A VARIATIONAL PROBLEM RELATED TO SEMICLASSICAL FOCUSING NLS

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ABSTRACT

We consider the variational problem of maximizing the weighted equilibrium Green's energy of a distribution of charges free to move in a subset of the upper half-plane, under a particular external field. We show that this problem admits a solution and that, under some conditions, this solution is regular in some strictly defined sense.

The external field in question defines a problem that appears in the analysis of the semiclassical focusing NLS problem, but also in the asymptotic analysis of multipoint Padé approximants.

This paper is a result of joint work with K.McLaughlin, P.Miller and E.Rakhmanov.

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0. INTRODUCTION

In a recent book by some of the authors [1], the problem of the so-called semiclassical asymptotics for the focusing nonlinear Schrödinger equation has been considered. More precisely, one is interested in studying the behavior of solutions of

$$i\hbar\partial_t\psi + \frac{\hbar^2}{2}\partial_x^2\psi + |\psi|^2\psi = 0, \tag{0}$$
$$under\ \psi(x,0) = \psi_0(x),$$

in the so-called semiclassical limit, i.e. as $\hbar \to 0$. Here $\psi_0(x)$ is a positive "bell-shaped" (i.e. $\psi_0(-x) = \psi_0(x)$, with one single maximum at 0, and such that $\psi_0''(0) < 0$) function that is rapidly decreasing for large $|x|$ (so that the inverse scattering theory can be applied).

Although this is a problem arising in PDE theory it is directly related to problems in approximation theory. For example, for a large class of initial data, the inverse scattering problem to which (0) can be reduced, for fixed h, is equivalent to the problem of constructing multipoint Padé approximants for a particular exponential function. Thus, the semiclassical problem studied in [1] is equivalent to a problem of finding an asymptotic expression for some Padé approxiamnts (see equation (2.6) in [1]).

In Chapter 8 of [1] it is noted that the above problem is related and can be reduced to a particular "electrostatic" variational problem of maximizing the equilibrium energy of a distribution of charges that are free to move under a given external electrostatic field (at least assuming that the WKB-approximated density of the eigenvalues admits a holomorphic extension in the upper half-plane). In fact, it is pointed out that the existence and regularity of a solution of that variational problem implies the existence of the so-called "g-function" necessary to justify the otherwise rigorous methods employed in [1].

In this paper, we indeed present the proof of existence and regularity of the problem introduced in Chapter 8 of [1]. The method used to prove the existence of

S-curves arising in the solution of the "max-min" energy problem was first outlined in [2] and further developed in [3] and [4], at least for logarithmic potentials. But, the concrete particular problem addressed in this paper involves additional technical issues. In particular, one needs to use information depending on the very particular external field we have, in order to show that the support of the solution does not touch the boundary of the underlying space. Only this ensures that variations can be taken.

The paper is organised as follows. In section 1, we introduce the variational problem. In section 2, we prove a "Gauss-Frostman" type theorem which shows that the variational problem that we try to solve is not vacuous. In section 3, we present the proof of continuity of a particularly defined "energy functional". In section 4, we present a proof of existence of a solution of the variational problem. In section 5, we show that the "max-min" solution of the variational problem does not "touch" the boundary of the underlying domain, except possibly at three special points. This enables us to eventually take variations and show that the max-min property implies regularity of the support of the solution and the S-property in section 6. By regularity, we mean that the support of the maximizing measure is a union of analytic arcs. This union is generically finite. Regularity and the S-property imply the existence of an appropriate g-function which ensures the validity of the analysis of [1].

1. THE SETTING

The space we have to work with is the upper half-plane minus a "spike": $\mathbb{H} = \{z : Imz > 0\} \setminus \{z : Rez = 0, 0 < Imz \leq A\}$. In the closure of this space, $\bar{\mathbb{H}}$, we consider the points ix_+ and ix_-, where $0 \leq x < A$ as different. In other words, we cut a slit in the upper half-plane along the segment $(0, iA)$ and distinguish between the two sides of the slit. The point infinity belongs to $\bar{\mathbb{H}}$, but not $\mathbb{H}$.

The set of distinguished points $\mathbb{A}$ contains the two points 0_+ and 0_-. We define $\mathbb{F}$ to be the set of all continua F in $\bar{\mathbb{H}}$ (i.e. connected compact sets containing more than one point) containing the distinguished points $0_+, 0_-$.

We need to introduce a topology on $\mathbb{F}$. We define the Hausdorff metric as follows. Let $F^\delta = \{z \in \mathbb{C} : \rho(z, F) < \delta\}$, where ρ is the usual distance between compact sets $\rho(E, F) = min_{z \in E, \zeta \in F} |z - \zeta|_C$ and $|z - \zeta|_C$ is the chordal distance (since we want $\bar{\mathbb{H}}$ to be compact). Then, define the Hausdorff distance

$$\rho_{\mathbb{H}}(E, F) = inf\{\delta : E \subset F^\delta, F \subset E^\delta\}. \tag{1}$$

It is easily seen (cf. [5], p.61, with trivial modifications) that, since $\bar{\mathbb{H}}$ is compact and complete, so is $\mathbb{F}$.

