DOUBLE SHORT EXACT SEQUENCES AND K_1 OF AN EXACT CATEGORY

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MPI 95-118

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ABSTRACT. We introduce a modification of Bass' K_1^{det}. Our group is defined in terms of double short exact sequences instead of automorphisms and gives a better approximation to Quillen's K_1 than K_1^{det}, for any exact category. We establish its relation to K_1 by dealing with loops in the G-construction.

1. INTRODUCTION

The wish to obtain an algebraic description for K_1 of any exact category $\mathfrak{A}$ resembling the formula for rings $K_1(R) = GL(R)/E(R)$ leads to the Bass universal determinant functor (see [Ba1][Ba2][Ge])

$$K_1^{det}(\mathfrak{A}) = K_0(\operatorname{Aut}\mathfrak{A})/\sim \quad ,$$

where $\operatorname{Aut}\mathfrak{A}$ denotes the category of pairs (A,α) with $A \in \mathfrak{A}$ and $\alpha \in \operatorname{Aut} A$ and the equivalence relation is generated by

$$(A,\alpha\beta) \sim (A,\alpha) + (A,\beta) \quad . \tag{1.1}$$

There is a natural map (see [Ge][Sh1])

$$K_1^{det}(\mathfrak{A}) \to K_1(\mathfrak{A}) \tag{1.2}$$

which proves to be an isomorphism if $\mathfrak{A}$ is semisimple, i.e., if every short exact sequence in $\mathfrak{A}$ splits [Sh1][We]. However, in general this map need not be either surjective or injective (see sect. 5 in [Ge]).
In this paper we introduce a modification of $K_1^{det}(\mathfrak{A})$, an abelian group $\mathcal{D}(\mathfrak{A})$ which admits natural homomorphisms

$$K_1^{det}(\mathfrak{A}) \to \mathcal{D}(\mathfrak{A}) \to K_1(\mathfrak{A})$$

with the properties
(i) the composite map is (1.2);
(ii) the map $\mathcal{D}(\mathfrak{A}) \to K_1(\mathfrak{A})$ is surjective for any exact category $\mathfrak{A}$;
(iii) both maps are isomorphisms if $\mathfrak{A}$ is semisimple.
While the automorphisms in $\mathfrak{A}$ serve as generators for $K_1^{det}(\mathfrak{A})$, the generators of $\mathcal{D}(\mathfrak{A})$ are given by the following notion generalizing that of an automorphism.

Key words and phrases. Exact category, K_1, G-construction, double short exact sequence.

Definition 1.1. *A double short exact sequence in $\mathfrak{A}$ (a d.s.e.s. for short) is a pair of short exact sequences (s.e.s. for short) $A \xrightarrow{f_1} B \xrightarrow{g_1} C$ and $A \xrightarrow{f_2} B \xrightarrow{g_2} C$ on the same objects. Given such a d.s.e.s., we will write*

$$l = (A \underset{f_2}{\overset{f_1}{\rightrightarrows}} B \underset{g_2}{\overset{g_1}{\rightrightarrows}} C). \tag{1.3}$$

If $(A, \alpha) \in \operatorname{Aut}\mathfrak{A}$, we put

$$l(\alpha) = (0 \rightrightarrows A \underset{\alpha}{\overset{1}{\rightrightarrows}} A), \tag{1.4}$$

which enables us to regard an automorphism as a particular case of a d.s.e.s.
In section 3, we associate a loop in the G-construction of $\mathfrak{A}$ to any d.s.e.s. Proceeding then to the classes of those loops in $\pi_1(G.\mathfrak{A}) \cong K_1(\mathfrak{A})$, we establish the following relations:
(i) if $f_1 = f_2$ and $g_1 = g_2$ in (1.3), then the corresponding element of $K_1(\mathfrak{A})$ vanishes;
(ii) for any diagram of d.s.e.s's of the form

$$\begin{array}{ccccc} A' & \rightrightarrows & A & \rightrightarrows & A'' \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ B' & \rightrightarrows & B & \rightrightarrows & B'' \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ C' & \rightrightarrows & C & \rightrightarrows & C'' \end{array}$$

where the first (the upper) arrows commute with the first (the left) ones and the second (the lower) arrows commute with the second (the right) ones, the alternating sum of the elements of $K_1(\mathfrak{A})$ associated to the horizontal d.s.e.s's is equal to the alternating sum for the vertical d.s.e.s's in the diagram (section 5).
We then define $\mathcal{D}(\mathfrak{A})$ to be the abelian group generated by all d.s.e.s's in $\mathfrak{A}$ modulo relations (i) and (ii) posed on those generators rather than on their images in $K_1(\mathfrak{A})$, thus getting a natural map $\mathcal{D}(\mathfrak{A}) \to K_1(\mathfrak{A})$ (section 6).
Double short exact sequences form an exact category $DSES(\mathfrak{A})$, and one shows that relations (i) and (ii) imply additivity for the generators of $\mathcal{D}(\mathfrak{A})$, i.e., we have a natural epimorphism

$$K_0(DSES(\mathfrak{A})) \to \mathcal{D}(\mathfrak{A}).$$

Relation (1.1) rewritten in terms of the elements (1.4) also follows from (i) and (ii), thus the composite map $K_0(\operatorname{Aut}\mathfrak{A}) \to K_0(DSES(\mathfrak{A})) \to \mathcal{D}(\mathfrak{A})$ factors through $K_1^{det}(\mathfrak{A})$.
In fact, we have more than surjectivity of the map $\mathcal{D}(\mathfrak{A}) \to K_1(\mathfrak{A})$: in section 3 we sketch out how every element of $K_1(\mathfrak{A})$ corresponds to a d.s.e.s. in $\mathfrak{A}$. (Note that in the semisimple case, every element of $K_1(\mathfrak{A})$ comes from an automorphism in $\mathfrak{A}$). We refer to [Ne] for the detailed proof which is based on a result of Sherman who produces all elements of $K_1(\mathfrak{A})$ by more complicated data. I would like to thank Chuck Weibel who told me about this result. Clayton Sherman kindly gave me his preprint and we had numerous stimulating discussions; I am grateful to him for his

interest to my work.
Dan Grayson considered the group

$$K_0(DSES(\mathfrak{A}))/K_0(SES(\mathfrak{A})) \cong K_0(DSES(\mathfrak{A}))/\text{relation(i)}$$

in connection with weight filtrations. He proved that for any ring R, this group (associated to $\mathfrak{A} = \mathcal{P}_R$) is isomorphic to $K_1(R)$ (unpublished). It is his proof of this fact, based on beautiful matrix tricks that made me believe that the use of double short exact sequences is a more appropriate instrument for describing K_1 than automorphisms (cf. Remark 6.4). I would like to thank Dan for attracting my attention to d.s.e.s's and for his interest to my work.
I would like to express my distinguished gratitude to Max-Planck-Institut für Mathematik in Bonn for its hospitality during the preparation of this paper.

