VARIATION OF MODULI SPACES AND DONALDSON INVARIANTS UNDER CHANGE OF POLARIZATION

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1. Introduction

Let S be a smooth projective surface over the complex numbers and let $c_1 \in H^2(S, \mathbb{Z})$ and $c_2 \in H^4(S, \mathbb{Z})$ be two classes. For an ample divisor H on S, one can study the moduli space $M_H(c_1, c_2)$ of H-semistable torsion-free sheaves E on S of rank 2 with $c_1(E) = c_1$ and $c_2(E) = c_2$. We want to study the change of $M_H(c_1, c_2)$ under variation of H. It is known that the ample cone of S has a chamber structure, and that $M_H(c_1, c_2)$ depends only on the chamber containing H. In this article we will try to understand how $M_H(c_1, c_2)$ changes, when H passes through a wall separating two chambers. The set-theoretic changes of the subspace consisting of locally free sheaves and of $M_H(c_1, c_2)$ have been treated in [Q1] and [Gö1] respectively. We show that the change of the moduli space when H passes through a wall, can be expressed as a sequence of operations similar to a flip. In fact the moduli spaces at each step can be identified as moduli spaces of torsion-free sheaves with a suitable parabolic structure of length 1. We assume that either the geometric genus $p_g(S)$ is 0 or that K_S is trivial. We shall also make suitable hypotheses on the wall, and walls fulfilling this condition we call good. This assumption is reasonably weak if the Kodaira dimension of S is at most 0, but gets stronger if e.g., S is of general type. When the polarization passes through a good wall, each of the steps above is realized by a smooth blow-up along a projective bundle over a product of Hilbert schemes of points on S, followed by a smooth blow-down of the exceptional divisor in another direction.

The change of moduli spaces can be viewed as a change of GIT quotients, treated in [Th2] and [D-H]. These results could in principle be applicable, although it would still take quite some work to do so. We have however chosen a more direct approach via elementary transforms of universal families, which is more in the spirit of [Th1], and which also immediately gives the change of the universal sheaves needed for the computation of Donaldson invariants.

In the case that K_S is trivial, i.e., S is an abelian or a $K3$ surface, we see that the change of $M_H(c_1, c_2)$, when H passes through a wall, is given by elementary transformations of symplectic varieties.

In the case that $p_g(S) = q(S) = 0$ we use these results in order to compute the change of the Donaldson polynomials under change of polarisation. The Donaldson polynomials of a C^∞-manifold M of dimension 4 are defined using a Riemannian metric on S, but in case $b^+(M) > 1$ they are

Key words and phrases. Moduli spaces, Donalson invariants, Hilbert scheme of points.

known to be independent of the metric, as long as it is generic. In case $b_+(M) = 1$, (which for an algebraic surface S corresponds to $p_g(S) = 0$), the invariants have been introduced and studied by Kotschick in [Ko]. In [K-M] Kotschick and Morgan show that the invariants only depend on the chamber of the period point of the metric in the positive cone of $H^2(M, \mathbb{R})$. They also compute the lowest order term of the change and conjecture the shape of a formula for the change.

The case we are studying corresponds to M being an algebraic surface S with $p_g(S) = q(S) = 0$ and a wall lying in the ample cone, in addition we assume that the wall is good.

In a first step we compute the change of the Donaldson invariants in terms of natural cohomology classes on Hilbert schemes of points on S and then we use some computations in the cohomology rings of these Hilbert schemes to determine the six lowest order terms of the change of the Donaldson invariants explicitly. The results are compatible with the conjecture of [K-M] (which in particular predicts that three of the terms above are zero.

Parallelly and independently similar results to ours have been obtained by other authors. Matsuki and Wentworth show in [M-W] that the change of moduli spaces of torsion-free sheaves of arbitrary rank on a projective variety under change of polarisation can be described as a sequence of flips. In [F-Q] Friedman and Qin obtain very similar results to ours.

2. Background material

In this paper let S be a projective surface over $\mathbb{C}$. By the Neron-Severi group $NS(S)$ of S we mean the group of divisors modulo homological equivalence, i.e., the image of $Div(S)$ in $H^2(S, \mathbb{Z})$ under the map sending the class of a divisor D to its fundamental cycle $[D]$. Let $Div^0(S)$ be its kernel. Let $c_1 \in H^2(S, \mathbb{Z})$ and $c_2 \in H^4(S, \mathbb{Z}) = \mathbb{Z}$ be elements which will be fixed throughout the paper.

Let H be a polarization of S. As we mostly shall consider stability and semistability in the sense of Gieseker and Maruyama we shall write H-stable (resp. H-semistable) instead of Gieseker stable (resp. semistable) with respect to H and H-slope stable (resp. H-slope semistable) instead of stable (resp. semistable) with respect to H in the sense of Mumford-Takemoto. Denote by $M_H(c_1, c_2)$ the moduli space of H-semistable torsion-free sheaves E on S of rank 2 with $c_1(E) = c_1$ and $c_2(E) = c_2$ and $M_H^s(c_1, c_2)$ the open subscheme of $M_H^s(c_1, c_2)$ of stable sheaves. Let $Spl(c_1, c_2)$ be the moduli space of simple torsion-free sheaves with $c_1(E) = c_1$ and $c_2(E) = c_2$ (see [A-K]).

Notation 2.1. For a sheaf $\mathcal{F}$ on a scheme X and a divisor D let $\mathcal{F}(D) := \mathcal{F} \otimes \mathcal{O}_X(D)$.

Many of our arguments will take place over products $S \times X$, where X is a scheme. We shall denote by $p : S \times X \longrightarrow S$ and $q_X : S \times X \longrightarrow X$ the two projections and if there is no danger of confusion, we will drop the index X. For a divisor D on X we denote $D_S := q_X^*(D)$. For a sheaf $\mathcal{F}$ on $S \times X$ and a divisor or divisor class, D on S we denote by $\mathcal{F}(D)$ the sheaf $\mathcal{F}(p^*(D))$.

If X is a smooth variety of dimension n, we denote the cup product of two elements α and β in $H^*(X, \mathbb{Z})$ by $\alpha \cdot \beta$ and the degree of a class $\alpha \in H^{2n}(X, \mathbb{Z})$ by $\int_X \alpha$. For $\alpha, \beta \in H^2(S, \mathbb{Z})$ let $\langle \alpha \cdot \beta \rangle := \int_S \alpha \cdot \beta$. We write α^2 for $\langle \alpha \cdot \alpha \rangle$ and, for $\gamma \in H^2(S, \mathbb{Z})$, we put $\langle \alpha, \gamma \rangle := \langle \alpha \cdot \check{\gamma} \rangle$, where $\check{\gamma}$ is the Poincaré dual of γ.

Convention 2.2. If Y, X are schemes and there is a "canonical" map $f : X \longrightarrow Y$, then for a cohomology class $\alpha \in H^*(Y, \mathbb{Z})$ (resp. for a vector bundle E on Y) we will very often also denote the pull-back via f by α (resp. E).

Definition 2.3. [OG2] Let B be a scheme. A family of sheaves, $\mathcal{F}$, on S parametrized by B is a B-flat sheaf on $S \times B$. Two families of sheaves $\mathcal{F}$ and $\mathcal{G}$ on S parametrized by B are called equivalent if there exists an isomorphism $\mathcal{F} \simeq \mathcal{G} \otimes q_B^* M$, for some line bundle M on B. Let $(B_j)_{j \in J}$ be an étale cover of B by schemes. Assume that for each $j \in J$ there is a family $\mathcal{F}_j$ of sheaves on X parametrized by B_j, and that for each pair $k, l \in J$ the pullbacks of $\mathcal{F}_k$ and $\mathcal{F}_l$ to $B_k \times_B B_l$ are equivalent. Then we will say that the above data defines a pseudo-family of sheaves on S parametrized by B. We will denote it by $\mathcal{F}$.

It is clear what is meant by a map of pseudo-families and by two pseudo-families being equivalent.

The main reason to introduce pseudo-families is that the moduli space $M_H^s(c_1, c_2)$ does not always carry a universal family of sheaves, but there will always be a universal pseudo-family.

By the universal property of $M_H^s(c_1, c_2)$ a pseudo family of H-stable torsion-free sheaves E on S with $c_1(E) = c_1$, $c_2(E) = c_2$ parametrized by B gives rise to a morphism $B \longrightarrow M_H^s(c_1, c_2)$.

Walls and chambers for torsion-free sheaves

We now recall some results about walls and chambers from [Q1], [Q2] and [Göl].

Definition 2.4. (for the first part see [Q1] Def I.2.1.5) Let C_S be the ample cone in $NS(S) \otimes \mathbb{R}$. For $\xi \in NS(S)$ let

$$W^\xi := C_S \cap \{x \in NS(S) \otimes \mathbb{R} \mid \langle x \cdot \xi \rangle = 0\}.$$

We shall call W^ξ a wall of type (c_1, c_2), and say that it is defined by ξ if the following conditions are satisfied:

(1) $\xi + c_1$ is divisible by 2 in $NS(S)$,
(2) $c_1^2 - 4c_2 \leq \xi^2 < 0$,
(3) there is a polarisation H with $\langle H \cdot \xi \rangle = 0$.

In particular $d_\xi := (4c_2 - c_1^2 + \xi^2)/4$ is a nonnegative integer. An ample divisor H is said to lie in the wall W if $[H] \in W$. If D is a divisor with $[D] = \xi$, we will also say that D defines the wall W.

A *chamber* of type (c_1, c_2) or simply a chamber, is a connected component of the complement of the union of all the walls of type (c_1, c_2). Two different chambers will be said to be *neighbouring chambers* if the intersection of their closures contains a nonempty open subset of a wall.

We will call a wall W *good*, if $D + K_S$ is not effective for any divisor D defining the wall W.

If D defines a wall, then neither D nor $-D$ can be effective because D is orthogonal to an ample divisor. In particular every wall will be good if $-K_S$ is effective or if $[K_S]$ is a torsion class. More generally, a wall W will be good if there exists an ample divisor H in W with $\langle K_S \cdot H \rangle \leq 0$.

Definition 2.5. Let $\mathrm{Hilb}^l(S)$ be the Hilbert scheme of subschemes of length l on S. For $\alpha \in NS(S)$ and $l \in \mathbb{Z}$, let $M(1,\alpha,l)$ be the moduli space of rank 1 torsion-free sheaves $\mathcal{I}_Z(F)$ on S with $c_1(\mathcal{I}_Z(F)) = [F] = \alpha$, $c_2(F) = length(Z) = l$. Let

$$T_\xi^{n,m} := \coprod_{2\alpha=c_1+\xi} M(1,\alpha,n) \times M(1,c_1-\alpha,m).$$

Let $N_2 \subset NS(S)$ be the subgroup of 2-torsion elements. There is a (noncanonical) isomorphism

$$T_\xi^{n,m} \simeq N_2 \times \mathrm{Hilb}^n S \times Div^0(S) \times \mathrm{Hilb}^m S \times Div^0(S),$$

which depends on the choice of an $\alpha \in NS(S)$ with $2\alpha = c_1 + \xi$ and on a representative F in $Div(S)$ for α.

For any extension

$$(\epsilon) \qquad 0 \longrightarrow A_1 \longrightarrow E \longrightarrow A_2 \longrightarrow 0$$

where A_1 and A_2 are torsion-free rank one sheaves, we define $\Delta(\epsilon) := \chi(A_1) - \chi(A_2)$. Then if $\alpha = c_1(A_1) - c_1(A_2)$, the Riemann-Roch theorem gives $\Delta(\epsilon) = 1/2\langle(c_1(E) - K_S)\cdot\alpha\rangle + c_2(A_2) - c_2(A_1)$. Furthermore for any divisor D we have $\Delta(\epsilon(D)) = \Delta(\epsilon) + \langle\alpha \cdot D\rangle$, where $\epsilon(D)$ denotes the extension ϵ twisted by the line bundle $\mathcal{O}(D)$. This follows immediately from the fact that $c_1(E(D)) = c_1(E) + 2[D]$.

Assume that ξ defines a wall of type (c_1, c_2), and that n and m are nonnegative integers with $n + m = d_\xi = c_2 - (c_1^2 - \xi^2)/4$. Let $\mathbf{E}_\xi^{n,m}$ be the set of sheaves lying in nontrivial extensions

$$(2.5.1) \qquad 0 \longrightarrow \mathcal{I}_{Z_1}(F_1) \longrightarrow E \longrightarrow \mathcal{I}_{Z_2}(F_2) \longrightarrow 0$$

where $(\mathcal{I}_{Z_1}(F_1), \mathcal{I}_{Z_2}(F_2))$ runs through $T_\xi^{n,m}$. It is easy to see that every sheaf $E \in \mathbf{E}_\xi^{n,m}$ is simple ([Göl], lemma 2.3). Let

$$V_\xi^{n,m} := \mathbf{E}_\xi^{n,m} \setminus \Big(\bigcup_{l,s} \mathbf{E}_{-\xi}^{l,s}\Big).$$

Notation 2.6. Assume that H_+ and H_- are ample divisors lying in neighbouring chambers separated by the wall W. Then we define

$$A^+(W) := \Big\{\xi \in NS(S) \,\Big|\, \xi \text{ defines } W \text{ and } \langle\xi \cdot H_+\rangle > 0\Big\}$$

and $A^-(W) := -A^+(W)$.

The following proposition mostly comprizes some of the results of [Göl], that are generalizations of the corresponding results of [Q1], [Q2] and will be important for the rest of the paper. Note that unlike [Göl] we assume walls to be defined by classes in $NS(S)$ and not by numerical equivalence classes, and that we look at moduli spaces with fixed first Chern class and not with fixed determinant. The proofs in [Göl] stay however valid with very few changes.

Proposition 2.7. (1) *For H not lying on a wall, $M_H(c_1,c_2) \setminus M_H^s(c_1,c_2)$ is independent of H and $M_H(c_1,c_2)$ depends only on the chamber of H.*

For the rest of the proposition we assume that we are in the situation of 2.6 and that $\xi \in A^+(W)$.

(2) *Every* $E \in \mathbf{E}_\xi^{n,m}$ *is* H_+ *slope-unstable and the sequence (2.5.1) is its Harder-Narasimhan filtration with respect to* H_+.

(3) $\mathrm{Hom}(\mathcal{I}_{Z_1}(F), E) = \mathbb{C}$. *Thus, for* $E \in \mathbf{E}_\xi^{n,m}$, *the sequence 2.5.1 is the unique extension*

$$0 \longrightarrow \mathcal{I}_{W_1}(F_1) \longrightarrow E \longrightarrow \mathcal{I}_{W_2}(G_2) \longrightarrow 0$$

with $\langle (2F_1 - c_1) \cdot H_+ \rangle > 0$.

(4) *In particular we see that, for* $\xi, \eta \in A^+(W)$, *the subsets* $\mathbf{E}_\xi^{n,m}$, $\mathbf{E}_\eta^{k,l}$ *of* $Spl(c_1, c_2)$ *do not intersect, unless* $\xi = \eta$ *and* $(n, m) = (k, l)$.

(5) *If* $E \in \mathbf{E}_\xi^{n,m}$ *then* E *is* H_-*-slope stable if and only if* $E \in V_\xi^{n,m}$ *and* H_-*-slope unstable otherwise.*

(6) *On the other hand let* E *be a torsion-free sheaf with* $c_1(E) = c_1$ *and* $c_2(E) = c_2$, *which is* H_-*-semistable and* H_+*-unstable. Then* E *is* H_-*-slope stable and* $E \in \mathbf{E}_\xi^{n,m}$ *for suitable numbers* n *and* m *and* $\xi \in A^+(W)$.

Proof. (1) is ([Gö1], theorem 2.9(1)). (2) is easy. (3) follows from (2) and ([Gö1], lemma 2.3). (4) follows from (2). (5) is ([Gö1], prop 2.5). (6) is ([Gö1], lemma 2.2). □

3. Parabolic structures and the passage through a wall

As mentioned in the previous section, $M_H(c_1, c_2)$ depends only on the chamber to which H belongs. If H' lies in a neighbouring chamber to H the moduli space $M_{H'}(c_1, c_2)$ will in most cases be birational to $M_H(c_1, c_2)$, although new components do occur in some cases. If the wall separating the two chambers is good, we will describe the birational transformation in detail by giving an explicit sequence of blow-ups and blow-downs with smooth centers which are known.

If the wall is good, but the transformation is not birational, our arguments give a description of the components which are added to or deleted from the moduli space.

For the rest of the paper we will assume that H_+ and H_- are ample divisors lying in neighbouring chambers separated by the wall W, and that H is an ample divisor in the wall W which lies in the closure of the chambers containing H_- and H_+ respectively and which does not lie in any other wall. Furthermore we shall assume that $M = H_+ - H_-$ is effective. By replacing H_+ by a high multiple if necessary, we can always achieve this.

Our aim is to divide the passage through a wall into a number of smaller steps. To this purpose we will introduce a finer notion of stability. The starting point is the observation that unlike slope stability, Gieseker stability is not invariant under tensorization by a line bundle.

Lemma 3.1. *There is a positive integer* n_0 *such that for all* $l \geq n_0$ *and all torsion-free rank 2 sheaves* E *on* S *with* $c_1(E) = c_1$, $c_2(E) = c_2$

(1) E *is* H_-*-stable (resp. semistable) if and only if* $E(-lM)$ *is* H*-stable (resp. semistable).*

(2) E *is* H_+*-stable (resp. semistable) if and only if* $E(lM)$ *is* H*-stable (resp. semistable).*

Proof. It will be enough to show (1). As H_- does not lie on a wall, it is easy to see that E is H_--(semi)stable if and only if $E(M)$ is. Also there are only finitely many $\xi \in NS(S)$ defining the wall W. Therefore lemma 3.1 follows immediately from lemma 3.2 and lemma 3.3 below. □

Lemma 3.2. (1) *Assume E is H_--semistable but H-unstable. Then $E \in \mathbf{E}_{\xi}^{n,m}$ for suitable n, m and $\xi \in A^+(W)$.*

(2) *Assume E is H_--unstable but H-semistable. Then $E \in \mathbf{E}_{-\xi}^{n,m}$ for suitable n, m and $\xi \in A^+(W)$.*

Proof. We just prove (1), the proof of (2) being analoguous. By assumption there is an extension

$$(\epsilon) \qquad 0 \longrightarrow \mathcal{I}_{Z_1}(F_1) \longrightarrow E \longrightarrow \mathcal{I}_{Z_2}(F_2) \longrightarrow 0$$

with $\Delta(\epsilon(lH_-)) \leq 0$ and $\Delta(\epsilon(lH)) > 0$ for $l >> 0$. In particular we have $\langle \eta \cdot H_- \rangle \leq 0 \leq \langle \eta \cdot H \rangle$ where $\eta := 2[F_1] - c_1$. If $0 < \langle \eta \cdot H \rangle$, there would be a wall separating H_- and H. So $\langle \eta \cdot H \rangle = 0$, and unless η is a torsion class, it defines a wall in which H lies. As H lies in a unique wall it must be W. Hence $\eta \in A^+(W)$, and $E \in \mathbf{E}_{\eta}^{n,m}$.

Assume that η is a torsion class. Then F_1 and F_2 are numerically equivalent, and it is easily verified that

$$\Delta(\epsilon(lH_-)) = \Delta(\epsilon(lH))$$

which is a contradicition. □

Lemma 3.3. *Given n, m, ξ. Then there exists an integer k_0 such that for all $k > k_0$ and all $E \in \mathbf{E}_{\xi}^{n,m}$, the sheaf $E(-kM)$ is H-stable if and only E is H_--slope stable. Otherwise E is both H_--slope unstable and H-unstable.*

Proof. Let $E \in E_{\xi}^{n,m}$. Then there is an extension

$$(\epsilon_1) \qquad 0 \longrightarrow \mathcal{I}_{Z_1}(F_1) \longrightarrow E \longrightarrow \mathcal{I}_{Z_2}(F_2) \longrightarrow 0$$

with $\xi = 2[F_1] - c_1$. Assume first that E is H_--slope-stable and let

$$(\epsilon_2) \qquad 0 \longrightarrow \mathcal{I}_{Y_1}(G_1) \longrightarrow E \longrightarrow \mathcal{I}_{Y_2}(G_2) \longrightarrow 0$$

be another extension. Put $\eta := 2[G_1] - c_1$. As E is H_--slope stable we have $\langle \eta \cdot H_- \rangle < 0$, and because there is no wall between H_- and H, we know that $\langle \eta \cdot H \rangle \leq 0$. For any integers k and l

$$\Delta(\epsilon_2(-kM + lH)) = \Delta(\epsilon_2) - k\langle \eta \cdot M \rangle + l\langle \eta \cdot H \rangle.$$

Hence if $\langle \eta \cdot H \rangle < 0$ for all extensions ϵ_2 above, then $E(-kM)$ will be H-stable for any k. Assume that $\langle \eta \cdot H \rangle = 0$. By assumption H is contained in a single wall W, so necessarily $\eta \in A^+(W)$. Hence by proposition 2.7(3), we get $\mathcal{I}_{Z_1}(F_1) = \mathcal{I}_{Y_1}(G_1)$. Therefore it suffices to see that for $k >> 0$ and any l we have the inequality

$$\Delta(\epsilon_1(-kM + lH)) < 0.$$

Now

$$\Delta(\epsilon_1(-kM + lH)) = \Delta(\epsilon_1) - k\langle \eta \cdot M \rangle,$$

which is negative for $k >> 0$ as $\langle \eta \cdot M \rangle > 0$.

To prove the converse, assume that E is not H_--slope stable. Then by proposition 2.7(5) there is an extension

$$(\epsilon_3) \qquad 0 \longrightarrow \mathcal{I}_{Y_1}(F_2) \longrightarrow E \longrightarrow \mathcal{I}_{Y_2}(F_1) \longrightarrow 0.$$

Because $2[F_2] - c_1 = -\xi$ we have

$$\Delta(\epsilon_3(-kM + lH)) = \Delta(\epsilon_3) + \langle -\xi \cdot (-kM + lH) \rangle = \Delta(\epsilon_3) + k\langle \xi \cdot M \rangle > 0$$

for $k >> 0$ $\square$

From now on until the end of this section we fix n_0 as in lemma 3.1, and we put $C := (n_0 + 1)M$.

Definition 3.4. Let a be a real number between 0 and 1. For any torsion-free sheaf E we define

$$P_a(E) = ((1-a)\chi(E(-C)) + a\chi(E(C)))/rk(E).$$

A torsion-free sheaf E on S is called a-semistable if and only if every subsheaf $E' \subset E$ satisfies $P_a(E'(lH)) \leq P_a(E(lH))$ for all $l >> 0$, and it is called a-stable if strict inequality holds.

In particular, by lemma 3.1, E is 0-semistable if and only if it is H_--semistable, and it is 1-semistable if and only if it is H_+-semistable.

For any extension

$$(\epsilon) \qquad 0 \longrightarrow A_1 \longrightarrow E \longrightarrow A_2 \longrightarrow 0$$

we define $\Delta_a(\epsilon) := P_a(A_1) - P_a(A_2)$. Then $\Delta_a(\epsilon) = \Delta(\epsilon) + (2a-1)\langle C \cdot \alpha \rangle$ where $\alpha = c_1(A_1) - c_1(A_2)$. Clearly $\Delta_a(\epsilon(D)) = \Delta_a(\epsilon) + \langle D \cdot \alpha \rangle$ for any divisor D. A sheaf E is a-stable (resp. a-semistable) if $\Delta_a(\epsilon(lH)) < 0$ (resp. ≤ 0) for all $l >> 0$ and for any extension ϵ whose middle term is E.

