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Minimal Degree of the Difference of Two Polynomials over $\mathbb{Q}$, and Weighted Plane Trees.

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Abstract

A *weighted bicolored plane tree* (or just *tree* for short) is a bicolored plane tree whose edges are endowed with a positive integral weights. The degree of a vertex is defined as the sum of the weights of the edges incident to this vertex. In ordinary plane trees the weights of all edges are equal to 1. A weighted plane tree is a graphical representation of a pair of coprime polynomials $P, Q \in \mathbb{C}[x]$ of the same degree (defined uniquely up to an affine transformations of the source and target variables) such that: (a) the multiplicities of the roots of P (respectively, of Q) are equal to the degrees of the black (respectively, white) vertices of the corresponding tree; (b) the degree of the difference $R = P - Q$ attains the minimum, possible for the given multiplicities of the roots of P and Q. In fact, the degree of P and Q is equal to the total weight of the tree in question while the degree of R is equal to its "overweight", meaning its total weight minus the number of edges. If, furthermore, such a tree is uniquely determined by the set of its black-and-white vertex degrees (we call such trees *unitrees*), then the corresponding polynomials are defined over $\mathbb{Q}$.

The search for the triples (P, Q, R) such that $\deg R = \min$, besides being an interesting problem in its own sake, is also related to some important questions in number theory. Dozens of number-theoretic papers, from 1965 [4] to 2010 [3], were dedicated to the study of these polynomials. Since the main interest of number-theorists lies in Diophantine equations, the most interesting triples (P, Q, R) are those defined over $\mathbb{Q}$.

In this paper we give a complete classification of the *unitrees*. Unitrees provide us with the most massive class of triples of polynomials defined over $\mathbb{Q}$. In section 6 we study further combinatorial invariants of the Galois action on trees as well as on the corresponding polynomial triples, which permit us to find more triples defined over $\mathbb{Q}$.

In a subsequent paper we compute all the triples (P, Q, R) corresponding to the unitrees.

Contents

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1 Introduction

In 1965, Birch, Chowla, Hall, and Schinzel [4] asked a question which soon became famous:

> Let A and B be two coprime polynomials with complex coefficients; what is the possible minimum degree of the difference $R = A^3 - B^2$?

It is reasonable to suppose that A^3 and B^2 have the same degree and the same leading coefficients. Let us take $\deg A = 2k$, $\deg B = 3k$, so that $\deg A^3 = \deg B^2 = 6k$. Then, the following was conjectured in [4]:

1. For $R = A^3 - B^2$ one always has $\deg R \geq k + 1$.
2. This bound is sharp: that is, it is attained for infinitely many values of k.

The first conjecture was proved the same year by Davenport [7]. The second one turned out to be much more difficult and remained open for 16 years: in 1981 Stothers [20] showed that the bound is in fact attained not only for infinitely many values of k but for all of them.

The above problem may be generalized in various ways. The following one was considered in 1995 by Zannier [22]. Let $\alpha, \beta \vdash n$ be two partitions of n,

$$\alpha = (\alpha_1, \dots, \alpha_p), \quad \beta = (\beta_1, \dots, \beta_q), \quad \sum_{i=1}^{p} \alpha_i = \sum_{j=1}^{q} \beta_j = n,$$

and let P and Q be two coprime polynomials of degree n having the following factorization pattern:

$$P(x) \,=\, \prod_{i=1}^{p} (x - a_i)^{\alpha_i}\,, \qquad Q(x) \,=\, \prod_{j=1}^{q} (x - b_j)^{\beta_j}\,. \tag{1}$$

In these expressions we consider the multiplicities α_i and β_j, $i = 1, 2, \dots, p$, $j = 1, 2, \dots, q$ as being given, while the roots a_i and b_j are not fixed, though they must all be distinct. The goal is to find the minimum possible degree of the difference $R = P - Q$. The lower bound

$$\deg R \,\geq\, (n + 1) - (p + q) \tag{2}$$

was first proved by in 1981 by Stothers [20]; its attainability was proved in 1995 by Zannier [22].

Definition 1.1 (Davenport–Zannier triple) Let P, Q, $R \in \mathbb{C}[x]$ be polynomials such that $\deg P = \deg Q = n$, P and Q are coprime and have a prescribed factorization pattern (1), while the degree of the polynomial $R = P - Q$ equals $(n+1)-(p+q)$. Then the triple (P,Q,R) is called a *Davenport–Zannier triple*, or, in a more concise way, a *DZ-triple*.

The main subject of this paper is an investigation of DZ-triples *with rational coefficients*, or *defined over* $\mathbb{Q}$, that is, triples P, Q, $R \in \mathbb{Q}[x]$.

The paper is organized as follows. A preliminary work is carried out in Section 2: there we reduce the problem about polynomials to a problem about *weighted bicolored plane trees*. A weighted bicolored plane tree is a bicolored tree whose edges are endowed with positive integral weights. The *degree of a vertex* is defined as the sum of the weights of the edges incident to this vertex. The sum of the weights of all the edges is called the *total weight* of the tree. We show that a DZ-triple exists if and only if there exists a weighted bicolored plane tree of the total weight n having p black vertices of degrees $\alpha_1, \ldots, \alpha_p$ and q white vertices of degrees $\beta_1, \ldots, \beta_q$.

In Section 3 we prove the existence theorem for weighted bicolored plane trees. Namely, we show that a tree with the above characteristics exists if and only if

$$p + q \leq \frac{n}{d} + 1\,, \tag{3}$$

where $d = \gcd(\alpha, \beta)$ denotes the greatest common divisor of the numbers $\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q$. Bound (2) is deduced from this result. We hope that our proof of (2) is simpler than the original one. In Section 4 we consider the problem of minimizing of $\deg R$ in the case where inequality (3) is not satisfied.

In Sections 5 and 6 we investigate DZ-triples defined over $\mathbb{Q}$. This case is the most interesting one since specializing x to a rational value we may obtain an important information concerning differences of integers with given factorization patterns. It seems that such was the initial motivation of Birch et al. in [4]: to "measure" the distance between cubes and squares of integers. This subject is still actively studied: see, for example, a recent paper by Beukers and Stewart [3] (2010) and the bibliography therein.

The framework of our paper is the theory of *dessins d'enfants* (see, e. g., Chapter 2 of [15]). In particular, we use the following corollary of this theory which gives a sufficient, though not necessary condition for a DZ-triple to be defined over $\mathbb{Q}$: there exists *exactly one* weighted bicolored plane tree having the degrees of its black, respectively, white, vertices being $\alpha_1, \ldots, \alpha_p$, respectively, $\beta_1, \ldots, \beta_q$. We will call such trees *unitrees.*

In Section 5 we give a complete classification of unitrees: see Theorem 5.3. The statement is rather long so we do not enunciate it here in the Introduction. The class of unitrees consists of ten infinite series of trees and ten sporadic trees which do not belong to any series.

While the results of Section 5 may be considered as conclusive, Section 6 represents only first steps in a vast programme of study of the Galois action on weighted trees and its combinatorial invariants. We mostly mention examples which lead to DZ-triples defined over $\mathbb{Q}$, postponing a discussion of DZ-triples defined over more general number fields to subsequent publications. Quite a few results were already established for ordinary trees; they wait to be generalized to weighted trees. In particular, essentially all known examples of DZ-triples over $\mathbb{Q}$ which have been found in dozens of previous publications can be obtained by our construction. We do not provide the corresponding references here since the polynomials themselves will be presented elsewhere.

Finally, in Section 7 we mention some further possible developments of the subject.

This paper deals only with the combinatorial aspect of the whole construction, mainly with the classification of unitrees. The computation of the corresponding DZ-triples is postponed to a separate publication since the techniques used for this purpose are very different form the ones used in this paper. In particular, a great deal of symbolic computations as well as certain polynomial identities are required. Note that for an individual tree, the computation of the corresponding DZ-triple is a difficult task but the verification of the result is easy. The situation becomes significantly more complicated for an infinite series of trees, since for a given series one has to (a) compute several examples; (b) guess the general pattern; (c) *prove* that the pattern thus found is correct.

2 From polynomials through Belyi functions to weighted trees

2.1 Function $f = P/R$

Let $\alpha, \beta \vdash n$ be two partitions of an integer n, $\alpha = (\alpha_1, \ldots, \alpha_p)$, $\beta = (\beta_1, \ldots, \beta_q)$, $\sum_{i=1}^p \alpha_i = \sum_{j=1}^q \beta_j = n$, and let $P, Q \in \mathbb{C}[x]$ be two polynomials of degree n having the factorizations

$$P(x) \;=\; \prod_{i=1}^{p} (x - a_i)^{\alpha_i}\,, \qquad Q(x) \;=\; \prod_{j=1}^{q} (x - b_j)^{\beta_j}\,. \tag{4}$$

We suppose all a_i, b_j, $i = 1, \ldots, p$, $j = 1, \ldots, q$ to be distinct. Let the difference $R = P - Q$ have the following factorization:

$$R(x) \;=\; \prod_{k=1}^{r} (x - c_k)^{\gamma_k}\,, \qquad \deg R = m = \sum_{k=1}^{r} \gamma_k\,. \tag{5}$$

Our goal is to minimize $m = \deg R$; obviously, $m \geq r$.

Consider the following rational function of degree n:

$$f \;=\; \frac{P}{R}\,;$$

note that

$$f - 1 \;=\; \frac{Q}{R}\,.$$

Definition 2.1 (Critical value) A point $y \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is called *critical value* of a rational function f if the equation $f(x) = y$ has multiple roots.

The expressions written above for the function $f = P/Q$ provide us with at least three critical values of f:

- $y = 0$, provided that not all α_i are equal to 1;
- $y = 1$, provided that not all β_j are equal to 1; and
- $y = \infty$, if only we do not consider the trivial case $\deg R = \deg P - 1$: when $\deg R < \deg P - 1$, the function f has a multiple pole at infinity.

Denote $y_1, \ldots, y_s$ the other critical values of f, if there are any, and let u_t be the number of preimages of y_t, $t = 1, \ldots, s$; by the definition of a critical value, $u_t < n$.

Lemma 2.2 (Number of roots of R) *The number r of distinct roots of the polynomial R is*

$$r \;=\; (n+1) - (p+q) + \sum_{t=1}^{s} (n - u_t)\,. \tag{6}$$

Proof. In fact, the equality (6) is a particular case of the Riemann–Hurwitz formula, but for the sake of completeness we give its proof here.

Let us draw a *star-tree* with the center at 0 and with its rays going to the critical values $1, y_1, \ldots, y_s$. Considered as a map on the sphere, this tree has $s+2$ vertices, $s+1$ edges, and a single outer face with its "center" at ∞. Now let us take the preimage of this tree under f.

We will get a graph drawn on the preimage sphere which has $n\,(s+1)$ edges since each edge is "repeated" n times in the preimage. Its vertices are the preimages of the points $0, 1, y_1, \ldots, y_s$, so their number is equal to $p + q + \sum_{l=1}^{s} u_t$. What takes place with the faces?

If we puncture at ∞ the single open face in the image sphere, we get a punctured disk without any ramification points inside. The only possible unramified covering of a punctured open disk is a disjoint collection of punctured disks; their number is equal to the number of poles of f, namely,

$r+1$ (r roots of R and ∞). Inserting a point into each puncture we get $r+1$ simply connected open faces in the preimage sphere. The fact that they are simply connected also implies that the graph drawn on the preimage sphere is connected. Thus, the preimage of our star-tree is a plane map.

What remains is to apply Euler's formula. □

Corollary 2.3 (Lower bound) *We have*

$$m \;=\; \deg R \;\geq\; (n+1) - (p+q)\,. \tag{7}$$

The proof follows from (6) and from the obvious inequality $m \geq r$, see (5).

Now it becomes clear what we should do in order to attain this bound:

- Make the number s of critical values other than 0, 1, and ∞, equal to zero, in order to eliminate the sum $\sum_{l=1}^{s}(n-u_l)$ in the right-hand side of (6) altogether. Note that this is impossible when $p+q > n+1$.
- Make all the roots of R simple, that is, make $\gamma_1 = \ldots = \gamma_r = 1$, in order to get $m = r$. Another formulation of the same statement: the partition $\gamma \vdash n$, $\gamma = (\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_r)$, corresponding to the multiplicities of the poles, must have the form of a *hook*: $\gamma = (n - r, \underbrace{1, 1, \ldots, 1}_{r \text{ times}}) = (n-r, 1^r)$.

The conditions which imply the existence of such a polynomial R will be obtained in Section 3.

2.2 Dessins d'enfants and Belyi functions

Considering rational functions with only three critical values puts us into the framework of the theory of *dessins d'enfants*. Here we give a brief summary of this theory (only in a planar setting); the missing details, proofs, and bibliography can be found, for example, in [15], Chapter 2.

Definition 2.4 (Belyi function) A rational function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is called *Belyi function* if it does not have critical values outside the set $\{0, 1, \infty\}$.

For such a function, the tree considered in the proof of Lemma 2.2 is reduced to the segment $[0,1]$. Let us take this segment, color the point 0 in black and the point 1 in white, and consider the preimage $D = f^{-1}([0,1])$; we will call this preimage a *dessin*.

Proposition 2.5 (Dessin) *The dessin $D = f^{-1}([0,1])$ is a connected graph drawn on the sphere, and its edges do not intersect outside the vertices. Therefore, D may also be considered as a plane map. This map has a bipartite structure: black vertices are preimages of* 0*, and white vertices are preimages of* 1*.*

The degrees of the black vertices are equal to the multiplicities of the roots of the equation $f(x) = 0$, and the degrees of the white ones are equal to the multiplicities of the roots of the equation $f(x) = 1$. The sum of the degrees in both cases is equal to $n = \deg f$, which is also the number of edges.

The map D being bipartite, the number of edges surrounding each face is even. It is convenient, in defining the face degrees, to divide this number by two.

Definition 2.6 (Face degree) We say that an edge is *incident* to a face if, while going around the face in the positive (trigonometric) direction we follow this edge from its black end toward the white one. Thus, only a half of the edges surrounding the face are considered to be incident to this face. Moreover, each edge is incident to exactly one face. The *degree* of a face is equal to the number of edges incident to it.

According to this definition and to the remarks preceding it, every edge is incident to a single black vertex, to a single white vertex, and to a single face. The sum of the face degrees is equal to $n = \deg f$.

Proposition 2.7 (Faces and poles) *Inside each face there is a single pole of f, and the multiplicity of this pole is equal to the degree of the face.*

Definition 2.8 (Passport of dessin) The triple $\pi = (\alpha, \beta, \gamma)$ of partitions $\alpha, \beta, \gamma \vdash n$ which correspond to the degrees of the black vertices, of the white vertices, and of the faces of a dessin, is called a *passport* of the dessin.

Definition 2.9 (Combinatorial orbit) A set of the dessins having the same passport is called a *combinatorial orbit* corresponding to this passport.

The construction which associates a map to a Belyi function works also in the opposite direction. Let M be a bicolored map drawn on the sphere. Then, the sphere may be endowed with a complex structure, thus becoming the Riemann complex sphere, and M can be drawn as a dessin D obtained via a Belyi function. The following statement is a particular case of the classical Riemann's existence theorem:

Proposition 2.10 (Existence of Belyi functions) *Every bicolored plane map can be realized as a dessin $D = f^{-1}([0,1])$ where f is a Belyi function. The function $f = f(x)$ is unique up to a linear fractional transformation of the variable x.*

Remark 2.11 (On pictures) Maps are topological objects considered up to an orientation preserving homeomorphism of the sphere. By contrast, dessins are *rigid*: they are defined up to an isomorphism of the Riemann complex sphere, that is, up to a linear fractional transformation. To draw a dessin D we should first compute its Belyi function f_D, then solve numerically a significant sample of equations $f_D(x) = y$ for $y \in [0,1]$, and then put the corresponding points on the complex plane. In this paper, the pictures show only the topological structure of the maps. We consider them as abstract *representations* of dessins, or, from the other perspective, as a way of "denoting" dessins. Proposition 2.10 ensures that the corresponding dessins do exist.

Now our main problem may be reformulated in purely combinatorial terms:

Proposition 2.12 (Bound attainability) *The lower bound* (2) *is attained if and only if there exists a bicolored plane map with the passport $\pi = (\alpha, \beta, \gamma)$ in which the partitions $\alpha = (\alpha_1, \dots, \alpha_p)$ and $\beta = (\beta_1, \dots, \beta_q)$ are given, and γ has the form $\gamma = (n - r, 1^r)$ where 1 is repeated $r = (n+1) - (p+q)$ times.*

In geometric terms, all the faces of our map except the outer one must be of degree 1. Recall that the number of faces, which is equal to $r + 1$, is prescribed by Euler's formula.

Convention 2.13 (Finite faces; small faces) We call all the faces except the outer one, *finite faces*, and the faces of degree 1, *small faces*.

Example 2.14 (Cubes and squares: a solution) Let us look once again at the problem posed by Birch et al. in [4]. In order to show that the lower bound is attained we must construct a map with the following properties: all its black vertices are of degree 3; all its white vertices are of degree 2; and all its finite faces are small.

In order to simplify our pictures we use the following convention.

Convention 2.15 (White vertices of degree 2) When all the white vertices are of degree 2, it is convenient, in order to simplify a graphical representation of such a map, to draw only black vertices and to omit the white ones, considering them as being implicit. In such a picture, a line connecting two black vertices contains an invisible white vertex in its middle, and is thus not an edge but a union of two edges.

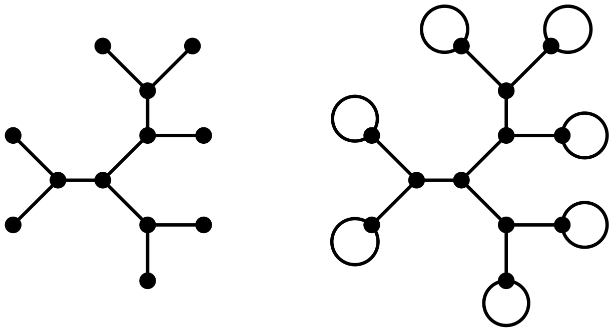

Figure 1: This map solves the problem which remained open for 16 years: there exist polynomials A and B, $\deg A = 2k$, $\deg B = 3k$, such that $\deg(A^3 - B^2) = k + 1$.

The construction of the maps we need in order to solve the problem about squares and cubes is very simple: we draw first a tree with all internal nodes being of degree 3, and then attach loops to its leaves, see Figure 1.

We see in this example a remarkable efficiency of the pictorial representation of problems concerning polynomials. If this representation were known in 1965, the proof of the conjecture would have taken 16 minutes instead of 16 years.

2.3 Number fields

As it was told in Proposition 2.10, a Belyi function $f(x)$ corresponding to a dessin is defined up to a linear fractional transformation of x. In this family of equivalent Belyi functions it is always possible to find one whose coefficients are algebraic numbers. If we act simultaneously on all the coefficients of such a function by an element of the *absolute Galois group* $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$, that is, by an automorphism σ of the field $\overline{\mathbb{Q}}$ of algebraic numbers, or, in other words, if we replace all the coefficients a_i of f by their algebraically conjugate numbers $\sigma(a_i)$, we obtain once again a Belyi function. Furthermore, one can prove that in such a way the action of $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ on Belyi functions descends on an action on dessins. There exist many combinatorial invariants of this action, the first and the simplest of them being the passport of the dessin. Thus, a combinatorial orbit (see Definition 2.9) may constitute a single Galois orbit, or may further split into a union of several Galois orbits. Every combinatorial orbit is finite, and therefore every Galois orbit is also finite.

One of the most important notions concerning the Galois action on dessins is that of the field of moduli.

Construction 2.16 (Field of moduli) Let D be a dessin, and let $\Gamma_D \leq \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ be its stabilizer. The orbit of D being finite, Γ_D is a subgroup of finite index in $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$. Let $H \leq \Gamma_D$ be the maximal normal subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ contained in Γ_D. According to the Galois correspondence between subgroups of $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ and algebraic extensions of $\mathbb{Q}$, there exists a number field K corresponding to H. This field is called the *field of moduli* of the dessin D. By construction, this field is unique: a dessin cannot have two different fields of moduli.

