The p-adic L-functions attached to Rankin convolutions of modular forms

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§ 1. Introduction

Let f be a newform in $S_k(N,\nu)$, i.e. of integral weight $k \geq 2$, level N and nebentypus character ν. Let $\overset{\circ}{\nu}$ denote the corresponding primitive character. f has a Fourier expansion

$$f(z)=\sum_{n=1}^{\infty} a_n q^n, \quad (q=e^{2\pi iz})$$

and the corresponding L-function

$$L(f,s)=\sum_{n=1}^{\infty} a_n n^{-s}$$

can be written as an Euler product of the form

$$L(f,s)=\prod_{\substack{r \\ \text{prime}}} (1-a_r r^{-s} + \overset{\circ}{\nu}(r) r^{k-1-2s})^{-1}.$$

For simplicity throughout this article we make the technical assumption, that f has rational Fourier coefficients, i.e. all $a_n \in \mathbb{Q}$. This implies that $\nu^2 = 1$. We denote by α_r and β_r the reciprocal zeros of the Euler polynominal at r, so that we get

$$1-a_r X + \nu(r) r^{k-1} X^2=(1-\alpha_r X).(1-\beta_r X).$$

We define the "imprimitive symmetric square" function attached to f by

$$D_\infty(f,s) = \prod_r [(1-\alpha_r^2 r^{-s}) \cdot (1-\alpha_r \beta_r r^{-s}) \cdot (1-\beta_r^2 r^{-s})]^{-1},$$

which can easily be transformed into the formula

$$D_\infty(f,s) = \frac{L_N(\nu^2, 2s+2-2k)}{L_N(\nu, s+1-k)} \sum_{n=1}^{\infty} a_n^2 n^{-s},$$

where L_N denotes the Dirichlet L-function with the Euler factors at primes dividing N removed.

The purpose of the present paper is to use algebraicity of special values of the function $D_\infty(f,s)$ and its twists $D_\infty(f,\chi,s)$ by certain Dirichlet characters χ to do p-adic interpolation and define in this way associated p-adic L-functions. It turns out that $D_\infty(f,\chi,s)$ is not quite the right object to consider. The two major "defects" are

a) that in general it does not satisfy a functional equation in a natural form for $s \longrightarrow 2k-1-s$,

b) that it is not necessarily entire (possibly there are poles at $s=k$, $k-1$).

This has already been remarked by Shimura [8], who proved meromorphic continuation of $D_\infty(f,\chi,s)$ to the whole s-plane with the only possibility of simple poles at $s=k$, $k-1$.

In § 2 we work out explicitely the modification of D_∞ by finitely many Euler factors such that the resulting <u>"primitive symmetric square"</u> function $\mathcal{D}_\infty(f,s)$ together with the twists $\mathcal{D}_\infty(f,\lambda,s)$ under consideration are <u>entire</u> functions satisfying a functional equation of canonical type (Theorem 1). This enables us then in § 3 to study algebraicity properties of all special values

$$\mathcal{D}_\infty(f,\chi,m) \quad \text{for} \quad m=1, \ldots, 2k-2 \text{ (Theorem 2).}$$

Note, that $m=k$, $k-1$ might be a <u>pole</u> of $D_\infty(f,\chi,s)$. If m is not a pole of $D_\infty(f,\chi,s)$, the algebraicity statement for $\mathcal{D}_\infty(f,\chi,m)$ easily reduces to Sturm's algebraicity results for $D_\infty(f,\chi,m)$ [10] . But if m is a pole of $\mathcal{D}_\infty(f,\chi,s)$ we must use the functional equation satisfied by $\mathcal{D}_\infty(f,\chi,s)$ to pass to $m'=2k-1-m$. There we exploit the fact that m' is not a pole of $D_\infty(f,\chi,s)$, thus showing "algebraicity for $\mathcal{D}_\infty(f,\chi,m')$" which eventually via the functional equation yields "algebraicity for $\mathcal{D}_\infty(f,\chi,m)$".

In § 4 we fix a prime $p \nmid 2Na_p$ and show the existence of p-adic L-functions $\mathcal{D}_{p,m}(f,s)$ for $m=1, \ldots, 2k-2$, which roughly speaking interpolate p-adically the special values $\mathcal{D}_\infty(f,\chi,m)$, where χ runs over all finite characters $\chi: 1+p\mathbb{Z}_p \longrightarrow \mathbb{C}^\times$ (Theorem 3). As a consequence of the functional equation of $\mathcal{D}_\infty(f,\chi,s)$ we will receive the functional equation of the p-adic L-functions:

$$\mathcal{D}_{p,m}(\mathfrak{f},s) = \mathcal{D}_{p,2k-1-m}(\mathfrak{f},2-s).$$

There is unpublished work of Hida treating p-adic interpolation of the special values of D_∞ by a different approach via p-adic modular forms. However, the methods of the present paper essentially grew out of a refinement of the techniques in B . Arnaud's Thèse [1] , where he shows that the integrals of characters against the proper (i.e. not smoothed) Pančiškin distribution are essentially p-integral.

The case of a newform of weight 2 is of particular interest. There, our primitive symmetric square is exactly the L-function attached to the system of l-adic representations $(\mathrm{Sym}^2 H_\ell^1(E))$ for the corresponding modular elliptic curve E . A detailled treatment of this case, in particular the connection with Iwasawa theory and the so-called main-conjecture are the subject of a forthcoming joint paper with J. Coates [3] .

§ 2. The primitive symmetric square

For the modification of $D_\infty(\mathfrak{f},\chi,s)$ it is convenient to introduce the notion of a minimal form.

Definition: A newform h of level M is called minimal (respectively r-minimal for a prime r), if h is not a twist h'_ψ of a newform h' of level $M'<M$ by a character ψ (respectively of conductor $c_\psi | r^\infty$).

Sometimes we write $\psi=\prod_r \psi_r$ where $c_{\psi_r} | r^\infty$. Let g be a minimal form associated with $\mathfrak{f}$, i.e. there is a character ε such that $g_\varepsilon=\mathfrak{f}$. Such a g always exists although it needs not to be unique. We suppose that g has level M and Fourier expansion

$$g = \sum_{n=1}^{\infty} b_n q^n.$$

We define Euler factors for primes $r|M$ by

$$\rho_r(\chi,s) := \begin{cases} 1-(\chi(r)r^{1-s})^2 & \text{if } b_r=0 \text{ and } \mathrm{ord}_r M \text{ even,} \\ 1-\chi(r)r^{1-s} & \text{otherwise.} \end{cases}$$

Proposition 2.1 and Definition: a) The "primitive symmetric square" function

$$\mathcal{D}_\infty(\mathfrak{f},\chi,s) := \prod_{r|M} \rho_r(\chi,s+2-k)^{-1} \frac{L_{Mc_\chi}(\chi^2,2s+2-2k)}{L(\chi\overset{\circ}{\nu},s+1-k)} \sum_{n=1}^{\infty} |b_n|^2 . \chi(n) n^{-s}$$

is independent of the choice of an associated minimal form g and differs from $D_\infty(f,\chi,s)$ only at finitely many Euler factors.

b) $\mathcal{D}_\infty(f,\chi,s)$ does not change, if we replace f by any twist f_ψ with a character ψ such that $(c_\psi,N)=1$. In particular one can assume that $N\equiv 0\,(4)$.

Proof: a) Suppose $g'\neq g$ is a second choice of level M' and $f=g'_{\varepsilon'}$. For an integer R and a prime r we put $R_r:=r^{\mathrm{ord}_r R}$. Since g is minimal iff g is r-minimal for all $r|N$ we may suppose $c_\varepsilon, c_{\varepsilon'}|r^\infty$ and show: $M_r = M'_r$ and

$$b_r = 0 \text{ iff } b'_r = 0 ,$$

where $g'=\sum_n b'_n q^n$. We know that $g' \sim g_{\varepsilon\bar{\varepsilon}'}$ (i.e. all but finitely many Fourier coefficients at primes coincide).

Case $g_{\varepsilon\bar{\varepsilon}'}$ is newform: Then $g'=g_{\varepsilon\bar{\varepsilon}'}$ and r-minimality of g' yields $M_r\geq M'_r$. If $b_r=0$, then $g=g'_{\bar{\varepsilon}\varepsilon'}$, hence $M'_r\geq M_r$ by r-minimality of g, so we have $M_r=M'_r$ in this case. If $b_r\neq 0$ and $M_r>M'_r$, then $c_{\varepsilon\bar{\varepsilon}'}\neq 1$ implies $b'_r=0$, hence $r^2|M'_r$ and $c_{\nu\bar{\varepsilon}'^2}|M'/r$. Note: $g'\in S_k(M',\nu\bar{\varepsilon}'^2)$. $b_r\neq 0$ yields $(c_{\nu\bar{\varepsilon}^2})_r = M_r$ or ($\nu_r=\varepsilon^2$ and $M_r = r$). The last case being impossible since $r^2|M'_r|M_r$, we arrive at $c_{(\bar{\varepsilon}\varepsilon')^2} = M_r$. Now apply Corollary 4.3 [2, p.235].

Put

$$\tilde{Q} := \begin{cases} c_{\varepsilon\bar{\varepsilon}'}, & \text{if } c_{\nu\bar{\varepsilon}^2\varepsilon\bar{\varepsilon}'} \geq M_r , \\ c_{\nu\bar{\varepsilon}^2\varepsilon\bar{\varepsilon}'}, & \text{if } \nu\bar{\varepsilon}^2\varepsilon\bar{\varepsilon}' \neq 1 \text{ and } c_{\nu\bar{\varepsilon}^2\varepsilon\bar{\varepsilon}'} < M_r . \end{cases}$$

Then $g' = g_{\varepsilon\bar{\varepsilon}'}$ is newform of level $\tilde{Q}.M$ if $\nu\bar{\varepsilon}\bar{\varepsilon}' \neq 1$ (otherwise it is not a newform). So $M_r' = \tilde{Q}M_r \geq M_r$, a contradiction. We get $M_r = M_r'$ also for $b_r \neq 0$. It remains to show that $b_r' = 0$ implies $b_r = 0$. If we assume $b_r \neq 0$, we get $(c_{\nu\bar{\varepsilon}^2})_r = M_r = M_r' \geq r^2$. Again by Corollary 4.3 from [2] we would arrive at $M_r' = \tilde{Q}M_r > M_r$, since $\varepsilon\bar{\varepsilon}' \neq 1$ (note: $b_r' = 0 \neq b_r$ implies $g' \neq g$) hence contradiction.