Remark 3.5. It is easy to see that $P_a(E(lH))$ is the parabolic Hilbert polynomial of the parabolic bundle $(E(C), E(-C), a)$, (i.e. with a filtration of length 1). Therefore E is a-semistable if and only if $(E(C), E(-C), a)$ is semistable. In [Ma-Yo] a coarse quasiprojective moduli space of stable parabolic sheaves with fixed Hilbert polynomial is constructed, and by [Yo] there exists a projective coarse moduli space for S-equivalence classes of semistable parabolic sheaves. In particular there exists a coarse moduli space $M_a(c_1, c_2)$ for a-semistable sheaves E on S with $c_1(E) = c_1$ and $c_2E = c_2$. We denote by $M_a^s(c_1, c_2)$ its open subscheme of stable sheaves.

Remark 3.6. We see that $M_{H_-}(c_1, c_2)$ and $M_0(c_1, c_2)$ respectively $M_{H_+}(c_1, c_2)$ and $M_1(c_1, c_2)$ are coarse moduli schemes for the same functor and therefore they are isomorphic.

Remark 3.7. The same proof as in the case of H-stable sheaves shows that $M_a^s(c_1, c_2)$ carries a universal pseudofamily. One checks easily that every $E \in M_a^s(c_1, c_2)$ is simple. As $M_a^s(c_1, c_2)$ and $Spl(c_1, c_2)$ both carry universal pseudofamilies, $\mathcal{V}$ and $\mathcal{W}$ respectively, there exists a morphism $f : M_a^s(c_1, c_2) \to Spl(c_1, c_2)$ such that $(\mathrm{id}_S \times f)^*(\mathcal{W}) = \mathcal{V}$. Let M be its image. By the same argument there exists a map $g : M \to M_a^s(c_1, c_2)$, with $(\mathrm{id}_S \times f)^*(\mathrm{id}_S \times g)^*(\mathcal{V}) = \mathcal{V}$. Hence f is an open embedding. In particular and what is the most important thing for us, the tangent space to $M_a^s(c_1, c_2)$ at a point E is $\mathrm{Ext}^1(E, E)$.

Definition 3.8. For all $a \in [0,1]$ let $A^+(a)$ be the set of $(\xi, n, m) \in A^+(W) \times \mathbb{Z}^2_{\geq 0}$ satisfying

$$n + m = c_2 - (c_1^2 - \xi^2)/4, \tag{3.8.1}$$

$$n - m = \langle \xi \cdot (c_1 - K_S) \rangle /2 + (2a-1)\langle \xi \cdot [C] \rangle. \tag{3.8.2}$$

A number a is called a *miniwall* if $A^+(a) \neq \emptyset$. A *minichamber* is a connected component of the complement of the set of all miniwalls in $[0,1]$. It is clear that there are finitely many minichambers. Two minichambers are called neighbouring minichambers if their closures intersect.

Remark 3.9. Note that $A^+(a)$ is the set of all ξ, n, m with $\xi \in A^+(W)$ for which there exists a (possibly split) extension

$$0 \longrightarrow A_1 \longrightarrow E \longrightarrow A_2 \longrightarrow 0 \tag{ϵ}$$

with $\xi = c_1(A_1) - c_1(A_2)$, $n = c_2(A_1)$, $m = c_2(A_2)$ and $\Delta_a(\epsilon) = 0$.

Lemma 3.10. *Let $0 \leq a_- < a_+ \leq 1$ and assume that neither a_- nor a_+ is a miniwall. Let E be a_--semistable and a_+-unstable. Then there exists a miniwall a between a_- and a_+ and an element $(\xi, n, m) \in A^+(a)$, such that $E \in \mathbf{E}_\xi^{n,m}$.*

Proof. By assumption E is a_+-unstable. Hence there is an extension

$$0 \longrightarrow A_1 \longrightarrow E \longrightarrow A_2 \longrightarrow 0 \tag{ϵ}$$

such that for all $l >> 0$ we have $\Delta_{a_+}(\epsilon(lH)) > 0$. Putting $\xi := c_1(A_1) - c_1(A_2)$ and using that E is a_--semistable, we obtain the following inequalities valid for all $l >> 0$

$$\Delta_{a_-}(\epsilon(lH)) = \Delta_{a_-}(\epsilon) + l\langle H \cdot \xi \rangle \leq 0 < \Delta_{a_+}(\epsilon(lH)) = \Delta_{a_+}(\epsilon) + l\langle H \cdot \xi \rangle.$$

In particular $\langle H \cdot \xi \rangle = 0$ and $\Delta_{a_-}(\epsilon) < 0 < \Delta_{a_+}(\epsilon)$.

Furthermore ξ is not a torsion class and ξ defines a wall on which H is lying, which therefore must be W. There clearly is an a such that $\Delta_a(\epsilon) = 0$. □

Lemma 3.11. *Let $a_- < a_+$ be in neighbouring minichambers separated by the miniwall a. Let $(\xi, n, m) \in A^+(a)$.*

(1) *Any $E \in \mathbf{E}_\xi^{n,m}$ is a_--stable, strictly a-semistable and b-unstable for all $b > a$.*
(2) *Any $E \in \mathbf{E}_{-\xi}^{m,n}$ is a_+-stable, strictly a-semistable and b-unstable for all $b < a$.*

Proof. By symmetry it is enough to show (1). Let $E \in \mathbf{E}_\xi^{n,m}$. Then E is given by an extension

$$0 \longrightarrow \mathcal{I}_{Z_1}(F_1) \longrightarrow E \longrightarrow \mathcal{I}_{Z_2}(F_2) \longrightarrow 0 \tag{ϵ}$$

with $\xi = 2[F_1] - c_1$ and $length(Z_1) = n$, $length(Z_2) = m$. Now if $b > a$ we have $\Delta_b(\epsilon(lH)) = \Delta_a(\epsilon(lH)) + 2(b-a)\langle C \cdot \xi \rangle = 2(b-a)\langle C \cdot \xi \rangle > 0$ since $\Delta_a(\epsilon) = 0$ and $\langle C \cdot \xi \rangle > 0$. Thus E is b-unstable. Assume that E is not a_--stable. Then it lies in an extension

$$0 \longrightarrow \mathcal{I}_{Y_1}(G_1) \longrightarrow E \longrightarrow \mathcal{I}_{Y_2}(G_2) \longrightarrow 0, \tag{ϵ_1}$$

for which $\Delta_{a_-}(\epsilon_1(lH)) \geq 0 > \Delta_a(\epsilon_1(lH))$ for $l >> 0$.

Hence we obtain $\langle(2G_1 - c_1)\cdot H\rangle \geq \langle(2F_1 - c_1)\cdot H\rangle$ and $P_{a_-}(\mathcal{I}_{Z_1}(F_1 + lH)) < P_{a_-}(\mathcal{I}_{Y_1}(G_1 + lH))$ and thus $\chi(\mathcal{I}_{Z_1}(F_1 + lH - C)) < \chi(\mathcal{I}_{Y_1}(G_1 + lH - C))$ or $\chi(\mathcal{I}_{Z_1}(F_1 + lH + C)) < \chi(\mathcal{I}_{Y_1}(G_1 + lH + C))$. Consequently $\mathrm{Hom}(\mathcal{I}_{Y_1}(G_1), \mathcal{I}_{Z_1}(F_1)) = 0$ and the obvious map $\mathcal{I}_{Y_1}(G_1) \longrightarrow \mathcal{I}_{Z_2}(F_2)$ is an injection. Hence $F_2 - G_1$ is effective. If $F_2 \neq G_1$, we would have $\langle(G_1 - F_2)\cdot H\rangle < 0$, and, by $\langle\xi\cdot H\rangle = 0$, we would get the contradiction $\langle(2G_1 - c_1)\cdot H\rangle < 0$. So $G_1 = F_2$. By the injectivity of $\mathcal{I}_{Y_1}(G_1) \longrightarrow \mathcal{I}_{Z_2}(F_2)$ and the fact that 2.5.1 is not split, we get $length(Z_2) < length(Y_1)$ which shoes that E is a_--stable. A similar argument shows that E is strictly a-semistable. $\square$

Remark 3.12. We can also easily see from the above arguments that in the situation of 3.11 any sheaf $E \in M_{a_-}(c_1, c_2)$, which does not lie in any $\mathbf{E}_\xi^{n,m}$ for $(\xi, n, m) \in A^+(a)$ is a_--stable (resp. semistable) if and only if it is a-stable (resp. semistable).

Remark 3.13.
(1) Looking at the proof of [Ma2] for the sufficient criterion for the existence of a universal family on $M_H(c_1, c_2)$, we see that the same proof also works for $M_a(c_1, c_2)$ and we get the same criterion, i.e. if c_1 is not divisible by 2 in $NS(S)$ or otherwise $4c_2 - c_2^2$ is not divisible by 8 and $M_a(c_1, c_2) = M_a^s(c_1, c_2)$, then $M_a^s(c_1, c_2)$ carries a universal family.
(2) From the results obtained so far it follows easily that, under the above conditions for the Chern classes, $M_a(c_1, c_2) = M_a^s(c_1, c_2)$ if and only if a is not a miniwall.

Proposition 3.14.
(1) *$M_0(c_1, c_2) = M_{H_-}(c_1, c_2)$ and $M_1(c_1, c_2) = M_{H_+}(c_1, c_2)$.*
(2) *If $b \in [0, 1]$ is not on a miniwall, the moduli space $M_b(c_1, c_2)$ depends only on the minichamber in which b is lying, and $M_b(c_1, c_2) \setminus M_b^s(c_1, c_2)$ is independent of b.*
(3) *Let $a_- < a_+$ be in neighbouring minichambers separated by the miniwall a. Then we have a set-theoretical decomposition*

$$M_{a_+}(c_1, c_2) = \left(M_{a_-}(c_1, c_2) \setminus \coprod_{(\xi,n,m)\in A^+(a)} \mathbf{E}_\xi^{n,m} \right) \sqcup \left(\coprod_{(\xi,n,m)\in A^+(a)} \mathbf{E}_{-\xi}^{m,n} \right),$$

and there are morphisms

$$\begin{array}{ccc} M_{a_-}(c_1, c_2) & & M_{a_+}(c_1, c_2) \\ & \searrow \psi_- \quad \swarrow \psi_+ & \\ & M_a(c_1, c_2) & \end{array}$$

which are open embeddings over

$$M_{a_-}(c_1, c_2) \setminus \coprod_{(\xi,n,m)\in A^+(a)} \mathbf{E}_\xi^{n,m} \ \textit{and} \ M_{a_+}(c_1, c_2) \setminus \coprod_{(\xi,n,m)\in A^+(a)} \mathbf{E}_{-\xi}^{m,n}.$$

Proof. (1), (2), (3) follow by putting together the results of this section. By lemma 3.11 all the points of $M_{a_-}(c_1, c_2)$ and $M_{a_+}(c_1, c_2)$ are a-semistable and hence we get the morphisms ψ_- and ψ_+. The statement that they be open embeddings over the indicated open subsets, follows from remark 3.12. $\square$

4. The normal bundles of the exceptional sets

Our aim in this and the next chapter is to describe the passage through a miniwall which corresponds to a good wall. We keep the assumptions from the beginning of the previous section. In addition to those we assume that either $p_g(S) = 0$ or K_S is trivial, and that the wall W is good.

Let a define a miniwall and let $(\xi, n, m) \in A^+(a)$. Let $a_- < a_+$ lie in neighbouring minichambers separated by a. For simplicity of notation we shall assume that $A^+(a) = \{(\xi, n, m)\}$. Because, for (ξ, n_1, m_1), (ξ_2, n_2, m_2) distinct elements of $A^+(a)$, the sets $\mathbf{E}_{\xi_1}^{n_1,m_1}$ and $\mathbf{E}_{\xi_2}^{n_2,m_2}$ are disjoint by proposition 2.7 and our arguments are local in a neighbourhood of each $\mathbf{E}_\eta^{l,s}$, this assumption can be made without loss of generality. Furthermore we assume for simplicity of notation that $NS(S)$ has no 2-torsion. Then the classes $(c_1 + \xi)/2$, $(c_1 - \xi)/2 \in NS(S)$ are well-defined and $T_\xi^{n,m} = M(1, (c_1 + \xi)/2, n) \times M(1, (c_1 - \xi)/2, m)$. Again this assumption is not important, as otherwise the components of $E_\xi^{n,m}$ and $E_{-\xi}^{m,n}$ are disjoint.

Notation 4.1. We shall write $M_- := M_{a_-}(c_1, c_2)$, $M_+ := M_{a_+}(c_1, c_2)$, $M_-^s := M_{a_-}^s(c_1, c_2)$, $M_+^s := M_{a_+}^s(c_1, c_2)$ and put $\mathbf{E}_- := \mathbf{E}_\xi^{n,m}$ and $\mathbf{E}_+ := \mathbf{E}_{-\xi}^{n,m}$.

Definition 4.2. Let $\mathcal{F}_1'$ (resp. $\mathcal{F}_2'$) be the pull-back of a universal sheaf over $S \times M(1, (c_1+\xi)/2, n)$ (resp. $S \times M(1, (c_1 - \xi)/2, m)$) to $S \times T$, where $T := M(1, (c_1 + \xi)/2, n) \times M(1, (c_1 - \xi)/2, m)$. Let $q = q_T : S \times T \to T$ be the projection. Let $\mathcal{A}_-' := \operatorname{Ext}^1_q(\mathcal{F}_2', \mathcal{F}_1')$ and $\mathcal{A}_+' := \operatorname{Ext}^1_q(\mathcal{F}_1', \mathcal{F}_2')$ and $\mathbb{P}_- := \mathbb{P}(\mathcal{A}_-')$, $\mathbb{P}_+ := \mathbb{P}(\mathcal{A}_+')$. Let π_- (resp. π_+) be the projections of $\mathbb{P}_-$ (resp. $\mathbb{P}_+$) to T and τ_- (resp. τ_+) the tautological sublinebundles of $\mathcal{A}_- := \pi_-^*(\mathcal{A}_-')$ (resp. $\mathcal{A}_+ := \pi_+^*(\mathcal{A}_+')$). Let $\mathcal{F}_1 := (\mathrm{id}_S \times \pi_-)^*\mathcal{F}_1'$ and $\mathcal{F}_2 := (\mathrm{id}_S \times \pi_-)^*\mathcal{F}_2'$.

Lemma 4.3. (1) *$\mathcal{A}_-'$ is locally free of rank $-\xi(\xi - K_S)/2 + n + m - \chi(\mathcal{O}_S)$ and its formation commutes with arbitrary base change.*

(2) *There is an isomorphism $\operatorname{Ext}^1(\mathcal{F}_2', \mathcal{F}_1') \longrightarrow H^0(T, \mathcal{A}_-')$, hence over $S \times \mathbb{P}_-$ there is a tautological extension*

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{E} \to \mathcal{F}_2(\tau_-) \longrightarrow 0. \tag{4.3.1}$$

There is a morphism $i_- : \mathbb{P}_- \longrightarrow M_-$ with image $\mathbf{E}_-$.

Proof. As ξ defines a wall, $\operatorname{Hom}_q(\mathcal{F}_2, \mathcal{F}_1)$ is fibrewise 0, and, as the wall is good, $F_1 - F_2 + K_S$ is not effective for $(F_1, F_2) \in T$, therefore by Serre duality for the extension groups [Mu2] also $\operatorname{Ext}^2_q(\mathcal{F}_2, \mathcal{F}_1) = 0$. So (1) follows by Riemann-Roch for the extension groups [Mu2]. Now we apply [La]. □

Proposition 4.4. (1) *If $p_g(S) = 0$ or if K_S is trivial, then $i_- : \mathbb{P}_- \longrightarrow M_-$ is a closed embedding and M_- is smooth along $\mathbf{E}_-$. The irreducible component of M_- containing $\mathbf{E}_-$ has the expected dimension.*

(2) *If $p_g(S) = 0$, then the normal bundle $N_{\mathbf{E}_-/M_-}$ of $\mathbf{E}_-$ in M_- is equal to $\mathcal{A}_+(\tau_-)$.*

(3) *If K_S is trivial, then $N_{\mathbf{E}_-/M_-} = Q^\vee(\tau_-)$, where Q is the universal quotient bundle on $\mathbb{P}_- = \mathbb{P}(\mathcal{A}_-)$.*

Proof. By proposition 2.7(3) and lemma 4.3 the map $\mathbb{P}_- \longrightarrow M_-$ is injective with image $\mathbf{E}_-$. We also see by proposition 2.7 that $\mathbf{E}_- \subset M_-^s$. In case K_S is trivial, $Spl(c_1, c_2)$ and thus also the open subscheme M_-^s are smooth by [Mu1].

In order to see that M_- is smooth along $\mathbf{E}_-$ in the case $p_g(S) = 0$, we have to show that $\operatorname{Ext}^2(E, E) = 0$ for any $E \in \mathbf{E}_-$. So let $E \in \mathbf{E}_-$ be given by a nontrivial extension (2.5.1)

$$(\epsilon) \qquad 0 \longrightarrow \mathcal{I}_{Z_1}(F_1) \longrightarrow E \longrightarrow \mathcal{I}_{Z_2}(F_2) \longrightarrow 0.$$

As the wall W is good, we obtain by Serre duality and the fact that $p_g(S) = 0$ that $\operatorname{Ext}^2(\mathcal{I}_{Z_i}(F_i), \mathcal{I}_{Z_j}(F_j)) = 0$ for $i = 1,2$ and $j = 1,2$. Hence applying $\operatorname{Ext}^2(\mathcal{I}_{Z_i}(F_i), \cdot)$ to (ϵ) we get $\operatorname{Ext}^2(\mathcal{I}_{Z_i}(F_i), E) = 0$ for $i = 1,2$ and this in turn shows that $\operatorname{Ext}^2(E, E) = 0$.

We now want to compute the normal bundle to $\mathbf{E}_-$.

First Case: $p_g(S) = 0$. Applying $\operatorname{Hom}_q(\cdot, \cdot)$ on both sides of the sequence (4.3.1) and denoting by π_i the composition of π_- with the projection to the i^{th} factor we get the following exact diagram of locally free sheaves on $\mathbb{P}_-$

$$\text{(4.4.1)} \qquad \begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_{\mathbb{P}_-/T} & \longrightarrow & \operatorname{Ext}^1_q(\mathcal{F}_2(\tau_-), \mathcal{E}) & \longrightarrow & \pi_2^* T_{M(1,(c_1-\xi)/2,m)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \psi & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Ext}^1_q(\mathcal{E}, \mathcal{F}_1) & \xrightarrow{\varphi} & i_-^*(T_{M_-}) & \longrightarrow & \operatorname{Ext}^1_q(\mathcal{E}, \mathcal{F}_2)(\tau_-) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi_1^* T_{M(1,(c_1+\xi)/2,n)} & \longrightarrow & \operatorname{Ext}^1_q(\mathcal{F}_1, \mathcal{E}) & \longrightarrow & \mathcal{A}_+(\tau_-) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

To identify the entries in this diagram we have used the following facts.

(1) $\operatorname{Hom}_q(\mathcal{F}_1, \mathcal{F}_1) = \operatorname{Hom}_q(\mathcal{F}_2, \mathcal{F}_2) = \mathcal{O}_{\mathbb{P}_-}$.

(2) If Q is the universal quotient on $\mathbb{P}(\mathcal{A}_-)$, then the relative tangent bundle is $T_{\mathbb{P}_-/T} = Q(-\tau_-)$, i.e. the cokernel of the natural map $\mathcal{O}_{\mathbb{P}_-} = \operatorname{Hom}_q(\mathcal{F}_1, \mathcal{F}_1) \longrightarrow \operatorname{Ext}^1_q(\mathcal{F}_2(\tau_-), \mathcal{F}_1)$.

(3) $\pi_2^*(T_{M(1,(c_1-\xi)/2,m)}) = \operatorname{Ext}^1_q(\mathcal{F}_2, \mathcal{F}_2)$ and $\pi_2^* T_{M(1,(c_1+\xi)/2,n)} = \operatorname{Ext}^1_q(\mathcal{F}_1, \mathcal{F}_1)$.

(4) By Mukai's sheafified Kodaira-Spencer map [Mu1] we have $i_-^* T_{M_-} = \operatorname{Ext}^1_q(\mathcal{E}, \mathcal{E})$. Mukai shows the result only if S is an abelian or K3-surface, but in his proof he only uses that $Spl(c_1, c_2)$ is smooth in a neighbourhood of $\mathbf{E}_-$, (which we have just seen) and $\operatorname{Ext}^1_q(\mathcal{E}, \mathcal{E})$ is locally free and compatible with base change.

To show that the sequences in the diagram are exact we just use standard techniques. It is enough to check the exactness fibrewise. One has repeatedly to make use of the fact that ξ defines a good wall, i.e. if $E \in \mathbf{E}_-$ is given by (2.5.1), then $F_1 - F_2$, $F_2 - F_1$, $F_1 - F_2 + K_S$, $F_2 - F_1 + K_S$ are not effective, which implies that $\operatorname{Hom}_q(\mathcal{F}_1, \mathcal{F}_2(\tau_-)) = \operatorname{Hom}_q(\mathcal{F}_2(\tau_-), \mathcal{F}_1) = \operatorname{Ext}^2_q(\mathcal{F}_1, \mathcal{F}_2(\tau_-)) = \operatorname{Ext}^2_q(\mathcal{F}_2(\tau_-), \mathcal{F}_1) = 0$. In addition we use that all $E \in \mathbf{E}_-$ are simple and that $\operatorname{Ext}^2_q(\mathcal{F}_2, \mathcal{F}_2) = \operatorname{Ext}^2_q(\mathcal{F}_1, \mathcal{F}_1) = 0$. We also use the vanishings from the proof of the smoothness of M_- along $\mathbf{E}_-$.

Second Case: K_S is trivial.

We apply essentially the same arguments as in the first case. Now however we have $\mathrm{Ext}^2_q(\mathcal{E},\mathcal{F}_1) = \mathrm{Ext}^2_q(\mathcal{F}_2(\tau_-),\mathcal{E}) = \mathrm{Ext}^2_q(\mathcal{E},\mathcal{E}) = \mathrm{Ext}^2_q(\mathcal{F}_1,\mathcal{F}_1) = \mathrm{Ext}^2_q(\mathcal{F}_2,\mathcal{F}_2) = \mathcal{O}_{\mathbb{P}_-}$, which follows easily from Mukai's results [Mu1]. We also notice that by Serre-duality $\mathcal{A}_+$ is canonically dual to $\mathcal{A}_-$. Using all this we again get the diagram (4.4.1) with the entry $\mathcal{A}_+(\tau_-)$ in the lower right corner replaced by the kernel of the natural map $\mathcal{A}_+(\tau_-) \to \mathcal{O}_{\mathbb{P}_-}$, i.e. $Q^\vee(\tau_-)$.

Claim: The image of the Kodaira-Spencer map $\kappa : T_{\mathbb{P}_-} \to \mathrm{Ext}^1_q(\mathcal{E},\mathcal{E})$ is $Im(\varphi)+Im(\psi)$ (see (4.4.1)).

