Feynman graphs to motives

Karen Yeats*
Boston University
111 Cummington St., Boston, MA, USA

Mathematische Arbeitstagung June 22-28, 2007
Max-Planck-Institut für Mathematik, Bonn, Germany

1 Introduction

The purpose of the talk is to give an overview of the papers [6] and [1]. There is nothing new herein, and probably nothing surprising to the expert. In a sentence, [6] is about how the primitives determine the game and [1] is about making it look like algebraic geometry and the search for motives.

2 Renormalization Hopf algebras

The basic objects here are graphs with specified edge and vertex types, and a power counting weight associated with each type of edge and vertex. They are most naturally considered as formed of labelled half-edges, with the labelling then forgotten, which automatically leads to dividing by the size of the automorphism group of the graph and gives a natural way to understand the external edges.

We say a graph is superficially divergent in 4 dimensions if

$$4\ell - \sum_e w(e) - \sum_v w(v) \geq 0$$

where ℓ is the first Betti number of the graph, sum indices of e and v indicate summing over internal edges and vertices respectively of the graph, and w is the power counting weight. The terminology comes from the fact that these values count the powers of the integration variable in the integrand of the Feynman integral.

The Hopf algebra as a vector space is the span of 1-particle irreducible (1PI) graphs (what a combinatorialist would call 2-edge-connected). The only interesting operation in this type of combinatorial Hopf algebra is the coproduct, as multiplication is disjoint union, addition is formal, and the antipode can be recursively defined from the rest. This leaves

*Great thanks to the people of the Arbeitstagung for the surprising and humbling opportunity to speak

$$\Delta(\Gamma) = \Gamma \otimes \mathbf{1} + \mathbf{1} \otimes \Gamma + \sum_{\substack{\gamma \subset \Gamma \\ \gamma \text{ divergent}}} \gamma \otimes \Gamma/\gamma$$

where $\mathbf{1}$ can be thought of as the empty graph, the sum runs over superficially divergent 1PI subgraphs and their disjoint unions, and Γ/γ denotes Γ with each connected component of γ contracted to a point.

Note that primitive graphs may be sums.

The Hopf algebra is graded by the first Betti number of the graph. Also playing an important role is the Hochschild cohomology, and most importantly the 1-cocycles

$$\Delta B_+ = (\mathrm{id} \otimes B_+)\Delta + B_+ \otimes \mathbf{1}$$

which prototypically are given by insertion of graphs into a given primitive graph.

3 The primitives determine the game

All non-empty graphs are in the image of some B_+. Specifying which B_+ goes where is the combinatorial content of Dyson–Schwinger equations, which are systems of equations:

$$X^r(x) = \mathbf{1} \pm \sum_k x^k B_+^k(X^r(x) Q^k(x))$$

where $Q(x) = \prod_r (X^r(x))^s$, r indexes the system, and the s_r are integers. These are generally non-linear recursive equations.

An important example is

$$X(x) = \mathbf{1} - xB_+\left(\frac{1}{X(x)}\right)$$

from [2]. Here there is one primitive and one B_+.

The solutions of Dyson–Schwinger equations give comodules. Compare to natural growth, which gives all comodules [4], but those which are straightforward here are complicated there and vice versa.

The analytic content comes from a Mellin transform for each primitive, $F^{k,r}(\rho)$. By using primitives which may be non-trivial sums we can avoid multivariate $F^{k,r}$.

In the Broadhurst and Kreimer example

$$F(\rho) = \frac{1}{q^2} \int d^4k \left(\frac{k \cdot q}{(k^2)^{1+\rho}(k+q)^2} - \left.\frac{k \cdot q}{(k^2)^{1+\rho}(k+q)^2}\right|_{q^2=1} \right)$$

where $k = (k_1, k_2, k_3, k_4), q = (q_1, q_2, q_3, q_4) \in \mathbb{R}^4$, $d^4k = dk_1 dk_1 dk_3 dk_4$, and squaring a vector is short for the standard inner product.

Combine the analytic and combinatorial information ($X^r \mapsto G^r$, $B_+^{r,k} \mapsto F^{r,k}$) into $G^r(x, L) = \sum \gamma_k^r(x) L^k$ with $\gamma_k^r(x) = \sum_{j \geq k} \gamma_{k,j}^r x^j$. This give the (analytic) Dyson–Schwinger equations.

In the example

$$G(x,L) = 1 - \frac{x}{q^2}\int d^4k \frac{k\cdot q}{k^2 G(x, \log k^2)(k+q)^2} - \cdots \bigg|_{q^2=1}$$

where $L = \log(q^2)$.

Using the scattering type formula [3] we can pick out the coefficient of L^n. Then, since the scattering type formula only

sees the linear part of the Hopf algebra, we can rewrite using the linearized coproduct to get

$$\gamma_k^r(x) = \frac{1}{k}\left(\gamma_1^r(x)\gamma_{k-1}^r(x) + \sum_j s^j \gamma_1^j(x) x\partial_x \gamma_{k-1}^r(x)\right)$$

Plugging $G^r(x,L) = \sum \gamma_k^r(x)L^k$ into the Dyson–Schwinger equation and using the usual trick $\partial_\rho^k x^{-\rho}|_{\rho=0} = (-1)^k \log^k(x)$ we get a messy triangular recursion for the coefficients of γ_1^r in terms of the coefficients of $F^{r,k}$

By suitable choice of primitives we may assume the $F^{r,k}$ are geometric series, in which case the messy recursion simplifies to

$$\gamma_1^r(x) = \sum_k p^r(k)x^k - \gamma_1^r(x)^2 - \sum_j s^j\gamma_1^j(x)x\partial_x\gamma_1^r(x)$$

where $p^r(k)$ is the value of the r-primitives at k loops.