<u>Case $g_{\varepsilon\bar{\varepsilon}'}$ not a newform:</u> Proposition 4.1 [2] tells us for r-minimal g with $b_r = 0$ that all twists g_ψ with $c_\psi | r^\infty$ are newforms. So we know $b_r \neq 0 = b_r'$ and hence

$$c_{\nu_r\bar{\varepsilon}^2} = M_r \quad \text{or} \quad (\nu_r = \varepsilon^2 \text{ and } M_r = r).$$

In the last case any twist of g by a character is a newform by Corollary 4.1 [2], so this is excluded here. By Corollary 4.3 we have for $c_{\nu_r\bar{\varepsilon}^2} = M_r$:

$$g_{\varepsilon\bar{\varepsilon}'} \text{ newform iff } \varepsilon\bar{\varepsilon}' \neq \bar{\nu}_r\varepsilon^2 .$$

Hence we get from our assumption : $\nu_r = \varepsilon\varepsilon'$. Now $g_{\varepsilon\bar{\varepsilon}'} = \sum_n b_n \varepsilon\bar{\varepsilon}'(n)q^n$ has character $\nu\bar{\varepsilon}^2\varepsilon^2\bar{\varepsilon}'^2 = (\nu\nu_r^{-1})\varepsilon\bar{\varepsilon}'$.

Apply the involution W_{M_r} of [2] ! There is a newform $h \in S_k(M, \nu\nu_r^{-1}\varepsilon\bar{\varepsilon}')$ and $\lambda_{M_r}(g) \in \bar{\mathbb{Q}}$ with $|\lambda_{M_r}(g)| = 1$ such that

$$g|W_{M_r} = \lambda_{M_r}(g) . h$$

and $h = \sum_n c_n q^n$ where

$$c_p = \begin{cases} \overline{(\nu_r \bar{\varepsilon}^2)(p)} . b_p & \text{if } p \neq r , \\ (\nu\nu_r^{-1})(p) . \bar{b}_p & \text{if } p = r . \end{cases}$$

By comparison of Fourier coefficients we see that

$$g' \sim g_{\varepsilon\bar{\varepsilon}'} \sim h ,$$

hence $g' = h$ and therefore $M_r' = M_r$. Furthermore we get

$$b_r' = 0 \text{ iff } r^2 | M_r' \text{ and } c_{\nu_r\bar{\varepsilon}^{2}} | M_r'/r ,$$

$$b_r = 0 \text{ iff } r^2 | M_r \text{ and } c_{\nu_r\bar{\varepsilon}^{2}} | M_r/r ,$$

which completes the proof of a).

b) It is clear that replacing f by f_ψ with $(c_\psi, N) = 1$ does not affect an associated minimal form g , since $f = g_\varepsilon$ implies $f_\psi = g_{\varepsilon\psi}$ and the assumption $(c_\psi, N) = 1$ guarantees that f_ψ is again a newform. □

In all what follows we suppose that $(c_\chi, N) = 1$ and define

$$\Gamma_\infty(\nu\chi, s) := (2\pi)^{-s}\Gamma(s)\pi^{-s/2}\Gamma\left(\frac{s-k+2-H_{\chi\nu}}{2}\right), \text{ where}$$

$$\nu: \quad \nu\chi(-1) = (-1)^{H_{\nu\chi}}, \quad H_{\nu\chi} = 0,1 .$$

$$B := \prod_{r|M} r^{\operatorname{ord}_r M - m(r)} \quad \text{where } m(r) := \begin{cases} \left[\dfrac{\operatorname{ord}_r M}{2}\right] & \text{if } b_r = 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$W_\chi := \chi^2(B)\,\frac{G(\chi)^2}{G(\chi\nu)\cdot G(\bar\chi)^2}\sqrt{\nu\chi(-1)c_{\chi\nu}}$$

where the Gauß sum $G(\chi)$ is given by

$$G(\chi) := \sum_{x=1}^{c_\chi} \chi(x)\exp(2\pi i x/c_\chi).$$

<u>Theorem 1 : The function</u>

$$R(\chi,s) := (B^2 c_\chi^{\;3} c_\nu^{\;-1})^{s/2}\,\Gamma_\infty(\nu\chi,s)\cdot\mathcal{D}_\infty(f,\chi,s)$$

<u>has analytic continuation to the whole complex plane where it satisfies the functional equation</u>

$$R(\chi,s) = W_\chi \cdot R(\bar\chi, 2k-1-s).$$

$R(\chi,s)$ <u>is entire except for odd</u> k <u>and trivial</u> χ, <u>in which case there are exactly two simple poles:</u> $s=k$, $k-1$.

The proof will occupy the rest of this § . We fix a minimal form g associated with $\mathfrak{f}$ and apply Theorem 2.2 from [5] to the newforms $F_1 := g \in S_k(M, \nu\bar{\varepsilon}^2)$ and $F_2 := g_{\bar{\chi}} \in S_k(Mc_\chi^2, \nu\bar{\varepsilon}^2\bar{\chi}^2)$. One easily checks that conditions A), B), C) of [5, p. 41] are satisfied. In the notation of that article we set

$$M' := \prod_{r \mid c_{\chi^2}} c_{\chi_r}^2 / c_{\chi_r^2}$$

$$M'' := M \prod_{\substack{r \mid c_\chi \\ \chi_r^2 = 1}} c_{\chi_r}^2 .$$

Li formulates her result in terms of the pseudo-eigenvalues $\lambda_r(F_i)$ under the action of W_r-operators.

<u>Lemma 2.2: For a prime</u> $r \mid M$ <u>such that</u> $b_r = 0$ <u>we have for</u>

$$n(r) := \max \left\{ n \in \mathbb{N};\ \frac{\lambda_r(g_\psi)}{\lambda_r(g_{\bar{\chi}\psi})} = \frac{\lambda_r(g)}{\lambda_r(g_{\bar{\chi}})}\ \forall \psi \text{ with } c_\psi \mid r^n \right\}$$

<u>that</u> $n(r) \geq \left[\frac{\mathrm{ord}_r M}{2}\right]$.

<u>Proof:</u> The twisting operator R_χ and W_r-operators behave like

$$g|R_\chi|W_r = \bar{\chi}(M_r)\cdot g|W_r|R_\chi$$

where $g|R_\chi = G(\bar{\chi})\cdot g_\chi$. Hence

$$\frac{\lambda_r(g)}{\lambda_r(g_{\bar{\chi}})} = \bar{\chi}(M_r) \ .$$

The same argument for
$g_\psi \in S_k(\mathrm{lcm}(M, c_\psi^2, c_\psi \cdot c_{\nu\bar{\varepsilon}^2}), \nu\bar{\varepsilon}^2\psi^2)$ instead of g yields

$$\frac{\lambda_r(g_\psi)}{\lambda_r(g_{\psi\bar{\chi}})} = \bar{\chi}(M(\psi)_r)$$

where $M(\psi)$ denotes the level of the newform g_ψ . Since for r-minimal g with $b_r=0$ by Theorem 4.3 of [2] one has $(c_{\nu\bar{\varepsilon}^2})_r \le \sqrt{M_r}$, we get for $c_\psi^2|M_r$: $M(\psi) = M$ by minimality of g , which proves the lemma.

The lemma justifies our definition of $m(r)$ which in Li's article is

$$m(r) := \begin{cases} \min(n(r), \left[\frac{\mathrm{ord}_r M}{2}\right]) & \text{if } b_r=0 , \\ 0 & \text{otherwise.} \end{cases}$$

For $r|M''$ Li defines

$$\Theta_r(\chi,s)=\begin{cases}1-\chi(r).|b_r|^2 r^{-s+2-k} & \text{if } M_r=r \text{ and } (\nu\bar{\varepsilon}^2)_r=\chi_r=1,\\ 1-\chi(r)|b_r|^2 r^{-(s+k-1)} & \text{if } M_r=(c_{\nu\bar{\varepsilon}^2})_r \text{ and } \chi_r=1,\\ 1-\chi^2(r)r^{-2s} & \text{if } b_r=0 \text{ and } \mathrm{ord}_r M \text{ even},\\ 1-\chi(r).r^{-s} & \text{if } b_r=0 \text{ and } \mathrm{ord}_r M \text{ odd},\\ 1 & \text{otherwise.}\end{cases}$$

Since for $r|M$ we have

$$|b_r|^2 = \begin{cases} r^{k-1} & \text{if } M_r=(c_{\nu\bar{\varepsilon}^2})_r\\ r^{k-2} & \text{if } M_r=r \text{ and } \nu_r=\varepsilon_r^2,\\ 0 & \text{otherwise,}\end{cases}$$

we can express the Θ's by the formula

$$\Theta_r(\chi,s) = \begin{cases}1-\chi^2(r)r^{-2s} & \text{if } b_r=0 \text{ and } \mathrm{ord}_r M \text{ even},\\ 1-\chi(r)r^{-s} & \text{otherwise,}\end{cases}$$

hence we get the

Remark 2.3:

$$\Theta_r(\chi,s) = \rho_r(\chi,s+1) .$$

We must introduce some more notation to formulate Li's result. For $r|M'$ define

$$Q_r := \begin{cases} M_r'^{\,2} & \text{if } M_r' > c_{\chi_r} , \\ c_{\chi_r}^{\,2} & \text{otherwise,} \end{cases}$$

and

$$\Lambda_r(\chi) := \bar{\chi}_r^2(-1) . \bar{\nu}\varepsilon^2(\chi\bar{\chi}_r)^2(c_{\chi_r}^2) . (\bar{\chi}\chi_r)(M_r')G(\chi_r^2) . \lambda_r(g_{\bar{\chi}})^2 \frac{Q_r}{c_{\chi_r}^2}$$

We set

$$\Psi_{g,g_{\bar{\chi}}}(s) := (2\pi)^{-2s}\Gamma(s)\Gamma(s+1) \prod_{r|M''} \Theta_r(\chi,s)^{-1} . L_{g,g_{\bar{\chi}}}(s) ,$$

where

$$L_{g,g_{\bar{\chi}}}(s) := L_{Mc_\chi}(\chi^2,2s) . \sum_{n=1}^{\infty} |b_n|^2 . \chi(n) n^{-(s+k-1)} .$$

<u>Proposition 2.4: (W. Li) The function</u> $\Psi_{g,g_{\bar{\chi}}}(s)$ <u>has analytic continuation to the whole complex plane, which is an entire function if</u> $\chi \neq 1$ <u>and which has only simple</u>

poles at $s=0$ and $s=1$ if χ is trivial. It satisfies the functional equation

$$\Psi_{g,g_{\bar{\chi}}}(s) = A_\chi(s) . \Psi_{\bar{g},\overline{g_{\bar{\chi}}}}(1-s) ,$$

where

$$A_\chi(s) := \prod_{r|M} (\chi^2(r) r^{1-2s})^{\mathrm{ord}_r M - m(r)} . \prod_{\substack{r|c_\chi \\ \chi_r^2 = 1}} (\chi^2(r) r^{1-2s})^{2\mathrm{ord}_r c_\chi}$$
$$. \prod_{r|c_{\chi^2}} G(\chi_r^2) \wedge_r(\chi) \left(\frac{c_{\chi_r^2}}{M'_r}\right)^{1-2s} Q_r^{-s}(\chi\bar{\chi}_r)(Q_r c_{\chi_r}^2/M'_r)$$

and $\bar{g} = \sum_n \bar{b}_n q^n$.