Note that, by what we have shown so far, the claim implies the theorem.

Proof of the Claim. For dimension reasons it is enough to show that $Im(\varphi)$ and $Im(\psi)$ both are contained in the image of κ. We show it for $Im(\varphi)$. It is enough to show this fibrewise. Let $F_1 \in M(1,(c_1+\xi)/2,n)$ and let $(\mathbb{P}_-)_{F_1}$ be the fibre of the projection $\pi_1 : \mathbb{P}_- \longrightarrow M(1,(c_1+\xi)/2,n)$ over F_1. Then $(\mathbb{P}_-)_{F_1}$ is the space of extensions

$$0 \to F_1 \to E \to G \to 0$$

with G running through $M(1,(c_1-\xi)/2,m)$. Let $x \in (\mathbb{P}_-)_{F_1}$ be given by an extension

$$(\lambda_x) \qquad\qquad 0 \to F_1 \to E \to G_1 \to 0.$$

We will want to show that $\kappa(T_{(\mathbb{P}_-)_{F_1}}(x)) = \varphi(\mathrm{Ext}^1(G_1,E))$.

The tangent space to $(\mathbb{P}_-)_{F_1}$ at x is the space of first order deformations of E together with an injection $F_1 \to E$. For $t \in T_{(\mathbb{P}_-)_{F_1}}(x)$ we get therefore the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & F_1 \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & G_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_1 \oplus F_1 \arrow[r] \arrow[d] & \widetilde{E} \arrow[r] \arrow[d] & \widetilde{G} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_1 \arrow[r] \arrow[d] \arrow[ur, "\gamma"] & E \arrow[r] \arrow[d] & G_1 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \qquad (*)$$

and we see that $T_{(\mathbb{P}_-)_{F_1}}(x)$ can be identified with the space of diagrams $(*)$. Furthermore $\kappa(t)$ is the extension class of the middle column of $(*)$. From $(*)$ we also get a sequence $0 \longrightarrow E \longrightarrow \widetilde{E}/\gamma(F_1) \longrightarrow G_1 \to 0$ such that $\widetilde{E}$ is defined by pull-back

$$\begin{tikzcd}
\widetilde{E}/\gamma(F_1) \arrow[r] & G_1 \\
\widetilde{E}. \arrow[u] \arrow[r] & E \arrow[u]
\end{tikzcd}$$

This gives a map $\theta : T_{(\mathbb{P}_-)_{F_1}}(x) \longrightarrow Ext^1(G_1,E)$, such that the restriction of κ to $T_{(\mathbb{P}_-)_{F_1}}(x)$ is $\varphi\circ\theta$. To finish the proof we have to see that θ is an isomorphism. We give an inverse. Let

$$0 \longrightarrow E \longrightarrow W \longrightarrow G_1 \longrightarrow 0$$

be an extension. We define $\widetilde{E}$ as the fibre product

$$\begin{array}{ccc} W & \longrightarrow & G_1 \\ \uparrow & & \uparrow \\ \widetilde{E}, & \longrightarrow & E \end{array}$$

and we see that it lies in a diagram $(*)$. $\square$

Remark 4.5. Assume $p_g(S) = 0$. From lemma 4.4 it follows that the dimension of $\mathbf{E}_-$ is at most the expected dimension $N = (4c_2 - c_1^2) - 3\chi(\mathcal{O}_S) + q(S)$. We have to distinguish two cases.

(1) $dim(\mathbf{E}_-) < N$ and $dim(\mathbf{E}_+) < N$. Then the change from M_- to M_+ is a birational transformation.

(2) $dim(\mathbf{E}_-) = N$ or $dim(\mathbf{E}_+) = N$. We can assume that $dim(\mathbf{E}_-) = N$. Then by lemma 4.4 $\mathbf{E}_-$ is a smooth connected component of M_-, which is isomorphic to $\mathbb{P}_-$. And, $\mathcal{A}_+(\tau_-)$ being the normal bundle to $\mathbf{E}_-$, we have $\mathcal{A}_+ = 0$ and therefore $\mathbf{E}_+ = \emptyset$. This happens if and only if $\langle \xi \cdot (\xi - K_S)\rangle/2 + d_\xi = \chi(\mathcal{O}_S)$. If we allow $NS(S)$ to contain 2-torsion, we see that all the connected components of $\mathbf{E}_-$ are connected components of M_-.

Assume for the following definition and corollary that we are in case (1) of 4.5, i.e. that the change from M_- to M_+ is birational.

Definition 4.6. Let $\widetilde{M}_-$ be the blow-up of M_- along $\mathbf{E}_-$ and D the exceptional divisor. Similarly let $\widetilde{M}_+$ be the blow up of M_+ along $\mathbf{E}_+$. Let π_D, π_{D-}, π_{D+} be the projections from D to T, $\mathbb{P}_-$, $\mathbb{P}_+$ respectively.

Corollary 4.7. (1) *If $p_g(S) = 0$ then D is isomorphic to $\mathbb{P}_- \times_T \mathbb{P}_+$ and with this identification $\mathcal{O}(D)|_D = \mathcal{O}(\tau_- + \tau_+)$.*

(2) *If K_S is trivial, then $\mathcal{A}_-$ and $\mathcal{A}_+$ are canonically dual and D is the incidence correspondence $\{(l, H) \in \mathbb{P}(\mathcal{A}_-) \times_T \mathbb{P}^\vee(\mathcal{A}_-) \,|\, l \subset H\}$ and $\mathcal{O}(D)|_D$ is the restriction of $\mathcal{O}(\tau_- + \tau_+)$.*

5. Blow-up construction

We keep the assumptions and notations of the last section. In addition we assume that we in case (1) of 4.5, i.e. the map $\widetilde{M}_- \longrightarrow M_-$ is birational. In this section we want to show that $\widetilde{M}_-$ and $\widetilde{M}_+$ are isomorphic. We shall construct a morphism $\varphi_+ : \widetilde{M}_- \to M_+$, which we shall show is the blow-up of M_+ along $\mathbf{E}_+$. Let $\varphi_- : \widetilde{M}_- \to M_-$ be the blow-up map and $j : D \to \widetilde{M}_-$ be the embedding. We denote $\widetilde{M}_-^s := \varphi_-^{-1} M_-^s$. Let $\mathcal{U}_-$ be a universal pseudo-family on $S \times M_-^s$ and $\mathcal{V}_- := (\mathrm{id}_S \times \varphi_-)^* \mathcal{U}_-$. We want to make an elementary transform of $\mathcal{V}_-$ along $D_S := S \times D$ to obtain a pseudo-family $\mathcal{V}_+$ of a_+-stable sheaves on $\widetilde{M}_-^s$ and thus the desired map φ_+. If $\mathcal{U}_-$ is a universal family, then also $\mathcal{V}_+$ will be one.

Notation 5.1. For a sheaf $\mathcal{H}$ on $S \times \mathbb{P}_-$ (resp. $S \times \mathbb{P}_+$) we will write $\mathcal{H}_D$ for $(\mathrm{id}_S \times \pi_{D-})^* \mathcal{H}$ (resp. $(\mathrm{id}_S \times \pi_{D+})^* \mathcal{H}$). We also write $\mathcal{F}_{1D}$ and $\mathcal{F}_{2D}$ instead of $(\mathcal{F}_1)_D$ and $(\mathcal{F}_2)_D$.

Definition 5.2. By the universal property of M_- and lemma 4.3 there is a line bundle λ on D such that there is an exact sequence

$$0 \to \mathcal{F}_{1D}(\lambda) \longrightarrow \mathcal{V}_-|_{D_S} \longrightarrow \mathcal{F}_{2D}(\tau_- + \lambda) \longrightarrow 0, \tag{5.2.1}$$

indeed there is already a sequence like this on $\mathbf{E}_-$. Let γ be the composition $\mathcal{V}_- \longrightarrow \mathcal{V}_-|_{D_S} \longrightarrow \mathcal{F}_{2D}(\tau_- + \lambda)$. Then we put $\mathcal{V}_+ := ker\gamma$. Because $\mathcal{V}_-$ is flat on $S \times \widetilde{M}_-^s$, and $\mathcal{F}_{2D}(\lambda + \tau_-)$ is flat on the Cartier divisor $S \times D$, $\mathcal{V}_+$ is flat over $S \times \widetilde{M}_-^s$.

The restrictions of $\mathcal{V}_+$ and $\mathcal{V}_-$ to $S \times \widetilde{M}_-^s \setminus D$ are naturally isomorphic. There are diagrams of sheaves on $S \times \widetilde{M}_-^s$

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{V}_-(-D_S) \arrow[r] \arrow[d, equal] & \mathcal{V}_+ \arrow[r] \arrow[d] & \mathcal{F}_{1D}(\lambda) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{V}_-(-D_S) \arrow[r] & \mathcal{V}_- \arrow[r] \arrow[d] & \mathcal{V}_-|_{D_S} \arrow[r] \arrow[d] & 0 \\
 & & \mathcal{F}_{2D}(\tau_- + \lambda) \arrow[r, equal] \arrow[d] & \mathcal{F}_{2D}(\tau_- + \lambda) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd} \tag{5.2.2}$$

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{V}_+(-D_S) \arrow[r] \arrow[d, equal] & \mathcal{V}_-(-D_S) \arrow[r] \arrow[d] & \mathcal{F}_{2D}(\lambda - \tau_+) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{V}_+(-D_S) \arrow[r] & \mathcal{V}_+ \arrow[r] \arrow[d] & \mathcal{V}_+|_{D_S} \arrow[r] \arrow[d] & 0 \\
 & & \mathcal{F}_{1D}(\lambda) \arrow[r, equal] \arrow[d] & \mathcal{F}_{1D}(\lambda) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd} \tag{5.2.3}$$

By the rightmost column of (5.2.3), $(\mathcal{V}_+)_x \in \mathbf{E}_+$ for all $x \in D$. Therefore by proposition 3.14 $\mathcal{V}_+$ is a pseudo-family of a_+-stable sheaves over $\widetilde{M}_-^s$ and defines a morphism $\varphi_+ : \widetilde{M}_-^s \to M_+^s$. We see from the definitions that the restriction of φ_+ to $\widetilde{M}_-^s \setminus D$ is an isomorphism to $M_+^s \setminus \mathbf{E}_+$, which coincides with the natural identification $\widetilde{M}_-^s \setminus D \simeq M_-^s \setminus \mathbf{E}_- \simeq M_+^s \setminus \mathbf{E}_+$. As $\mathbf{E}_- \subset M_-^s$ and $\mathbf{E}_+ \subset M_+^s$, we see that φ_+ extends to a morphism $\widetilde{M}_- \longrightarrow M_+$, which we still denote by φ_+.

Theorem 5.3. *$\varphi_+ : \widetilde{M}_- \longrightarrow M_+$ is the blow up of M_+ along $\mathbf{E}_+$.*

Proof. By the above $\varphi_+(D) \subset \mathbf{E}_+$. We want to show that $\varphi_+|_D$ is the projection $\pi_{D+} : D \longrightarrow \mathbf{E}_+$. For this we have to show that the extension

$$0 \longrightarrow \mathcal{F}_{2D}(\lambda - \tau_+) \longrightarrow \mathcal{V}_+|_{S\times D} \longrightarrow \mathcal{F}_{1D}(\lambda) \longrightarrow 0$$

from the rightmost column of (5.2.3) is the pull-back via π_{D+} of the tautological extension on $\mathbb{P}_+$ (defined analogously to 4.3.1) tensorized with $\mathcal{O}_D(\lambda - \tau_+)$. It is enough to show this fibrewise.

Let $x = (x_-, x_+) \in D \subset \mathbb{P}_- \times_T \mathbb{P}_+$ and let $V_- := (\mathcal{V}_-)_x$ and $V_+ := (\mathcal{V}_+)_x$ be given by extensions

$$0 \longrightarrow F_1 \longrightarrow V_- \longrightarrow F_2 \longrightarrow 0, \tag{5.3.1}$$

$$0 \longrightarrow F_2 \longrightarrow V_+ \longrightarrow F_1 \longrightarrow 0. \tag{5.3.2}$$

Then $\pi_D(x)$ is the point $(F_1, F_2) \in T$ and $x_- \in (\mathbb{P}_-)_{(F_1,F_2)} = \mathbb{P}(\mathrm{Ext}^1(F_2, F_1))$ is the extension class of (5.3.1). Then we have to show that $x_+ \in (\mathbb{P}_+)_{(F_1,F_2)} = \mathbb{P}(\mathrm{Ext}^1(F_1, F_2))$ is the extension class of (5.3.2).

Let $R := \operatorname{Spec} \mathbb{C}[\epsilon]/(\epsilon^2)$ and let $t : R \to \widetilde{M}_-$ be a tangent vector to $\widetilde{M}_-$ at x, which is not tangent to D. Then t factors through $2D$ (i.e. the subscheme defined by $\mathcal{I}_D^2$). If we restrict the diagrams (5.2.2), (5.2.3) to $2D_S$, we see that the image of the map $\mathcal{V}_-(-D_S)|_{2D_S} \longrightarrow \mathcal{V}_+|_{2D_S}$ is $\mathcal{I}_{D_S}\mathcal{V}_-/\mathcal{I}_{D_S}^2\mathcal{V}_-$ and the image of the composition $\mathcal{V}_+(-D_S)|_{2D_S} \longrightarrow \mathcal{V}_-(-D_S)|_{2D_S} \longrightarrow \mathcal{V}_+|_{2D_S}$ is $\mathcal{I}_{D_S}\mathcal{F}_{1D}(\lambda) \cdot /\mathcal{I}_{D_S}^2\mathcal{F}_{1D}(\lambda)$. Therefore, by pulling back the diagrams (5.2.2), (5.2.3) to $S \times R$ via $(\mathrm{id}_S \times t)$ and pushing down with the projection $p : S \times R \to S$, we get the diagrams

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & V_- \arrow[r] \arrow[d, equal] & \widetilde{V}_+ \arrow[r] \arrow[d] & F_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & V_- \arrow[r] & \overline{V}_- \arrow[r] \arrow[d] & V_- \arrow[r] \arrow[d] & 0 \\
 & & F_2 \arrow[r, equal] \arrow[d] & F_2 \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd} \tag{5.3.3}$$

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & F_1 \arrow[r] \arrow[d, equal] & V_- \arrow[r] \arrow[d] & F_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_1 \arrow[r] & \widetilde{V}_+ \arrow[r] \arrow[d] & V_+ \arrow[r] \arrow[d] & 0 \\
 & & F_1 \arrow[r, equal] \arrow[d] & F_1 \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd} \tag{5.3.4}$$

The extension class $\delta \in \mathbb{P}(\mathrm{Ext}^1(V_-, V_-))$ of the middle row of (5.3.3) is the class of the image of t under $d\varphi_- : T_{\widetilde{M}_-}(x) \longrightarrow T_{M_-}(\varphi_-(x)) = \mathrm{Ext}^1(V_-, V_-)$. The image of the composition

$$T_{M_-}(x) \longrightarrow \varphi_-^*(T_{M_-}(x)) \xrightarrow{\rho} \varphi_-^*(N_{\mathrm{E}_-/M_-}(x))$$

is the tautological subline-bundle of $\varphi_-^*(N_{\mathbf{E}_-/M_-}(x)) = \mathrm{Ext}^1(F_1, F_2)$ and the kernel is $T_D(x)$. Therefore the image of $\rho(\delta)$ in $(\mathbb{P}_+)_{(F_1,F_2)} = \mathbb{P}(\mathrm{Ext}^1(F_1,F_2))$ is x_+. By (4.4.1) the map ρ is the composition

$$\mathrm{Ext}^1(V_-,V_-) \xrightarrow{\rho_1} \mathrm{Ext}^1(F_1,V_-) \xrightarrow{\rho_2} \mathrm{Ext}^1(F_1,F_2)$$

given by applying $\mathrm{Hom}(\cdot,\cdot)$ on both sides to the sequence $0 \to F_1 \to V_- \to F_2 \to 0$. By (5.3.3) $\rho_1(\delta)$ is the extension class of the first row of (5.3.3) giving $\widetilde{V}_+$, and then, by (5.3.4), $\rho(\delta)$ is the extension class of (5.3.2). So we see that $\varphi_+|_D$ is the projection to $\mathbf{E}_+$.

If for the moment we call $\overline{\varphi}_+ : \widetilde{M}_+ \to M_+$ the blow-up of M_+ along $\mathbf{E}_+$ and $\overline{D}$ the exceptional divisor, we get analogously that $\overline{D} \simeq \mathbb{P}_- \times_T \mathbb{P}_+$, (or the incidence correspondence in $\mathbb{P}_- \times_T \mathbb{P}_+$ in case K_S is trivial). In the same way as above we can construct a morphism $\overline{\varphi}_- : \widetilde{M}_+ \longrightarrow M_-$ such that $\overline{\varphi}_-|_{\overline{D}}$ is the projection to $\mathbf{E}_-$ and $\overline{\varphi}_-|_{\widetilde{M}_+\setminus\overline{D}}$ is just the natural identification $\widetilde{M}_+ \setminus \overline{D} \simeq M_+ \setminus \mathbf{E}_+ \simeq M_- \setminus \mathbf{E}_-$. Therefore we have morphisms $\varphi_- \times \varphi_+ : \widetilde{M}_- \to M_- \times M_+$, $\overline{\varphi}_- \times \overline{\varphi}_+ : \widetilde{M}_+ \to M_- \times M_+$, which by the above are injective and easily seen to be injective on tangent vectors. Furthermore $(\varphi_- \times \varphi_+)(\widetilde{M}_- \setminus D) = (\overline{\varphi}_- \times \overline{\varphi}_+)(\widetilde{M}_+ \setminus \overline{D})$. Therefore $\widetilde{M}_-$ and $\widetilde{M}_+$ are isomorphic and in fact both isomorphic to the closure of the graph of the obvious rational map $M_- \to M_+$. □

In the following theorem we put together the main results we have obtained so far.

Theorem 5.4. *Let S be a surface with either $p_g(S) = 0$ or K_S trivial. Let $c_1 \in NS(S)$, $c_2 \in \mathbb{Z}$ and put $N := 4c_2 - c_1^2 - 3\chi(\mathcal{O}_S) + q(S)$. Let W be a good wall of type (c_1, c_2) and let H_-, H_+ be ample divisors on S in neighbouring chambers separated by W. Then for all $a \in [0,1]$ there exist spaces $M_a(c_1,c_2)$ and a finite set of miniwalls dividing $[0,1]$ into finitely many minichambers such that the following holds:*

1. *$M_0(c_1,c_2) = M_{H_-}(c_1,c_2)$, $M_1(c_1,c_2) = M_{H_+}(c_1,c_2)$.*
2. *If a_1, a_2 are in the same minichamber then $M_{a_1}(c_1,c_2) = M_{a_2}(c_1,c_2)$.*
3. *If $a_- < a < a_+$ and a is the unique miniwall between a_- and a_+ then $M_{a_+}(c_1,c_2)$ is obtained from $M_{a_-}(c_1,c_2)$ as follows: We blow up $M_{a_-}(c_1,c_2)$ along the disjoint smooth subvarieties $\mathbf{E}_\xi^{n,m}$, with $(\xi,n,m) \in A^+(a)$ (see 3.8) which fulfill $0 \le dim(\mathbf{E}_\xi^{n,m}) < N$ and blow-down the exceptional divisors to $\mathbf{E}_{-\xi}^{m,n}$ respectively. Then we remove the $\mathbf{E}_\xi^{n,m}$ with $(\xi,n,m) \in \mathcal{A}^+(a)$ and $dim(\mathbf{E}_\xi^{n,m}) = N$ (which are unions of connected components of $M_{a_-}(c_1,c_2)$) and take the disjoint union with all $\mathbf{E}_{-\xi}^{m,n}$ with $(\xi,n,m) \in \mathcal{A}^+(a)$ and $\mathbf{E}_\xi^{n,m} = \emptyset$.*
4. *If H is an ample divisor on W which lies in the closure of both of the chambers containing H_- and H_+, then, for all $b \in [0,1]$, the space $M_b(c_1,c_2)$ is a moduli space of H-semistable sheaves on S with a suitable parabolic structure.*

In [Mu1] Mukai defines elementary transforms of a symplectic variety X as follows. Assume X contains a subvariety P, which has codimension n and is a $\mathbb{P}_n$-bundle over a variety Y. Let $\widetilde{X}$ be the blow-up of X along P. Then the exceptional divisor E is isomorphic to the incidence correspondence in $P \times_Y P'$, where P' is the dual projective bundle to P. One can then blow down E to P' to obtain a smooth symplectic variety X'. We will for the moment call Y the center of such an elementary transformation.

So by the above we obtain the following:

Corollary 5.5. *Let S be a K3-surface or an abelian surface. Let H_-, H_+ be polarisations which both do not lie on a wall. Then $M_{H_+}(c_1, c_2)$ is obtained from $M_{H_-}(c_1, c_2)$ by a series of elementary transforms, whose centers are of the form $M(1, (c_1+\xi)/2, n) \times M(1, (c_1-\xi)/2, m)$ for ξ defining a wall between H_- and H_+ and (n, m) running through the nonnegative integers with $n + m = (4c_2 - c_1^2 + \xi^2)/4$.*

Remark 5.6. If $q(S) \neq 0$ we can also, for $A \in Pic(S)$, $c_2 \in \mathbb{Z}$ and an ample divisor H, study the moduli space $\widetilde{M}_H(A, c_2)$ of rank 2 torsion-free sheaves E on S with $det(E) = A$ and $c_2(E) = c_2$. Then there is a morphism $M_H(c_1, c_2) \longrightarrow Pic^0(S)$, whose fibres are the various $\widetilde{M}_H(A, c_2)$ for A with $c_1(A) = c_1$. Then, by restricting our arguments to the fibres, we get that theorem 5.4 also holds with the obvious changes for $\widetilde{M}_H(A, c_2)$.

6. The change of the Donaldson invariants in terms of Hilbert schemes

In this section we assume that $q(S) = 0$. Let $\gamma_{c_1,c_2,g}$ be the Donaldson polynomial with respect to a Riemannian metric g associated to the principal $SO(3)$-bundle P on S whose second Stiefel-Whitney class $w_2(P)$ is the reduction of c_1 mod 2 and whose first Pontrjagin class is $p_1(P) = (c_1^2 - 4c_2)$. Then $\gamma_{c_1,c_2,g}$ is a homogeneous polynomial on $H_*(S, \mathbb{Q})$ of weight $2N = 2(4c_2 - c_1^2 - 3\chi(\mathcal{O}_S))$, where the elements of $H_i(S, \mathbb{Q})$ have weight $4 - i$. In case $p_g(S) > 0$ it is known that $\gamma_{c_1,c_2,g}$ does not depend on the metric (as long as it is generic).

In [Ko] the invariant has been introduced for 4-manifolds M with $b_+(M) = 1$. In [K-M] it has been shown that in case $b_+(M) = 1$, $b_1(M) = 0$ it depends only on the chamber of the period point of the metric in the positive cone of $H^2(M, \mathbb{R})$.