Below we list some properties of the fields of moduli. Let $\mathcal{D} = \{D_1, \ldots, D_s\}$ be an orbit of the Galois action on dessins.

- The field of moduli is the same for all the elements of the orbit.
- The degree of K as an extension of $\mathbb{Q}$ is equal to $s = |\mathcal{D}|$.
- The coefficients of Belyi functions corresponding to the dessins $D \in \mathcal{D}$, if they are chosen as algebraic numbers, always belong to an extension L of K.
- The action of the group $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ on dessins $D \in \mathcal{D}$ may be reduced to the action of $\mathrm{Gal}(K|\mathbb{Q})$.

- The action of $\mathrm{Gal}\,(L|K)$ on Belyi functions may change a position of a dessin $D \in \mathcal{D}$ on the complex sphere but does not change its combinatorial structure; in other words, as a map, the dessin in question remains the same.

In the absolute majority of cases the situation is much simpler: the field of moduli of an orbit is the smallest number field to which the coefficients of the corresponding Belyi functions belong. However, in some specially constructed examples we need a larger field $L \supset K$ to be able to find Belyi functions. There exists a simple *sufficient condition* which ensures that the coefficients do belong to K, see [6]: this condition is the existence of a *bachelor.*

Definition 2.17 (Bachelor) A *bachelor* is a black vertex (a white vertex; a face) such that there is no other black vertex (no other white vertex; no other face) of the same degree.

Remark 2.18 (Positioning of bachelors) If a dessin contains several bachelors then up to three of them can be placed at rational points, that is, at points in $\mathbb{Q} \cup \{\infty\}$, and this will not prevent the Belyi function for the dessin in question to be defined over the field of moduli.

For the dessins we study in this paper a bachelor always exists: it is the outer face (since all the other faces are of degree 1).

Recalling that the degree of K is equal to the size of the orbit we conclude that *if a combinatorial orbit consists of a single element, then it is also a Galois orbit, and its moduli field is* $\mathbb{Q}$.

Summarizing what was stated above we may affirm the following:

Proposition 2.19 (Coefficients in $\mathbb{Q}$) *If for a given passport $\pi = (\alpha, \beta, \gamma)$, where the partition γ is of the form $\gamma = (n-r, 1^r)$, there exists a unique bicolored plane map, then there exists a corresponding Belyi function with rational coefficients, and therefore there also exists a DZ-triple with rational coefficients.*

The two main statements of this section are Proposition 2.12, which concerns the existence, and Proposition 2.19, which concerns the definablity over $\mathbb{Q}$. Note that Proposition 2.12 is of the "if and only if" type while Proposition 2.19 provides only an "if"-type condition.

2.4 How do the weighted trees come in?

Though the weighted trees are, in our opinion, natural and interesting objects to be studied in their own sake, and, in particular, their enumeration is not yet carried out, in our paper they are used as a merely technical tool which is easy to manipulate. In Figure 2, left, it is shown how a typical bicolored map with all its finite faces being small looks like. (Recall that according to Definition 2.6 a face of degree 1 is surrounded by two edges, but only one of these edges is incident to the face.) It is convenient to symbolically represent such a map in a form of a tree (see Figure 2, right) by replacing several multiple edges which connect neighboring vertices, by a single edge with a weight equal to the number of these multiple edges. In this way, the operations of cutting and gluing subtrees, exchange of weights between edges, etc., become easier to implement and to understand.

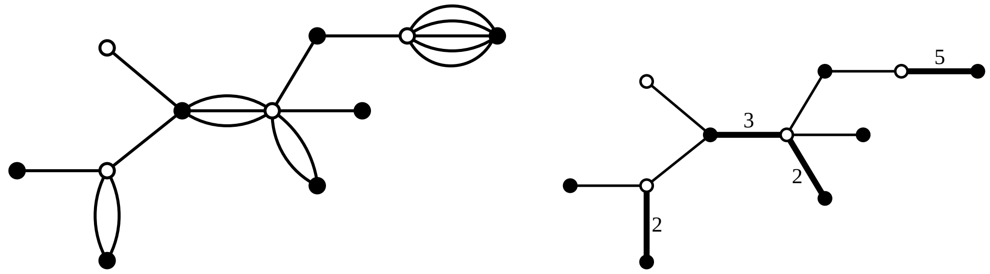

Figure 2: The passage from a map with all its finite faces being small, to a weighted tree. The weights which are not explicitly indicated are equal to 1; the edges of the weight bigger than 1 are drawn thick.

Definition 2.20 (Weighted tree) A *weighted bicolored plane tree*, or *weighted tree*, or just *tree* for short, is a bicolored plane tree whose edges are endowed with positive integral *weights*. The sum of the weights of the edges of a tree is called the *total weight* of the tree. The *degree* of a vertex is the sum of the weights of the edges incident to this vertex. The *weight distribution* of a weighted tree is a partition $\mu \vdash n$, $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ where $m = p + q - 1$ is the number of edges, and μ_i, $i = 1, \ldots, m$ are the weights of the edges.

Definition 2.21 (Passport and extended passport of a tree) The pair (α, β) of partitions $\alpha, \beta \vdash n$ of the total weight n of a tree, corresponding to the degrees of the black vertices and of the white vertices of a weighted tree, is called a *passport* of this tree. *Extended passport* of a tree associates to each vertex of the tree a set of weights of the edges incident to this vertex.

The adjective *plane* in this definition means that our trees are considered not as mere graphs but as plane maps, with their particular embedding into the sphere being given. More precisely, this means that the cyclic order of branches around each vertex of the tree is fixed, and changing this order will give a different tree. *All the trees considered in this paper will be endowed with the "plane" structure*; therefore, the adjective "plane" will often be omitted.

Example 2.22 (Tree of Figure 2) The total weight of the tree shown in Figure 2 is $n = 18$; its passport is $(\alpha, \beta) = (5^2 2^3 1^2, 7^1 6^1 4^1 1^1)$; the face degree distribution is $\gamma = 10^1 1^8$, and the weight distribution is $\mu = 5^1 3^1 2^2 1^6$. The extended passport of this tree looks as follows:

$$\begin{array}{ll} \text{black vertices:} & (5), (3,1,1), (2), (2), (1,1), (1), (1) \\ \text{white vertices:} & (3,2,1,1), (5,1), (2,1,1), (1) \end{array}$$

Now Proposition 2.12 may be reformulated as follows:

Theorem 2.23 (Bound attainability) *The lower bound* (2) *is attained if and only if there exists a weighted tree with the passport* (α, β).

Definition 2.24 (Ordinary trees) Weighted trees whose weight distribution is $\mu = 1^n$ will be called *ordinary trees*.

Ordinary trees correspond to *Shabat polynomials*: they are particular cases of Belyi functions, with a single pole at infinity.

3 Existence theorem

The question we study in this section is as follows: for a given pair of partitions $\alpha, \beta \vdash n$, does there exist a weighted tree of the total weight n with the passport (α, β)? Equivalently, does there exist a rational function with three critical values, and with the multiplicities of the preimages of these critical values being, first, two given partitions $\alpha, \beta \vdash n$, and then, the third partition being equal to $\gamma = (n - r, 1^r)$?

This question is a particular case of a more general problem of *realizability* of ramified coverings: does there exist a ramified covering of a given Riemann surface with the given "local data" (that is, with given multiplicities of the preimages of ramification points)? The problem goes back to the classical paper by Hurwitz [11] (1891). Though many particular cases are well investigated, the problem in its full generality remains unsolved. Among numerous publications dedicated to the realizability we would like to mention early works by Husemoller [12] (1962) and Thom [21] (1965); an important paper by Edmonds, Kulkarni, and Stong [8] (1984); and recent publications [18] (2009), [5] (2008), and [17] (2009).

The main result of this section is the following theorem:

Theorem 3.1 (Realizability of a passport by a tree) *Let* $\alpha, \beta \vdash n$ *be two partitions of* n, $\alpha = (\alpha_1, \ldots, \alpha_p)$, $\beta = (\beta_1, \ldots, \beta_q)$, *and let* $\gcd(\alpha, \beta) = d$. *Then a weighted tree with the passport* (α, β) *exists if and only if*

$$p + q \le \frac{n}{d} + 1 \,. \tag{8}$$

The case where condition (8) is not satisfied is investigated in the next section.

Theorem 3.1 and Theorem 4.1 below are equivalent to the main result (Theorem 1) of U. Zannier [22]. In his paper, Zannier remarks that it would be interesting to apply the theory of dessins d'enfants to this problem in a more direct way, and mentions a remark by G. Jones that such an approach might produce a simpler proof. This is indeed the case, as we will see in this section. Besides that, this theory enables us to find a huge class of DZ-triples over $\mathbb{Q}$ (in a way, "almost all" of them), see Section 5; and it also gives us a more direct access to Galois theory, see Section 6. We have already seen a first glimpse of the power of the "dessin method" in Example 2.14.

3.1 Forests

A *forest* is a disjoint union of trees.

Proposition 3.2 (Forest) *Any pair (α, β) of partitions $\alpha, \beta \vdash n$ can be realized as a passport of a forest of weighted trees.*

Proof. If there are two equal parts $\alpha_i = \beta_j$ in the partitions α and β, we make a separate edge with the weight $w = \alpha_i = \beta_j$ and proceed with the new passport (α', β'), where α' and β' are obtained from α and β by eliminating their parts α_i and β_j, respectively.

If there are no equal parts, suppose, without loss of generality, that there are two parts $\alpha_i > \beta_j$. Then we do the following (see Figure 3):

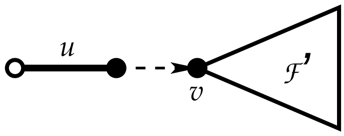

Figure 3: Construction of a forest in the case $\alpha_i > \beta_j$. Here $u = \beta_j$ and $v = \alpha_i - \beta_j$. The forest $\mathcal{F}'$ corresponds to the passport (α', β') where β' is obtained from β by eliminating the part β_j, and α' is obtained from α by replacing α_i with $\alpha_i - \beta_j$.

(a) make an edge with the weight $u = \beta_j$;

(b) consider the new passport (α', β') where β' is obtained from β by eliminating the part β_j, and α' is obtained from α by replacing α_i with $v = \alpha_i - \beta_j$;

(c) construct inductively a forest $\mathcal{F}'$ of the total weight $n - u$ corresponding to the passport (α', β'); by definition, this forest must have a black vertex of degree v;

(d) glue the edge of weight u to the forest $\mathcal{F}'$ by fusing two vertices, as is shown in Figure 3, and get a forest $\mathcal{F}$ corresponding to (α, β).

3.2 Sewing a tree from a forest

Theorem 3.3 (Existence) *Suppose that* $\gcd(\alpha, \beta) = 1$. *Then the passport* (α, β) *can be realized by a weighted tree if and only if* $p + q \leq n + 1$.

Proof. According to Proposition 3.2 we may suppose that we already have a forest corresponding to the passport (α, β). Now suppose that there are two edges of weights s and u, $s < u$, which belong to different trees. Then we may sew them together by the operation shown in Figure 4. The degrees of the vertices in the new, connected figure are the same as in the old, disconnected one. Figure 5 shows that the operation works in the same way when there are subtrees attached to the ends of the adjoined edges.

We repeat this sewing operation until it becomes impossible to continue. The latter may happen in two ways. Either we have got a connected tree, and then we are done. Or there is no more

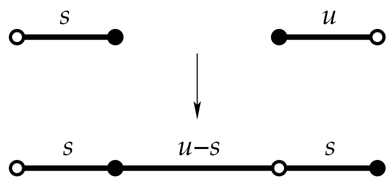

Figure 4: Sewing two edges.

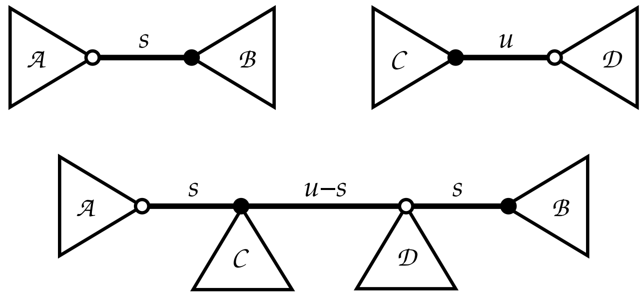

Figure 5: Sewing two trees.

edges with different weights while the forest remains disconnected. Then, taking into account that $\gcd(\alpha, \beta) = 1$, we conclude that the weights of all edges are equal to 1, that is, we have got a forest consisting of ordinary trees. In this case, the number of vertices $p+q$ is strictly greater than $n+1$, which contradicts the condition of the theorem. $\square$

We will use the inverse operation, namely, the ripping of a connected tree in two, in the proof of Proposition 5.11 (see Figure 24). Note that the side edges in Figures 4 and 5 have the same weight s.

3.3 When all weights are multiples of $d > 1$

Now suppose that $\gcd(\alpha, \beta) = d > 1$.

Lemma 3.4 (gcd > 1) *The degrees of all vertices of a forest are divisible by $d > 1$ if and only if the weights of all edges are also divisible by d.*

Proof. In one direction it is evident: the degrees of the vertices are sums of weights, and therefore, if all the weights are multiples of d, then the same is true for the degrees.

In the opposite direction, if all the vertex degrees are divisible by d, then it is true, in particular, for the leaves. Cut any leaf off the tree to which it belongs, and the statement is reduced to the same one for a smaller forest. $\square$

Thus, dividing by d all the vertex degrees, that is, all the elements of the partitions α and β, we return to the situation of Theorem 3.3, with the same numbers p and q, and with the total weight equal to n/d. This finishes the proof of Theorem 3.1. $\square$

The reader must appreciate the simplicity of the above proof: number theorists were approaching this result for 30 years. Once again, the credit goes to the pictorial representation of polynomials with the desired properties.

4 Weak bound

When condition (8) of Theorem 3.1 is satisfied then, according to Theorem 2.23, the main bound (2) is attained. If this condition is *not satisfied*, then the following holds:

Theorem 4.1 (Weak bound) *Let* $\gcd(\alpha, \beta) = d$*, and let* $p + q > \dfrac{n}{d} + 1$*. Then*

$$\deg R \geq \frac{(d-1)\,n}{d}, \tag{9}$$

and this bound is attained.

4.1 Polynomials and cacti

We will need the following proposition which was proved in 1965 by Thom [21], and then reproved in many other publications. For the reader's convenience we provide a short proof based on "Dessins d'enfants" theory following [15], Corollary 1.6.9.

Proposition 4.2 (Realizability of polynomials) *Let* $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ *be a set of* $k \geq 1$ *partitions* $\lambda_i \vdash n$ *of number* n*. Denote by* p_i *the number of parts of* λ_i*,* $i = 1, 2, \ldots, k$*. Let* $y_1, y_2, \ldots, y_k \in \mathbb{C}$ *be arbitrary complex numbers. Then a necessary and sufficient condition for the existence of a polynomial* $T \in \mathbb{C}[x]$ *of degree* n*, all of whose finite critical values are contained in the set* $\{y_1, y_2, \ldots, y_k\}$*, with the multiplicities of the roots of the equations* $T(x) = y_i$ *corresponding to the partitions* λ_i*,* $i = 1, 2, \ldots, k$*, is the following equality:*

$$\sum_{i=1}^{k} p_i \;=\; (k-1)\,n + 1\,, \tag{10}$$

or, equivalently,

$$\sum_{i=1}^{k} (n - p_i) \;=\; n - 1\,. \tag{11}$$

Proof. Let D be a simple closed curve on the y-plane passing through the points $y_1, y_2, \ldots, y_k$, and let C be its preimage $C = T^{-1}(D)$. Then C is a tree-like map often called *cactus*: it does not contain any cycles except n "copies" of D glued together at the vertices which are preimages of y_i; the number of copies of D glued together at a vertex is equal to the multiplicity of the corresponding root of the equation $T(x) = y_i$; see Figure 6. Equation (10) may then be interpreted as Euler's formula for the cactus since the cactus has $\sum_{i=1}^{k} p_i$ vertices, kn edges, and $n + 1$ faces (n copies of D and the outer face). As to equation (11), it means that the degree of the derivative $T'(x)$ is $n - 1$.

These observations prove that conditions (10)–(11) are necessary. Notice that the partition $\lambda = 1^n$ may be eliminated from Λ (if it belongs to it), and may also be added to it, and this does not invalidate equalities (10) and (11).

The proof that (10) and (11) are also sufficient is divided into two parts. The first part is purely combinatorial and consists in constructing a cactus (at least one) with the vertex degrees corresponding to Λ. The second part is just a reference to Riemann's existence theorem which relates combinatorial data to the complex structure, as it was already the case in Proposition 2.10.

The proof of the *existence of a cactus* in question is similar to that of Proposition 3.2; namely, it consists in "cutting a leaf". Here a *leaf* means a copy of D which is attached to C at a single vertex (see Figure 7). This cutting operation must be carried out not with the cactus itself (since it is not yet constructed) but with its passport Λ: it is easy to verify that (10) implies that all partitions $\lambda_i \in \Lambda$ except maybe one contain a part equal to 1. We eliminate these parts, and diminish by 1 a part in the remaining partition. In this way we obtain a valid passport Λ' of degree $n - 1$; then we construct inductively a smaller cactus; and then glue to it an nth copy of D. We leave further details to the reader. $\square$

Note that for rational functions, and even for Laurent polynomials, a similar statement is not valid, see [17]: conditions based on the Euler formula remain necessary but they are no longer sufficient. See also Example 4.7.

Another approach to the proof of Proposition 4.2 is to use an enumerative formula due to Goulden and Jackson [10] which gives the number of cacti corresponding to a given list of partitions Λ. Let us write a partition $\lambda \vdash n$ in the power notation: $\lambda = 1^{d_1} 2^{d_2} \ldots n^{d_n}$ where d_i is the

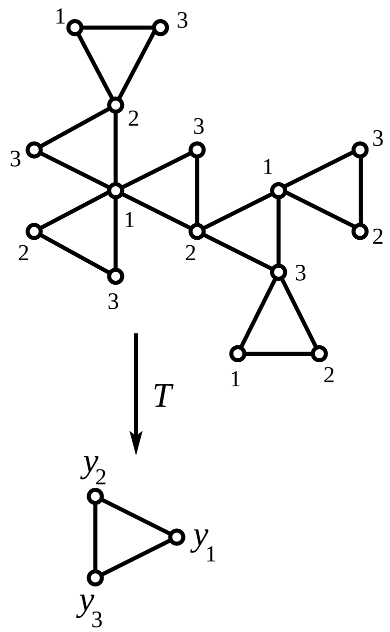

Figure 6: A cactus. In this example there are three finite critical values y_1, y_2, y_3; therefore, D is a triangle. A cactus is of degree 7, and therefore it contains seven triangles. Vertices which are preimages of y_1, respectively of y_2 and of y_3, are labeled by 1, respectively by 2 and by 3. In this example we have $\Lambda = (3^1 2^1 1^2, 2^2 1^3, 2^1 1^5)$. Namely, $\lambda_1 = 3^1 2^1 1^2$ shows how many triangles are glued together at vertices labeled by 1, while partitions λ_2 and λ_3 correspond to labels 2 and 3.