This is Theorem 2.2 from [5] in our special case $F_1 = g$, $F_2 = g_{\bar{\chi}}$:

Lemma 2.5: $A_\chi(s) = (Bc_\chi^2)^{1-2s} \chi^2(B) \left(\frac{G(\chi)}{G(\bar{\chi})}\right)^2$.

Proof: For the primes $r|M$ their contribution to $A_\chi(s)$ gives us straight away the factors B^{1-2s} and $\chi^2(B)$. For the primes $r|c_{\chi^2}$ we reduce everything to the proof of

<u>Lemma 2.6:</u>

$$\Lambda_r(\chi) = G(\bar\chi_r^2)\,\frac{G(\chi_r)^2}{G(\bar\chi_r)^2}\begin{cases}(\bar\chi\chi_r)(c_{\chi_r}^2) & \text{if } r \neq 2\,,\\ 4(\bar\chi\chi_r)(4c_{\chi_r}^2) & \text{if } r = 2\,.\end{cases}$$

We continue the proof of Lemma 2.5 and show Lemma 2.6 later. The contribution of primes $r|c_{\chi^2}$ to $A_\chi(s)$ now is easy to calculate as

$$G(\chi_r^2)\cdot G(\bar\chi_r^2)\left(\frac{G(\chi_r)}{G(\bar\chi_r)}\right)^2\begin{cases}4(\bar\chi\chi_2)(4c_{\chi_2}^2)\cdot(c_{\chi_2}/2)^{1-2s}\cdot(4c_{\chi_2}^2)^{-s}\cdot(\chi\bar\chi_2)^2(2c_{\chi_2}^3) & \\ \qquad\qquad \text{if } r=2\,, & \\ (\bar\chi\chi_r)(c_{\chi_r}^2)c_{\chi_r}^{1-2s}c_{\chi_r}^{-2s}(\chi\bar\chi_r)^2(c_{\chi_r}^3) \quad \text{otherwise}, & \end{cases}$$

$$=\left(\frac{G(\chi_r)}{G(\bar\chi_r)}\right)^2\cdot\,(c_{\chi_r}^2)^{1-2s}\,(\chi\bar\chi_r)(c_{\chi_r}^4)\,.$$

By the decomposition formula for Gauß sums

$$G(\chi) = \prod_{r|c_\chi}(\chi\bar\chi_r)(c_{\chi_r})\cdot G(\chi_r)$$

and by the identity $G(\chi_r) = G(\bar\chi_r)$ for quadratic characters χ_r we arrive at

$$\prod_{r|c_\chi}(c_{\chi_r}^2)^{1-2s}(\chi\bar\chi_r)(c_{\chi_r}^4)\;\cdot\left(\frac{G(\chi_r)}{G(\bar\chi_r)}\right)^2 = (c_\chi^2)^{1-2s}\left(\frac{G(\chi)}{G(\bar\chi)}\right)^2 \quad ,$$

which immediately gives the desired formula in Lemma 2.5.

Proof of Lemma 2.6: The proof is easily reduced to show

$$\lambda_r(g_{\bar{\chi}}) = (\chi\bar{\chi}_r)(c_{\chi_r}^{2}).\bar{\nu}\varepsilon^2(c_{\chi_r}).\bar{\chi}_r(-1)\ \frac{G(\bar{\chi}_r)}{G(\chi_r)}\ .$$

By Theorem 4.1 of [2,p.231] we have

$$\lambda_r(g_{\bar{\chi}_r}) = \bar{\nu}\varepsilon^2(c_{\chi_r}).\bar{\chi}_r(-1).G(\bar{\chi}_r)/G(\chi_r)$$

and by Proposition 3.4 of [2]

$$g_{\bar{\chi}_r}|R_{\bar{\chi}\chi_r}|W_r = (\chi\bar{\chi}_r)(c_{\chi_r}^{2}).g_{\bar{\chi}_r}|W_r|R_{\bar{\chi}\chi_r}\ ,$$

so that by comparison of first Fourier coefficients we get

$$G(\bar{\chi}\chi_r).\lambda_r(g_{\bar{\chi}}) = (\chi\bar{\chi}_r)(c_{\chi_r}^{2}).\lambda_r(g_{\bar{\chi}_r}).G(\bar{\chi}\chi_r)\ ,$$

hence the desired formula for $\lambda_r(g_{\bar{\chi}})$.

As a conclusion from Proposition 2.4 we get the statement of Theorem 1 up to holomorphy.

Proposition 2.7: $R(\chi,s)$ has analytic continuation to a meromorphic function on $\mathbb{C}$ satisfying the predicted functional equation.

Proof: Firstly we note that $\Psi_{\bar{g},\overline{g_{\bar{\chi}}}}(s)=\Psi_{g,g_{\bar{\chi}}}(\bar{s})$ since

$\overline{g_{\bar\chi}} = (\bar g)_\chi$ and $\Psi_{g,g_{\bar\chi}}(s)$ does not change when we replace $g = \sum_n b_n q^n$ by $\bar g = \sum_n \bar b_n q^n$, which is obvious by definition. So putting

$$R^*(\chi,s) := (Bc_\chi^2)^s . \Psi_{g,g_{\bar\chi}}(s)$$

we can reformulate a slightly weaker form of Proposition 2.4 via Lemma 2.5 as follows:

Lemma 2.8: $R^*(\chi,s)$ has analytic continuation to a meromorphic function on $\mathbb{C}$ which satisfies the functional equation

$$R^*(\chi,s) = W_\chi^* . R^*(\bar\chi, 1-s)$$

with the root number $W_\chi^* := \chi^2(B) . (G(\chi)/G(\bar\chi))^2$.

Now divide $R^*(\chi,s)$ by the Dirichlet L-function

$$Z(\mathring{\nu\chi},s) := c_{\mathring{\nu\chi}}^{s/2} . \pi^{-s/2} . L(\mathring{\nu\chi},s) . \begin{cases} \Gamma(s/2) & \text{if } \mathring{\nu\chi}(-1)=1 , \\ \Gamma(\frac{s+1}{2}) & \text{if } \mathring{\nu\chi}(-1)=-1 , \end{cases}$$

and use its functional equation

$$Z(\mathring{\nu\chi},s) = \frac{G(\mathring{\nu\chi})}{\sqrt{\mathring{\nu\chi}(-1)\, c_{\mathring{\nu\chi}}}} Z(\mathring{\bar\nu\bar\chi},1-s) .$$

(where $\mathring{\nu}$ = primitive character associated with ν)

We get by Remark 2.3

$$\frac{R^*(\chi,s)}{\overset{\circ}{Z}(\nu\chi,s)} = \pi . c_\nu^{-s/2} . (B^2 c_\chi^3)^{-1/2} . R(\chi,s+k-1) \quad ,$$

hence the predicted functional equation for $R(\chi,s)$ follows with the root number

$$W_\chi = W_\chi^* \cdot \frac{\sqrt{\nu\chi(-1)c_{\nu\chi}}}{\overset{\circ}{G}(\nu\chi)} = \chi^2(B)\frac{G(\chi)}{G(\bar{\chi})^2} \cdot \frac{\sqrt{\nu\chi(-1)c_{\nu\chi}}}{\nu(c_\chi)\chi(c_\nu)\overset{\circ}{G}(\nu)} \quad .$$

We still have to show the entireness of $R(\chi,s)$. The proof is based on the following result of Shimura [8] .

<u>Proposition 2.9: (Shimura) Let</u> $h \in S_k(N',\mu)$ <u>be a newform with Fourier expansion</u> $h(z) = \sum_{n=1}^\infty d_n q^n$ <u>and let</u> χ <u>be a (primitive) Dirichlet character. Then the function</u>

$$R(h,\chi,s) := \Gamma_\infty(\mu\chi,s) \frac{L_{N'c_\chi}(\chi^2\mu^2, 2s-2k+2)}{L_{N'}(\chi\mu, s-k+1)} \sum_{n=1} d_n^2 \chi(n) n^{-s}$$

<u>can be continued to a meromorphic function on</u> $\mathbb{C}$ <u>, which is holomorphic except for possible simple poles at</u> $s=k$ <u>and</u> $s=k-1$ <u>. There is a pole at</u> $s=k$ <u>if and only if</u>

(i) $\mu\chi$ is an odd quadratic character,

(ii) $\int h(z)\, \overline{h^\rho_{\bar{\chi}}(z)}\, y^{k-2} dxdy \neq 0$, <u>where the integral</u>

<u>is taken over a fundamental domain of</u>

$\Gamma_\circ(N'c_\chi^2) \setminus \mathcal{H}$ and $h^{\rho}_{\bar\chi}(z) := \sum_n \bar\chi(n)\bar d_n q^n$.

Corollary 2.10: If $(c_\chi, N')=1$ then $R(h,\chi,s)$ has no pole at $s=k$, except $\chi=1$, k is odd and $h=h^\rho$.

Proof: Since $(c_\chi, N')=1$, the form $h^\rho_{\bar\chi}$ is a newform of level $N'c_\chi^2$. Therefore the Petersson product

$$< h, h^\rho_{\bar\chi}> = \int h(z)\overline{h^\rho_{\bar\chi}(z)}\, y^{k-2}dxdy$$

vanishes as long as $h \neq h^\rho_{\bar\chi}$. This is guaranteed by excluding the case k odd, $\chi=1$, $h=h^\rho$, since the Petersson product of two newforms is non zero if and only if they coincide.