The algebro-geometric analogues of the Donaldson polynomials are defined as follows:

Definition 6.1. ([OG1], [OG2]) Assume that $M_H(c_1, c_2)$ is a fine moduli space, i.e. $M_H(c_1, c_2) = M_H^s(c_1, c_2)$, and there is a universal sheaf $\mathcal{U}$ on $S \times M_H(c_1, c_2)$. We define a linear map

$$\nu_{c_1,c_2,H} : H_i(S, \mathbb{Q}) \to H^{4-i}(M_H(c_1, c_2), \mathbb{Q}); \quad \nu_{c_1,c_2,H}(\alpha) := (c_2(\mathcal{U}) - \frac{1}{4}c_1^2(\mathcal{U}))/\alpha,$$

where $/$ denotes the slant product. We assume furthermore that $M_H(c_1, c_2)$ is of the expected dimension $N := 4c_2 - c_1^2 - 3\chi(\mathcal{O}_S)$. Given classes $\alpha_s \in H_{2j_s}(S, \mathbb{Q})$, for $s = 1, \dots, k$ with $2k - \sum_s j_s = N$, we set

$$\Phi_{c_1,c_2,H}(\alpha_1, \dots, \alpha_k) := \int\limits_{M_H(c_1,c_2)} \nu_{c_1,c_2,H}(\alpha_1) \cdot \dots \cdot \nu_{c_1,c_2,H}(\alpha_k).$$

As c_1, c_2 are fixed in our paper, we will write $\nu_H = \nu_{c_1,c_2,H}$ and $\Phi_H = \Phi_{c_1,c_2,H}$.

Let $pt \in H_0(S, \mathbb{Z})$ be the class of a point in S. Knowing Φ_H is equivalent to knowing the numbers

$$\Phi_{H,l,r}(\alpha) := \int\limits_{M_H(c_1,c_2)} \nu_H(\alpha)^l \cdot \nu_H(pt)^r.$$

for all l, r with $l + 2r = N$ and all $\alpha \in H_2(S, \mathbb{Q})$.

Definition 6.2. Following [OG2], we call $M_H(c_1, c_2)$ admissible if the following holds:

(1) H does not lie on a wall of type (c_1, c_2);
(2) $dim(M_H(c_1, c_2)) = N$,
(3) if c_1 is divisible by 2 in $NS(S)$, then $N > (4c_2 - c_1^2)/2$;
(4) $dim(M_H(c_1, k)) + 2(c_2 - k) < N$ for all $k < c_2$.

For admissible $M_H(c_1, c_2)$ the results of [Mo] and [Li] give

$$\Phi_H|_{H^2(S,\mathbb{Q})} = (-1)^{(c_1^2+(c_1 \cdot K_S))}\gamma_{c_1,c_2,g(H)}|_{H^2(S,\mathbb{Q})},$$

where $g(H)$ is the Fubini-Study metric associated to H. Furthermore if $c_2 >> 0$, then $\Phi_H = (-1)^{(c_1^2+(c_1 \cdot K_S))}\gamma_{c_1,c_2,g(H)}$.

We now want to determine how Φ_H changes, when H passes through a wall. We assume that if c_1 is divisible by 2 in $NS(S)$ then $(4c_2 - c_1^2)$ is not divisible by 8. Then, by the criterion of [Ma2], $M_H(c_1, c_2)$ is a fine moduli space, unless H lies on a wall.

Now we assume that we are in the situation of section 3, i.e. H_- and H_+ are ample divisors lying in neighbouring chambers separated by W, and H a polarization on the wall W not lying on any other wall and lying in the closure of both the chambers containing H_- and H_+. We assume furthermore that W is a good wall. For $b \in [0, 1]$ we have $M_b(c_1, c_2)$ as in section 3.

Definition 6.3. By remark 3.13 we see that, for b not on a miniwall, $M_b(c_1, c_2) = M_b^s(c_1, c_2)$ and there is a universal sheaf on $M_b(c_1, c_2)$.

Assume that $b \in [0, 1]$ does not lie on a miniwall. Then analoguosly to the definition of Φ_H and $\Phi_{H,l,r}$ in 6.1, we may define Φ_b and $\Phi_{b,l,r}$ by always replacing $M_H(c_1, c_2)$ by $M_b(c_1, c_2)$.

We notice that $\Phi_{H_-} = \Phi_0$ and $\Phi_{H_+} = \Phi_1$ and it is obvious that Φ_b only depends on the minichamber containing b. We therefore have to determine the change of Φ_b when b passes through a miniwall.

We will make the same assumptions as in section 4, i.e. let a be a miniwall and let $(\xi, n, m) \in A^+(a)$. Let $a_- < a_+$ lie in neighbouring minichambers separated by a. To simplify the notation we will for the moment assume that $A^+(a) = \{(\xi, n, m)\}$ and that $H^2(S, \mathbb{Z})$ contains no 2-torsion. We also assume that either $p_g(S) = 0$ or K_S is trivial.

Notation 6.4. We use the notations and definitions of sections 4 and 5. If the change is birational, i.e. we are not in case (1) of 4.5, we shall write $\widetilde{M}$ instead of $\widetilde{M}_-$. Let $d := d_\xi = n+m$, $e_- = rk(\mathcal{A}_-)$, $e_+ = rk(\mathcal{A}_+)$, then $N = 2d + e_- + e_+ - 1$ if $p_g(S) = 0$ and $N = 2d + e_- + e_+ - 2$ if K_S is trivial. We put $\nu_+ := \nu_{a_+}$, $\nu_- := \nu_{a_-}$, $\Phi_+ := \Phi_{a_+}$, $\Phi_- := \Phi_{a_-}$, $\Phi_{+,l,r} := \Phi_{a_+,l,r}$ and $\Phi_{-,l,r} := \Phi_{a_-,l,r}$.

Note that the condition $q(S) = 0$ implies $Pic(S) \simeq NS(S)$. For $\beta \in NS(S)$ we may therefore denote by $\mathcal{O}_S(\beta)$ the corresponding line bundle. Let q_1, q_2 be the two projections of $T = \mathrm{Hilb}^n(S) \times \mathrm{Hilb}^m(S)$.

Remark 6.5. (1) If the change is birational, then by the projection formula Φ_+, $\Phi_{+,l,r}$ (resp. Φ_-, $\Phi_{-,l,r}$) coincide with the numbers which are defined analogously by replacing $M_a(c_1, c_2)$ by $\widetilde{M}$ and the universal sheaf by $\mathcal{V}_+$ (resp. $\mathcal{V}_-$).

(2) Assume $p_g(S)=0$ and say $\mathbf{E}_+=\emptyset$. Let $\mathcal{E}$ be the universal sheaf on $\mathbf{E}_-$ from (4.3.1), then we can define $\sigma_-: H_i(S,\mathbb{Q}) \longrightarrow H^{4-i}(\mathbf{E}_-,\mathbb{Q})$ and δ_- and $\delta_{-,l,r}$ in the same way as ν_- and Φ_- and $\Phi_{-,l,r}$ by replacing M_- by $\mathbf{E}_-$ and the universal sheaf on M_- by $\mathcal{E}$. Then $\Phi_+ - \Phi_- = -\delta_-$.

Definition 6.6. Let $Z_n(S) \subset S \times \mathrm{Hilb}^n(S)$ be the universal subscheme. In $S \times \mathrm{Hilb}^n(S) \times \mathrm{Hilb}^m(S)$, we put $\mathcal{Z}_1 := (\mathrm{id}_S \times q_1)^{-1}(Z_n(S))$, $\mathcal{Z}_2 := (\mathrm{id}_S \times q_2)^{-1}(Z_m(S))$ and denote by $\mathcal{I}_{\mathcal{Z}_1}$, $\mathcal{I}_{\mathcal{Z}_2}$ the corresponding idealsheaves. Let $F_1 := \mathcal{O}_S((c_1+\xi)/2)$, $F_2 := \mathcal{O}_S((c_1-\xi)/2)$. By our assumptions $T = \mathrm{Hilb}^n(S) \times \mathrm{Hilb}^m(S)$ and $\mathcal{F}_1' = \mathcal{I}_{\mathcal{Z}_1}(F_1)$, $\mathcal{F}_2' = \mathcal{I}_{\mathcal{Z}_2}(F_2)$. Let $h_n : \mathrm{Hilb}^n(S) \longrightarrow S^{(n)}$ be the Hilbert-Chow morphism [Fo], where $S^{(n)}$ is the n-fold symmetric power of S with the quotient map $\varphi_n : S^n \longrightarrow S^{(n)}$. For $i=1,\dots,n$ we denote by $p_i : S^n \to S$ the projection to the i^{th} factor. We denote $\Delta_i := \{(x,x_1,\dots,x_n) \in S \times S^n \mid x = x_i\}$ and $Y_n := (\mathrm{id}_S \times \varphi_n)(\Delta_1)$. We have linear maps

$$\begin{aligned} \iota_n : H_i(S,\mathbb{Q}) &\longrightarrow H^{4-i}(\mathrm{Hilb}^n(S),\mathbb{Q}); \qquad \iota_n(\alpha) = [Z_n(S)]/\alpha \quad \text{and} \\ \bar{\iota}_n : H_i(S,\mathbb{Q}) &\longrightarrow H^{4-i}S^{(n)},\mathbb{Q}); \qquad \bar{\iota}_n(\alpha) = [Y_n]/\alpha. \end{aligned}$$

For $\alpha \in H^i(S,\mathbb{Q})$ put $\alpha_{n,m} := [\mathcal{Z}_1]/\alpha + [\mathcal{Z}_2]/\alpha = q_1^*(\iota_n(\alpha)) + q_2^*(\iota_n(\alpha)) \in H^{4-i}(T,\mathbb{Q})$.

The map ι_n is in fact easy to describe:

Lemma 6.7. (1) $[Z_n(S)] = (\mathrm{id}_S \times h_n)^*([Y_n])$.

(2) $(\mathrm{id}_S \times \varphi_n)^*([Y_n]) = \sum_i [\Delta_i]$

(3) *For $\alpha \in H^i(S,\mathbb{Q})$ we have $\iota_n(\alpha) = h_n^*(\bar{\iota}_n(\alpha))$ and $\varphi_n^*(\bar{\iota}_n(\alpha)) = \sum_{i=1}^n p_i^*(\check{\alpha})$, where $\check{\alpha}$ is the Poincaré dual of α.*

Proof. (1). Out of codimension 3 on $S \times \mathrm{Hilb}^n(S)$ we have $\mathcal{O}_{Z_n(S)} = (\mathrm{id}_S \times h_n)^*(\mathcal{O}_{Y_n})$. So we get $[Z_n(S)] = (\mathrm{id}_S \times h_n)^*([Y_n])$. Out of codimension 3 we also have $(\mathrm{id}_S \times \varphi_n)^*(\mathcal{O}_{Y_n}) = \bigoplus_i \mathcal{O}_{\Delta_i}$. Therefore (2) follows in the same way as (1). (3) follows immediately from (1) and (2). □

Remark 6.8. For the total Chern classes we have $c((\mathrm{id}_S \times j)^*\mathcal{V}_-) = (\mathrm{id}_S \times j)^* c(\mathcal{V}_-)$ and $c((\mathrm{id}_S \times j)^*\mathcal{V}_+) = (\mathrm{id}_S \times j)^* c(\mathcal{V}_+)$, where, as above, $j : D \longrightarrow \widetilde{M}$ is the embedding of the exceptional divisor.

Proof. We have to see that $\mathrm{Tor}_k(\mathcal{V}_-, \mathcal{O}_{S\times D}) = 0$ for all $k > 0$ (and similarly for $\mathcal{V}_+$). This follows however easily from the flatness of $\mathcal{V}_-$ over $\widetilde{M}_-$. □

Lemma 6.9. (1) *Assume that we are in case (1) of 4.5, i.e. the change of moduli is birational. Then, for $\alpha \in H_2(S,\mathbb{Q})$, we have*

$$\begin{aligned} \nu_+(\alpha) - \nu_-(\alpha) &= -\frac{1}{2}\langle \xi, \alpha\rangle [D], \\ \nu_+(pt) - \nu_-(pt) &= \frac{1}{4} j_*([\tau_-] - [\tau_+]). \end{aligned}$$

(2) *If $\mathbf{E}_+ = \emptyset$ then*

$$\begin{aligned} \sigma_-(\alpha) &= \frac{1}{2}\langle \xi, \alpha\rangle [\tau_-], \\ \sigma_-(pt) &= -\frac{1}{4}[\tau_-]^2. \end{aligned}$$

Proof. By (5.2.2) we have the sequence

$$0 \longrightarrow \mathcal{V}_- \longrightarrow \mathcal{V}_+(D_S) \longrightarrow \mathcal{F}_{1D}(\lambda + \tau_- + \tau_+) \longrightarrow 0.$$

Using Riemann-Roch without denominators [Jo] we get

$$\begin{aligned} c_1(\mathcal{F}_{1D}(\lambda + \tau_- + \tau_+)) &= [D_S] \\ c_2(\mathcal{F}_{1D}(\lambda + \tau_- + \tau_+)) &= -c_1(\mathcal{F}_{1D}(\lambda)), \end{aligned}$$

and thus

$$\begin{aligned} c_1(\mathcal{V}_+(D_S)) &= c_1(\mathcal{V}_-) + [D_S], \\ c_2(\mathcal{V}_+(D_S)) &= c_2(\mathcal{V}_-) + [D_S] \cdot c_1(\mathcal{V}_-) - (\mathrm{id}_S \times j)_*(c_1(\mathcal{F}_{1D}(\lambda))) \\ 4c_2(\mathcal{V}_+) - c_1(\mathcal{V}_+)^2 &= 4c_2(\mathcal{V}_+(D_S)) - c_1(\mathcal{V}_+(D_S))^2 \\ &= 4c_2(\mathcal{V}_-) - c_1(\mathcal{V}_-)^2 + 2[D_S] \cdot c_1(\mathcal{V}_-) - [D_S]^2 - 4(\mathrm{id}_S \times j)_*(c_1(\mathcal{F}_{1D}(\lambda))). \end{aligned}$$

Let $\alpha \in H_2(S, \mathbb{Q})$. As $[D_S]$ is the pull-back of $[D]$ from $\widetilde{M}$, we have

$$([D_S] \cdot c_1(\mathcal{V}_-))/\alpha = [D](c_1(\mathcal{V}_-)/\alpha) = \langle c_1, \alpha \rangle [D].$$

Furthermore $(\mathrm{id}_S \times j)_* c_1(\mathcal{F}_{1D}(\lambda))/\alpha = j_*(c_1(\mathcal{F}_1(\lambda))/\alpha)$, where the second slant product is taken on $S \times D$ and $c_1(\mathcal{F}_1) = \pi_D^*(c_1(\mathcal{F}_1')) = p^*([F_1])$. So we get $(\mathrm{id}_S \times j)_*(c_1(\mathcal{F}_1)/\alpha) = \langle F_1 \cdot \alpha \rangle [D]$. As λ is the pull-back of a divisor on D, we have $(\mathrm{id}_S \times j)_* c_1(\lambda)/\alpha = 0$ and similarly $[D_S]^2/\alpha = 0$. So we get $\nu_+(\alpha) - \nu_-(\alpha) = -\frac{1}{2}\langle \xi, \alpha \rangle [D]$

By $c_1(\mathcal{F}_1) = p^*([F_1])$, $c_1(\mathcal{F}_2) = p^*([F_2])$, we get $c_1(\mathcal{F}_1')/pt = c_1(\mathcal{F}_2')/pt = 0$. Then the sequence

$$0 \longrightarrow \mathcal{F}_{1D}(\lambda) \longrightarrow \mathcal{V}_-|_D \longrightarrow \mathcal{F}_{2D}(\tau_- + \lambda) \longrightarrow 0,$$

and remark 6.8 give

$$\begin{aligned} (c_1(\mathcal{V}_-) \cdot [D_S])/pt &= (\mathrm{id}_S \times j)_*(c_1(\mathcal{V}_-|_D))/pt = j_*([\tau_-] + 2[\lambda]), \\ (c_1(\mathcal{F}_{1D}(\lambda)))/pt &= j_*(c_1(\mathcal{F}_1(\lambda))/pt) = j_*([\lambda]), \\ [D_S]^2/pt &= [D]^2. \end{aligned}$$

So we get

$$\begin{aligned} \nu_+(pt) - \nu_-(pt) &= \frac{1}{4}([D]^2 + j_*([2\tau_- + 4\lambda]) - 4j_*([\lambda])) \\ &= \frac{1}{4} j_*([\tau_-] - [\tau_+]). \end{aligned}$$

(2) can be shown using essentially the same arguments. $\square$

Lemma 6.10. *Let $l + 2r = N$.*

(1) *If we are in case (1) of* 4.5, *then*

$$\Phi_{+,l,r}(\alpha) - \Phi_{-,l,r}(\alpha) = \sum_{b=0}^{l} \sum_{c=0}^{r} (-1)^{r-c+1} 2^{b+2c-N} \binom{l}{b} \binom{r}{c} \langle \xi, \alpha \rangle^{l-b} \int_D \left(\alpha_{n,m}^b pt_{n,m}^c \sum_{s+t=N-b-2c-1} (-\tau_+)^s \tau_-^t \right)$$

(2) $\mathbf{E}_+ = \emptyset$, *then*

$$\Phi_{+,l,r}(\alpha) - \Phi_{-,l,r}(\alpha)$$
$$= \sum_{b=0}^{l}\sum_{c=0}^{r}(-1)^{r-c+1}2^{b+2c-N}\binom{l}{b}\binom{r}{c}\langle\xi,\alpha\rangle^{l-b}\int_{\mathbf{E}_-}\left(\alpha_{n,m}^{b}pt_{n,m}^{c}\tau_-^{N-b-2c}\right)$$

Proof. (1) By remark 6.8 we get for $\alpha \in H_i(S,\mathbb{Q})$ that $[D]\cdot\nu_+(\alpha) = j_*((4c_2(\mathcal{V}_+|_D) - c_1(\mathcal{V}_+|_D)^2)/4\alpha$ (and similar for ν_-). By the sequences

$$0 \longrightarrow \mathcal{F}_{2D}(-\tau_+ + \lambda) \longrightarrow \mathcal{V}_+|_{D_S} \longrightarrow \mathcal{F}_{1D}(\lambda) \longrightarrow 0$$
$$0 \longrightarrow \mathcal{F}_{1D}(\lambda) \longrightarrow \mathcal{V}_-|_{D_S} \longrightarrow \mathcal{F}_{2D}(\tau_- + \lambda) \longrightarrow 0$$

we get

$$4c_2(\mathcal{V}_+|_D) - c_1(\mathcal{V}_+|_D)^2 = 4(c_2(\mathcal{F}_{1D}) + c_2(\mathcal{F}_{2D})) - (c_1(\mathcal{F}_{2D}) - c_1(\mathcal{F}_{1D}) - [\tau_+])^2$$
$$4c_2(\mathcal{V}_-|D) - c_1(\mathcal{V}_-|D)^2 = 4(c_2(\mathcal{F}_{1D}) + c_2(\mathcal{F}_{2D})) - (c_1(\mathcal{F}_{1D}) - c_1(\mathcal{F}_{2D}) - [\tau_-])^2$$

By the above we have $c_1(\mathcal{F}_{1D}) = p^*([F_1])$, $c_1(\mathcal{F}_{2D}) = p^*([F_2])$, $c_2(\mathcal{F}_{1D}) = (\mathrm{id}_S\times\pi_D)^*(c_2(\mathcal{I}_{\mathcal{Z}_1})) = (\mathrm{id}_S\times\pi_D)^*([\mathcal{Z}_1])$ and $c_2(\mathcal{F}_{2D}) = (\mathrm{id}_S\times\pi_D)^*([\mathcal{Z}_2])$, where, as above, $\pi_D : D \longrightarrow T$ is the projection. So we have

$$\begin{aligned}4c_2(\mathcal{V}_+|_D) - c_1(\mathcal{V}_+|_D)^2 &= 4(\mathrm{id}_S\times\pi_D)^*([\mathcal{Z}_1]+[\mathcal{Z}_2]) - (p^*(\xi)+[\tau_+])^2,\\ 4c_2(\mathcal{V}_-|_D) - c_1(\mathcal{V}_-|_D)^2 &= 4(\mathrm{id}_S\times\pi_D)^*([\mathcal{Z}_1]+[\mathcal{Z}_2]) - (p^*(\xi)-[\tau_-])^2,\end{aligned}$$

and thus for $\alpha \in H_2(S,\mathbb{Q})$:

$$\begin{aligned}j^*(\nu_+(\alpha)) &= \alpha_{n,m} + \frac{1}{2}\langle\xi,\alpha\rangle[-\tau_+]\\ j^*(\nu_-(\alpha)) &= \alpha_{n,m} + \frac{1}{2}\langle\xi,\alpha\rangle[\tau_-]\\ j^*(\nu_+(pt)) &= pt_{n,m} - \frac{1}{4}[\tau_+]^2\\ j^*(\nu_-(pt)) &= pt_{n,m} - \frac{1}{4}[\tau_-]^2\end{aligned}$$

We write

$$\begin{aligned}&\Phi_{+,l,r}(\alpha) - \Phi_{-,l,r}(\alpha) =\\ &= \int_{\widetilde{M}}\left(\nu_+(\alpha)^l(\nu_+(pt)^r - \nu_-(pt)^r) + \nu_+(pt)^r(\nu_+(\alpha)^l - \nu_-(\alpha)^l)\right)\\ &= \int_D\left(\frac{1}{4}([-\tau_+]+[\tau_-])j^*\left(\sum_{s+t=r-1}\nu_+(pt)^s\nu_-(pt)^t\nu_+(\alpha)^l\right)\right.\\ &\qquad\left. - \frac{1}{2}\langle\xi,\alpha\rangle\, j^*\left(\sum_{s+t=l-1}\nu_+(\alpha)^s\nu_-(\alpha)^t\nu_-(pt)^r\right)\right).\end{aligned}$$

Now the claim follows after a straightforward computation. (2) follows easily from lemma 6.9(2). □

Proposition 6.11. (1) *If S is a $K3$ surface and $N > 0$, then $\Phi_+ = \Phi_-$.*

(2) *If $p_g(S)=0$, then for $\alpha\in H_2(S,\mathbb{Q})$ and l,r with $l+2r=N$ we have*

$$\begin{aligned}&\Phi_{+,l,r}(\alpha)-\Phi_{-,l,r}(\alpha)\\ &=\sum_{b=0}^{l}\sum_{c=0}^{r}(-1)^{r-c+e_-}2^{b+2c-N}\binom{l}{b}\binom{r}{c}\langle\xi,\alpha\rangle^{l-b}\int_T\left(\alpha_{n,m}^b pt_{n,m}^c s_{2d-b-2c}(\mathcal{A}'_+\oplus\mathcal{A}'^{\vee}_-)\right)\end{aligned}$$

Proof. (1) It easy to show using Riemann-Roch, that the condition $N>0$ implies $e_->1$ and $e_+>1$. Therefore, as $\alpha_{n,m}$ and $pt_{n,m}$ are pull-backs from T, it is enough to show that for $k\le e_-+e_+-2$ we have

$$(\pi_D)_*\left(\sum_{s+t=k}(-\tau_+)^s\tau_-^t\right)=0.$$

Now D is the projectivisation $\mathbb{P}(Q)$ where $Q=\mathcal{A}_-/\tau_-$ over $\mathbb{P}_-=\mathbb{P}(\mathcal{A}'_-)$. Therefore

$$\begin{aligned}(\pi_D)_*\left(\sum_{s+t=k}(-\tau_+)^s\tau_-^t\right)&=(\pi_-)_*\left(\sum_{s+t=k}s_{s-e_++2}(Q)\tau_-^t\right)\\ &=(\pi_-)_*(s_{k-e_++2}(\mathcal{A}_-))\\ &=(\pi_-)_*\pi_-^*(s_{k-e_++2}(\mathcal{A}'_-))=0.\end{aligned}$$

Here $\pi_-:\mathbb{P}_-\longrightarrow T$ is the projection. (2) We just note that $\pi_+)_*((-\tau_+)^k)=s_{k-e_++1}(\mathcal{A}'_+)$ and $(\pi_-)_*(\tau_-^k)=(-1)^{e_-+1}s_{k-e_-+1}(\mathcal{A}'^{\vee}_-)$. Then the result follows immediately from the definitions and lemma 6.10. □

For the rest of the chapter we assume that $p_g(S)=q(S)=0$. On the other hand we allow $NS(S)=H^2(S,\mathbb{Z})$ to contain torsion.