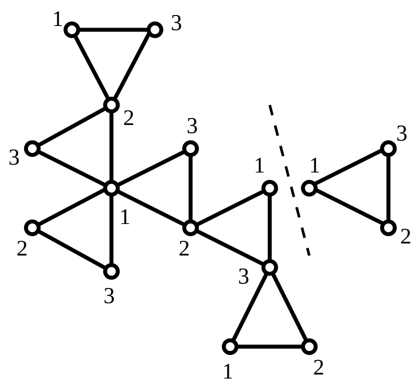

Figure 7: Cutting off a leaf from a cactus. A leaf exists since every partition in Λ, except maybe one, contains a part equal to 1: this is a consequence of (10).

number of parts of λ equal to i, so that $\sum_{i=1}^{n} d_i = p$ (here p is the total number of parts in λ), and $\sum_{i=1}^{n} i d_i = n$. Denote

$$N(\lambda) \;=\; \frac{(p-1)!}{d_1!\,d_2!\,\ldots\,d_n!}\,.$$

Then the following is true:

Proposition 4.3 (Enumerative formula) *For a given* $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ *satisfying conditions* (10)–(11) *we have*

$$\sum \frac{1}{|\mathrm{Aut}\,(C)|} \;=\; n^{k-2} \prod_{i=1}^{k} N(\lambda_i) \tag{12}$$

where the sum is taken over the cacti C *with the passport* Λ*, and* $|\mathrm{Aut}\,(C)|$ *is the size of the automorphism group of* C*.*

Notice that Proposition 4.3 provides another proof of Proposition 4.2. Indeed, the right-hand side of formula (12) is always positive. □

Remark 4.4 (Enumeration of ordinary trees) Taking $k = 2$ in Proposition 4.2 we may put the critical values y_1 and y_2 to 0 and 1, and replace the simple closed curve D passing through these points by the segment $[0, 1]$. Then a cactus becomes an *ordinary* bicolored plane tree with the passport $\Lambda = (\lambda_1, \lambda_2)$. In this case the number of trees (with the weights $1/|\mathrm{Aut}\,(C)|$) is also given by formula (12). This fact will be useful in the future: in order to verify that a given ordinary tree is a unitree we can just compute the number given by (12).

4.2 Weak bound for polynomials

Proposition 4.5 ($p + q \geq n + 1$) *Suppose that* $\gcd(\alpha, \beta) = 1$ *and the number* $p + q$ *of parts in* α *and* β *is* $p + q \geq n + 1$*. Then there exist polynomials* P *and* Q *having the factorization pattern* (4) *such that* $P - Q = \mathrm{Const}$*.*

Proof. We have $n + 1 \leq p + q \leq 2n$, therefore $1 \leq (2n + 1) - (p + q) \leq n$. Let us take an arbitrary partition $\lambda_3 \vdash n$ having $(2n + 1) - (p + q)$ parts, and also take $\lambda_1 = \alpha$ and $\lambda_2 = \beta$. Then for $\Lambda = (\lambda_1, \lambda_2, \lambda_3)$ conditions (10)–(11) are satisfied. Hence, there exists a polynomial $T(x)$ satisfying all the conditions of Proposition 4.2, with three critical values y_1, y_2, y_3 which may be chosen arbitrarily. Taking $P(x) = T(x) - y_1$ and $Q(x) = T(x) - y_2$ we obtain two polynomials which factorize as in (4) and whose difference is

$$R(x) \;=\; P(x) - Q(x) \;=\; y_2 - y_1 \;=\; \mathrm{Const}\,.$$

Thus, the obvious lower bound $\deg R \geq 0$ is indeed attained. □

Let us now consider the case $\gcd(\alpha, \beta) = d > 1$. We have $P = f^d$ and $Q = g^d$. Therefore, $R = f^d - g^d$ factors into d factors $f - \zeta g$, where ζ runs over the d-th roots of unity. If one of the factors, which we may without loss of generality assume to be $f - g$, has degree $< n/d$, then the leading coefficients of f and g coincide. Hence, the leading coefficients of f and ζg for $\zeta \neq 1$ do not coincide, and all the remaining $d - 1$ factors have degree exactly n/d.

If $p + q \leq \dfrac{n}{d} + 1$, then

$$\min\deg\,(f - g) \;=\; \Big(\frac{n}{d} + 1\Big) - (p + q)\,,$$

and therefore

$$\min\deg\,(f^d - g^d) \;=\; \Big[\Big(\frac{n}{d} + 1\Big) - (p + q)\Big] + (d - 1)\cdot\frac{n}{d} \;=\; (n + 1) - (p + q)\,,$$

which is consistent with Theorem 3.1.

On the other hand, if $p+q > \dfrac{n}{d}+1$ then $\min\deg\,(f-g)=0$, and therefore

$$\min\deg R \;=\; 0+(d-1)\cdot\frac{n}{d}\,.$$

This finishes the proof of Theorem 4.1. $\square$

We see that the only case when the main lower bound $(n+1)-(p+q)$ is not attained, in spite of the fact that $p+q\le n+1$, is when both conditions $d=\gcd(\alpha,\beta)>1$ and

$$\frac{n}{d}+1<p+q\le n+1$$

are satisfied.

Example 4.6 (Weak bound) Let us take $n=6$, $\alpha=4^1 2^1$, and $\beta=2^3$, so that $p=2$, $q=3$, and $d=2$. The usual expression

$$(n+1)-(p+q)=(6+1)-(2+3)=2$$

gives us a bound which cannot be attained. The correct answer is given by Theorem 3.1:

$$\min\deg R \;=\; (d-1)\cdot\frac{n}{d} \;=\; (2-1)\cdot\frac{6}{2} \;=\; 3\,.$$

And, indeed, there is only one plane map with two black vertices, one of degree 4 and one of 2, and with three white vertices of degree 2, see Figure 8. This map has *two finite faces*; one of these faces is of degree 2, so that the sum of degrees of finite faces is not 2 but 3.

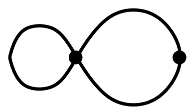

Figure 8: This is the only plane map having the passport $(4^1 2^1, 2^3)$. We see that one of the faces is of degree 2.

Remark 4.7 (Non-realizable planar data) Let us take α and β such that $\gcd(\alpha,\beta)=d>1$ and $\dfrac{n}{d}+1<p+q\le n+1$. Let us also take $r=(n+1)-(p+q)$ and $\gamma=(n-r,1^r)$. Then the passport $\pi=(\alpha,\beta,\gamma)$ satisfies the Euler relation: there are $p+q$ vertices, n edges, and $r+1$ faces, so that

$$(p+q)-n+(r+1) \;=\; (p+q)-n+[(n+1)-(p+q)+1] \;=\; 2.$$

However, a plane map with these data does not exist.

The principal vocation of this paper is to use combinatorics for the study of polynomials. But here in this particular example we deduce a non-trivial statement about plane maps from a trivial property of polynomials, namely, from the fact that the degree of a polynomial cannot be negative.

5 Classification of unitrees

This section contains the main results of the paper: here we give a complete classification of the passports satisfying the following property: a weighted bicolored plane tree having this passport is unique. As we have explained before, in Proposition 2.19, Belyi functions corresponding to such trees are defined over $\mathbb{Q}$.

Ordinary unitrees were classified by Adrianov in 1989. However, his initial proof was never published since it looked too cumbersome. Then, in 1992, appeared the paper [10] by Goulden and Jackson with the enumerative formula (12), which opened a possibility for another proof, by carefully analizing this formula and looking for the cases in which it gives a number ≤ 1 (recall that (12) counts each tree C with the weight $1/|\mathrm{Aut}(C)|$). Our situation is more difficult in two ways. First, we deal with weighted trees; and, second, we don't have an enumerative formula at our disposal. For these (and many other) reasons such a formula would be very welcome.

To be more specific, we need a formula which, *for a given passport*, would give us, in an explicit way, the number of the corresponding weighted bicolored plane trees. An additional difficulty here ensues from the fact that the same passport may correspond not only to trees but also to forests.

Assumption 5.1 (Passports from now on) In the remaining part of the paper we will consider only the passports (α, β) such that $\gcd(\alpha, \beta) = 1$ and $p + q \leq n + 1$.

Definition 5.2 (Unitree) A weighted bicolored plane tree which is unique for its passport is called a *unitree*.

5.1 Statement of the main result

The classification of unitrees is summarized in the following theorem:

Theorem 5.3 (Complete list of unitrees) *Up to an exchange of black and white, the complete list of unitrees consists of the following* 20 *cases:*

- *Five infinite series A, B, C, D, E of trees shown in* Figures 9, 10, 11, 12, and 13, *involving two integral weight parameters s and t which are supposed to be coprime (thus either $s \neq t$, or $s = t = 1$). Note that*
  - *for the diameter ≥ 5, only the trees of the types B and E exist;*
  - *for the diameter 4, the trees of types B, D, and E exist;*
  - *for the diameter 3, the trees of types B, C, and E exist.*
- *Five infinite series F, G, H, I, J shown in* Figures 14 *and* 15.
- *Ten sporadic trees K, L, M, N, O, P, Q, R, S, T shown in* Figures 16, 17, 18, *and* 19.

Remark 5.4 (Non-disjoint) The above series are not disjoint. For example, the trees of the series C with $k = l = 1$ also belong to the series B. If $s > t$, C_1 becomes a part of E_3, up to a renaming of variables; etc.

Remark 5.5 (Adrianov's list) The list of ordinary unitrees compiled by Adrianov in 1989 consists of: series A, B, and C_1 with $s = t = 1$; series F, H, and I; and the sporadic tree Q.

Statement 5.6 (Three criteria) *The following three criteria are obvious but very useful and will be quite often used in the proof.*

1. *All the branches going out of a vertex of a unitree, except maybe one, are isomorphic as rooted weighted trees, with roots being the edges incident to the vertex in question.* This property must be true for every vertex of a unitree.
2. *If a passport (α, β) corresponds to a unitree then the weight distribution μ* (see Definition 2.20, page 9) *corresponding to this passport is also unique.*
3. *If a passport (α, β) corresponds to a unitree then the corresponding extended passport* (see Definition 2.21, page 9) *is also unique. Furthermore, the first sentence of this statement implies that the weights of all edges adjacent to a vertex, except maybe one, are equal.*

Concerning the first criterium, it suffices to note that if there exist three or more non-isomorphic branches going out of a vertex, then they can be cyclically ordered in at least two different ways.

Remark 5.7 (White vertices of degree 2) Notice that quite a few of our trees have all their white vertices being of degree 2, and thus, according to Convention 2.15, we can make these vertices implicit and draw the pictures as usual plane maps. The corresponding maps are shown in Figure 20.

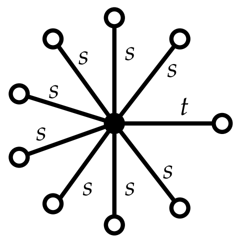

Figure 9: Series A: stars. The edge of the weight s is repeated $k \geq 0$ times.

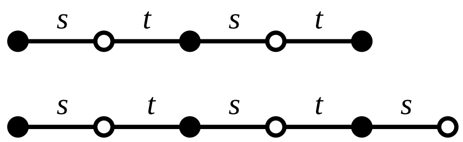

Figure 10: Series B: periodic chains of an arbitrary length. We distinguish the chains of even and odd length since they have passports of different type.

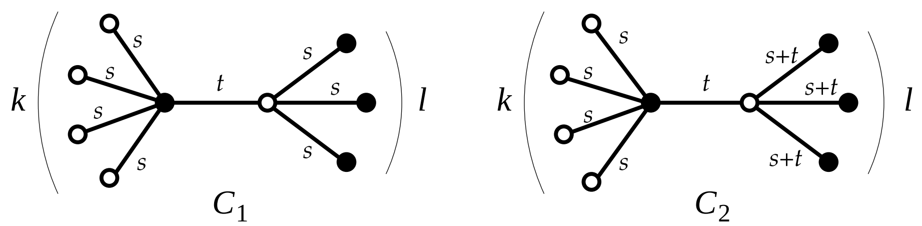

Figure 11: Series C_1 and C_2: brushes of diameter 3. Here $k, l \geq 1$.

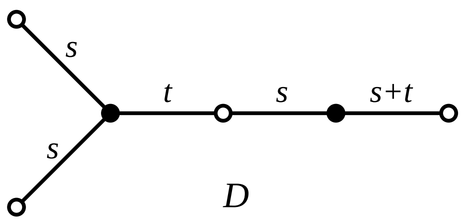

Figure 12: Series D: brushes of diameter 4. There are exactly two leaves of weight s and exactly one leaf of weight $s + t$.

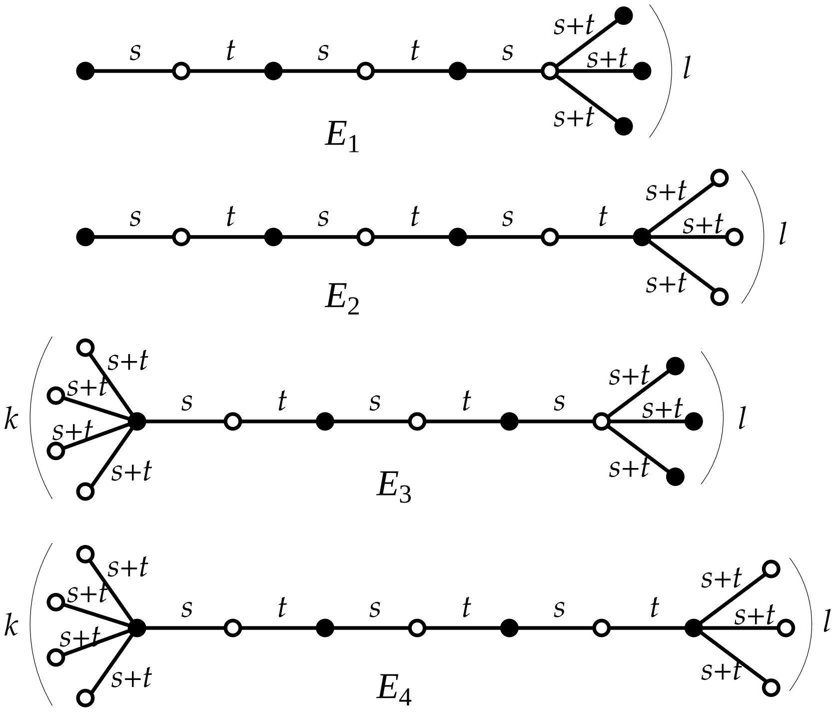

Figure 13: Series E: brushes of an arbitrary length. If there is a leaf of weight s, it is "solitary" on one of the ends of the brush; otherwise, all the leaves are of the weight $s+t$. The parameters $k, l \geq 1$.

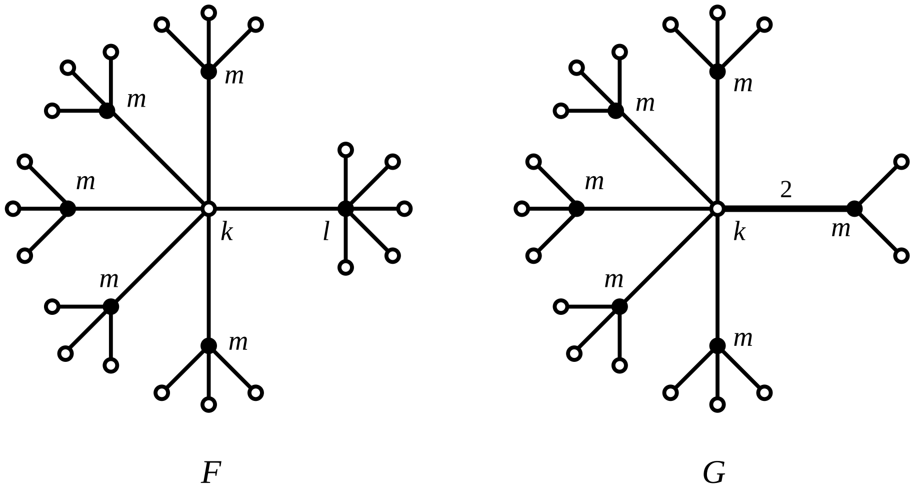

Figure 14: Two series of unitrees of diameter 4. In the trees of the series F all the edges are of weight 1; the degrees of vertices (except the leaves) are indicated. In the trees of the series G, there is exactly one edge of weight 2, all the other edges being of weight 1; note that this time the degrees of the black vertices are all equal.

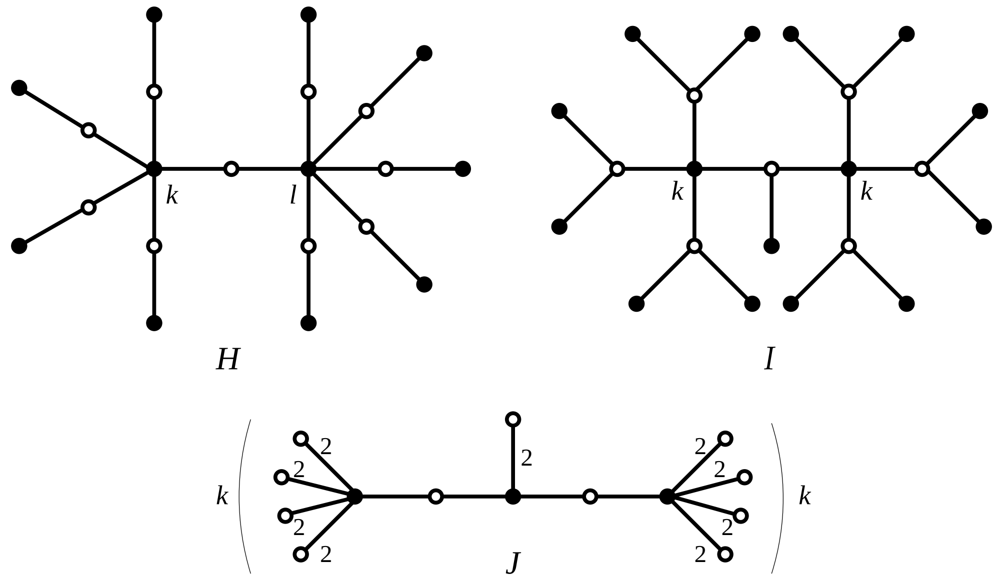

Figure 15: Three series of unitrees of diameter 6. In H and I, all edges are of weight 1. In H the black vertices which are not leaves are of degrees k and l which may be non-equal; in I they are of the same degree k. In H all white vertices are of degree 2; in I, they are all of degree 3. In J, the number of leaves of the weight 2 on the left and on the right is the same.

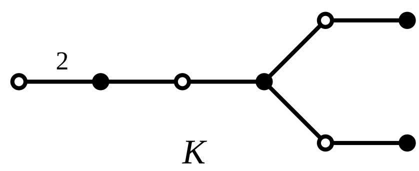

Figure 16: A sporadic unitree of diameter 5.

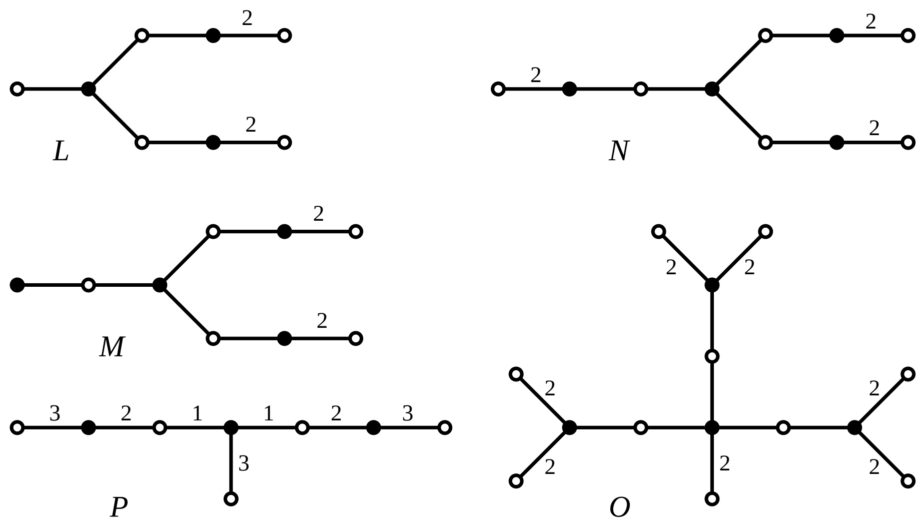

Figure 17: Five sporadic unitrees of diameter 6.