We return to our special situation, where $f=g_\varepsilon$. Define a quadratic character $\tilde\varepsilon$ and a (primitive) character ε' by

$$\tilde\varepsilon := \prod_{r|M,\varepsilon_r^2=1} \varepsilon_r \ , \quad \varepsilon=\varepsilon'.\tilde\varepsilon \ .$$

Consider the newform $h:=g_{\varepsilon'} \in S_k(N', \nu_{N'})$ with $h=\sum_{n=1}^\infty d_n q^n$,

where $\nu_{N'}$ is the character mod N' associated with $\mathring{\nu}$. Note: $f=h_{\tilde\varepsilon}$ so that by Proposition 2.1 $\mathcal{D}_\infty(f,\chi,s)=\mathcal{D}_\infty(h,\chi,s)$. We want to relate $R(h,\chi,s)$ with $R(\chi,s)$ and exploit Proposition 2.9. We write

$$\frac{L_{N'c_\chi}(\chi^2\nu^2, 2s-2k+2)}{L_{N'}(\chi\mathring{\nu}, s-k+1)} \cdot \sum_{n=1}^\infty d_n^2\, \chi(n) n^{-s} = \prod_r R^{(r)}(r^{-s})^{-1}$$

and

$$\mathcal{D}_\infty(h,\chi,s) = \prod_r \mathcal{D}^{(r)}(h,r^{-s})^{-1} .$$

By Shimura's Lemma [9,p.790] we can describe the Euler factors $R^{(r)}(r^{-s})$ by:

$$R^{(r)}(X) = (1-\alpha'^{2}_{r}\chi(r)X).(1-\beta'^{2}_{r}\chi(r)X).(1-\nu_{N'}(r)\chi(r)r^{k-1}X)$$

where

$$1-d_r X+\nu_{N'}(r)r^{k-1}X^2 = (1-\alpha'_r X).(1-\beta'_r X)$$

is the Euler polynominal at r associated with h. The same procedure applied to

$$\mathcal{D}_\infty(h,\chi,s) = \prod_{r|M}\rho_r(\chi,s+2-k)^{-1}\frac{L_{Mc_\chi}(\chi^2,2s+2-2k)}{L(\nu\chi,s+1-k)}\sum_{n=1}^{\infty} b_n\bar{b}_n\,\chi(n)n^{-s}$$

delivers

$$\sum_{n=1}^{\infty}|b_n|^2\chi(n)n^{-s} = \prod_r \frac{(1-|\gamma_r{}^2\delta_r{}^2|\chi(r)^2r^{-2s})}{(1-|\gamma_r|^2\chi(r)r^{-s})(1-|\delta_r|^2\chi(r)r^{-s})(1-\gamma_r\bar{\delta}_r\chi(r)r^{-s}).(1-\bar{\gamma}_r\delta_r\chi(r)r^{-s})}$$

where

$$1-b_rX+\bar{\varepsilon}^2\nu_M(r)r^{k-1}X^2 = (1-\gamma_rX).(1-\delta_rX)$$

is the Euler polynominal of g at r . We observe that all Fourier coefficients of h are rational, since the field $\mathbb{Q}(d_1,d_2,\dots)$ is generated by the d_n with $(n,N')=1$ and because by definition of $\tilde{\varepsilon}$ $(n,N')=1$ implies $(n,c_{\tilde{\varepsilon}})=1$, thus $d_n=\tilde{\varepsilon}(n).a_n\in\mathbb{Q}$. Now $b_n.\varepsilon'(n)=d_n$ shows for $r\nmid c_{\varepsilon'}$ that

$$(1-\gamma_rX)(1-\delta_rX)=1-\varepsilon'(r)d_rX+\bar{\varepsilon}'^2(r)\nu_M(r)r^{k-1}X^2=(1-\bar{\varepsilon}'(r)\alpha_r'X)(1-\bar{\varepsilon}'(r)\beta_r'X)$$

and since d_r is rational

$$(1-\bar{\gamma}_r\chi(r)X)(1-\bar{\delta}_r\chi(r)X)=(1-\varepsilon'(r)\chi(r)\alpha_r'X)(1-\varepsilon'(r)\chi(r)\beta_r'X) \quad ,$$

so the corresponding quotient above simplifies to

$$\frac{1-1_M(r)\chi(r)^2r^{2k-2-2s}}{(1-\alpha_r'^2\chi(r)r^{-s})(1-\beta_r'^2\chi(r)r^{-s})(1-\nu_M(r)\chi(r)r^{k-1-s})^2}$$

We get for $r\nmid N'$ (i.e. $r\nmid M.c_{\varepsilon'}$) :

$$\mathcal{D}^{(r)}(h,\chi) = R^{(r)}(X) \quad .$$

If $r|M$ and $r\nmid c_{\varepsilon'}$, then $\gamma_r=b_r,\delta_r=0$ and

$$\mathcal{D}^{(r)}(h,r^{-s}) = \frac{\rho_r(\chi,s+2-k)}{1-\chi(r)\mathring{\nu}(r)r^{k-s-1}}(1-\alpha_r'^2\chi(r)r^{-s}) \quad ,$$

hence

$$\mathcal{D}^{(r)}(h,X)=\begin{cases}(1-\chi(r)^2r^{2k-2}X^2)\ /\ (1-\chi(r)\mathring{\nu}(r)r^{k-1}X) & \text{if } b_r=0 \text{ and } \mathrm{ord}_r M \text{ even.}\\ (1-\chi(r)r^{k-1}X)\,(1-\alpha_r'^2\chi(r)X)\ /\ (1-\chi(r)\mathring{\nu}(r)r^{k-1}X) & \text{otherwise,}\end{cases}$$

whereas

$$R^{(r)}(X) = 1-\alpha_r'^2\chi(r)X \ .$$

If $r|c_{\varepsilon'}$, $r\nmid M$, then $\nu_r=\varepsilon_r'^2$ since $r\nmid c_{\nu\bar{\varepsilon}^2}$. We get $d_r=\varepsilon'(r).b_r=0$ hence

$$R^{(r)}(X) = 1$$

and

$$\mathcal{D}^{(r)}(h,X)=\frac{(1-\chi(r)r^{k-1}X)^2(1-\chi(r)\bar{\nu}\varepsilon^2(r)\gamma_r^2X)(1-\chi(r)\nu\bar{\varepsilon}^2(r)\bar{\gamma}_r^2X)}{(1-\chi(r)\mathring{\nu}(r)r^{k-1}X)}$$

where $|\gamma_r|^2 = r^{k-1}$.

If $r|(M,c_{\varepsilon'})$, then again $d_r=\varepsilon'(r).b_r=0$ and therefore

$$R^{(r)}(X) = 1 \ ,$$

whereas $\gamma_r = b_r$ and $\delta_r = 0$ yield

$$\mathcal{D}^{(r)}(h,r^{-s})=\rho_r(\chi,s+2-k)(1-|b_r|^2\chi(r)r^{-s})/(1-\chi(r)\mathring{\nu}(r)r^{k-1-s}) ,$$

hence

$$\mathcal{D}^{(r)}(h,X)=\begin{cases}(1-\chi(r)^2r^{2k-2}X^2)/(1-\chi(r)\mathring{\nu}(r)r^{k-1}X) & \text{if } b_r=0 \text{ and } \mathrm{ord}_rM \text{ even},\\ (1-\chi(r)r^{k-1}X)(1-|b_r|^2\chi(r)X)/(1-\chi(r)\mathring{\nu}(r)r^{k-1}X) & \text{otherwise}.\end{cases}$$

<u>Conclusion:</u> <u>For all but finitely many primes</u> r <u>we have</u>

$$\mathcal{D}^{(r)}(h,X) = R^{(r)}(X)$$

<u>and moreover</u>

$$R(\chi,s) = R(h,\chi,s) \cdot Q(\chi,s) ,$$

<u>where</u> $Q(\chi,s)$ <u>is a product of rational functions</u> Q_r <u>in</u> r^{-s} <u>whose zeros and poles are on the lines</u> $\mathrm{Re}(s)=k-1$, $k-2$. <u>Moreover for</u> $\sigma\in\mathrm{Aut}(\bar{\mathbb{Q}})$ <u>and any integer</u> $m\neq k-1$, $k-2$ <u>we have</u> $Q(\chi,m)^\sigma=Q(\chi^\sigma,m)$ <u>if</u> $\nu_r=1$ for $r\nmid M$.

Now we can complete the proof of the entireness of $R(\chi,s)$. By Proposition 2.9, Corollary 2.10 we know that $R(\chi,s)$ is holomorphic outside the lines $\mathrm{Re}(s)=k-1$, $k-2$, hence by the functional equation (Proposition 2.7) is holomorphic everywhere.

§ 3. Algebraicity of special values

As before let f be a newform in $S_k(N,\nu)$. The aim of this section is to study algebraicity properties of the special values $\mathcal{D}_\infty(f,\chi,m)$ for $m=1,2,\ldots,2k-2$.

Remark 3.1: Such results for the imprimitive symmetric square $D_\infty(f,\chi,s)$ were first proven by Sturm [10] . However, since the Euler factors of $\mathcal{D}_\infty(f,\chi,s)$ which do not appear in $D_\infty(f,\chi,s)$ may vanish at $s=k-1$ but never vanish at $s=k$ we will deduce also algebraicity statements at $m=k-1$ from the functional equation for $\mathcal{D}_\infty(f,\chi,s)$.

We normalize the Petersson inner product for forms f_i of weight k for $\Gamma_0(N)$ such that $f_1 f_2$ is a cusp form via

$$< f_1,f_2 >_N = \int_{\Gamma_0(N)\backslash H} f_1(z)\overline{f_2(z)}y^{k-2}dxdy .$$

Let ω be the primitive character such that

$$\omega(d) = \nu(d).\chi(d)\left(\frac{-1}{d}\right)^k\left(\frac{\chi(-1)}{d}\right) \quad \text{for} \quad (d,4Nc_\chi) = 1 .$$

We define the quantities

$$Z_0(f,\chi,m):=\frac{\pi^{-m}}{<f,f>}G(\bar{\omega})\mathcal{D}_\infty(f,\chi,m) \quad \text{for} \quad \chi(-1)=(-1)^{m+1},1\leq m\leq k-1,$$

$$Z_1(f,\chi,m):=\frac{\pi^{k-2m-1}}{<f,f>}G(\bar{\chi}^2)\mathcal{D}_\infty(f,\chi,m) \quad \text{for } \chi(-1)=(-1)^m,k\leq m\leq 2k-2$$

under the assumptions of Theorem 1. By Proposition 2.1 we can assume that $4|N$.

<u>Theorem 2: Suppose</u> $\chi^2 \neq 1$. <u>If</u> $\nu_r = 1$ <u>for</u> $r \nmid M$, <u>then the</u> $Z_i(\mathfrak{f},\chi,m)$ <u>are</u> Aut($\mathbb{C}$)-equivariant, <u>i.e. for any automorphism</u> $\sigma \in \mathrm{Aut}(\mathbb{C})$

$$Z_i(\mathfrak{f},\chi,m)^\sigma = Z_i(\mathfrak{f},\chi^\sigma,m) \quad ,$$

otherwise we know that at least that $Z_i(\mathfrak{f},\chi,m)$ is algebraic.

<u>Remark 3.2:</u> If χ has the "wrong" parity $\chi(-1)=(-1)^m$, then $Z_0(\mathfrak{f},\chi,m)=0$ for $m=1,\ldots\ldots,k-1$, hence the theorem is trivially true in these cases, since the Γ-factors in the functional equation for $\mathcal{D}_\infty(\mathfrak{f},\chi,s)$ imply that $\mathcal{D}_\infty(\mathfrak{f},\chi,s)$ must vanish at $s=m$ in these cases.