Definition 6.12. Let $\xi\in H^2(S,\mathbb{Z})$ be a class defining a good wall of type (c_1,c_2). Let $d_\xi:=(4c_1-c_1^2+\xi^2)/4$, $e_\xi:=-\langle\xi\cdot(\xi-K_S)\rangle/2+d_\xi+1$ and

$$T_\xi:=\mathrm{Hilb}^{d_\xi}(S\sqcup S)=\coprod_{n+m=d_\xi}\mathrm{Hilb}^n(S)\times\mathrm{Hilb}^m(S).$$

Let $q_\xi:S\times T_\xi\longrightarrow T_\xi$ be the projection. Let V_ξ be the sheaf $p^*(\mathcal{O}_S(-\xi)\oplus\mathcal{O}_S(-\xi+K_S))$ on $S\times T_\xi$. Let $\mathcal{Z}_1^\xi$ (resp.$\mathcal{Z}_2^\xi$) be the subscheme of $S\times T_\xi$ which restricted to each component $S\times\mathrm{Hilb}^n(S)\times\mathrm{Hilb}^m(S)$ is the subscheme $\mathcal{Z}_1$ (resp. $\mathcal{Z}_2$) from 6.6. Let $\mathcal{I}_{\mathcal{Z}_1^\xi},\mathcal{I}_{\mathcal{Z}_2^\xi}$ be the corresponding ideal sheaves. For $\alpha\in H_i(S,\mathbb{Q})$ let $\tilde\alpha\in H^{4-i}(T_\xi,\mathbb{Q})$ be the class whose restriction to each component $\mathrm{Hilb}^n(S)\times\mathrm{Hilb}^m(S)$ of T_ξ is $\alpha_{n,m}$. Then for all l,r with $l+2r=N$ we define a map $H_2(S,\mathbb{Q})\longrightarrow\mathbb{Q}$ by

$$\begin{aligned}\delta_{\xi,l,r}(\alpha)\;:=\;&\sum_{b=0}^{l}\sum_{c=0}^{r}(-1)^{r-c+e_\xi}2^{b+2c-N}\binom{l}{b}\binom{r}{c}\langle\xi,\alpha\rangle^{l-b}\\ &\int_{T_\xi}\left(\tilde\alpha^b\widetilde{pt}^c s_{2d_\xi-2c-b}(\mathrm{Ext}^1_q(\mathcal{I}_{\mathcal{Z}_1^\xi},\mathcal{I}_{\mathcal{Z}_2^\xi}\otimes V_\xi)\right)\end{aligned}$$

Theorem 6.13. *Let S be a surface with $p_g(S)=q(S)=0$. Let $c_1\in H^2(S,\mathbb{Z})$ and $c_2\in\mathbb{Z}$. Assume that, if c_1 is divisible by 2 in $H^2(S,\mathbb{Z})$ then $(4c_2-c_1^2)$ is not divisible by 8. Let W be a good wall of type (c_1,c_2) and let H_- and H_+ be ample divisors on S lying in neighbouring chambers separated by W.*

Let n_2 be the number of 2-torsion points in $H^2(S,\mathbb{Z})$. Then for all l,r with $l+2r = N = (4c_2 - c_1^2) - 3$ we have

$$\Phi_{H_+,l,r} - \Phi_{H_-,l,r} = n_2 \sum_{\xi \in A^+(W)} \delta_{\xi,l,r}.$$

Here, as above,

$$A^+(W) = \{\xi \in H^2(S,\mathbb{Z}) \mid Z \text{ defines the wall } W \text{ and } \langle \xi \cdot H_+ \rangle > 0\}.$$

Therefore we get for a class $\alpha \in H_2(S,\mathbb{Q})$

$$(\gamma_{c_1,c_2,g(H_+)} - \gamma_{c_1,c_2,g(H_-)})(\underbrace{pt,\dots,pt}_{r},\underbrace{\alpha,\dots,\alpha}_{l}) = (-1)^{(c_1^2 + \langle c_1 \cdot K_S \rangle)} n_2 \sum_{\xi \in A^+(W)} \delta_{\xi,l,r}(\alpha).$$

Proof. If $H^2(S,\mathbb{Z})$ contains no 2-torsion, and $a_- < a_+$ are in neighbouring minichambers separated by a miniwall a with $A^+(a) = \{(\xi,n,m)\}$, then proposition 6.11 computes $\Phi_{a_+,l,r} - \Phi_{a_-,l,r}$. By Serre duality and the definitions we see that in the notations of proposition 6.11 $\mathcal{A}'_+ \oplus \mathcal{A}'_-{}^\vee = \mathrm{Ext}^1_q(\mathcal{I}_{Z_1}, \mathcal{I}_{Z_2} \otimes V_\xi)$. Thus, if for all miniwalls a the set $A^+(a)$ consists of only one element, the theorem follows.

If $N_2 \subset H^2(S,\mathbb{Z})$ is the subgroup of 2-torsion, then $T_\xi^{n,m} \simeq N_2 \times \mathrm{Hilb}^n(S) \times \mathrm{Hilb}^m(S)$. So the exceptional divisor in $\widetilde{M}$ has n_2 isomorphic components (or we add n_2 isomorphic connected components to $\widetilde{M}$ or subtract them), and each component gives the same contribution to $\Phi_{a_+,l,r} - \Phi_{a_-,l,r}$.

Assume that $A^+(a) = \{(\xi_1,n_1,m_1),\dots,(\xi_2,n_2,m_2)\}$. Then, as we have seen above, the $\mathbf{E}_{\xi_i}^{n_i,m_i}$ are disjoint, and, as the change $\Phi_{a_+,l,r} - \Phi_{a_-,l,r}$ can be computed on the exceptional divisor (or the added components), it is just the sum of the contributions for all (ξ_i,n_i,m_i). The result now follows by adding up the contributions of all the miniwalls. $\square$

By the results we have obtained so far, in order to compute explicitly the change of the Donaldson invariants, when the polarisation passes through a good wall $W = W^\xi$, we have first to determine the Chern classes of the bundles $\mathrm{Ext}^1_{q_\xi}(\mathcal{I}_{Z_1^\xi}, \mathcal{I}_{Z_2^\xi} \otimes V_\xi)$ on T_ξ, and then make explicit computations in the cohomology ring of $\mathrm{Hilb}^d(S \sqcup S)$.

In the rest of this section we will again use the assumptions and notations from 6.4, and will adress the first question, i.e. we express the Chern classes of the vector bundles $\mathrm{Ext}^1_q(\mathcal{I}_{Z_1}, \mathcal{I}_{Z_2} \otimes V)$ on $T = \mathrm{Hilb}^n(S) \times \mathrm{Hilb}^m(S)$, (where we have written $V := V_\xi$) in terms of those of "standard bundles".

Definition 6.14. Using the projections $p : S \times T \longrightarrow S$ and $q : S \times T \longrightarrow T$ we associate to a vector bundle U of rank r on S the vector bundles $[U]_1 := q_*(\mathcal{O}_{Z_2} \otimes p^*(U))$ and $[U]_2 := q_*(\mathcal{O}_{Z_1} \otimes p^*(U))$ of ranks rn (resp. rm) on T.

For a Cohen-Macaulay scheme Z, we denote by ω_Z its dualizing sheaf.

Lemma 6.15.

$$\mathrm{Ext}^2_q(\mathcal{O}_{Z_1}, \mathcal{O}_{Z_2} \otimes p^*V) = q_*(\omega_{Z_1} \otimes \omega_T^{-1} \otimes \mathcal{O}_{Z_2} \otimes p^*V)$$

and $\mathrm{Ext}^i_q(\mathcal{O}_{Z_1}, \mathcal{O}_{Z_2} \otimes p^*V) = 0$ for $i \neq 2$.

Proof. Let

$$0 \longrightarrow B_2 \longrightarrow B_1 \longrightarrow \mathcal{O}_{S\times T} \longrightarrow \mathcal{O}_{\mathcal{Z}_1} \longrightarrow 0 \tag{6.15.1}$$

be a locally free resolution on $S \times T$. We apply $\mathcal{H}om(\cdot, \mathcal{O}_{\mathcal{Z}_2} \otimes p^*V)$ to obtain the complex

$$0 \longrightarrow \mathcal{O}_{\mathcal{Z}_2} \otimes p^*V \longrightarrow B_1^* \otimes \mathcal{O}_{\mathcal{Z}_2} \otimes p^*V \longrightarrow B_2^* \otimes \mathcal{O}_{\mathcal{Z}_2} \otimes p^*V \longrightarrow 0,$$

whose cohomologies are the $\mathcal{E}xt^i(\mathcal{O}_{\mathcal{Z}_1}, \mathcal{O}_{\mathcal{Z}_2} \otimes p^*V)$. We can arrive at this complex differently, namely by first dualizing and then tensorizing by $\mathcal{O}_{\mathcal{Z}_2} \otimes p^*V$. By dualizing and using that $\mathcal{Z}_1$ is Cohen-Macauley we obtain

$$0 \longrightarrow \mathcal{O}_{S\times T} \longrightarrow B_1^* \longrightarrow B_2^* \longrightarrow \omega_{\mathcal{Z}_1} \otimes \omega_T^{-1} \longrightarrow 0.$$

Tensorizing by $\mathcal{O}_{\mathcal{Z}_2} \otimes p^*V$ gives the sequence

$$0 \longrightarrow \mathcal{O}_{\mathcal{Z}_2} \otimes p^*V \longrightarrow B_1^* \otimes \mathcal{O}_{\mathcal{Z}_2} \otimes p^*V \longrightarrow B_2^* \otimes \mathcal{O}_{\mathcal{Z}_2} \otimes p^*V \longrightarrow \omega_{\mathcal{Z}_1} \otimes \omega_T^{-1} \otimes \mathcal{O}_{\mathcal{Z}_2} \otimes p^*V \longrightarrow 0,$$

which is exact by the corollaire on p. V.20 in [Se] because $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are Cohen-Macaulay and intersect properly. Hence $\mathcal{E}xt^2(\mathcal{O}_{\mathcal{Z}_1}, \mathcal{O}_{\mathcal{Z}_2} \otimes p^*V) = \omega_{\mathcal{Z}_1} \otimes \omega_T^{-1} \otimes \mathcal{O}_{\mathcal{Z}_2} \otimes p^*V$ and $\mathcal{E}xt^i(\mathcal{O}_{\mathcal{Z}_1}, \mathcal{O}_{\mathcal{Z}_2} \otimes p^*V) = 0$ for $i < 2$. As $\mathcal{Z}_2$ and $\mathcal{Z}_1$ are flat of dimension 0 over T, the result follows by applying q_*. $\square$

Proposition 6.16. *In the Grothendieck ring of sheaves on T we have the equality*

$$\begin{aligned}\mathrm{Ext}^1_q(\mathcal{I}_{\mathcal{Z}_1}, \mathcal{I}_{\mathcal{Z}_2} \otimes p^*V) &= [V]_2 + ([V^\vee(K_S)]_1)^\vee + (H^1(S, \mathcal{O}_S(-\xi)) \oplus H^1(S, \mathcal{O}_S(-\xi + K_S))) \otimes \mathcal{O}_T \\ &\quad - q_*(\omega_{\mathcal{Z}_1} \otimes \omega_T^{-1} \otimes \mathcal{O}_{\mathcal{Z}_2} \otimes p^*V).\end{aligned}$$

Proof. Case n=0: We will use repeatedly that ξ defines a good wall, so in particular $q_*(p^*V) = R^2q_*(p^*V) = 0$. We apply $\mathrm{Hom}_q(\mathcal{O}_{S\times T}, \cdot)$ to the sequence

$$0 \longrightarrow \mathcal{I}_{\mathcal{Z}_2} \otimes p^*V \longrightarrow p^*V \longrightarrow \mathcal{O}_{\mathcal{Z}_2} \otimes p^*V \longrightarrow 0$$

to obtain

$$0 \longrightarrow [V]_2 \longrightarrow \mathrm{Ext}^1_q(\mathcal{O}_{S\times T}, \mathcal{I}_{\mathcal{Z}_2} \otimes p^*V) \longrightarrow R^1q_*(p^*V) \longrightarrow 0. \tag{6.16.1}$$

The surjectivity follows as $\mathcal{Z}_2$ is flat of dimension 0 over T and the injectivity by $q_*p^*V = 0$.
General case: We apply $\mathrm{Hom}_q(\cdot, \mathcal{I}_{\mathcal{Z}_2} \otimes p^*V)$ to the sequence $0 \longrightarrow \mathcal{I}_{\mathcal{Z}_1} \longrightarrow \mathcal{O}_{S\times T} \longrightarrow \mathcal{O}_{\mathcal{Z}_1} \longrightarrow 0$ to get

$$0 \to \mathrm{Ext}^1_q(\mathcal{O}_{S\times T}, \mathcal{I}_{\mathcal{Z}_2} \otimes p^*V) \to \mathrm{Ext}^1_q(\mathcal{I}_{\mathcal{Z}_1}, \mathcal{I}_{\mathcal{Z}_2} \otimes p^*V) \to \mathrm{Ext}^2_q(\mathcal{O}_{\mathcal{Z}_1}, \mathcal{I}_{\mathcal{Z}_2} \otimes p^*V) \to 0. \tag{6.16.2}$$

The exactness on the left follows from the fact that $q_*(\mathcal{I}_{\mathcal{Z}_2} \otimes p^*V) = 0$ and so $\mathrm{Ext}^1_q(\mathcal{O}_{\mathcal{Z}_1}, \mathcal{I}_{\mathcal{Z}_2} \otimes p^*V)$ is torsion-free being a subsheaf of the locally free sheaf $R^1q_*(\mathcal{I}_{\mathcal{Z}_2} \otimes p^*V)$. Its support is contained in $q(\mathcal{Z}_1 \cap \mathcal{Z}_2)$ and thus it is the zero sheaf.

We apply $\mathrm{Hom}_q(\mathcal{O}_{\mathcal{Z}_1}, \cdot)$ to $0 \longrightarrow \mathcal{I}_{\mathcal{Z}_2} \otimes p^*V \longrightarrow p^*V \longrightarrow \mathcal{O}_{\mathcal{Z}_2} \otimes p^*V \longrightarrow 0$ and use lemma 6.15 to obtain

$$0 \longrightarrow \mathrm{Ext}^2_q(\mathcal{O}_{\mathcal{Z}_1}, \mathcal{I}_{\mathcal{Z}_2} \otimes p^*V) \longrightarrow \mathrm{Ext}^2_q(\mathcal{O}_{\mathcal{Z}_1}, p^*V) \longrightarrow q_*(\omega_{\mathcal{Z}_1} \otimes \omega_T^{-1} \otimes \mathcal{O}_{\mathcal{Z}_2} \otimes p^*V) \longrightarrow 0. \tag{6.16.3}$$

By duality $\mathrm{Ext}^2_q(\mathcal{O}_{\mathcal{Z}_1}, p^*V) = q_*(\mathcal{O}_{\mathcal{Z}_1} \otimes p^*(V^\vee(K_S))^\vee = [V^\vee(K_S)]_1^\vee$. Thus the result follows by putting 6.16.1 to 6.16.3 together. $\square$

7. Explicit computations on Hilbert schemes of points

The aim of this section is to make theorem 6.13 more explicit. We want to compute the contributions δ_ξ to the change of the Donaldson invariants for a class ξ defining a good wall, in terms of cohomology classes and intersection numbers on S. We do not succeed in determining δ_ξ completely. It turns however out that δ_ξ can be developed in terms of powers of ξ and we will compute the six lowest order terms (as predicted by the conjecture of Kotschick and Morgan half of them are zero).

Notation 7.1. In this section we fix a class $\xi \in H^2(S, \mathbb{Z})$ which defines a good wall of type (c_1, c_2) and will therefore drop ξ in our notation. In particular we write $d := d_\xi$, $e := e_\xi$ and $T := \mathrm{Hilb}^d(S \sqcup S)$. As usual let p and q be the projections of $S \times T$ to S and T respectively. We write $V := \mathcal{O}_S(-\xi) \oplus \mathcal{O}_S(-\xi + K_S)$, $\mathcal{Z}_1 := \mathcal{Z}_1^\xi$, $\mathcal{Z}_2 := \mathcal{Z}_2^\xi$ and $\delta_{l,r} := \delta_{\xi,l,r}$. We put $\Gamma := q_*(\omega_{\mathcal{Z}_1} \otimes \omega_{S\times T}^{-1} \otimes \mathcal{O}_{\mathcal{Z}_2} \otimes p^*V)$.

We see by theorem 6.13 that, in order to compute the change $\delta_{l,r}$, it is enough to compute $\int_T s(\mathrm{Ext}^1_q(\mathcal{I}_{\mathcal{Z}_1}, \mathcal{I}_{\mathcal{Z}_2} \otimes p^*V)) \cdot \gamma$ for all classes $\gamma \in H^*(T, \mathbb{Q})$ which are pull-backs from $S^{(d)}$ via the natural map $\mathrm{Hilb}^d(S \sqcup S) \longrightarrow (S \sqcup S)^{(d)} \longrightarrow S^{(d)}$. By proposition 6.16 we have

$$\begin{aligned}\int_T s(\mathrm{Ext}^1_q(\mathcal{I}_{\mathcal{Z}_1}, \mathcal{I}_{\mathcal{Z}_2} \otimes p^*V)) \cdot \gamma \quad &= \quad \int_T s([V^\vee(K_S)]_1^\vee \oplus [V]_2) \cdot \gamma \\ &\quad + \int_T (c(\Gamma) - 1) s([V^\vee(K_S)]_1^\vee \oplus [V]_2) \cdot \gamma.\end{aligned}$$

In the first part of this section we compute the first of these two integrals. As said in the beginning of this section, we only want to compute the terms of lowest order of the change of the Donaldson invariants. This corresponds to restricting our attention to a big open subset of the Hilbert scheme of points.

Notation 7.2. A point $\sigma \in S^{(n)}$ is a formal linear combination $\sum_i m_i x_i$ of points on S with positive integer coefficients and $\sum_i m_i = n$. The support $supp(\sigma)$ is the set of points x_i. For all $i \leq n$ let

$$S_i^{(n)} := \{\sigma \in S^{(n)} \mid \#supp(\sigma) \geq n - i + 1\}.$$

Furthermore, for any variety X with a canonical morphism $f : X \longrightarrow S^{(n)}$, we denote $f^{-1}S_i^{(n)}$ by X_i. For the universal family $Z_n(S) \subset S \times \mathrm{Hilb}^n(S)$ we denote by $Z_n(S)_i$ the preimage of $\mathrm{Hilb}^n(S)_i$.

In order to compute the first integral we will use an inductive approach, which is based on results of [E1],[F-G] and which is similar to computations in [Gö2] on the Hilbert scheme of 3 points.

Definition 7.3. ([E1],[F-G]) Let $S^{[n-1,n]} \longrightarrow S \times \mathrm{Hilb}^{n-1}(S)$ be the blow-up along the universal family $Z_{n-1}(S)$, and let F_n the exceptional divisor. Contrary to our conventions in the previous section for any vector bundle E on S we will denote by $E[n]$ the vector bundle $q_*(\mathcal{O}_{Z_n(S)} \otimes p^*E)$ on $\mathrm{Hilb}^n(S)$.

Theorem 7.4. *([E1]) $S^{[n-1,n]}$ is smooth. There is a natural morphism $S^{[n-1,n]} \longrightarrow \mathrm{Hilb}^n(S)$, and on $S^{[n-1,n]}$ we have an exact sequence*

$$0 \longrightarrow V(-F_n) \longrightarrow V[n] \longrightarrow V[n-1] \longrightarrow 0, \tag{7.4.1}$$

where we have used convention 2.2.

It is easy to see that the induced map $S^{[n-1,n]} \longrightarrow S \times \mathrm{Hilb}^n(S)$ factors through $Z_n(S) \subset S \times \mathrm{Hilb}^n(S)$, and that the map $S^{[n-1,n]} \longrightarrow Z_n(S)$ is an isomorphism over the open set $Z_n(S)_1$. We denote by $S_i^{[n-1,n]}$ the preimage of $Z_n(S)_i$.

Lemma 7.5. *Let $N_n^\vee$ be the conormal sheaf of $Z_n(S)$ in $S \times \mathrm{Hilb}^n(S)$. Then we have an exact sequence on $S_2^{[n-1,n]}$*

$$0 \longrightarrow N_n^\vee \longrightarrow T_S^\vee \longrightarrow \mathcal{O}_{F_n}(-F_n) \longrightarrow 0.$$

Here we have used the convention 2.2. In particular on $S_2^{[n-1,n]}$ we get

$$s(N_n^\vee) = s(T_S^\vee)\frac{1-F_n}{1-2F_n}.$$

Proof. It is easy to see that $S_2^{[n-1,n]} \longrightarrow \mathrm{Hilb}^n(S)_2$ is a branched n-fold cover, étale out of F_n and with ramification of order 1 along F_n. So the result follows in the same way as in the proof of ([F-G], lemma 2.10). □

Lemma 7.6. *Let i be a positive integer and assume that $\alpha_1, \alpha_2 \in A^*(\mathrm{Hilb}^n(S))$ have the same pull-back to $\mathrm{Hilb}^n(S)_i$. Then*

$$\int\limits_{\mathrm{Hilb}^n(S)} \alpha_1 \cdot \beta = \int\limits_{\mathrm{Hilb}^n(S)} \alpha_2 \cdot \beta$$

for all $\beta \in H^{4n-4i-2}(S^{(n)}, \mathbb{Q})$. The same result holds if we replace $\mathrm{Hilb}^n(S)_i$ by $S_i^{[n-1,n]}$.

Proof. Let $j : \mathrm{Hilb}^n(S) \setminus \mathrm{Hilb}^n(S)_i \longrightarrow Hilb^n(S)$ be the inclusion. We get $\alpha_1 = \alpha_2 + j_*(\mu)$ for a class $\mu \in A^*(\mathrm{Hilb}^n(S) \setminus \mathrm{Hilb}^n(S)_i)$. As the codimension of the complement of $S_i^{(n)}$ in $S^{(n)}$ is $2i$, the result follows by the projection formula. □

Notation 7.7. For all $l \geq 1$ we denote by Δ_l the "small" diagonal $\{(x, \dots, x) \mid x \in S\}$ and by $[\Delta_l]$ its cohomology class.

