on $\mathbb{A}$. Then for $|t|<|x|<|t|^{-1}$ there holds*

$$\kappa^2\int_{\mathbb{T}}\biggl(\int_{C_w}\Delta(z,w,x;t)f(z)\frac{\mathrm{d}z}{z}\biggr)\frac{\mathrm{d}w}{w}=f(x), \tag{4.8}$$

where

$$\Delta(z,w,x;t)=\frac{\Gamma(tw^{\pm}x^{\pm},t^{-1}w^{\pm}z^{\pm})}{\Gamma(t^{\pm 2},z^{\pm 2},w^{\pm 2})}. \tag{4.9}$$

The poles of the integrand at $z=t^{-1}w^{\pm}p^{\mu}q^{\nu}$ for $(\mu,\nu)\neq(0,0)$ are of course also in the interior of C_w, but since these all have absolute value less than $|t|$ (thanks to $M<|t|^2$) they do not lie in $\mathbb{A}$.

If one drops the condition that $f(z)=f(z^{-1})$ then the right hand side of (4.8) should be symmetrized, giving $(f(x)+f(x^{-1})/2$ instead of $f(x)$.

Since the kernel $\Delta(z,w,x;t)$ factorizes as $\Delta(z,w,x;t)=\delta(z,w;t^{-1})\delta(w,x;t)$ with

$$\delta(z,w;t)=\frac{\Gamma(tw^{\pm}z^{\pm})}{\Gamma(t^2,z^{\pm 2})}$$

the identity (4.8) may also be put as the following elliptic integral transform. If

$$\hat{f}(w;t)=\kappa\int_{C_w}\delta(z,w;t^{-1})f(z;t)\frac{\mathrm{d}z}{z} \tag{4.10a}$$

then

$$f(x;t)=\kappa\int_{\mathbb{T}}\delta(w,x;t)\hat{f}(w;t)\frac{\mathrm{d}w}{w} \tag{4.10b}$$

provided all the conditions and definitions of Theorem 4.1 are assumed. The theorem may thus be formally viewed as the inversion of the integral operator $\delta(w;t)$ defined by

$$\delta(w;t)f = \kappa \int_{C_w} \delta(z,w;t^{-1}) f(z) \frac{\mathrm{d}z}{z}.$$

The external variable w enters the kernel $\delta(z,w;t)$ through the term $\Gamma(tw^{\pm}z^{\pm})$, which reflects only a part of the elliptic beta integral structure (4.2). In this sense, we have a universal integral transformation, playing a central role in the context of integral Bailey pairs [25]. In particular, after taking the limit $p \to 0$, we obtain a q-hypergeometric integral transformation which does not distinguish the Askey-Wilson and Rahman integrals. In this respect, our integral transformation essentially differs from the one introduced in [14] on the basis of the full kernel of the Askey-Wilson integral.

An example of a pair $(f, \hat{f})$ is given by f of (4.6) and

$$\hat{f}(z) = \prod_{i=1}^{4} \Gamma(s_i z^{\pm}) \prod_{1\le i<j\le 4} \Gamma(t^2 s_i s_j).$$

For later comparison we eliminate s_4 and apply (2.3b). After normalizing the above pair of functions we find the new pair

$$f(z) = \frac{\prod_{i=1}^{3} \Gamma(S s_i^{-1}, t s_i z^{\pm})}{\Gamma(tSz^{\pm})} \quad \text{and} \quad \hat{f}(z) = \frac{\prod_{i=1}^{3} \Gamma(t^2 S s_i^{-1}, s_i z^{\pm})}{\Gamma(t^2 S z^{\pm})} \tag{4.11}$$

with $S = s_1 s_2 s_3$ and $\max\{|s_1|, |s_2|, |s_3|, |t^{-2}S^{-1}pq|\} < 1$. Writing $f(z;t,s)$ and $\hat{f}(z;t,s)$ instead of $f(z)$ and $\hat{f}(z)$, gives

$$\hat{f}(z;t,s) = f(z^{-1};t^{-1},ts) \quad \text{and} \quad f(z;t,s) = \hat{f}(z^{-1};t^{-1},ts) \tag{4.12}$$

with $s = (s_1, s_2, s_3)$ and $ts = (ts_1, ts_2, ts_3)$. The reason for writing z^{-1} and not z on the right is that it is the above form that generalizes to A_n, see Section 5.2.1. For the pair $(f, \hat{f})$ of (4.11) we can also deform the respective contours of integration in (4.10) and more symmetrically write

$$\hat{f}(z;t,s) = \kappa \int_{C_{w;t^{-1},s}} \delta(z,w;t^{-1}) f(z;t,s) \frac{\mathrm{d}z}{z}$$

and

$$f(z;t,s) = \kappa \int_{C_{w;t,ts}} \delta(z,w;t) \hat{f}(w;t,s) \frac{\mathrm{d}z}{z},$$

with $C_{w;t,s}$ a contour that has the points $t^{-1}w^{\pm}p^{\mu}q^{\nu}$, $ts_i p^{\mu}q^{\nu}$ and $t^{-1}S^{-1}p^{\mu+1}q^{\nu+1}$ in its interior, and where t and $s = (s_1, s_2, s_3)$ can be chosen freely. This can also be captured in just a single equation as

$$f(z;t,s) = \kappa \int_{C_{w;t,ts}} \delta(z,w;t) f(z;t^{-1},ts) \frac{\mathrm{d}z}{z}.$$

4.3. **Proof of Theorem 4.1.** Consider the integral over z in (4.8) for fixed $w \in \mathbb{T}$ such that $w^2 \neq 1$. By deforming the integration contour from C_w to $\mathbb{T}$ the simple poles at $z = t^{-1}w^{\pm}$ ($tw^{\pm}$) move from the interior (exterior) to the exterior (interior) of the contour of integration. Calculating the respective residues using the $f(z) = f(z^{-1})$ and $\Delta(z,w,x;t) = \Delta(z^{-1},w,x;t)$ symmetries and the limit (2.6), yields

$$\kappa \int_{C_w} \Delta(z,w,x;t) f(z) \frac{\mathrm{d}z}{z} = \kappa \int_{\mathbb{T}} \Delta(z,w,x;t) f(z) \frac{\mathrm{d}z}{z} + \frac{\Gamma(tw^{\pm}x^{\pm})}{\Gamma(t^2)} \biggl(\frac{f(t^{-1}w)}{\Gamma(w^2, t^2w^{-2})} + \frac{f(t^{-1}w^{-1})}{\Gamma(w^{-2}, t^2w^2)} \biggr). \tag{4.13}$$

Since $1/\Gamma(1) = \Gamma(pq) = 0$ both sides vanish identically for $w^2 = 1$ so that the above is true for all $w \in \mathbb{T}$.

Next, by (4.13),

$$\begin{aligned} I(x;t) &:= \kappa^2 \int_{\mathbb{T}} \int_{C_w} \Delta(z,w,x;t) f(z) \frac{\mathrm{d}z}{z} \frac{\mathrm{d}w}{w} \\ &= \kappa^2 \int_{\mathbb{T}^2} \Delta(z,w,x;t) f(z) \frac{\mathrm{d}z}{z} \frac{\mathrm{d}w}{w} + 2\kappa \int_{\mathbb{T}} \frac{\Gamma(tw^{\pm}x^{\pm})}{\Gamma(t^2, w^2, t^2w^{-2})} f(t^{-1}w) \frac{\mathrm{d}w}{w}, \end{aligned}$$

where we have made the substitution $w \to w^{-1}$ in the integral over w corresponding to the last term on the right of (4.13).

To proceed we replace $w \to tz$ in the single integral on the right and invoke Fubini's theorem to interchange the order of integration in the double integral. Hence

$$I(x;t) = \kappa^2 \int_{\mathbb{T}^2} \Delta(z,w,x;t) f(z) \frac{\mathrm{d}w}{w} \frac{\mathrm{d}z}{z} + 2\kappa \int_{t^{-1}\mathbb{T}} \frac{\Gamma(x^{\pm}z^{-1}, t^2x^{\pm}z)}{\Gamma(t^2, z^{-2}, t^2z^2)} f(z) \frac{\mathrm{d}z}{z}.$$

where $a\mathbb{T}$ denotes the positively oriented circle of radius $|a|$. If we deflate $t^{-1}\mathbb{T}$ to $\mathbb{T}$ the pole at $z = x$ (if $1 < |x| < |t|^{-1}$) or $z = x^{-1}$ (if $|t| < |x| < 1$) moves from the interior to the exterior of the integration contour. By the symmetry of f we find

$$I(x;t) = \kappa^2 \int_{\mathbb{T}^2} \Delta(z,w,x;t) f(z) \frac{\mathrm{d}w}{w} \frac{\mathrm{d}z}{z} + 2\kappa \int_{\mathbb{T}} \frac{\Gamma(x^{\pm}z^{-1}, t^2x^{\pm}z)}{\Gamma(t^2, z^{-2}, t^2z^2)} f(z) \frac{\mathrm{d}z}{z} + f(x) \tag{4.14}$$

irrespective of whether $|t| < |x| < 1$ or $1 < |x| < |t|^{-1}$.

When $|x| = 1$ we require the Sokhotsky–Plemelj definition of the Cauchy integral in the case of a pole singularity on the integration contour C:

$$\int_C \frac{f(z)}{z-x} \, \mathrm{d}z = \frac{1}{2} \int_{C_+} \frac{f(z)}{z-x} \, \mathrm{d}z + \frac{1}{2} \int_{C_-} \frac{f(z)}{z-x} \, \mathrm{d}z,$$

where $f(z)$ is holomorphic on C, and $C_{\pm}$ are contours which include/exclude the point $x \in C$ by an infinitesimally small deformations of C in the vicinity of x. By the $x \to x^{-1}$ symmetry of our integral it thus follows that (4.14) is true for all $|t| < |x| < |t|^{-1}$.

To complete the proof we need to show that the integrals on the right-hand side of (4.14) vanish. To achieve this we use that for $z \in \mathbb{T}$

$$\begin{aligned} &\kappa \int_{\mathbb{T}} \Delta(z,w,x;t) \frac{\mathrm{d}w}{w} \\ &\quad = \kappa \int_C \Delta(z,w,x;t) \frac{\mathrm{d}w}{w} - \frac{\Gamma(x^{\pm}z^{-1}, t^2x^{\pm}z)}{\Gamma(t^2, z^{-2}, t^2z^2)} - \frac{\Gamma(x^{\pm}z, t^2x^{\pm}z^{-1})}{\Gamma(t^2, z^2, t^2z^{-2})}, \end{aligned} \tag{4.15}$$

where C is a contour such that the points $w = tx^{\pm}p^{\mu}q^{\nu}$ and $w = t^{-1}z^{\pm}p^{\mu}q^{\nu}$ lie in its interior. The two ratios of elliptic gamma functions on the right correspond to the residues of the poles at $w = t^{-1}z^{\pm}$ and $w = tz^{\pm}$ which, for $|z| = 1$ and $|t| < 1$, lie in the exterior and interior of $\mathbb{T}$, respectively. Note that we again have implicitly assumed $z^2 \neq 1$ in the calculation of the respective residues, but that (4.15) is true for all $z \in \mathbb{T}$.

Since $\Gamma(pq) = 0$ it follows from the elliptic beta integral (4.2) with $t_5t_6 = pq$ that the integral on the right vanishes, resulting in

$$\kappa \int_{\mathbb{T}} \Delta(z, w, x; t) \frac{\mathrm{d}w}{w} = -\frac{\Gamma(x^{\pm}z^{-1}, t^2x^{\pm}z)}{\Gamma(t^2, z^{-2}, t^2z^2)} - \frac{\Gamma(x^{\pm}z, t^2x^{\pm}z^{-1})}{\Gamma(t^2, z^2, t^2z^{-2})}.$$

Substituting this in the first term on the right of (4.14) and making a $z \to z^{-1}$ variable change establishes the desired cancellation of integrals in (4.14), thereby establishing the theorem.

5. Inversion formulas. II. The root systems A_n and C_n

5.1. **Main results.** To state our multi-dimensional inversion theorems we first extend the (what will be referred to as A_1 or C_1) symmetry $f(z) = f(z^{-1})$ to functions of n variables. Let g be a symmetric function of $n+1$ independent variables. Then a function $f(z) = f(z_1, \dots, z_n) := g(z_1, \dots, z_{n+1})$ is said to have A_n symmetry. (Recall our convention that $z_1 \cdots z_{n+1} = 1$.) Similarly, we say that $f(z) = f(z_1, \dots, z_n)$ has C_n symmetry if f is symmetric under signed permutations. That is, $f(z) = f(\mathrm{w}(z))$ for $\mathrm{w} \in S_n$ and $f(z_1, \dots, z_n) = f(z_1^{\sigma_1}, \dots, z_n^{\sigma_n})$ where each $\sigma_i \in \{-1, 1\}$. For example f has A_2 symmetry if $f(z_1, z_2) = f(z_2, z_1)$ and $f(z_1, z_2) = f(z_1, z_3) = f(z_1, z_1^{-1}z_2^{-1})$, and f has C_2 symmetry if $f(z_1, z_2) = f(z_2, z_1)$ and $f(z_1, z_2) = f(z_1, z_2^{-1})$. The integrands of the integrals (3.4) and (3.5) provide examples of functions that are A_n or C_n symmetric.

Below we will also use the root system analogues of κ of equation (4.1);

$$\kappa^{\mathscr{A}} = \frac{(p;p)_\infty^n (q;q)_\infty^n}{(2\pi \mathrm{i})^n (n+1)!} \quad \text{and} \quad \kappa^{\mathscr{C}} = \frac{(p;p)_\infty^n (q;q)_\infty^n}{(2\pi \mathrm{i})^n 2^n n!}. \tag{5.1}$$

Finally, we need to discuss a somewhat technically involved issue. The $n = 1$ inversion formula (4.8) features the integration contour C_w which is a deformation of the contour $\mathbb{T}$ such that the poles of the integrand at $t^{-1}w^{\pm}p^{\mu}q^{\nu}$ are in the interior of C_w. Now the A_n beta integral (3.4) is computed by iteratively integrating over the n components of z. Let us choose to integrate z_n first then z_{n-1} and so on. When doing the z_i integral the integrand will have poles which are independent of $z_1, \dots, z_{i-1}$ and poles which depend on these variables through their product $Z_{i-1} := z_1 \cdots z_{i-1}$. For example, when doing the z_n integral over $\mathbb{T}$ we need to compute the residues of the poles at $z_n = t_ip^{\mu}q^{\nu}$ and $z_{n+1} = s_i^{-1}p^{-\mu}q^{-\nu}$, i.e., at $z_n = s_ip^{\mu}q^{\nu}Z_{n-1}^{-1}$. Just as in the $n = 1$ case we wish to utilize the A_n beta integral in which $(t_1, \dots, t_{n+1})$ is substituted by $(t^{-1}w_1, \dots, t^{-1}w_{n+1})$ with $w_i \in \mathbb{T}$. (Compare this with (4.3).) Hence we need to again analytically continue the integral (3.4) by appropriately deforming the integration contours. Because of the above-discussed poles depending on the remaining integration variables this deformation — which will be denoted by C_w^n — cannot be of the form $C_1 \times C_2 \times \cdots \times C_n$ with each of the one-dimensional contours C_i independent of z. Rather what we get is that C_n depends on t and w as well as on Z_{n-1}. Then C_{n-1} will depend on t, w, and Z_{n-2}

and so on. Of course, this is all assuming the above order of integrating out the components of z, but, evidently, all ordering are in fact equivalent.

We would like an efficient description of the deformed contours that is independent of the chosen order of integration and that reflects the A_n symmetry present in the problem. However, since we want to avoid the complexities of genuine higher-dimensional residue calculus, we adopt a convention that C_w^n does not explicitly describe each of the one-dimensional contours composing it. Rather, we encode C_w^n by indicating which poles of the integrand are to be taken in the interior and exterior at each stage of the iterative computation of the integral over z.