<u>Proof of Theorem 2:</u> We start by quoting Sturm's results adjusted to our notation. If one defines quantities $Z_i(\mathfrak{f},\chi,m)$ by the same formula as $Z_i(\mathfrak{f},\chi,m)$ except that $\mathcal{D}_\infty$ is replaced by D_∞ then Sturm's result says (under the conditions $4|N$ and $\chi^2 \neq 1$) that the $Z_i(\mathfrak{f},\chi,m)$ are Aut($\mathbb{C}$)-equivariant (see Theorem 1 [10]). As we saw at the end of § 2 the two functions $\mathcal{D}_\infty$ and D_∞ only differ by a product $Q(\chi,s)$ of Euler factors with zeros and poles on the lines $\mathrm{Re}(s)=k-1$, $k-2$. Moreover if $\nu_r=1$ for $r \nmid M$,

then $Q(\chi,m)$ is $\mathrm{Aut}(\mathbb{C})$-equivariant for $m \neq k-1$, $k-2$, hence this proves already Theorem 2 for $m \neq k-1$, $k-2$. For $m=k-1$ we apply the functional equation to $Z_1(\mathfrak{f},\bar{\chi},k)$. We get

$$R(\bar{\chi},k) = W_{\bar{\chi}} R(\chi,k-1) ,$$

so

$$(B^2c_\chi^3c_\nu^{-1})^{k/2}(2\pi)^{-k}\Gamma(k)\pi^{-k/2}\Gamma(1)\mathcal{D}_\infty(\mathfrak{f},\bar{\chi},k) =$$

$$= W_{\bar{\chi}}(B^2c_\chi^3c_\nu^{-1})^{k-1/2}(2\pi)^{-k+1}\Gamma(k-1).\pi^{-k/2}\Gamma(\tfrac{1}{2})\mathcal{D}_\infty(\mathfrak{f},\chi,k-1) .$$

This enables us to write

$$Z_\circ(\mathfrak{f},\chi,k-1) = Z_1(\mathfrak{f},\bar{\chi},k)\ \frac{G(\bar{\omega})}{G(\chi^2)}\ \chi^2(B)\ \frac{G(\bar{\chi}\nu)G(\chi)^2}{G(\bar{\chi})^2}\ R_m$$

with some $R_m \in \mathbb{Q}^*$. Note, that $\chi\nu(-1)=(-1)^{m+k}=1$ here, so that in particular $\omega=\nu\chi$. Since $(c_\chi,c_\nu)=1$ we can decompose

$$G(\bar{\omega}) = G(\bar{\chi}\nu) = \bar{\chi}(c_\nu)\nu(c_\chi)G(\bar{\chi}).G(\nu)$$

so that by the wellknown automorphism rule for Gauß sums

$$G(\chi)^{\sigma_t} = \bar{\chi}(t)^{\sigma_t}\ G(\chi^{\sigma_t})$$

(for any automorphism σ_t which sends roots of unity to their t^{th} power) we get $\mathrm{Aut}(\mathbb{C})$-equivariance of $Z_\circ(\mathfrak{f},\chi,k-1)$. In case, that we only know algebraicity of $Z_1(\mathfrak{f},\bar{\chi},k)$ we can at least conclude that $Z_\circ(\mathfrak{f},\chi,k-1)$ is also algebraic. For $m=k-2$ one argues in the same way by going back to the $\mathrm{Aut}(\mathbb{C})$-equivariance of $Z_1(\mathfrak{f},\bar{\chi},k+1)$.

§ 4. P-adic interpolation

In this section we want to interpolate p-adically the algebraic numbers $Z_i(\mathfrak{f},\chi,m)$ given by the special values of $\mathcal{D}_\infty(\mathfrak{f},\chi,s)$ in the critical strip $m=1,\ldots\ldots,2k-2$. We deal first with the special values of the imprimitive function $D_\infty(\underline{f},\chi,s)$ for $m=1,\ldots\ldots,k-1$ and $\chi(-1)=(-1)^{m+1}$. For the rest of this paper we fix a rational prime $p\nmid 2Na_p$ and embeddings i_p and i_∞ of an algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$ in $\bar{\mathbb{Q}}_p$ and in $\mathbb{C}$:

$$\bar{\mathbb{Q}}_p \xleftarrow{\quad i_p \quad} \supset \bar{\mathbb{Q}} \subset \xrightarrow{\ i_\infty\ } \mathbb{C} .$$

By our assumption the Euler polynominal

$$1-a_pX + \nu(p)p^{k-1}X^2 = (1-\alpha_pX)(1-\beta_pX)$$

has a reciprocal root, say α_p , which is a p-adic unit.

<u>Theorem 3:</u> <u>For any odd</u> $m=1,\ldots\ldots,k-1$ <u>with</u> $2(k-m)\not\equiv 0(p-1)$ <u>there is a constant</u> $C(m)\in\bar{\mathbb{Q}}^*$ <u>and a power series</u> $\mathcal{G}_m(T)\in\mathbb{Z}_p[[T]]$ <u>such that for any non trivial finite character</u> $\chi: 1+p\mathbb{Z}_p \longrightarrow \mathbb{C}^*$ <u>we have</u>

$$i_p^{-1}\left(\mathcal{G}_m(i_\infty^{-1}(\chi(1+p)-1))\right)=i_\infty^{-1}\left(C(m)\left(\frac{p^{m-1}}{\alpha_p^2}\right)^{\mathrm{ord}_p c_\chi}\frac{G(\chi)}{\pi^m\langle\mathfrak{f},\mathfrak{f}\rangle}D_\infty(\mathfrak{f},\bar{\chi},m)\right) .$$

<u>Proof:</u> We choose an integer u prime to p and, following

Pančiškin [6] we define a distribution $\mu_{u,m}$ on $\Gamma:=1+p\mathbb{Z}_p$ by demanding

$$\int_\Gamma \chi d\mu_{u,m}=(1-\chi(u)^2u^{2(k-m)})\left(\frac{p^{m-1}}{\alpha_p^{\,2}}\right)^{m_\chi}\frac{G(\bar{\chi})}{\pi^m\langle f,f\rangle}D_\infty(f,\chi,m)$$

for non trivial characters χ of Γ of conductor $c_\chi=p^{m_\chi}$ and

$$\int_\Gamma d\mu_{u,m}=(1-u^{2(k-m)}).(1-\frac{p^{m-1}}{\alpha_p^{\,2}})(1-\beta_p^{\,2}p^{-m})(1-\nu(p)p^{k-1-m})\frac{1}{\pi^m\langle f,f\rangle}D_\infty(f,m) .$$

Note, that we always assume $N\equiv 0(4)$, so that by Sturm [10] also the last integral is algebraic. Theorem 3 is a consequence of

<u>Theorem 4: Pančiškin's distribution</u> $\mu_{u,m}$ <u>is bounded for any odd</u> $m=1,\ldots\ldots,k-1$.

We continue with the proof of Theorem 3. By Theorem 4 there is a constant $C(m)$ such that for any compact open $\mathcal{U}\subset\Gamma$ the value $C(m).\mu_{u,m}(\mathcal{U})$ is a p-integral algebraic number. Thus we get a measure $\mu^*_{u,m}$ on $\mathbb{Z}_p$ via the standard isomorphism

$$\mathbb{Z}_p\longrightarrow\Gamma\ ,\ s\longmapsto(1+p)^s\ .$$

For the corresponding element $G_{u,m}(T)$ in the Iwasawa algebra $\mathbb{Z}_p[[T]]$ we then have

$$G_{u,m}(\chi(1+p)-1) = \int_{\Gamma} \chi d\mu_{u,m} = \int_{\mathbb{Z}_p} \chi(1+p)^s d\mu^*_{u,m}(s) \ .$$

Since $2(k-m) \not\equiv 0(p-1)$ we can chose a $u \in \mathbb{Z}$ such that

$$u^{2(k-m)} \not\equiv 1(p) \ .$$

Therefore the factor $1-\bar{\chi}(u)^2 u^{2(k-m)}$ is always a p-adic unit so that it can be interpolated by a unit

$$H_{u,m}(T) \in \mathbb{Z}_p[[T]]^* \text{ , i.e.}$$

$$H_{u,m}(\chi(1+p)-1) = 1-\bar{\chi}(u)^2 u^{2(k-m)} \ .$$

Eventually we find that

$$G_m(T) := G_{u,m}(T) . H_{u,m}(T)^{-1} \in \mathbb{Z}_p[[T]]$$

is the power series with the required properties, which completes the proof of Theorem 3.

<u>Proof of Theorem 4:</u> We have to show that for any $y \equiv 1(p)$ the values

$$\mu_{u,m}(y+p^r\mathbb{Z}_p) = p^{-r} \ [\int_{\Gamma} d\mu_{u,m} +$$

$$\sum_{\substack{\chi \\ 2\le m_\chi \le r}} \bar{\chi}(y)(1-\chi(u)^2 u^{2(k-m)})\left(\frac{p^{m-1}}{\alpha_p^2}\right)^{m_\chi} \frac{G(\bar{\chi})}{\pi^m \langle f, f\rangle} D_\infty(f,\chi,m)]$$

have p-adic absolut value bounded independent of y and r. To begin with we define two modified forms

$$f_0(z) := f(z) - \beta_p . f(pz) \quad ,$$

$$f_1(z) := f(z) - \alpha_p . f(pz) \quad ,$$

which have the properties

(i) $\quad D_\infty(f_0,\chi,s) = D_\infty(f,\chi,s)$ for $m_\chi \ge 1$,

(ii) $\quad \int_\Gamma d\mu_{u,m} = (1-u^{2(k-m)}).(1-\frac{p^{m-1}}{\alpha_p^2}) \frac{1}{\pi^m \langle f,f\rangle} D_\infty(f_0,m)$,

(iii) $\quad f_1|T(p) = \beta_p . f_1$, $f_0|T(p) = \alpha_p . f_0$ for Hecke operator $T(p)$,

(iv) $\quad f_0^\rho = f_1$ (ρ=complex conjugation on Fourier coefficients).

