Hecke's integral formula

by

Eduardo Friedman

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str.26
D-5300 Bonn 3
Federal Republic of Germany

Department of Mathematics
University of Pennsylvania
Philadelphia, PA 19104
U.S.A.

MPI/88-44

Hecke's integral formula

Eduardo Friedman*

Summary: We simplify the multiple integral appearing in Hecke's formula for the Dedekind zeta function by turning it into a single integral. Hecke's expression then turns out to be but one of an infinite family of formulas each of which is equivalent to the functional equation and meromorphic continuation of the zeta function. As a corollary we obtain a formula for the regulator and a lower bound for the number of integral ideals of norm at most $|D|^{\frac{1}{2}}(\log\log|D|)^{n/2}$, where D is the discriminant and n the degree. We also give a practical formula for quick calculations with any Dirichlet series that has a functional equation and an analytic continuation.

§ 1. Introduction

Hecke [He] gave in 1917 the first formula for the (completed) Dedekind zeta function $\xi_k(s)$ valid in the entire complex plane:

$$\xi_k(s) = \frac{2^{r_1}hR}{w(s-1)s} + \sum_{a}\left[g\left[\frac{\sqrt{|D|}}{Na},s\right] + g\left[\frac{\sqrt{|D|}}{Na},1-s\right]\right] . \qquad (1.1)$$

* Research supported by the Max-Planck-Institut für Mathematik and the National Science Foundation.

We shall spare the reader for as long as we can from a detailed description of Hecke's formula (1.1), but we must at least point out that $g(x,s)$ is given by an (r_1+r_2)-dimensional integral, $N = \mathrm{Norm}_{k/\mathbb{Q}}$ and that a runs over all integral ideals of the number field k. Hecke, of course, was interested in (1.1) mainly to prove the functional equation and meromorphic continuation of $\zeta_k(s)$. In fact, he wrote (1.1) in a slightly different form using theta functions. This, while theoretically preferable, is even more cumbersome than (1.1). When Siegel needed to use Hecke's formula to prove his part of the Brauer-Siegel theorem, he simplified it somewhat by replacing the theta functions by the ideals [Si 1].

The main problem with (1.1) is that $g(x,s)$ is very hard to compute if r_1+r_2 is not tiny. In applications, this has led to discarding the sum [Si 2] using only $g(x,s) > 0$ if s is real, or to bounding $g(x,s)$ from below rather coarsely [Si 1]. Sometimes one needs something more precise than this. Suppose we try to compute hR using (1.1). Since all terms make sense for $s > 1$, we can isolate the residue to obtain

$$\frac{2^{r_1} hR}{w} = s(s-1) \sum_{a} [\text{a mess}], \qquad s > 1. \quad (1.2)$$

If you examine the term in brackets more closely, you soon get the feeling that the infinite sum (1.2) should be very well approximated by the early terms (if s is chosen large) and that the remaining terms should all be positive. To prove this, as well as to have a useful tool for computing hR, one needs a workable formula for $g(x,s)$.

Theorem 1.

$$g(x,s) = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\left[\frac{x}{2^{r_2}\pi^{n/2}}\right]^z \frac{\Gamma(z/2)^{r_1}\Gamma(z)^{r_2}}{z-s}\,dz\ , \tag{1.3}$$

<u>where</u> $\delta > 0$, $\delta > \operatorname{Re} s$, <u>but otherwise the integral is independent of</u> δ, <u>and</u> $g(x,s)$ <u>is defined by</u> (2.0) <u>below</u>.

If one writes the integral (1.3) as sum of residues by shifting δ to $-\infty$, one finds that $g(x,s)$ is given by a rapidly convergent power series in x^{-1} and $\log x$ whose coefficients depend on s in a straight-forward manner. From this the behavior as $x \to +\infty$ can be read off as in [F]. The behavior as $x \to 0^+$ follows directly from (1.3) and old results about integrals of this kind going back to Barnes (See [Br] [F], the references there and Proposition 2.3 c) below).

On substituting (1.3) into Hecke's (1.1), one finds that the resulting formula is actually a consequence of the functional equation and meromorphic continuation of $\zeta_k(s)$ (see Proof 2 of Proposition 2.1 below). Thus Hecke's formula is *equivalent* to these properties of $\zeta_k(s)$. One also finds that $g(x,s)$ may be replaced by many other functions.

From Theorem 1 we can obtain a cleaner version of (1.2).

Corollary 1.

$$\frac{2^{r_1}hR}{w} = \sum_{\mathfrak{a}} H\left[\frac{\sqrt{|D|}}{N\mathfrak{a}}\right]\ , \tag{1.4}$$

where

$$H(x) = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\left[\frac{x}{\pi^{n/2}2^{r_2}}\right]^{z}\Gamma(z/2)^{r_1}\Gamma(z)^{r_2}(2z-1)\,dz, \qquad \delta > 0 \tag{1.5}$$

and $\mathfrak{a}$ runs over all integral ideals of the number field. Also

$$\lim_{x\to 0^+} H(x)\exp\left[n\pi\, x^{-2/n}\right]x^{(3-r_1-r_2)/n} = 4\pi\left[\frac{2^{1+r_1}}{n}\right]^{1/2} \tag{1.6}$$

and

$$\lim_{x\to+\infty}\frac{H(x)}{(\log x)^{r_1+r_2-1}} = \frac{2^{r_1}}{(r_1+r_2-1)!}\ . \tag{1.7}$$

We add that $H(x)$ is also given by a quickly convergent power series in x^{-1} and $\log x$ [F] .