2. Review of the G-construction

In [GG] Gillet and Grayson attached a simplicial set $G.\mathfrak{A}$ to any exact category $\mathfrak{A}$ and proved that $|G.\mathfrak{A}|$ is homotopy equivalent to $\Omega|Q\,\mathfrak{A}| \sim \Omega|S.\mathfrak{A}|$, the equivalence being natural in $\mathfrak{A}$. Thus one can take the formula

$$K_m(\mathfrak{A}) = \pi_m(G.\mathfrak{A}), \quad m \geq 0,$$

for a definition of the higher K-groups of $\mathfrak{A}$.

An n-simplex in $G.\mathfrak{A}$ is a pair of triangular diagrams in $\mathfrak{A}$ of the form

$$\begin{array}{ccccccc} & & & & & & P_{n/n-1} \\ & & & & & & \uparrow \\ & & & & & & \cdots \\ & & & & P_{2/1} & \to \cdots \to & P_{n/1} \\ & & & & \uparrow & & \uparrow \\ & & P_{1/0} & \to & P_{2/0} & \to \cdots \to & P_{n/0} \\ & & \uparrow & & \uparrow & & \uparrow \\ P_0 & \to & P_1 & \to & P_2 & \to \cdots \to & P_n \end{array} \qquad \begin{array}{ccccccc} & & & & & & P_{n/n-1} \\ & & & & & & \uparrow \\ & & & & & & \cdots \\ & & & & P_{2/1} & \to \cdots \to & P_{n/1} \\ & & & & \uparrow & & \uparrow \\ & & P_{1/0} & \to & P_{2/0} & \to \cdots \to & P_{n/0} \\ & & \uparrow & & \uparrow & & \uparrow \\ P'_0 & \to & P'_1 & \to & P'_2 & \to \cdots \to & P'_n \end{array} \qquad (2.1)$$

subject to the conditions:
(i) the quotient index subtriangles in both diagrams coincide;
(ii) all the squares commute;
(iii) all the sequences of the form $P_j \to P_k \to P_{k/j}$, $P'_j \to P'_k \to P_{k/j}$, and $P_{j/i} \to P_{k/i} \to P_{k/j}$ with $i \leq j \leq k$ are s.e.s's in $\mathfrak{A}$.
In particular, a vertex in $G.\mathfrak{A}$ is a pair of objects (P, P'), and an edge connecting (P_0, P'_0) to (P_1, P'_1) is a pair of s.e.s's $(P_0 \to P_1 \to P_{1/0},\ P'_0 \to P'_1 \to P_{1/0})$ with equal cokernels. We will also write (s, s') for an edge, where s and s' denote s.e.s's with equal cokernels.
The i-th face of (2.1) amounts to deleting all the objects whose indices contain i.

For instance, the faces of a generic 2-simplex

$$
t = \left[\begin{array}{ccccccccc}
 & & & & P_{2/1} & & & & P_{2/1} \\
 & & & & \uparrow & & & & \uparrow \\
 & & P_{1/0} & \longrightarrow & P_{2/0} & & P_{1/0} & \longrightarrow & P_{2/0} \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
P_0 & \longrightarrow & P_1 & \longrightarrow & P_2 & P_0' \longrightarrow & P_1' & \longrightarrow & P_2'
\end{array} \right] \tag{2.2}
$$

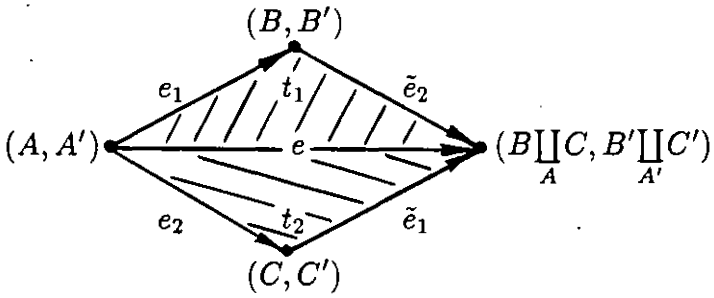
are given by

$$
\begin{aligned}
d_0 t &= (P_1 \to P_2 \to P_{2/1}, \quad P_1' \to P_2' \to P_{2/1}) \\
d_1 t &= (P_0 \to P_2 \to P_{2/0}, \quad P_0' \to P_2' \to P_{2/0}) \\
d_2 t &= (P_0 \to P_1 \to P_{1/0}, \quad P_0' \to P_1' \to P_{1/0}).
\end{aligned}
$$

The following push-out procedure is useful for constructing homotopies of loops in $G.\mathfrak{A}$. It is by applying this procedure that Gillet and Grayson obtained in [GG] the first algebraic description for the elements of $K_1(\mathfrak{A})$ in terms of loops in $G.\mathfrak{A}$ (of a rather complicated form). Given two edges

$$
\begin{aligned}
e_1 &= (A \to B \to M, \quad A' \to B' \to M) \\
e_2 &= (A \to C \to N, \quad A' \to C' \to N)
\end{aligned} \tag{2.3}
$$

with the same initial vertex (A, A'), and a choice of push-out objects $B \coprod_A C$ and $B' \coprod_{A'} C'$ in $\mathfrak{A}$, we can construct in $G.\mathfrak{A}$ a picture of the form



(2.4)

which consists of two 2-simplices t_1 and t_2. For instance, t_1 is given by

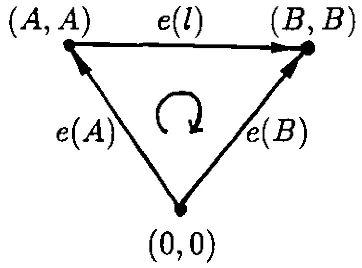
$$
t_1 = \left[\begin{array}{ccccccccc}
 & & & & N & & & & N \\
 & & & & \uparrow & & & & \uparrow \\
 & & M & \longrightarrow & M \oplus N & & M & \longrightarrow & M \oplus N \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
A & \longrightarrow & B & \longrightarrow & B \coprod_A C & A' \longrightarrow & B' & \longrightarrow & B' \coprod_{A'} C'
\end{array} \right]
$$

where all the arrows are induced by those in (2.3) in the obvious way. The second 2-simplex t_2 is given by similar data.

3. ATTACHING LOOPS IN $G.\mathfrak{A}$ TO DOUBLE SHORT EXACT SEQUENCES

Let 0 denote a distinguished zero object in $\mathfrak{A}$ and let (0,0) be the base point in $G.\mathfrak{A}$. For $A \in \mathfrak{A}$, we denote by $e(A)$ the standard edge from $(0,0)$ to (A, A) given by $(0 \to A \xrightarrow{1} A, \quad 0 \to A \xrightarrow{1} A)$.