Our external field ϕ is the function defined as follows. Let $G(z; \eta)$ be the Green's function for the upper half-plane

$$G(z; \eta) = log \frac{|z - \eta^*|}{|z - \eta|} \tag{2}$$

and $d\mu^0(\eta)$ be the nonnegative measure $-\rho^0(\eta) d\eta$ on the segment $[0, iA]$ oriented from 0 to iA. The star denotes complex conjugation. Then

$$\phi(z) = -\int G(z; \eta) d\mu^0(\eta) - Re(i\pi J \int_z^{iA} \rho^0(\eta) d\eta + 2iJ(zx + z^2 t)). \tag{3}$$

Here we assume $t \geq 0$, so that $\sigma = J$ in formula (8.3) of [1]. J is a priori 1 or -1, but eventually in [1] we set $J = sgn(x)$. So, from now on in this paper we also set $J = sgn(x)$.

The WKB density ρ^0 is assumed to admit a holomorphic extension in the complex plane. Hence ϕ is a harmonic function in $\mathbb{H}$; it is also continuous in $\bar{\mathbb{H}}$ except at infinity.

Next, let $\mathbb{M}$ be the set of all positive measures on $\mathbb{H}$, such that both the (free) energy

$$E(\mu) = \int\int G(x,y)d\mu(x)d\mu(y), \ \mu \in \mathbb{M} \tag{4}$$

and $\int \phi d\mu$ are finite. Also, let

$$V^{\mu}(z) = \int G(z,x)d\mu(x), \ \mu \in \mathbb{M}. \tag{5}$$

The weighted energy of the field ϕ is

$$E_{\phi}(\mu) = E(\mu) + 2\int \phi d\mu, \tag{6}$$

for any $\mu \in \mathbb{M}$.

Now, given any $F \in \mathbb{F}$, the equilibrium measure λ^F supported in F is defined by

$$E_{\phi}(\lambda^F) = min_{\mu \in M(F)} E_{\phi}(\mu), \tag{7}$$

where $M(F)$ is the set of measures in $\mathbb{M}$ which are supported in F. It is not always true that such an equilibrium measure exists. The Gauss-Frostman theorem ([6], p.135) guarantees the existence of the equilibrium measure when F does not touch the boundary of the domain $\mathbb{H}$. This is not the case here. Still, as we show in the next section, in the particular case of our special external field, for any given x, t and for any continuum F not containing infinity, the weighted energy is bounded below and λ^F exists. So, in particular, we do know that the supremum of the equilibrium weighted energies over all continua is finite.

2. A GAUSS-FROSTMAN THEOREM

We claim that for any continuum $G \in \mathbb{H}$, containing 0_- and 0_+, not containing the point ∞, and approaching 0_+, 0_- non-tangentially to the real line, the weighted energy is bounded below and the equilibrium measure λ^G exists.

We begin by considering the equilibrium measure on the contour F that wraps itself around the spike, call it λ^F. We have seen (section 6.2.1 of [1]) that such a measure exists and that it minimizes energy on the contour F, which is hence bounded below. Note that the field ϕ is independent of time on F, so λ^F is also independent of time.

Note that $\rho(z=0) = \rho^0(z=0)$, where ρ^0 is the WKB density of eigenvalues. This can be seen, for example, from formula (6.3) of [1] and the fact that C_I lies on the imaginary axis when $t=0$, together with the fact that $\rho(\eta)$ is imaginary.

From the fact that $\bar{\rho} = \rho - \rho^0$ is constrained to be in the Hölder class with exponent 1/2 ((5.14) of [1]), we deduce that $\bar{\rho} = O(z^{1/2})$ (at worst) near $z=0$. In fact, from formula (6.3) of [1], we can expand in powers of $z^{1/2}$ near zero.

Instead of the definition of the weighted energy, we use the following formula (easily verified).

$$E_\phi(\mu) - E_\phi(\lambda^F) = E(\mu - \lambda^F) + 2\int (V^{\lambda^F} + \phi) d(\mu - \lambda^F), \tag{8}$$

where μ is some positive measure on the continuuum G. Here $V^{\lambda^F}(u) = \int G(u,v) d\lambda^F(v)$, where $G(u,v) = log|(u-v^*)/(u-v)|$ is the Green function for the upper half-plane.

To show that $E_\phi(\mu)$ is bounded below, all we need is show that the difference $E_\phi(\mu) - E_\phi(\lambda^F)$ is bounded below.

Note that since $V^{\lambda^F} + \phi = 0$, *on* $supp(\lambda^F)$, the integral in (8) can be written as $\int (V^{\lambda^F} + \phi) d\mu$.

We have

$$
\begin{aligned}
V^{\lambda^F}(z)+\phi(z) &= \int_0^{b_0(x)} log\frac{|z+iu|}{|z-iu|}(-\rho)_{t=0}du+\phi = \\
&= -Re[\int_0^{b_0(x)} log\frac{|z+iu|}{|z-iu|}\bar{\rho}_{t=0}du]+O(|z|) = \\
&= -Re[\int_0^{b_0(x)} log\frac{|z+iu|}{|z-iu|}u^{1/2}du]+O(|z|) = \\
&= O(|z|) \ near \ z=0,
\end{aligned}
$$

by using the formula (8.3) of [1] for ϕ. Re denotes real parts. So we can write $V^{\lambda^F}+\phi \geq c(A,x)|z|$ in a neighborhood of $z=0$, where $c(A,x)$ will be some negative constant independent of z. Note that the dependence on t is not suppressed, but it is of order $O(|z^2|)$.

It is now not hard to see that the $O(|z|)$ decay implies our result.