It is well-known that one can calculate hR to any desired accuracy by computing the splitting of sufficiently many primes. In this sense Corollary 1 is not new. However, our few experiments seem to indicate that formula (1.4) provides a simple and practical way to compute hR as long as $|D|$ is not immense compared with the minimum discriminant for the given (r_1,r_2) . Of course, if one knows that $h = 1$, then (1.4) gives a formula for R . For example, any totally real quintic field of discriminant less than 368000 has $h = 1$ (This follows from Odlyzko's lower bounds). When $h > 1$ is possible, the calculation of R (and therefore, of h) is trickier:

Corollary 2.

$$\frac{2^{r_1}R}{w} = \frac{1}{2}\Big[\sum_{\mathfrak{c}} H\Big[\frac{\sqrt{|D|}}{N\mathfrak{c}}\Big] + \sum_{\mathfrak{b}} H\Big[\frac{\sqrt{|D|}}{N\mathfrak{b}}\Big] \Big] \quad , \tag{1.8}$$

where $\mathfrak{c}$ (resp., $\mathfrak{b}$) runs over all integral ideals in the principal class (resp., in the class of the different) and $H(x)$ is given by (1.5).

The problem with (1.8) is that one must be able to list all principal integral ideals of norm less than $|D|^{\frac{1}{2}+\varepsilon}$. Nevertheless, in [F] we applied Corollary 2 to find the smallest regulator of any number field.

Corollary 3 Let $M(x)$ be the number of integral ideals in the number field $k(\neq \mathbb{Q})$ having norm $\leq x$. Then $M\Big[(\sqrt{|D|}(\log|D|)^{n/2}\Big] \geq ChR$ where $C > 0$ depends only on $n = [k:\mathbb{Q}]$. If k contains no quadratic extension of $\mathbb{Q}$ (or if no quadratic field contained in k has a Siegel zero) then $M\Big\{\sqrt{|D|}(\log\log|D|)^{n/2}\Big] \geq ChR$.

An old result due to Dedekind, as improved by Weber and Landau [La 2], is

$$|M(x)-\kappa x| < C_1(\log|D|)^n |D|^{\frac{1}{n+1}} x^{\frac{n-1}{n+1}}, \qquad x \geq 0 \ , \tag{1.9}$$

where $\kappa = \dfrac{2^{r_1}(2\pi)^{r_2}hR}{w\sqrt{|D|}}$ and C_1 (like all C_i below) depends only on n . But (1.9) says almost nothing if $x < C_2\sqrt{|D|}(\log|D|)^{\frac{n+1}{2}}$. In fact, for such x (1.9) is implied by the inequalities [La 1] [La 2]

$$\kappa < C_3(\log|D|)^{n-1} \quad \text{and} \quad M(x) < C_4 x(\log x)^{n-1} . \qquad (1.10)$$

Note that $M\left[\sqrt{|D|}(\log|D|)^{n/2}\right] > C_5 h(\log|D|)^{\frac{1}{2}}$ is easy since any ideal class has an integral ideal of norm less than $\sqrt{|D|}$ and the rational integers provide $(\log|D|)^{\frac{1}{2}}$ principal integral ideals of norm less than $(\log|D|)^{n/2}$. Corollary 3 is an improvement on this if k is not a CM-field, since then $R > C_6 \log|D|$ [Re].

So far we have dealt only with the zeta function. Similar methods yield the following, which we state somewhat vaguely here:

Proposition 2.3. _Suppose_ $L(s) = \sum_{n\geq 1} a_n n^{-s}$ _and_ $\tilde{L}(s) = \sum_{n\geq 1} b_n n^{-s}$ _are two Dirichlet series. Let_ $\Lambda(s) = B(s)L(s)$, $\tilde{\Lambda}(s) = B(s)\tilde{L}(s)$ _with_ $B(s) = C^s \prod_{j=1}^{M} \Gamma(\beta_j s + b_j)$, $\beta_j > 0$, $M \geq 1$, $C > 0$. _Assume_ Λ _and_ $\tilde{\Lambda}$ _are entire functions and satisfy_ $\Lambda(k-s) = W\tilde{\Lambda}(s)$ _for some_ $k > 0$ _and_ $W \in \mathbb{C}$. _Then, for any_ s,

$$\Lambda(s) = \sum_{n\geq 1} \left[a_n f(\tfrac{C}{n}, s) + W b_n f(\tfrac{C}{n}, k-s)\right] , \qquad (1.11)$$

where $f(x,s)$ _is given by an integral analogous to_ (1.3) _and decreases exponentially as_ $x \to 0^+$. _Moreover,_ $f(x,s)$ _is also given by a rapidly convergent power series in_ x^{-1} _and_ $\log x$ _if the_ β_j _are rational_.

It is a pleasure to acknowledge E. Calabi's help with Theorem 1.