Let $l = (A \underset{f_2}{\overset{f_1}{\rightrightarrows}} B \underset{g_2}{\overset{g_1}{\rightrightarrows}} C)$ be a d.s.e.s., and let $e(l)$ denote the edge from (A, A) to (B, B) given by l. We associate to l a loop $\mu(l)$ in $G.\mathfrak{A}$ of the form



(3.1)

and let $m(l)$ be its class in $K_1(\mathfrak{A}) = \pi_1(G.\mathfrak{A})$.

Lemma 3.1. *(trivial) The loop $\mu(l)$ bounds a 2-simplex in $G.\mathfrak{A}$ if and only if $f_1 = f_2$ and $g_1 = g_2$. In this case, this 2-simplex is uniquely determined and given by*

$$t(l) = \begin{bmatrix} & & C & & & C \\ & & \uparrow{\scriptstyle g} & & & \uparrow{\scriptstyle g} \\ & A \xrightarrow{\;f\;} & B & & A \xrightarrow{\;f\;} & B \\ & {\scriptstyle 1}\uparrow & \uparrow{\scriptstyle 1} & & {\scriptstyle 1}\uparrow & \uparrow{\scriptstyle 1} \\ 0 \longrightarrow & A \xrightarrow{\;f\;} & B & 0 \longrightarrow & A \xrightarrow{\;f\;} & B \end{bmatrix}$$

where we put $f = f_1 = f_2$ and $g = g_1 = g_2$.

Restricting ourselves to the d.s.e.s's of the form $l(\alpha)$ for $(A, \alpha) \in \operatorname{Aut}\mathfrak{A}$ (cf. (1.4)), we obtain the map $\alpha \mapsto m(l(\alpha))$. This is one of the various equivalent ways to attach an element of K_1 to an automorphism. The loop $\mu(l(\alpha))$ is actually a 2-edge loop of the form

for the edge $e(0)$ is degenerate. One checks that every 2-edge loop of this form is homotopic to $\mu(l(\alpha))$ for some α, thus the elements of $K_1(\mathfrak{A})$ representable by automorphisms are precisely those representable by 2-edge loops in $G.\mathfrak{A}$.

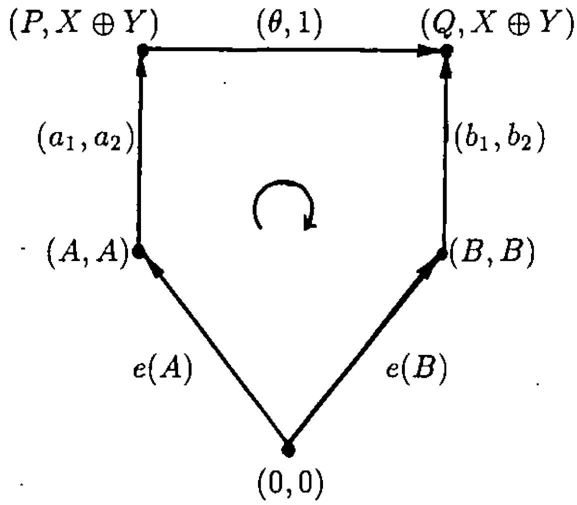
It is known that not every element of $K_1(\mathfrak{A})$ can be represented in general by an automorphism. Moreover, $K_1(\mathfrak{A})$ is not generated by such elements for some $\mathfrak{A}$ (Proposition 5.1 in [Ge]). However, we prove that every element of $K_1(\mathfrak{A})$ can be represented by a loop of the form (3.1).

Theorem 3.2. *Let $\mathfrak{A}$ be an arbitrary exact category. Then for any element $x \in K_1(\mathfrak{A})$ there exists a double short exact sequence l such that $x = m(l)$.*

We sketch out how this assertion can be deduced from a result of Sherman. The reader is referred to [Ne] for details.
Consider data of the form

$$j = (A \xrightarrow{\alpha} X \xrightarrow{\gamma} C,\ B \xrightarrow{\beta} Y \xrightarrow{\delta} D;\quad \theta : A \oplus Y \oplus C \xrightarrow{\sim} X \oplus B \oplus D) \tag{3.2}$$

where $A \to X \to C$ and $B \to Y \to D$ are short exact sequences in $\mathfrak{A}$ and θ is an isomorphism. Given such data, we will sometimes denote $A \oplus Y \oplus C$ by P and $X \oplus B \oplus D$ by Q for short. Sherman associates to j a loop $\nu(j)$ in $G.\mathfrak{A}$ of the form



where $(\theta, 1)$ denotes the edge $(P \xrightarrow{\theta} Q \to 0,\ X \oplus Y \xrightarrow{1} X \oplus Y \to 0)$ and the s.e.s's yielding the vertical edges are given by

$$a_1 = (A \xrightarrow{\begin{pmatrix}1\\0\\0\end{pmatrix}} A \oplus Y \oplus C \xrightarrow{\begin{pmatrix}0&0&1\\0&1&0\end{pmatrix}} C \oplus Y)$$

$$a_2 = (A \xrightarrow{\begin{pmatrix}\alpha\\0\end{pmatrix}} X \oplus Y \xrightarrow{\gamma \oplus 1_Y} C \oplus Y)$$

$$b_1 = (B \xrightarrow{\begin{pmatrix}0\\1\\0\end{pmatrix}} X \oplus B \oplus D \xrightarrow{\begin{pmatrix}1&0&0\\0&0&1\end{pmatrix}} X \oplus D)$$

$$b_2 = (B \xrightarrow{\begin{pmatrix}0\\\beta\end{pmatrix}} X \oplus Y \xrightarrow{1_X \oplus \delta} X \oplus D).$$

Let $n(j)$ denote the corresponding element in $K_1(\mathfrak{A})$. The following assertion was proved by Sherman in [Sh1] for abelian categories and then in [Sh2] for arbitrary exact categories.

Theorem 3.3. *For any $x \in K_1(\mathfrak{A})$ there exists j of the form (3.2) such that $x = n(j)$.*

We associate to j a pair of short exact sequences

$$s_1 = (A \oplus B \xrightarrow{f} A \oplus Y \oplus C \xrightarrow{p} C \oplus D)$$
$$s_2 = (A \oplus B \xrightarrow{g} X \oplus B \oplus D \xrightarrow{q} C \oplus D) \quad .$$

where

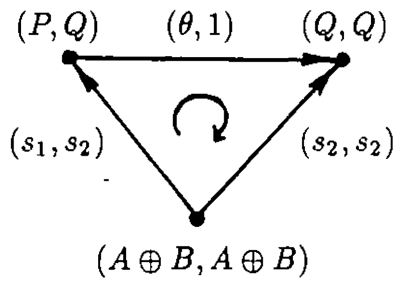
$$f = \begin{pmatrix} 1 & 0 \\ 0 & \beta \\ 0 & 0 \end{pmatrix}, \quad g = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad p = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \delta & 0 \end{pmatrix}, \quad q = \begin{pmatrix} \gamma & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Replacing $A \oplus Y \oplus C$ in s_1 by $X \oplus B \oplus D$ via θ we obtain the double short exact sequence

$$l(j) = (A \oplus B \underset{g}{\overset{\theta \circ f}{\rightrightarrows}} X \oplus B \oplus D \underset{q}{\overset{p \circ \theta^{-1}}{\rightrightarrows}} C \oplus D).$$

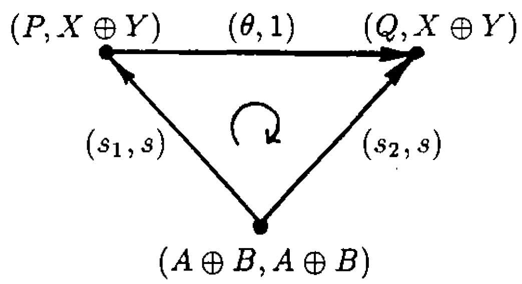
It now suffices to show that the loops $\mu(l(j))$ and $\nu(j)$ are freely homotopic since the group $\pi_1(G.\mathfrak{A}) = K_1(\mathfrak{A})$ is abelian.