Let $p, q, t \in \mathbb{C}$ such that $M < |t|^{n+1} < 1$ and denote

$$\mathbb{A} = \{z \in \mathbb{C}^n |\ |z_j| < |t|^{-1} + \epsilon,\ j \in [n+1]\} \tag{5.2}$$

for infinitesimally small but positive ϵ. Let f be an A_n symmetric function holomorphic on $\mathbb{A}$ and let the generalization of the kernel (4.9) to the root system pair $(\mathrm{A}_n,\mathrm{A}_n)$ be given by

$$\begin{aligned}\Delta^{(\mathscr{A},\mathscr{A})}&(z, w, x; t) \\ &= \frac{\prod_{i,j=1}^{n+1} \Gamma(tw_i x_j, t^{-1} w_i^{-1} z_j^{-1})}{\Gamma(t^{n+1}, t^{-n-1}) \prod_{1 \le i < j \le n+1} \Gamma(z_i z_j^{-1}, z_i^{-1} z_j, w_i w_j^{-1}, w_i^{-1} w_j)}.\end{aligned} \tag{5.3}$$

Then for $w \in \mathbb{T}^n$ we write

$$\int_{C_w^n} \Delta^{(\mathscr{A},\mathscr{A})}(z, w, x; t) f(z) \frac{\mathrm{d}z}{z}, \tag{5.4}$$

where — by abuse of notation — we write '$C_w^n \subset \mathbb{A}$' as the 'deformation of the (oriented) n-torus $\mathbb{T}^n$' such that for all $i \in [n+1]$

$$z_j = t^{-1} w_i^{-1} \begin{cases} \text{lies in the interior of } C_w^n \text{ if } j \in [n] \\ \text{lies in the exterior of } C_w^n \text{ if } j = n+1. \end{cases} \tag{5.5}$$

More precisely, we consider C_w^n as an iteratively defined n-dimensional structure encoding which poles of the integrand are to be taken in the interior/exterior at each stage of the iterative integration over z. That is, if we again fix the order of integration as before then, when integrating over z_j, the poles (these will occur regardless of which components of z are already integrated out) at $z_j = t^{-1} w_i^{-1} p^\mu q^\nu$ are all in the interior of C_j because (i) for $(\mu, \nu) \neq (0, 0)$ we have $|t^{-1} w_i^{-1} p^\mu q^\nu| < |t|^{-1} M < |t|^n$, but for $z \in \mathbb{A}$ each z_j is bounded (in absolute value) from below by $|t|^n$, (ii) for $(\mu, \nu) = (0, 0)$ we have to satisfy (5.5). The poles at $z_j = t^{n-j+1} w_{i_1} \cdots w_{i_{n-j+1}} p^{-\mu} q^{-\nu} Z_{j-1}^{-1}$ (corresponding to the pole $z_{n+1} = t^{-1} w_i^{-1} p^\mu q^\nu$ with $z_n, \ldots, z_{j+1}$ integrated out) will be in the exterior of C_j because (i) for $(\mu, \nu) \neq (0, 0)$ we have $|t^{n-j+1} w_{i_1} \cdots w_{i_{n-j+1}} p^{-\mu} q^{-\nu} Z_{j-1}^{-1}| > |t^{n-j+1} Z_{j-1}^{-1}| M^{-1} > |t^{-j} Z_{j-1}^{-1}| > |t|^{-1}$ (by (5.2)) and $C_w^n \subset \mathbb{A}$), but for $z \in \mathbb{A}$, $|z_j| < |t|^{-1}$, (ii) for $(\mu, \nu) = (0, 0)$ we have to satisfy (5.5). Of course, because f is holomorphic on $\mathbb{A}$ its poles are either trivially in the interior or exterior of each contour C_j.

The most rigorous definition of the integration domain $C_w^n \subset \mathbb{A}$ is obtained by considering it as a deformation of $\mathbb{T}^n$ allowing for an analytical continuation of the integral

$$\int_{\mathbb{T}^n} \frac{\prod_{i,j=1}^{n+1} \Gamma(t_i z_j^{-1})}{\prod_{1\le i<j\le n+1} \Gamma(z_i z_j^{-1}, z_i^{-1} z_j)} \, f(z) \frac{dz}{z}$$

from the restricted values of parameters $|t_i| < 1$, $i \in [n+1]$, to the region $t_i = t^{-1} w_i^{-1}$ with $|t| < 1$ and $|w_i| = 1$, $w_1 \cdots w_{n+1} = 1$. Clearly, this defines the z-dependent part of the integral (5.4). However, for making our computations efficient we will not reformulate our results in this coordinate independent way but characterize C_w^n by locations of the appropriate poles.

With a trivial modification of the above notation we can now formulate two generalizations of (4.8) corresponding to the root system pairs $(\mathrm{A}_n, \mathrm{A}_n)$ and $(\mathrm{A}_n, \mathrm{C}_n)$. In the next section we shall also formulate a conjecture for the pair $(\mathrm{C}_n, \mathrm{A}_n)$.

Theorem 5.1 (**(A,A) inversion formula**). *Let $q, p, t \in \mathbb{C}$ such that $M < |t|^{n+1} < 1$ and let $\mathbb{A}$ be defined as in (5.2). For fixed $w \in \mathbb{T}^n$ let $C_w^n \subset \mathbb{A}$ denote a deformation of the (oriented) n-torus $\mathbb{T}^n$ such that (5.5) holds for all $i \in [n+1]$. Let f be an A_n symmetric function holomorphic on $\mathbb{A}$. Then for $x \in \mathbb{C}^n$ such that $|x_j| < |t|^{-1}$ for all $j \in [n+1]$ and*

$$|x_j| > 1 \quad \text{for } j \in [n], \tag{5.6}$$

there holds

$$(\kappa^{\mathscr{A}})^2 \int_{\mathbb{T}^n} \biggl(\int_{C_w^n} \Delta^{(\mathscr{A},\mathscr{A})}(z,w,x;t) f(z) \frac{\mathrm{d}z}{z} \biggr) \frac{\mathrm{d}w}{w} = f(x), \tag{5.7}$$

where $\Delta^{(\mathscr{A},\mathscr{A})}(z,w,x;t)$ is given in (5.3).

Because of the A_n symmetry in x of (5.7) the condition (5.6) can of course be replaced by the condition that all but one of $|x_1|, \dots, |x_{n+1}|$ exceeds one. In fact, we have strong evidence that the condition (5.6) is not necessary. However, the proof of the theorem becomes significantly more complicated if (5.6) is dropped and in the absence of (5.6) we have only been able to complete the proof for $n \le 2$.

Theorem 5.2 (**(A,C) inversion formula**). *Let $p, q, t \in \mathbb{C}$ such that $M < |t|^{n+1} < 1$ and let $\mathbb{A}$ be defined as in (5.2). Let $C_n^w \subset \mathbb{A}$ be a deformation of $\mathbb{T}^n$ such that for all $i \in [n]$*

$$z_j = t^{-1} w_i^{\pm} \begin{cases} \text{lies in the interior of } C_w^n \text{ if } j \in [n] \\ \text{lies in the exterior of } C_w^n \text{ if } j = n+1. \end{cases} \tag{5.8}$$

Let f be an A_n symmetric function holomorphic on $\mathbb{A}$. Then for $x \in \mathbb{C}^n$ such that $|x_j| < |t|^{-1}$ for all $j \in [n+1]$ and such that (5.6) holds, we have

$$\kappa^{\mathscr{A}} \kappa^{\mathscr{C}} \int_{\mathbb{T}^n} \biggl(\int_{C_w^n} \Delta^{(\mathscr{A},\mathscr{C})}(z,w,x;t) f(z) \frac{\mathrm{d}z}{z} \biggr) \frac{\mathrm{d}w}{w} = f(x). \tag{5.9}$$

where

$$\Delta^{(\mathscr{A},\mathscr{C})}(z,w,x;t) = \frac{\prod_{i=1}^n \prod_{j=1}^{n+1} \Gamma(t w_i^{\pm} x_j, t^{-1} w_i^{\pm} z_j^{-1})}{\prod_{i=1}^n \Gamma(w_i^{\pm 2}) \prod_{1\le i<j\le n} \Gamma(w_i^{\pm} w_j^{\pm})} \times \frac{1}{\prod_{1\le i<j\le n+1} \Gamma(z_i z_j^{-1}, z_i^{-1} z_j, t^2 z_i z_j, t^{-2} z_i^{-1} z_j^{-1})}. \tag{5.10}$$

Again the condition (5.6) is probably unnecessary, but without it our proof of Theorem 5.2 requires some rather intricate modifications for $n \geq 3$.

If one drops the condition that f is an A_n symmetric function then it is immediate from the A_n symmetry of the left-hand side that $f(x)$ on the right of (5.7) and (5.9) should be replaced by the A_n symmetric

$$\frac{1}{(n+1)!} \sum_{\mathrm{w} \in S_{n+1}} g(x_{\mathrm{w}_1}, \dots, x_{\mathrm{w}_{n+1}}),$$

where $g(x_1, \dots, x_{n+1}) = f(x_1, \dots, x_n)$.

The proofs of the two inversion theorems are very similar, the only significant difference being that (5.7) requires the A_n elliptic beta integral and (5.9) the C_n elliptic beta integral to establish the vanishing of certain unwanted terms arising in the expansion of the integral over C_w^n as a sum of integrals over $\mathbb{T}^n, \dots, \mathbb{T}, \mathbb{T}^0$. We therefore content ourselves with only presenting the details of the proof of Theorem 5.2.

First, however, let us state the root systems analogues of some of the equations of Section 4.2. This will lead us to discover a new elliptic beta integral for the root system A_n. Loosely speaking this new integral may be viewed as the inverse of the C_n beta integral (3.5) with respect to the kernel $\Delta^{(\mathscr{A},\mathscr{C})}$.

5.2. **Consequences of Theorems 5.1 and 5.2.**

5.2.1. *Integral identities.* Introducing

$$\nabla^{\mathscr{A}}(z,w;t) = \frac{\prod_{i,j=1}^{n+1} \Gamma(t w_i^{-1} z_j^{-1})}{\Gamma(t^{n+1}) \prod_{1\leq i<j\leq n+1} \Gamma(z_i z_j^{-1}, z_i^{-1} z_j)}, \tag{5.11}$$

the kernel $\Delta^{(\mathscr{A},\mathscr{A})}(z,w,x;t)$ may be factored as

$$\Delta^{(\mathscr{A},\mathscr{A})}(z,w,x;t) = \nabla^{\mathscr{A}}(z,w;t^{-1}) \nabla^{\mathscr{A}}(w^{-1},x^{-1};t),$$

with $w^{-1} = (w_1^{-1}, \dots, w_n^{-1})$ and $x^{-1} = (x_1^{-1}, \dots, x_n^{-1})$.

Defining the elliptic integral transform

$$\hat{f}^{\mathscr{A}}(w;t) = \kappa^{\mathscr{A}} \int_{C_w^n} \nabla^{\mathscr{A}}(z,w;t^{-1}) f^{\mathscr{A}}(z;t) \frac{\mathrm{d}z}{z}$$

the claim of Theorem 5.1 is equivalent to the inverse transformation

$$f^{\mathscr{A}}(x;t) = \kappa^{\mathscr{A}} \int_{\mathbb{T}^n} \nabla^{\mathscr{A}}(w^{-1},x^{-1};t) \hat{f}^{\mathscr{A}}(w;t) \frac{\mathrm{d}w}{w}.$$

From the A_n elliptic beta integral one readily obtains the following example of a pair $(f^{\mathscr{A}}, \hat{f}^{\mathscr{A}})$:

$$f^{\mathscr{A}}(z) = \prod_{i=1}^{n+2} \Gamma(S s_i^{-1}) \prod_{j=1}^{n+1} \frac{\prod_{i=1}^{n+2} \Gamma(t s_i z_j)}{\Gamma(t S z_j)}$$

$$\hat{f}^{\mathscr{A}}(z) = \prod_{i=1}^{n+2} \Gamma(t^{n+1} S s_i^{-1}) \prod_{j=1}^{n+1} \frac{\prod_{i=1}^{n+2} \Gamma(s_i z_j^{-1})}{\Gamma(t^{n+1} S z_j^{-1})},$$

with $S = s_1 \cdots s_{n+2}$ and $\max\{|s_1|, \dots, |s_{n+2}|, |t^{-n-1} S^{-1} pq|\} < 1$. Here the conditions on the s_i ensure that $f^{\mathscr{A}}(z)$ is holomorphic on $\mathbb{A}$ as follows from a reasoning similar to the one presented immediately after Theorem 4.1. We also remark that

$f^{\mathscr{A}}$ and $\hat{f}^{\mathscr{A}}$ are again related by the simple symmetry (4.12) provided we now take $s = (s_1, \dots, s_{n+2})$.

Next we turn our attention to Theorem 5.2 and define

$$\delta^{\mathscr{C}}(w, x; t) = \frac{\prod_{i=1}^{n} \prod_{j=1}^{n+1} \Gamma(tw_i^{\pm} x_j)}{\prod_{i=1}^{n} \Gamma(w_i^{\pm 2}) \prod_{1 \le i < j \le n} \Gamma(w_i^{\pm} w_j^{\pm})}$$

and

$$\delta^{\mathscr{A}}(z, w; t) = \frac{\prod_{i=1}^{n} \prod_{j=1}^{n+1} \Gamma(tw_i^{\pm} z_j^{-1})}{\prod_{1 \le i < j \le n+1} \Gamma(z_i z_j^{-1}, z_i^{-1} z_j, t^{-2} z_i z_j, t^2 z_i^{-1} z_j^{-1})},$$

so that

$$\Delta^{(\mathscr{A}, \mathscr{C})}(z, w, x; t) = \delta^{\mathscr{A}}(z, w; t^{-1}) \delta^{\mathscr{C}}(w, x; t). \tag{5.12}$$

Then, according to Theorem 5.2, if

$$\hat{f}^{\mathscr{C}}(w; t) = \kappa^{\mathscr{A}} \int_{C_w^n} \delta^{\mathscr{A}}(z, w; t^{-1}) f^{\mathscr{A}}(z; t) \frac{\mathrm{d}z}{z} \tag{5.13a}$$

then

$$f^{\mathscr{A}}(x; t) = \kappa^{\mathscr{C}} \int_{\mathbb{T}^n} \delta^{\mathscr{C}}(w, x; t) \hat{f}^{\mathscr{C}}(w; t) \frac{\mathrm{d}w}{w}. \tag{5.13b}$$

We note that for $n = 1$ this simplifies to (4.10a) and (4.10b) up to factors of $\Gamma(t^{\pm 2})$.

Let us now choose $\hat{f}^{\mathscr{C}}(w; t)$ as

$$\hat{f}^{\mathscr{C}}(w; t) = \prod_{i=1}^{n} \prod_{j=1}^{n+3} \Gamma(w_i^{\pm} s_j) \tag{5.14a}$$

with $\max\{|s_1|, \dots, |s_{n+3}|\} < 1$ and $t^{n+1} s_1 \cdots s_{n+3} = pq$. Provided that

$$\max\{|x_1|, \dots, |x_{n+1}|\} < |t|^{-1}$$

we can evaluate the integral (5.13b) by the C_n elliptic beta integral (3.5) with $t_j \to tx_j$ for $j \in [n+1]$ and $t_{j+n+1} \to s_j$ for $j \in [n+3]$, to find

$$f^{\mathscr{A}}(x; t) = \prod_{i=1}^{n+1} \prod_{j=1}^{n+3} \Gamma(tx_i s_j) \prod_{1 \le i < j \le n+1} \Gamma(t^2 x_i x_j) \prod_{1 \le i < j \le n+3} \Gamma(s_i s_j). \tag{5.14b}$$

Substituting this in (5.13a) and observing that $f^{\mathscr{A}}$ satisfies the conditions imposed by the theorem, we obtain the new elliptic beta integral

$$\kappa^{\mathscr{A}} \int_{C_w^n} \prod_{j=1}^{n+1} \frac{\prod_{i=1}^{n} \Gamma(t^{-1} w_i^{\pm} z_j^{-1}) \prod_{i=1}^{n+3} \Gamma(ts_i z_j)}{\prod_{1 \le i < j \le n+1} \Gamma(z_i z_j^{-1}, z_i^{-1} z_j, t^{-2} z_i^{-1} z_j^{-1})} \frac{\mathrm{d}z}{z}$$
$$= \prod_{i=1}^{n} \prod_{j=1}^{n+3} \Gamma(s_j w_i^{\pm}) \prod_{1 \le i < j \le n+3} \frac{1}{\Gamma(s_i s_j)}.$$

By appropriately deforming the contour of integration this may be analytically continued to $|t| > 1$. Then replacing $t \to t^{-1}$ and $w_i \to t_i$ and $z_i \to z_i^{-1}$, the result can be written as an integral over $\mathbb{T}^n$;

$$\kappa^{\mathscr{A}} \int_{\mathbb{T}^n} \prod_{j=1}^{n+1} \frac{\prod_{i=1}^n \Gamma(tt_i^{\pm} z_j) \prod_{i=1}^{n+3} \Gamma(t^{-1} s_i z_j^{-1})}{\prod_{1\le i<j\le n+1} \Gamma(z_i z_j^{-1}, z_i^{-1} z_j, t^2 z_i z_j)} \frac{\mathrm{d}z}{z} = \prod_{i=1}^{n} \prod_{j=1}^{n+3} \Gamma(t_i^{\pm} s_j) \prod_{1\le i<j\le n+3} \frac{1}{\Gamma(s_i s_j)}$$

for $s_1 \cdots s_{n+3} = t^{n+1}pq$ and $\max\{|t|, |tt_1^{\pm}|, \dots, |tt_n^{\pm}|, |t^{-1}s_1|, \dots, |t^{-1}s_{n+3}|\} < 1$. Alternatively, we may choose to replace $t_i \to t^{-1}t_i$ and $s_i \to ts_i$ and then eliminate t^2 using $t^2 = pqS^{-1}$ with $S = s_1 \cdots s_{n+3}$. By (2.3b) this results in our next theorem.