Write $\mu = M\sigma$, where $M>0$ is the total mass of μ and σ is a probability measure (on G). Choose ϵ such that for $|u|<\epsilon$ we have $V^{\lambda^F}+\phi \geq c(A,x)|u|$. Then

$$
\begin{aligned}
&E_\phi(\mu)-E_\phi(\lambda^F) \geq \int G(u,v)d(\mu-\lambda^F)(u)d(\mu-\lambda^F)(v) \\
&+2\int_{|v|\geq\epsilon}(V^{\lambda^F}+\phi)(v)d\mu(v)+\int_{|v|<\epsilon}2c(A,x)|v|d(\mu-\lambda^F)(v).
\end{aligned} \tag{9}
$$

The first integral of the RHS can be written as $\int_{|u|\geq\epsilon,|v|\geq\epsilon}+2\int_{|u|<\epsilon,|v|\geq\epsilon}+\int_{|u|,|v|<\epsilon}$. The sum of the first integral plus the second term of the RHS of (9) is bounded below, by standard Gauss-Frostman. It remains to consider

(10)

$$
\begin{aligned}
&(\int_G+\int_{|v|\geq\epsilon})[\int_{|u|<\epsilon}G(u,v)d(\mu-\lambda^F)(u)]d(\mu-\lambda^F)(v)+\int_{|v|<\epsilon}2c(A,x)|v|d(\mu-\lambda^F)(v) \\
&\geq (\int_G+\int_{|v|\geq\epsilon})[\int_{|u|<\epsilon}G(u,v)d(M\sigma-\lambda^F)(u)+2c(A,x)|v|]d(M\sigma-\lambda^F)(v).
\end{aligned}
$$

Now, it is easy to see that for M large, and since F is non-tangential to the real line,

$$
\int_{|u|<\epsilon}G(u,v)d(M\sigma-\lambda^F)(u)+2c(A,x)|v| \geq |vlogv|+2c(A,x)|v|. \tag{11}
$$

By adjusting ϵ if necessary, we have $|vlogv| > 2|c(A,x)||v|$, and hence the integral in (11) is positive. Integrating again wrt. $Md\sigma - d\lambda^F$, again for M large, we see that the integral of (10) is positive.

Since when M is bounded above we have our estimates trivially, we get boundedness below over all positive M.

We have thus proved one part of our (generalised) Gauss-Frostman Theorem.

THEOREM 1. Let ϕ be given by (3). Let F be a continuum in $\bar{\mathbb{H}} \setminus \infty$ and suppose that F is contained in some sector $\pi < \alpha < arg(\lambda) < \beta < 0$ as $\lambda \to 0$. Let $M(F)$ be the set of measures $\in \mathbb{M}$ which are supported in F. (So, in particular their free energy is finite and $\phi \in L_1(\mu)$.) We have

$$inf_{\mu \in M(F)} E_\phi(\mu) > -\infty. \tag{12}$$

Furthermore the equilibrium measure on F exists, that is there is a non-negative measure λ^F with support contained in F such that $E_\phi[F] = E_\phi(\lambda^F) = inf_{\mu \in M(F)} E_\phi(\mu)$.
PROOF: The proof that (12) implies the existence of an equilibrium measure is a well known theorem. For our particular field ϕ given by (3) it is easy to prove. Indeed, the identity

$$E(\mu - \nu) = 2E_\phi(\mu) + 2E_\phi(\nu) - 4E_\phi\left(\frac{\mu + \nu}{2}\right)$$

implies that any sequence μ_n minimizing $E_\phi(\mu)$ is a Cauchy sequence in (unweighted) energy. Since the space of positive measures is complete (see for example [7], Theorem 1.18, p.90), there is a measure μ_0 such that $E(\mu_n - \mu_0) \to 0$. We then have $E(\mu_n) \to E(\mu_0) < +\infty$ and hence $\mu_n \to \mu_0$ weakly (see e.g. [7], p.82-88; this is a standard result).

The fact that $\phi \in L_1(\mu_0)$ is trivial for our particular field.

3. CONTINUITY OF ENERGY

We consider the functional that takes a given continuum F to the equilibrium energy on this continuum:

$$\mathbb{E} : F \to E_\phi[F] = E_\phi(\lambda^F) = \inf_{\mu \in M(F)} (E(\mu) + 2\int \phi d\mu) \tag{13}$$

and we want to show that it is continuous, if ϕ is continuous in $\bar{\mathbb{H}}$. Note that this is not the case for the field given by (3), since it has a singularity at ∞. We will see how to circumvent this difficulty later. For the present section, ϕ is just assumed to be a continuous function in $\bar{\mathbb{H}}$.

So, suppose $G \in \mathbb{F}$, with $\rho_{\mathbb{H}}(F, G) < \rho$, a small positive constant. Let $\lambda = \lambda_\phi^F$ be the equilibrium measure on F and $\mu = \lambda_\phi^G$ be the equilibrium measure on G.

We consider the Green's balayage of μ on F, say $\hat{\mu}$. Then *supp* $\hat{\mu} \in F$ and

$$\begin{gathered} V^{\hat{\mu}} = V^{\mu} \ on \ F, \\ \int u d\hat{\mu} = \int u d\mu, \end{gathered} \tag{14}$$

for any function u that is harmonic in $\mathbb{H} \setminus F$ and continuous in $\bar{\mathbb{H}}$.

Similarly consider the balayage of λ to G. We trivially have

$$E_\phi[G] \leq E_\phi(\hat{\lambda}),$$

$$E_\phi[F] \leq E_\phi(\hat{\mu}).$$

LEMMA 1. Suppose K is a compact set with positive capacity, μ some positive measure in the plane and $\hat{\mu}$ is the Green's balayage to K. Then

$$\begin{gathered} V^{\hat{\mu}} = V^{\mu} - V^{\mu}_{K^c}, \\ E(\hat{\mu}) = E(\mu) - E_{K^c}(\mu), \end{gathered} \tag{15}$$

where $E_{K^c}(\mu)$ is the unweighted Green energy with respect to $K^c = \{Imz > 0\} \setminus K$.