We put $N.c_\chi^2 = N_\chi$ for $m_\chi \ge 1$ and want to give an integral expression for $D_\infty(f_0,\chi,s)$ following Shimura [8] (see also [10] and [11]):

$$(4\pi)^{-s/2}\Gamma(s/2)D_\infty(f_0,\chi,s)$$

$$= \int_{\Gamma_\circ(N_\chi)\backslash H} f_0(z)\,\overline{\theta_{\bar\chi}(z)}\, y^{-3/2} L_N(\chi^2, 2s+2-2k)\, E\,(z, s+1, 2k-1, \omega)\; dxdy$$

where the theta-series is given by

$$\theta_\chi(z) := \frac{1}{2} \sum_{n=-\infty}^{\infty} \chi(n) q^{n^2} ,$$

and the Eisenstein series is (in Shimura's notation)

$$E(z,s,\lambda,\omega) := y^{s/2} \sum_{\gamma\in W_\chi} \omega(d_\gamma)\, j(\gamma,z)^\lambda |j(\gamma,z)|^{-2s}$$

with W_χ any set of representatives for $\Gamma_\infty\backslash\Gamma_\circ(N_\chi)$, $\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix};\ t \in \mathbb{Z} \right\}$. Hence we can rewrite the integral as Petersson inner product

$$(4\pi)^{-s/2}\Gamma(s/2) D_\infty(f_0, \chi, s) =$$

$$= \langle f_0(z), \theta_{\bar\chi}(z) L_N(\bar\chi^2, 2s+2-2k) E(z, s+2-2k, 1-2k, \bar\omega) \rangle_{N_\chi} .$$

We want to consider

$$h^*(z,\chi,s) := h(z,\chi,s)\Big|_k \begin{pmatrix} 0 & -1 \\ N_\chi & 0 \end{pmatrix}$$

for

$$h(z,\chi,s) := \theta_\chi(z) . L_N(\chi^2, 2s+2-2k) E(z, s+2-2k, 1-2k, \omega) .$$

By definition

$$h^*(z,\chi,s) = \theta_\chi(-\frac{1}{N_\chi z}) L_N(\chi^2, 2s+2-2k) E(-\frac{1}{N_\chi z}, s+2-2k, 1-2k, \omega). (z\sqrt{N_\chi})^{-k}$$

Sturm [10] has shown that for m=1,.....k-1 the functions $h=h(z,\chi,m)$ are (nonholomorphic) generalized modular forms (cf. [10,p.234], [9,p.794f]). By Lemma 7 of [9] such forms can be uniquely written as

$$h = g_\circ + \sum_{\nu=1}^{r} \delta_{k-2\nu}^{(\nu)} g_\nu \qquad (r < \frac{k}{2})$$

where g_ν is a (holomorphic) modular form of level N_χ, weight $k-2\nu$ with the same nebentypus character as h and where the differential operator $\delta_{k-2\nu}^{(\nu)}$ is defined by

$$\delta_\lambda^{(\nu)} = \delta_{\lambda+2\nu-2} \cdot \ldots \cdot \delta_{\lambda+2}\delta_\lambda \quad (\nu \geq 1)$$

with

$$\delta_\lambda = \frac{1}{2\pi i}\left(\frac{\lambda}{2iy} + \frac{\partial}{\partial z}\right) \quad \text{for} \quad \lambda \in \mathbb{N} .$$

By Lemma 6 of [9] we get

$$\langle f_0(z), h(z,\bar{\chi},m)\rangle_{N_\chi} = \langle f_0(z), g_0(z,\bar{\chi},m)\rangle_{N_\chi} ,$$

i.e. the special value $D_\infty(f_0,\chi,m)$ only depends on the holomorphic projection g_0 of h. Since the Petersson inner product is the same if we apply the operator

$$W_{N_\chi} = \begin{pmatrix} 0 & -1 \\ N_\chi & 0 \end{pmatrix}$$

to both arguments, we also have

$$<\delta_0(z),\ h(z,\bar{\chi},m)>_{N_\chi} = <\delta_0^*(z),\ h^*(z,\bar{\chi},m)>_{N_\chi} .$$

Taking holomorphic projection of the generalized modular form $h^*(z,\bar{\chi},m)$ leads to

$$<\delta_0(z),\ h(z,\bar{\chi},m)>_{N_\chi} = <\delta_0(z),(h^*)_0{}^*>_{N_\chi} .$$

<u>Lemma 4.1:</u> <u>There are linear forms</u> $F_n(X_0,\ldots\ldots,X_r) \in \mathbb{Z}[X_0,\ldots,X_r]$ <u>which depend only on</u> k <u>and</u> n <u>such that</u>

$$C.(h^*)_0 = \sum_{n=0}^{\infty} F_n(c_{n,o},\ldots\ldots,c_{n,r})q^n$$

<u>and</u>

$$F_n(X_0,\ldots\ldots X_r) \equiv C.X_0 \bmod n$$

<u>for a fixed constant</u> $C \in \mathbb{Z}$, <u>where</u>

$$h^*(z,\chi,m) = \sum_{j=0}^{r} (4\pi y)^{-j} \sum_{n=0}^{\infty} c_{n,j}q^n .$$

<u>Proof:</u> From the formula

$$h = g_0 + \sum_{\nu=1}^{r} \delta^{(\delta)}_{k-2\nu}\, g_\nu$$

we get

$$g_r^* = g_r|_{k-2r} W_{N_\chi} = (-1)^r \sum_{n=0}^{\infty} c_{n,r} q^n$$

by comparison of the coefficients of y^{-r} . Using the identity

$$\delta_\lambda^{(\nu)} = \sum_{j=0}^{\nu} \binom{\nu}{j} \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda+\nu-j)} (-4\pi y)^{-j} \left(\frac{1}{2\pi i} \frac{\partial}{\partial z}\right)^{\nu-j}$$

we arrive at

$$h^* - \frac{\Gamma(k-2r)}{\Gamma(k-r)} \delta_{k-2r}^{(r)} g_r^*$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{r-1} \left(c_{n,j} - (-1)^{j+r} \frac{\Gamma(k-2r)}{\Gamma(k-r-j)} \binom{r}{j} n^{r-j} c_{n,r} \right) (4\pi y)^{-j} q^n \ .$$

This being the first step towards reducing h^* to its holomorphic part $(h^*)_0$ we can continue now with h^* replaced by

$${}_{(1)}h^* := h^* - \frac{\Gamma(k-2r)}{\Gamma(k-r)} \delta_{k-2r}^{(r)} g_r^*$$

which has the form

$${}_{(1)}h^* = \sum_{j=0}^{r-1} (4\pi y)^{-j} \sum_n c_{n,j}^{(1)} q^n \ .$$

After r steps we arrive at $_{(r)}h^*=(h^*)_0$ and we see from the formula for the first step, that $C.c_{n,o}^{(r)}$ will be an integral linear combination of $c_{n,o},\ldots\ldots,c_{n,r}$ where the coefficients of $c_{n,1},\ldots\ldots,c_{n,r}$ are divisible by n. This proves the lemma.

The Fourier coefficients $c_{n,j}=c_{n,j}(\chi)$ of $h^*(z,\chi,m)$ can be explicitely determined from [8,p.86ff] and [7,p.457]:

$$h^*(z,\chi,m)=\theta_{\bar{\chi}}(\tfrac{N}{4}z)G(\chi)(-i\tfrac{N}{2}z)^{1/2}L_N(\chi^2,2m+2-2k)E'(z,m+2-2k,1-2k,\omega)(z\sqrt{N_\chi})^{-1/2}$$

where

$$E'(z,m+2-2k,1-2k,\omega) = E(z,m+2-2k,1-2k,\omega)\Big|_{k-\frac{1}{2}}\begin{pmatrix}0 & -1\\ N_\chi & 0\end{pmatrix}.$$

Proposition 1 in [8] says:

$$N_\chi^{\frac{2m+3-2k}{4}}\, i^{\frac{1}{2}-k}\, y^{k-1-\frac{m}{2}}\, L_N(\chi^2,2m+2-2k).E'(z,m+2-2k,1-2k,\omega) =$$

$$\frac{\Gamma(m+\frac{1}{2}-k)}{\Gamma(\frac{m+1}{2})\,\Gamma(\frac{m}{2}+1-k)}\, i^{1/2-k}(2y)^{k-m-1/2}\, 2\pi.L_N(\omega^2,2m+2-2k) +$$

$$\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} e^{2\pi inx}\tau_n(y,\tfrac{m+1}{2},\tfrac{m}{2}+1-k).L_N(\omega_n,m+1-k).\beta(n,m+2-2k)$$

where ω_n denotes the primitive character given by

$$\omega_n(a):=(\tfrac{-1}{a})^{k+1}(\tfrac{nN}{a})\,\omega(a)\quad ,$$

the functions τ_n are defined by

$$i^{\alpha-\beta}(2\pi)^{-\alpha-\beta}\Gamma(\alpha)\Gamma(\beta)\tau_n(y,\alpha,\beta) = \begin{cases} n^{\alpha+\beta-1}e^{-2\pi ny}\sigma(4\pi ny,\alpha,\beta) & \text{if } n>0 \\ |n|^{\alpha+\beta-1}e^{-2\pi|n|y}\sigma(4\pi|n|y,\beta,\alpha) & \text{if } n<0, \\ \Gamma(\alpha+\beta-1).(4\pi y)^{1-\alpha-\beta} & \text{if } n=0 \end{cases}$$

with the hypergeometric function

$$\sigma(y,\alpha,\beta) = \int_0^\infty (u+1)^{\alpha-1}u^{\beta-1}e^{-yu}du \ ,$$

and where

$$\beta(n,s) = \sum_{a,b} \mu(a)\omega_n(a)\omega^2(b)a^{1-k-s}b^{3-2k-2s} \ ,$$

the sum being extended over all integers $a,b>0$ prime to $N.p$ such that $(ab)^2$ divides n . (μ = Moebius function). We remark, that we can restrict the sum above to positive n , since for $n<0$ the character ω_n has the same parity as k , i.e. $\omega_n(-1)=(-1)^k$, and therefore $L_N(\omega_n, m+1-k)$ vanishes for odd $m=1,\ldots\ldots,k-1$. For $n>0$ the values of τ_n are

$$\tau_n(y,\tfrac{m+1}{2},\tfrac{m}{2}+1-k) = n^{m-k+1/2}e^{-2\pi ny}i^{k-3/2}(2\pi)^{m-k+3/2}.$$

$$.\,\Gamma(\tfrac{m+1}{2})^{-1}\Gamma(\tfrac{m}{2}+1-k)^{-1}\sum_{\chi=0}^{\frac{m-1}{2}}\binom{\frac{m-1}{2}}{\chi}\Gamma(\tfrac{m}{2}+1-k+\chi)(4\pi ny)^{k-\frac{m}{2}-1-\chi}$$

and so we can express

$$L_N(\chi^2,2m+2-2k)E'(z,m+2-2k,1-2k,\omega)=\sum_{j=0}^{\frac{m-1}{2}}\sum_{n=0}^{\infty}(4\pi y)^{-j}d_{j,n}q^n$$

where $d_{j,o}=0$ except

$$d_{\frac{m-1}{2},0}=B_{\frac{m-1}{2}}.(\pi)^{\frac{m+1}{2}}.2^{k-m/2}N_\chi^{(2k-2m-3)/4}.L_N(\omega^2,2m+2-2k)$$

and for $n>0$:

$d_{j,n}=$

$$(-1)^{k-1}2^{k-\frac{1}{2}}\pi^{\frac{m+1}{2}}.N_\chi^{(2k-3-2m)/4}\binom{\frac{m-1}{2}}{j}B_j n^{\frac{m-1}{2}-j}.L_N(\bar{\omega}_n,m+1-k).\beta(n,m+2-2k)$$

where $$B_j := \frac{\Gamma(\frac{m}{2}+1-k+j)}{\Gamma(\frac{m+1}{2})\Gamma(\frac{m}{2}+1-k)} \in \mathbb{Q}^* .$$

Now we obviously get the Fourier coefficients $c_{n,j}=c_{n,j}(\chi)$ of $h^*(z,\chi,m)$ by multiplying the q-expansion of the theta series with the Fourier expansion of $L_N(\chi^2,\dots).E'(z,\dots.)$ above as follows:

<u>Lemma 4.2:</u> <u>There are constants</u> C', $C_j=C_j(k,m,N)\in\mathbb{Q}^*.\pi^{(m+1)/2}$ <u>for</u> $j=0,\dots.\frac{m-1}{2}$ <u>such that</u>

$$c_{n,j}(\chi)=C_j.c_\chi^{k-m-2}.G(\chi).\sum_{\substack{n_1,n_2>0\\ \frac{N}{4}n_1^2+n_2=n}}\bar{\chi}(n_1)n_2^{\frac{m-1}{2}-j}.L_N(\omega_{n_2},m+1-k).\beta(n_2,m+2-2k)$$

for $j \neq \frac{m-1}{2}$ and

$$c_{n,\frac{m-1}{2}}(\chi) = \begin{Bmatrix} c_{\chi}^{k-m-2}.C'.G(\chi).\bar{\chi}(n_1).L_N(\omega^2, 2m+2-2k) & \text{if } n=\frac{N}{4}n_1^2 \\ 0 & \text{otherwise} \end{Bmatrix} +$$

$$C_{\frac{m-1}{2}} \sum_{\substack{n_1,n_2>0 \\ \frac{N}{4}n_1^2+n_2=n}} \bar{\chi}(n_1).L_N(\omega_{n_2}, m+1-k).\beta(n_2, m+2-2k)$$

The case of the trivial character $\chi=\chi_0$ being similar to the nontrivial case we omit the details and just state the result:

Lemma 4.3: With the same constants as in Lemma 4.2 we have for $j \neq \frac{m-1}{2}$:

$$c_{n,j}(\chi_0) = C_j.p^{(k-m-1)/2}\Big[\sum_{\substack{n_1,n_2>0 \\ \frac{N}{4}pn_1^2+n_2=n}} n_2^{\frac{m-1}{2}-j} L_{Np}(\omega_{pn_2}, m+1-k).$$

$$.\beta(n_2, m+2-2k)+\tfrac{1}{2}n^{\frac{m-1}{2}-j} L_{Np}(\omega_{pn}, m+1-k)\,\beta(n, m+2-2k)\Big],$$

and for $j=\frac{m-1}{2}$:

$$c_{n,\frac{m-1}{2}}(\chi_0) = p^{(k-m-1)/2}\Big[\begin{Bmatrix} C'.L_{Np}(\chi_0, 2m+2-2k) & \text{if } n=\frac{N}{4}pn_1^2, \\ 0 & \text{otherwise} \end{Bmatrix}.\begin{Bmatrix} \frac{1}{2} & \text{if } n=0 \\ 1 & n>0 \end{Bmatrix}$$

$$C_{\frac{m-1}{2}} \cdot \sum_{\substack{n_1,n_2>0 \\ \frac{N}{4}pn_1^2+n_2=n}} L_{Np}(\omega_{pn_2}, m+1-k)\ \beta(n_2, m+2-2k)] .$$

We are in particular interested in the behaviour of the $c_{n,0}(\chi)$'s .

<u>Lemma 4.4: There is a global factor</u> $C\in\mathbb{Q}^*$ <u>such that the following congruence holds for any</u> $n,r\in\mathbb{N}$ <u>and</u> y <u>prime to</u> p:

$$C.[p^{\frac{k}{2}-1} \cdot \sum_{\substack{\chi\neq\chi_0 \\ m_\chi\leq r}} \bar{\chi}(y).(1-\bar{\chi}(u)^2u^{2(k-m)}).G(\chi)c_\chi^{-(k-1-m)}c_{np^{2r-1},0}(\bar{\chi})$$

$$+(1-u^{2(k-m)})(c_{n.p^{2r},0}(\chi_0)-p^{m-1}c_{np^{2r-2},0}(\chi_0))]\equiv 0(p^r) .$$

<u>Proof:</u> By Lemma 4.2 the first sum in the brackets becomes:

$$p^{\frac{k}{2}-1} \cdot C_0 \sum_{\substack{\chi\neq\chi_0 \\ m_\chi\leq r}} \bar{\chi}(y).(1-\bar{\chi}(u)^2u^{2(k-m)})G(\chi)c_\chi G(\bar{\chi}) .$$

$$\cdot \sum_{\substack{n_j>0 \\ \frac{N}{4}n_1^2+n_2=np^{2r-1}}} \chi(n_1).L_N.(\bar{\omega}_{n_2}, m+1-k).\beta(n_2, m+2-2k)n_2^{\frac{m-1}{2}}$$

and by Lemma 4.3 the χ_0 - part in the case $m\neq 1$ is given by

$$p^{\frac{k}{2}-1}.C_0.(1-u^{2(k-m)}).[\sum_{\substack{n_j>0 \\ \frac{N}{4}pn_1^2+n_2=np^{2r}}} (n_{2/p})^{\frac{m-1}{2}} L_{Np}(\omega_{pn_2},m+1-k)\,\beta(n_2,m+2-2k)$$

$$+\frac{1}{2}(np^{2r-1})^{\frac{m-1}{2}} L_{Np}(\omega_{pn},m+1-k)\,\beta(n,m+2-2k)$$

$$-p^{m-1}\sum_{\substack{n_j>0 \\ \frac{N}{4}pn_1^2+n_2=np^{2r-2}}} (n_{2/p})^{\frac{m-1}{2}} L_{Np}(\omega_{pn_2},m+1-k)\,\beta(n_2,n+2-2k)$$

$$-\frac{1}{2}p^{m-1}(np^{2r-3})^{\frac{m-1}{2}} L_{Np}(\omega_{pn},m+1-k)\,\beta(n,m+2-2k)$$

$$=p^{\frac{k}{2}-1}.C_0.(1-u^{2(k-m)})\sum_{\substack{n_j>0 \\ \frac{N}{4}n_1^2+n_2=np^{2r-1} \\ p\nmid n_i}} n_2^{\frac{m-1}{2}} L_{Np}(\omega_{n_2},m+1-k)\,\beta(n_2,m+2-2k)$$

where we have used that β only depends on the part of n prime to p and that ω_n only depends on the square free part of n. For the case $m=1$ in a similar way we arrive at the same expression. Now it is obviously sufficient to make a fixed choice of data $(n_1,n_2,a,b) \in \mathbb{Z}^4_{>0}$ with $p\nmid n_i$, $(ab)^2|n_2$, $(ab,Np) = 1$ and to prove for any such choice the congruence

$$\sum_{\substack{\chi \\ m_\chi \leq r}} \bar{\chi}(y).(1-\bar{\chi}(u)^2 u^{2(k-m)}) L_p(\bar{\omega}_{n_2}, m+1-k)\cdot (1-\bar{\omega}_{n_2}(p) p^{k-m-1} \bar{\omega}_{n_2}(a)\bar{\omega}^{2}(b)) \equiv 0 (p^r) \quad ,$$

since the expression in the lemma is an integral linear combination of these sums. We remark that as well we could have omitted the factors $\bar{\omega}_{n_2}(a)\bar{\omega}^{2}(b)$ just by changing y. Since $\omega^2=\chi^2$ and

$$\omega_{n_2}(t) = \left(\frac{-n_2 N}{t}\right) \nu(t) \ \chi(t)$$

we are reduced to show

$$\sum_{\substack{\chi \\ m_\chi \leq r}} \bar{\chi}(y).(1-\bar{\omega}_{n_2}(u_2)^2 u_2^{2(k-m)}) L(\bar{\omega}_{n_2}, m+1-k).(1-\bar{\omega}_{n_2}(p) p^{k-m-1}) \equiv 0 (p^r) \quad ,$$

where we have chosen $u_2 \equiv u \bmod p^r$ such that $(u_2, n_2 N)=1$. This again can be reduced to prove, in terms of Bernoulli numbers

$$\sum_{\substack{\chi \\ m_\chi \leq r}} \bar{\chi}(y).(1-\bar{\omega}_{n_2}(u_2) u_2^{k-m}) \frac{B_{k-m,\bar{\omega}_{n_2}}}{k-m} (1-\bar{\omega}_{n_2}(p) p^{k-m-1}) \equiv 0 (p^r) \quad .$$

But this congruence is exactly the condition for the smoothed Bernoulli distribution

$$E_{k-m,u_2}(y+p^r \mathbb{Z}_p) := p^{r(k-m-1)} \frac{-1}{k-m}\left[B_{k-m}\left(\left\langle\frac{y}{p^r}\right\rangle\right) - u_2^{k-m} B_{k-m}\left(\left\langle\frac{u_2^{-1} y}{p^r}\right\rangle\right)\right]$$

(cf. [4] p.45) to be a measure, which proves Lemma 4.4.

<u>Lemma 4.5:</u> <u>The statement of Lemma 4.4 remains true if we replace the coefficients</u> $c_{n,0}(\chi)$ <u>of</u> $h^*(z,\chi,m)$ <u>by the coefficients</u> $c_n(\chi)$ <u>of the holomorphic projection</u>

$$h^*_0(z,\chi,m) = \sum_{n=0}^{\infty} c_n(\chi).q^n .$$

<u>Proof:</u> By Lemma 4.1 we have

$$C.c_n(\chi) = F_n(c_{n,0}(\chi),\ldots\ldots,c_{n,\frac{m-1}{2}}(\chi))$$

where

$$F_n(X_0,\ldots\ldots,X_{\frac{m-1}{2}}) \equiv CX_0 \bmod n.\mathbb{Z}[X_0,\ldots\ldots,X_{\frac{m-1}{2}}] .$$

The expression in brackets of Lemma 4.4 remains at least integral if we replace the $c_{n,0}(\chi)$ by the $c_{n,j}(\chi)$, so that from F_n being a linear form we get

$$C.C_0^{-1}[p^{k/2-1} \sum_{\substack{\chi \neq \chi_0 \\ m_\chi \leq r}} \bar{\chi}(y).(1-\bar{\chi}(u)^2 u^{2(k-m)})G(\chi)c_\chi^{-(k-1-m)} c_{np^{2r-1}}(\chi)$$

$$+(1-u^{2(k-m)}).(c_{np^{2r}}(\chi_0)-p^{m-1}c_{np^{2r-2}}(\chi_0))]$$

$$C.C_0^{-1}[p^{k/2-1}.\sum_{\substack{\chi\neq\chi_0\\ m_\chi\le r}}\bar{\chi}(y).(1-\bar{\chi}(u)^2u^{2(k-m)})G(\chi).c_\chi^{-(k-1-m)}c_{np^{2r-1},0}(\chi)$$

$$+(1-u^{2(k-m)}).(c_{np^{2r},0}(\chi_0)-p^{m-1}.c_{np^{2r-2},0}(\chi_0))\ \mathrm{mod}\ np^{2r-2}\ ,$$

so Lemma 4.4 yields the desired congruence for $r\geq 2$, which proves Lemma 4 for some constant C .