Theorem 5.3. *Let $t_1, \dots, t_n, s_1, \dots, s_{n+3} \in \mathbb{C}$ and $S = s_1 \cdots s_{n+3}$. For*

$$\max\{|t_1|, \dots, |t_n|, |s_1|, \dots, |s_{n+3}|, |pqS^{-1}t_1^{-1}|, \dots, |pqS^{-1}t_n^{-1}|\} < 1 \tag{5.15}$$

there holds

$$\frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \prod_{j=1}^{n+1} \frac{\prod_{i=1}^n \Gamma(t_i z_j) \prod_{i=1}^{n+3} \Gamma(s_i z_j^{-1})}{\prod_{i=1}^n \Gamma(St_i z_j^{-1})} \prod_{1\le i<j\le n+1} \frac{\Gamma(Sz_i^{-1}z_j^{-1})}{\Gamma(z_i z_j^{-1}, z_i^{-1} z_j)} \frac{\mathrm{d}z}{z} = \frac{(n+1)!}{(p;p)_\infty^n (q;q)_\infty^n} \prod_{i=1}^{n} \prod_{j=1}^{n+3} \frac{\Gamma(t_i s_j)}{\Gamma(St_i s_j^{-1})} \prod_{1\le i<j\le n+3} \Gamma(Ss_i^{-1}s_j^{-1}).$$

We will give an independent proof of this new A_n elliptic beta integral in Section 6. Somewhat surprising, even when p tends to zero the corresponding beta integral is new;

$$\frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \prod_{j=1}^{n+1} \frac{\prod_{i=1}^n (St_i z_j^{-1};q)_\infty}{\prod_{i=1}^n (t_i z_j;q)_\infty \prod_{i=1}^{n+3} (s_i z_j^{-1};q)_\infty} \prod_{1\le i<j\le n+1} \frac{(z_i z_j^{-1}, z_i^{-1} z_j;q)_\infty}{(Sz_i^{-1}z_j^{-1};q)_\infty} \frac{\mathrm{d}z}{z} = \frac{(n+1)!}{(q;q)_\infty^n} \prod_{i=1}^{n} \prod_{j=1}^{n+3} \frac{(St_i s_j^{-1};q)_\infty}{(t_i s_j;q)_\infty} \prod_{1\le i<j\le n+3} \frac{1}{(Ss_i^{-1}s_j^{-1};q)_\infty}$$

for $\max\{|t_1|, \dots, |t_n|, |s_1|, \dots, |s_{n+3}|\} < 1$. When s_{n+3} tends to zero this reduces to a limiting case of Gustafson's $\mathrm{SU}(n)$ q-beta integral [11, Theorem 2.1].

Formally the further limit $q \to 1^-$ can be taken by replacing $z_j \to q^{u_j}$, $t_j \to q^{a_j}$ and $s_j \to q^{b_j}$ and choosing $q = \exp(-\pi/\alpha)$ for α positive and real. The integral can then be conveniently expressed in terms of the q-Gamma function

$$\Gamma_q(x) = (1-q)^{1-x} \frac{(q;q)_\infty}{(q^x;q)_\infty}.$$

By

$$\Gamma(x) = \lim_{q\to 1^-} \Gamma_q(x),$$

with $\Gamma(x)$ the classical gamma function, the limit when α tends to infinity is readily obtained.

Theorem 5.4. *Let $a_1, \dots, a_n, b_1, \dots, b_{n+3} \in \mathbb{C}$ and $B = b_1 + \cdots + b_{n+3}$ such that*

$$\min\{\mathrm{Re}(a_1), \dots, \mathrm{Re}(a_n), \mathrm{Re}(b_1), \dots, \mathrm{Re}(b_{n+3})\} > 0.$$

Denote by $\Gamma(x)$ *the classical instead of elliptic gamma function. Then*

$$\begin{aligned}\frac{1}{(2\pi\mathrm{i})^n}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\cdots\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\prod_{j=1}^{n+1}&\frac{\prod_{i=1}^{n}\Gamma(a_i+u_j)\prod_{i=1}^{n+3}\Gamma(b_i-u_j)}{\prod_{i=1}^{n}\Gamma(B+a_i-u_j)}\\&\times\prod_{1\le i<j\le n+1}\frac{\Gamma(B-u_i-u_j)}{\Gamma(u_i-u_j,u_j-u_i)}\,\mathrm{d}u_1\cdots\mathrm{d}u_n\\&=(n+1)!\prod_{i=1}^{n}\prod_{j=1}^{n+3}\frac{\Gamma(a_i+b_j)}{\Gamma(B+a_i-b_j)}\prod_{1\le i<j\le n+3}\Gamma(B-b_i-b_j)\end{aligned}$$

with $u_1+\cdots+u_{n+1}=0$.

In the large b_{n+3} limit this coincides with the large α_n limit of [11, Theorem 5.1]. A more rigorous justification of Theorem 5.4 can be given using the technique of Section 6.

The above discussion of the new elliptic beta integral suggests that Theorem 5.2 should have the following companion.

Conjecture 5.1 ((C,A) inversion formula). *Let* $p,q,t\in\mathbb{C}$ *such that* $M<|t|^{n+1}<1$. *For fixed* $w\in\mathbb{T}^n$ *let* C_w *denote a contour inside the annulus* $\mathbb{A}=\{z\in\mathbb{C}|\ |t|-\epsilon<|z|<|t|^{-1}+\epsilon\}$ *for infinitesimally small but positive* ϵ, *such that* C_w *has the points* $t^{-1}w_j$ *for* $j\in[n+1]$ *in its interior, and set* $C_w^n=C_w\times\cdots\times C_w$. *For* f *a* C_n *symmetric function holomorphic on* $\mathbb{A}^n$, *and* $x\in\mathbb{C}^n$ *such that* $|t|<|x_j|<|t|^{-1}$, *we have*

$$\kappa^{\mathscr{A}}\kappa^{\mathscr{C}}\int_{\mathbb{T}^n}\bigg(\int_{C_w^n}\Delta^{(\mathscr{C},\mathscr{A})}(z,w,x;t)f(z)\frac{\mathrm{d}z}{z}\bigg)\frac{\mathrm{d}w}{w}=f(x), \tag{5.16}$$

where

$$\begin{aligned}\Delta^{(\mathscr{C},\mathscr{A})}(z,w,x;t)=&\frac{\prod_{i=1}^{n}\prod_{j=1}^{n+1}\Gamma(tx_i^{\pm}w_j^{-1},t^{-1}z_i^{\pm}w_j)}{\prod_{i=1}^{n}\Gamma(z_i^{\pm2})\prod_{1\le i<j\le n}\Gamma(z_i^{\pm}z_j^{\pm})}\\&\times\frac{1}{\prod_{1\le i<j\le n+1}\Gamma(w_iw_j^{-1},w_i^{-1}w_j,t^{-2}w_iw_j,t^2w_i^{-1}w_j^{-1})}.\end{aligned}$$

Note that

$$\Delta^{(\mathscr{C},\mathscr{A})}(z,w,x;t)=\delta^{\mathscr{C}}(z,w;t^{-1})\delta^{\mathscr{A}}(w,x;t)$$

to be compared with (5.12). Hence, if this conjecture were true then

$$\hat{f}^{\mathscr{A}}(w;t)=\kappa^{\mathscr{C}}\int_{C_w^n}\delta^{\mathscr{C}}(z,w;t^{-1})f^{\mathscr{C}}(z;t)\frac{\mathrm{d}z}{z} \tag{5.17a}$$

would imply

$$f^{\mathscr{C}}(x;t)=\kappa^{\mathscr{A}}\int_{\mathbb{T}^n}\delta^{\mathscr{A}}(w,x;t)\hat{f}^{\mathscr{A}}(w;t)\frac{\mathrm{d}w}{w} \tag{5.17b}$$

which is the image of (5.13) under the interchange of the root systems A and C. If we take

$$f^{\mathscr{C}}(z;t)=\prod_{i=1}^{n}\prod_{j=1}^{n+3}\Gamma(ts_jz_i^{\pm}) \tag{5.18a}$$

for $\max\{|s_1|,\dots,|s_{n+3}|\}<1$ and $t^2 s_1\cdots s_{n+3}=pq$, then the integral (5.17a) can be calculated using a deformation of the C_n elliptic beta integral (3.5), and

$$\hat{f}^{\mathscr{A}}(w;t)=\prod_{i=1}^{n+1}\prod_{j=1}^{n+3}\Gamma(w_is_j)\prod_{1\le i<j\le n+1}\Gamma(t^{-2}w_iw_j)\prod_{1\le i<j\le n+3}\Gamma(t^2s_is_j). \tag{5.18b}$$

Substituting this in (5.17b) once more yields the integral of Theorem 5.3 (up to simple changes of variables).

Eliminating s_{n+3} in the functions listed in (5.14) and (5.18), and making the s dependence explicit, we get the formal symmetry relations

$$\hat{f}^{\mathscr{C}}(z;t,s)=f^{\mathscr{C}}(z;t^{-1},ts)\quad\text{and}\quad \hat{f}^{\mathscr{A}}(z;t,s)=f^{\mathscr{A}}(z;t^{-1},ts)$$

where $s=(s_1,\dots,s_{n+2})$ and $ts=(ts_1,\dots,ts_{n+2})$.

As already mentioned at the end of Section 5.1, the proofs of Theorems 5.1 and 5.2 hinge on the vanishing of certain unwanted elliptic hypergeometric integrals. Key to this are specializations of the the A_n and C_n elliptic beta integrals. If a similar approach is taken with respect to Conjecture 5.1 one encounters unwanted integrals for which the vanishing condition is not evident.

5.2.2. *Series identities.* Using residue calculus one can reduce elliptic beta integrals to summation identities for elliptic hypergeometric series, see e.g., [6, 24] for examples of this procedure. Below we will give the main steps of such a calculation for the elliptic beta integral of Theorem 5.3.

First let us write the integral in question in the form

$$\kappa^{\mathscr{A}}\int_{\mathbb{T}^n}\rho(z;s,t)\frac{\mathrm{d}z}{z}=1, \tag{5.19}$$

with integration kernel $\rho(z;s,t)$ for $s\in\mathbb{C}^{n+3}$ and $t\in\mathbb{C}^n$ given by

$$\begin{aligned}\rho(z;s,t)=\prod_{j=1}^{n+1}\frac{\prod_{i=1}^n\Gamma(t_iz_j)\prod_{i=1}^{n+3}\Gamma(s_iz_j^{-1})}{\prod_{i=1}^n\Gamma(St_iz_j^{-1})}\prod_{1\le i<j\le n+1}\frac{\Gamma(Sz_i^{-1}z_j^{-1})}{\Gamma(z_iz_j^{-1},z_i^{-1}z_j)}\\ \times\prod_{i=1}^{n}\prod_{j=1}^{n+3}\frac{\Gamma(St_is_j^{-1})}{\Gamma(t_is_j)}\prod_{1\le i<j\le n+3}\frac{1}{\Gamma(Ss_i^{-1}s_j^{-1})}.\end{aligned} \tag{5.20}$$

The following poles of $\rho(z;s,t)$ lie in the interior of $\mathbb{T}^n$:

$$\begin{aligned} z_{n+1}^{-1}&=t_ip^{\mu}q^{\nu} && i\in[n]\\ z_j&=s_ip^{\mu}q^{\nu} && i\in[n+3],\ j\in[n]\\ z_{n+1}^{-1}&=S^{-1}t_i^{-1}p^{\mu+1}q^{\nu+1} && i\in[n]\\ z_iz_j&=Sp^{\mu}q^{\nu} && 1\le i<j\le n.\end{aligned} \tag{5.21}$$

Now we replace (5.15) by

$$\begin{aligned}\max\{|t_1|,\dots,|t_n|,|s_{n+1}|,\dots,|s_{n+3}|,\\ |pqS^{-1}t_1^{-1}|,\dots,|pqS^{-1}t_n^{-1}|\}<1<\min\{|s_1|,\dots,|s_n|\}.\end{aligned}$$

Accordingly, we deform $\mathbb{T}^n$ to C^n such that the set of poles in the interior of C^n is again given by (5.21). Then, obviously,

$$\kappa^{\mathscr{A}}\int_{C^n}\rho(z;s,t)\frac{\mathrm{d}z}{z}=1. \tag{5.22}$$

Next the integral over C^n is expanded as a sum over integrals over $\mathbb{T}^{n-m}$ (for the details of such an expansion, see Section 5.3). Let N_i be a fixed non-negative integer and assume that $1<|s_iq^{N_i}|<|q|^{-1}$ for $i\in[n]$ and $|p|<\min\{|s_1|^{-1},\dots,|s_n|^{-1}\}$. Then the only poles of the integrand crossing the contour in its deformation from C^n to $\mathbb{T}^n$ are the poles at $z_j=s_iq^{\lambda_i}$ for $\lambda_i\in\mathbb{Z}_{N_i+1}$, $i\in[n]$ and $j\in[n+1]$, and the expansion takes the form

$$\int_{C^n}\rho(z;s,t)\frac{\mathrm{d}z}{z}=\sum_{m=0}^{n}\sum_{\lambda^{(m)}}\int_{\mathbb{T}^{n-m}}\rho_{\lambda^{(m)}}(z^{(n-m)};s,t)\frac{\mathrm{d}z^{(n-m)}}{z^{(n-m)}},\tag{5.23}$$

with $z^{(i)}=(z_1,\dots,z_i)$, $\lambda^{(i)}=(\lambda_1,\dots,\lambda_i)\in\mathbb{Z}^i$, and where the sum over $\lambda^{(m)}$ is subject to the restriction that $0\le\lambda_i\le N_i$ for all $i\in[m]$. The function $\rho_{\lambda^{(m)}}(z^{(n-m)};s,t)$ is obtained from $\rho_{\lambda^{(0)}}(z^{(n)};s,t)=\rho(z;s,t)$ by computing the relevant residues. For the present purposes we only need the explicit form of the kernel for $m=n$. Writing λ for $\lambda^{(n)}$ and $\rho_\lambda(z^{(0)};s,t)$ as $\rho_\lambda(s,t)$, it is given by

$$\begin{aligned}
\rho_\lambda(s,t)&=(2\pi\mathrm{i})^n(n+1)!\,\mathrm{Res}_{z_1=s_1q^{\lambda_1}}\Big(\cdots\Big(\mathrm{Res}_{z_n=s_nq^{\lambda_n}}\Big(\frac{\rho(z;s,t)}{z}\Big)\Big)\cdots\Big)\\
&=\frac{1}{\kappa^{\mathscr{A}}}\prod_{i=1}^{n}\bigg(\frac{\Gamma(S'^{-1}t_i,SS's_i^{-1})}{\Gamma(S'^{-1}s_i^{-1},SS't_i)}\prod_{j=n+1}^{n+3}\frac{\Gamma(St_is_j^{-1},s_i^{-1}s_j)}{\Gamma(Ss_i^{-1}s_j^{-1},t_is_j)}\bigg)\\
&\quad\times\prod_{i=1}^{n}\frac{\theta(S's_iq^{\lambda_i+|\lambda|})}{\theta(S's_i)}\prod_{1\le i<j\le n}\frac{\theta(q^{\lambda_i-\lambda_j}s_is_j^{-1})}{\theta(s_is_j^{-1})}\frac{1}{(qS^{-1}s_is_j)_{\lambda_i+\lambda_j}}\\
&\quad\times\prod_{j=1}^{n+3}\frac{(S's_j)_{|\lambda|}}{\prod_{i=1}^{n}(qs_is_j^{-1})_{\lambda_i}}\prod_{i,j=1}^{n}(s_it_j,qS^{-1}s_it_j^{-1})_{\lambda_i}\\
&\quad\times\prod_{i=1}^{n}\frac{(SS's_i^{-1})_{|\lambda|-\lambda_i}\,q^{i\lambda_i}}{(SS't_i,qS't_i^{-1})_{|\lambda|}}
\end{aligned}$$

for $S'=s_1\cdots s_n$ and $|\lambda|=\lambda_1+\dots+\lambda_n$. The other kernels arise as m-fold residues and their explicit form is quite involved. In the remainder we will only use the fact that $\rho_{\lambda^{(m)}}(z^{(n-m)};s,t)$ contains the factor $\prod_{i=1}^{n-m}1/\Gamma(t_is_i)$.