PROOF OF LEMMA 1: The first identity follows from the fact that $V^{\hat{\mu}} - V^{\mu}$ vanishes on K and the real line, and is harmonic in K^c and superharmonic in $\{Imz > 0\}$.

Integrating $E(\hat{\mu}) = \int V^{\hat{\mu}} d\hat{\mu} = \int V^{\mu} d\hat{\mu} - \int V^{\mu}_{K^c} d\hat{\mu} = \int (V^{\mu} - V^{\mu}_{K^c}) d\mu = E(\mu) - E_{K^c}(\mu)$. The proof of the Lemma follows.

In particular, since unweighted energies are nonnegative,

$$E(\hat{\mu}) \leq E(\mu). \tag{16}$$

PROOF OF CONTINUITY: Let u_ϕ be a function harmonic in $\mathbb{H} \setminus F$ such that $u_\phi = \phi$ on F and $u_\phi = 0$ on $\partial\mathbb{H} \setminus F$. By the definition of balayage one has $\int \phi d\hat{\mu} = \int u_\phi d\mu$.

We have $E_\phi[F] \leq E_\phi(\hat{\mu}) = E(\hat{\mu}) + 2\int \phi d\hat{\mu} \leq E(\mu) + 2\int \phi d\mu + 2\int (u_\phi - \phi) d\mu = E_\phi(\mu) + 2\int (u_\phi - \phi) d\mu = E_\phi[G] + 2\int (u_\phi - \phi) d\mu$.

In a small neighbourhood of $F, \bar{F}_\rho = \{z : \rho(z, F) \leq \rho\}$, we have

$$|2 \int (u_\phi - \phi)(y) d\mu(y)| \leq C max_{y \in \bar{F}_\rho} |u_\phi(y) - \phi(y)|. \tag{17}$$

We assumed here that the equilibrium measures on continua near F are bounded above. This is easy to see. Suppose, first, that the point ∞ is not in F. Indeed, on the support of the equilibrium measure λ, we have

$$V^\lambda + \phi = 0. \tag{18}$$

If the equilibrium measures on continua near F were unbounded, then so would be the potentials V^λ. But ϕ is definitely bounded near F. This contradicts the above equality.

If $\infty \in F$, we refer to Theorem 3 below. In fact, since we have assumed that ϕ is continuous in this section, ∞ is not a special point. But for ϕ given by (3) ∞ is a point of discontinuity. This case is considered in the next section.

Now given $y \in \bar{F}_\rho$, choose $z \in F$ such that $|z - y| = \rho$. The above expression (17) is then

$$\leq C max_{y \in \bar{F}_\rho} |u_\phi(y) - \phi(y) - u_\phi(z) + \phi(z)| \leq o(1) + C max_{y \in \bar{F}_\rho} |u_\phi(y) - u_\phi(z)|.$$

It remains to bound $|u_\phi(y) - u_\phi(z)|$ by an $o(1)$ quantity.

The next Theorem is due to Milloux and can be found in [8].

THEOREM 2. Suppose D is an open disc of radius R, with center z_0; let y be a point in D, F a continuum in $\mathbb{C}$, containing z_0, and Ω be the connected component of $D \setminus F$ containing y. Let $w(z)$ be a function harmonic in Ω such that

$$\begin{aligned} w(z) &= 0,\ z \in F \cap \partial\Omega, \\ w(z) &= 1,\ z \in \partial\Omega \setminus F. \end{aligned} \tag{19}$$

Then $w(y) \leq C(\frac{|y-z_0|}{R})^{1/2}$.

PROOF: See [8], p.347.

Now, select a disc of radius $\rho^{1/2}$, centered on z. We have $|u_\phi(y) - u_\phi(z)| = o(1)$ on the part of F lying in the disc, while $|u_\phi(y) - u_\phi(z)|$ is bounded by some positive constant M on the disc boundary.

LEMMA 2. Let Ω be a domain, $\partial\Omega = F_1 \cup F_2$ and

$$\begin{aligned} w_1 &= 0,\ z \in F_1, \\ &= 1,\ z \in F_2; \\ w_2 &= 1,\ z \in F_1, \\ &= 0,\ z \in F_2. \end{aligned}$$

Suppose u is harmonic in Ω and

$$\begin{aligned} u(z) &\leq \epsilon,\ z \in F_1, \\ u(z) &\leq M,\ z \in F_2. \end{aligned}$$

Then $u(z) \leq \epsilon w_2(z) + M w_1(z)$.

PROOF OF LEMMA 2: Maximum principle.

Now, using Milloux's Theorem, we get $|u_\phi(y) - u_\phi(z)| \leq o(1) w_2(z) + M w_1(z) \leq o(1) + MC\frac{|y-z|}{\rho}^{1/2} \leq o(1) + MC\rho^{1/2}$. Hence result.

4. PROOF OF EXISTENCE OF A MAXIMIZING CONTINUUM

We now recall that the energy continuity proof was based on the continuity of ϕ. In our case, ϕ is discontinuous at ∞. Still we can prove that the energy is upper semicontinuous and that will be enough.

THEOREM 3. For the external field given by (3), the energy functional defined in (8) is upper semicontinuous.

PROOF: Let F be a continuum. If ∞ is not in F, then we're done. If $\infty \in F$, let $\lambda = \lambda^F$ be the equilibrium measure. We can assume that on the equilibrium measure ϕ is bounded by 0. Indeed, on the support of the equilibrium measure λ, we have

$$V^\lambda + \phi = 0.$$

But $V^\lambda \geq 0$, so $\phi \leq 0$.