Lemma 3.4. *(easy)* *The loop $\mu(l(j))$ is freely homotopic to the loop*



(3.3)

Lemma 3.5. *(easy)* *The loop $\nu(j)$ is freely homotopic to the loop*



(3.4)

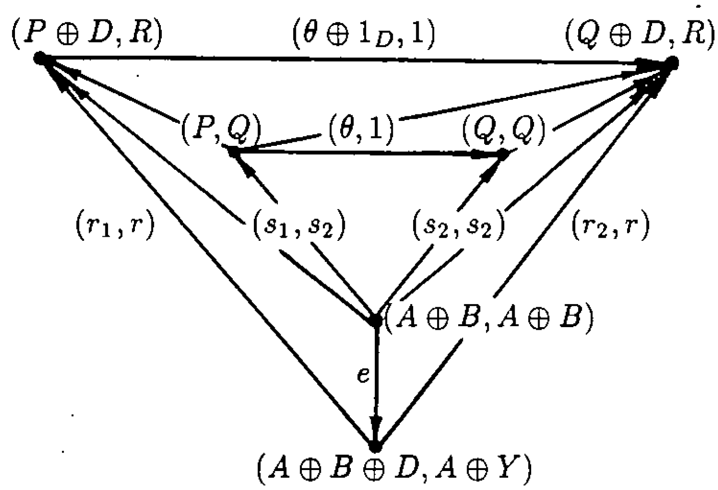
where $s = (A \oplus B \xrightarrow{\alpha \oplus \beta} X \oplus Y \xrightarrow{\gamma \oplus \delta} C \oplus D)$.

Lemma 3.6. *The loops (3.3) and (3.4) are freely homotopic.*

Sketch of the proof. Let e denote the edge

$$e = (A \oplus B \xrightarrow{\binom{1_{A\oplus B}}{0}} A \oplus B \oplus D \xrightarrow{(0,0,1)} D, \quad A \oplus B \xrightarrow{1_A \oplus \beta} A \oplus Y \xrightarrow{(0,\delta)} D).$$

We construct the following configuration of six 2-simplices

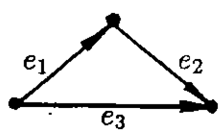


(3.5)

It consists of three push-outs of the form (2.4) and yields a free homotopy between the loop (3.3) and the outer loop (we can call (3.5) the push-out of the loop (3.3) along the edge e). Then we take the push-out of the loop (3.4) along e, but in the latter case we have to replace the object $(X \oplus Y) \coprod_{A\oplus B} (A \oplus Y) \cong X \oplus (Y \coprod_B Y)$ by $X \oplus Y \oplus D$ by means of the isomorphism $Y \coprod_B Y \cong Y \oplus D$ in order to obtain the same outer loop. One should take care about this change of objects, because there are two natural isomorphisms $Y \coprod_B Y \cong Y \oplus D$ that differ by the natural involution of $Y \coprod_B Y$. The explicit data of the form (2.2) yielding the required six 2-simplices in the latter homotopy are given in [Ne]. ■

4. Admissible triples of edges in the G-construction

Suppose we are given a triple of edges $\tau = (e_1, e_2, e_3)$ that forms a triangular contour in $G.\mathfrak{A}$



i.e., these edges are given by data of the form

$$\begin{aligned} e_1 &= (P_0 \xrightarrow{\alpha_{0,1}} P_1 \xrightarrow{\alpha_{1,1/0}} P_{1/0}\,, P'_0 \xrightarrow{\alpha'_{0,1}} P'_1 \xrightarrow{\alpha'_{1,1/0}} P_{1/0}) \\ e_2 &= (P_1 \xrightarrow{\alpha_{1,2}} P_2 \xrightarrow{\alpha_{2,2/1}} P_{2/1}\,, P'_1 \xrightarrow{\alpha'_{1,2}} P'_2 \xrightarrow{\alpha'_{2,2/1}} P_{2/1}) \\ e_3 &= (P_0 \xrightarrow{\alpha_{0,2}} P_2 \xrightarrow{\alpha_{2,2/0}} P_{2/0}\,, P'_0 \xrightarrow{\alpha'_{0,2}} P'_2 \xrightarrow{\alpha'_{2,2/0}} P_{2/0}) \end{aligned} \qquad (4.1)$$

Such a triple τ is said to be an *admissible triple* if

$$\alpha_{0,2} = \alpha_{1,2} \circ \alpha_{0,1} \quad \text{and} \quad \alpha'_{0,2} = \alpha'_{1,2} \circ \alpha'_{0,1} \quad . \tag{4.2}$$

If τ is admissible, consider the diagrams

$$\begin{array}{ccccc} P_0 & \xrightarrow{\alpha_{0,1}} & P_1 & \xrightarrow{\alpha_{1,1/0}} & P_{1/0} \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle \alpha_{1,2}} & & \downarrow{\scriptstyle \alpha_{1/0,2/0}} \\ P_0 & \xrightarrow{\alpha_{0,2}} & P_2 & \xrightarrow{\alpha_{2,2/0}} & P_{2/0} \\ \downarrow & & \downarrow{\scriptstyle \alpha_{2,2/1}} & & \downarrow{\scriptstyle \alpha_{2/0,2/1}} \\ 0 & \longrightarrow & P_{2/1} & \xrightarrow{1} & P_{2/1} \end{array} \qquad \begin{array}{ccccc} P'_0 & \xrightarrow{\alpha'_{0,1}} & P'_1 & \xrightarrow{\alpha'_{1,1/0}} & P_{1/0} \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle \alpha'_{1,2}} & & \downarrow{\scriptstyle \alpha'_{1/0,2/0}} \\ P'_0 & \xrightarrow{\alpha'_{0,2}} & P'_2 & \xrightarrow{\alpha'_{2,2/0}} & P_{2/0} \\ \downarrow & & \downarrow{\scriptstyle \alpha'_{2,2/1}} & & \downarrow{\scriptstyle \alpha'_{2/0,2/1}} \\ 0 & \longrightarrow & P_{2/1} & \xrightarrow{1} & P_{2/1} \end{array}$$

where the vertical arrows at the right $\alpha_{1/0,2/0}, \alpha_{2/0,2/1}, \alpha'_{1/0,2/0}$, and $\alpha'_{2/0,2/1}$ are uniquely determined by the arrows in (4.1). Then

$$l(\tau) = (P_{1/0} \underset{\alpha'_{1/0,2/0}}{\overset{\alpha_{1/0,2/0}}{\rightrightarrows}} P_{2/0} \underset{\alpha'_{2/0,2/1}}{\overset{\alpha_{2/0,2/1}}{\rightrightarrows}} P_{2/1})$$

is said to be the *double short exact sequence associated to the admissible triple* τ. We will use the same letter τ to denote the loop $e_1 e_2 e_3^{-1}$ in $G.\mathfrak{A}$.