The next step in the computation is to let t_i tend to $q^{-N_i}s_i^{-1}$ in (5.23) for all $i\in[n]$. Since for $m\neq n$ the factor $\prod_{i=1}^{n-m}1/\Gamma(t_is_i)$ vanishes in this limit, the only contribution to the sum over m comes from the term $m=n$. For later reference we state this explicitly;

$$\kappa^{\mathscr{A}}\lim_{\substack{t_i\to q^{-N_i}s_i^{-1}\\ \forall\, i\in[n]}}\int_{C^n}\rho(z;s,t)\frac{\mathrm{d}z}{z}=\kappa^{\mathscr{A}}\lim_{\substack{t_i\to q^{-N_i}s_i^{-1}\\ \forall\, i\in[n]}}\sum_{\lambda}\rho_\lambda(s,t),\tag{5.24}$$

with λ_i ranging from 0 to N_i in the sum on the right. Thanks to (5.22) the left-hand side is equal to 1, leading to the following elliptic hypergeometric series identity.

Theorem 5.5. *For $S = s_1 \cdots s_{n+3}$, $S' = s_1 \cdots s_n$ we have*

$$\sum_{\lambda} \prod_{i=1}^{n} \frac{\theta(S's_i q^{\lambda_i+|\lambda|})}{\theta(S's_i)} \prod_{1\le i<j\le n} \frac{\theta(q^{\lambda_i-\lambda_j} s_i s_j^{-1})}{\theta(s_i s_j^{-1})} \frac{1}{(qS^{-1}s_i s_j)_{\lambda_i+\lambda_j}}$$
$$\times \prod_{j=1}^{n+3} \frac{(S's_j)_{|\lambda|}}{\prod_{i=1}^{n}(qs_i s_j^{-1})_{\lambda_i}} \prod_{i,j=1}^{n} (q^{-N_j} s_i s_j^{-1}, q^{N_j+1} S^{-1} s_i s_j)_{\lambda_i}$$
$$\times \prod_{i=1}^{n} \frac{(SS's_i^{-1})_{|\lambda|-\lambda_i}\, q^{i\lambda_i}}{(q^{-N_i} SS's_i^{-1}, q^{N_i+1}S's_i)_{|\lambda|}}$$
$$= \prod_{i=1}^{n} \biggl(\frac{(qS's_i)_{N_i}}{(qS^{-1}S'^{-1}s_i)_{N_i}} \prod_{j=n+1}^{n+3} \frac{(qS^{-1}s_i s_j)_{N_i}}{(qs_i s_j^{-1})_{N_i}} \biggr),$$

where the sum is over $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that $0 \le \lambda_i \le N_i$ for all $i \in [n]$.

The above result is equivalent to the sum proven by Rosengren in [18, Corollary 6.3], which is an elliptic version of the Schlosser's D_n Jackson sum [20, Theorem 5.6]. In view of our derivation it appears more appropriate to associate Theorem 5.5 with the root system A_n.

An alternative way to modify (5.15) is to take

$$\max\{|s_1|, \ldots, |s_{n+3}|, |pqS^{-1}t_1^{-1}|, \ldots, |pqS^{-1}t_n^{-1}|\} < 1 < \min\{|t_1|, \ldots, |t_n|\}. \tag{5.25}$$

Again deforming $\mathbb{T}^n$ to C^n so as to let (5.21) be the set of poles in the interior of C^n we once more get an integral identity of the form (5.22). Assuming that $|p| < \min\{|t_1|^{-1}, \ldots, |t_n|^{-1}\}$ and $1 < |t_i q^{N_i}| < |q|^{-1}$ for $i \in [n]$, the poles crossing the contour in its deformation from C^n back to $\mathbb{T}^n$ now correspond to the poles at $z_j = t_i^{-1} q^{-\lambda_i}$ for $\lambda_i \in \{0, \ldots, N_i\}$, $i \in [n]$ and $j \in [n+1]$. Appropriately redefining the integration kernels the expansion (5.23) still takes place. In particular, this time

$$\begin{aligned}
\rho_\lambda(s,t) &= (2\pi \mathrm{i})^n (-1)^n (n+1)! \\
&\qquad \times \operatorname{Res}_{z_1 = t_1^{-1} q^{-\lambda_1}} \biggl(\cdots \Bigl(\operatorname{Res}_{z_n = t_n^{-1} q^{-\lambda_n}} \Bigl(\frac{\rho(z;s,t)}{z} \Bigr) \Bigr) \cdots \biggr) \\
&= \frac{1}{\kappa^{\mathscr{A}}} \prod_{i=1}^{n} \frac{1}{\Gamma(T^{-1}t_i^{-1})} \prod_{j=1}^{n+3} \biggl(\Gamma(T^{-1}s_j) \prod_{i=1}^{n} \Gamma(St_i s_j^{-1}) \biggr) \\
&\quad \times \prod_{1\le i\le j\le n} \frac{1}{\Gamma(St_i t_j)} \prod_{1\le i<j\le n+3} \frac{1}{\Gamma(Ss_i^{-1}s_j^{-1})} \\
&\quad \times \prod_{i=1}^{n} \frac{\theta(Tt_i q^{\lambda_i+|\lambda|})}{\theta(Tt_i)} \prod_{1\le i<j\le n} \frac{\theta(q^{\lambda_i-\lambda_j} t_i t_j^{-1})}{\theta(t_i t_j^{-1})} (St_i t_j)_{\lambda_i+\lambda_j} \\
&\quad \times \prod_{j=1}^{n+3} \frac{\prod_{i=1}^{n}(t_i s_j)_{\lambda_i}}{(qTs_j^{-1})_{|\lambda|}} \prod_{i,j=1}^{n} \frac{1}{(qt_i t_j^{-1}, St_i t_j)_{\lambda_i}} \\
&\quad \times \prod_{i=1}^{n} \frac{(Tt_i, qTS^{-1}t_i^{-1})_{|\lambda|}\, q^{i\lambda_i}}{(qTS^{-1}t_i^{-1})_{|\lambda|-\lambda_i}}
\end{aligned}$$

for $T=t_1\cdots t_n$. Letting s_i tend to $q^{-N_i}t_i^{-1}$ for $i\in[n]$ we once again find that all but the $m=n$ term vanishes in the sum over m in (5.23). After the identification of $(s_{n+1},s_{n+2},s_{n+3})$ with (b_1,b_2,b_3) this yields the following companion to Theorem 5.5.

Theorem 5.6. *For $T=t_1\cdots t_n$ and $A=qTb_1^{-1}b_2^{-1}b_3^{-1}$ we have*

$$\begin{aligned}
\sum_{\lambda}\prod_{i=1}^n &\frac{\theta(Tt_iq^{\lambda_i+|\lambda|})}{\theta(Tt_i)}\prod_{1\le i<j\le n}\frac{\theta(q^{\lambda_i-\lambda_j}t_it_j^{-1})}{\theta(t_it_j^{-1})}(q^{1-|N|}A^{-1}t_it_j)_{\lambda_i+\lambda_j}\\
&\times\prod_{i,j=1}^n\frac{(q^{-N_j}t_it_j^{-1})_{\lambda_i}}{(qt_it_j^{-1},q^{1-|N|}A^{-1}t_it_j)_{\lambda_i}}\prod_{j=1}^3\frac{\prod_{i=1}^n(t_ib_j)_{\lambda_i}}{(qTb_j^{-1})_{|\lambda|}}\\
&\times\prod_{i=1}^n\frac{(Tt_i,q^{|N|}ATt_i^{-1})_{|\lambda|}\,q^{i\lambda_i}}{(q^{N_i+1}Tt_i)_{|\lambda|}(q^{|N|}ATt_i^{-1})_{|\lambda|-\lambda_i}}\\
=\prod_{i,j=1}^n&\frac{(At_i^{-1}t_j^{-1})_{|N|-N_i}}{(At_i^{-1}t_j^{-1})_{|N|}}\prod_{1\le i<j\le n}\frac{(At_i^{-1}t_j^{-1})_{|N|}}{(At_i^{-1}t_j^{-1})_{|N|-N_i-N_j}}\\
&\times\prod_{i=1}^n\prod_{j=1}^3\frac{(At_i^{-1}b_j)_{|N|}}{(At_i^{-1}b_j)_{|N|-N_i}}\prod_{i=1}^n(qTt_i)_{N_i}\prod_{j=1}^3\frac{1}{(qTb_j^{-1})_{|N|}},
\end{aligned}$$

where the sum is over all $\lambda=(\lambda_1,\dots,\lambda_n)$ such that $0\le\lambda_i\le N_i$ for each $i\in[n]$, and $N=N_1+\cdots+N_n$.

The above result corresponds to the elliptic analogue of Bhatnagar's D_n summation [5], and can be transformed into the identity of Theorem 5.5 by changing the summation indices from λ_i to $N_i-\lambda_i$ for all $i\in[n]$. Actually, the described residue calculus with the simplest choices $N_i=0$ (i.e., when there remains only one trivial term in the sum of Theorem 5.6) will be used in Section 6 in the alternative proof of Theorem 5.3. The full sums of Theorems 5.5 and 5.6 then follow from the application of general residue calculus.

5.3. **Proof of Theorem 5.2.** We begin by introducing some notation and definitions. For $a\in\mathbb{Z}_{n+1}$ let $Z_0=1$, $Z_a=z_1\cdots z_a$, $\overline{W}_0=1$, $\overline{W}_a=w_{n-a+1}\cdots w_n$, $z^{(a)}=(z_1,\dots,z_a)$ and $w^{(a)}=(w_1,\dots,w_a)$. Note that $Z_n=z_{n+1}^{-1}$, $\overline{W}_n=w_{n+1}^{-1}$, $z^{(n)}=z$ and $w^{(n)}=w$. Dropping the superscript $(\mathscr{A},\mathscr{C})$ in $\Delta(z,w,x;t)$ we recursively define

$$\Delta(z^{(n-a)},w,x;t)=\operatorname{Res}_{z_{n-a+1}=t^{-1}w_{n-a+1}}\frac{\Delta(z^{(n-a+1)},w,x;t)}{z_{n-a+1}} \tag{5.26a}$$

$$=-\operatorname{Res}_{z_{n-a+1}=t^a\overline{W}_a^{-1}Z_{n-a}^{-1}}\frac{\Delta(z^{(n-a+1)},w,x;t)}{z_{n-a+1}}. \tag{5.26b}$$

for $a\in[n]$. The equality of the two expressions on the right easily follows from the A_n symmetry of $\Delta(z,w,x;t)$ in the z-variables. Indeed, the above recursions imply that $\Delta(z^{(n-a)},w,x;t)=\Delta(\sigma(z^{(n-a)}),w,x;t)$ for $\sigma\in S_{n-a}$ and that $\Delta(z^{(n-a)},w,x;t)$ is invariant under the variable change $z_i\to t^a\overline{W}_a^{-1}Z_{n-a}^{-1}$ (and hence $Z_{n-a}\to t^a\overline{W}_a^{-1}z_i^{-1}$) for arbitrary fixed $i\in[n-a]$. These two symmetries of course generate a group of dimension $(n-a+1)!$ isomorphic to S_{n-a+1}, and for $a=0$ correspond to the A_n symmetry of $\Delta(z,w,x;t)$. By the C_n symmetry of $\Delta(z,w,x;t)$ in the w-variables it also follows that $\Delta(z^{(n-a)},w,x;t)$ has C_{n-a} symmetry in the variables $(w_1,\dots,w_{n-a})$ and C_a symmetry in the variables $(w_{n-a+1},\dots,w_n)$. Hence for $k\in[n-a+1]$ and $\sigma\in\{-1,1\}$

$$\begin{aligned}(5.27)\quad \operatorname{Res}_{z_{n-a+1}=t^{-1}w_k^{\sigma}} &\frac{\Delta(z^{(n-a+1)},w,x;t)}{z_{n-a+1}}\\ &= -\operatorname{Res}_{z_{n-a+1}=t^a w_k^{-\sigma}\overline{W}_{a-1}^{-1}Z_{n-a}^{-1}} \frac{\Delta(z^{(n-a+1)},w,x;t)}{z_{n-a+1}}\\ &= \Delta(z^{(n-a)},w,x;t)|_{w_{n-a+1}\leftrightarrow w_k^{\sigma}}.\end{aligned}$$

Using (2.6) and induction, the explicit form for $\Delta(z^{(n-a)},w,x;t)$ is easily found to be

(5.28)

$$\begin{aligned}\Delta(z^{(n-a)},w,x;t) &= \frac{1}{(p;p)_\infty^a (q;q)_\infty^a}\frac{\prod_{i=1}^n\prod_{j=1}^{n+1}\Gamma(tw_i^{\pm}x_j)}{\prod_{i=1}^n\Gamma(w_i^{\pm2})\prod_{1\le i<j\le n}\Gamma(w_i^{\pm}w_j^{\pm})}\\ &\times\prod_{i=n-a+1}^n\biggl[\Gamma(w_i^{-2})\prod_{j=1}^{n-a}\Gamma(w_i^{-1}w_j^{\pm})\biggr]\prod_{n-a+1\le i<j\le n}\frac{\Gamma(w_i^{-1}w_j^{-1})}{\Gamma(w_iw_j)}\\ &\times\prod_{j=1}^{n-a+1}\frac{\prod_{i=1}^{n-a}\Gamma(t^{-1}w_i^{\pm}z_j^{-1})]}{\prod_{i=n-a+1}^n\Gamma(tw_i^{\pm}z_j)}\\ &\times\prod_{1\le i<j\le n-a+1}\frac{1}{\Gamma(z_iz_j^{-1},z_i^{-1}z_j,t^2z_iz_j,t^{-2}z_i^{-1}z_j^{-1})},\end{aligned}$$

where, to keep the expression from spilling over, we have set $z_{n-a+1}:=t^a\overline{W}_a^{-1}Z_{n-a}^{-1}$. Note that this also makes all of the claimed symmetries of $\Delta(z^{(n-a)},w,x;t)$ manifest.

After these preliminaries we can state our first intermediate result. Let

$$(5.29)\qquad I(x;t):=\int_{\mathbb{T}^n}\int_{C_w^n}\Delta(z,w,x;t)f(z)\frac{\mathrm{d}z}{z}\frac{\mathrm{d}w}{w}.$$

Proposition 5.1. *There holds*

$$\begin{aligned}(5.30)\quad I(x;t)=\sum_{a=0}^n(4\pi\mathrm{i})^a\binom{n}{a}\frac{(n+1)!}{(n-a+1)!}&\int_{\mathbb{T}^n}\int_{\mathbb{T}^{n-a}}\Delta(z^{(n-a)},w,x;t)\\ &\times f(z^{(n-a)},t^{-1}w_{n-a+1},\dots,t^{-1}w_n)\frac{\mathrm{d}z^{(n-a)}}{z^{(n-a)}}\frac{\mathrm{d}w}{w}.\end{aligned}$$

We break up the proof into two lemmas, the first of which requires some more notation. Write $\mathbb{A}$ of (5.2) as $\mathbb{A}^n$ and, more generally, define $\mathbb{A}^{n-a}$ for $a\in\mathbb{Z}_{n+1}$ as

$$\begin{aligned}(5.31)\quad \mathbb{A}^{n-a}=\{z\in\mathbb{C}^{n-a}|\ |t|^n-\epsilon<|z_j|<|t|^{-1}+\epsilon,\ j\in[n-a],&\\ \text{and } |Z_{n-a}^{-1}|<|t|^{-a-1}+\epsilon\}&\end{aligned}$$

for infinitesimally small but positive ϵ. For $a=0$ the condition $|t|^n-\epsilon<|z_j|$ becomes superfluous and we recover (5.2). We will sometimes somewhat loosely say that $z_j=u$ is not in $\mathbb{A}^{n-a}$ when we really mean that $z^{(n-a)}=(z_1,\dots,z_{n-a})$

with $z_j = u$ is not in $\mathbb{A}^{n-a}$. Since, $|Z_{n-a-1}^{-1}| = |Z_{n-a}^{-1}||z_{n-a}| \le |t|^{-a-2} + \epsilon$ we have the filtration $\mathbb{A}^0 \subset \mathbb{A}^1 \subset \cdots \subset \mathbb{A}^{n-a}$, so that $f(z^{(n-a)}, t^{-1}w_{n-a+1}, \dots, t^{-1}w_n)$ is holomorphic on $\mathbb{A}^{n-a}$.