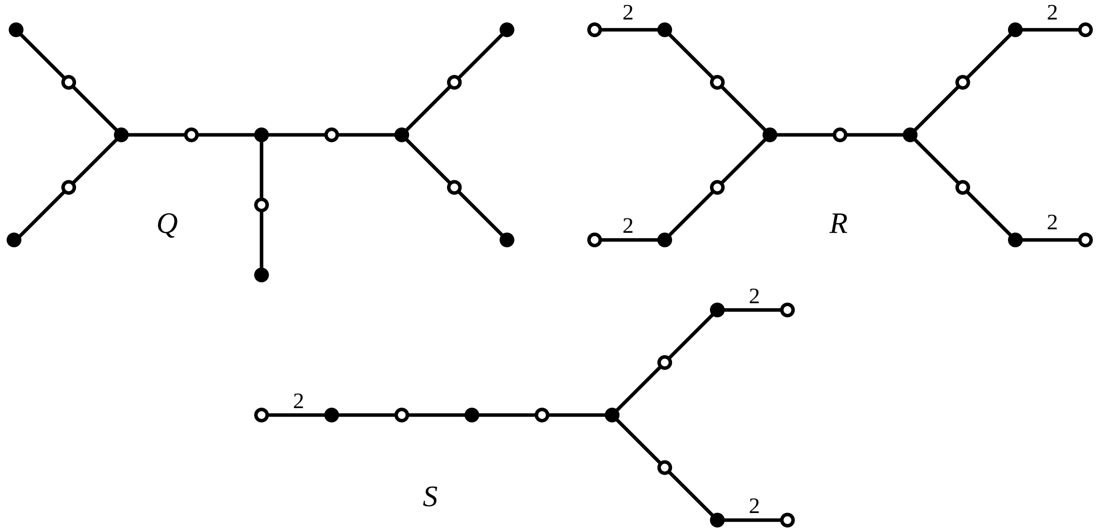

Figure 18: Three sporadic unitrees of diameter 8.

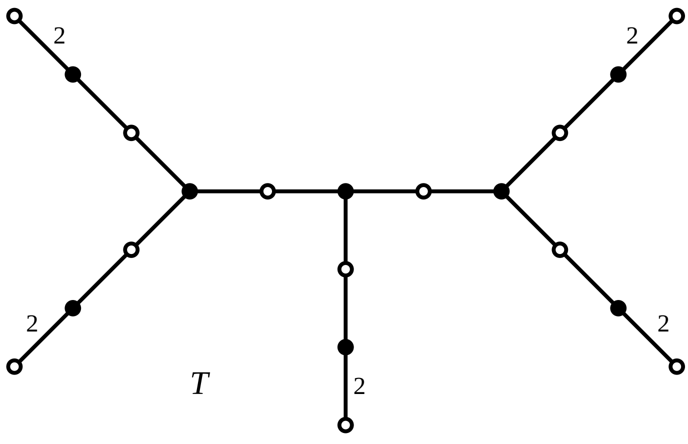

Figure 19: A sporadic unitree of diameter 10.

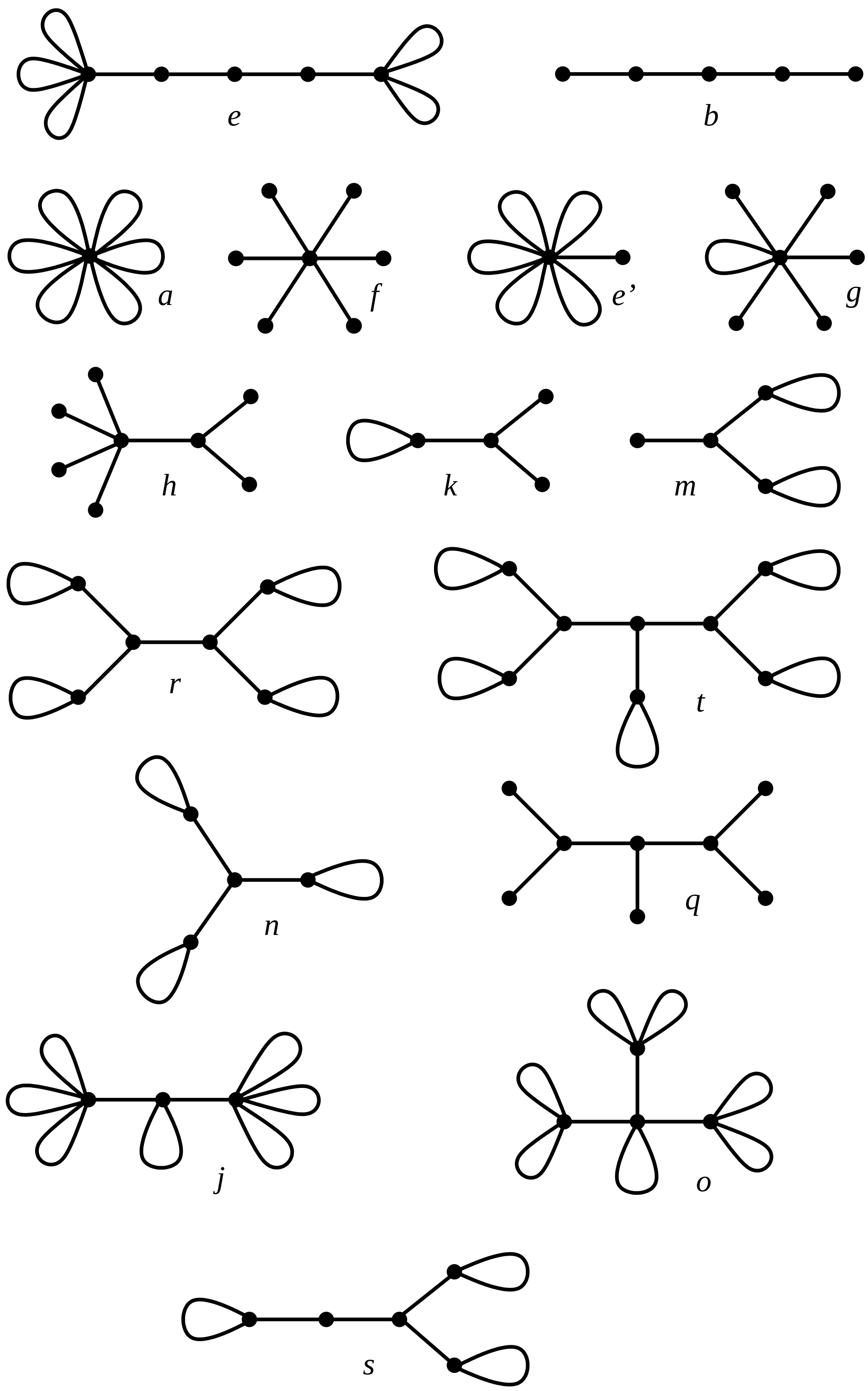

Figure 20: Unimaps. Small letters correspond to the capital letters by which we have previously denoted the unitrees; for example, the series e here is a particular case of the series E (when $s = t = 1$). Note that a, b, e', f, and g are particular cases of e. Note also that the series h and j and the sporadic unimaps k, m, n, o, q, r, s are not "particular cases" but just coincide with H, J, K, etc., respectively.

Before proceeding with the proof we must make an important "disclaimer":

Remark 5.8 (Weighted trees vs. weighted maps) When we speak about unitrees we mean their uniqueness among the weighted trees and not among the weighted maps. For example, we affirm that the tree on the left in Figure 21 is a unitree, and this is so in spite of the fact that there exists another weighted map with the same set of black-and-white vertex degrees. Indeed, in order to belong to the same combinatorial orbit, maps must have not only the same set of black-and-white vertex degrees but also the same set of face degrees. This is not the fact for the two pictures shown in Figure 21: the tree on the left represents a map with the face degree partition $\gamma = 4^1 1^6$, while for the map on the right the face degree partition is $\gamma = 3^1 2^1 1^5$. In particular, this map has a finite face of degree greater than 1 and therefore does not correspond to the polynomials we are looking for.

Notice that for the maps of Figure 20 all their finite faces are of degree 1.

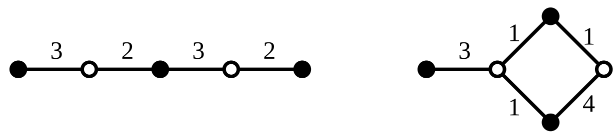

Figure 21: The tree on the left and the map on the right have the same set of black-and-white vertex degrees. Still, the tree is a unitree since it is unique among the weighted *trees*.

The strategy of the proof of Theorem 5.3 is as follows. We propose various transformations of trees changing the trees themselves while preserving their passports: this is a way to show that the combinatorial orbit of a given tree consists of more that one element. The trees which survive such a surgery are (a) those to which the transformation in question cannot be applied, and (b) those for which the transformed tree turns out to be isomorphic to the initial one. In this way we gradually eliminate all the trees which are not unitrees. Then, at the final stage, we show that all the trees which pass through this series of sieves are indeed unitrees.

5.2 Weight distribution

Sometimes, we can change not the topology of the tree but the distribution of its weights, while remaining in the same combinatorial orbit.

Lemma 5.9 (Condition on weights) *Let s, t, u be the weights of three successive edges of a unitree, as in* Figure 22, left. *If $s \le u$, then either $u = s$, or $u = s + t$. Similarly, if $s \ge u$, then either $u = s$, or $u = s - t$.*

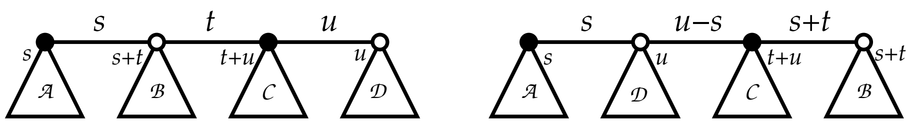

Figure 22: Weight exchange. If $s < u$ but $u \ne s + t$ then exactly two parts of the weight distribution μ have changed.

Proof. Rotating if necessary the tree under consideration, without loss of generality we may assume that $s < u$. Then we can construct the tree shown in Figure 22, right, replacing the weight u with $s + t$, the weight t with $u - s$, and exchanging the places of the subtrees $\mathcal{B}$ and $\mathcal{D}$. We see that the vertex degrees of the new tree are the same as in the initial one, while the weights of two edges have changed, unless $u = s + t$. Thus, if $s < u$ but $u \ne s + t$ then exactly two parts of the weight distribution μ have changed, which contradicts to the second assertion of Statement 5.6. $\square$

Corollary 5.10 (Adjacent edges) *If in a unitree there are two adjacent edges of the same weight s, and at least one of them is not a leaf, then $s = 1$.*

Proof. An edge which is not a leaf must be the middle edge of a path of length 3, see Figure 23. According to Lemma 5.9 we have either $x = s$ or $x = 2s$. If there is an edge of weight y attached to the middle edge of the path, like in Figure 23, then we have either $y = x$ or $y = x + s$, so the possible values for y are s, $2s$, or $3s$. Dealing in the same way with the other edges of the tree, we see that the weights of all of them are multiples of s. According to Assumption 5.1 this means that $s = 1$. $\square$

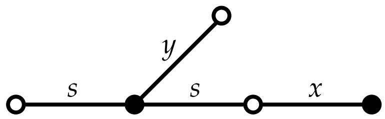

Figure 23: Two adjacent edges of the same weight s; one of them is not a leaf.

Proposition 5.11 (Path s, t, s) *Suppose that a unitree contains a path of three successive edges having the weights s, t, s. Then the only possible weights for all the edges of this tree are s, t, or $s + t$.*

Proof. Let us make an operation inverse to the one used in the proof of Theorem 3.3, that is, "rip" the tree along the edge of the weight t, as in Figure 24.

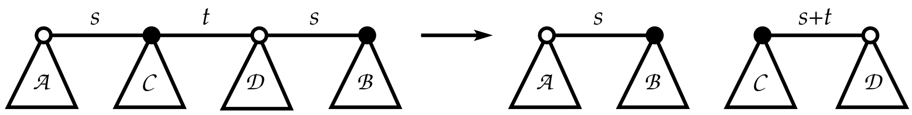

Figure 24: "Ripping" a tree: an operation inverse to that of the proof of Theorem 3.3.

Now suppose that in one of the subtrees $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ there exists an edge of a weight $x \neq s, t, s + t$, and try to sew the two trees together in a different way.

1. Suppose that in one of the subtrees $\mathcal{A}$ or $\mathcal{B}$ there exists an edge of a weight $x \neq s + t$. Sew it to the edge of the weight $s + t$ by the procedure explained in the proof of Theorem 3.3.
   (a) If $x < s + t$ then the weights of the four edges participating in the operations of ripping and sewing, instead of being s, t, s, x become $s + t - x, x, x, s$. Removing from the two sets the coinciding elements s and x, we get, on the one hand, s, t, and, on the other hand, $s + t - x, x$. These sets coincide only when $x = s$ or $x = t$.
   (b) If $x > s + t$ then, instead of s, t, s, x, we obtain $x - s - t, s + t, s + t, s$. These two sets cannot coincide at all since x is greater than every term in the second set.
2. Suppose that in one of the subtrees $\mathcal{C}$ or $\mathcal{D}$ there exists an edge of a weight $x \neq s$. Sew it to the edge of the weight s by the same procedure as above.
   (a) If $x > s$ then, instead of s, t, s, x, we get $x - s, s, s, s + t$. Removing s, s from both sets we get, on the one hand, t, x, and on the other hand, $x - s, s + t$. These sets coincide only when $x = s + t$.
   (b) If $x < s$ then the new set of weights is $s - x, x, x, s + t$; this set cannot coincide with s, t, s, x since $s + t$ is greater than every term in the second set.

Thus, the hypothesis that there exists an edge of a weight $x \neq s, t, s + t$ leads to a contradiction, which finishes the proof. $\square$

Remark 5.12 Notice that the operation of ripping-and-sewing introduced in the proof of Proposition 5.11 often leads to another tree even in the case when the weights of all the edges of the tree under consideration are s, t, or $s+t$. Below we will often use this operation and call it sts-operation.

Proposition 5.13 (Path s, t, $s+t$, I) *Suppose that a unitree contains a path of three successive edges having the weights s, t, $s+t$, and suppose also that $s \neq t$. Then the edge of the weight $s+t$ is a leaf.*

Remark 5.14 (What is a leaf) By abuse of language, we call *leaf* both a vertex which has only one edge (of whatever weight) incident to it, and also the edge itself incident to this vertex. We hope that this terminology will not lead to a confusion.

Proof of Proposition 5.13. The trees shown in Figure 25 obviously have the same passport. Suppose that the edge of the weight $s+t$ is not a leaf, so that the subtree $\mathcal{D}$ in the left tree is non-empty. Denote by x the weigh of any edge of $\mathcal{D}$ which is adjacent to the vertex q (these edges have the same weight according to the first assertion of Statement 5.6). By Lemma 5.9, the possible values of x are either t or $s+2t$. In the first case, we get a sub-path containing three edges of the weights t, $s+t$, t, and, according to Proposition 5.11, the only possible edge weights in such a tree could be t, $s+t$, and $s+2t$. But this contradicts to the supposition that we have already an edge of the weight s with $s \neq t$.

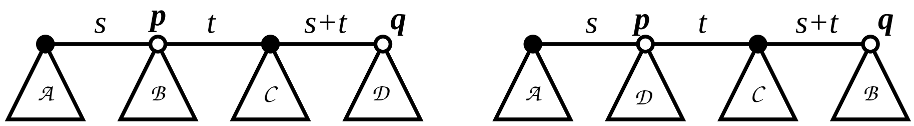

Figure 25: Suppose that the subtree $\mathcal{D}$ is not empty.

If, on the other hand, $x = s+2t$, then there are at least three *non-isomorphic* subtrees attached to the vertex p in the right tree, since there are edges of three different weight s, t, and $s+2t$ incident to this vertex. This situation violates the first assertion of Statement 5.6. □

Remark 5.15 ($s=t$) If $s=t$ then both arguments in the above proof are no longer valid, though it is still difficult to make left and right trees of Figure 25 isomorphic. Recall that if $s=t$ then $s=t=1$, see Corollary 5.10. And, indeed, the paths of length 4 with the weights $1,1,2,1$ are present in the series G (Figure 14), and the paths with the weights $1,1,2,3$ are present in the tree P (Figure 17).

Proposition 5.16 (Path s, t, $s+t$, II) *Suppose that a unitree contains a path of three successive edges having the weights s, t, $s+t$, and suppose also that the vertex adjacent to the edges of weights s and t has the valency $s+t$. Then the edge of the weight $s+t$ is a leaf.*

Proof. Keeping notation of Figure 25, it is enough to observe that if the subtree $\mathcal{B}$ is empty, then the two trees cannot be isomorphic since the right one has more leafs than the left one. This statement remains valid also for $s=t$. □

5.3 Brushes

A brush is a chain with two bunches of leaves attached to its ends: see formal definition below. Typical representatives of brushes are the trees shown in Figure 13.

In this section we classify all brush unitrees.

Definition 5.17 (Profound vertices and crossroads) A vertex of a tree is *profound* if, after having removed all the leaves from the tree, this vertex does not become a leaf. A vertex of a tree is a *crossroad* if it is profound and has at least three branches going out of it.

Definition 5.18 (Brush) A tree is called a *brush* if it does not contain crossroads.

Proposition 5.19 (Brush unitrees) *A brush unitree belongs either to one of the series A, B, C, D, E* (Figures 9, 10, 11, 12, and 13)*, or to the series F with the degree of the central vertex* $k = 2$*, or to the series G with the degree of the central vertex* $k = 3$ (Figure 14)*.*

Proof. In this part of the proof we only eliminate brush trees which are not unitrees. The uniqueness of the remaining brush trees will be proved later, in Section 5.7.

When all the edges of a tree are leaves we get the series A consisting of stars (Figure 9). According to the first criterium of Statement 5.6, only one of the leaves may have a weight different from the others. Similarly, trees of diameter three lead to the series C. The weight distribution is obtained according to Lemma 5.9.

Let us consider now the trees of diameter ≥ 5, and then return to the diameter 4 case. It follows from Lemma 5.9 and Proposition 5.16, that if s and t are weights of any two adjacent edged which are not leafs, then the weights s and t alternate along all the path connecting vertices from which leafs grow. Furthermore, since this path contains at least three edges, it follows from Proposition 5.11 that the only possible weight of a leaf which is not obtained by the further alternance of s and t is $s+t$. Now look at Figure 26, where an sts-operation is applied to a brush tree having $k \geq 2$ leaves of weight s on one of its ends. The tree thus obtained, shown on the right, is distinct from the initial one since it contains a crossroad. A similar surgery can be made if there are $l \geq 2$ leaves of the weight s or t (depending on the pairity of the diameter) on the right end of the tree. Thus, for the diameters ≥ 5 only the types B and E survive: if a bunch of leaves at an end of the tree contains two or more leaves then the weight of these leaves is $s+t$.

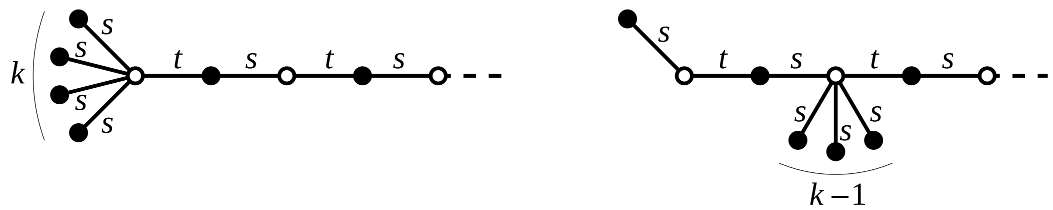

Figure 26: These trees have the same passport. They are not isomorphic since the right tree contains a crossroad while the left one does not. Therefore, if $k \geq 2$ then the weight of the leaves must be $s+t$ and not s (or t).

The above argument ("the new tree contains a crossroad") fails for the brush trees of diameter 4. Therefore, the diameter 4 case needs a special consideration. Let us take a diameter of the tree, that is, a chain of length 4 going from one of its ends to the other. By Lemma 5.9 the sequence of the weights of its three first edges can be either s, t, s, or s, t, $s+t$, or $s+t$, t, s. Consider first the case s, t, s. Using Lemma 5.9 again we see that a tree has one of two forms shown in Figure 27 on the left.

The first of these forms (above, left) can be transformed in a way shown on the right. The new tree is not equal to the initial one, unless either it belongs to the type B (where $k = l = 1$), or $s = t = 1$, which is a particular case of the series F, where the degree of the central vertex is $k = 2$. For the second form (below, left), the operation shown on the right cannot be applied when $k = 1$, that is, when the tree belongs to the series E_1; and it does not change the tree when $k = 2$ and $l = 1$, which corresponds to the series D.