Lemma 4.1. *(main lemma about admissible triples) If τ is an admissible triple, then the loop $\tau = e_1 e_2 e_3^{-1}$ is freely homotopic to*

$$(P_2, P'_2) \oplus \mu(l(\tau)) .$$

Remark 4.2. Suppose we are given a choice of direct sums $A \oplus B$ for all pairs of objects (A, B) in $\mathfrak{A}$. Then given a vertex (A, A') in $G.\mathfrak{A}$, we have the simplicial shift map

$$(A, A') \oplus -- : G.\mathfrak{A} \to G.\mathfrak{A}$$

which takes a simplex of the form (2.1) to

$$\begin{array}{ccccc} & & & & P_{n/(n-1)} \\ & & & & \uparrow \\ & & & & \cdots \\ & & P_{1/0} & \to \cdots \to & P_{n/0} \\ & & \uparrow & & \uparrow \\ A \oplus P_0 & \to & A \oplus P_1 & \to \cdots \to & A \oplus P_n \end{array} \qquad \begin{array}{ccccc} & & & & P_{n/(n-1)} \\ & & & & \uparrow \\ & & & & \cdots \\ & & P_{1/0} & \to \cdots \to & P_{n/0} \\ & & \uparrow & & \uparrow \\ A' \oplus P'_0 & \to & A' \oplus P'_1 & \to \cdots \to & A' \oplus P'_n \end{array}$$

Any edge connecting (A, A') to (B, B') yields a simplicial homotopy between the maps $(A, A') \oplus --$ and $(B, B') \oplus --$. Thus the homotopy class of such a shift map depends on the component of the vertex only, and this gives rise to the natural action of $K_0(\mathfrak{A})$ on $G.\mathfrak{A}$ up to homotopy. In particular, if (A, A') is in the base point component, then $(A, A') \oplus --$ is homotopic to the identity map.

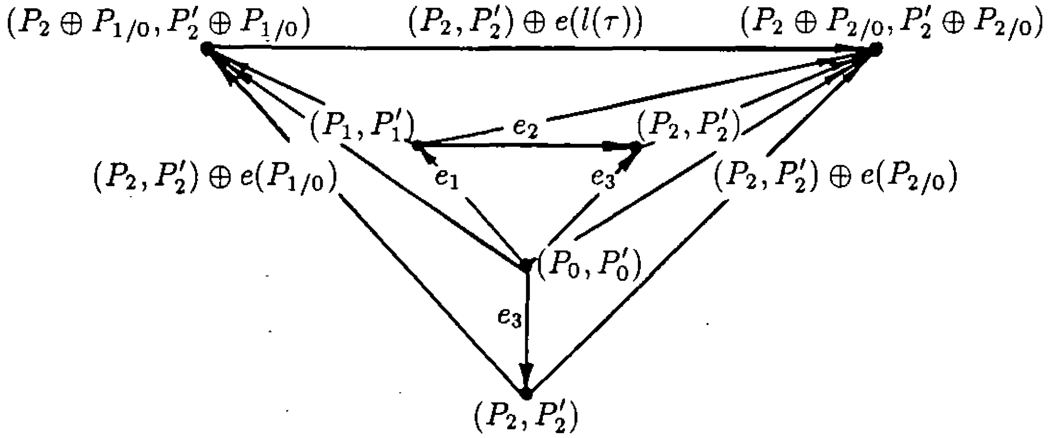
Corollary 4.3. *If an admissible triple τ lies in the base point component of $G.\mathfrak{A}$, then the loop τ is freely homotopic to $\mu(l(\tau))$.*

Remark 4.4. The admissibility condition (4.2) is necessary for τ to be the contour of a (uniquely determined) 2-simplex. In fact, $e_1, e_2,$ and e_3 bound a 2-simplex if and only if

$$\alpha_{1/0,2/0} = \alpha'_{1/0,2/0} \quad \text{and} \quad \alpha_{2/0,2/1} = \alpha'_{2/0,2/1} \tag{4.3}$$

i.e., if the two s.e.s's in $l(\tau)$ coincide (compare to Lemma 3.1). In this case, if a loop in $G.\mathfrak{A}$ contains an edge (or two edges) of τ and we replace this edge by the other two (respectively, by the other one), then we will obtain a homotopic loop. If (4.3) does not hold, then nevertheless Lemma 4.1 enables us to speak of "homotopies of loops modulo elements of the form $\mu(l)$" induced by admissible triples.

Proof of the lemma. First we apply successively the push-out procedure (2.4) to the edge e_3 along the edges e_1 and e_2, and then to the edge e_3 along itself. It follows from (4.2) that the resulting edges will coincide. Replacing $P_2 \coprod_{P_0} P_1$ and $P_2 \coprod_{P_0} P_2$ (respectively, $P'_2 \coprod_{P'_0} P'_1$ and $P'_2 \coprod_{P'_0} P'_2$) by $P_2 \oplus P_{1/0}$ and $P_2 \oplus P_{2/0}$ (respectively, by $P'_2 \oplus P_{1/0}$ and $P'_2 \oplus P_{2/0}$) by means of the natural isomorphisms, we obtain a free homotopy of the form



This homotopy connects τ to $(P_2, P'_2) \oplus \mu(l(\tau))$ (we depict the two copies of e_3 as different edges in order to make the procedure more visible). It is an easy exercise to write down explicitly the data of the form (2.2) for the six 2-simplices that form this homotopy. The lemma is proved. ■