Very rarely is anyone clever enough to solve these differential equations. In [2] Broadhurst and Kreimer found a closed form

solution in terms of the complementary error function for the running example of this section. More often we must console ourselves with better understanding the resulting asymptotic series.

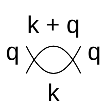
Even very special cases are of great interest

$$2\gamma_1(x) = \frac{x}{3} + \frac{x^2}{4} - \gamma_1(x)^2 + \gamma_1(x)x\partial_x\gamma_1(x).$$

Always the primitives control the game.

4 Making it look like algebraic geometry

For this section we'll restrict ourselves to primitive graphs with one type of undirected edge. In this context a graph with ℓ independent cycles will have 2ℓ edges.

From the graph 

we have the integral

$$\int d^4k \frac{1}{k^2(k+q)^2}$$

where $k = (k_1, k_2, k_3, k_4), q = (q_1, q_2, q_3, q_4) \in \mathbb{R}^4$, $d^4k = dk_1dk_1dk_3dk_4$, and squaring a vector is short for the standard inner product.

In general we get
$$\int \prod_e d^4k_e \frac{1}{k_e^2} \prod_v \delta(\sum_{e\sim v} k_e)$$
where e and v as subscripts refer to summation over edges and vertices respectively, and $e \sim v$ means the edge e is adjacent to the vertex v. The δ factors simply force conservation of momentum.

The integral is divergent in the cases that interest us. The Hopf algebra encodes the physicists' ad-hoc method to fix the divergence. For primitive graphs no proper subset of the variables gives a divergent integral so the problem is much simpler; a single subtraction suffices. We have choice of how to regularize. For the sake of this talk we'll go right to the residue to get the projectivised period integral
$$I = \int_{\mathbb{P}^{4\ell-1}(\mathbb{R})} \frac{\Omega_{4\ell-1}}{q_1 \cdots q_{2\ell}}$$
where
$$\Omega_{n-1} = \sum_{i=1}^{n} (-1)^i Z_i dZ_1 \wedge \cdots \wedge dZ_{i-1} \wedge dZ_{i+1} \wedge \cdots \wedge dZ_{n-1}$$
In the example $q_1 = q_2 = k_1^2+k_2^2+k_3^2+k_4^2$ with q set to 0 (no external momenta), and $k_i = Z_i/Z_n$.

Now we use the fact that
$$\frac{1}{A^2} = \int_0^\infty dae^{-A^2a},$$
Schwinger parameters, on all the q_i to turn I into Gaussian integrals and integrals over new variables, one per edge. Do the Gaussian integrals to get
$$I = C \int_\sigma \frac{\Omega_{2\ell-1}}{\Psi_\Gamma^2}$$
where C is an explicit constant made with some 2πs, and Ψ_Γ is the Kirchoff polynomial of the graph,
$$\Psi_\Gamma = \sum_{\substack{\text{spanning}\\ \text{tree } T}} \prod_{e \notin T} a_e$$
where the a_e are variables, one for each edge. So we are now interested in the variety of the Kirchoff polynomial and its period integral I.

5 Motives

These periods are known to be $\mathbb{Q}$-linear combinations of multiple zeta values in many individual cases and in a very few families, as found by Broadhurst, Kreimer, Bierenbaum, Weinzierl, ... We no longer believe that we have non-multiple zeta value examples; the conjecture is that multiple zeta values suffice.

Multiple zeta values should come from motives.

The dream is that motivic theory can show that all Feynman integrals are multiple zeta values and that given a graph we can use the tool box of algebraic geometry to predict which multiple zeta values occur. We are not very far towards the dream.

To calculate the cohomology of the graph variety we need to blow up where it meets the chain of integration, namely wherever we have a 1PI subgraph (not necessarily superficially divergent). This suggests a different Hopf algebra, where the subgraphs in the coproduct are taken to be all 1PI subgraphs. This Hopf algebra plays well with Kontsevich graph homology.

In practice all which can be computed, even now a few years later, are the wheels which give single zeta values. Even these involve some difficult computations due to Hélène Esnault. She uses the symmetry of the triangular wedges making up the wheels.

We should be able to use combinatorial properties of the graph directly to compute the cohomology and hence the zeta values. Every attempt leads to tantalizing almost-patterns, but no answers.

References

[1] Spencer Bloch, Hélǹe Esnault, Dirk Kreimer, On Motives Associated to Graph Polynomials. Commun.Math.Phys. **267**, (2006), 181-225. (Also arXiv:math/0510011).

[2] D.J. Broadhurst and D. Kreimer, Exact solutions of Dyson-Schwinger equations for iterated one-loop integrals and propagator-coupling duality. Nucl.Phys. B **600**, (2001), 403-422. (Also arXiv:hep-th/0012146).

[3] A. Connes and D. Kreimer. Hopf algebras, renormalization and noncommutative geometry. Commum. Math. Phys. **199** (1998), 203-242. (Also arXiv:hep-th/9808042)

[4] L. Foissy, Finite-dimensional comodules over the Hopf algebra of rooted trees. J. Algebra *255*, (2002), 89-120.

[5] Dirk Kreimer and Karen Yeats, An Étude in non-linear Dyson-Schwinger Equations. Nucl. Phys. B Proc. Suppl., **160**, (2006), 116-121. (Also arXiv:hep-th/0605096.)

[6] Dirk Kreimer and Karen Yeats, Recursion and Growth Estimates in Renormalizable Quantum Field Theory. arXiv:hep-th/0612179.