Assume now that the sequence of weights of the edges of a diameter starts from s, t, $s+t$, so that the edge of the weight $s+t$ is not a leaf. In this case by Proposition 5.13 we necessarily have $s = t = 1$ and a tree either belongs to the series G, where the degree of the central vertex is $k = 3$, or has the form shown in Figure 28 on the left. In the last case, however, the tree under consideration is not a unitree, which can be seen by a transformation shown on the right.

Finally, if the sequence of weights of edges of a diameter starts from $s+t$, s, t, then either a tree belongs to the series E_4, or the sequence of weights of the edges of a diameter is $s+t$, s, t, $s-t$. In the latter case, however, Proposition 5.13 yields that $s-t = t = 1$, implying that a tree is the one shown in Figure 28 on the left. □

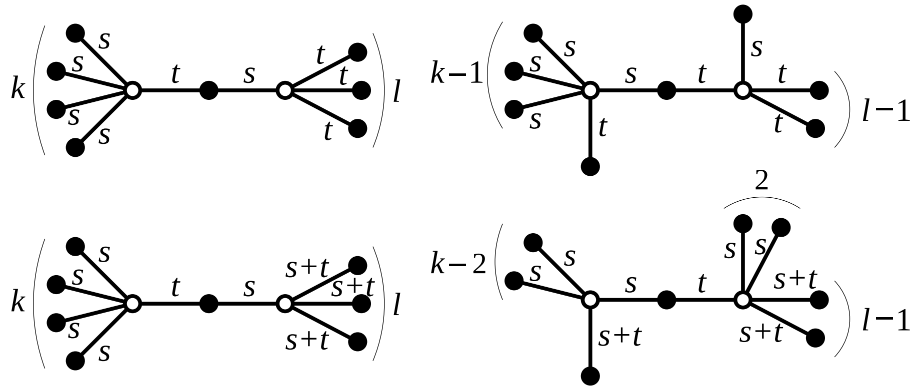

Figure 27: The trees on the upper level have the same passport; they are different if $s \neq t$ and at least one of the parameters k, l is greater than 1. The trees on the lower level also have the same passport; they are different if either $k > 2$, or $l > 1$, or both.

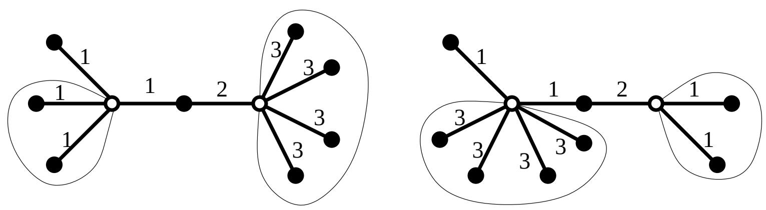

Figure 28: The bunch of leaves of the weight 1 which is transplanted from left to right can be empty, if there is only one leaf on the left.

5.4 Trees with repeatig branches of height 2

From now on we will assume that the trees we consider are not brushes, that is, they contain at least one crossroad. Recall that a crossroad is a profound vertex at which three or more branches meet, see Definition 5.17. A typical tree with a crossroad is shown in Figure 29. According to Statement 5.6, all the branches attached to the crossroad, except maybe one, are isomorphic as rooted trees, with their root edges (shown in thick lines in the figure) being incident to the crossroad. We call these branches *repeating*; in the figure they are denoted by the same letter $\mathcal{R}$; the subtrees of $\mathcal{R}$ attached to the root are denoted by $\mathcal{R}'$. The subtrees $\mathcal{R}'$, are non-empty since otherwise the vertex to which $\mathcal{R}$ and $\mathcal{N}$ are attached would not be profound. The number of branches of the type $\mathcal{R}$ can be two or more, but the majority of the transformations given below involve only two branches; therefore, in the majority of pictures we will draw only two repeating branches.

By convention, we suppose that *the branch $\mathcal{N}$ is always non-empty.* If all the branches meeting at the crossroad are isomorphic to $\mathcal{R}$ and might therefore be all considered as repeating, we take an arbitrary one of them and, somewhat artificially, declare it to be the "non-repeating" branch $\mathcal{N}$. The subtree $\mathcal{N}'$ has a right to be empty.

The roots of repeating branches are adjacent edges which are not leaves. Therefore, according to Corollary 5.10, their weights must be equal to 1. Finally, the *height* of a repeating or non-repeating branch is the distance from its root vertex (that is, the crossroad) to its farthest leaf.

In the previous section we classified the brush unitrees, which are by definition unitrees without crossroads. In this section we establish a complete list of all possible trees whose repeating branches all have the height 2.

Proposition 5.20 (Branches of height 2) *A unitree whose repeating branches are all of the height 2 belongs to one of the types F, G, H, or K.*

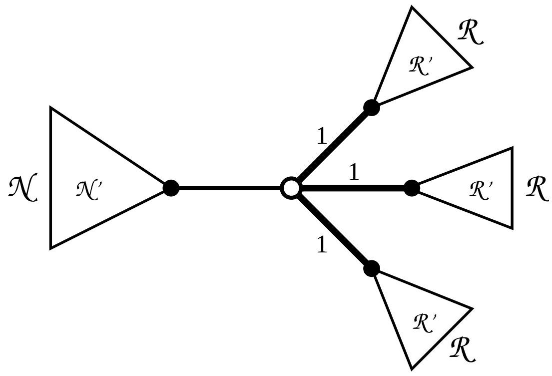

Figure 29: A typical tree with a crossroad. The branch $\mathcal{N}$ may, or may not be isomorphic to $\mathcal{R}$.

Proof. First of all observe that weights of leaves of repeating branches cannot be equal to 2 since otherwise the transformation shown in Figure 30 leads to a non-isomorphic tree (the tree on the right has fewer leaves than the one on the left). Thus, the weights of these leaves are equal to 1.

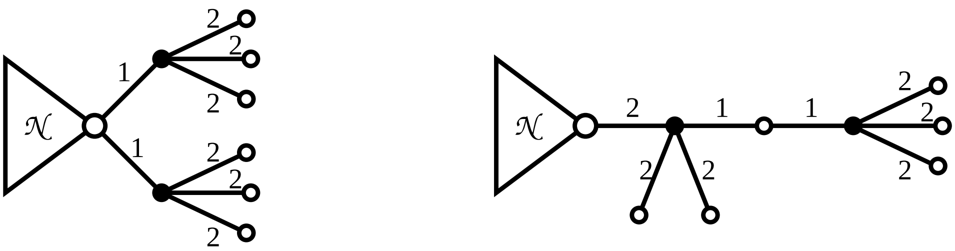

Figure 30: A transformation of repeating branches of height 2 with leaves of weight 2.

Consider the non-repeating branch $\mathcal{N}$. According to Lemma 5.9, its root edge can only be of weight 1 or 2. The case $\mathcal{N}' = \varnothing$ (that is, when $\mathcal{N}$ consists of a single edge, see Figure 29) is a particular case of the series F and G. Suppose then that $\mathcal{N}' \neq \varnothing$.

If the non-repeating branch is of the height 2, that is, if it is a root edge with a bunch of leaves attached to it, then these leaves could be of weight 1 or 2 when the root edge is of weight 1, and they could be of weight 1 or 3 when the root edge is of weight 2. However, the leaves of the weights 2 and 3 are impossible, as can be seen in Figure 31. In this figure, we take all the repeating branches but one and re-attach them to one of the leaves of the non-repeating branch. This transformation produces a tree with fewer leaves. Thus, all the leaves of the non-repeating branch must be of weight 1.

In addition, when the root edge of the non-repeating branch is of weight 2 this branch cannot be isomorphic to the repeating branches. This observation implies that the degree of the black vertex lying on this branch must be equal to the degrees of the black vertices of repeating branches since otherwise an exchange of leaves between repeating and non-repeating branches could be possible. Thus, the only remaining possibilities are the trees of the types F and G, see Figure 14 (Page 18).

Consider now the case of a non-repeating branch of height ≥ 3. Suppose first that the vertex q, which is the nearest neighbor of the crossroad vertex p when we move along the non-repeating branch, is itself a crossroad. According to our supposition, the tree does not have repeating branches of the height greater than 2. Hence, the repeating branches growing out of q are of height 2. But then a leaf $\mathcal{L}$ of such a branch can be interchanged with a repeating branch $\mathcal{U}$ growing out of p, see Figure 32. This operation would create at least three different trees attached to p: one of them would be of height 1, another one of height 2, and a third one of height 4. Thus, this possibility is ruled out.

Suppose next that the vertex q is not a crossroad. Then the tree looks like the one in Figure 33,

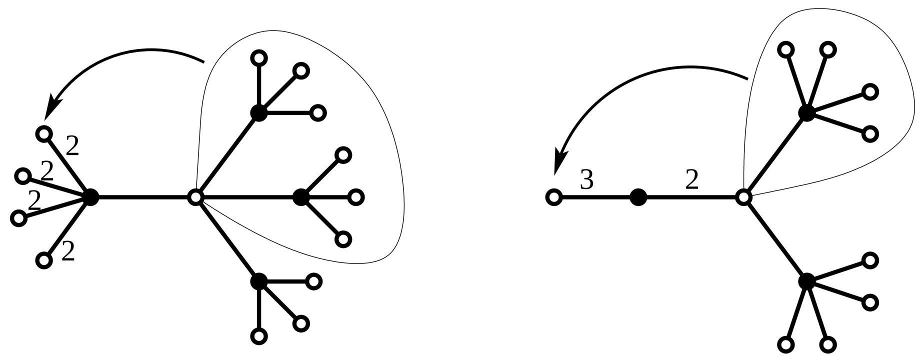

Figure 31: Non-repeating branch of height 2. These transformations show that the weight of its leaves cannot be 2 or 3.

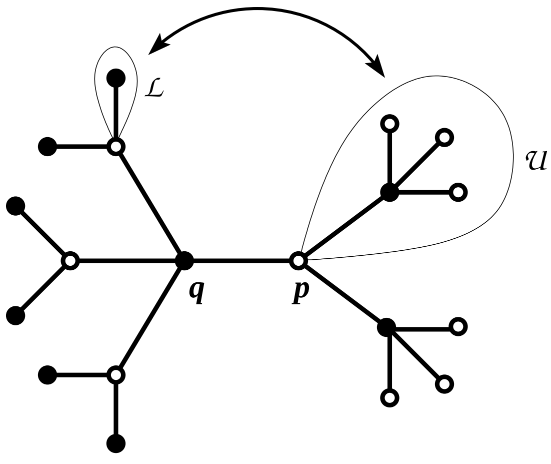

Figure 32: A leaf $\mathcal{L}$ can be interchanged with a repeating branch $\mathcal{U}$.

top left, where $\mathcal{A}$ is non-empty. A priori, there are four possibilities for the values (s, t), namely, $(1, 1)$, $(1, 2)$, $(2, 1)$, and $(2, 3)$. The case $(2, 3)$ can be immediately ruled out since the edge of the weight 3 should be a leaf, but we have supposed that $\mathcal{A} \neq \varnothing$.

The cases $(s, t) = (1, 1)$ or $(2, 1)$ can be treated together. When $t = 1$ we can re-attach $\mathcal{A}$ to one of the leaves of the repeating branches, as is shown in the same figure on the top right. Among the branches attached to the vertex p of the tree thus obtained there is only one branch of a height greater than 2: it is $\mathcal{W}$. Therefore, all the remaining branches are repeating, so we may conclude that $s = 1$ (the case $s = 2$ is impossible), and all the repeating branches have only one leaf. Thus, the tree looks like the one on the bottom left in Figure 33, where two possibilities may occur: either $u = 1$; or $u = 2$, and then, according to Proposition 5.16, $\mathcal{A}' = \varnothing$, so the edges of the weight $u = 2$ are leaves.

In the first case, we can exchange the repeating branches attached to the vertices p and r. Therefore, they all must be equal, and we get a tree of the type H, see Figure 15 (Page 19). In the second case, we can interchange one of the leaves of weight 2 with two repeating branches attached to p. The only tree which does not change after this transformation is the one which has exactly one leaf of weight 2 and exactly two repeating branches attached to p, that is, the tree K, see Figure 16 (Page 19).

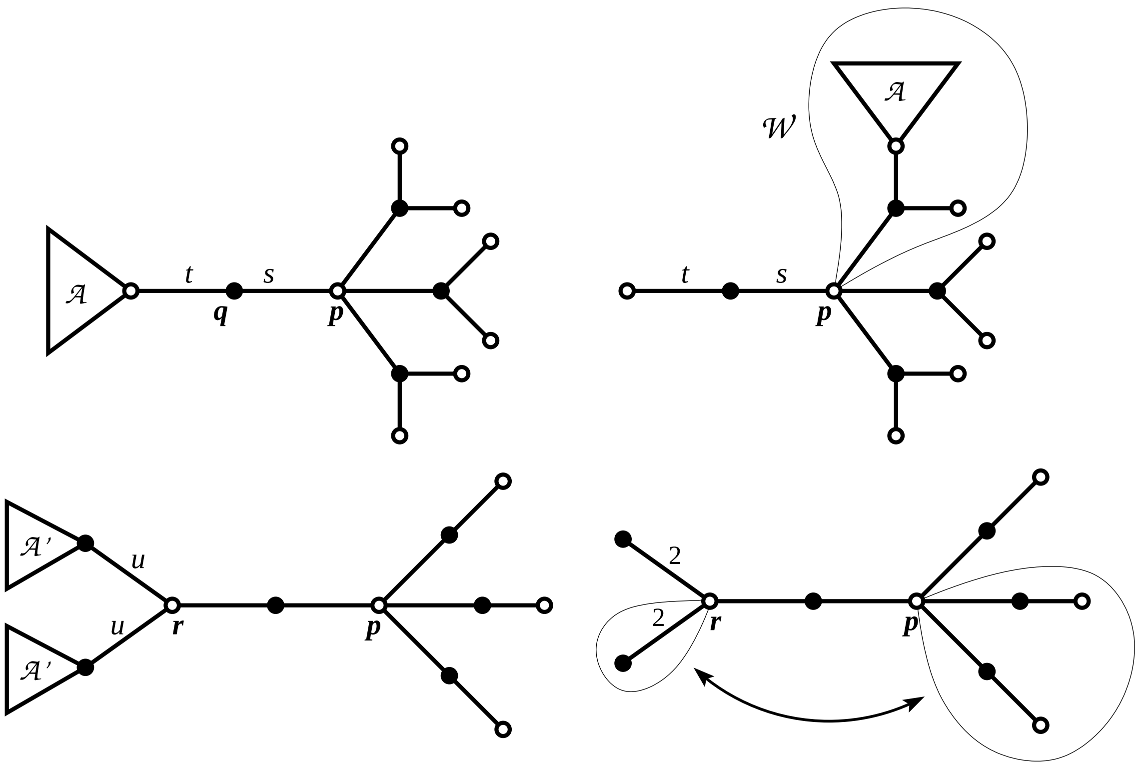

Figure 33: Illustration to the proof of Proposition 5.20.

There remains the last case to be ruled out: when the tree shown in Figure 33, top left, has $s = 1$ and $t = 2$; see also Figure 34, left. In this case we can interchange the subtree $\mathcal{A}$ with all but one repeating branches, see the tree on the right of Figure 34. We see that $\mathcal{A}$ must consist of several copies of the branch $\mathcal{U}$ since otherwise $\mathcal{A}$ should consist of copies of the longer branch at p, and we would get repeating branches of the height greater than 2. Then, we may take the left tree of Figure 34, cut all the repeating branches form p and re-attach them to one of the leaves of $\mathcal{A}$. This operation will necessarily produce a different tree since the only edge of the weight 2 will be now at distance 1 from the leaf while it was at distance 2 in the initial tree. Therefore, this case is impossible.

Proposition 5.20 is proved. $\square$

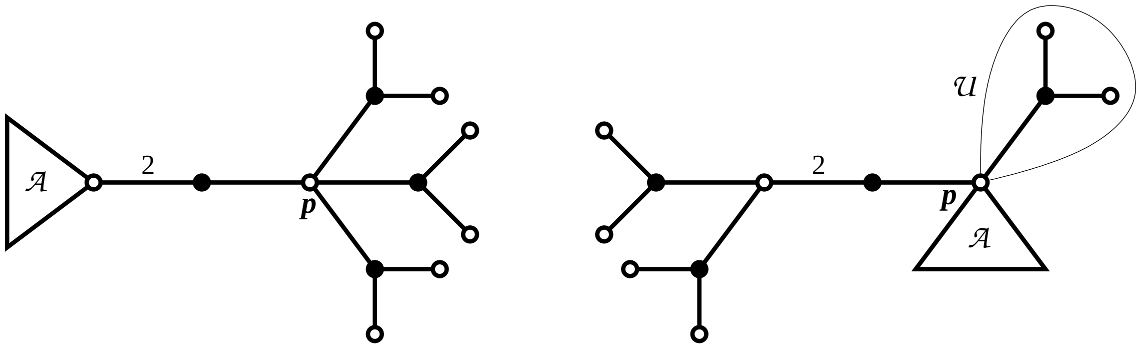

Figure 34: Illustration to the proof of Proposition 5.20.

5.5 Trees with repeating branches of the type $(1, s, s+1)$

If a unitree has a crossroad from which grow repeating branches of height > 2, then these branches "start" either with a path 1, s, $s+1$, or with a path 1, t, 1, where s and t here may be equal to either 1 or 2 (see Figures 36 and 43). In this subsection we classify unitrees which have no crossroads of the second type. We start with the following lemma.

Lemma 5.21 *If a unitree has a repeating branch of the type* $(1, s, s+1)$*, then this branch has one of the two forms shown in* Figure 35. *Furthermore, in the second case the unitree is necessarily the tree* P.

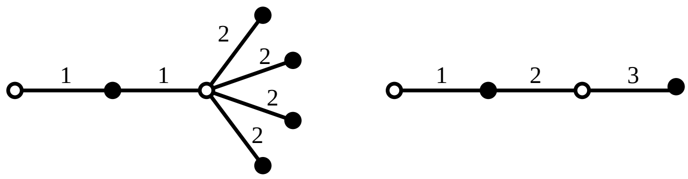

Figure 35: Illustration to the proof of Lemma 5.21.

Proof. First of all observe that the subtrees $\mathcal{A}$ and $\mathcal{C}$ in Figure 36 can be interchanged, and if one of them was empty while the other was not, this operation would change the number of leaves, so that the tree in question could not be a unitree. We will show now that the assumption that both trees $\mathcal{A}$ and $\mathcal{C}$ are not empty also leads to a contradiction (so that, in fact, both of them are empty).

Since the tree $\mathcal{V}$ is isomorphic to a subtree of $\mathcal{U}$ and is therefore distinct from $\mathcal{U}$, if $\mathcal{A}$ is not empty, then it consists of a certain number of copies of $\mathcal{U}$ or of $\mathcal{V}$. The first case is impossible since $\mathcal{A}$ is a subtree of $\mathcal{U}$. Therefore, $\mathcal{A}$ consists of a certain number of copies of $\mathcal{V}$ implying that $\mathcal{C}$ is a proper subtree of $\mathcal{A}$. Now, interchanging $\mathcal{A}$ and $\mathcal{C}$ in every repeating branch we may prove in the same way that that $\mathcal{A}$ is a proper subtree of $\mathcal{C}$, implying the contradiction needed. Thus, $\mathcal{A}$ and $\mathcal{C}$ are empty. In particular, $\mathcal{B}$ is merely a collection of leaves of weight $s+1$.