We define classes $t_{1-}, t_{2-}, t_{3-} \in H^*(S, \mathbb{Q})$ by

$$\begin{aligned}
t_{1-} &:= 1 + (2\xi - K_S) + (3\xi^2 - 3\xi K_S + K_S^2), \\
t_{2-} &:= 3 + (18\xi - 13K_S) + (63\xi^2 - 91\xi K_S + 33K_S^2 + 5s_2(S)), \\
t_{3-} &:= 27 + (270\xi - 237K_S).
\end{aligned}$$

Here $s_i(S) := s_i(T_S)$ is the i^{th} Segre class of S. We define t_{1+}, t_{2+}, t_{3+} by replacing K_S by $(-K_S)$ in the definition of t_{1-}, t_{2-}, t_{3-} respectively and put $t_i := t_{i-} + t_{i+}$, i.e.

$$\begin{aligned} t_1 &= 2 + 4\xi + 6\xi^2 + 2K_S^2, \\ t_2 &= 6 + 36\xi + 126\xi^2 + 66K_S^2 + 10s_2(S), \\ t_3 &= 54 + 540\xi. \end{aligned}$$

Lemma 7.8. *Let $\gamma \in H^{4n-2k}(S^{(n)}, \mathbb{Q})$ with $k \leq 5$. Then*

$$n \int_{\mathrm{Hilb}^n(S)} s(V[n]) \cdot \gamma = \sum_{l=1}^{3} \int_{S^l \times \mathrm{Hilb}^{n-l}(S)} (-1)^{l-1} [\Delta_l] p_1^* t_{l-} \cdot s(V[n-l]) \cdot \gamma,$$

where $p_1 : S^l \longrightarrow S$ is the projection to the first factor.

Proof. By theorem 7.4 we have the identity $s(V[n]) = s(V(-F_n)) s(V[n-1])$ on $S^{[n-1,n]}$ and furthermore

$$s(V(-F_n)) = \sum_{i,j \geq 0} \binom{i+j+1}{i+1} s_i(V) F_n^j.$$

So we get

$$\begin{aligned} &(7.8.1) \qquad n \int_{\mathrm{Hilb}^n(S)} s(V[n]) \cdot \gamma = \int_{S \times \mathrm{Hilb}^{n-1}(S)} s(V) s(V[n-1]) \cdot \gamma \\ &(7.8.2) \qquad + \sum_{i,j \geq 0} \int_{S^{[n-1,n]}} F_n \binom{i+j+2}{i+1} s_i(V) F_n^j s(V[n-1]) \cdot \gamma. \end{aligned}$$

By using $V = \mathcal{O}_S(-\xi) \oplus \mathcal{O}_S(-\xi + K_S)$, we see immediately that $s(V) = t_{1-}$. We denote for all i by f_i the composition

$$S^{[n-i,n-i+1]} \to S \times \mathrm{Hilb}^{n-i}(S) \to S \times S^{(n-i)} \to S^i \times S^{(n-i)} \to S^{(n)},$$

where the second map is induced by the diagonal map $S \longrightarrow S^i$ and put we $\gamma_i := f_i^*(\gamma)$. The integral (7.8.2) can be expressed as an integral over F_n. We push it forward to $Z_{n-1}(S) \subset S \times \mathrm{Hilb}^{n-1}(S)$ and pull back to $S^{[n-2,n-1]}$. Note that f_2 maps $S_i^{[n-2,n-1]}$ to $S_{i+1}^{(n)}$. So we get, in view of lemma 7.5 and lemma 7.6,

$$(7.8.3) \qquad (7.8.2) = -\int_{S^{[n-2,n-1]}} \sum_{i,j>0} \binom{i+j+3}{i+1} s_i(V) s_j(N_{n-1}^\vee) s(V[n-1]) \cdot \gamma_2.$$

We now again use the identity $s(V[n-1]) = s(V(-F_{n-1})) s(V[n-2])$ on $S^{[n-2,n-1]}$ and obtain

$$(7.8.4) \qquad (7.8.3) = \int_{S \times \mathrm{Hilb}^{n-2}(S)} \sum_{i+j+l \leq 2} \binom{i+l+3}{i+1} s_i(V) s_j(V) s_l(T_S^\vee) s(V[n-2]) \cdot \gamma_2$$

$$(7.8.5) \qquad + \int_{S^{[n-2,n-1]}} \sum_{i,j,l} \binom{i+l+3}{i+1} s_i(V) s(V[n-2]) \big(s_j(V(F_{n-1})) s_l(N_{n-1}^\vee) - s_j(V) s_l(T_S^\vee) \big) \cdot \gamma_2$$

By explicit calculation and the definition of V, we get for the first integral

$$\begin{aligned}
&\sum_{i+j+l\leq 2}\binom{i+l+3}{i+1}s_i(V)s_j(V)s_l(T_S^\vee)\\
&= 3+9s_1(V)-4K_S+13s_2(V)+6s_1(V)^2-14s_1(V)K_S+5s_2(S)\\
&= t_{2-}.
\end{aligned}$$

Now we compute the integral (7.8.5). We use the formula

$$s(N_{n-1}^\vee)=s(T_S^\vee)\frac{1-F_{n-1}}{1-2F_{n-1}}$$

and the notation

$$2^{[l]}=\begin{cases}1 & l<0;\\ 2^l & l\geq 0.\end{cases}$$

to obtain

$$\begin{aligned}
(7.8.5) \quad = \quad -\int_{S^{[n-2,n-1]}}\sum_{i,j_1,j_2,k_1,k_2} & F_{n-1}\binom{i+k_1+k_2+3}{i+1}\binom{j_1+j_2+1}{j_1+1}2^{[k_2-1]}\cdot\\
&\cdot s_i(V)s_{j_1}(V)s_{k_1}(T_S^\vee)F_{n-1}^{j_2+k_2-1}s(V[n-2])\cdot\gamma_2.
\end{aligned}$$

This can again be expressed as an integral over F_{n-1}. We push forward to $Z_{n-2}(S)\subset S\times\mathrm{Hilb}^{n-2}(S)$ and then pull back to $S^{[n-3,n-2]}$. Note that f_3 maps $S_i^{[n-3,n-2]}$ to $S_{i+2}^{(n)}$. Therefore using lemma 7.5 and lemma 7.6 to see that we can replace the push-forward of F_{n-1}^l by the pull-back of $s_{l-2}(T_S^\vee)$ via the projection $S^{[n-3,n-2]}\longrightarrow S\times\mathrm{Hilb}^{n-3}(S)$. We then push forward to $S\times\mathrm{Hilb}^{n-3}(S)$ and notice that by theorem 7.4 and lemma 7.6 we can replace the push-forward of $s(V[n-2])$ by $s(V)s(V[n-3])$. Putting all this together we obtain

$$\begin{aligned}
(7.8.5) \quad = \quad \int_{S\times Hilb^{n-3}(S)}\sum_{i+j_1+j_2+k_1+k_2+l\leq 3} & \binom{i+k_1+k_2+3}{i+1}\binom{j_1+j_2+1}{j_1+1}2^{[k_2-1]}\cdot\\
&\cdot s_i(V)s_{j_1}(V)s_l(V)s_{k_1}(T_S^\vee)s_{j_2+k_2-2}(T_S^\vee)s(V[n-3])\cdot\gamma_3.
\end{aligned}$$

We obtain, again by direct calculation,

$$\begin{aligned}
&\sum_{i+j_1+j_2+k_1+k_2+l\leq 3}\binom{i+k_1+k_2+3}{i+1}\binom{j_1+j_2+1}{j_1+1}2^{[k_2-1]}s_i(V)s_{j_1}(V)s_l(V)s_{k_1}(T_S^\vee)s_{j_2+k_2-2}(T_S^\vee)\\
&= 27+135s_1(V)-102K_S=27+270\xi-237K_S.
\end{aligned}$$

This completes the proof. $\square$

Remark 7.9. Let $\gamma\in H^{4n-2k}(S^{(n)},\mathbb{Q})$ with $k\leq 5$. Then the same proof shows

$$n\int_{\mathrm{Hilb}^n(S)}s((V^\vee(K_S)[n])^\vee)\cdot\gamma=\sum_{l=1}^{3}\int_{S^l\times\mathrm{Hilb}^{n-l}(S)}(-1)^{l-1}[\Delta_l]p_1^*t_{l+}\cdot s((V^\vee(K_S)[n-l])^\vee)\cdot\gamma.$$

We will now introduce a compact notation for some symmetric cohomology classes on S^n that will also help us in organizing our combinatorical calculations.

Definition 7.10. We denote by $\mathfrak{S}_n$ the symmetric group on n letters, which acts on S^n by permuting the factors. For $\alpha \in H^{2*}(S^k,\mathbb{Q})$ and $\beta \in H^{2*}(S^l,\mathbb{Q})$ we define $\alpha*\beta \in H^{2*}(S^{k+l},\mathbb{Q})^{\mathfrak{S}_{k+l}}$ by putting

$$\alpha*\beta := \frac{1}{(k+l)!}\sum_{\sigma\in\mathfrak{S}_{k+l}} (p_{\sigma(1)}\times\ldots\times p_{\sigma(k)})^*\alpha\cdot(p_{\sigma(k+1)}\times\ldots\times p_{\sigma(k+l)})^*\beta.$$

It is easy to see that $*$ is a commutative and associative operation. We will denote

$$\alpha^{*k} := \underbrace{\alpha*\alpha*\ldots*\alpha}_{k}.$$

Remark 7.11. The following elementary properties of $*$ will be very important for our further computations:

(1) For $\alpha \in H^{2*}(S^k,\mathbb{Q})$, $\beta\in H^{2*}(S^l,\mathbb{Q})$ and $w \in H^*(S^{k+l},\mathbb{Q})^{\mathfrak{S}_{k+l}}$ it follows immediately from the symmetry of w that

$$\begin{aligned}\int_{S^{k+l}}(\alpha*\beta)\cdot w &= \int_{S^{k+l}}(p_1\times\ldots\times p_k)^*\alpha\cdot(p_{k+1}\times\ldots\times p_{k+l})^*\beta\cdot w\\ &= \sum_{(w_1,w_2)}\int_{S^k}\alpha w_1\cdot\int_{S^l}\beta w_2.\end{aligned}$$

Here $w=\sum_{(w_1,w_2)} w_1\cdot w_2$ is the Künneth decomposition. Analogous results hold if more then two factors are multiplied via $*$.

(2) Let 1 denote the neutral element of the ring $H^*(S,\mathbb{Q})$. Then 1^{*k} is the neutral element of $H^*(S^k,\mathbb{Q})$.

(3) It is also easy to see from the definitions that $*$ fulfills the distributive law $\alpha*(\beta_1+\beta_2) = \alpha*\beta_1+\alpha*\beta_1$. In fact $+$ and $*$ make $\bigoplus_{n\geq 0} H^{2*}(S^n,\mathbb{Q})^{\mathfrak{S}_n}$ a commutative ring.

(4) In particular the binomial formula holds:

$$\sum_{k+l=n}\binom{n}{k}\alpha^{*k}*\beta^{*l} = (\alpha+\beta)^{*n}.$$

Notation 7.12. For a class $\alpha\in H^*(S,\mathbb{Q})$ and a postive integer i we denote by $(\alpha)_i := [\Delta_i]p_1^*\alpha \in H^*(S^i,\mathbb{Q})^{\mathfrak{S}_i}$, where $[\Delta_i]$ is the (small) diagonal $\{(x,\ldots,x)\mid x\in S\}$ in S^i. In particular $(\alpha)_1=\alpha$. We will in the future write $(\alpha)_i(\beta)_j$ instead of $(\alpha)_i*(\beta)_j$ and $(\alpha)_i^m$ instead of $(\alpha)_i^{*m}$. Furthermore we write $\alpha^{*m}\beta^{*l}$ and $\alpha^{*m}(\beta)_i$ instead of $\alpha^{*m}*\beta^{*l}$ and $\alpha^{*m}*(\beta)_i$.

Proposition 7.13. *Let $\gamma\in H^{4d-2k}S^{(d)},\mathbb{Q})$ with $k\leq 5$ and $w\in H^{4d-2k}(S^d,\mathbb{Q})$ its pull-back to S^d. Then*

$$\begin{aligned}d!\int_{\mathrm{Hilb}^d(S\sqcup S)} & s([V^\vee(K_S)]_1^\vee\oplus[V]_2)\cdot\gamma\\ &= \int_{S^d}\left(t_1^{*d}-\binom{d}{2}(t_2)_2t_1^{*(d-2)}+2\binom{d}{3}(t_3)_3t_1^{*(d-3)}+3\binom{d}{4}(t_2)_2^2t_1^{*(d-4)}\right)\cdot w.\end{aligned}$$

Proof. Let n, m be nonnegative integers with $n+m=d$. Let $\gamma_1 \cdot \gamma_2 \in H^*(S^{(n)}) \times S^{(m)}, \mathbb{Q}) \setminus 0$ be a Künneth component of the pull-back of γ via $S^{(n)} \times S^{(m)} \longrightarrow S^{(d)}$. Let $w_1 \cdot w_2 \in H^{4n-2l}(S^n, \mathbb{Q})^{\mathfrak{S}_n} \times H^{4n-2r}(S^m, \mathbb{Q})^{\mathfrak{S}_m}$ be the pull-back of $\gamma_1 \cdot \gamma_2$. Then $0 \leq l, r \leq 5$. By an easy induction using lemma 7.8, remark 7.9 and remark 7.11 and ignoring all terms of codimension ≥ 6 we get

$$n! \int\limits_{\mathrm{Hilb}^n(S)} s((V^\vee(K_S)[n])^\vee) \cdot \gamma_1 = \int_{S^n} P_n \cdot w_1$$

$$m! \int\limits_{\mathrm{Hilb}^m(S)} s(V[m]) \cdot \gamma_2 = \int_{S^m} Q_m \cdot w_2,$$

where

$$P_n = t_{1+}^{*n} - \sum_{i=2}^{n}(i-1)(t_{2+})_2 t_{1+}^{*(n-2)} + \sum_{i=2}^{n-2}\sum_{j=i+2}^{n}(i-1)(j-1)(t_{2+})_2^2 t_{1+}^{*(n-4)} + \sum_{i=3}^{n}(i-1)(i-2)(t_{3+})_3 t_{1+}^{*(n-3)}.$$

and Q_m is defined analogously to P_n replacing n, t_{1+}, t_{2+} and t_{3+} by m, t_{1-}, t_{2-} and t_{3-} respectively.

Applying again remark 7.11 we obtain

$$n!m! \int\limits_{\mathrm{Hilb}^n(S)\times\mathrm{Hilb}^m(S)} s((V^\vee(K_S)[n])^\vee) s(V[m]) \cdot \gamma = \int\limits_{S^d} (P_n * Q_m) \cdot w,$$

and thus

$$d! \int\limits_{\mathrm{Hilb}^d(S\sqcup S)} s([V^\vee(K_S)]_1^\vee) s([V]_2) \cdot \gamma = \sum_{n+m=d} \binom{d}{n} \int\limits_{S^d} (P_n * Q_m) \cdot w$$

Finally we have

$$\begin{aligned}
&\sum_{n+m=d} \binom{d}{n} P_n * Q_m \\
&= \sum_{n+m=d} \binom{d}{n} \Bigg(t_{1+}^{*n} t_{1-}^{*m} - \binom{n}{2}(t_{2+})_2 t_{1+}^{*(n-2)} * t_{1-}^{*m} - \binom{m}{2}(t_{2-})_2 t_{1+}^{*n} t_{1-}^{*(m-2)} \\
&\quad + 3\binom{n}{4}(t_{2+})_2^2 t_{1+}^{*(n-4)} t_{1-}^{*m} + \binom{n}{2}\binom{m}{2}(t_{2+})_2 (t_{2-})_2 t_{1+}^{*(n-2)} t_{1-}^{*(m-2)} + 3\binom{m}{4}(t_{2-})_2^2 t_{1+}^{*n} t_{1-}^{*(m-4)} \\
&\quad + 2\binom{n}{3}(t_{3+})_3 t_{1+}^{*(n-3)} t_{1-}^{*m} + 2\binom{m}{3}(t_{3-})_3 t_{1+}^{*n} t_{1-}^{*(m-3)} \Bigg) \\
&= t_1^{*d} - \binom{d}{2}(t_2)_2 t_1^{*(d-2)} + 3\binom{d}{4}(t_2)_2^2 t_1^{*(d-4)} + 2\binom{d}{3}(t_3)_3 t_1^{*(d-3)}.
\end{aligned}$$

□

Now we want to compute the second integral

$$\int_T (c(\Gamma) - 1) s([V^\vee(K_S)]_1^\vee) s([V]_2) \cdot \gamma$$

for $\gamma \in H^{4d-2k}(S^{(d)}, \mathbb{Q})$ with $k \leq 5$, The conventions of 2.2 stay in effect.

Definition 7.14. Let n, m be nonnegative integers with $n + m = d$. We consider the following diagram

$$\begin{array}{ccccc} S \times \mathrm{Hilb}^n(S) \times \mathrm{Hilb}^m(S) & \xrightarrow{q} & \mathrm{Hilb}^n(S) \times \mathrm{Hilb}^m(S) & \xrightarrow{g} & S^{(n)} \times S^{(m)} \\ \uparrow \tilde{\varphi} & & \varphi \uparrow & & \varphi' \uparrow \\ S \times \tilde{S}^n \times \tilde{S}^m & \xrightarrow{\tilde{q}} & \tilde{S}^n \times \tilde{S}^m & \xrightarrow{\tilde{g}} & S^n \times S^m \end{array}$$

Here, as above, q and $\tilde{q}$ are the projections, $g : \mathrm{Hilb}^n(S) \times \mathrm{Hilb}^m(S) \longrightarrow S^{(n)} \times S^{(m)}$ is the product of the Hilbert-Chow morphisms and $\varphi' : S^n \times S^m \longrightarrow S^{(n)} \times S^{(m)}$ is the product of the quotient maps, and all the other varieties and maps are defined via pull-back. For $i = 1, 2$ we put $\widetilde{Z}_i := \tilde{\varphi}^{-1}(\mathcal{Z}_i)$ and $\widetilde{Z}_{1,2} := \widetilde{Z}_1 \cap \widetilde{Z}_2$, (i.e. the scheme-theoretic intersection).

Notation 7.15. We denote by

$$(S^{(n)} \times S^{(m)})_* := \Big\{(\sigma_+, \sigma_-) \in S^{(n)} \times S^{(m)} \;\Big|\; \#supp(\sigma_+) \geq n-1,\ \#supp(\sigma_-) \geq m-1,\ \#supp(\sigma_+ + \sigma_-) \geq d-2\Big\}.$$

Furthermore for all X with a natural morphism $f : X \longrightarrow S^{(n)} \times S^{(m)}$ we denote $X_* := f^{-1}(S^{(n)} \times S^{(m)})_*$. We put

$$\Gamma_{n,m} := \varphi^*(\Gamma|_{(\mathrm{Hilb}^n(S) \times \mathrm{Hilb}^m(S))_*})$$

(see 7.1 for the definition of Γ). For $1 \leq i \leq n$ (resp. $1 \leq j \leq m$) we denote by p_{i+} (resp. p_{j-}) the projection from $S^n \times S^m$ onto the i^{th} factor of S^n (resp. the j^{th} factor of S^m).

For $\epsilon = +, -$, $\eta = +, -$ we put

$$\begin{aligned} \Delta^{\epsilon}_{0,i} &:= \Big\{ (x, x_1^+, \ldots x_n^+, x_1^-, \ldots x_m^-) \in S \times S^n \times S^m \;\Big|\; x = x_i^{\epsilon} \Big\} \\ \Delta^{\epsilon\eta}_{i,j} &:= \Big\{ (x_1^+, \ldots x_n^+, x_1^-, \ldots x_m^-) \in S^n \times S^m \;\Big|\; x_i^{\epsilon} = x_j^{\eta} \Big\} \\ \Delta^{\epsilon\eta}_{0,i,j} &:= \Big\{ (x, x_1^+, \ldots x_n^+, x_1^-, \ldots x_m^-) \in S \times S^n \times S^m \;\Big|\; x = x_i^{\epsilon} = x_j^{\eta} \Big\} \end{aligned}$$

We will also denote by $\Delta^{+-}_{i,j}$, $\Delta^{+-}_{0,i,j}$, $\Delta^{\epsilon}_{0,i}$ the pull-backs $\tilde{g}^{-1}(\Delta^{+-}_{i,j})$, $(\mathrm{id}_S \times \tilde{g})^{-1}(\Delta^{+-}_{0,i,j})$, $(\mathrm{id}_S \times \tilde{g})^{-1}(\Delta^{\epsilon}_{0,i})$. We denote $D_{i,j} := \tilde{g}^{-1}(\Delta^{++}_{i,j})$ and $E_{i,j} := \tilde{g}^{-1}(\Delta^{--}_{i,j})$. $D_{i,j}$ and $E_{i,j}$ are divisors (see below), we denote $F_i := \sum_{j<i} D_{i,j}$ and $G_i := \sum_{j<i} E_{i,j}$.

Remark 7.16. The following easy facts will be used throughout the computation.

(1) It is well known that $(\mathrm{Hilb}^n(S) \times \mathrm{Hilb}^m(S))_*$ is obtained from $(S^n \times S^m)_*$ by blowing up all the $\Delta^{++}_{i,j}$ and $\Delta^{--}_{i,j}$ and taking the quotient by the action of the product of the symmetric groups $\mathfrak{S}_n \times \mathfrak{S}_m$. It follows that in fact $(\tilde{S}^n \times \tilde{S}^m)_*$ is just the blow up of $(S^n \times S^m)_*$ along the (disjoint) smooth subvarieties $(\Delta^{++}_{i,j})_*$ and $(\Delta^{--}_{i,j})_*$ and the $(D_{i,j})_*$ and $(E_{i,j})_*$ are the exceptional divisors.

(2) It is also easy to see that $(\widetilde{Z}_1)_* = \bigcup_{i=1}^n (\Delta_{0,i}^+)_*$, $(\widetilde{Z}_2)_* = \bigcup_{j=1}^m (\Delta_{0,j}^-)_*$ and therefore

$$(\widetilde{Z}_{1,2})_* = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} (\Delta_{0,i,j}^{+-})_* .$$

(We mean here the scheme theoretic union, i.e. the scheme defined by the intersection of the ideals).