This means that we can change ϕ to $\phi' = min(\phi, 0)$, which is an upper semicontinuous function.

Let ϕ_n be a sequence of continuous functions such that $\phi_n \downarrow \phi'$. Each functional $E_{\phi_n}[F]$ is continuous, and $E_{\phi_n}[F] \downarrow E_{\phi'}[F]$. So, $E_{\phi'}[F]$ is upper semicontinuous.

THEOREM 4. For the external field given by (3), there exists a contour F such that the equilibrium measure λ^F exists and $E_\phi[F] = E_\phi(\lambda^F) = max_{F \in \mathbb{F}} min_{\mu \in M(F)} E_\phi(\mu)$.

PROOF: We know (section 6.2 of [1]) that there is at least one contour (namely the one wrapped around the spike $(0, iA)$) for which the equilibrium measure exists and $E_\phi(\lambda^F) > -\infty$, for all time. Hence the supremum over continua in $\mathbb{F}$ is finite (and trivially nonpositive), since $\mathbb{F}$ is compact. Call it L.

We can now take a sequence F_n such that $E_\phi[F_n] \to L$. Choose a convergent subsequence of continua $F_n \to F$. By upper semicontinuity of the weighted energy functional,

$$\bar{lim} E_\phi[F_n] \leq E_\phi[F] \leq L = lim E_\phi[F_n].$$

So $L = E_\phi[F]$. The theorem is proved.

5. ACCEPTABILITY OF THE CONTOUR

We have thus shown that a solution of the maximum-minimum problem exists. We do not know yet that it is an analytic contour. All we know is that it is a "continuum". The fact that it is indeed an analytic contour and that the support of the measure is a countable union of analytic arcs will follow from the analyticity properties of the external field.

It is important however, that the maxi-minimizing continuum does not touch the boundary of the space $\mathbb{H}$ except of course at the points $0^-, 0^+$, and perhaps at ∞. This is to guarantee that variations with respect to the maximizing contour can be properly taken.

The proof of the acceptability of the continuum requires three things.

(i) The continuum does not touch the real negative axis.

(ii) The continuum does not touch the real positive axis.

(iii) The continuum does not touch the "spike" $\{z : Rez = 0, 0 < Imz \leq A\}$, except of course at time $t = 0$.

(iv) It also remains to consider the point at infinity. We cannot prove that the continuum does not hit this point. In fact, our numerics ([1], Chapter 6) show that, depending on the sign of J it may well do so. However, we will eventually show that even if we allow the continuum to pass through infinity, a modification of the argument in [1] will work.

Without loss of generality, we will consider the case $x \geq 0$ and hence $J = 1$.

PROOFS: (i) If $z < 0$, then $\phi(z) > 0$. But we can always delete the strictly positive measure lying in a region where the field is positive and make the energy smaller. So even the solution of the "inner" minimizing problem has to lie away from the real negative axis.

(ii) If $z > 0$, then a short calculation shows that $\frac{d\phi}{dImz} > 0$.

It is crucial here that if $u \in \mathbb{R}$ then $G(u, v) = 0$, while if both u, v are off the real line $G(u, v) > 0$. Hence, for any configuration that involves a contour including points on the real line, we can find a configuration with no points on the real line, by pushing measures up away from the real axis, which has greater (unweighted *and* weighted) energy. So, suppose the maximizing continuum touches the axis. We can always push the measures up away from the real axis and end up with a continuum that has greater minimal energy, thus arriving at a contradiction.

(iii) We must show that the maximizing continuum stays away from the linear segment that connects 0 and iA. We can see that this is indeed the case by comparing to what we know is true near time zero. Begin by observing that for t small and the appropriate choice of $J = sgn(x)$, the contour (which at time zero coincides with the piece of the boundary of $\mathbb{H}$ that joins points 0_- and 0_+) is "repelled" by the "spike" $[0, iA]$. (See [1] section 6.2.)

Now suppose that for some (not necessarily small) time t_0 the maximizing continuum is forced to touch the "spike". We will arrive at a contradiction.

An easy straightforward calculation shows that the external field ϕ given by (3) satisfies

$$\phi(\epsilon u + iv; x, t_0) = \phi(iv; x, \epsilon t_0) + O(\epsilon^2),$$

for u, v real.

We also note that the measure solving the maximin problem can be expressed via an integral formula with respect to its associated potential and hence the external field (e.g. see [6] section II.1 and use $V^\lambda + \phi = 0$ on $supp(\lambda)$). This expression depends on the values of ϕ in a small neighborhood of the point where we are calculating the measure. This implies that the situation near the "spike" at time t_0 can be reduced to the situation at time ϵt_0, modulo a field difference $O(\epsilon^2)$. Now we know ([1] section 6.2) that for small times, the contour does not touch the spike (except at 0). In fact, the proof of this fact in [1] also holds if we perturb the field by the same $O(\epsilon^2)$ term for $u + iv$ not on the spike. Hence, even for t_0, the contour has to be at a positive distance from the spike, except of course at $u + iv = 0$.