Assume now that $s = 2$. Then our tree must look as in Figure 37, top left, where the number of repeating branches at the vertex p might be two or more. Let us take two of these branches, apply the transformation shown on top right, and look what takes place at the vertex q. According to the first assertion of Statement 5.6, all the subtrees growing out of this vertex, except maybe one, must be isomorphic. This can only happen when $k = 1$, and the subtree growing from q to the left is isomorphic to the one growing from q to the right. Therefore, before the transformation there were exactly two (and not more) repeating branches at p, and the subtree $\mathcal{N}$ was reduced to a single leaf of weight 3. The resulting situation is shown in Figure 37, bottom left. In this case, our transformation can still be applied, but it leads to a tree isomorphic to the initial one. The unitree thus obtained is P, see Figure 17. □

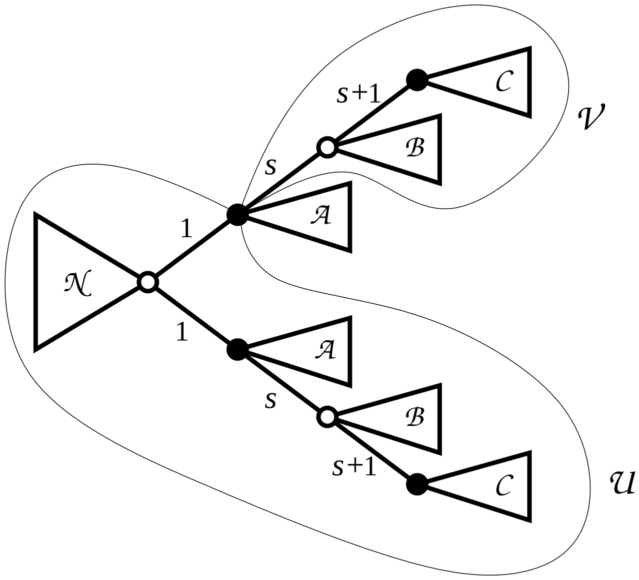

Figure 36: Illustration to the proof of Lemma 5.21: the subtrees $\mathcal{A}$ and $\mathcal{C}$ are empty; the subtree $\mathcal{B}$ is a bunch of leaves of weight $s+1$.

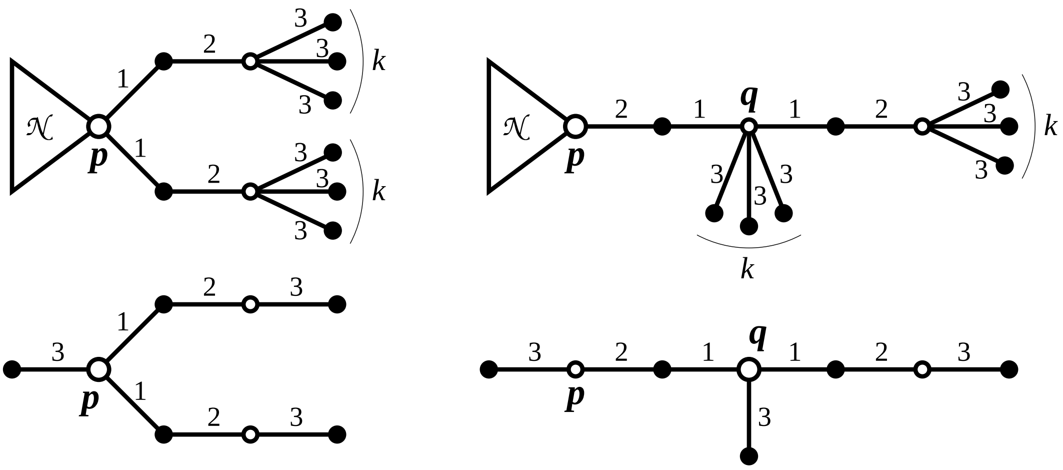

Figure 37: Illustration to the proof of Lemma 5.21: transformations of repeating branches of height 3 with the weight sequence $1, 2, 3$.

Proposition 5.22 (Branches of the type $(1, s, s+1)$**)** *A unitree which has at least one crossroad of type* $(1, s, s+1)$ *but no crossroads of type* $(1, t, 1)$ *belongs to one of the types J, L, N, M, O, P, R, or S.*

Proof. In view of Lemma 5.21 we may assume that the repeating branches have the form shown on Figure 35 on the left. Suppose first that the number of the repeating branches is three or more, and apply the transformation shown in Figure 38, that is, interchange the positions of a leaf of weight 2 and of a pair of repeating branches. If the number of the repeating branches was more than three then the principle "all branches except maybe one are isomorphic" would be violated at the vertex p. The same principle would be violated at the vertex q if the number of leaves in a repeating branch was more than two. Therefore, the number of repeating branches is three, and our transformation looks as is shown in Figure 38, bottom. Now, returning to the vertex p and applying once again the same principle we see that the non-repeating branch $\mathcal{N}$ is either a single leaf of weight 2, and then we get the tree O (Figure 17); or, otherwise, $\mathcal{N}$ should be equal to the long branch growing out of p; but then the same principle would be violated at q.

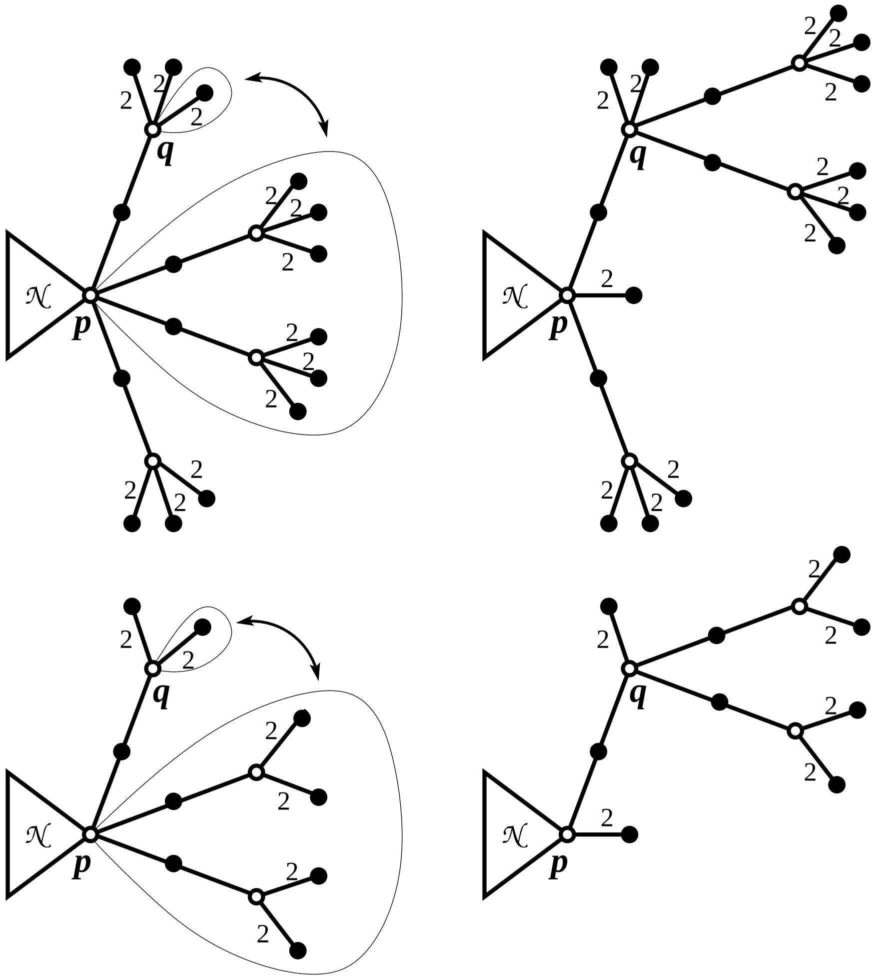

Figure 38: Illustration to the proof of Proposition 5.22: transformations of repeating branches of height 3 with the weight sequence $1, 1, 2$.

Suppose next that the number of repeating branches is two. Then the starting edge of the non-repeating branch $\mathcal{N}$, is either of weight 2, and then, according to Proposition 5.16, it is a leaf, and we get the tree J (Figure 17), or it is of weight 1. In the latter case it does not have to be a leaf, though this situation imposes another constraint: the repeating branches must have only one leaf, otherwise the transformation shown in Figure 39 can be applied.

What remains is to study more attentively the structure of the non-repeating branch $\mathcal{N}$. Here we will consider the following cases:

- The height of $\mathcal{N}$ is 1, that is, $\mathcal{N}' = \varnothing$.
- The height of $\mathcal{N}$ is 2.
- The height of $\mathcal{N}$ is 3 or more, and $\mathcal{N}$ starts with a path having the weights 1, s, 1 where s is equal to 1 or 2.

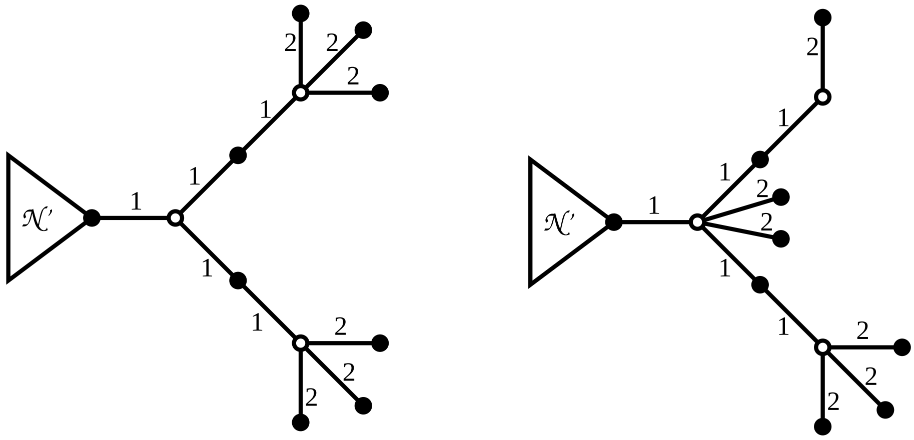

Figure 39: Illustration to the proof of Proposition 5.22: two repeating branches.

- The height of $\mathcal{N}$ is 3 or more, and $\mathcal{N}$ starts with a path having the weights 1, s, $s+1$ where s is equal to 1 or 2.

The height of $\mathcal{N}$ equal to 1 case is trivial: we get the tree L (Figure 17).

The height of $\mathcal{N}$ equal to 2 case is illustrated in Figure 40. If the weight of the leaves of the non-repeating branch is equal to 2 then, whatever is their number, the reattachment shown on the left changes the tree since the new tree has one leaf less than the initial one. If the weight of the leaves of the non-repeating branch is equal to 1 then the reattachment shown on the right also changes the tree unless there is only one leaf in the non-repeating branch. The latter case gives us the tree M (Figure 17).

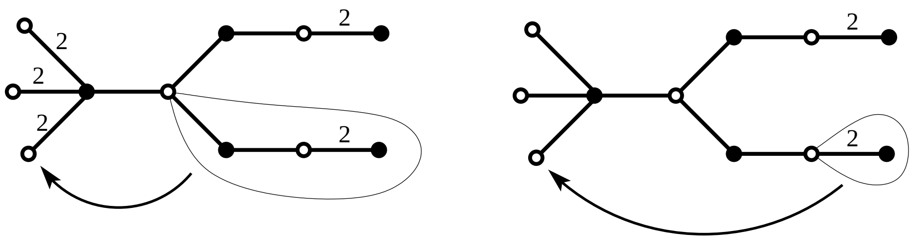

Figure 40: Illustration to the proof of Proposition 5.22: non-repeating branch of height 2.

Assume now that the height of the non-repeating branch is ≥ 3, and this branch contains a path with the weights 1, s, 1, see the upper tree in Figure 41. First of all, we remark that the subtree $\mathcal{A}$ may be interchanged with a chain of length 2 attached to the vertex p. Then, according to the principle "all branches except maybe one are isomorphic", two situations may occur. First, we could thus create two repeating branches $\mathcal{U}$ attached to the vertex q, see the tree in the middle. But such a tree would contain repeating branches of the type $(1, s, 1)$ which contradicts to our supposition. The other possibility is that $\mathcal{A}$ is equal to the chain which was attached to p. Then we get a vertex r (see the lower tree) which is of degree 2 and is incident to two edges of weights 1 and 1. Therefore, according to Proposition 5.16, the edge of weight s, which is not a leaf, cannot have weight 2; hence, $s = 1$. Finally, we affirm that $\mathcal{C} = \varnothing$, otherwise it could be reattached to the vertex p and we would get three different trees attached to q. The resulting tree is shown in Figure 41, bottom. If $\mathcal{B} = \varnothing$ we get the tree S (Figure 18). If $\mathcal{B} \neq \varnothing$ then, again according to the principle "all branches except maybe one are isomorphic", $\mathcal{B}$ must be equal to a chain of weights 1, 1, 2, and we get the tree R (Figure 18), since otherwise a tree would contain repeating branches of the type $(1, s, 1)$. (Note that in the last case we obtain the tree T which is considered in Proposition 5.23 which treats the case of repeating branches of the type $(1, t, 1)$.)

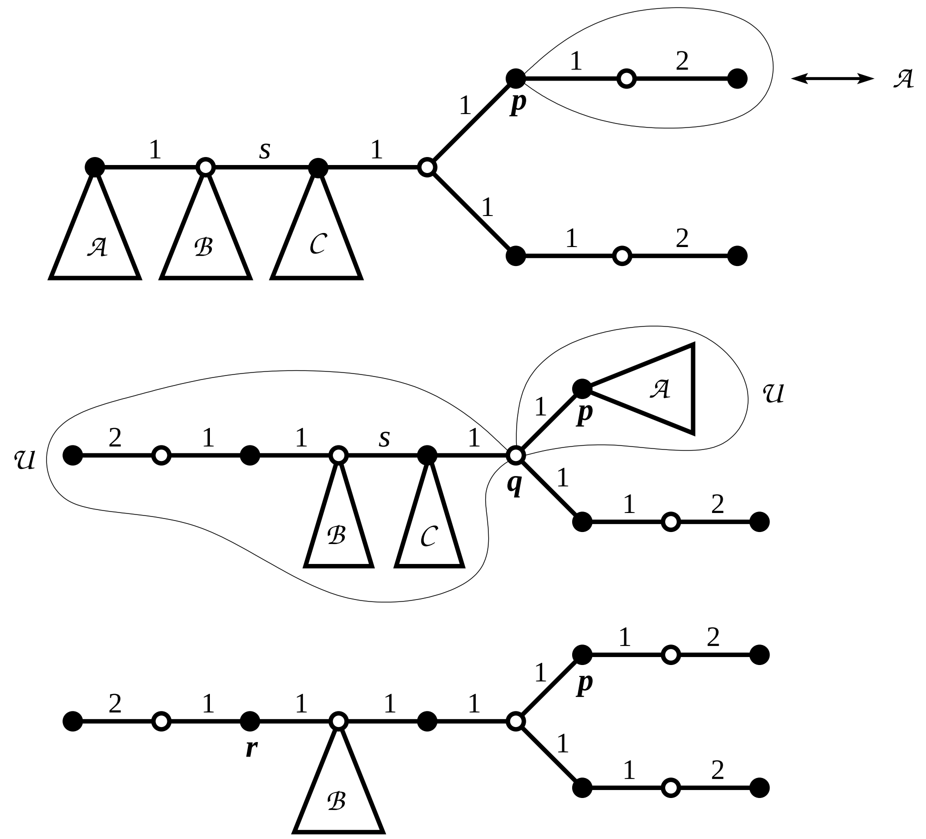

Figure 41: Illustration to the proof of Proposition 5.22: non-repeating branch containing a path with the weights 1, s, 1.

Finally, consider the case when the non-repeating branch is of height ≥ 3 and contains a path with the weights 1, s, $s+1$, see Figure 42. We affirm that in this case $s = 1$ and all the three subtrees $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are empty, so that we get the tree N of Figure 17 (and what we call "non-repeating branch" is in this case equal to the repeating branches). Indeed, the tree contains a path with the weights 1, 1, 1; therefore, according to Proposition 5.11, the only possible weights are 1 and 2, so $s = 1$. Now, if $\mathcal{A} \neq \varnothing$ then it could be reattached to the vertex q thus producing a tree with one more leaf. Therefore, $\mathcal{A} = \varnothing$. Then, $\mathcal{C}$ is also empty since $\mathcal{A}$ and $\mathcal{C}$ can be interchanged. Finally, if $\mathcal{B} \neq \varnothing$ then there are two possibilities. Either $\mathcal{B}$ is a bunch of leaves of weight 2; but then it can be reattached to the vertex p. Or $\mathcal{B}$ is a number of copies of the long branch growing out of the vertex r; but then, once again, we would create repeating branches of the type $(1, t, 1)$.

Proposition 5.22 is proved. $\square$

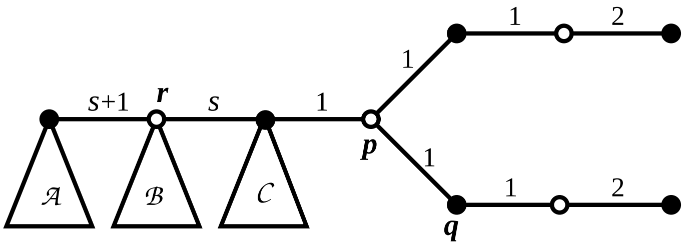

Figure 42: Illustration to the proof of Proposition 5.22: non-repeating branch containing a path with the weights 1, s, $s+1$.

5.6 Trees with repeating branches of type $(1, t, 1)$

In this subsection we classify unitrees which have crossroads of the type $(1, t, 1)$.

Proposition 5.23 (Branches of type $(1, t, 1)$) *A unitree which has at least one crossroad of type $(1, t, 1)$ belongs to one of the types I, Q, or T.*

Proof. First of all, observe that by the first assertion of Statement 5.6 the subtree $\mathcal{B}$ is a collection of copies of the subtree $\mathcal{U}$, and the subtree $\mathcal{A}$ is a collection of copies of the subtree $\mathcal{V}$ (see Figure 43). Further, an *sts*-operation, applied to the first tree of Figure 43 gives the second tree shown in this figure. This image implies that there are only two repeating branches growing from the vertex p, otherwise the tree would certainly change. Now, looking at the vertex q of the second tree of Figure 43 we see that either $\mathcal{N} = \mathcal{U}$ or $\mathcal{N} = \mathcal{W}$. If $\mathcal{N} = \mathcal{W}$ then the initial tree would look like the third tree of the same figure. Then we could once again apply an *sts*-transformation and make the long branch even longer, and one of the repeating branches, shorter (see the fourth tree of the figure), which would give us three branches of three different heights attached to p. Hence, $\mathcal{N} = \mathcal{U}$. In particular, we have proved that whenever a unitree has a crossroads of type $(1, t, 1)$ the corresponding non-repeating branch is a subtree of the repeating branch.

There is a natural notion of a *center* of a tree. We cut off all the leaves, then repeat this operation again and again until there remains either a vertex (then this vertex is the center) or an edge (and then its ends are both called centers). It is clear that any transformation of a unitree must send a center to a center.

Now, it follows from what have already been shown before that the tree we consider here has only one center, and in the first tree the center is p while in the second one the center is r. Hence, the vertex p of the first tree must correspond to the vertex r of the second one, and thus we must have $t = 1$ and $\mathcal{B} = \mathcal{N} = \mathcal{U}$. Therefore, the tree has the form shown in Figure 44, with the same number $l \geq 1$ of branches growing out of the vertices u and v.