§ 2. Proofs

We formalize our notation, which we suggest the reader skip for now.

k = number field

$n = [k:\mathbb{Q}]$

D = discriminant

h = class number

R = regulator

(r_1, r_2) = number of (real, complex) places

$N\mathfrak{a} = \mathrm{Norm}_{k/\mathbb{Q}}(\mathfrak{a})$ = norm of an (integral) ideal $\mathfrak{a}$

$T \subset Cl(k)$ = a subset of the ideal class group of k

$\zeta_T(s) = \sum\limits_{A \in T} \sum\limits_{\mathfrak{a} \in A} (N\mathfrak{a})^{-s}$ $\left(= \sum\limits_{\mathfrak{a} \in T} N\mathfrak{a}^{-s}\right.$ by abuse of notation) = the sum of the ideal-class zeta functions corresponding to the ideal classes A in T

$\bar{\mathfrak{d}}$ = ideal class of the different

$T' = \{A^{-1}\bar{\mathfrak{d}} \mid A \in T\}$

$$\Lambda_T(s) = \left[\frac{|D|^{\frac{1}{2}}}{\pi^{n/2} 2^{r_2}}\right]^s \Gamma\left(\tfrac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} \zeta_T(s)$$

$m = r_1 + r_2$

$$e_j = \begin{cases} 1 & \text{if } j \leq r_1 \\ 2 & \text{if } j > r_1 \end{cases}$$

For $x > 0$ define

$$g(x,s) = \underset{\substack{\prod_{j=1}^{m} y_j^{e_j} \geq 1 \\ y_j > 0}}{\iint\cdots\cdots\int} \prod_{j=1}^{m} \left[y_j^{e_j s/2} \exp(-\pi e_j y_j \; x^{-2/n}) \frac{dy_j}{y_j} \right] . \qquad (2.0)$$

Proof of Theorem 1.

For $x > 0$ and δ as in Theorem 1, let

$$a_j = e_j \pi \; x^{-2/n}$$
$$f_j(t) = e^{\delta t/2} \exp\left[-a_j \exp(t/e_j)\right] .$$

Since $\delta > 0$, $f_j(t)$ has a Fourier transform and satisfies the Fourier inversion formula. Write $s = \alpha + \delta$, so $\operatorname{Re} \alpha < 0$. We calculate now:

$$g(x,s) = \underset{\substack{\prod_{j=1}^{m} y_j^{e_j} \geq 1 \\ y_j > 0}}{\int\cdots\cdots\int} \prod_{j=1}^{m} \left[y_j^{e_j \alpha/2} \; y_j^{\delta e_j/2} \exp(-a_j y_j) \frac{dy_j}{y_j} \right]$$

$$= 2^{-r_2} \underset{\substack{-\infty < u_j < \infty \\ \sum_{j=1}^{m} u_j > 0}}{\int\cdots\cdots\cdots\int} \exp\left[\frac{\alpha}{2} \sum_{j=1}^{m} u_j\right] \prod_{j=1}^{m} f_j(u_j) du_j$$

(We have put $e^{u_j} = y_j^{e_j}$ for $1 \le j \le m$.)

$$= 2^{-r_2} \int_{v=0}^{\infty} e^{\frac{\alpha}{2}v} \int_{u_2=-\infty}^{\infty} \cdots\cdots \int_{u_m=-\infty}^{\infty} f_1\left[v - \sum_{j=2}^{m} u_j\right] \prod_{j=2}^{m} f_j(u_j)du_j \, dv$$

(where $v = \sum_{j=1}^{m} u_j$ and u_j (for $2 \le j \le m$) are new variables)

$$= 2^{-r_2} \int_{v=0}^{\infty} e^{\frac{\alpha}{2}v} \left[f_1 * \ldots * f_m\right](v) \, dv$$

(convolution, Calabi's observation)

$$= \frac{1}{2\pi} \int_{v=0}^{\infty} e^{\frac{\alpha}{2}v} \int_{w=-\infty}^{\infty} e^{iwv} \prod_{j=1}^{m} \left[a_j^{-e_j(\frac{\delta}{2} - iw)} \Gamma(e_j(\frac{\delta}{2} - iw))\right] dw \, dv$$

(Fourier inversion)

$$= \frac{-1}{2\pi} \int_{w=-\infty}^{\infty} e^{iwv} \left[\frac{x}{2^{r_2}\pi^{n/2}}\right]^{2(\frac{\delta}{2} - iw)} \prod_{j=1}^{m} \Gamma(e_j(\frac{\delta}{2} - iw)) \quad \frac{dw}{\frac{\alpha}{2}+iw}$$

(we reversed the integrals, which is possible since $\operatorname{Re} \alpha < 0$)

$$= \frac{-1}{2\pi i} \int_{\frac{\delta}{2}-i\infty}^{\frac{\delta}{2}+i\infty} \left[\frac{x}{2^{r_2}\pi^{n/2}}\right]^{2z} \frac{\Gamma(z)^{r_1}\Gamma(2z)^{r_2}}{\frac{\alpha+\delta}{2} - z} \, dz$$

(put $z = \frac{\delta}{2} - iw$)

$$= \frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} \left[\frac{x}{2^{r_2}\pi^{n/2}}\right]^{z} \frac{\Gamma(z/2)^{r_1}\Gamma(z)^{r_2}}{z - s} \, dz \quad .$$

Hence Theorem 1 is proved.

For $\mathrm{Re}\ s > 0$ it is often convenient to shift the contour to the left and so rewrite $g(x,s)$ in the form

$$g(x,s) = \left[\frac{x}{2^{r_2}\pi^{n/2}}\right]^s \Gamma(s/2)^{r_1}\Gamma(s)^{r_2} +$$

$$+ \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} \left[\frac{x}{2^{r_2}\pi^{n/2}}\right]^z \frac{\Gamma(z/2)^{r_1}\Gamma(z)^{r_2}}{z-s}\,dz\ ,\quad 0<\tau<\mathrm{Re}\ s\ . \tag{2.1}$$

Proposition 2.1. <u>Suppose</u> $T \subset \mathrm{Cl}(k)$ <u>satisfies</u> $\zeta_T(s) = \zeta_{T'}(s)$ <u>for all</u> s . <u>Then for</u> $\mathrm{Re}\ s > 1$,

$$\frac{2^{r_1}R|T|}{w} = s(s-1)\sum_{\mathfrak{a}} G\left[\frac{\sqrt{|D|}}{N\mathfrak{a}}\ ,s\right]\ ,$$