We must also generalize the definition of C_w^n given in Theorem 5.2. To simplify notations we drop the explicit w dependence of C_w^n and write C_w^n as C_0^n, with 0 a new label (not related to the w-variables). More generally we define $C_m^{n-a} \subset \mathbb{A}^{n-a}$ for $m \in \mathbb{Z}_n$ and $a \in \mathbb{Z}_{m+1}$ as deformations of $\mathbb{T}^{n-a}$ such that for fixed $w \in \mathbb{C}^{n-a}$ and $i \in [n-a]$,

(5.32a) $\quad z_j = t^{-1}w_i^{\pm}$ lies in the interior of C_m^{n-a} for $j \in \{m-a+1, \dots, n-a\}$

(5.32b) $\quad z_j = t^{-1}w_i^{\pm}$ lies in the exterior of C_m^{n-a} for $j \in [m-a]$

(5.32c) $\quad Z_{n-a}^{-1} = t^{-a-1}w_i^{\pm}\overline{W}_a$ lies in the exterior of C_m^{n-a}.

For $m = a = 0$ this definition simplifies to (5.8). If C_m^{n-a} and $\hat{C}_m^{n-a}$ both satisfy (5.32) they will be referred to as homotopic.

Finally, we introduce the shorthand notation

$$\mathcal{F}(z^{(n-a)}, w) := \Delta(z^{(n-a)}, w, x; t) f(z^{(n-a)}, t^{-1}w_{n-a+1}, \dots, t^{-1}w_n),$$

where the x and t dependence of $\mathcal{F}$ have been suppressed.

Lemma 5.1. *For $m \in \mathbb{Z}_n$*

$$I(x;t) = \sum_{a=0}^{m} (4\pi \mathrm{i})^a \binom{m}{a} \frac{n!}{(n-a)!} \int_{\mathbb{T}^n} \int_{C_m^{n-a}} \mathcal{F}(z^{(n-a)}, w) \frac{\mathrm{d}z^{(n-a)}}{z^{(n-a)}} \frac{\mathrm{d}w}{w}. \tag{5.33}$$

Proof. We prove this by induction on m. Since for $m = 0$ we recover definition (5.29) of $I(x;t)$, we only need to establish the induction step. Writing the expression on the right of (5.33) for fixed m as $I_m(x;t)$, the problem is to show that $I_m(x;t) = I_{m+1}(x;t)$ for $m \in \mathbb{Z}_{n-1}$.

Since

$$\Delta(z^{(n-a)}, w, x; t) = 0 \text{ if } w_i = w_j \text{ for } 1 \le i < j \le n-a, \tag{5.34}$$

we may without loss of generality assume that all components of $w^{(n-a)}$ are distinct when considering the $z^{(n-a)}$-integration for fixed w.

In the integral over $z^{(n-a)}$ we deform C_m^{n-a} to $B_m^{n-a} \subset \mathbb{A}^{n-a}$ such that

$$\begin{aligned} z_j = t^{-1}w_i^{\pm} &\text{ lies in the interior of } B_m^{n-a} \text{ for } j \in \{m-a+1, \dots, n-a-1\} \\ z_j = t^{-1}w_i^{\pm} &\text{ lies in the exterior of } B_m^{n-a} \text{ for } j \in [m-a] \cup \{n-a\} \\ Z_{n-a}^{-1} = t^{-a-1}w_i^{\pm}\overline{W}_a &\text{ lies in the exterior of } B_m^{n-a}. \end{aligned}$$

To see how this deformation changes the integral over $z^{(n-a)}$ we need to investigate the location of the poles of the integrand. From (2.2) and (5.28) it follows that $\Delta(z^{(n-a)}, w, x; t)$ has poles at

$$\begin{aligned} z_j &= t^{-1}w_i^{\pm}p^{\mu}q^{\nu} && \text{for } i, j \in [n-a] \\ Z_{n-a}^{-1} &= t^{-a-1}w_i^{\pm}\overline{W}_a p^{\mu}q^{\nu} && \text{for } i \in [n-a] \\ z_j &= t^{-1}w_i^{\pm}p^{\mu+1}q^{\nu+1} && \text{for } i \in \{n-a+1, \dots, n\},\ j \in [n-a] \\ Z_{n-a}^{-1} &= t^{-a-1}w_i^{\pm}\overline{W}_a p^{\mu+1}q^{\nu+1} && \text{for } i \in \{n-a+1, \dots, n\}. \end{aligned}$$

We note in particular that the terms in the last line of (5.28) do not imply any poles for $\Delta(z^{(n-a)}, w, x; t)$ thanks to the reflection equation (2.5). Of the poles listed above only those corresponding to the first two lines with $(\mu,\nu)=(0,0)$ are in $\mathbb{A}^{n-a}$, i.e., the poles at $z_j=t^{-1}w_i^{\pm}$ for $i,j\in[n-a]$ and at $Z_{n-a}^{-1}=t^{-a-1}w_i^{\pm}\overline{W}_a$ for $i\in[n-a]$. Indeed, for $(\mu,\nu)\neq(0,0)$, all of the above z_j satisfy (since $w\in\mathbb{T}^n$) $|z_j|\leq|t|^{-1}M<|t|^n$, incompatible with (5.31). Likewise, for $(\mu,\nu)\neq(0,0)$, the above listed poles for Z_{n-a}^{-1} satisfy $|Z_{n-a}^{-1}|\leq|t|^{-a-1}M<|t|^{n-a}$. But from (5.31) we have $|z_j|<|t|^{-1}$ for all $j\in[n-a]$ which implies that $|Z_{n-a}^{-1}|>|t|^{n-a}$.

Comparing the definitions C_m^{n-a} and B_m^{n-a} we thus see that the only difference between the integral over the former and the latter is that the poles at $z_{n-a}=t^{-1}w_k^{\pm}$ for $k\in[n-a]$ have moved to the exterior of B_m^{n-a} To compensate for this discrepancy we need to calculate the residues (denoted by $R_{k,\pm}$) of the integrand at $z_{n-a}=t^{-1}w_k^{\pm}$ and integrate this over $B_{m;k,\pm}^{n-a-1}\subset\mathbb{A}^{n-a-1}$. Here $B_{m;k,\pm}^{n-a-1}$ corresponds to "C_m^{n-a} restricted to $z_{n-a}=t^{-1}w_k^{\pm}$." That is, for fixed $w\in\mathbb{T}^{n-a}$ and $i\in[n-a-1]$, $i\neq k$,

$$\begin{aligned} z_j=t^{-1}w_i^{\pm} &\text{ lies in the interior of } B_{m;k,\sigma}^{n-a-1} \\ &\qquad\qquad\qquad \text{for } j\in\{m-a+1,\dots,n-a-1\} \\ z_j=t^{-1}w_i^{\pm} &\text{ lies in the exterior of } B_{m;k,\sigma}^{n-a-1} \text{ for } j\in[m-a] \\ Z_{n-a-1}^{-1}=t^{-a-2}w_i^{\pm}w_k^{\sigma}\overline{W}_a p^{\mu}q^{\nu} &\text{ lies in the exterior of } B_{m;k,\sigma}^{n-a-1}. \end{aligned}$$

The exclusion of $i=k$ is justified by the fact that $R_{k,\sigma}$ is free of poles at the above when $i=k$ thanks to (5.34). By (5.27) with $a\to a+1$ and the fact that f is holomorphic on $\mathbb{A}^{n-a}$ we get

$$\begin{aligned} I_m(x;t)=\sum_{a=0}^{m}(4\pi\mathrm{i})^a\binom{m}{a}\frac{n!}{(n-a)!}\biggl[\int_{\mathbb{T}^n}\int_{B_m^{n-a}}\mathcal{F}(z^{(n-a)},w)\frac{\mathrm{d}z^{(n-a)}}{z^{(n-a)}}\frac{\mathrm{d}w}{w} \\ +2\pi\mathrm{i}\sum_{k=1}^{n-a}\sum_{\sigma\in\{\pm1\}}\int_{\mathbb{T}^n}\int_{B_{m;k,\sigma}^{n-a-1}}\mathcal{F}(z^{(n-a-1)},w)|_{w_{n-a}\leftrightarrow w_k^{\sigma}}\frac{\mathrm{d}z^{(n-a-1)}}{z^{(n-a-1)}}\frac{\mathrm{d}w}{w}\biggr]. \end{aligned}$$

We now change integration variables in both terms inside the square brackets. In the first term we substitute $z_{n-a}\leftrightarrow z_{m-a+1}$ and in the second term (or rather its summand for fixed σ) we substitute $w_k^{\sigma}\leftrightarrow w_{n-a}$. Using the permutation symmetry of f and noting that $B_m^{n-a}|_{z_{n-a}\leftrightarrow z_{m-a+1}}$ is homotopic to C_{m+1}^{n-a} and $B_{m;k,\sigma}^{n-a-1}|_{w_k\leftrightarrow w_{n-a}}=B_{m;n-a,1}^{n-a-1}$ is homotopic to C_{m+1}^{n-a-1}, this yields

$$\begin{aligned} I_m(x;t)=\sum_{a=0}^{m}(4\pi\mathrm{i})^a\binom{m}{a}\frac{n!}{(n-a)!}\biggl[\int_{\mathbb{T}^n}\int_{C_{m+1}^{n-a}}\mathcal{F}(z^{(n-a)},w)\frac{\mathrm{d}z^{(n-a)}}{z^{(n-a)}}\frac{\mathrm{d}w}{w} \\ +(4\pi\mathrm{i})(n-a)\int_{\mathbb{T}^n}\int_{C_{m+1}^{n-a-1}}\mathcal{F}(z^{(n-a-1)},w)\frac{\mathrm{d}z^{(n-a-1)}}{z^{(n-a-1)}}\frac{\mathrm{d}w}{w}\biggr]. \end{aligned}$$

Finally changing the summation index $a\to a-1$ in the sum corresponding to the second term inside the square brackets, and using the standard binomial recursion leads to the desired $I_m(x;t)=I_{m+1}(x;t)$. □

From Lemma 5.1 with $m=n-1$ it follows that

$$I(x;t)=\sum_{a=0}^{n-1}(4\pi\mathrm{i})^a\binom{n-1}{a}\frac{n!}{(n-a)!}\int_{\mathbb{T}^n}\int_{C^{n-a}}\mathcal{F}(z^{(n-a)},w)\frac{\mathrm{d}z^{(n-a)}}{z^{(n-a)}}\frac{\mathrm{d}w}{w},$$

where the label $n-1$ of C_{n-1}^{n-a} has been dropped, having served its purpose.

The second lemma needed to prove Proposition 5.1 should thus read as follows.

Lemma 5.2. *There holds*

$$\begin{aligned}(5.35)\qquad &\sum_{a=0}^{n-1}(4\pi\mathrm{i})^a\binom{n-1}{a}\frac{n!}{(n-a)!}\int_{\mathbb{T}^n}\int_{C^{n-a}}\mathcal{F}(z^{(n-a)},w)\frac{\mathrm{d}z^{(n-a)}}{z^{(n-a)}}\frac{\mathrm{d}w}{w}\\ &\qquad=\sum_{a=0}^{n}(4\pi\mathrm{i})^a\binom{n}{a}\frac{(n+1)!}{(n-a+1)!}\int_{\mathbb{T}^n}\int_{\mathbb{T}^{n-a}}\mathcal{F}(z^{(n-a)},w)\frac{\mathrm{d}z^{(n-a)}}{z^{(n-a)}}\frac{\mathrm{d}w}{w}.\end{aligned}$$

Proof. The only difference between the two integrals over $z^{(n-a)}$ in (5.35) is that C^{n-a} — given by (5.32) with $m=n-1$ — has the poles of the the integrand at $z_{n-a}=t^{-1}w_k^{\pm}$ for $k\in[n-a]$ in its interior and the poles at $Z_{n-a}^{-1}=t^{-a-1}w_k^{\pm}\overline{W}_a$ for $k\in[n-a]$ (i.e., $z_{n-a}=t^{a+1}w_k^{\pm}\overline{W}_a^{-1}Z_{n-a-1}^{-1}$) in its exterior, whereas $\mathbb{T}^{n-a}$ has the latter in its interior and the former in its exterior. Hence, applying (5.27) and

$$\begin{aligned}f(z^{(n-a-1)},t^{a+1}w_k^{-\sigma}\overline{W}_a^{-1}Z_{n-a-1}^{-1},t^{-1}w_{n-a+1},\dots,t^{-1}w_n)&\\ =f(z^{(n-a-1)},t^{-1}w_k^{\sigma},t^{-1}w_{n-a+1},\dots,t^{-1}w_n)&\end{aligned}$$

as follows from the A_n symmetry of f, we get

$$\begin{aligned}\int_{\mathbb{T}^n}\int_{C^{n-a}}\mathcal{F}(z^{(n-a)},w)\frac{\mathrm{d}z^{(n-a)}}{z^{(n-a)}}\frac{\mathrm{d}w}{w}&\to\int_{\mathbb{T}^n}\int_{\mathbb{T}^{n-a}}\mathcal{F}(z^{(n-a)},w)\frac{\mathrm{d}z^{(n-a)}}{z^{(n-a)}}\frac{\mathrm{d}w}{w}\\ +4\pi\mathrm{i}\sum_{k=1}^{n-a}\sum_{\sigma\in\{\pm1\}}&\int_{\mathbb{T}^n}\int_{\mathbb{T}^{n-a-1}}\mathcal{F}(z^{(n-a-1)},w)|_{w_{n-a}\leftrightarrow w_k^{\sigma}}\frac{\mathrm{d}z^{(n-a-1)}}{z^{(n-a-1)}}\frac{\mathrm{d}w}{w}\\ &=\int_{\mathbb{T}^n}\int_{\mathbb{T}^{n-a}}\mathcal{F}(z^{(n-a)},w)\frac{\mathrm{d}z^{(n-a)}}{z^{(n-a)}}\frac{\mathrm{d}w}{w}\\ &\quad+8\pi\mathrm{i}(n-a)\int_{\mathbb{T}^n}\int_{\mathbb{T}^{n-a-1}}\mathcal{F}(z^{(n-a-1)},w)\frac{\mathrm{d}z^{(n-a-1)}}{z^{(n-a-1)}}\frac{\mathrm{d}w}{w}.\end{aligned}$$

Here the last expression on the right follows after the variable change $w_k^{\sigma}\leftrightarrow w_{n-a}$ in the second double integral. We wish to emphasize that one of the factors 2 in $4\pi\mathrm{i}\sum_k\dots$ is due to the fact that the poles at $z_{n-a}=t^{-1}w_k^{\pm}$ and $z_{n-a}=t^{a+1}w_k^{\mp}\overline{W}_a^{-1}Z_{n-a-1}^{-1}$ yields the same contribution by virtue of (5.27). In what follows we further examine the contributions arising from $z_{n-a}=t^{-1}w_i^{\pm}$.