If $\mathcal{C} = \varnothing$ then we get a tree of the type I. Thus we may assume that $\mathcal{C}$ is non-empty. Observe first that $l = 1$. Indeed, if $l > 1$ then $\mathcal{V}$ is a repeating branch. Furthermore, since $\mathcal{C}$ is non-empty, $\mathcal{V}$ is either $(1, t, t+1)$-branch or $(1, t, 1)$-branch. The first case is impossible by Lemma 5.21, while

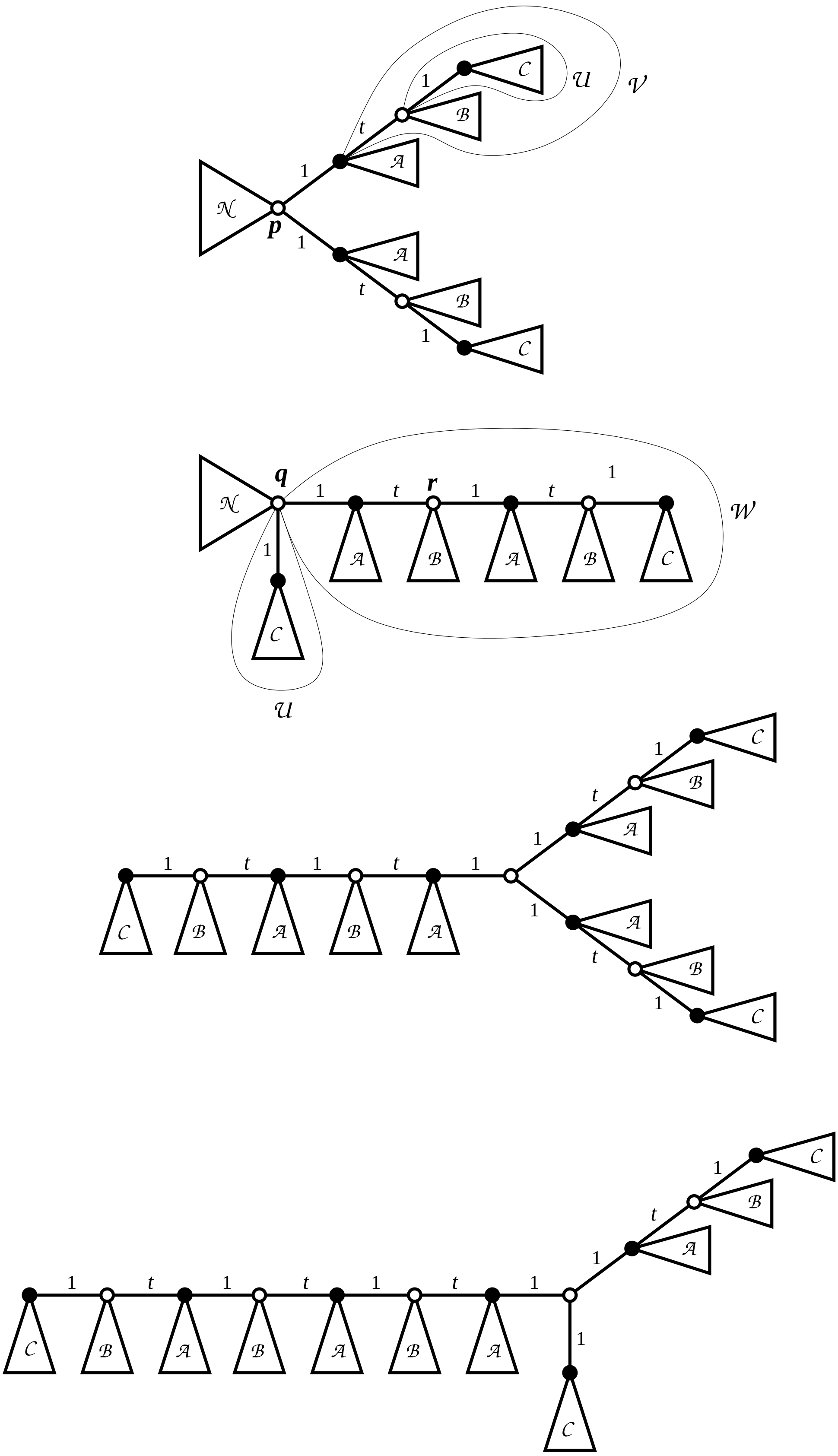

Figure 43: Illustration to the proof of Proposition 5.23.

the second case is impossible since, as we have shown in the previous paragraph, in this case the corresponding non-repeating branch should be a subtree of $\mathcal{V}$, which is not the case.

If $\mathcal{C}$ is a collection of leaves, then the transformation of Figure 30 shows that all leaves are of weight 1. Moreover, $\mathcal{C}$ contains not more than one leaf since otherwise we could transport all the other leaves to the vertex v, which would change the tree. Therefore, in this case we get the tree Q. Finally, if $\mathcal{C}$ is not a collection of leaves, then $\mathcal{W}$ is a repeating branch, and Lemma 5.21 implies that $\mathcal{W}$ has the form shown in Figure 35 on the left, where the number of leaves is equal to one since otherwise we could transport all the leaves but one to the vertex w. Therefore, in this case we get the tree T.

Proposition 5.23 is proved. □

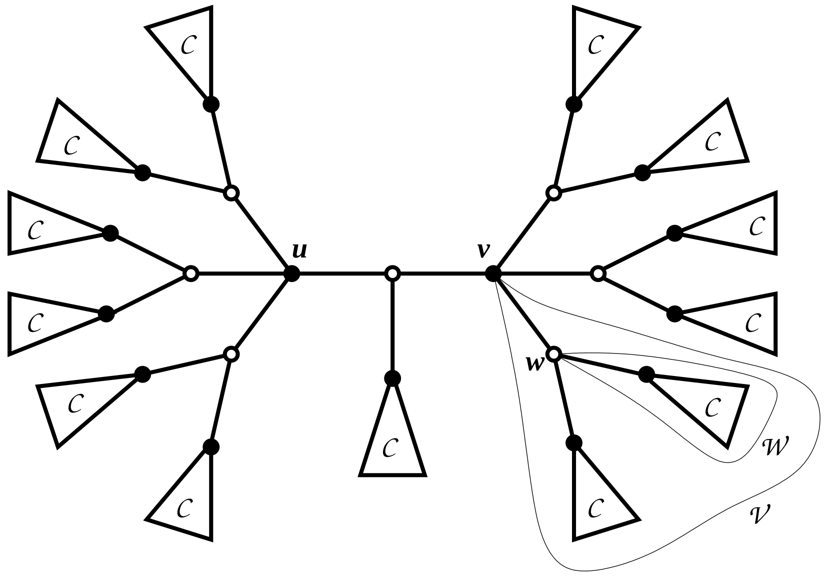

Figure 44: Illustration to the proof of Proposition 5.23.

5.7 Proof of the uniqueness of unitrees

Our main tool will be "cutting and gluing leaves", though these operations will be carried out not with the trees themselves but with their passports; the trees must be kept in mind for an intuitive understanding of the proof.

(A) There is only one black vertex (see Figure 9); therefore, all white vertices must be adjacent to it, which means that they are leaves. The uniqueness is evident.

(B) Let us consider, for example, the case of an odd length, and examine not the tree itself but its passport. We see that there are two vertices of degree s, a black one and a white one, while all the other vertices are of degree $s + t$. These latter vertices cannot be leaves (otherwise there would exist vertices of bigger degree), and a tree must have at least two leaves. We conclude that the leaves are the vertices of degree s. Even more so, we may affirm that the only way to obtain a tree with this passport is to glue a leaf of the weight s to a vertex of degree t and of the opposite color of a smaller tree. Cutting such a leaf from the initial tree, or, more exactly, from its passport, we obtain a new passport corresponding to a chain of a smaller (and even) length, which we may inductively suppose to be a unitree (the base of induction is a tree consisting of a single edge). Now we see that there exists only one way to re-attach the cut-off leaf of degree s in order to create a vertex of degree $s + t$: it must be attached to the *only one* existing vertex of degree t, which is by the way a leaf of the smaller tree.

The proof for an even length repeats the previous one almost word to word, only the leaves are now of the same color and of degrees s and t.

The same pattern will be repeated many times in this proof. If the structure of a passport implies the existence of a leaf of degree, say, s, then the only way to construct a corresponding tree is to glue an edge of the weight s to a vertex of the opposite color of a smaller tree. In the initial (bigger) tree this edge can only be attached to a vertex of degree bigger than s. If the smaller tree is a unitree, and if there is only one way to attach the new edge to it, then the bigger tree is also a unitree. In certain cases more than one way of attaching a new edge may exist, but they all lead to isomorphic trees.

(C) Consider the passport of a tree type C_1, and suppose that $k \geq l$, so that the biggest degree is $ks+t$; then the corresponding black vertex cannot be a leaf. The candidates for being leaves are: black and white vertices of degree s, and maybe the white vertex of degree $ls+t$ (if $k > l$). Since a tree has at least two leaves, there exists a leaf of degree s. Cut it off, and we get either a smaller instance of C_1, or that of B (if there remains only one leaf on the left and on the right), or that of A (if all the leaves were cut off on one of the sides). Then, after constructing the corresponding unitree of type C_1 or B or A, we can glue back the cut-off leaf to the only vertex of the bigger degree and of the opposite color.

(There is a little subtlety here. The planar structure of our trees means that we must choose not only a vertex to which we attach a new edge but also an angle into which it will be placed. If there are m edges incident to a vertex, there also are m angles between them, and therefore m ways of placing the new edge. But, obviously, in our case all these ways give the same plane tree, see Figure 11.)

Reasoning in the same way for C_2, we may be sure that there exists at least one leaf of degree s or $s+t$. Cut it off, and we either return to the pattern C_2, or get one of the patterns A or E_2 (whose uniqueness will be established later). Whatever is the case, all these trees have smaller total weight and thus may be inductively supposed to be unitrees. The place of the re-attachment of the cut-off leaf is uniquely determined: it is the only vertex of the opposite color which, before cutting the leaf in question, had a bigger degree in the initial tree.

(D) Two black vertices of degree $2s+t$ cannot be leaves. Therefore, there exists a leaf of degree s or $s+t$. Cut it off, and we get either C_1 or E_1. If, for example, the cut-off leaf was of degree s, then it should be reattached to the chain with the weights $s, t, s, s+t$ and create a black vertex of degree $2s+t$. It is clear that we must attach this leaf to the black vertex of degree $s+t$.

(E) Consider first the cases E_3 and E_4. All the vertices except two are of degree $s+t$; the two remaining ones are of degrees $(k+1)s+kt$ and $(l+1)s+lt$ for E_3, and $(k+1)s+kt$ and $ls+(l+1)t$ for E_4. Without loss of generality we may suppose that $(k+1)s+kt$ is the bigger of the two; therefore, it cannot be a leaf. For E_4, the "second best" vertex cannot be a leaf either since it has the same color; for E_3, if $k > l$, the vertex of degree $(l+1)s+lt$ might in principle be a leaf. Whatever is the case, there must exist another leaf, and it must be of degree $s+t$. Cut it off, and we obtain a smaller tree, with the possible pattern transitions as follows: $E_4 \to E_4$; $E_3 \to E_3$; $E_4 \to E_2$; or $E_3 \to E_1$, the latter two maybe with renaming the variables.

Now, for the cases E_1 and E_2 the situation is similar. All the vertices except two are of degree $s+t$. The vertex of the biggest degree cannot be a leaf. Therefore, there exists a leaf of degree s or $s+t$. Cut it off, and we get a smaller tree of the type E_1 from E_2, or E_2 from E_1, or we may arrive to the patterns A or B.

(F) Formula (12) gives 1 when $m \neq l$, and gives $1/k$ when $m = l$.

(G) The total weight is km while the number of edges is $km-1$; thus, there exists exactly one edge of weight 2 while all the other edges are of weight 1. The only white vertex to which the edge of weight 2 can be attached is the vertex of degree k, since all the other white vertices are of degree 1. The rest is obvious.

(H) Formula (12) gives 1 when $k \neq l$, and gives $1/2$ when $k = l$.

(I) Formula (12) gives 1.

(J) All white vertices are of degree 2; therefore, a weight of an edge can only be 1 or 2. There are only three black vertices, their degrees being 4, $2k+1$, $2k+1$. Therefore, the black vertices cannot be leaves since such leaves could not be attached to a white vertex; thus, all the leaves are white. A white vertex which is not a leaf must have two black neighbors; therefore, there are exactly two white vertices which are not leaves: they are "intermediate" white vertices between the black ones. The edges incident to them are both of weight 1. The black vertex of degree 4 cannot have two incident edges of weight 2 since such a tree would not be connected; it cannot have four incident edges of weight 1 since such a tree should need more than three black vertices. Therefore, the weights of the edges attached to this vertex must be 2, 1, 1. The rest is obvious.

(P) All black vertices are of degree 5, all white ones are of degree 3. Therefore, black vertices cannot be leaves. The number of (white) leaves cannot be 2 since all the chain trees were already classified. Two leaves cannot be attached to the same black vertex since the degree of such vertex would become ≥ 6. Thus, there are three white leaves, and they are attached to three different black vertices. What remains is to take three separate edges of weight 3 and to glue them into a tree through two remaining white vertices of degree 3. The uniqueness of such a gluing is easily verified.

(Q) Formula (12) gives 1.

(K, L, M, N, O, R, S, T) The proof of all these cases follows the same pattern.

Let us take, for example, the tree O. Its passport is $(5^4, 2^{10})$. Therefore, the number of vertices is $4 + 10 = 14$ and the number of edges is 13, while the total weight is $5 \cdot 4 = 2 \cdot 10 = 20$. Thus, the overweight is 7, and it must be distributed among the edges.

Now, no edge can have a weight greater than 2 since the degrees of white vertices are all equal to 2. Therefore, the tree O has seven edges of weight 2. Moreover, *all of them are leaves*; indeed, if something were attached to the white end such an edge then this white end would have a degree greater than 2.

The same reasoning may be carried out for all the above cases, with their respective overweights and numbers of leaves of weight 2.

Now, let us cut off all the leaves of weight 2. What remains is an ordinary tree, and we must verify that it is a unitree. Usually it is immediately obvious since the ordinary tree in question is very small; otherwise, we may apply formula (12), or else we may remark that, for example, for the tree T what remains after cutting off the leaves is the tree Q.

The last step consists in proving that there is only one way to glue back to this ordinary unitree the leaves of weight 2 that were previously cut off. For example, in the case O the ordinary unitree has black vertices of degrees 3, 1, 1, 1, and we have, by gluing to them seven edges of weight 2, make these degrees equal to 5, 5, 5, 5. Obviously, there is only one way to do that. In fact, in certain cases there are several ways of gluing but they give the same result because of a symmetry of the underlying ordinary tree.

Theorem 5.3 is proved. □

6 Other combinatorial Galois invariants

The theory of dessins d'enfants studies combinatorial invariants of the Galois action on dessins. These invariants have various levels of generality. The most general one is the passport; the subject of this paper is precisely the case when the passport alone guarantees the definability of a dessin over $\mathbb{Q}$. But our exposition would be incomplete if we did not mention several other, less general but quite often more interesting combinatorial Galois invariants.

6.1 Compositions

The following proposition shows a very general mechanism of constructing Belyi functions with all its finite poles being simple.

Proposition 6.1 (Composition) *Let $f = f(x)$ be a Belyi function such that the corresponding dessin D_f has all its finite faces being of degree 1, and let $A = A(t)$ be a polynomial whose all critical values are vertices of D_f. Then the function*

$$F(t) = f(A(t)), \qquad \text{or, otherwise,} \qquad F : \overline{\mathbb{C}} \xrightarrow{A} \overline{\mathbb{C}} \xrightarrow{f} \overline{\mathbb{C}}$$

is a Belyi function, and the finite faces of the dessin D_F corresponding to F are all of degree 1*. If, furthermore, the coefficients of both f and A are rational, then, obviously, the coefficients of F are also rational.*

Proof. Since A is a polynomial, the only poles of $F = f \circ A$, except infinity, are the preimages of the simple poles of f, i. e., the preimages of the centers of the small faces of D_f. Since A is not ramified over these simple poles, they remain simple for F, and each of them is "repeated" $\deg A$ times. $\square$

Example 6.2 (Composition) Consider the following functions:

$$f \;=\; -\,\frac{64\,x^3(x-1)}{8\,x+1}\,, \qquad A \;=\; \frac{1}{5^5}\cdot(t^2+4)^3(3\,t+8)^2\,.$$

Here f is a Belyi function corresponding to the upper left dessin in Figure 45, and A is a Belyi function corresponding to the lower left dessin. (Since A is a polynomial and therefore has a single pole at ∞, the dessin D_A has a single face and therefore is a tree. Belyi functions which are polynomials are called *Shabat polynomials*.)

Substituting $x = A(t)$ in f we obtain a Belyi function F corresponding to the dessin shown on the right of Figure 45. It is obvious that the *combinatorial* orbit of the dessin D_F consists of more than one element: for example, the petals attached to the vertices of degrees 9 and 6 can be cyclically arranged in many different ways. Still, $F \in \mathbb{Q}(x)$ by construction.

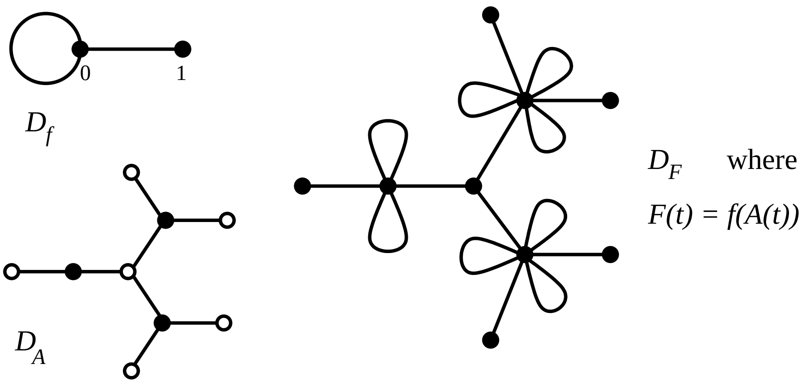

Figure 45: The pictures corresponding to $f(x)$ and $F(t)$ are drawn according to Convention 2.15; in the picture corresponding to $A(t)$ the black vertices are those sent to 0, and the white ones are those sent to 1.

Another example, based on the same function f, is as follows. We have

$$f-1 \;=\; -\,\frac{(8\,x^2-4\,x-1)^2}{8\,x+1}\,,$$

so the white vertices of the dessin D_f (which are not shown explicitly in Figure 45) are the roots of $8\,x^2-4\,x-1$, that is, they are equal to $(1\pm\sqrt{3})/4$. Now, the critical values of the polynomial

$$B \;=\; \frac{1}{3}\,t^3-\frac{3}{4}\,t+\frac{1}{4}\,,$$

that is, the values of B at the roots of $B' = t^2-3/4$, are equal to exactly $(1\pm\sqrt{3})/4$. Therefore, the composition $G(t) = f(B(t))$ is once again a Belyi function, and all its poles except infinity are simple. The corresponding dessin is shown in Figure 46.

It is obvious that the dessin D_G is not the only one having the passport $(3^3 1^3, 4^2 2^2)$. For example, the two "vertical" edges can both be put above, or both be put below the horizontal axis, or they can be cut off and attached to the leftmost white vertex of degree 2 (the one on the loop), or to the rightmost one (the one on the horizontal segment). Nevertheless, the dessin thus obtained is defined over $\mathbb{Q}$ by construction.

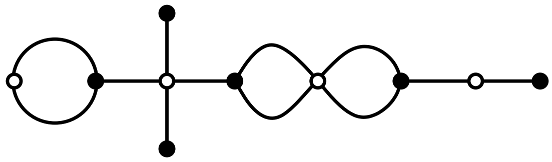

Figure 46: The dessin D_G corresponding to the function $G(t) = f(B(t))$; this time not all white vertices are of degree 2, therefore we show them explicitly.

Now the dessins D_F and D_G, being defined over $\mathbb{Q}$, may themselves serve for a similar construction: if, for example, C is a polynomial with coefficients in $\mathbb{Q}$ whose critical values are vertices of D_G, then the function $H = C \circ G = C \circ B \circ f$ is a Belyi function corresponding to a dessin D_H, all of whose finite faces are small. In such a composition, only f has to be a Belyi function while the subsequent terms may have more than three critical values.

Note however that the dessins D_f and D_A serving as building blocks for the above example both belong to the classification we intend to establish: it is their passports that guarantee that they are defined over $\mathbb{Q}$. As to the polynomial B, it is obtained from the Shabat polynomial corresponding to a chain tree with three edges after a simple change of variables; thus, it also corresponds to our classification.

Remark 6.3 (Primitive monodromy groups) It is known that a covering is a composition of two or more coverings of smaller degrees if and only if its monodormy group is imprimitive. Primitive permutation groups are, in a vast majority of cases, equal to S_n or A_n. Thereby, primitive groups other than S_n or A_n are of great interest since they may serve as an additional Galois invariant in the absence of the composition. Note that in our case the monodromy group must contain a permutation of the cycle structure $(n-r, 1^r)$. Motivated by our study of weighted trees, G. A. Jones classified all primitive permutation groups containing such a permutation, see [13]. In particular, it is shown that in all such cases $r \leq 2$.

6.2 A sporadic example

The world of dessins is rich with various specific cases. Let us consider, for example, the set of dessins shown in Figure 47. They constitute a combinatorial orbit for the passport (α, β, γ) where $\alpha = 3^{10}$, $\beta = 2^{15}$, and $\gamma = 24^1 1^6$. We might naïvely suppose that this combinatorial orbit constitutes also a Galois orbit; if this were the case, this orbit would be defined over a field of degree 4 (since it has four elements). However, the reality is more complicated, and more interesting.