<u>where</u> $|T|$ = <u>number of ideal classes in</u> T , $\mathfrak{a}$ <u>ranges over all integral ideals whose class is in</u> T <u>and</u>

$$G(x,s) = \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty}\left[\frac{x}{2^{r_2}\pi^{n/2}}\right]^z\Gamma(z/2)^{r_1}\Gamma(z)^{r_2}\left[\frac{1}{s-z}+\frac{1}{1-s-z}\right]dz\ ,$$

$$0<\tau<\mathrm{Re}\ s\ .$$

<u>Proof 1.</u> Hecke's formula [L;p. 254 , although $\mathfrak{d}^{-1}A^{-1}$ must be corrected to $\mathfrak{d}A^{-1}$] reads, when $\zeta_T = \zeta_{T'}$,

$$\Lambda_T(s) = \frac{2^{r_1}|T|R}{ws(s-1)} + \sum_{\mathfrak{a}\in T}\left[\ g\left[\frac{\sqrt{|D|}}{N\mathfrak{a}},s\right] + g\left[\frac{\sqrt{|D|}}{N\mathfrak{a}},1-s\right]\ \right]\ , \tag{2.2}$$

where $\sum_{a\in T}$ is short-hand for $\sum_{A\in T}\sum_{a\in A}$. Substitute the definition

$$\Lambda_T(s) = \Gamma(s/2)^{r_1}\Gamma(s)^{r_2}\sum_{a\in T}\left[\frac{\sqrt{|D|}}{Na\pi^{n/2}2^{r_2}}\right]^s \qquad (2.3)$$

into (2.2) and use (2.1) and Theorem 1 to obtain the Proposition.

<u>Proof 2.</u> Proposition 2.1 follows from the functional equation $\Lambda_T(1-s) = \Lambda_{T'}(s) = \Lambda_T(s)$ (since $\zeta_T = \zeta_{T'}$) and the fact that $\Lambda_T(s)$ can be continued to $\mathbb{C}$ except for simple poles at $s = 0$ and 1. Indeed, let

$$I = \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty}\Lambda_T(z)\left[\frac{1}{s-z}+\frac{1}{1-s-z}\right]dz\ , \qquad (\mathrm{Re}\ s > \tau > 1)\ . \qquad (2.4)$$

Standard estimates [L; p.266] show that we can shift the integral to $\mathrm{Re}\ z = \frac{1}{2}$, picking up a residue at $z = 1$:

$$I = \frac{2^{r_1}|T|R}{w(s-1)s} + \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\Lambda_T(z)\left[\frac{1}{s-z}+\frac{1}{1-s-z}\right]dz\ . \qquad (2.5)$$

But the integral on the right of (2.5) vanishes for the trivial reason that the values of the integrand at $z = \frac{1}{2} + it$ and $z = \frac{1}{2} - it$ cancel. If we now write (2.4) as a sum of integrals and use (2.3), we obtain Proposition 2.1 except that τ is restricted to $\tau > 1$. However the integral is independent of the value of τ as long as $0 < \tau < \mathrm{Re}\ s$ (but note that the sum and integral in Proposition 2.1 can be reversed only if $\tau > 1$).

Note that this second proof combined with Theorem 1 shows that Hecke's formula (1.1) is a consequence of the functional equation and meromorphic continuation of $\zeta_k(s)$.

Proposition 2.2. <u>Assume</u> $T \subset Cl(k)$ <u>satisfies</u> $\zeta_T(s) = \zeta_{T'}(s)$ <u>for all</u> s . <u>Then</u>

$$\frac{2^{r_1} R|T|}{w} = \sum_{\mathfrak{a}} H\left[\frac{\sqrt{|D|}}{N\mathfrak{a}}\right] , \tag{2.6}$$

<u>where</u> H <u>is defined by</u> (1.5) <u>and</u> $\mathfrak{a}$ <u>runs over all integral ideals whose class is in</u> T .

<u>Proof</u>. Take $s \to +\infty$ in Proposition 2.1. Alternatively, adapt proof 2 above or see [F].

<u>Proof of Corollaries 1 and 2</u>.
Take $T = Cl(k)$ for Corollary 1 and
$T = \{\text{principal class}\} \cup \{\text{class of different}\}$ for Corollary 2 and apply Proposition 2.2. For the asymptotic behavior of $H(x)$, see [F] and Proposition 2.4 below.

<u>Proof of Corollary 3</u>.
From Corollary 1 we have $H(x) < C_1\exp(-3n\ x^{-2/n})$ for all $x > 0$, where $C_1 > 0$ depends only on n . From Corollary 1, for any $y \geq 1$

$$\frac{2^{r_1}hR}{w} = \sum_{a} H\left[\frac{\sqrt{|D|}}{Na}\right] < C_1 M(\sqrt{|D|}y) + \sum_{Na>\sqrt{|D|}y} H\left[\frac{\sqrt{|D|}}{Na}\right]$$

$$< C_1\left[M(\sqrt{|D|}y) + \sum_{Na>\sqrt{|D|}y} \exp\left[-3n\left[\frac{\sqrt{|D|}}{Na}\right]^{-2/n}\right] \right] .$$