Now we can finish the proof of Theorem 4 by showing that for some $C(m)\in\mathbb{Q}^*$ and with

$$M_r:=\sum_{\substack{\chi\neq\chi_0\\ m_\chi\le r}}\bar{\chi}(y).(1-\chi(u)^2u^{2(k-m)})\left(\frac{p^{m-1}}{\alpha_p^2}\right)^{m_\chi}\frac{G(\bar{\chi})}{\pi^m<\mathfrak{f},\mathfrak{f}>}D_\infty(\mathfrak{f}_0,\chi,m)$$

$$+\ (1-u^{2(k-m)}).(1-\frac{p^{m-1}}{\alpha_p^2})\frac{1}{\pi^m<\mathfrak{f},\mathfrak{f}>}\cdot D_\infty(\mathfrak{f}_0,m)$$

the product $C(m).M_r$ is divisible by p^r for any $r\in\mathbb{N}$. As we saw earlier we have

$$D_\infty(\mathfrak{f}_0,\chi,m)=(4\pi)^{-m/2}\Gamma(m/2)<\mathfrak{f}_0(z),h(z,\bar{\chi},m)>_{N_\chi}\quad\text{for}\quad\chi\neq\chi_0$$

and

$$D_\infty(\mathfrak{f}_0,m)=(4\pi)^{-m/2}\Gamma(m/2)<\mathfrak{f}_0(z),h(z,\chi_0,m)>_{Np}\ ,$$

where we can replace h by $(h^*)_0$. For $\chi\neq\chi_0$ apply the trace $tr = Tr_{\Gamma_0(N_\chi)\setminus\Gamma_0(Np)}$ to the modular form $(h^*)_0{}^*$ without really affecting the inner product

$$D_\infty(\delta_0,\chi,m) = (4\pi)^{-m/2}\,\Gamma(m/2)\,\langle\delta_0, tr((h^*)_0{}^*)\rangle_{Np} .$$

Since $\Gamma_0(N_\chi)\setminus\Gamma_0(Np)$ is represented by the matrices $\begin{pmatrix}1 & 0\\ Npi & 1\end{pmatrix}$ for $i \bmod p^{2m_\chi-1}$ one easily sees for

$$tr((h^*)_0{}^*) = \sum_{i \bmod 2m_\chi-1} (h^*)_0{}^*|_k \begin{pmatrix}1 & 0\\ Npi & 1\end{pmatrix}$$

that

$$tr((h^*)_0{}^*|_k \begin{pmatrix}0 & -1\\ Np & 0\end{pmatrix} = \sum_i (h^*)_0|_k \begin{pmatrix}1 & -i\\ 0 & p^{2m_\chi-1}\end{pmatrix}$$

$$= p^{-(2m_\chi-1)(\frac{k}{2}-1)}\,(h^*)_0|T(p)^{2m_\chi-1} .$$

Therefore we get with $W_{Np} := \begin{pmatrix}0 & -1\\ Np & 0\end{pmatrix}$:

$$\langle\delta_0, h(z,\chi,m)\rangle_{N_\chi} = p^{-(2m_\chi-1)(\frac{k}{2}-1)}\langle\delta_0|_k W_{Np}, (h^*)_0|T(p)^{2m_\chi-1}\rangle_N .$$

Since W_{Np} normalizes $\Gamma_1(Np)$ and

$$W_{Np}^{-1}\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}W_{Np} = \begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix},$$

we see that the adjoint $T(p)^*$ of $T(p)$ on the level Np is given as

$$T(p)^* = W_{NP}^{-1} \circ T(p) \circ W_{Np} ,$$

which for any $r \geq m_\chi$ via $\delta_0 | T(p) = \alpha_p \cdot \delta_0$ implies:

$$\alpha_p^{2(r-m_\chi)} <\delta_0, h(z,\chi,m)>_{N_\chi} = p^{-(2m_\chi-1)(\frac{k}{2}-1)} <\delta_0 |_k W_{Np}, (h^*)_0 | T(p)^{2r-1}>_{Np}.$$

Similar we get with $h^* = h(z,\chi_0,m)|_k W_{Np}$

$$\alpha_p^{2r}\left(1-\frac{p^{m-1}}{\alpha_p^2}\right) <\delta_0, h(z,\chi_0,m)>_{Np}$$

$$= <\delta_0 |_k W_{Np}, (h^*)_0 | T(p)^{2r} - p^{m-1} . (h^*)_0 | T(p)^{2r-2}>_{Np} .$$

So if we define the modular forms

$$F_{r,y}(z) := \sum_{\chi \neq \chi_0} \chi(y) . (1-\bar{\chi}(u)^2 u^{2(k-m)} . p^{(m-1)m_\chi - (2m_\chi-1) . (\frac{k}{2}-1)} \frac{G(\chi)}{\pi^m <\delta,\delta>}$$

$$(h^*)_0(z,\bar{\chi},m) | T(p)^{2r-1}$$

$$+ (1-u^{2(k-m)}) \frac{1}{\pi^m <\delta,\delta>} ((h^*)_0(z,\chi_0,m) | T(p)^{2r}$$

$$- p^{m-1} . (h^*)_0(z,\chi_0,m) | T(p)^{2r-2})$$

we simply have

$$M_r = (4\pi)^{-m/2} . \Gamma(m/2) < \mathfrak{f}_0 |_k W_{Np}, F_{r,y}(z) >_{Np} .$$

Since the effect of $T(p)$ on Fourier coefficients is given by

$$T(p) : (h^*)_0 = \sum_{n=0}^{\infty} c_n q^n \longrightarrow \sum_{n=0}^{\infty} c_{np} q^n$$

we conclude from Lemma 4.5 and the Aut($\mathbb{C}$)-equivariance of $Z_0(\mathfrak{f}, \chi, m)$

<u>Lemma 4.6:</u> <u>The modular forms</u> $F'_{r,y} := C.p^{-r}.F_{r,y}(z).\pi^m < \mathfrak{f}, \mathfrak{f} >$ <u>have p-integral Fourier coefficients going to</u> $\mathbb{Z}_p$ <u>under</u> $i_p : \bar{\mathbb{Q}} \longrightarrow \bar{\mathbb{Q}}_p$.

-

The space $M_k(Np)$ of weight k modular forms of level Np having a $\mathbb{Q}$-structure, we also know that the forms $F'_{r,y}$ all lie in a finite dimensional $\bar{\mathbb{Q}}$-vector space, hence by Lemma 4.6 in a $\mathbb{Z}_p$-lattice. Therefore the values of the linear form

$$L_m : M_k(Np) \longrightarrow \mathbb{C}, F \longrightarrow (4\pi)^{-m/2} . \Gamma(m/2) \ < \mathfrak{f}_0 |_k W_{Np}, F >_{Np}$$

restricted to the set $\left\{ F'_{r,y} ; y, r \in \mathbb{N}, y \equiv 1 (p) \right\}$ must also lie in a $\mathbb{Z}_p$-lattice, hence they have in particular bounded p-adic absolut value, which proves that Pančiškin's distributions $\mu_{u,m}$ are in fact measures (i.e. bounded).

<u>Remark 4.7:</u> a) If the assumption $2(k-m) \not\equiv 0 (p-1)$ of Theorem 3

is not fulfilled we still may define an element

$$\mathcal{G}_m(T) \in \text{Quot}\ (\mathbb{Z}_p[[T]])$$

such that for all but finitely many characters χ we have

$$i_p^{-1}(\mathcal{G}_m(i_\infty^{-1}(\chi(1+p)-1))) = i_\infty^{-1}(C(m).(\frac{p^{m-1}}{\alpha_p^2})^{\text{ord}_p c_\chi} \frac{G(\chi)}{\pi^m\langle f,f\rangle} D_\infty(f,\bar{\chi},m)),$$

so that we get in any case a p-adic L-function by putting

$$D_{p,m}(f,s) := \mathcal{G}_m((1+p)^{1-s}-1) \quad \text{for} \quad s \in \mathbb{Z}_p\ .$$

b) By avoiding those χ where one of the "missing Euler factors" of the imprimitive symmetric square vanishes one also finds (by p-adic interpolation of these factors) an element

$$\tilde{\mathcal{G}}_m(T) \in \text{Quot}\ (\bar{\mathbb{Z}}_p[[T]])$$

such that we get p-adic interpolation of the special values of the primitive symmetric square by

$$i_p^{-1}(\tilde{\mathcal{G}}_m(i_\infty^{-1}(\chi(1+p)-1))) = i_\infty^{-1}(C(m)\,(\frac{p^{m-1}}{\alpha_p^2})^{\text{ord}_p c_\chi}\, \frac{G(\chi)}{\pi^m\langle f,f\rangle}\, \mathcal{D}_\infty(f,\bar{\chi},m))$$

for all but finitely many χ . I would expect that in fact $\tilde{\mathcal{G}}_m$ is a power series in $\mathbb{Z}_p[[T]]$ and that this equality holds for all χ with the appropriate change of the right hand side for $\chi=\chi_0$.

We define the associated L-function as

$$\mathcal{D}_{p,m}(\mathfrak{f},s) := \tilde{G}_m((1+p)^{1-s}-1) .$$

It is clear that by the functional equation satisfied by $\mathcal{D}_\infty$ we also get a measure on Γ describing the p-adic interpolation of the special values in the right half of the critical strip $m=k,\ldots\ldots,2k-2$. We define

$$\tilde{G}_m(T) := \tilde{G}_{2k-1-m}\left(\frac{-T}{1+T}\right) \quad \text{for} \quad m=k,\ldots\ldots,2k-2 .$$

<u>Proposition 4.8:</u> <u>For any even</u> $m=k,\ldots\ldots,2k-2$ <u>there is a constant</u> $C=C(m,\mathfrak{f}) \in \mathbb{Q}^*$ <u>such that for all but finitely many</u> χ :

$$i_p^{-1}(\tilde{G}_m(i_\infty^{-1}(\chi(1+p)-1)))=i_\infty^{-1}\left(C\chi\left(\frac{B^2}{c_\nu}\right)\left(\frac{p^{2m-k-1}}{\alpha_p^2}\right)^{m_\chi} G(\chi)^2\mathcal{D}_\infty(\mathfrak{f},\bar{\chi},m)\right)$$

where B denotes the integer which appears in the functional equation in Theorem 1.

The proof just consists of applying the functional equation for $\mathcal{D}_\infty$ relating the values at m and $2k-1-m$, using the fact that for $p\nmid a_p$ we have $\nu(p)=1$, and to follow the definition of $\tilde{G}_m$ for $m=k,\ldots\ldots,2k-2$. As an immediate consequence of the definition we see that the corresponding functional equation of the p-adic L-functions reads:

$$\mathcal{D}_{p,m}(\mathfrak{z},s) = \mathcal{D}_{p,2k-1-m}(\mathfrak{z},2-s) .$$

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