5. The main relation for the elements $m(l)$

Proposition 5.1. *(main lemma about d.s.e.s's) Suppose we are given a diagram of the form*

$$\begin{array}{ccccc}
A' & \overset{a_1}{\underset{a_2}{\rightrightarrows}} & A & \overset{x_1}{\underset{x_2}{\rightrightarrows}} & A'' \\
{\scriptstyle f_1} \downdownarrows {\scriptstyle f_2} & & {\scriptstyle g_1} \downdownarrows {\scriptstyle g_2} & & {\scriptstyle h_1} \downdownarrows {\scriptstyle h_2} \\
B' & \overset{b_1}{\underset{b_2}{\rightrightarrows}} & B & \overset{y_1}{\underset{y_2}{\rightrightarrows}} & B'' \\
{\scriptstyle u_1} \downdownarrows {\scriptstyle u_2} & & {\scriptstyle v_1} \downdownarrows {\scriptstyle v_2} & & {\scriptstyle w_1} \downdownarrows {\scriptstyle w_2} \\
C' & \overset{c_1}{\underset{c_2}{\rightrightarrows}} & C & \overset{z_1}{\underset{z_2}{\rightrightarrows}} & C''
\end{array} \tag{5.1}$$

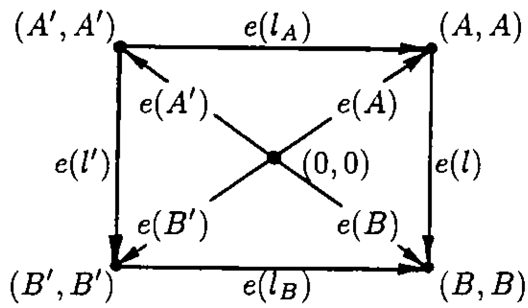
which consists of six double short exact sequences and is subject to the condition: the first arrows commute with the first ones and the second arrows commute with the second ones, i.e., for $i = 1,2$

$$b_i f_i = g_i a_i\,, \quad y_i g_i = h_i x_i\,, \quad c_i u_i = v_i b_i\,, \quad z_i v_i = w_i y_i\,. \tag{5.2}$$

Let l_A, l_B, and l_C (respectively, l', l, and l'') denote the horizontal (respectively, the vertical) d.s.e.s's in (5.1). Then in $K_1(\mathfrak{A})$ we have

$$m(l_A) - m(l_B) + m(l_C) = m(l') - m(l) + m(l'')\,.$$

Proof. It follows from the picture



that the loop $\mu(l')\mu(l_B)\mu(l)^{-1}\mu(l_A)^{-1}$ is freely homotopic to the outer loop α given by $e(l')e(l_B)e(l)^{-1}e(l_A)^{-1}$. As the group $\pi_1(G.\mathfrak{A}) = K_1(\mathfrak{A})$ is abelian, it suffices to show that α is freely homotopic to the loop $\mu(l_C)\mu(l'')^{-1}$. For $i = 1,2$, choose a particular object D_i to be the push-out of $B' \xleftarrow{f_i} A' \xrightarrow{a_i} A$ and consider the push-out diagram

$$\begin{array}{ccccc}
A' & \xrightarrow{a_i} & A & \xrightarrow{x_i} & A'' \\
\downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle \tilde{f}_i} & & \Vert \\
B' & \xrightarrow{\tilde{a}_i} & D_i & \xrightarrow{\tilde{x}_i} & A'' \\
\downarrow{\scriptstyle u_i} & & \downarrow{\scriptstyle \tilde{u}_i} & & \\
C' & = & C' & &
\end{array} \tag{5.3}$$

Let $d_i : D_i \to B$ denote the map uniquely determined by

$$d_i \circ \tilde{f}_i = g_i \quad \text{and} \quad d_i \circ \tilde{a}_i = b_i\,. \tag{5.4}$$

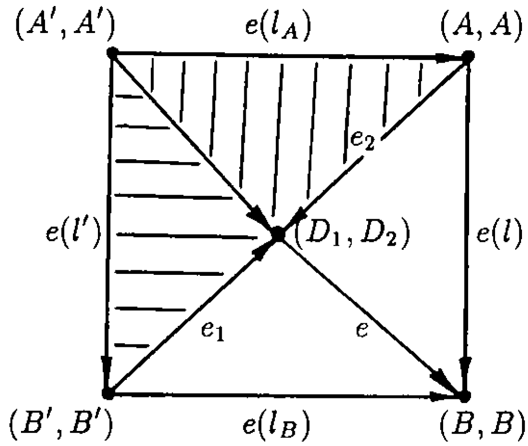
One checks that we have short exact sequences

$$D_i \xrightarrow{d_i} B \xrightarrow{p_i} C'' \qquad (i = 1,2)$$

where

$$p_i = w_i y_i = z_i v_i\,. \tag{5.5}$$

These data lead to the following picture



where the shaded area is the push-out of $e(l_A)$ and $e(l')$ (cf. (2.4)), and the edges e, e_1, and e_2 are given by

$$\begin{aligned} e &= (D_1 \xrightarrow{d_1} B \xrightarrow{p_1} C'', \quad D_2 \xrightarrow{d_2} B \xrightarrow{p_2} C'') \\ e_1 &= (B' \xrightarrow{\tilde{a}_1} D_1 \xrightarrow{\tilde{x}_1} A'', \quad B' \xrightarrow{\tilde{a}_2} D_2 \xrightarrow{\tilde{x}_2} A'') \\ e_2 &= (A \xrightarrow{\tilde{f}_1} D_1 \xrightarrow{\tilde{u}_1} C', \quad A \xrightarrow{\tilde{f}_2} D_2 \xrightarrow{\tilde{u}_2} C'). \end{aligned}$$

It follows that the free homotopy class of α is equal to the difference of the classes of $\alpha_2 = e_2 e\, e(l)^{-1}$ and $\alpha_1 = e_1 e\, e(l_B)^{-1}$. In view of Corollary 4.3, it now suffices to prove the following

Lemma 5.2. *$(e_1,\ e,\ e(l_B))$ and $(e_2,\ e,\ e(l))$ are admissible triples, the associated double short exact sequences being l'' and l_C respectively.*

Proof. The admissibility condition (4.2) follows in both cases from (5.4). The computation of the associated d.s.e.s's amounts to the diagrams

$$\begin{array}{ccccc} B' & \xrightarrow{\tilde{a}_i} & D_i & \xrightarrow{\tilde{x}_i} & A'' \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle d_i} & & \downarrow{\scriptstyle h_i} \\ B' & \xrightarrow{b_i} & B & \xrightarrow{y_i} & B'' \\ \downarrow & & \downarrow{\scriptstyle p_i} & & \downarrow{\scriptstyle w_i} \\ 0 & \longrightarrow & C'' & \xrightarrow{1} & C'' \end{array} \qquad \begin{array}{ccccc} A & \xrightarrow{\tilde{f}_i} & D_i & \xrightarrow{\tilde{u}_i} & C' \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle d_i} & & \downarrow{\scriptstyle c_i} \\ A & \xrightarrow{g_i} & B & \xrightarrow{v_i} & C \\ \downarrow & & \downarrow{\scriptstyle p_i} & & \downarrow{\scriptstyle z_i} \\ 0 & \longrightarrow & C'' & \xrightarrow{1} & C'' \end{array}$$