Abel summation [Ap; p. 77] and (1.10) yield

$$\sum_{Na>\sqrt{|D|}y} \exp\left[-3n\left[\frac{Na}{\sqrt{|D|}}\right]^{2/n}\right] < C_2\sqrt{|D|}(\log|D|)^{n-1}\exp(-3n\ y^{2/n}) .$$

If $y = (\log|D|)^{n/2}$, this last term tends to zero as $|D| \to +\infty$. This proves the first claim in Corollary 3. If $y = (\log\log|D|)^{n/2}$ we get

$$\frac{2^{r_1}hR}{w} < C_1 M\left[\sqrt{|D|}(\log\log|D|)^{n/2}\right] + C_3\sqrt{|D|}(\log|D|)^{-2n-1} . \qquad (2.7)$$

If k has no quadratic subfield, Stark [St; p. 135] showed $hR/w > C_4\sqrt{|D|}(\log|D|)^{-1}$. The same holds *if* for any quadratic subfield $k_0 \subset k$ we have $\xi_{k_0}(x) < 0$ for $1 - (C_5\log|D_k|)^{-1} \leq x < 1$, cf. [St; p. 148]. Corollary 3 now follows from (2.7).

The trick used in the second proof of Proposition 2.1 can be generalized to yield a practical method to compute with Dirichlet series satisfying a functional equation of the usual kind. We formalize this as follows. Suppose

i) $L(s) = \sum_{n\geq 1} a_n n^{-s}$ and $\tilde{L}(s) = \sum_{n\geq 1} b_n n^{-s}$ converge absolutely in some half-plane $\operatorname{Re} s > V$,

ii) there is an integer $M \geq 1$, positive numbers β_j and complex numbers b_j $(1 \leq j \leq M)$, positive numbers C and k and a complex number W such that if we set

$$\Lambda(s) = C^s L(s) \prod_{j=1}^{M} \Gamma(\beta_j s+b_j) \ ,$$

$$\tilde{\Lambda}(s) = C^s \tilde{L}(s) \prod_{j=1}^{M} \Gamma(\beta_j s+b_j) \ ,$$

then $\tilde{\Lambda}(s)$ and $\tilde{\Lambda}(s)$ have an *analytic* continuatiom to $\mathbb{C}$ and satisfy the identity $\Lambda(k-s) = W\,\tilde{\Lambda}(s)$.

Proposition 2.3. <u>Assume</u> i) <u>and</u> ii) above. <u>Then</u>

$$\Lambda(s) = \sum_{n\geq 1} \left[a_n f(\tfrac{C}{n},s) + Wb_n f(\tfrac{C}{n},k-s)\right] \qquad (2.8)$$

<u>where</u> $f : \mathbb{R}_+ \times \mathbb{C} \to \mathbb{C}$ <u>has the following properties</u>:

$$\text{a) } f(x,s) = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} x^z\left[\prod_{j=1}^{M}\Gamma(\beta_j z+b_j)\right]\frac{dz}{z-s}, \qquad \delta > N(s), \tag{2.9}$$

<u>where</u> $N(s) = \max\left[\max_{1\le j\le M}\{\mathrm{Re}(-b_j/\beta_j)\},\ \mathrm{Re}\ s\right]$ <u>and the integral (which converges very rapidly and can easily be computed numerically) is independent of</u> δ ($> N(s)$) .

b) <u>Assume</u> $\beta_j \in \mathbb{Q}$ $(1 \le j \le M)$. <u>Then</u>

$$f(x,s) = \sum_{j=1}^{M}\sum_{I} A_{I,j}^{(s)}\, x^I \frac{(\log x)^{j-1}}{(j-1)!} + x^s\prod_{j=1}^{M}\Gamma(\beta_j s+b_j) \tag{2.10}$$

<u>holds for</u> $s \neq -b_j/\beta_j$ $(1 \le j \le M)$. <u>Here</u> I <u>ranges over all the poles of</u> $\prod_{j=1}^{M}\Gamma(\beta_j z+b_j)$ (<u>i.e.</u>, $I = -\frac{b_j+p}{\beta_j}$ <u>for any</u> $1 \le j \le M$ <u>and for any integer</u> $p \ge 0$), <u>the</u> $A_{I,j}^{(s)}$ <u>are defined by</u>

$$\frac{1}{z-s}\prod_{j=1}^{M}\Gamma(\beta_j z+b_j) = \sum_{j=1}^{M} A_{I,j}^{(s)}/(z-I)^j + \text{(a function analytic at } z = I\text{)} \tag{2.11}$$

<u>and can be calculated recursively. For any pole</u> I_0 <u>satisfying</u> $\mathrm{Re}\ I_0 < \mathrm{Re}(s-1)$, $\mathrm{Re}\ I_0 < 0$, <u>a finite form of</u> (2.10) <u>is</u>

$$\left| f(x,s) - x^s\prod_{j=1}^{M}\Gamma(\beta_j s+b_j) - \sum_{j=1}^{M}\sum_{I\ge I_0} A_{I,j}^{(s)}\, x^I\frac{(\log x)^{j-1}}{(j-1)!}\right| <$$

$$< \left[\frac{c_1}{\Gamma(c_2|I_0|)}\right]^M x^{\mathrm{Re}(I_0-\alpha)}, \tag{2.12}$$

<u>where</u> α, c_1, $c_2 > 0$ <u>depend on the</u> β_j <u>and</u> b_j .