The reason for putting an arrow instead of an equal sign in the above is that the expression on the right is overcounting poles and needs an additional correction term. Indeed, we have computed the residues of $\mathcal{F}(z^{(n-a)},w)/z_{n-a}$ at its poles $z_{n-a}=t^{-1}w_k^{\pm}$. By exploiting the symmetry (5.27) and by making a variable change in the w-variables this effectively boiled down to picking up the residue at $z_{n-a}=t^{-1}w_{n-a}$ exactly $2(n-a)$ times. This residue, given by $\mathcal{F}(z^{(n-a-1)},w)$, has poles at

$$(5.36)\qquad Z_{n-a-1}=t^{a+2}w_i^{-\sigma}\overline{W}_{a+1}^{-1},\quad k\in[n-a-1].$$

These poles are in the interior of $\mathbb{T}^{n-a-1}$ and thus contribute to the integral over $\mathcal{F}(z^{(n-a-1)},w)$. But (5.36) times $z_{n-a}=t^{-1}w_{n-a}$ yields $Z_{n-a}=t^{a+1}w_i^{-\sigma}\overline{W}_a^{-1}$, or, equivalently, $Z_{n-a}^{-1}=t^{-a-1}w_i^{\sigma}\overline{W}_a$. According to (5.32) with $m=n-1$ these poles lie in the exterior of C^{n-a} and hence the poles (5.36) should not be contributing at all! Consequently we need to subtract the further term

$$\begin{aligned}
&-2(2\pi\mathrm{i})^2(n-a)\sum_{k=1}^{n-a-1}\sum_{\sigma\in\{\pm1\}}\\
&\qquad\times\int_{\mathbb{T}^n}\int_{\mathbb{T}^{n-a-2}}\mathcal{F}(z^{(n-a-2)},w)|_{w_{n-a-1}\leftrightarrow w_k^\sigma}\frac{\mathrm{d}z^{(n-a-2)}}{z^{(n-a-2)}}\frac{\mathrm{d}w}{w}\\
&\quad=-(4\pi\mathrm{i})^2(n-a)(n-a-1)\int_{\mathbb{T}^n}\int_{\mathbb{T}^{n-a-2}}\mathcal{F}(z^{(n-a-2)},w)\frac{\mathrm{d}z^{(n-a-2)}}{z^{(n-a-2)}}\frac{\mathrm{d}w}{w},
\end{aligned}$$

where we have used the second equality in (5.27) with $a\to a+2$, and the A_n symmetry of f. Therefore

$$\begin{aligned}
\int_{\mathbb{T}^n}\int_{C^{n-a}}\mathcal{F}(z^{(n-a)},w)\frac{\mathrm{d}z^{(n-a)}}{z^{(n-a)}}\frac{\mathrm{d}w}{w}&=\int_{\mathbb{T}^n}\int_{\mathbb{T}^{n-a}}\mathcal{F}(z^{(n-a)},w)\frac{\mathrm{d}z^{(n-a)}}{z^{(n-a)}}\frac{\mathrm{d}w}{w}\\
&\quad+2(4\pi\mathrm{i})(n-a)\int_{\mathbb{T}^n}\int_{\mathbb{T}^{n-a-1}}\mathcal{F}(z^{(n-a-1)},w)\frac{\mathrm{d}z^{(n-a-1)}}{z^{(n-a-1)}}\frac{\mathrm{d}w}{w}\\
&\quad+(4\pi\mathrm{i})^2(n-a)(n-a-1)\int_{\mathbb{T}^n}\int_{\mathbb{T}^{n-a-2}}\mathcal{F}(z^{(n-a-2)},w)\frac{\mathrm{d}z^{(n-a-2)}}{z^{(n-a-2)}}\frac{\mathrm{d}w}{w}.
\end{aligned}$$

Substituting this in the left-hand side of (5.35), shifting $a\to a-1$ and $a\to a-2$ in the sums corresponding to the integrals over $\mathbb{T}^{n-a-1}$ and $\mathbb{T}^{n-a-2}$ and using the binomial identity

$$\binom{n-1}{a}+2\binom{n-1}{a-1}+\binom{n-1}{a-2}=\binom{n+1}{a}=\frac{n+1}{n-a+1}\binom{n}{a}$$

yields the wanted right-hand side of (5.35), completing the proof. ☐

In the integral on the right-hand side of (5.30) we make the variable changes $z_i\to z_{i+a}$ for $i\in[n-a]$ followed by $t^{-1}w_{n-a+i}\to z_i$ for $i\in[a]$. By the permutation symmetry of f this gives

$$\begin{aligned}
(5.37)\quad I(x;t)=\sum_{a=0}^{n}(4\pi\mathrm{i})^a\binom{n}{a}\frac{(n+1)!}{(n-a+1)!}&\\
\times\int_{\mathbb{T}^{n-a}}\int_{(t^{-1}\mathbb{T})^a\times\mathbb{T}^{n-a}}\Delta(z,w^{(n-a)},x;t)f(z)\frac{\mathrm{d}z}{z}\frac{\mathrm{d}w^{(n-a)}}{w^{(n-a)}}&,
\end{aligned}$$

with

$$\Delta(z,w^{(n-a)},x;t):=\Delta((z_{a+1},\dots,z_n),(w_1,\dots,w_{n-a},tz_1,\dots,tz_a),x;t)$$

given by the somewhat unwieldy expression

$$\begin{aligned}\Delta(z,w^{(n-a)},x;t)&=\frac{1}{(p;p)_\infty^a(q;q)_\infty^a}\prod_{1\le i<j\le n-a}\frac{1}{\Gamma(w_i^{\pm}w_j^{\pm})}\\&\times\prod_{j=1}^{a}\frac{\prod_{i=1}^{n+1}\Gamma(t^2x_iz_j,x_iz_j^{-1})}{\Gamma(t^2z_j^2)\prod_{i=a+1}^{n+1}\Gamma(z_iz_j^{-1},t^2z_iz_j)}\\&\times\prod_{i=1}^{n-a}\frac{\prod_{j=1}^{n+1}\Gamma(tw_i^{\pm}x_j)\prod_{j=a+1}^{n+1}\Gamma(t^{-1}w_i^{\pm}z_j^{-1})}{\Gamma(w_i^{\pm 2})\prod_{j=1}^{a}\Gamma(tw_i^{\pm}z_j)}\\&\times\prod_{1\le i<j\le a}\frac{1}{\Gamma(z_iz_j^{-1},z_i^{-1}z_j,t^2z_iz_j,t^2z_iz_j)}\\&\times\prod_{a+1\le i<j\le n+1}\frac{1}{\Gamma(z_iz_j^{-1},z_i^{-1}z_j,t^2z_iz_j,t^{-2}z_i^{-1}z_j^{-1})}.\end{aligned}\tag{5.38}$$

It is easily checked that the only poles of $\Delta(z,w^{(n-a)},x;t)$ in the z_j-plane located on the annulus $1<|z_j|<|t|^{-1}$ for $j\in[a]$ are given by $z_j=x_i$ for $i\in[n]$. For example, the pole at $z_{n+1}=t^{-1}w_i^{\pm}p^\mu q^\nu$ corresponds to a pole in the z_j-plane ($j\in[a]$) at $z_j=tw_i^{\pm}(\prod_{k=1;k\neq j}^{n}z_k)^{-1}p^{-\mu}q^{-\nu}$ with absolute value $|z_j|=|t|^a|p|^{-\mu}|q|^{-\nu}$. For $(\mu,\nu)=(0,0)$ this implies $|z_j|<1$ and for $(\mu,\nu)\neq(0,0)$ this implies $|z_j|>|t|^aM^{-1}>|t|^{n-1}M^{-1}>|t|^{-1}$. As another example, the pole at $z_j^2=t^{-2}p^{\mu+1}q^{\nu+1}$ has absolute value $|z_j|^2\le|t^{-2}pq|\le|t^{-2}|M^2\le|t|^{2n}\le 1$, et cetera. Consequently, in deflating the contours $t^{-1}\mathbb{T}$ to $\mathbb{T}$ the poles at $z_j=x_i$ for $j\in[a]$ and $i\in[n]$ move from the interior to the exterior but no other poles of the integrand cross the contours of integration. Recursively defining the necessary residues as

$$\begin{aligned}&\Delta^{k_1,\dots,k_b}(z^{(n-b)},w^{(n-a)},x;t)\\&\quad=\operatorname{Res}_{\zeta=x_{k_1}}\frac{\Delta^{k_2,\dots,k_b}((z_1,\dots,z_{a-b},\zeta,z_{a-b+2},\dots,z_{n-b}),w^{(n-a)},x;t)}{\zeta}\end{aligned}\tag{5.39}$$

for $b\in[a]$ and $k_1,\dots,k_b\in[r]$ with $k_i\neq k_j$ we get our third lemma.

Lemma 5.3. *There holds*

$$\begin{aligned}I(x;t)=\sum_{a=0}^{n}\sum_{b=0}^{a}\frac{2^an!(n+1)!(2\pi\mathrm{i})^{a+b}}{(a-b)!(n-a)!(n-a+1)!}\sum_{1\le k_1<\dots<k_b\le n}\int_{\mathbb{T}^{n-a}}\int_{\mathbb{T}^{n-b}}\\\times\Delta^{k_1,\dots,k_b}(z^{(n-b)},w^{(n-a)},x;t)f(z^{(n-b)},x_{k_1},\dots,x_{k_b})\frac{\mathrm{d}z^{(n-b)}}{z^{(n-b)}}\frac{\mathrm{d}w^{(n-a)}}{w^{(n-a)}}.\end{aligned}$$

Proof. Since $\Delta(z,w^{(n-a)},x;t)$ exhibits permutation symmetry in the variables z_1, $\dots,z_a$ it follows that

$$\Delta^{k_1,\dots,k_b}(z^{(n-b)},w^{(n-a)},x;t)=\Delta^{\mathrm{w}(k_1,\dots,k_b)}(z^{(n-b)},w^{(n-a)},x;t)$$

for $\mathrm{w}\in S_a$. Hence $\Delta^{k_1,\dots,k_b}(z^{(n-b)},w^{(n-a)},x;t)$ is invariant under permutation of the variables $x_{k_1},\dots,x_{k_b}$.

After this preliminary comment we will show by induction on c that

$$\begin{aligned}(5.40)\quad &\int_{(t^{-1}\mathbb{T})^a\times\mathbb{T}^{n-a}} \Delta(z,w^{(n-a)},x;t)f(z)\frac{\mathrm{d}z}{z}\\ &\quad=\sum_{b=0}^{c}(2\pi\mathrm{i})^b\binom{c}{b}\sum_{\substack{k_1,\dots,k_b=1\\ k_i\neq k_j}}^{n}\int_{(t^{-1}\mathbb{T})^{a-c}\times\mathbb{T}^{n-a+c-b}}\\ &\qquad\times\Delta^{k_1,\dots,k_b}\big(z^{(n-b)},w^{(n-a)},x;t\big)f\big(z^{(n-b)},x_{k_1},\dots,x_{k_b}\big)\frac{\mathrm{d}z^{(n-b)}}{z^{(n-b)}}\end{aligned}$$

for $c\in\mathbb{Z}_{a+1}$. Since this is trivially true for $c=0$ we only need to establish the induction step. Let $L(a,c)$ denote the left hand side of (5.40). Then

$$\begin{aligned}L(a,c)&=\sum_{b=0}^{c}(2\pi\mathrm{i})^b\binom{c}{b}\sum_{\substack{k_1,\dots,k_b=1\\ k_i\neq k_j}}^{n}\int_{(t^{-1}\mathbb{T})^{a-c-1}\times\mathbb{T}^{n-a+c-b+1}}\\ &\quad\times\Delta^{k_1,\dots,k_b}\big(z^{(n-b)},w^{(n-a)},x;t\big)f\big(z^{(n-b)},x_{k_1},\dots,x_{k_b}\big)\frac{\mathrm{d}z^{(n-b)}}{z^{(n-b)}}\\ &\quad+\sum_{b=0}^{c}(2\pi\mathrm{i})^{b+1}\binom{c}{b}\sum_{\substack{k_0,k_1,\dots,k_b=1\\ k_i\neq k_j}}^{n}\int_{(t^{-1}\mathbb{T})^{a-c-1}\times\mathbb{T}^{n-a+c-b}}\\ &\quad\times\Delta^{k_0,\dots,k_b}\big(z^{(n-b-1)},w^{(n-a)},x;t\big)f\big(z^{(n-b-1)},x_{k_0},\dots,x_{k_b}\big)\frac{\mathrm{d}z^{(n-b-1)}}{z^{(n-b-1)}}.\end{aligned}$$

In the seccond sum we shift the summation index $b\to b+1$ and relabel the k_i as k_{i+1}. By the standard binomial recurrence we then find that $L(a,c)=L(a,c+1)$ as desired.

To now obtain the expansion of Lemma 5.3 we choose $c=a$ in (5.40) and use the permutation symmetry of the integrand in the x_{k_i} to justify the simplification

$$\sum_{\substack{k_1,\dots,k_b=1\\ k_i\neq k_j}}^{n}(\dots)\to b!\sum_{1\le k_1<\dots<k_b\le n}(\dots).$$

Substituting the resulting expression in (5.37) completes the proof. □

By changing the order of the sums as well as the order of the integrals, the identity of Lemma 5.3 can be rewritten as

$$\begin{aligned}I(x;t)&=\sum_{b=0}^{n}\sum_{1\le k_1<\dots<k_b\le n}\sum_{a=b}^{n}\frac{2^a n!(n+1)!(2\pi\mathrm{i})^{a+b}}{(a-b)!(n-a)!(n-a+1)!}\\ &\quad\times\int_{\mathbb{T}^{n-b}}\bigg[\int_{\mathbb{T}^{n-a}}\Delta^{k_1,\dots,k_b}\big(z^{(n-b)},w^{(n-a)},x;t\big)\frac{\mathrm{d}w^{(n-a)}}{w^{(n-a)}}\bigg]\\ &\qquad\times f\big(z^{(n-b)},x_{k_1},\dots,x_{k_b}\big)\frac{\mathrm{d}z^{(n-b)}}{z^{(n-b)}}.\end{aligned}$$

In what may well be considered the second part of the proof of Theorem 5.2 we will prove the following proposition.

Proposition 5.2. *For $b \in \mathbb{Z}_n$ and fixed $1 \le k_1 < \cdots < k_b \le n$ we have*

(5.41)
$$\sum_{a=b}^{n} \frac{(4\pi\mathrm{i})^a}{(n-a+1)!}\binom{n-b}{n-a}\int_{\mathbb{T}^{n-b}}\biggl[\int_{\mathbb{T}^{n-a}} \Delta^{k_1,\dots,k_b}(z^{(n-b)},w^{(n-a)},x;t)\frac{\mathrm{d}w^{(n-a)}}{w^{(n-a)}}\biggr] \times f(z^{(n-b)},x_{k_1},\dots,x_{k_b})\frac{\mathrm{d}z^{(n-b)}}{z^{(n-b)}}=0.$$

From this result it follows that the only non-vanishing contribution to $I(x;t)$ comes from $b=a=n$ and $(k_1,\dots,k_n)=(1,\dots,n)$. As we shall see shortly,

$$\Delta(z^{(0)},w^{(0)},x;t) := \Delta^{1,\dots,n}(z^{(0)},w^{(0)},x;t) = \frac{1}{(p;p)_\infty^{2n}(q;q)_\infty^{2n}}, \tag{5.42}$$

leading to $\kappa^{\mathscr{A}}\kappa^{\mathscr{C}} I(x;t)=f(x)$ as claimed by the theorem.

Proof of Proposition 5.2. It is sufficient to prove the proposition for $(k_1,\dots,k_b)=(1,\dots,b)$. Other choices of the k_i (for fixed b) simply follow by a relabelling of the x_i variables. The advantage of this particular choice of k_i is that many of the expressions below significantly simplify.