(3) For $i \neq j$ we have (scheme-theoretically)

$$\begin{aligned} (\Delta_{0,i}^+)_* \cap \Delta_{0,j}^+ &= (\Delta_{0,i}^+)_* \cap D_{i,j}, \\ (\Delta_{0,i}^-)_* \cap \Delta_{0,j}^- &= (\Delta_{0,i}^-)_* \cap E_{i,j}. \end{aligned}$$

Lemma 7.17. (1) *Let X be a smooth variety, and let Y and Z be Cohen-Macauley subschemes of X such that the ideal $\mathcal{I}_{Z/(Y \cup Z)}$ of Z in $Y \cup Z$ is $\mathcal{O}_Y(-D)$ for a divisor D on Y. Then in the Grothendieck ring of X we have*

$$\begin{aligned} \mathcal{O}_{Y \cup Z} &= \mathcal{O}_Y(-D) + \mathcal{O}_Z \quad \textit{and} \\ \omega_X^{-1} \otimes \omega_{Y \cup Z} &= \omega_X^{-1} \otimes \omega_Y(D) + \omega_X^{-1} \otimes \omega_Z . \end{aligned}$$

(2) *Let $f : X \longrightarrow Y$ be a morphism between smooth varieties. Let $Z \subset Y$ be a Cohen-Macauley subscheme of codimension 2 and assume $W := f^{-1}(Z)$ has pure codimension 2 in X. Then*

$$f^*(\omega_Y^{-1} \otimes \omega_Z) = \omega_X^{-1} \otimes \omega_W .$$

(3) *Let X be a smooth variety and Y and Z Cohen-Macauley subschemes of codimension 2 intersecting properly. Then in the Grothendieck ring of X we have*

$$\mathcal{O}_Y \otimes \mathcal{O}_Z = \mathcal{O}_{Y \cap Z} .$$

Proof. (1) The first identity follows from the standard exact sequence

$$0 \longrightarrow \mathcal{O}_Y(-D) \longrightarrow \mathcal{O}_{Y \cup Z} \longrightarrow \mathcal{O}_Z \longrightarrow 0. \tag{$*$}$$

Now we dualize $(*)$ and use that for a two codimensional Cohen-Macauley subscheme $W \subset X$ we have

$$\mathcal{E}xt^i(\mathcal{O}_W, \mathcal{O}_X) = \begin{cases} 0 & i < 2, \\ \omega_X^{-1} \otimes \omega_X & i = 2 \end{cases}$$

to obtain the sequence

$$0 \longrightarrow \omega_X^{-1} \otimes \omega_Z \longrightarrow \omega_X^{-1} \otimes \omega_{Y \cup Z} \longrightarrow \omega_X^{-1} \otimes \omega_Y(F) \longrightarrow 0$$

and thus the second identity.

(2) We take a locally free resolution

$$0 \longrightarrow B \longrightarrow A \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_Z \longrightarrow 0.$$

Pulling it back we obtain the sequence

$$0 \longrightarrow f^*B \longrightarrow f^*A \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_W \longrightarrow 0,$$

which stays exact by the Hilbert-Birch theorem (see e.g. [P-S] lemma 3.1). Dualizing we obtain the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow f^*A \longrightarrow f^*B \longrightarrow \omega_X^{-1} \otimes \omega_Z \longrightarrow 0.$$

We can also arrive at this sequence differently, by first dualizing and then pulling back. This way we obtain the sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow f^*A \longrightarrow f^*B \longrightarrow f^*(\omega_X^{-1} \otimes \omega_Z) \longrightarrow 0,$$

and (2) follows.

(3) By the corollaire on p. 20 in [Se] we have $\mathrm{Tor}_i(\mathcal{O}_Y, \mathcal{O}_Z) = 0$ for $i > 0$, and (3) follows. □

Lemma 7.18. *In the Grothendieck ring of $(\widetilde{S}^n \times \widetilde{S}^m)_*$ we have the equality*

$$\varphi^*(\Gamma_{n,m}) = \sum_{i=1}^{n}\sum_{j=1}^{m}\Big(\mathcal{O}_{\Delta^{+-}_{i,j}}(F_i - G_j - p_{i+}^*\xi) + \mathcal{O}_{\Delta^{+-}_{i,j}}(F_i - G_j - p_{i+}^*(\xi + K_S))\Big).$$

Proof. Using remark 7.16(2) and remark 7.16(3) and applying lemma 7.17(1) inductively we obtain in the Grothendieck ring of $S \times (\widetilde{S}^n \times \widetilde{S}^m)_*$ the equalities

$$\mathcal{O}_{\widetilde{Z}_1} = \sum_{i=1}^{n} \mathcal{O}_{\Delta^+_{0,i}}(-F_i) \tag{7.18.1}$$

$$\mathcal{O}_{\widetilde{Z}_2} = \sum_{j=1}^{m} \mathcal{O}_{\Delta^-_{0,j}}(-G_j) \quad \text{and} \tag{7.18.2}$$

$$\begin{aligned}
\widetilde{\varphi}^*(\omega^{-1}_{S\times \mathrm{Hilb}^n(S)\times \mathrm{Hilb}^m(S)} \otimes \omega_{Z_1}) &= \omega^{-1}_{S\times\widetilde{S}^n\times\widetilde{S}^m} \otimes \omega_{\widetilde{Z}_1} \\
&= \sum_{i=1}^{n} \omega^{-1}_{S\times\widetilde{S}^n\times\widetilde{S}^m}\omega_{\Delta^+_{0,i}}(F_i) \\
&= \sum_{i=1}^{n} \mathcal{O}_{\Delta^+_{0,i}}(-p_{i+}^*K_S + F_i),
\end{aligned}$$

where in the third and the last line we have used lemma 7.17(2). Now using lemma 7.17(3) and tenzorizing by p^*V we obtain in the Grothendieck ring of $S \times (\widetilde{S}^n \times \widetilde{S}^m)_*$ the equality

$$(\mathrm{id}_S \times \varphi)^*(\omega_T^{-1} \otimes \omega_{Z_1} \otimes \mathcal{O}_{Z_2} \otimes p^*V) = \sum_{i=1}^{n}\sum_{j=1}^{m}\Big(\mathcal{O}_{\Delta^{+-}_{0,i,j}}(F_i - G_j - p_{i+}^*\xi) + \mathcal{O}_{\Delta^{+-}_{0,i,j}}(F_i - G_j - p_{i+}^*(\xi + K_S))\Big).$$

The morphism $\varphi : (\widetilde{S}^n \times \widetilde{S}^m)_* \longrightarrow \mathrm{Hilb}^n(S) \times \mathrm{Hilb}^m(S)$ is flat. Therefore we get by ([Ha] prop.III.9.3)

$$\begin{aligned}
&\varphi^* q_*(\omega_T^{-1} \otimes \omega_{Z_1} \otimes \mathcal{O}_{Z_2} \otimes p^*V) \\
&= \tilde{q}_*(\mathrm{id}_S \times \varphi)^*(\omega_T^{-1} \otimes \omega_{Z_1} \otimes \mathcal{O}_{Z_2} \otimes p^*V) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} \tilde{q}_*((\mathcal{O}_{\Delta^{+-}_{0,i,j}}(F_i - G_j - p_{i+}^*\xi) + \mathcal{O}_{\Delta^{+-}_{0,i,j}}(F_i - G_j - p_{i+}^*(\xi + K_S)) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m}(\mathcal{O}_{\Delta^{+-}_{i,j}}(F_i - G_j - p_{i+}^*\xi) + \mathcal{O}_{\Delta^{+-}_{i,j}}(F_i - G_j - p_{i+}^*(\xi + K_S)),
\end{aligned}$$

in the Grothendieck ring of $(\widetilde{S}^n \times \widetilde{S}^m)_*$. The last identity follows from the fact that the projection $\tilde{q}|_{\Delta^{+-}_{0,i,j}} : \Delta^{+-}_{0,i,j} \longrightarrow \Delta^{+-}_{i,j}$ is an isomorphism. □

Lemma 7.19. *Let X be a smooth variety and let $i : Y \longrightarrow X$ be the closed embedding of a smooth subvariety of codimension 2 with conormal bundle $N^\vee$. Let D be a divisor on Y. Then*

$$c(i_*(\mathcal{O}_Y(-D))) = 1 - i_*\Big(\sum_{k,l\geq 0} \binom{k+l+1}{l} D^l s_k(N^\vee)\Big).$$

Proof. This is a straightforward application of Riemann-Roch without denominators [Jo]. □

Lemma 7.20. *Let $1 \leq k \leq 5$. Then $c_k(\Gamma_{n,m}|_{(\widetilde{S}^n\times\widetilde{S}^m)_*})$ is the part of degree k of*

$$\begin{aligned}
&-\sum_{(i,j)}[\Delta^{+-}_{i,j}]\Big(2 + 4p^*_{i+}\xi + 4(G_j - F_i) + p^*_{i+}(6\xi^2 + 3s_2(S) - K_S^2) + 12p^*_{i+}\xi(G_j - F_i)\\
&+6(F_i^2 + G_j^2) + p^*_{i+}(24\xi^2 + 12s_2(S) - 4K_S^2)(G_j - F_i) + 24p^*_{i+}\xi(G_j^2 + F_i^2) + 8(G_j^3 - F_i^3)\Big)\\
&\qquad + \sum_{(i,j)\neq(i_1,j_1)} [\Delta^{+-}_{i,j}][\Delta^{+-}_{i_1,j_1}](4 + 8p^*_{i+}\xi + 8p^*_{i_1+}\xi)
\end{aligned} \tag{7.20.1}$$

Here (i,j) and (i_1,j_1) run through $\{1,\dots,n\}\times\{1,\dots,m\}$.

Proof. We compute on $(\widetilde{S}^n \times \widetilde{S}^m)_*$. We notice that $[\Delta^{+-}_{i,j}]$ is just the pull-back of the corresponding class in $S^n \times S^m$ via $\tilde{g}$ and the conormal bundle of $\Delta^{+-}_{i,j}$ is just the pull-back of the conormal bundle, i.e. $p^*_{i+}(T^\vee_S)$. Furthermore we note that on $(\widetilde{S}^n \times \widetilde{S}^m)_*$ we have $[\Delta^{+-}_{i,j}]\cdot F_i \cdot G_j = 0$. Therefore we obtain by lemma 7.19 after some calculation that for $1 \leq k \leq 5$ the Chern class $c_k(\mathcal{O}_{\Delta^{+-}_{i,j}}(-p^*_{i+}\xi + F_i - G_j))$ is the part of degree k of

$$\begin{aligned}
-[\Delta^{+-}_{i,j}]\Big(&1 + p^*_{i+}(2\xi - K_S) + 2(G_j - F_i) + p^*_{i+}(3\xi^2 - 3\xi K_S + s_2(S))\\
&+p^*_{i+}(6\xi - 3K_S)(G_j - F_i) + 3(G_j^2 + F_i^2)\\
&+p^*_{i+}(12\xi^2 - 12K_S\xi + 4s_2(S))(G_j - F_i) + p^*_{i+}(12\xi - 6K_S)(G_j^2 + F_i^2) + 4(G_j^3 - F_i^3)\Big).
\end{aligned}$$

Analogously we obtain that $c_k(\mathcal{O}_{\Delta^{+-}_{i,j}}(-p^*_{i+}(\xi + K_S) + F_i - G_j))$ is the part of degree k of

$$\begin{aligned}
1 - [\Delta^{+-}_{i,j}]\Big(&1 + p^*_{i+}(2\xi + K_S) + 2(G_j - F_i) + p^*_{i+}(3\xi^2 + 3\xi K_S + s_2(S))\\
&+p^*_{i+}(6\xi + 3K_S)(G_j - F_i) + 3(G_j^2 + F_i^2)\\
&+p^*_{i+}(12\xi^2 + 12K_S\xi + 4s_2(S))(G_j - F_i) + p^*_{i+}(12\xi + 6K_S)(G_j^2 + F_i^2) + 4(G_j^3 - F_i^3)\Big).
\end{aligned}$$

We notice that $[\widetilde{\Delta}^{+-}_{i,j}]^2 = [\Delta^{+-}_{i,j}]p^*_{i+}(c_2(S))$. Thus, by multiplying out, we get that $c_k(\mathcal{O}_{\Delta^{+-}_{i,j}}(-p^*_{i+}\xi -$

$F_i + G_j) \oplus \mathcal{O}_{\Delta^{+-}_{i,j}}(-p^*_{i+}(\xi + K_S) - F_i + G_j))$ is the part of degree k of

$$\begin{aligned}1 - [\Delta^{+-}_{i,j}]\Big(&2 + 4p^*_{i+}\xi + 4(G_j - F_i)\\ &+p^*_{i+}(6\xi^2 + 3s_2(S) - K_S^2) + 12p^*_{i+}\xi(G_j - F_i) + 6(G_j^2 + F_i^2)\\ &+p^*_{i+}(24\xi^2 + 12s_2(S) - 4K_S^2)(G_j - F_i) + 24p^*_{i+}\xi(G_j^2 + F_i^2) + 8(G_j^3 - F_i^3)\Big).\end{aligned}$$

Now we take the product over all i, j. We use that on $(\tilde{S}^n \times \tilde{S}^m)_*$ we have $[\Delta^{+-}_{i_1,j_1}] \cdot [\Delta^{+-}_{i_2,j_2}] \cdot F_i = [\Delta^{+-}_{i_1,j_1}] \cdot [\Delta^{+-}_{i_2,j_2}] \cdot G_j = 0$ unless $\{i_1, j_1\} = \{i_2, j_2\}$, and obtain the result. □

Remark 7.21. (1) In the Grothendieck ring of $(\widetilde{S}^n \times \widetilde{S}^m)_*$ we have

$$\begin{aligned}\varphi^*([V]_2|_{\mathrm{Hilb}^n(S)\times\mathrm{Hilb}^m(S)}) &= \sum_{j=1}^m p^*_{j-}V(-G_j),\\ \varphi^*([V^\vee(K_S)]^\vee|_{\mathrm{Hilb}^n(S)\times\mathrm{Hilb}^m(S)}) &= \sum_{i=1}^n p^*_{i+}(V(-K_S))(F_i).\end{aligned}$$

(2) Therefore, for $l \le 3$, $s_l(\varphi^*([V]_2|_{\mathrm{Hilb}^n(S)\times\mathrm{Hilb}^m(S)}))$ is the part of degree l of

$$\begin{aligned}\prod_{j=1}^m p^*_{j-}t_{1-} + \sum_{1\le j\le j_1\le m}\Big(&+2E_{j,j_1} + p^*_{j-}(10\xi - 5K_S)E_{j,j_1} + 3E^2_{j,j_1})\\ &+p^*_{j-}(30\xi^2 - 30\xi K_S + 9K_S^2)E_{j,j_1} + p^*_{j-}(18\xi - 9K_S)E^2_{j,j_1} + 4E^3_{j,j_1}\Big)\prod_{j_2\notin\{j,j_1\}} p^*_{j_2-}t_{1-},\end{aligned} \tag{7.21.1}$$

and $s_l(\varphi^*([V^\vee(K_S)])^\vee|_{\mathrm{Hilb}^n(S)\times\mathrm{Hilb}^m(S)})$ is the part of degree l of

$$\begin{aligned}\prod_{i=1}^n p^*_{i+}t_{1+} + \sum_{1\le i\le i_1\le n}\Big(&-2D_{i,i_1} - p^*_{i+}(10\xi + 5K_S)D_{i,i_1} + 3D^2_{i,i_1}\\ &-p^*_{i+}(30\xi^2 + 30\xi K_S + 9K_S^2)D_{i,i_1} + p^*_{i+}(18\xi + 9K_S)D^2_{i,i_1} - 4D^3_{i,i_1}\Big)\prod_{i_2\notin\{i,i_1\}} p^*_{i_2+}t_{1+}.\end{aligned} \tag{7.21.2}$$

Proof. (1) follows from the formulas 7.18.1, 7.18.2 by tensorizing with p^*V (resp. $p^*(V^\vee(K_S))$) and pushing down via $\tilde{q}_*$. (2) is just a straightforward computation using that $E_{i,j} \cdot E_{k,l} = D_{i,j} \cdot D_{k,l} = 0$ for $\{i, j\} \ne \{k, l\}$. □

Remark 7.22. Let $k \le 5$ and $\gamma \in H^{4d-2k}(\tilde{S}^n \times \tilde{S}^m, \mathbb{Q})$ and assume that $\alpha_1, \alpha_2 \in A^*(\tilde{S}^n \times \tilde{S}^m)$ have the same pull-back to $(\tilde{S}^n \times \tilde{S}^m)_*$. Then, for all $i \le n$, $j \le m$, we get analogously to lemma 7.6

$$\int_{\tilde{S}^n\times\tilde{S}^m} \Delta^{+-}_{i,j} \cdot (\alpha_1 - \alpha_2) \cdot \gamma = 0.$$

Proposition 7.23. *Let $\gamma \in H^{4d-2k}(S^{(d)},\mathbb{Q})$ with $k \le 5$, and let $w \in H^{4d-2k}(S^d,\mathbb{Q})$ be the pull-back of γ to S^d. Then*

$$\begin{aligned} d! \int_{\mathrm{Hilb}^d(S\sqcup S)} & (c(\Gamma)-1)s([V^\vee(K_S)]_1^\vee)s([V]_2)\cdot\gamma \\ &= \int_{S^d}\Big(- d(d-1)(2+12\xi+42\xi^2+3s_2(S)+K_S^2)_2 t_1^{*(d-2)} \\ &\qquad +d(d-1)(d-2)(30+260\xi)_3 t_1^{*(d-3)} \\ &\qquad +2d(d-1)(d-2)(d-3)(2+12\xi)_2^2 t_1^{*(d-4)}\Big)\cdot w, \end{aligned}$$

and, with

$$\begin{aligned} R_d \;:=\; & t_1^{*d} - d(d-1)(5+30\xi+105\xi^2+8s_2(S)+34K_S^2)_2 t_1^{*(d-2)} \\ & +d(d-1)(d-2)(48+440\xi)_3 t_1^{*(d-3)} \\ & +\frac{d(d-1)(d-2)(d-3)}{2}(5+30\xi)_2^2 t_1^{*(d-4)}, \end{aligned}$$

we get

$$d! \int_{\mathrm{Hilb}^d(S\sqcup S)} c(\Gamma)s([V^\vee(K_S)]_1^\vee)s([V]_2)\cdot\gamma = \int_{S^d} R_d\cdot w.$$

Proof. We fix n and m with $n+m=d$ and start by computing on $(\widetilde{S}^n\times\widetilde{S}^m)$. Using remark 7.22 we can restrict our attention to $(\widetilde{S}^n\times\widetilde{S}^m)_*$. We multiply out the formulas (7.21.1),(7.21.2) and (7.20.1) and push down to $S^n\times S^m$. We shall use the following facts: On $(\tilde{S}^n\times\tilde{S}^m)_*$ any of D_i and E_j, gives zero when multiplied by $[\Delta_{i,j}^{+-}][\Delta_{i_1,j_1}^{+-}]$. Furthermore $\tilde{g}_*(D_{i,j})=\tilde{g}_*E_{i,j}=0$, $\tilde{g}_*(D_{i,i_1}^2)=-[\Delta_{i,i_1}^{++}]$, $\tilde{g}_*(E_{j,j_1})^2=-[\Delta_{j,j_1}^{--}]$, $\tilde{g}_*(D_{i,i_1}^3)=[\Delta_{i,i_1}^{++}]p_{i+}^*(K_S)$, $\tilde{g}_*(E_{j,j_1}^3)=[\Delta_{j,j_1}^{--}]p_{j-}^*(K_S)$.

Below we collect the result of the push-down in ten terms according to the factors that they contain *before* the push-down. All the summands contain at least one diagonal factor $[\Delta_{i,j}^{+-}]$ and at most two diagonal factors $[\Delta_{i,j}^{+-}]$, $[\Delta_{i_1,j_1}^{+-}]$. The first seven terms come from summands containing precisely one factor $[\Delta_{i,j}^{+-}]$. So to define these summandss we can fix i and j. The first term corresponds to summands not containing any exceptional divisor D_{i,i_1} or E_{j,j_1}. The second to seventh summands correspond in that order to the push-downs of the terms containing only powers of D_{i,i_1} with $i_1<i$, E_{j,j_1} with $j_1<j$, D_{i,i_1} with $i_1>i$, E_{j,j_1} with $j_1>j$, D_{i_1,i_2} with $i\notin\{i_1,i_2\}$ and E_{j_1,j_2} with $j\notin\{j_1,j_2\}$. Notice that on $(\widetilde{S}^n\times\widetilde{S}^m)_*$ the class $[\Delta_{i,j}^{+-}]D_{i_1,i_2}E_{j_1,j_2}$ is zero for all i_1,i_2,j_1,j_2 and $[\Delta_{i,j}^{+-}]D_{i_1,i_2}D_{i_3,i_4}=[\Delta_{i,j}^{+-}]E_{j_1,j_2}E_{j_3,j_4}=0$ unless $\{i_1,i_2\}=\{i_3,i_4\}$ (resp. $\{j_1,j_2\}=\{j_3,j_4\}$). The last three summands correspond to terms containing two diagonal factors $[\Delta_{i,j}^{+-}][\Delta_{i_1,j_1}^{+-}]$. In that order they correspond to the possibilties that $j=j_1$, that $i=i_1$ and finally that $i\neq i_1$ and $j\neq j_1$. After a long but elementary computation we get that, if $k\le 5$, $\tilde{g}_*(\varphi^*(c_k(\Gamma_{n,m})-1)s([V^\vee(K_S)[n]^\vee)s(V[m]))$

is the part of degree k of

$$\sum_{(i,j)}\Bigg(-[\Delta^{+-}_{i,j}]p^*_{i+}(2+12\xi+42\xi^2+3s_2(S)+K_S^2)\prod_{i_1\neq i}p^*_{i_1+}(t_{1+})\prod_{j_1\neq j}p^*_{j_1-}(t_{1-})$$

$$+\sum_{i_1<i}[\Delta^{+-}_{i,j}][\Delta^{++}_{i,i_1}]p^*_{i+}(20+160\xi+90K_S)\prod_{i_2\notin\{i,i_1\}}p^*_{i_2+}t_{1+}\prod_{j_1\neq j}p^*_{j_1-}t_{1-}$$

$$+\sum_{j_1<j}[\Delta^{+-}_{i,j}][\Delta^{--}_{j,j_1}]p^*_{i+}(20+160\xi-90K_S)\prod_{i_1\neq i}p^*_{i_1+}t_{1+}\prod_{j_2\notin\{j,j_1\}}p^*_{j_2-}t_{1-}$$

$$+\sum_{i_1>i}[\Delta^{+-}_{i,j}][\Delta^{++}_{i,i_1}]p^*_{i+}(6+60\xi+20K_S)\prod_{i_2\notin\{i,i_1\}}p^*_{i_2+}t_{1+}\prod_{j_1\neq j}p^*_{j_1-}t_{1-}$$

$$+\sum_{j_1>j}[\Delta^{+-}_{i,j}][\Delta^{--}_{j,j_1}]p^*_{j-}(6+60\xi-20K_S)\prod_{i_1\neq i}p^*_{i_1+}t_{1+}\prod_{j_2\notin\{j,j_1\}}p^*_{j_2-}t_{1-}$$

$$+\sum_{i_1\neq i}\sum_{i_2\neq i,i_2<i_1}\Bigg([\Delta^{+-}_{i,j}][\Delta^{++}_{i_1,i_2}]p^*_{i+}(2+12\xi)p^*_{i_1+}(3+18\xi+13K_S)$$

$$\cdot\prod_{i_3\notin\{i,i_1,i_2\}}p^*_{i_3+}t_{1+}\prod_{j_1\neq j}p^*_{j_1-}t_{1-}\Bigg)+\sum_{j_1\neq j}\sum_{j_2\neq j,j_2<j_1}\Bigg([\Delta^{+-}_{i,j}][\Delta^{--}_{j_1,j_2}]p^*_{j-}(2+12\xi)$$

$$\cdot p^*_{j_1-}(3+18\xi-13K_S)\prod_{i_1\neq j}p^*_{i_1+}t_{1+}\prod_{j_3\notin\{j,j_1,j_2\}}p^*_{j_3-}t_{1-}\Bigg)$$

$$+\sum_{i_1<i}[\Delta^{+-}_{i,j}][\Delta^{+-}_{i_1,j}]p^*_{i+}(4+40\xi+4K_S)\prod_{i_2\notin\{i,i_1\}}p^*_{i_2+}t_{1+}\prod_{j_1\neq j}p^*_{j_1-}t_{1-}$$

$$+\sum_{j_1<j}[\Delta^{+-}_{i,j}][\Delta^{+-}_{i,j_1}]p^*_{j-}(4+40\xi-4K_S)\prod_{i_1\neq i}p^*_{i_1+}t_{1+}\prod_{j_2\notin\{j,j_1\}}p^*_{j_2-}t_{1-}$$

$$+\sum_{i_1<i}\sum_{j_1\neq j}[\Delta^{+-}_{i,j}][\Delta^{+-}_{i_1,j_1}]p^*_{i+}(2+12\xi)p^*_{i_1+}(2+12\xi)\prod_{i_2\notin\{i,i_1\}}p^*_{i_2+}t_{1+}\prod_{j_2\notin\{j,j_1\}}p^*_{j_2-}t_{1-}\Bigg).$$