c) $\lim\limits_{x\to 0^+} f(x,s)\, x^R \exp(Qx^{-1/\mu}) = S$, (uniformly for $|s|$ bounded)

where μ, Q, R, S depend on the β_j and b_j (but not on s) as follows:

$$\mu = \sum_{j=1}^{M} \beta_j > 0$$

$$Q = \mu\beta^{1/\mu} > 0 \text{ , where } \beta = \prod_{j=1}^{M} \beta_j^{-\beta_j} > 0$$

$$R = \mu^{-1}\left[\sum_{j=1}^{M} b_j - \frac{M+1}{2}\right]$$

$$S = (2\pi)^{(M-1)/2} \mu^{-\frac{1}{2}} \beta^R \prod_{j=1}^{M} \beta_j^{(b_j - \frac{1}{2})} \neq 0 .$$

Remark. This is one of those propositions whose statement is longer than its proof (except for c) which is, fortunately, in the literature). In most applications to number theory, β_j can be taken to be $\frac{1}{2}$ for all j . One can then change variables in (2.9) to get $\beta_j = 1$, which simplifies everything. Although $f(x,s)$ would still look a bit ugly, it would be as nice as a Bessel function (see [F] for a closely related example).

Proof.

Let

$$\Lambda_\pm(z) = \Lambda(z) \pm W\tilde{\Lambda}(z)$$

$$\rho_\pm(s,z) = \frac{1}{z-s} \pm \frac{1}{k-z-s} .$$

Note that

$$\Lambda_{\pm}(k-z) = \pm \Lambda_{\pm}(z) ,$$
$$\rho_{\pm}(s,k-z) = \pm \rho_{\pm}(s,z) .$$

Assume first $\mathrm{Re}\ s > \frac{k}{2}$ and take $\delta > V$ (= abscissa of convergence), $\delta > \mathrm{Re}\ s$. Let

$$I_{+} = \frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} \Lambda_{+}(z)\rho_{-}(s,z)dz ,$$

$$I_{-} = \frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} \Lambda_{-}(z)\rho_{+}(z)dz .$$

Note that

$$I_{\pm} = \sum_{n\geq 1} (a_n \pm W b_n) \frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} \left[\frac{C}{n}\right]^{z} \left[\prod_{j=1}^{M} \Gamma(\beta_j z+b_j)\right] \left[\frac{1}{z-s} \mp \frac{1}{k-s-z}\right] dz . \tag{2.13}$$

By standard estimates [La; p. 266], we may shift the contour in the integral defining $I_{\pm}$ to $\mathrm{Re}\ s = \frac{k}{2}$. By assumption, Λ and $\tilde{\Lambda}$ are entire so that we only pick up a residue at $z = s$:

$$I_{\pm} = \Lambda_{\pm}(s) + \frac{1}{2\pi i} \int_{\frac{k}{2}-i\infty}^{\frac{k}{2}+i\infty} \Lambda_{\pm}(z)\rho_{\mp}(s,z)dz = \Lambda_{\pm}(s), \tag{2.14}$$

for the integral again vanishes (because the integrand at $\frac{k}{2}$ + it = − integrand at $\frac{k}{2}$ − it) . From (2.13) and (2.14):

$$\Lambda(s) = \tfrac{1}{2}(\Lambda_+(s) + \Lambda_-(s)) = \tfrac{1}{2}(I_+ + I_-) =$$

$$\sum_{n\geq 1} \frac{1}{2\pi i} \int_{\tau-i\infty}^{\tau+i\infty} \left[\frac{C}{n}\right]^z \left[\prod_{j=1}^{M} \Gamma(\beta_j z+b_j)\right] \left[\frac{a_n}{z-s} + \frac{Wb_n}{z-(k-s)}\right] dz \ .$$

This is (2.8) and (2.9), except that we have required $\mathrm{Re}\ s > \frac{k}{2}$ and $\delta > \max(V, \mathrm{Re}\ s)$ instead of $\delta > N(s)$ in (2.9). However, (2.9) is independent of δ ($> N(s)$) since there are no poles of the integrand for $\mathrm{Re}\ z > N(s)$. The restriction $\mathrm{Re}\ s > \frac{k}{2}$ can be dropped because both sides of (2.8) are entire functions of s (for the right-hand side of (2.8) this follows from c)).

To prove b), shift the line of integration in (2.9) leftward to $\mathrm{Re}\ z = \mathrm{Re}(I_0-\alpha)$ where $\alpha > 0$ is chosen (independently of I_0) so that no pole I lies in $\mathrm{Re}(I_0-2\alpha) < \mathrm{Re}\ z < \mathrm{Re}(I_0)$. Such an α exists because the β_j are rational. If $\mathrm{Re}\ I_0 < \mathrm{Re}(s-1)$ and $\mathrm{Re}(I_0) < 0$, then

$$f(x,s) = \sum_{I\geq I_0} \mathrm{Res}_{z=I}\left[\frac{x^z}{z-s} \prod_{j=1}^{M} \Gamma(\beta_j z+b_j)\right] + x^s \prod_{j=1}^{M} \Gamma(\beta_j s+b_j) +$$

$$+ \frac{1}{2\pi i} \int_{\mathrm{Re}\ z=\mathrm{Re}(I_0-\alpha)} x^z \left[\prod_{j=1}^{M} \Gamma(\beta_j z+b_j)\right] \frac{dz}{z-s} \ . \qquad (2.15)$$

The integral in (2.15) is readily estimated, to be at most $x^{\mathrm{Re}(I_0-\alpha)}\left[c_1 / \Gamma(c_2|I_0|)\right]^M$ for some $c_1, c_2 > 0$ which depend on the β_j and b_j (see [F] for a similar calculation). Now (2.12) follows from (2.15) and definition (2.11). Equation (2.10) follows from (2.12) on taking $\mathrm{Re}(I_0) \to -\infty$. Since all the β_j are rational we may, after a change of variable in (2.9), assume $\beta_j \in \mathbb{Z}$ $(1 \le j \le M)$. Then the $A^{(s)}_{I,j}$ can be recursively calculated using $\Gamma(z-1) = \frac{\Gamma(z)}{z-1}$. This proves b).