The first ingredient needed for the proof is the actual computation of

$$\Delta(z^{(n-b)},w^{(n-a)},x;t) := \Delta^{1,\dots,b}(z^{(n-b)},w^{(n-a)},x;t).$$

From definition (5.39), and the equations (5.38) and (2.6) it is not hard to show that

$$\begin{aligned}
\Delta(z^{(n-b)},w^{(n-a)},x;t) &= \frac{1}{(p;p)_\infty^{a+b}(q;q)_\infty^{a+b}}\prod_{1\le i<j\le n-a}\frac{1}{\Gamma(w_i^{\pm}w_j^{\pm})}\\
&\times \prod_{i=1}^{b}\frac{\prod_{j=b+1}^{n+1}\Gamma(x_i^{-1}x_j,t^2x_ix_j)}{\prod_{j=1}^{n-b+1}\Gamma(x_i^{-1}z_j,t^2x_iz_j)}\prod_{j=1}^{a-b}\frac{\prod_{i=b+1}^{n+1}\Gamma(t^2x_iz_j,x_iz_j^{-1})}{\Gamma(t^2z_j^2)\prod_{i=a-b+1}^{n-b+1}\Gamma(z_iz_j^{-1},t^2z_iz_j)}\\
&\times \prod_{i=1}^{n-a}\frac{\prod_{j=b+1}^{n+1}\Gamma(tw_i^{\pm}x_j)\prod_{j=a-b+1}^{n-b+1}\Gamma(t^{-1}w_i^{\pm}z_j^{-1})}{\Gamma(w_i^{\pm 2})\prod_{j=1}^{a-b}\Gamma(tw_i^{\pm}z_j)}\\
&\times \prod_{1\le i<j\le a-b}\frac{1}{\Gamma(z_iz_j^{-1},z_i^{-1}z_j,t^2z_iz_j,t^2z_iz_j)}\\
&\times \prod_{a-b+1\le i<j\le n-b+1}\frac{1}{\Gamma(z_iz_j^{-1},z_i^{-1}z_j,t^2z_iz_j,t^{-2}z_i^{-1}z_j^{-1})},
\end{aligned}$$

where, given $z^{(n-b)}$,

$$z_{n-b+1} := X_b^{-1}Z_{n-b}^{-1}, \tag{5.43}$$

i.e., $z_1\cdots z_{n-b+1}=X_b^{-1}=x_1^{-1}\cdots x_b^{-1}$. For $b=n$ this implies $z_1=X_n^{-1}=x_{n+1}$ so that $\Delta(z^{(0)},w^{(0)},x;t)$ is given by (5.42).

In the C_n elliptic elliptic beta integral (3.5) we deform $\mathbb{T}^n$ to $C^n=C\times\cdots\times C$ with $C=C^{-1}\subset\mathbb{C}$ the usual positively oriented Jordan curve, such that the points $t_ip^{\mu}q^{\nu}$ for $i\in[2n+4]$ are in the interior of C. With $\mathbb{T}^n$ replaced by such C^n the integral (3.5) holds for all $t_1,\dots,t_{2n+4}$ subject only to $t_1\cdots t_{2n+4}=pq$. Then choosing $t_1 \cdots t_{2n+2} = 1$ and $t_{2n+3}t_{2n+4} = pq$ and using (2.3b) and $\Gamma(pq) = 0$, we get

$$\int_{C^n} \prod_{j=1}^{n} \frac{\prod_{i=1}^{2n+2} \Gamma(t_i z_j^{\pm})}{\Gamma(z_j^{\pm 2})} \prod_{1 \le i < j \le n} \frac{1}{\Gamma(z_i^{\pm} z_j^{\pm})} \frac{\mathrm{d}z}{z} = 0,$$

where $C^n = C \times \cdots \times C$ such that C has the points $t_i p^{\mu} q^{\mu}$ for $i \in [2n+2]$ in its interior. Replacing z by w, n by $n-b$, and $t_i \to t x_{i+b}$, $t_{i+n-b+1} \to t^{-1} z_i^{-1}$ for $i \in [n-b+1]$, with z_{n-b+1} defined by (5.43) to ensure that

$$\begin{aligned} 1 = t_1 \cdots t_{2n+2} \to x_{b+1} \cdots x_{n+1} z_1^{-1} \cdots z_{n-b+1}^{-1} \\ = x_{b+1} \cdots x_{n+1} X_b = x_1 \cdots x_{n+1} = 1, \end{aligned}$$

yields,

$$\int_{C^{n-b}} \Delta(z^{(n-b)}, w^{(n-b)}, x; t) \frac{\mathrm{d}w^{(n-b)}}{w^{(n-b)}} = 0$$

for $b \in \mathbb{Z}_n$. Here $C^{n-b} = C \times \cdots \times C$ such that C has the points $t x_{i+b} p^{\mu} q^{\nu}$ and $t^{-1} z_i^{-1} p^{\mu} q^{\nu}$ for $i \in [n-b+1]$ in its interior. Obviously, we then also have

$$\int_{\mathbb{T}^{n-b}} \int_{C^{n-b}} \Delta(z^{(n-b)}, w^{(n-b)}, x; t) f(z^{(n-b)}, x^{(b)}) \frac{\mathrm{d}w^{(n-b)}}{w^{(n-b)}} \frac{\mathrm{d}z^{(n-b)}}{z^{(n-b)}} = 0, \tag{5.44}$$

to be compared with (5.41).

Deforming C^{n-b} to $\mathbb{T}^{n-b}$ by successively deforming the 1-dimensional contours C to $\mathbb{T}$, the poles of the integrand at $t^{-1} z_i^{-1}$ $(t z_i)$ for $i \in [n-b+1]$ move from the interior (exterior) of C to the exterior (interior) of $\mathbb{T}$, but no other poles cross the contours of integration. The function $\Delta(z^{(n-b)}, w^{(n-a)}, x; t)$ has been obtained from $\Delta(z, w, x; t)$ by computing residues corresponding to poles in the z-variables, see (5.26) and (5.39). Presently we are at a stage of the calculation that requires the computation of residues with respect to the above-listed poles in the w-variables. Amazingly, this does not lead to a further generalization of $\Delta(z, w, x; t)$. Indeed, the following truly remarkable relation holds for $a \in \{b, \dots, n\}$:

$$\begin{aligned} &\mathrm{Res}_{w_{n-a} = t^{-1} z_{a-b+1}^{-1}} \frac{\Delta(z^{(n-b)}, w^{(n-a)}, x; t)}{w_{n-a}} \\ &\quad = -\mathrm{Res}_{w_{n-a} = t z_{a-b+1}} \frac{\Delta(z^{(n-b)}, w^{(n-a)}, x; t)}{w_{n-a}} = \Delta(z^{(n-b)}, w^{(n-a-1)}, x; t). \end{aligned} \tag{5.45}$$

Moreover, since $\Delta(z^{(n-b)}, w^{(n-a)}, x; t)$ exhibits permutation symmetry in the variables $z_{a-b+1}, \dots, z_{n-b+1}$ we also have

$$\begin{aligned} &\mathrm{Res}_{w_{n-a} = t^{-1} z_j^{-1}} \frac{\Delta(z^{(n-b)}, w^{(n-a)}, x; t)}{w_{n-a}} \\ &\quad = -\mathrm{Res}_{w_{n-a} = t z_j} \frac{\Delta(z^{(n-b)}, w^{(n-a)}, x; t)}{w_{n-a}} \\ &\quad\quad = \Delta(z^{(n-b)}, w^{(n-a-1)}, x; t)|_{z_{a-b+1} \leftrightarrow z_j} \end{aligned} \tag{5.46}$$

for $j \in \{a-b+1, \dots, n-b+1\}$. These results imply our next lemma.

Lemma 5.4. *For $b \in \mathbb{Z}_n$ there holds*

$$\sum_{a=b}^{n} \frac{(4\pi\mathrm{i})^a}{(n-a)!} \binom{n-b}{n-a} \biggl[\int_{\mathbb{T}^{n-b}} + \frac{1}{n-a+1} \sum_{c=1}^{a-b} \int_{\mathbb{T}^{c-1}\times(X_b^{-1}\mathbb{T})\times\mathbb{T}^{n-b-c}} \biggr] \\ \times \int_{\mathbb{T}^{n-a}} \Delta(z^{(n-b)}, w^{(n-a)}, x; t) f(z^{(n-b)}, x^{(b)}) \frac{\mathrm{d}w^{(n-a)}}{w^{(n-a)}} \frac{\mathrm{d}z^{(n-b)}}{z^{(n-b)}} = 0.$$

Before proving the lemma we complete the proof of Proposition 5.2.

If we can show that for all $c \in [a-b]$ no poles of the integrand in the z_c-plane cross the contour of integration when $X_b^{-1}\mathbb{T}$ is inflated to $\mathbb{T}$ then the identity of Lemma 5.4 simplifies to

$$\sum_{a=b}^{n} \frac{(4\pi\mathrm{i})^a}{(n-a+1)!} \binom{n-b}{n-a} \int_{\mathbb{T}^{n-b}} \int_{\mathbb{T}^{n-a}} \Delta(z^{(n-b)}, w^{(n-a)}, x; t) \\ \times f(z^{(n-b)}, x^{(b)}) \frac{\mathrm{d}w^{(n-a)}}{w^{(n-a)}} \frac{\mathrm{d}z^{(n-b)}}{z^{(n-b)}} = 0,$$

where we divided out an overall factor $(n-b+1)$. Since this is (5.41) with $(k_1, \dots, k_b) = (1, \dots, b)$ we are done with the proof of Proposition 5.2 if we can show that $\Delta(z^{(n-b)}, w^{(n-a)}, x; t)$ is free of poles in the annulus $|X_b|^{-1} \le |z_j| \le 1$ for $j \in [a-b]$. Since $\Delta(z^{(n-b)}, w^{(n-a)}, x; t)$ has permutation symmetry in the variables $z_1, \dots, z_{a-b}$, it is enough to check this condition for $j = a-b$. The rest is a matter of case-checking, where it should be noted that, since $b \le n-1$, $|t|^{n-1} \le |t|^b < |X_b|^{-1}$. For example, the pole at $z_{a-b} = t^{-2} x_{i+b}^{-1} p^{-\mu} q^{-\nu}$ (for $i \in [n-b+1]$) has absolute value $|z_{a-b}| > |t|^{-1} > 1$ for $i \ne n-b+1$ and absolute value $|z_{a-b}| > |t|^{-n-2} > 1$ for $i = n-b+1$. The pole at $z_{a-b} = x_i p^\mu q^\nu$ (for $i = b+1, \dots, n+1$) has absolute value $|z_{a-b}| > |t|^{-1} > 1$ if $(\mu,\nu) = (0,0)$ and $i \ne n+1$, has absolute value $|z_{a-b}| = |x_{n+1}| < |X_b|^{-1}$ if $(\mu,\nu) = (0,0)$ and $i = n+1$, has absolute value $|z_{a-b}| < |t|^{-1} M < |t|^{n-1} < |X_b|^{-1}$ if $(\mu,\nu) \ne (0,0)$ and $i \ne n+1$, and has absolute value $|z_{a-b}| < M < |t|^n < |X_b|^{-1}$ if $(\mu,\nu) \ne (0,0)$ and $i = n+1$. ☐

Proof of Lemma 5.4. We will show by induction on d that for $b \in \mathbb{Z}_n$ and $d \in \{b, \dots, n\}$ there holds

$$L_{b,d} := \sum_{a=b}^{d} \frac{(4\pi\mathrm{i})^a}{(d-a)!} \binom{n-b}{n-a} \biggl[\int_{\mathbb{T}^{n-b}} + \frac{1}{n-a+1} \sum_{c=1}^{a-b} \int_{\mathbb{T}^{c-1}\times(X_b^{-1}\mathbb{T})\times\mathbb{T}^{n-b-c}} \biggr] \\ \times \int_{\mathbb{T}^{d-a}\times C_a^{n-d}} \Delta(z^{(n-b)}, w^{(n-a)}, x; t) f(z^{(n-b)}, x^{(b)}) \frac{\mathrm{d}w^{(n-a)}}{w^{(n-a)}} \frac{\mathrm{d}z^{(n-b)}}{z^{(n-b)}} = 0.$$

Here $C_a^{n-d} = C_a \times \cdots \times C_a$ with C_a a contour that has the points

$$t x_{i+b} p^\mu q^\nu \text{ for } i \in [n-b+1], \tag{5.47a}$$

$$t^{-1} z_{i+a-b}^{-1} p^\mu q^\nu \text{ for } i \in [n-a+1], \tag{5.47b}$$

$$t^{-1} z_i p^{\mu+1} q^{\nu+1} \text{ for } i \in [a-b], \tag{5.47c}$$

in its interior. Since for $d = b$ we recover (5.44) it suffices to establish that $L_{b,d} = (d-b+1) L_{b,d+1}$ for $d \in \{b, \dots, n-1\}$.

Without loss of generality we may assume that $|t|^{2n+1} < |C_a| < |t|^{-2n-1}$. This is compatible with the fact that the points listed in (5.47a) and (5.47b) must lie in the interior of C_a, and guarantees that the points listed in (5.47c) lie in its interior. When integrating over $w^{(n-a)}$ for fixed $z^{(n-b)}$ we may also assume that $z_{a-b+1},\dots,z_{n-b+1}$ are all distinct.

We now deform C_a^{n-d} to $C_a^{n-d-1}\times\mathbb{T}$. The poles of $\Delta(z^{(n-b)},w^{(n-a)},x;t)$ in the w_{n-a}-plane coincide with the points listed in (5.47) and their reciprocals. The usual inspection of their moduli shows that under the assumption that $|x_i|<|t|^{-1}$, the only poles crossing the contour are those corresponding to $w_{n-a}=t^{-1}z_{k+a-b}^{-1}$ for $k\in[n-a+1]$. Hence, by (5.45) and (5.46),

(5.48)
$$\begin{aligned}
I_{a,d}(z^{(n-b)},x;t) &:= \int_{\mathbb{T}^{d-a}\times C_a^{n-d}} \Delta(z^{(n-b)},w^{(n-a)},x;t)\frac{\mathrm{d}w^{(n-a)}}{w^{(n-a)}}\\
&= I_{a,d+1}(z^{(n-b)},x;t)\\
&\quad + 4\pi\mathrm{i}\sum_{k=a-b+1}^{n-b+1}\int_{\mathbb{T}^{d-a}\times C_{a;k}^{n-d-1}} \Delta(z^{(n-b)},w^{(n-a-1)},x;t)|_{z_{a-b+1}\leftrightarrow z_k}\frac{\mathrm{d}w^{(n-a-1)}}{w^{(n-a-1)}}.
\end{aligned}$$

Here the integration variables w_{d-a+1} and w_{n-a} in the first integral on the right have been permuted in order to simplify $\mathbb{T}^{d-a}\times C_a^{n-d-1}\times\mathbb{T}$ to $\mathbb{T}^{d-a+1}\times C_a^{n-d-1}$, leading to $I_{a,d+1}$. Furthermore, $C_{a;k}^{n-d-1}=C_{a;k}\times\cdots\times C_{a;k}$ with $C_{a;k}$ a contour that has the points listed in (5.47) in its interior be it that in (5.47b) and (5.47c) the respective conditions $i\neq k$ and $i=k$ need to be added. This latter condition is automatically satisfied if we demand that $|t|^{2n+1}<|C_{a;k}|<|t|^{-2n-1}$ which also implies that $C_{a;a-b+1}$ may be identified with C_{a+1} so that $C_{a;a-b+1}^{n-d-1}=C_{a+1}^{n-d-1}$.