By virtue of (5.2-4), we have $y_i\, d_i\, \tilde{a}_i = y_i\, b_i = 0 = h_i\, \tilde{x}_i\, \tilde{a}_i$ and $y_i\, d_i\, \tilde{f}_i = y_i\, g_i = h_i\, x_i = h_i\, \tilde{x}_i\, \tilde{f}_i$. It follows from the universal property of push-outs that $y_i\, d_i = h_i\, \tilde{x}_i$, i.e., the upper right square in the first diagram commutes, and similarly for the second diagram. Commutativity of the other squares follows easily from (5.4-5). Lemma 5.2 and Proposition 5.1 are proved. ■

6. THE GROUP $\mathcal{D}(\mathfrak{A})$

Definition 6.1. *Let $\mathfrak{A}$ be an exact category. We define $\mathcal{D}(\mathfrak{A})$ to be the abelian group with generators $\langle l \rangle$ for all double short exact sequences l in $\mathfrak{A}$ subject to the relations*
(i) $\langle l \rangle = 0$ if l has the form

$$l = (A \underset{f}{\overset{f}{\rightrightarrows}} B \underset{g}{\overset{g}{\rightrightarrows}} C)\,; \tag{6.1}$$

(ii) for any diagram of the form

$$\begin{array}{ccccc} A' & \rightrightarrows & A & \rightrightarrows & A'' \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ B' & \rightrightarrows & B & \rightrightarrows & B'' \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ C' & \rightrightarrows & C & \rightrightarrows & C'' \end{array} \tag{6.2}$$

consisting of six double short exact sequences

$$l_A = (A' \rightrightarrows A \rightrightarrows A'')\,, \quad l_B = (B' \rightrightarrows B \rightrightarrows B'')\,, \quad l_C = (C' \rightrightarrows C \rightrightarrows C'')$$

$$l' = (A' \rightrightarrows B' \rightrightarrows C')\,, \quad l = (A \rightrightarrows B \rightrightarrows C)\,, \quad l'' = (A'' \rightrightarrows B'' \rightrightarrows C'')\,,$$

such that the first arrows commute with the first ones and the second arrows commute with the second ones, we have

$$\langle l_A \rangle - \langle l_B \rangle + \langle l_C \rangle = \langle l' \rangle - \langle l \rangle + \langle l'' \rangle\,.$$

Theorem 6.2. *Let $\mathfrak{A}$ be an exact category.*

1. *There is a well-defined homomorphism*

$$m : \mathcal{D}(\mathfrak{A}) \longrightarrow K_1(\mathfrak{A})$$

given on generators by $\langle l \rangle \mapsto m(l)$. This homomorphism is surjective; moreover, every element of $K_1(\mathfrak{A})$ is the image of some generator $\langle l \rangle$.

2. *There is a well-defined homomorphism*

$$K_1^{det}(\mathfrak{A}) \longrightarrow \mathcal{D}(\mathfrak{A}) \tag{6.3}$$

given on generators by $(A, \alpha) \mapsto \langle l(\alpha) \rangle$, where $l(\alpha) = (0 \rightrightarrows A \underset{\alpha}{\overset{1}{\rightrightarrows}} A)$ for $A \in \mathfrak{A}$ and $\alpha \in Aut A$. The composite map

$$K_1^{det}(\mathfrak{A}) \longrightarrow \mathcal{D}(\mathfrak{A}) \longrightarrow K_1(\mathfrak{A}) \tag{6.4}$$

coincides (up to sign) with the standard map that associates an object of $K_1(\mathfrak{A})$ to any automorphism in $\mathfrak{A}$ (cf. [Ge][Sh1]).

3. If the category $\mathfrak{A}$ is semisimple, then both maps in (6.4) are isomorphisms.

Proof. 1. This follows from Lemma 3.1, Proposition 5.1, and Theorem 3.2.
2. Let $DSES(\mathfrak{A})$ denote the category of d.s.e.s's in $\mathfrak{A}$, the morphisms being the diagrams of the form

$$\begin{array}{ccccc} A' & \overset{a_1}{\underset{a_2}{\rightrightarrows}} & A & \overset{x_1}{\underset{x_2}{\rightrightarrows}} & A'' \\ f\big\downarrow & & g\big\downarrow & & h\big\downarrow \\ B' & \overset{b_1}{\underset{b_2}{\rightrightarrows}} & B & \overset{y_1}{\underset{y_2}{\rightrightarrows}} & B'' \end{array}$$

such that $b_i\, f = g\, a_i$ and $y_i\, g = h\, x_i$ for $i = 1, 2$. A s.e.s. in $DSES(\mathfrak{A})$ is a diagram of the form

$$\begin{array}{ccccc} A' & \rightrightarrows & A & \rightrightarrows & A'' \\ f\big\downarrow & & g\big\downarrow & & h\big\downarrow \\ B' & \rightrightarrows & B & \rightrightarrows & B'' \\ u\big\downarrow & & v\big\downarrow & & w\big\downarrow \\ C' & \rightrightarrows & C & \rightrightarrows & C'' \end{array} \tag{6.5}$$

such that $A' \to B' \to C'$, $A \to B \to C$ and $A'' \to B'' \to C''$ are s.e.s's in $\mathfrak{A}$; thus $DSES(\mathfrak{A})$ becomes an exact category. The exact inclusion functor $\operatorname{Aut}\mathfrak{A} \longrightarrow DSES(\mathfrak{A})$ given by $\alpha \mapsto l(\alpha)$ yields the map $K_0(\operatorname{Aut}\mathfrak{A}) \longrightarrow K_0(DSES(\mathfrak{A}))$. We can regard (6.5) as a particular case of (6.2), where the vertical d.s.e.s's are of the form (6.1). Thus relations (i) and (ii) in the definition of $\mathcal{D}(\mathfrak{A})$ imply that

$$\langle l_A \rangle - \langle l_B \rangle + \langle l_C \rangle = 0$$

whenever l_A, l_B and l_C denote the rows of any diagram of the form (6.5). It follows that there is a natural epimorphism

$$K_0(DSES(\mathfrak{A})) \longrightarrow \mathcal{D}(\mathfrak{A}) .$$

The diagram

$$\begin{array}{ccccc} 0 & \rightrightarrows & 0 & \rightrightarrows & 0 \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ 0 & \rightrightarrows & A & \overset{1}{\underset{\alpha}{\rightrightarrows}} & A \\ \downdownarrows & & 1 \downdownarrows 1 & & 1 \downdownarrows \beta \\ 0 & \rightrightarrows & A & \overset{1}{\underset{\beta\alpha}{\rightrightarrows}} & A \end{array} \tag{6.6}$$

shows that the composite map $K_0(\operatorname{Aut}\mathfrak{A}) \longrightarrow K_0(DSES(\mathfrak{A})) \longrightarrow \mathcal{D}(\mathfrak{A})$ factors through relation (1.1), i.e., we have a well-defined map $K_1^{det}(\mathfrak{A}) \longrightarrow \mathcal{D}(\mathfrak{A})$.