The proof of c) is rather long but is given in full detail in [Br], especially §2.2 and §10.1. The uniformity in s is not explicitly stated in [Br], but it follows from the proof. To apply Braaksma's results one must first change variables in (2.9) from z to $-z$ and write $\frac{1}{-z-s} = \frac{\Gamma(-z-s)}{\Gamma(-z-s+1)}$. Braaksma's equation (2.21) then shows that his function $H(z)$ includes our $f(x,s)$ as special case ($n = 0$, $m = M+1$, $p = 1$, $q = M+1$ in his notation). We shall sketch his method of proof in a related case below.

<u>Remarks</u>.

a) One can write down infinitely many formulas similar to (2.8) by replacing $\rho_{\pm}(s,z)$ by any function enjoying the formal properties of $\rho_{\pm}$ used in the proof of (2.8). Our choice, which agrees with Hecke's by Theorem 1, is the simplest but not necessarily the best [F; §4]. In this sense, Hecke's is but one of an infinite family of formulas.

b) If $\tilde{\Lambda}(s)$ is allowed to have finitely many poles, one may proceed in two ways.

1) Add the polar parts to (2.8) and give a quickly convergent formula for each of the residues similar to Proposition 2.1,

or

2) Multiply $\rho_{\pm}(s,z)$ by a polynomial $P(s)$ chosen so that $\Lambda(s)P(s)$ is analytic and $P(k-s) = P(s)$. This will slightly change $f(x,s)$, but not its essential properties.

For the practical application of Corollary 1 (or of formula (1.11)) to the numerical calculation of hR (or of $\Lambda(s)$) one needs asymptotic formulas with all constants explicit. These constants can be computed by following Braaksma's proof. We shall sketch this below for the function $H(x)$ appearing in Corollary 1.

Proposition 2.4. <u>Let</u> $H(x)$ <u>be defined by</u> (1.5) <u>and assume</u> $r_1+r_2 \geq 3$. <u>Then</u>

$$0 < H(x) < 8\pi(2^{1+r_1}/n)^{1/2} x^{(r_1+r_2-3)/n} \exp(-n\pi\, x^{-2/n})$$

$$(0 < x \leq x_0) \quad , \qquad (2.16)$$

<u>where</u> x_0 <u>must satisfy the following: If we let</u> $w_0 = 2\pi\, x_0^{-2/n}$, <u>we must have</u> $w_0 \geq 2$, $\frac{nw_0}{r_1+r_2+1} > \frac{12}{5}$ <u>and</u>

$$\frac{w_0}{2} - \frac{1}{4} > \left[\frac{nw_0}{2G(\alpha)}\right]^{\alpha} \frac{J(\alpha)}{n} + \left[\frac{nw_0}{2G(\alpha+1)}\right]^{\alpha} \frac{J(\alpha+1)}{4G(\alpha+1)} ,$$

<u>where</u> $\alpha = \frac{1}{2}(r_1+r_2-1)$, $\quad G(t) = \frac{nw_0}{2} - t$,

$$J(t) = \frac{T_1(t)\exp\left[\frac{1}{12G(t)}\right]\left[\frac{2G(t)}{nw_0}\right]^{1/2}}{\left[\frac{5}{6} - \frac{4t}{nw_0}\right]^{1/2}} +$$

$$+ \frac{(2/\pi)^{1/2}T_{1.1}(t)\exp\left[\frac{1.1}{12G(t)} + \frac{nw_0}{2}\left[\frac{1}{2}\log 2 - \frac{\pi G(t)}{2nw_0}\right]\right]}{G(t)^{1/2}\left[\frac{\pi}{4} - \frac{t}{2G(t)}\right]} ,$$

$$T_j(t) = \frac{nw_0}{2}\left[\exp\left[\frac{j(r_1 + \frac{r_2}{2})}{6w_0} + \frac{2E(t)}{nw_0} + \frac{j}{6nw_0-12t+12}\right] - 1\right], \quad (j=1 \ \underline{\text{or}}\ 1.1)$$

$$E(t) = (t-1)\left[t - \frac{1}{2} + \frac{(t-1)}{2}\left[1 + \frac{4(t-1)}{3(nw_0-2t+2)}\right]\right] .$$

<u>Remarks</u>.

1) The restriction $r_1+r_2 \geq 3$ ($\Leftrightarrow \alpha \geq 1$) is only to avoid even messier expressions. If $r_1 = 0$ or $r_2 = 0$ and $r_1+r_2 \leq 2$, then $H(x)$ can be given in terms of Bessel or exponential functions (see [F] for the case $(r_1,r_2) = (2,0)$). Thus Proposition 2.4 essentially excludes only $(r_1,r_2) = (1,1)$.

2) One could take a coarse bound of the kind $w_0 > cn$ for some absolute constant c , but this would be very wasteful unless n is large.

Sketch of Proof.

For $a = 0$ or 1, let

$$G_a(y) = \frac{1}{2\pi i}\int_{w-i\infty}^{w+i\infty} y^{-s}\, s^a\, \Gamma(s/2)^{r_1}\Gamma(s)^{r_2} ds ,$$

$$w = \frac{2y^{2/n}}{n} > 0 .$$

Then $H(x) = 2G_1(\pi^{n/2}2^{r_2}x^{-1}) - G_0(\pi^{n/2}2^{r_2}x^{-1})$.