Next consider

$$J_{a,b,d} := \int_{\mathbb{T}^{n-b}} I_{a,d}(z^{(n-b)},x;t)f(z^{(n-b)},x^{(b)})\frac{\mathrm{d}z^{(n-b)}}{z^{(n-b)}} \tag{5.49a}$$

and

$$K_{a,b,c,d} := \int_{\mathbb{T}^{c-1}\times(X_b^{-1}\mathbb{T})\times\mathbb{T}^{n-b-c}} I_{a,d}(z^{(n-b)},x;t)f(z^{(n-b)},x^{(b)})\frac{\mathrm{d}z^{(n-b)}}{z^{(n-b)}}. \tag{5.49b}$$

We will compute these integrals using (5.48), starting with $J_{a,b,d}$. By permuting the integration variables z_k and z_{a-b+1} for $k\in\{a-b+1,\dots,n-b\}$, and by using the permutation symmetry of f, the sum over k in (5.48) (with the term $k=n-b+1$ excluded) simply gives rise to $(n-a)$ times $J_{a+1,b,d+1}$. The last term in the sum is to be treated somewhat differently. Recalling the definition of z_{n-b+1} in (5.43) we make the variable change $z_{a-b+1}\leftrightarrow z_{n-b+1}=Z_{n-b}^{-1}X_b^{-1}$ in the integral corresponding to this remaining term. By the A_n symmetry of f this leaves the integrand unchanged. Consequently,

$$J_{a,b,d}=J_{a,b,d+1}+4\pi\mathrm{i}(n-a)J_{a+1,b,d+1}+4\pi\mathrm{i}K_{a+1,b,a-b+1,d+1}. \tag{5.50}$$

We carry out exactly the same variables changes to compute $K_{a,b,c,d}$ for $c\in[a-b]$. Note that since $c\leq a-b$ these changes do not interfere with the structure of the contours of (5.49b). The only notable difference will be that when permuting $z_{a-b+1}\leftrightarrow z_{n-b+1}=Z_{n-b}^{-1}X_b^{-1}$ corresponding to the last term in the sum over k in (5.48), the z_{a-b+1} contour of integration will not change from $\mathbb{T}$ to $X_b^{-1}\mathbb{T}$

as before, but will remain just $\mathbb{T}$. Indeed, when $z_{a-b+1} \to X_b^{-1}Z_{n-b}^{-1}$ we find $|z_{a-b+1}| \to |X_b^{-1}Z_{n-b}^{-1}| = 1$ since $|z_c| = |X_b^{-1}|$. Therefore

$$K_{a,b,c,d} = K_{a,b,c,d+1} + 4\pi\mathrm{i}(n-a+1)K_{a+1,b,c,d+1}. \tag{5.51}$$

The remainder of the proof is elementary. By definition,

$$L_{b,d} = \sum_{a=b}^{d} \frac{(4\pi\mathrm{i})^a}{(d-a)!}\binom{n-b}{n-a}\biggl[J_{a,b,d} + \frac{1}{n-a+1}\sum_{c=1}^{a-b} K_{a,b,c,d}\biggr].$$

Substituting (5.50) and (5.51) and shifting the summation index from a to $a-1$ in all terms of the summand that carry the subscript $a+1$ yields $L_{b,d} = (d-b+1)L_{b,d+1}$. □

6. Alternative proof of the new A_n elliptic beta integral

The proof of Theorem 5.3 presented in this section adopts and refines a technique for proving elliptic beta integrals via q-difference equations that was recently developed in [26]. We note that this method is different from the q-difference approach of Gustafson [11]. The latter employs q-difference equations depending solely on the "external" parameters in beta integrals (like the t_i and s_i in (3.4) and (3.5)) and not on the integration variables themselves. Although Gustafson's method can also be applied to the integral of Theorem 5.3 by virtue of the fact that both sides of the integral identity satisfy

$$\begin{aligned}\sum_{i=1}^{n+1} \frac{\theta(q^{-1}Ss_i^{-1}s_{n+2}^{-1})}{\theta(Ss_{n+2}^{-1}s_{n+3}^{-1})} \prod_{\substack{j=1\\ j\neq i}}^{n+1} \frac{\theta(q^{-1}s_j^{-1}s_{n+3})}{\theta(s_is_j^{-1})} \\ \times I(s_1,\dots,s_{i-1},qs_i,s_{i+1},\dots,s_{n+2},q^{-1}s_{n+3}) = I(s_1,\dots,s_{n+3}),\end{aligned}$$

the proof would require a non-trivial vanishing hypothesis similar to those formulated in [7].

After these preliminary remarks we turn our attention to the actual proof of Theorem 5.3. We begin by noting that the kernel $\rho(z;s,t)$ defined in (5.20) satisfies a q-difference equation involving the integration variables. For $v=(v_1,\dots,v_n)\in\mathbb{C}^n$ and $a\in\mathbb{C}$ let $\pi_{i,a}(v) = (v_1,\dots,v_{i-1},av_i,v_{i+1},\dots,v_n)$.

Lemma 6.1. *We have*

$$\rho(z;s,t) - \rho(z;s,\pi_{1,q}(t)) = \sum_{i=1}^{n}\bigl[g_i(z;s,t) - g_i(\pi_{i,q^{-1}}(z);s,t)\bigr], \tag{6.1}$$

where $g_i(z;s,t) = \rho(z;s,t)f_i(z;s,t)$ and

$$\begin{aligned} f_i(z;s,t) = t_1z_{n+1}\theta(St_1^2)\prod_{j=1}^{n}\frac{\theta(t_1z_j)}{\theta(z_jz_{n+1}^{-1},St_jz_{n+1}^{-1})}\prod_{j=2}^{n}\frac{\theta(t_jz_i,q^{-1}St_jz_i^{-1})}{\theta(q^{-1}t_jz_{n+1})} \\ \times\prod_{j=1}^{n+3}\frac{\theta(s_jz_{n+1}^{-1})}{\theta(t_1s_j)}\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{\theta(q^{-1}z_j^{-1}z_{n+1},Sz_j^{-1}z_{n+1}^{-1})}{\theta(z_iz_j^{-1},q^{-1}Sz_i^{-1}z_j^{-1})}.\end{aligned} \tag{6.2}$$

Proof. From (2.4) and $\theta(x) = -x\,\theta(x^{-1})$ it readily follows that

$$\frac{\rho(z;s,\pi_{1,q}(t))}{\rho(z;s,t)} = \prod_{i=1}^{n+1} \frac{\theta(t_1 z_i)}{\theta(St_1 z_i^{-1})} \prod_{j=1}^{n+3} \frac{\theta(St_1 s_j^{-1})}{\theta(t_1 s_j)}$$

and

$$\begin{aligned}\frac{\rho(\pi_{i,q^{-1}}(z);s,t)}{\rho(z;s,t)} &= -\frac{z_i \theta(q^{-2} z_i z_{n+1}^{-1})}{q z_{n+1} \theta(z_i^{-1} z_{n+1})} \prod_{j=1}^{n} \frac{\theta(t_j z_{n+1}, q^{-1} S t_j z_{n+1}^{-1})}{\theta(q^{-1} t_j z_i, S t_j z_i^{-1})} \\ &\quad\times \prod_{j=1}^{n+3} \frac{\theta(s_j z_i^{-1})}{\theta(q^{-1} s_j z_{n+1}^{-1})} \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{\theta(q^{-1} z_i z_j^{-1}, q^{-1} z_j z_{n+1}^{-1}, S z_i^{-1} z_j^{-1})}{\theta(z_i^{-1} z_j, z_j^{-1} z_{n+1}, q^{-1} S z_j^{-1} z_{n+1}^{-1})}.\end{aligned}$$

Dividing both sides of (6.1) by $\rho(z;s;t)$ we thus obtain the theta function identity

$$\begin{aligned}1 - \prod_{i=1}^{n+1} \frac{\theta(t_1 z_i)}{\theta(St_1 z_i^{-1})} \prod_{j=1}^{n+3} \frac{\theta(St_1 s_j^{-1})}{\theta(t_1 s_j)} &= \sum_{i=1}^{n} \frac{t_1 z_i \theta(St_1^2)}{\theta(St_1 z_i^{-1})} \prod_{j=1}^{n+3} \frac{\theta(s_j z_i^{-1})}{\theta(t_1 s_j)} \prod_{\substack{j=1\\ j\neq i}}^{n+1} \frac{\theta(t_1 z_j)}{\theta(z_i^{-1} z_j)} \\ + t_1 z_{n+1} \theta(St_1^2) \sum_{i=1}^{n} \prod_{j=1}^{n} &\frac{\theta(t_1 z_j)}{\theta(z_j z_{n+1}^{-1}, S t_j z_{n+1}^{-1})} \prod_{j=2}^{n} \frac{\theta(t_j z_i, q^{-1} S t_j z_i^{-1})}{\theta(q^{-1} t_j z_{n+1})} \\ &\quad\times \prod_{j=1}^{n+3} \frac{\theta(s_j z_{n+1}^{-1})}{\theta(t_1 s_j)} \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{\theta(q^{-1} z_j^{-1} z_{n+1}, S z_j^{-1} z_{n+1}^{-1})}{\theta(z_i z_j^{-1}, q^{-1} S z_i^{-1} z_j^{-1})}.\end{aligned}$$

Observing that the left-hand side and the first sum on the right are independent of $t_2, \dots, t_n$ suggests that the above identity is a linear combination of

(6.3)
$$\sum_{i=1}^{n+1} \frac{t_1 z_i \theta(St_1^2)}{\theta(St_1 z_i^{-1})} \prod_{j=1}^{n+3} \frac{\theta(s_j z_i^{-1})}{\theta(t_1 s_j)} \prod_{\substack{j=1\\ j\neq i}}^{n+1} \frac{\theta(t_1 z_j)}{\theta(z_i^{-1} z_j)} = 1 - \prod_{i=1}^{n+1} \frac{\theta(t_1 z_i)}{\theta(St_1 z_i^{-1})} \prod_{j=1}^{n+3} \frac{\theta(St_1 s_j^{-1})}{\theta(t_1 s_j)},$$

and

$$\sum_{i=1}^{n} \prod_{j=2}^{n} \frac{\theta(t_j z_i, q^{-1} S t_j z_i^{-1})}{\theta(q^{-1} t_j z_{n+1}, S t_j z_{n+1}^{-1})} \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{\theta(q^{-1} z_j^{-1} z_{n+1}, S z_j^{-1} z_{n+1}^{-1})}{\theta(z_i z_j^{-1}, q^{-1} S z_i^{-1} z_j^{-1})} = 1. \tag{6.4}$$

To prove (6.3) we first note that the conditions $S = s_1 \dots s_{n+3}$ and $z_1 \cdots z_{n+1} = 1$ may be replaced by the single condition $S z_1 \cdots z_{n+1} = s_1 \cdots s_{n+3}$. Since this requires departing from the convention that $z_1 \cdots z_{n+1} = 1$ it is perhaps better to state this generalization with n replaced by $n-1$, i.e,

$$\sum_{i=1}^{n} \frac{a z_i \theta(Ba^2)}{\theta(Ba z_i^{-1})} \prod_{j=1}^{n+2} \frac{\theta(b_j z_i^{-1})}{\theta(a b_j)} \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{\theta(a z_j)}{\theta(z_i^{-1} z_j)} = 1 - \prod_{i=1}^{n} \frac{\theta(a z_i)}{\theta(Ba z_i^{-1})} \prod_{j=1}^{n+2} \frac{\theta(Bab_j^{-1})}{\theta(a b_j)}, \tag{6.5}$$

for $B z_1 \cdots z_n = b_1 \cdots b_{n+2}$.

To prove (6.5) we bring all terms to one side and write the resulting identity as $f(a) = 0$. Using $\theta(px) = -x^{-1}\theta(x)$ it is easily checked that $f(pa) = f(a)$.

The function f has poles at $a = b_j^{-1} p^m$ and $a = B^{-1} z_i p^m$ for $m \in \mathbb{Z}$, $j \in [n+2]$ and $i \in [n]$. If we can show that the residues at these poles vanish then $f(a)$ must be constant by Liouville's theorem. That this constant must then be zero easily follows by taking $a = z_1^{-1}$.

By the permutation symmetry of f in $z_1, \dots, z_n$ and $b_1, \dots, b_{n+2}$, and by the periodicity of f it suffices to consider the poles at $a = b_1^{-1}$ and $a = B^{-1}z_1$. By $\theta(x) = -x\theta(x^{-1})$ the residue of f at the latter pole is easily seen to vanish. Equating the residue of f at $a = b_1^{-1}$ to zero, and replacing $(b_2, \dots, b_{n+2}) \to (a_1, \dots, a_{n+1})$ and $Bb_1^{-1} \to A$, yields the A_n elliptic partial fraction expansion [18, Equation (4.3)]

$$\sum_{i=1}^{n} \prod_{j=1}^{n+1} \frac{\theta(a_j z_i^{-1})}{\theta(A a_j^{-1})} \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{\theta(A z_j^{-1})}{\theta(z_i^{-1} z_j)} = 1$$

for $Az_1 \cdots z_n = a_1 \cdots a_{n+1}$.

The task of proving (6.4) is even simpler. From [9, Lemma 4.14] we have the elliptic partial fraction expansion

$$\sum_{i=1}^{n} \prod_{j=1}^{n-1} \frac{\theta(b_j z_i, b_j z_i^{-1})}{\theta(a b_j, a^{-1} b_j)} \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{\theta(a z_j, a^{-1} z_j)}{\theta(z_i^{-1} z_j, z_i z_j)} = 1.$$

Making the substitutions

$$z_i \to q^{-1/2} S^{1/2} z_i^{-1}, \quad b_j \to q^{-1/2} S^{1/2} t_j, \quad a \to q^{-1/2} S^{-1/2} z_{n+1}$$

results in (6.4). $\square$

By integrating the identity of Lemma 6.1 over $\mathbb{T}^n$ and by rescaling some of the integration variables, we obtain

$$I(s,t) - I(s, \pi_{1,q}(t)) = \kappa^{\mathscr{A}} \sum_{i=1}^{n} \biggl(\int_{\mathbb{T}^n} - \int_{\mathbb{T}^{i-1} \times (q^{-1}\mathbb{T}) \times \mathbb{T}^{n-i}} \biggr) g_i(z; s, t) \frac{\mathrm{d}z}{z}, \tag{6.6}$$

where $I(s,t)$ denotes the integral on the left of (5.19). A careful inspection learns that $g_i(z; s, t)$ has poles in the z_i-plane at

$$z_i = \begin{cases} S z_j^{-1} p^{\mu} q^{\nu-1} \\ S t_j p^{-\mu-1} q^{-\nu-2+\delta_{j,1}} \\ t_j^{-1} p^{-\mu} q^{-\nu-1} \\ s_j p^{\mu} q^{\nu} \end{cases}, \quad z_{n+1} = \begin{cases} S z_j^{-1} p^{\mu} q^{\nu+1} & j \in [n]/\{i\} \\ S t_j p^{-\mu-1} q^{-\nu} & j \in [n] \\ t_j^{-1} p^{-\mu} q^{1-\nu-\delta_{j,1}} & j \in [n] \\ s_j p^{\mu} q^{\nu+1} & j \in [n+3], \end{cases}$$

where $z_{n+1} = a$ stands for $z_i^{-1} = a z_1 \cdots z_{i-1} z_{i+1} \cdots z_n$. Imposing the conditions (which are stronger than (5.15))

$$\max\{|t_1|, |q^{-1}t_2|, \dots, |q^{-1}t_n|, |s_1|, \dots, |s_{n+3}|, \\ |pS^{-1}t_1^{-1}|, \dots, |pS^{-1}t_n^{-1}|, |q^{-1}S|\} < 1 \tag{6.7}$$

it follows that none of the listed poles of g_i lies on the annulus $1 \le |z_i| \le |q|^{-1}$. Consequently, when (6.7) holds the right-hand side of (6.6) vanishes and

$$I(s,t) = I(s, \pi_{1,q}(t)). \tag{6.8}$$

Expanding $I(s,t)$ in a Taylor series in p we have

$$I(s,t;q,p) = \sum_{j=0}^{\infty} I_j(s,t;q) p^j, \tag{6.9}$$

with $I_j(s,t;q)$ holomorphic in s and t for

$$\max\{|t_1|,\dots,|t_n|,|s_1|,\dots,|s_{n+3}|\}<1.$$

(The remaining conditions of (5.15) involving the parameter p ensure convergence of the series (6.9), but, obviously, bear no relation to the analyticity of $I_j(s,t;q)$.)

Thanks to (6.8) we have the termwise q-difference $I_j(s,t;q)=I_j(s,\pi_{1,q}(t))$ when

$$\max\{|t_1|,|q^{-1}t_2|,\dots,|q^{-1}t_n|,|s_1|,\dots,|s_{n+3}|,|q^{-1}S|\}<1.$$

Since this may be iterated and since the limiting point $t_1=0$ of the sequence $t_1,t_1q,t_1q^2,\dots$ lies inside the domain of analyticity of $I_j(s,t)$, we conclude that $I_j(s,t)$ is independent of t_1. Lifting this to $I(s,t)$ and exploiting the symmetry in the t_i it follows that $I(s,t)$ is independent of t for

$$\max\{|q^{-1}t_1|,\dots,|q^{-1}t_n|,|s_1|,\dots,|s_{n+3}|,|pS^{-1}t_1^{-1}|,\dots,|pS^{-1}t_n^{-1}|,|q^{-1}S|\}<1$$

and thus, by analytic continuation, for (5.15).

In order to compute $I(s,t)=I(s)$ we repeat the reasoning of Section 5.2 and note that when (5.15) is replaced by (5.25) then $I(s)$ is given by the integral on the left of (5.22). We also know that this integral does not depend on t and hence, by (5.24),

$$I(s)=\lim_{\substack{t_i\to s_i^{-1}\\ \forall\, i\in[n]}} I(s)=\kappa^{\mathscr{A}}\lim_{\substack{t_i\to s_i^{-1}\\ \forall\, i\in[n]}}\sum_\lambda \rho_\lambda(s,t).$$

The expression on the right equals to 1 thanks to the trivial fact that for $N_i=0$ we have $\lambda=0$ and the conditions $t_i=s_i^{-1}$ yield $\kappa^{\mathscr{A}}\rho_0(s,t)=1$. As a final remark, we note that application of the full residue calculus with $N_i\neq 0$ to our A_n integral results in Theorems 5.5 and 5.6. Therefore, the considerations of the present section provide an alternative proof of the corresponding sums as well.

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