Namely: the dessin a is the only one having a rotational symmetry of order 3 around a black vertex. Therefore, the singleton $\{a\}$ constitutes a Galois orbit. The dessins b and c are the only ones which are centrally symmetric, the center being a white vertex (we recall that the white vertices, being all of degree 2, are not shown explicitly in the picture). Therefore, the set $\{b, c\}$ also constitutes a Galois orbit.

The dessin d is not symmetric and does not have any other specific combinatorial feature. But it remains solitary, and therefore it constitutes a Galois orbit all by itself. Since the orbits $\{a\}$ and $\{d\}$ consist of a single element, their Belyi functions are defined over $\mathbb{Q}$. Thus, the dessin d is defined over $\mathbb{Q}$ for no other reason than the fact that *it remains alone after all the other Galois orbits being taken away*.

This example is markworthy by the reason that different authors returned to it many times. The Belyi function for a was computed by Birch and already appeared in [4] (1965); the one for d was computed 35 years later by Elkies [9] (2000). The Belyi functions for b and c were computed by Shioda [19] (2004): they are defined over $\mathbb{Q}(\sqrt{-3})$.

Our combinatorial approach does not make the computational part of the work any easier. Its advantage is elsewhere. It consists in the fact that, before any computation, we may be sure of the following:

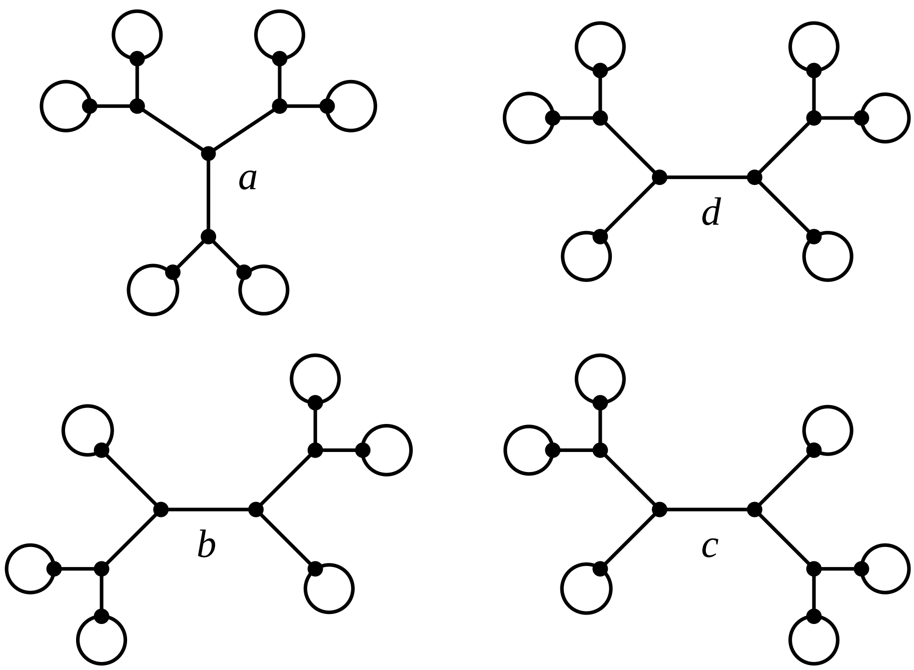

Figure 47: This combinatorial orbit, corresponding to the passport (α, β, γ) where $\alpha = 3^{10}$, $\beta = 2^{15}$, $\gamma = 24^1 1^6$, splits into three Galois orbits: $\{a\}$, $\{b, c\}$, and $\{d\}$. The dessins a and d are defined over $\mathbb{Q}$.

- there exist exactly four non-equivalent Belyi functions with the passport $(3^{10}, 2^{15}, 24^1 1^6)$; here "non-equivalent" means that they cannot be obtained from one another by a linear fractional change of variables;
- Belyi functions corresponding to a and d are defined over $\mathbb{Q}$;
- Belyi function corresponding to a is a rational function in x^3 (because of the threefold symmetry of the dessin a);
- Belyi functions for the orbit $\{b, c\}$ are defined over an imaginary quadratic field (since b and c are sent to each other by the complex conjugation).

More examples similar to this one are given in below.

6.3 Sporadic examples of Beukers and Stewart [3]

One of the main sources of inspiration for our study was the paper [3] by Beukers and Stewart. In their paper, the authors study only the case of powers. Namely, they look for polynomials A and B, *defined over* $\mathbb{Q}$, for which the degree $\deg(A^p - B^q)$ attains its minimum. The degrees of polynomials in question are $\deg A = qr$, $\deg B = pr$ where the parameter r may be greater than 1. The passport of the corresponding tree is (p^{qr}, q^{pr}).

The authors find, as also we do, several infinite series of DZ-triples (which they call Davenport pairs), and several sporadic examples. The first such example, for which $(p, q, r) = (5, 2, 2)$, corresponds to our sporadic tree O. The next one, $(p, q, r) = (5, 3, 1)$, corresponds to the sporadic tree P. However, the subsequent examples do not correspond to anything we have found up to now. What is taking place?

It turns out that here we encounter once again the phenomenon that we already explained in Section 6.2.

Example 6.4 $((p,q,r) = (7,3,1))$ There exist two trees corresponding to the passport $(7^3, 3^7)$: they are shown in Figure 48. We see that one of the trees is symmetric, with the symmetry of order 3, while the other one is not. Therefore, this combinatorial orbit splits into two Galois orbits, and hence both trees are defined over $\mathbb{Q}$. The left-hand one corresponds to the example given in [3].

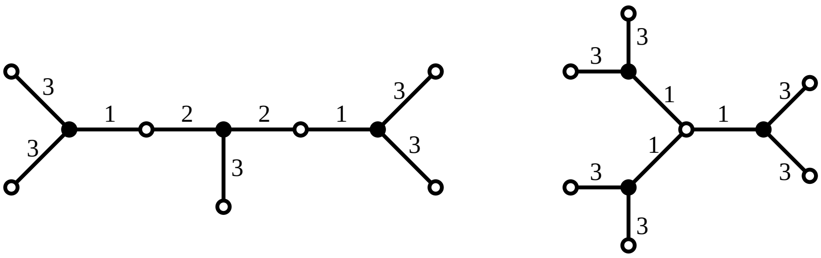

Figure 48: Two trees corresponding to the passport $(7^3, 3^7)$; one of them is symmetric, the other one is not.

Remark 6.5 (Symmetric trees) The automorphism group of a plane tree is always cyclic. If it is $\mathbb{Z}_k$ then the Belyi function for the corresponding map is $F(x) = f(x^k)$ where f is the Belyi function for the map corresponding to a single branch of the tree (the vertex of this branch, which will become the center of the symmetric tree, must be put to the origin). Therefore, symmetric examples do not have much interest since they can easily be reduced to smaller ones.

Example 6.6 ($(p,q,r) = (8,3,1)$ and $(10,3,1)$) The situation for the passports $(8^3, 3^8)$ and $(10^3, 3^{10})$ is similar to the previous one. For the first passport there are two trees, and only one of them is symmetric (see Figure 49); therefore, both are defined over $\mathbb{Q}$. For the second passport there are three trees (see Figure 50). One of them is symmetric with the symmetry of order 2; one is symmetric with the symmetry of order 3; and one is asymmetric. Therefore, all the three trees are defined over $\mathbb{Q}$. In both cases polynomials given in [3] correspond to asymmetric trees.

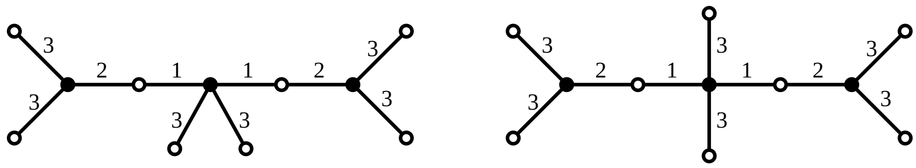

Figure 49: Two trees corresponding to the passport $(8^3, 3^8)$.

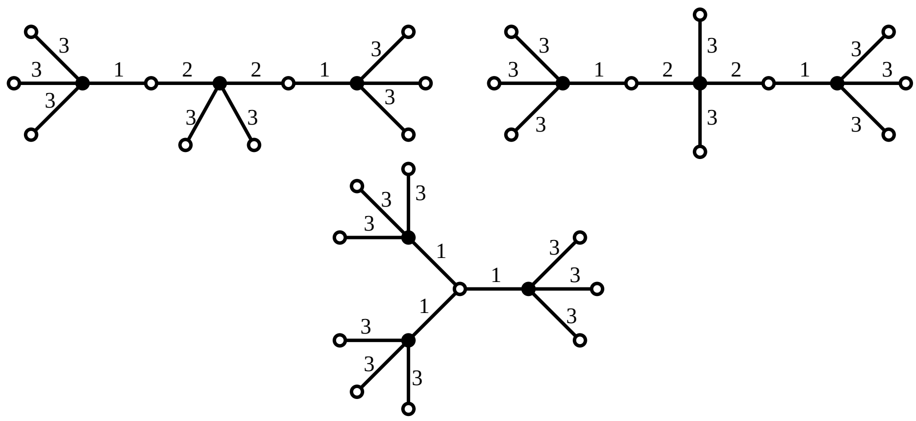

Figure 50: Three trees corresponding to the passport $(10^3, 3^{10})$.

Example 6.7 (Further sporadic DZ-triples) The next example given in [3] corresponds to the passport $(5^4, 4^5)$. This time, there are three trees: one of them is symmetric with the symmetry of order 2; another one is symmetric with the symmetry of order 4; the third tree is asymmetric. All the three are therefore defined over $\mathbb{Q}$.

For the passport $(6^5, 5^6)$ there are four trees. One of them is symmetric with the symmetry of order 5; two are symmetric with the symmetry of order 2; the remaining tree is asymmetric. Therefore, the combinatorial orbit containing four trees splits into three Galois orbits. The asymmetric tree corresponds to the sporadic example given in [3].

We leave it to the reader to draw the trees in question.

Remark 6.8 (No clear reason) At the time of this writing, we do not have any clear explanation for the next example from [3], with the passport $(9^5, 5^9)$. Since the corresponding trees have 13 edges, no symmetry is possible. Maybe one of the phenomena described for ordinary trees in the next section takes place here. In any case, one cannot hope to reduce the whole body of Galois theory to combinatorics.

7 Further questions and developments

Here we discuss briefly some possibilities for further research.

7.1 Extending the results for ordinary trees to weighted trees

In this section we mention, very briefly, some known results about ordinary trees which might eventually be generalized to weighted trees. A general reference for the results mentioned in this section is Chapter 2 of [15].

Enumeration of weighted trees. It would be very interesting to find an enumerative formula which would count the number of weighted trees with a given passport. However, this problem may turn out to be very difficult because of the fact that the same passport can be realized by a tree and by a forest. Therefore, an inclusion-exclusion procedure will be necessary, preventing a nice closed formula of Goulden–Jackson's type.

Inverse enumeration problem. The problem is formulated as follows: *For a given $m \geq 1$, classify all passports and corresponding weighted trees such that there exist exactly m trees with this passport.* In our paper, we have solved this problem for $m = 1$. The following result was proved in [1]:

Theorem 7.1 (Combinatorial orbits of size m) *For any $m \geq 1$ the combinatorial orbits of ordinary trees containing exactly m elements are classified as follows:*

- *the series of chain trees (only for $m = 1$);*
- *a finite number of series of diameter 4;*
- *a finite number of series of diameter 6;*
- *a finite number of sporadic orbits with at most $12m + 1$ edges.*

Our results for weighted trees and for $m = 1$ fall into line with this pattern, only the chains must be replaced with brushes, and the bound $12m + 1$ must be increased. It would be interesting to see if a similar theorem is valid for the general case. Maybe the bound must be imposed not on the number of edges but on the total weight of a tree.

Primitive monodromy groups of weighted trees. There is a finite number of ordinary trees whose monodromy group is primitive and not equal to either S_n or A_n. The complete list of such trees is given in [2]. In the same way, there is a finite number of cacti whose monodromy group is primitive and not equal to either S_n or A_n. Their detailed study may be found in [14]. We are sure that a similar result must be valid also for weighted trees. A complete list of such weighted trees remains to be found. See in this respect Remark 6.3.

Splitting combinatorial orbits, case 1. For the following three passports

- $(4^1 1^{n-4}, p^2 q^2)$: a series of trees of diameter 4; here $n = 2p + 2q$ and $p \neq q$;
- $(4^p, q^2 1^{n-2q})$: a series of trees of diameter 6; here $n = 4p$;
- $(4^3 1^8, 2^{10})$: sporadic trees of diameter 8; here $n = 20$

the combinatorial orbits consist of two (ordinary) trees, but one of these trees is symmetric while the other one is not. Therefore, both trees are defined over $\mathbb{Q}$.

For the weighted trees, we have seen similar examples in the previous section: the corresponding passports were $(7^3, 3^7)$ and $(8^3 3^8)$. An infinite series of weighted trees with the same property is shown in Figure 51. It would be interesting to produce a complete classification of such cases.

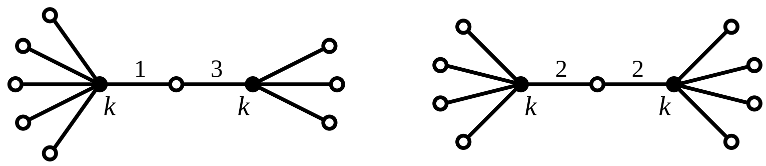

Figure 51: The combinatorial orbit consists of two trees, but it splits into two Galois orbits since one of the trees is symmetric while the other one is not. The degrees of the black vertices are both equal to $k \geq 3$; all leaves are of the weight 1.

Splitting combinatorial orbits, case 2. Consider the following passport for ordinary trees: $(5^1 1^{n-5}, p^2 q^3)$ (here $n = 2p + 3q$ and $p \neq q$). It is easy to see that there exist exactly two trees having this passport, and neither of them is symmetric. A simple computation shows that these trees are defined over the field $\mathbb{Q}\,(\sqrt{\Delta})$ where $\Delta = 3\,(p+2q)\,(2p+3q)$. Now, if we take, for example, $p = 6s^2 - 3t^2$ and $q = 2t^2 - 3s^2$, choosing s and t in such a way that p and q become positive and not equal, we get Δ to be a perfect square. Therefore, both trees are defined over $\mathbb{Q}$, and this splitting of the combinatorial orbit into two Galois orbits does not have any specific combinatorial reason: it is due to certain Diophantine relations between vertex degrees. Once again, it would be interesting to extend this scheme to weighted trees.

Splitting combinatorial orbits, case 3. The following example is maybe the most spectacular one. We consider the (ordinary) trees corresponding to the passport $(7^1 1^{n-7}, p^2 q^5)$ (here $n = 2p+5q$ and $p \neq q$). It is easy to see that there exist exactly three ordinary trees having this passport. Therefore, they are defined over a cubic extension of $\mathbb{Q}$; the cubic polynomial generating this field may be written explicitly. Now, we ask the following question: is it possible that this polynomial has a rational root? If yes, then the combinatorial orbit in question will split into two Galois orbits, one of them defined over $\mathbb{Q}$, and the other one quadratic.

It turns out that the search of polynomials having a rational root can be reduced to the search of rational points on a particular elliptic curve. The curve in question contains infinitely many rational points. We have computed the first 11 solutions. The smallest one corresponds to trees having $n = 686$ edges ($p = 33$, $q = 124$); the 11th solution corresponds to trees having $n \approx 3.45 \cdot 10^{134}$ edges. Similar constructions certainly must also exist for weighted trees.

7.2 Other questions

Relaxing the minimum degree condition. Let us revisit the initial problem about the minimum degree of the polynomial $A^3 - B^2$, see page 2. When there are no DZ-triples defined over $\mathbb{Q}$, we may relax the condition of the $\deg R$ being the least possible and thus obtain solutions with bigger $\deg R$ but, in return, defined over $\mathbb{Q}$. Two example of this kind are shown in Figures 52 and 53. In the first one, $k = 6$ but $\deg R = 9$ instead of 7 since one of the faces is of degree 3 instead of 1. In the second example, $k = 7$ but $\deg R = 9$ instead of 8 since, instead of two black vertices of degree 3 we have here one black vertex of degree 6.

The second example deserves a closer attention. Though a computation of the Belyi function in this case is not difficult, it is still interesting to analize this example in purely combinatorial terms. A smaller map shown on the right in Figure 53 is a unimap s of Figure 20, which is also equal to the unitree S in Figure 18. Therefore, it is defined over $\mathbb{Q}$. Its black vertex of degree 2 is a bachelor (Definition 2.17); therefore, it can be placed at any rational position (Remark 2.18), for example, at the point $x = 0$. Then it remains to insert x^3 instead of x in its Belyi function, and we get a Belyi function for the bigger "triple" dessin. This example shows that the possibilities of the combinatorial approach to this problem are far from being exhausted.

In general, it would be interesting to establish an upper bound on the difference between the minimum degree attainable in $\mathbb{C}[x]$, and the one attainable in $\mathbb{Q}[x]$.

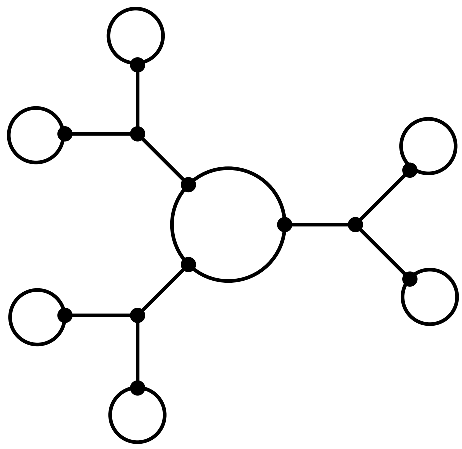

Figure 52: This map represents two polynomials A and B, of degrees $2k = 12$ and $3k = 18$ respectively, such that $\deg(A^3 - B^2) = 9$. Thus, the degree of the difference does not attain its minimal value $k+1 = 7$, but in return both A and B are defined over $\mathbb{Q}$.

Unimaps. Besides the maps with all finite faces of degree 1, there are other classes of maps for which the question of uniqueness is interesting. One such example is the class of maps with exactly two faces: their Belyi functions are Laurent polynomials. The existence questions for such maps were completely settled in [17]; the uniqueness remains to be studied. Some new Galois phenomena related to the non-existence of bachelors appear in this case, see [6].

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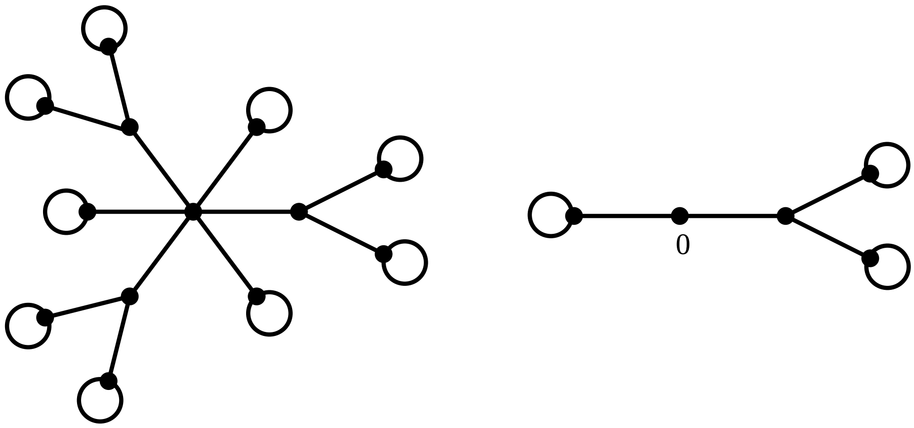

Figure 53: This map represents two polynomials A and B, of degrees $2k = 14$ and $3k = 21$ respectively, such that $\deg(A^3 - B^2) = 9$. Thus, the degree of the difference does not attain its minimal value $k+1 = 8$, but in return both A and B are defined over $\mathbb{Q}$.

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