The aim is to write

$$G_a(y) = \text{main term} + \text{explicitly bounded term} = M_a(y) + B_a(y) \quad (2.17)$$

and work out when $y \geq y_0 \Rightarrow 2M_1(y) - M_0(y) > |2B_1(y)| + |B_0(y)|$ (For then $|H(x) -$ main term$| <$ main term if $x \leq x_0 = \pi^{n/2}2^{r_2}y_0^{-1}$. This is equivalent to (2.16)). To do this, write

$$s^a\Gamma(s)^{r_2}\Gamma(s/2)^{r_1} =$$

$$= 2^{r_2 s}2^{r_1/2}(2\pi)^{\alpha}\left[\frac{n}{2}\right]^{\alpha-a+\frac{1}{2}} n^{-ns/2}\Gamma\left(\frac{ns}{2} - \alpha + a\right)\left[1 + \frac{R_a(s)}{\frac{ns}{2} - \alpha + a - 1}\right] \quad (2.18)$$

where $\operatorname{Re} s = w \geq 2$ and

$$|R_a(s)| \leq \left\{ \begin{array}{ll} T_1(\alpha+1-a) & \text{if } |\operatorname{Im} s| \leq w - \frac{2}{n}(\alpha - a) \\ T_{1,1}(\alpha+1-a) & \text{otherwise} \end{array} \right\} .$$

This is proved using an explicit form of Stirling's formula for $\log \Gamma(s)$:

$$\log \Gamma(s) = (s-\tfrac{1}{2})\log s - s + \tfrac{1}{2}\log(2\pi) + \frac{Q(s)}{12s} \quad , \qquad (2.19)$$

where

$|Q(s)| \leq 1$ if $\mathrm{Re}\ s > 0$ and $|\mathrm{Im}\ s| \leq \mathrm{Re}\ s$ [W-W; p. 252] ,

$|Q(s)| \leq 1.1$ if $|\mathrm{Im}\ s| \geq \mathrm{Re}\ s \geq 1$ [N; p. 208, (17)] .

From (2.18) we find, say for $a = 1$,

$$G_1(y2^{r_2}n^{-n/2}) = 2^{r_1/2}(2\pi)^{\alpha}\left[\frac{n}{2}\right]^{\alpha-\frac{1}{2}}\left[\frac{1}{2\pi i}\int_{W-i\infty}^{W+i\infty} y^{-s}\Gamma(\frac{ns}{2} - \alpha+1)ds + \right.$$

$$\left. + \frac{1}{2\pi i}\int_{W-i\infty}^{W+i\infty} y^{-s}\Gamma(\frac{ns}{2} - \alpha)R_1(s)ds \right] .$$

The first integral, being an inverse Mellin transform of a Mellin transform, is $\frac{2}{n}\, y^{-2(\alpha-1)}\exp(-y^{2/n})$. This gives $M_1(y)$ in (2.17). The second integral is bounded by integrating an upper bound for $|y^{-s}\Gamma(\frac{ns}{2} - \alpha)R_1(s)|$. A bound for $R_1(s)$ was given above while a bound for $|y^{-s}\Gamma(\frac{ns}{2} - \alpha)|$ follows from (2.19). The resulting integral is then treated as in [Br; pp. 308-309]. This gives the messy but explicitly bounded term needed in (2.17).

References

[Ap] Apostol, T.: Introduction to Analytic Number Theory. Berlin: Springer 1976.

[Br] Braaksma, B.L.J.: Asymptotic expansions and analytic continuation for a class of Barnes-integrals. Compos. Math. 15, 239-341 (1963).

[F] Friedman, E.: Analytic formulas for the regulator of a number field, pre-print (1987).

[He] Hecke, E.: Über die Zetafunktion beliebiger algebraischer Zahlkörper. Gött. Nachr. 1917. Mathematische Werke, 159-171.

[L] Lang, S.: Algebraic Number Theory. Berlin: Springer, 1986.

[La 1] Landau, E.: Abschätzungen von Charaktersummen, Einheiten und Klassenzahlen. Gött. Nachr. 1918. Collected Works, vol. 7, 114-132.

[La 2] Landau, E.: Verallgemeinerung eines Polyaschen Satzes auf algebraische Zahlkörper. Gött. Nachr. 1918. Collected Works, vol. 7, 161-171.

[N] Nielsen, N.: Handbuch der Theorie der Gammafunktion. Leipzig: Teubner 1906 (reprinted by Chelsea in 1965).

[Re] Remak, R.: Über Grössenbeziehungen zwischen Diskriminanten und Regulator eines algebraischen Zahlkörpers. Compos. Math. 10, 245-285 (1952).

[Si 1] Siegel, C.L.: Über die Klassenzahl quadratischer Zahlkörper. Acta Arithmetica I (1935). Gesammelte Abhandlungen, vol. 1, 406-409.

[Si 2] Siegel, C.L.: Abschätzung von Einheiten. Gött. Nachr. 1969. Gesammelte Abhandlungen, vol. 4, 66-81.

[St] Stark, H.: Some effective cases of the Brauer-Siegel theorem. Invent. Math. **23**, 135-152 (1974).

[W-W] Whittaker, E. and Watson, G.: A Course in Modern Analysis. Cambridge University Press, 1969.

Eduardo Friedman
Department of Mathematics
University of Pennsylvania
Philadelphia, PA 19104, U.S.A.