(iv) At first, it might seem that the maximizing continuum (which is eventually seen to be a contour) should not pass through infinity. Indeed, the transformations (2.17) and (4.1) of [1] implicitly assume that the contour C lies in $\mathbb{C}$. Otherwise, one would lose the appropriate normalization for M at infinity. However, one has to simply notice that infinity is just an arbitrary choice of normalizing point, once we view our Riemann-Hilbert problems in the compact Riemann Sphere. The important observation is that the composition of transformations (2.17) and (4.1) (which are purely formal, i.e. no estimates are required and no approximation is needed) does not introduce any bad (essential) singularities. In the end, the asymptotic behavior of $\tilde{N}^{\sigma}$ is still the identity as $z \to \infty$ in the lower half-plane and non-singular as $z \to \infty$ in the upper half-plane. So, in the end it is acceptable for a contour to go through the point infinity.

6. A SOLUTION OF THE VARIATIONAL PROBLEM HAS THE S-PROPERTY

In the following section, we prefer to consider our variational problem as a problem in the whole complex plane, not just the upper half-plane. We will thus be able to make direct use of the results of [3] on which this section is based.

We let (see Chapter 8 of [1])

$$V(z) = -\int_{-iA}^{iA} log(z-\eta)\rho^0(\eta)d\eta - sgn(x)(2ixz + 2itz^2 + i\pi\int_z^{iA}\rho^0(\eta)d\eta) \tag{20}$$

and $V_R = ReV, V_I = ImV$ be the real and imaginary parts of V, respectively. As in section 1, $d\mu^0(\eta) = -\rho^0(\eta)d\eta$ is a non-negative measure in the upper-half plane, expressing the WKB-derived distribution of eigenvalues of the Zakharov-Shabat operator associated to the focusing NLS problem, in the semiclassical limit. In the lower half-plane the measure is extended simply by $\mu^0(\eta^*) = -\mu^0(\eta)$.

The space of measures on which the new variational problem is defined will now consist of signed measures, satisfying $\mu(z^*) = -\mu(z)$ and having total mass zero. In the upper half-plane the measures are positive, while in the lower half-plane the measures are negative. The potentials are logarithmic (rather than Green's) potentials and the energy of a measure is

$$E_{V_R}(\mu) = 1/2\int_{supp\mu}\int_{supp\mu} log\frac{1}{|u-v|}d\mu(u)d\mu(v) + \int_{supp\mu} V_R(u)d\mu(u). \tag{21}$$

It is immediate that the solution of the variational problem introduced in section 1 is the restriction of a variational problem defined in the full complex plane minus the slit $[-iA, iA]$, under external field given by V_R and with weighted energy defined by (21). A priori, the symmetry $\mu(z^*) = -\mu(z)$ has to be imposed. However, it follows from the symmetry of the external field. Indeed, consider the solution $\mu = \lambda^F$ of the max-min problem in the whole complex plane, without imposing symmetry. Then, in particular the "min" part of the problem is satisfied, so μ is an equilibrium measure. Taking first variation one gets $V_R(z) + Re\int_{supp\mu} log(z-\eta)d\mu(\eta) = 0$. It is not a priori clear that μ obeys any symmetry or that $supp\mu$ is symmetric. But it does follow (see later) that $supp\mu$ is a union of analytic arcs. So, differentiating the above relation we see that V_R' is the real part of the Hilbert transform of the measure μ. Now (see [9] for example) μ is then the Hilbert transform of V_R', on a symmetric extension of the contour $supp\mu$. The symmetry for μ then follows from the symmetry for V_R, that is $V_R(z^*) = -V_R(z)$.

The fact that the total mass of μ is zero also follows. We now have $supp\mu = C \cup C*$ for some contour C in the upper half-plane. So, $\int_{C\cup C*} log|z-\eta|\rho(\eta)d\eta = \int_C log\frac{|z-\eta|}{|z-\eta*|}(-\rho(\eta))d\eta$. As z goes to infinity the RHS goes to zero, hence the LHS goes to zero and the zero mass condition follows.

Having established that we can focus on the doubled-up variational problem, we proceed to show regularity and the S-property. Having established in section 5 that the contour solving the variational problem does not touch the real line or the slit $[-iA, iA]$ we can take small variations of measures and contours, never intersecting the slit or the real line, and keeping the points $0_-, 0_+$ fixed. This ensures that the measures involved stay always positive or negative. The fact that ∞ can belong to the contour is not a problem. Our variations will keep it automatically fixed.

The first step is to show that the solution of the variational problem satisfies a crucial relation.

THEOREM 5. If $\mu = \lambda^F$ is the (signed) equilibrium measure minimizing the weighted logarithmic energy (21) under external field $V_R = ReV$ where V is given by (20), then

$$Re\Big(\int_{supp\mu} \frac{d\mu(u)}{u-z} + V'(z)\Big)^2 = Re(V'(z))^2 - 2Re\int_{supp\mu} \frac{V'(z)-V'(u)}{z-u} d\mu(u) + \frac{1}{z^2}\Big[Re\Big\{\int_{supp\mu} 2(u+z)V'(u)\; d\mu(u)\Big\}\Big]. \tag{22}$$

PROOF: We first need to prove the following.

THEOREM 6. Let Γ be a critical point of E_{V_R}, or more precisely the functional taking a continuum Γ to $E_{V_R}(\lambda^\Gamma)$, and assume that Γ is not tangent to $\mathbb{R}\cup[-iA, iA]$. Let $\mu = \lambda^\Gamma$ and O_Γ be an open set containing the interior of Γ and $h \in C^1(O_\Gamma)$. We have

$$Re\Big[\int\int \frac{h(u)-h(v)}{u-v} d\mu(u)d\mu(v)\Big] = 2Re\Big[\int V'(u)h(u)d\mu(u)\Big]. \tag{23}$$

PROOF: Consider the family of measures $\{\mu^\tau, \tau \in \mathbb{C}, |\tau| < \tau_0\}$ defined by $d\mu^\tau(z^\tau) = d\mu(z)$ where $z^\tau = z + \tau h(z))$, or equivalently, $\int f(z)d\mu^\tau(z) = \int f(z^\tau)d\mu(z),\ f \in L_1^\mu(O)$.