Now we want to translate this result into the notation 7.12. Using remark 7.11 and notation 7.12 we see that for $w\in H^{4d-2k}(S^d,\mathbb{Q})^{\mathfrak{S}_d}$ and $a\in H^*(S,\mathbb{Q})$ we have

$$\int_{S^d}[\Delta^{+-}_{i,j}]p^*_{i+}a\prod_{i_1\neq i}p^*_{i_1+}t_{1+}\prod_{j_1\neq i}p^*_{j_1-}t_{1-}\cdot w \;=\; \int_{S^d}(a)_2t_{1+}^{*(n-1)}t_{1-}^{*(m-1)}\cdot w.$$

Now assume $j\neq j_1$. Then

$$\int_{S^d}[\Delta^{+-}_{i,j}][\Delta^{+-}_{i,j_1}]p^*_{i+}a\prod_{i_1\neq i}p^*_{i_1+}t_{1+}\prod_{j_2\notin\{j,j_1\}}p^*_{j_2-}t_{1-}\cdot w \;=\; \int_{S^d}(a)_3t_{1+}^{*(n-1)}t_{1-}^{*(m-2)}\cdot w.$$

We also see that $[\Delta^{+-}_{i,j}][\Delta^{+-}_{i,j_1}]=[\Delta^{+-}_{i,j}][\Delta^{--}_{j,j_1}]$ and $[\Delta^{+-}_{i,j}][\Delta^{+-}_{i_1,j}]=[\Delta^{+-}_{i,j}][\Delta^{++}_{i,i_1}]$. If $i\neq i_1$ and $j\neq j_1$ we get similarly

$$\int_{S^d}[\Delta^{+-}_{i,j}][\Delta^{+-}_{i_1,j_1}]p^*_{i+}a_1p^*_{i_1+}a_2\prod_{i_2\notin\{i,i_1\}}p^*_{i_2+}t_{1+}\prod_{j_2\notin\{j,j_1\}}p^*_{j_2-}t_{1-}\cdot w \;=\; \int_{S^d}(a_1)_2(a_2)_2t_{1+}^{*(n-2)}t_{1-}^{*(m-2)}\cdot w.$$

We can translate our result into this notation and simplify it by collecting the terms number $2, 4, 8$ and the terms $3, 5, 9$ respectively. So we get for $w \in H^{4d-2k}(S^d, \mathbb{Q})^{\mathfrak{S}_d}$ with $k \leq 5$:

$$
\begin{aligned}
&\int_{S^d} \tilde{g}_*(\varphi^*((c(\Gamma)-1)s([V^\vee(K_S)[n]^\vee)s(V[m]))) \cdot w \\
&= \int_{S^d} \Bigg(-nm(2+12\xi+42\xi^2+3s_2(S)+K_S^2)_2 t_{1+}^{*(n-1)} t_{1-}^{*(m-1)} \\
&\quad + \binom{n}{2} m(30+260\xi+114K_S)_3 t_{1+}^{*(n-2)} t_{1-}^{*(m-1)} \\
&\quad + \binom{m}{2} n(30+260\xi-114K_S)_3 t_{1+}^{*(n-1)} t_{1-}^{*(m-2)} \\
&\quad + 2\binom{n}{2}\binom{m}{2}(2+12\xi)_2^2 t_{1+}^{*(n-2)} t_{1-}^{*(m-2)} \\
&\quad + mn\binom{n-1}{2}(2+12\xi)_2(3+18\xi+13K_S)_2 t_{1+}^{*(n-3)} t_{1-}^{*(m-1)} \\
&\quad + nm\binom{m-1}{2}(2+12\xi)_2(3+18\xi-13K_S)_2 t_{1-}^{*(m-3)} t_{1+}^{*(n-1)} \Bigg) \cdot w
\end{aligned}
$$

Now we sum over all m, n and keep in mind that the map $\varphi : (\widetilde{S}^n \times \widetilde{S}^m)_* \longrightarrow \mathrm{Hilb}^n(S) \times \mathrm{Hilb}^m(S)$ has degree $m!n!$. So we obtain

$$
\begin{aligned}
&d! \int_{\mathrm{Hilb}^d(S \sqcup S)} (c(\Gamma)-1)s([V^\vee(K_S)]_1^\vee)s([V]_2) \cdot \gamma \\
&= \sum_{n+m=d} \binom{d}{n} \int_{S^d} \tilde{g}_*(\varphi^*((c(\Gamma)-1)s([V^\vee(K_S)[n]^\vee)s(V[m]))) \cdot w \\
&= \sum_{n+m=d} \int_{S^d} \Bigg(-d(d-1)\binom{d-2}{n-1}(2+12\xi+42\xi^2+3s_2(S)+K_S^2)_2 t_{1+}^{*(n-1)} t_{1-}^{*(m-1)} \\
&\quad + d(d-1)(d-2)\binom{d-3}{n-2}(30+260\xi)_3 t_{1+}^{*(n-2)} t_{1-}^{*(m-1)} \\
&\quad + 2d(d-1)(d-2)(d-3)\binom{d-4}{n-2}(2+12\xi)_2^2 t_{1+}^{*(n-2)} t_{1-}^{*(m-2)} \Bigg) \cdot w \\
&= \int_{S^d} \Big(-d(d-1)(2+12\xi+42\xi^2+3s_2(S)+K_S^2)_2 t_1^{*(d-2)} \\
&\quad + d(d-1)(d-2)(30+260\xi)_3 t_1^{*(d-3)} \\
&\quad + 2d(d-1)(d-2)(d-3)(2+12\xi)_2^2 t_1^{*(d-4)} \Big) \cdot w.
\end{aligned}
$$

This shows the first formula. The second follows by combining this formula with proposition 7.13. □

Now we have described the intersection numbers $\int_T s(\mathrm{Ext}_q^1(\mathcal{I}_{Z_1}, \mathcal{I}_{Z_2} \otimes p^*V) \cdot \gamma$, and are in a position to finish our computation of the leading terms of the change of the Donaldson invariants $\delta_{l,r}(\alpha)$. We first want to compute a formula for the change of $\delta_{N,0}(\alpha)$ and then compute how one has to modify this formula to get $\delta_{l,r}(\alpha)$. The reason that the computation of $\delta_{N,0}(\alpha)$ is easier, is the following fact:

Remark 7.24. Let l, j, k be positive integers, $\alpha \in H^2(S,\mathbb{Q})$, $\beta \in H^{2i}(S,\mathbb{Q})$ and $\gamma \in H^*(S^k,\mathbb{Q})^{\mathfrak{S}_k}$. Then we get

$$\int_{S^{k+j}} (\beta)_j * \gamma \cdot (p_1^*\alpha + \ldots + p_{k+j}^*\alpha)^l = j^{2-i} \int_{S^{k+j}} \beta * pt^{*(j-1)} * \gamma \cdot (p_1^*\alpha + \ldots + p_{k+j}^*\alpha)^l \tag{7.24.1}$$

Proof. For the diagonal $\Delta_j \subset S^j$ and a class $\alpha \in H^2(S,\mathbb{Q})$, we have $(p_1^*\alpha + \ldots + p_j^*\alpha) \cdot [\Delta_j] = jp_1^*(\alpha)[\Delta_j]$. By remark 7.11 the left hand side of (7.24.1) is equal to

$$\left(\int_{S^k} \Delta_j p_1^*\beta \cdot (p_1^*\alpha + \ldots + p_k^*\alpha)^{2-i}\right)\left(\int_{S^j} \gamma \cdot (p_1^*\alpha + \ldots + p_j^*\alpha)^{l+i-2}\right).$$

So the result follows. □

Notation 7.25. We denote by q_S the quadratic form on $H_2(S,\mathbb{Z})$ and, for $\gamma \in H^2(S,\mathbb{Q})$, we let L_γ be the linear form on $H_2(S,\mathbb{Q})$ given by $\alpha \mapsto \langle\gamma,\alpha\rangle$. For a class $\beta \in H_i(S,\mathbb{Q})$ we denote $\bar{\beta} := p_1^*\check{\beta} + \ldots + p_d^*\check{\beta} \in H^{4-i}(S^d,\mathbb{Q})$, where as above, $\check{\beta}$ is the Poincaré dual of β. Note that by lemma 6.7 and definition 6.12 $\varphi^*(\widetilde{\beta}|_{\mathrm{Hilb}^n(S)\times\mathrm{Hilb}^m(S)})$ is the pullback of $\bar{\beta}$. Let $N = 4c_2 - c_1^2 - 3$ again be the expected dimension of $M_H(c_1,c_2)$.

Lemma 7.26. *For all $x, y \geq 0$ and all $\alpha \in H_2(S,\mathbb{Q})$ we have*

$$\int_{S^d} \xi^{*x} pt^{*y} 1^{*(d-x-y)} \cdot \bar{\alpha}^{2d-x-2y} = \frac{(2d-x-2y)!}{2^{d-x-y}} q_S(\alpha)^{d-x-y} \langle\xi,\alpha\rangle^x.$$

Proof. By remark 7.11 we have

$$\int_{S^d} \xi^{*x} pt^{*y} 1^{*(d-x-y)} \cdot \bar{\alpha}^{N-m} = \int_{S^d} p_1^*\xi \cdot \ldots \cdot p_x^*\xi \cdot p_{x+1}^* pt \cdot \ldots \cdot p_{x+y}^* pt \cdot (p_1^*\check{\alpha} + \ldots + p_d^*\check{\alpha})^{N-m},$$

and it is easy to see that this is just $\frac{(2d-x-2y)!}{2^{d-x-y}} q_S(\alpha)^{d-x-y}\langle\xi,\alpha\rangle^x$. □

Theorem 7.27. *In the polynomial ring on $H^*(S,\mathbb{Q})$ we have*

$$\delta_{\xi,N,0} \equiv (-1)^{e_\xi} \sum_{k=0}^{2} \frac{N!}{(N-2d+2k)!(d-k)!} Q_k(N,d,K_S^2) L_{\xi/2}^{N-2d+2k} q_S^{d-k} \quad \textit{modulo } L_\xi^{N-2d+6},$$

where, by convention $\frac{1}{m!} = 0$ for $m < 0$ and

$$\begin{aligned} Q_0(N,d,K_S^2) &= 1 \\ Q_1(N,d,K_S^2) &= 2N + 2K_S^2 - 2d + 8 \\ Q_2(N,d,K_S^2) &= 2N^2 - 4dN + 4NK_S^2 + 21N + 2d^2 - 4dK_S^2 - 18d + 2(K_S^2)^2 + 18K_S^2 + 49. \end{aligned}$$

Proof. Let $R_d \in H^*(S^d,\mathbb{Q})^{\mathfrak{S}_d}$ be the class from proposition 7.23 with

$$d! \int_{\mathrm{Hilb}^d(S\sqcup S)} c(\Gamma)s([V^\vee(K_S)]_1^\vee)s([V]_2) \cdot \gamma = \int_{S^d} R_d \cdot w.$$

By remark 7.24 there is a class U_d' which is a linear combination of classes of the form $\xi^{*x}pt^{*y}1^{*(d-x-y)}$ with $\int_{S^d} R_d \cdot \bar{\alpha}^b = \int_{S^d} U_d' \cdot \bar{\alpha}^b$ for all $\alpha \in H_2(S,\mathbb{Q})$. We write $U_d' := \sum_{x,y\geq 0} u_{x,y}\xi^{*x}pt^{*y}1^{*(d-x-y)}$ and $U_d := \sum_{x+y\leq 2} u_{x,y}\xi^{*x}pt^{*y}1^{*(d-x-y)}$ By definition 6.12 and theorem 6.13 we see that

$$\delta_{\xi,N,0}(\alpha) = \sum_{i=0}^{2d} A_i \cdot \langle \xi, \alpha\rangle \int_{S^d} \{U_d'\}_i \bar{\alpha}^{2d-i},$$

where $\{\ \}_i$ denotes the part of degree i, and the A_i are suitable rational numbers. Thus $\delta_{\xi,N,0}$ modulo L_ξ^{n-2d+6} is already determined by U_d. As S is a surface with $p_g(S) = q(S) = 0$, we have $12 = 12\chi(\mathcal{O}_S) = K_S^2 + c_2(S)$ and thus we can replace $s_2(S)$ by $2K_S^2 - 12$. So, using proposition 7.23, we obtain after a short calculation that

$$\begin{aligned}
U_d \quad = \quad & 2^d 1^{*d} + 2^{d+1}d \cdot 1^{*(d-1)}*\xi + 2^{d+1}d(d-1)1^{*(d-2)}\xi^{*2} \\
& +2^d d(3\xi^2 + K_S^2 - 5d + 5)1^{*(d-1)}*pt \\
& +2^d d(d-1)(6\xi^2 + 2K_S^2 - 10d + 5)1^{*(d-2)}*\xi*pt \\
& +2^{d-2}d(d-1)(18(\xi^2)^2 + 12\xi^2K_S^2 + 2(K_S^2)^2 - 60d\xi^2 - 20dK_S^2 + 50d^2 \\
& +15\xi^2 - 10K_S^2 - 34d - 36)pt^{*2}1^{*(d-2)},
\end{aligned}$$

where we view ξ^2 and K_S^2 as integers and not as cohomology classes.

Now we apply definition 6.12 and lemma 7.26. Then, after some computation, we get the result with $Q_0(N,d,K_S^2)$, $Q_1(N,d,K_S^2)$, $Q_2(N,d,K_S^2)$ replaced by

$$\begin{aligned}
P_0(N,d,K_S^2,\xi^2) &= 1 \\
P_1(N,d,K_S^2,\xi^2) &= 8N - 26d + 6\xi^2 + 2K_S^2 + 26 \\
P_2(N,d,K_S^2,\xi^2) &= 18(\xi^2)^2 + 12(\xi^2)(K_S^2) + 2(K_S^2)^2 + 48N\xi^2 - 156d\xi^2 \\
& \quad -52dK_S^2 + 338d^2 + 16K_S^2N + 32N^2 - 208dN + 207\xi^2 \\
& \quad +54K_S^2 + 264N - 882d + 508.
\end{aligned}$$

We notice that by definition $d = (4c_2 - c_1^2 + \xi^2)/4$ and $N = 4c_2 - c_1^2 - 3$ and thus $\xi^2 = 4d - N - 3$. Substituting this into the $P_i(N,d,K_S^2,\xi^2)$ we obtain the result. □

We see that the result is compatible with the conjecture of Kotschick and Morgan. In fact it suggests a slightly sharper statement.

Conjecture 7.28. *In the polynomial ring on $H^2(S,\mathbb{Q})$ we have*

$$\delta_{\xi,N,0} = (-1)^{e_\xi} \sum_{k=0}^{d} \frac{N!}{(N-2d+2k)!(d-k)!} Q_k(N,d,K_S^2) L_{\xi/2}^{N-2d+2k} q_S^{d-k},$$

where $Q_k(N,d,K_S^2)$ is a polynomial of degree k in N,d,K_S^2, which is independent of S and ξ.

Now we want to compute $\delta_{l,r}$ in general. We shall see that there is reasonably simple relationship between the formula for $\delta_{N,0}$ and that for $\delta_{l,r}$ (with $l + 2r = N$), which is however obscured by the existence of a correction term coming from the failure of remark 7.24 for classes of the form $\bar{\alpha}^{k-2}\bar{pt}$ (instead of $\bar{\alpha}^k$).

Lemma 7.29. (1) *For all $x, y \geq 0$, all $c \leq r$ and all $\alpha \in H_2(S,\mathbb{Q})$ we have with $m := 2d-2c-x-2y$:*

$$\int_{S^d} \xi^{*x} pt^{*y} 1^{*(d-x-y)} \cdot \bar{pt}^c \bar{\alpha}^m = \frac{(d-x-y)!}{(d-x-y-c)!} \frac{m!}{2^{d-x-y-c}} q_S(\alpha)^{d-x-y-c} \langle \xi, \alpha \rangle^x$$

(2)

$$\int_{S^d} (1)_2 1^{*(d-2)} \cdot \bar{pt}\, \bar{\alpha}^{2d-4} = \frac{4d-6}{d-1} \int_{S^d} pt{*}1^{*(d-1)} \cdot \bar{pt}\, \bar{\alpha}^{2d-4}$$

Proof. (1) By remark 7.11 we have

$$\int_{S^d} \xi^{*x} pt^{*y} \cdot \bar{pt}^c \bar{\alpha}^m = \int_{S^d} p_1^*\xi \cdots p_x^*\xi \cdot p_{x+1}^* pt \cdots p_{x+y}^* pt \cdot (p_1^* \check{\alpha} + \ldots + p_d^* \check{\alpha})^m \cdot (p_1^* \check{pt} + \ldots + p_d^* \check{pt})^c$$

and it is elementary to show that this is just

$$\frac{(d-x-y)!}{(d-x-y-c)!} \frac{m!}{2^{d-x-y-c}} q_S(\alpha)^{d-x-y-c} \langle \xi, \alpha \rangle^x.$$

(2) By remark 7.11 and remark 7.24 we have

$$\begin{aligned}
&\int_{S^d} (1)_2 1^{*(d-2)} \cdot \bar{pt}\, \bar{\alpha}^{2d-4} \\
&= \int_{S^d} [\Delta_{1,2}] \cdot (p_1^* \check{\alpha} + \ldots + p_d^* \check{\alpha})^{2d-4} \cdot (p_1^* \check{pt} + \ldots + p_d^* \check{pt}) \\
&= 2 \int_{S^{d-2}} (p_1^* \check{\alpha} + \ldots + p_{d-2}^* \check{\alpha})^{2d-4} + (d-2) \int_{S^{d-1}} [\Delta_{1,2}] \cdot (p_1^* \check{\alpha} + \ldots + p_d^* \check{\alpha})^{2d-4} \\
&= (4d-6) \int_{S^{d-2}} \bar{\alpha}^{2d-4} \\
&= (4d-6) \int_{S^{d-1}} pt * 1^{*(d-2)} \bar{\alpha}^{2d-4} \\
&= \frac{4d-6}{d-1} \int_{S^d} pt{*}1^{*(d-1)} \cdot \bar{pt}\, \bar{\alpha}^{2d-4}
\end{aligned}$$

□

Theorem 7.30. *Let l, r be nonnegative integers with $l + 2r = N$. Then in the polynomial ring on $H^*(S,\mathbb{Q})$ we get*

$$\delta_{\xi,l,r} \equiv \sum_{c=0}^{2} \frac{(-1)^{r-c+e_\xi}}{2^{-3c+2r}} \binom{r}{c} \sum_{k=c}^{2} \frac{l!}{(l-2d+2k)!(d-k)!} Q_{k-c,c}(l,d,K_S^2,\xi^2) L_{\xi/2}^{l-2d+2k} q_S^{d-k}$$

modulo ξ^{N-2d+6}, where

$$Q_{m,c}(l,d,K_S^2,\xi^2) \quad = \quad P_m(l,d,K_S^2,\xi^2) + 21mc \quad \text{for } m+c \leq 2$$

Here the $P_i(N,d,K_S^2,\xi^2)$ are the polynomials from the proof of theorem 7.27.

Proof. For $i \leq r$ and a class $\gamma \in H^*(S^d,\mathbb{Q})^{\mathfrak{S}_d}$ we denote by $W_{l,r,c}(\gamma)$ the map that associates to $\alpha \in H_2(S,\mathbb{Q})$ the number

$$\sum_{b=0}^{l} (-1)^{r-c+e_\xi} 2^{b+2c-N} \binom{l}{b} \binom{r}{c} \langle \xi, \alpha \rangle^{l-b} \int_{S^d} \gamma \cdot \bar{\alpha}^b \bar{pt}^c.$$

Let $R_d, U_d \in H^*(S^d, \mathbb{Q})^{\mathfrak{S}_d}$ be the classes from the proof of 7.27. By thm 6.13 and proposition 7.23 we get

$$\delta_{l,r} \equiv \sum_{c=0}^{r} W_{l,r,c}(R_d).$$

By lemma 7.24 we see that $W_{l,r,0}(R_d) \equiv W_{l,r,0}(U_d)$ modulo L_ξ^{N-2d+6}. Furthermore we get modulo L_ξ^{N-2d+6}

$$\begin{aligned} W_{l,r,k}(R_d) &\equiv 0 \text{ for } k > 2 \\ W_{l,r,2}(R_d) &\equiv W_{l,r,2}(2^d 1^{*d}) \\ W_{l,r,1}(R_d) &\equiv W_{l,r,1}(\bar{R}_d), \end{aligned}$$

where $\bar{R}_d = t_1^{*d} - 5 \cdot 2^{d-2} d(d-1)(1)_2 1^{*(d-2)}$. By lemma 7.29(2) $W_{l,r,1}(\bar{R}_d) \equiv W_{l,r,1}(\bar{U}_d)$ where

$$\bar{U}_d = t_1^{*d} - 5 \cdot 2^{d-2} d(4d-6) pt * 1^{*(d-1)}.$$

Now the result follows by applying lemma 7.29(1) and some computation. □

Remark 7.31. Using $\xi^2 = 4d - 2r - l - 3$ we get equivalently

$$\begin{aligned} Q_{0,0}(l,d,K_S^2,\xi^2) &= Q_{0,1}(l,d,K_S^2,\xi^2) = Q_{0,2}(l,d,K_S^2,\xi^2) = 1 \\ Q_{1,0}(l,d,K_S^2,\xi^2) &= 2l - 2d - 12r + 2K_S^2 + 8 \\ Q_{2,0}(l,d,K_S^2,\xi^2) &= 72r^2 - 24rl + 24dr - 24K_S^2 r + 2l^2 - 4dl + 4K_S^2 l \\ &\quad +2d^2 - 4dK_S^2 + 2(K_S^2)^2 - 198r + 21l - 18d + 18K_S^2 + 49 \\ Q_{1,1}(l,d,K_S^2,\xi^2) &= 2l - 2d - 12r + 2K_S^2 + 29. \end{aligned}$$

Remark 7.32. We see that our results contain as a special case the formulas for the change for $d \leq 2$. In the case that $d = 3$ we notice that the spaces $X_\bullet$ and $Y_\bullet$ (for X and Y schemes with a natural morphism to $S^{(d)}$ and $S^{(n)} \times S^{(m)}$ respectively) just coincide with X respectively Y. Therefore our computations are valid on the whole of $\mathrm{Hilb}^d(S \sqcup S)$ and our methods will also give complete formulas for the change of the Donaldson invariants in case $d = 3$. We however do not carry out the elementary but long computations here.

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