With $\hat{h} = \hat{h}(u,v) = \frac{h(u)-h(v)}{u-v}$, we have $\frac{u^\tau - v^\tau}{u-v} = 1 + \tau\hat{h}$, so that $log\frac{1}{|u^\tau - v^\tau|} - log\frac{1}{|u-v|} = -log|1+\tau\hat{h}| = -Re(\tau\hat{h}) + O(\tau^2)$.

Integrating with respect to $d\mu(u)d\mu(v)$ we arrive at

$$E(\mu^\tau) - E(\mu) = -Re\Big[\tau\int \hat{h}d\mu(u)d\mu(v)\Big] + O(\tau^2), \tag{24}$$

where $E(\mu)$ denotes the free logarithmic energy of the measure μ. Also,

$\int V_R d\mu^\tau - \int V_R d\mu = 2\int(V_R(u^\tau) - V_R(u))d\mu(u) = 2Re[\tau\int V'(u)h(u)d\mu(u)] + O(\tau^2)$. Combining with the above,

$$E_{V_R}(\mu^\tau) - E_{V_R}(\mu) = Re\Big(-\tau\int \hat{h}d\mu(u)d\mu(v) + 2\tau\int V'hd\mu\Big) + O(\tau^2). \tag{25}$$

So, if μ is a critical point of the map $\mu \to E_{V_R}(\mu)$ the linear part of the increment is zero. In other words given a continuous function h and a measure μ the function $E_{V_R}(\mu^\tau)$ of τ is differentiable at $\tau = 0$ and that the derivative is

$$Re(-H(\mu)),\ where\ H(\mu) = \int\int \hat{h}d\mu^2 - 2\int V'hd\mu. \tag{26}$$

But what we really want is the derivative of the energy as a function of the equilibrium measure. This function can be shown to be differentiable and its derivative can be set to zero at a critical continuum.

Indeed, we need to show

LEMMA 3. At a critical Γ,

$$\frac{d}{d\tau}E_{V_R}((\lambda^{\Gamma})^{\tau})|_{\tau=0} = \frac{d}{d\tau}E_{V_R}(\lambda^{\Gamma_\tau})|_{\tau=0} = 0.$$

Here λ^{Γ} means the equilibrium measure on the contour Γ. In the relation above $\Gamma_\tau = supp(\lambda^{\Gamma})^{\tau}$. The first derivative is of a function of general measures. The second derivative is of a function of equilibrium measures.

PROOF: Define the measure σ_τ with support Γ and such that $(\sigma_\tau)^{\tau} = \lambda^{\Gamma_\tau}$.

LEMMA 4. With H defined as above, we have

$$ReH(\sigma_\tau) \to ReH(\lambda^{\Gamma}),$$

as $\tau \to 0$.

PROOF. By (25)-(26), we have

$$E_{V_R}((\lambda^{\Gamma})^{\tau}) - E_{V_R}(\lambda^{\Gamma}) = -Re(\tau H(\lambda^{\Gamma}) + O(\tau^2)),$$

$$E_{V_R}(\lambda^{\Gamma_\tau}) - E_{V_R}(\sigma_\tau) = -Re(\tau H(\sigma_\tau) + O(\tau^2)).$$

On the other hand, $E_{V_R}(\sigma_\tau) \geq E_{V_R}(\lambda^{\Gamma})$, and $E_{V_R}((\lambda^{\Gamma})^{\tau}) \geq E_{V_R}(\lambda^{\Gamma_\tau})$. It follows that

$$E_{V_R}(\sigma_\tau) - Re(\tau H(\sigma_\tau)) + O(\tau^2) = E_{V_R}(\lambda^{\Gamma_\tau}) \leq E_{V_R}((\lambda^{\Gamma})^{\tau}) = E_{V_R}(\lambda^{\Gamma}) - Re(\tau H(\lambda^{\Gamma})) + O(\tau^2).$$

Hence $E_{V_R}(\sigma_\tau) \to E_{V_R}(\lambda^{\Gamma})$. It also follows immediately that $ReH(\sigma_\tau) \to ReH(\lambda^{\Gamma})$. This proves Lemma 4.

To complete the proof of Lemma 3, we note that $0 \geq E_{V_R}(\lambda^{\Gamma_\tau}) - E_{V_R}(\lambda^{\Gamma}) \geq E_{V_R}(\lambda^{\Gamma_\tau}) - E_{V_R}(\sigma_\tau) = -Re(\tau H(\sigma_\tau)) + O(\tau^2)$. Hence the derivative of $E_{V_R}((\lambda^{\Gamma})^{\tau})$ at $\tau = 0$ is (by Lemma 4) $ReH(\lambda^{\Gamma})$. This proves Lemma 3 and Theorem 6.

PROOF OF THEOREM 5. Consider the Schiffer variation, i.e. take $h(u) = \frac{u^2}{u-z}$ where z is some fixed point not in Γ. We have

$$\hat{h} = \hat{h}(u,v) = \frac{h(u) - h(v)}{u - v} = 1 - \frac{z^2}{(u-z)(v-z)},$$

and therefore

$$Re[\int\int \hat{h}(u,v)d\mu(u)d\mu(v)] = Re(\int\int d\mu(u)d\mu(v) - z^2[\int_{supp\mu} \frac{d\mu(u)}{u-z}]^2).$$

Next

$$\begin{aligned} &Re[\int 2V'(u)h(u)d\mu(u)] \\ &= 2Re[\int (u+z)V'(u)d\mu(u) + z^2\int \frac{V'(u)d\mu(u)}{u-z}] \\ &= 2Re[\int 2V'(u)(u+z)d\mu(u) + \\ &2z^2\int \frac{V'(u)-V'(z)}{u-z}d\mu(u) + 2z^2V'(z)\int\frac{d\mu(u)}{u-z}]. \end{aligned}$$

Theorem 5 now follows from Theorem 6.

Theorem 5 above implies that the support of μ is the level set of the real part of a function that is analytic except at countably many branch points. In fact, $supp\mu$ is characterized by $\int log\frac{1}{|u-z|}d\mu(u) + V_R(z) = 0$. From Theorem 5 we get

$$Re[\int \frac{d\mu(u)}{u-z} + V'(z)] = Re[(R_\mu(z))^{1/2}] \tag{27}$$

where

$$\begin{aligned} R_\mu(z) = (V'(z))^2 - 2\int_{supp\mu} \frac{V'(z)-V'(u)}{z-u}d\mu(u) \\ +\frac{1}{z^2}(\int_{supp\mu} 2(u+z)V'(u)\ d\mu(u)). \end{aligned} \tag{28}$$

This is a function analytic in $\mathbb{H}\cup\mathbb{H}^*$, with possibly a pole at zero. By integrating, we have that $supp(\mu)$ is characterized by

$$Re\int^z (R_\mu)^{1/2}dz = constant.$$

At worst R_μ has countably many zeros in $\mathbb{H}\cup\mathbb{H}^*$ and $\int^z(R_\mu)^{1/2}dz$ is analytic except possibly at the zeros of R_μ. So $Re\int^z(R_\mu)^{1/2}dz$ is the real part of a function analytic except at countably many branch points. Hence the support of μ is the level set of the real part of a function that is analytic except at countably many branch points. So it has to be a countable union of analytic arcs.

Now, it is important to note that

$$\begin{aligned} R_\mu(z) \sim -sgn(x)[16t^2z^2 + \pi^2(\rho^0(z))^2],\ as\ z\to\infty, \\ R_\mu(z) \sim \frac{1}{z^2}\int 2uV'(u)d\mu(u),\ as\ z\to 0. \end{aligned} \tag{29}$$

This implies that R_μ is generically not zero at the point ∞. Hence it can only have finitely many zeros on the support of μ, otherwise they would have to accumulate near ∞, and hence R_μ would be 0 there. "Generically" can mean both in terms of the initial data, but also in terms of x, t.

Also, R_μ cannot have an accumulation point of zeros at $z = 0$, because even if the pole at 0 can be removed, it is easy to see that R_μ can be holomorphically continued across $z = 0$.

We have thus arrived at a justification of an "infinite gap ansatz", and generically a "finite gap ansatz", concerning the semiclassical limit of focusing NLS. It only remains to verify the "S-property". The justification of the "g-function" ansatz in [1] will follow immediately.

THEOREM 7. (The S-property)

Let $X(z) = \int_{supp\mu} log(\frac{1}{u-z})d\mu(u)$, $X_R(z) = ReX(z) = \int_{supp\mu} log(\frac{1}{|u-z|})d\mu(u)$, $W_\mu = X'$. We have, at any interior point of $supp\mu$,

$$\frac{d}{dn_+}(V_R + X_R) = \frac{d}{dn_-}(V_R + X_R),$$

where the two derivatives above denote the normal derivatives, on the $+$ and $-$ sides respectively.

REMARK. Once Theorem 7 is proved it follows by the Cauchy-Riemann equations that $(V_I + ImX)_+ + (V_I + ImX)_-$ is constant on each connected component of $supp\mu$, which means that $Im\tilde{\phi}$ is constant on connected components of the contour, where $\tilde{\phi}$ is as defined in formula (4.13) of [1]. This proves the existence of the appropriate "g-functions" in [1].

PROOF OF THEOREM 7. From Theorem 5, we have

$$|W_\mu(z) + V'(z)| = |R_\mu|^{1/2}.$$

Using the definition for X, the above relation becomes

$$|\frac{d}{dz}(X + V)| = |R_\mu|^{1/2}.$$

Now, $Re(X+V)=0$ on the support of the equilibrium measure. So, in particular $Re(X+V)$ is constant along the equilibrium measure. Hence $|\frac{d}{dz}(X+V)|$ must be equal to the modulus of the normal derivative across the equilibrium measure. So,

$$|\frac{d}{dn_{\pm}}(V_R + ReX)| = |\frac{d}{dz}(X+V)| = |R_{\mu}(z)|^{1/2}.$$

Hence,

$$|\frac{d}{dn_{+}}(V_R + ReX)| = |\frac{d}{dn_{-}}(V_R + ReX)|.$$

But it is easy to see that both LHS and RHS quantities inside the modulus sign are negative. This is because $V_R + ReX = 0$ on $supp\mu$ and negative on each side of $supp\mu$. Hence result.

We recapitulate our results in the folllowing theorem.

THEOREM 8. The semiclassical limit of (0) with bell-shaped initial data generically admits a finite genus description in the sense of [1], assuming that the spectral density of eigenvalues ($\rho^0(\eta)$ in relations (3.1) and (3.2) of [1]) admits an analytic continuation in the complex plane.

REMARK: In general the spectral density ρ^0 can admit branch singularities in the upper complex plane. We believe that even in such a case the finite gap genus should be justifiable, at least generically.

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