Let $\tilde{l}(\alpha) = (A \overset{\alpha}{\underset{1}{\rightrightarrows}} A \rightrightarrows 0)$ for $A \in \mathfrak{A}$ and $\alpha \in \operatorname{Aut} A$. It follows from the diagram

$$\begin{array}{ccccc}
0 & \rightrightarrows & 0 & \rightrightarrows & 0 \\
\downdownarrows & & \downdownarrows & & \downdownarrows \\
A & \overset{\alpha}{\underset{1}{\rightrightarrows}} & A & \rightrightarrows & 0 \\
1 \downdownarrows 1 & & 1 \downdownarrows \alpha & & \downdownarrows \\
A & \overset{1}{\underset{1}{\rightrightarrows}} & A & \rightrightarrows & 0
\end{array}$$

that $\langle \tilde{l}(\alpha) \rangle = \langle l(\alpha) \rangle$ in $\mathcal{D}(\mathfrak{A})$, hence $m(\tilde{l}(\alpha)) = m(l(\alpha))$. Sherman proves (p.234 of [Sh1]) that if $\mathfrak{A}$ is semisimple, then α maps to the class of the loop $\mu(\tilde{l}(\alpha))$ in $BS^{-1}S(\mathfrak{A})$ (instead of $G.\mathfrak{A}$) by the map $K_1^{det}(\mathfrak{A}) \longrightarrow K_1(\mathfrak{A})$ of [Ge]. It is a boring but trivial exercise to follow those arguments and replace $BS^{-1}S$ by $G.\mathfrak{A}$ and $Q\mathfrak{A}$ by $S.\mathfrak{A}$ in the general case. We leave this to the reader.

3. The composite map in (6.4) is an isomorphism if $\mathfrak{A}$ is semisimple (Theorem 3.3 of [Sh1] or Proposition 1 of [We]). Thus the map $K_1^{det}(\mathfrak{A}) \longrightarrow \mathcal{D}(\mathfrak{A})$ is injective in this case. Further, any d.s.e.s. is isomorphic to a d.s.e.s. of the form

$$l = (A \overset{\binom{1}{0}}{\underset{\binom{a}{c}}{\rightrightarrows}} A \oplus C \overset{(0,1)}{\underset{(x,y)}{\rightrightarrows}} C). \tag{6.7}$$

Let $(u, v) : A \oplus C \to A$ be a splitting for $\begin{pmatrix} a \\ c \end{pmatrix}$, and put $\alpha = \begin{pmatrix} u & v \\ x & y \end{pmatrix}$. Then $\alpha \in \operatorname{Aut}(A \oplus C)$, and the diagram

$$\begin{array}{ccccc}
0 & \rightrightarrows & 0 & \rightrightarrows & 0 \\
\downdownarrows & & \downdownarrows & & \downdownarrows \\
A & \overset{\binom{1}{0}}{\underset{\binom{a}{c}}{\rightrightarrows}} & A \oplus C & \overset{(0,1)}{\underset{(x,y)}{\rightrightarrows}} & C \\
1 \downdownarrows 1 & & 1 \downdownarrows \alpha & & 1 \downdownarrows 1 \\
A & \overset{\binom{1}{0}}{\underset{\binom{1}{0}}{\rightrightarrows}} & A \oplus C & \overset{(0,1)}{\underset{(0,1)}{\rightrightarrows}} & C
\end{array} \tag{6.8}$$

shows that $\langle l \rangle = \langle l(\alpha) \rangle$ in $\mathcal{D}(\mathfrak{A})$. Thus the map $K_1^{det}(\mathfrak{A}) \longrightarrow \mathcal{D}(\mathfrak{A})$ is surjective. This completes the proof of the theorem.■

Remark 6.9. One can check that in the semisimple case, the assignment $\alpha = \begin{pmatrix} u & v \\ x & y \end{pmatrix}$ to a d.s.e.s. l of the form (6.7) gives rise to a well-defined homomorphism $\mathcal{D}(\mathfrak{A}) \longrightarrow K_1^{det}(\mathfrak{A})$. This map is easily seen to be the inverse isomorphism for (6.3). We note that this argument does not appeal to the comparison between K_1^{det} and Quillen's K_1.

Remark 6.4. Let $SES(\mathfrak{A})$ denote the exact category of s.e.s's in $\mathfrak{A}$. The exact inclusion functor

$$SES(\mathfrak{A}) \longrightarrow DSES(\mathfrak{A}), \quad (A \xrightarrow{f} B \xrightarrow{g} C) \mapsto (A \underset{f}{\overset{f}{\rightrightarrows}} B \underset{g}{\overset{g}{\rightrightarrows}} C)$$

has two natural splittings that take $(A \underset{f_2}{\overset{f_1}{\rightrightarrows}} B \underset{g_2}{\overset{g_1}{\rightrightarrows}} C)$ to $(A \xrightarrow{f_1} B \xrightarrow{g_1} C)$ and $(A \xrightarrow{f_2} B \xrightarrow{g_2} C)$ respectively. Thus we can regard $K_0(SES(\mathfrak{A}))$ as a direct summand in $K_0(DSES(\mathfrak{A}))$, and the group $K_0(DSES(\mathfrak{A}))/K_0(SES(\mathfrak{A}))$ is obviously the same as the quotient of $K_0(DSES(\mathfrak{A}))$ by relation (i) in Definition 6.1. We have therefore the natural epimorphism

$$K_0(DSES(\mathfrak{A}))/K_0(SES(\mathfrak{A})) \longrightarrow \mathcal{D}(\mathfrak{A}). \tag{6.9}$$

Grayson proves that if R is a ring and $\mathfrak{A} = \mathcal{P}_R$ denotes the category of finitely generated projective modules over R, then the composite map

$$K_0(DSES(\mathfrak{A}))/K_0(SES(\mathfrak{A})) \longrightarrow \mathcal{D}(\mathfrak{A}) \longrightarrow K_1(\mathfrak{A})$$

is an isomorphism (unpublished). It then follows from Theorem 6.2 (iii) that the map (6.9) is also an isomorphism in this case. Grayson's proof involves more complicated matrix tricks of the type (6.8).
We do not know if (6.9) is an isomorphism for any $\mathfrak{A}$. We do not either know if the map $K_0(\operatorname{Aut}\mathfrak{A}) \longrightarrow K_0(DSES(\mathfrak{A}))/K_0(SES(\mathfrak{A}))$ factors in general through $K_1^{det}(\mathfrak{A})$ since the trick like (6.6) is *a priori* not possible for the latter group.

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Low-dimensional Representations of Aut(F_2)

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*Supported in part by the NSERC Grant A-5285.
†Supported in part by an NSERC grant.

1 Introduction

Let $F_2 = \langle x, y\rangle$ be the free group of rank 2 with generators x and y. We will denote the automorphism group $\mathrm{Aut}(F_2)$ by Φ_2. There is a well known open problem concerning the linearity of this group : Is it true that Φ_2 has a faithful linear representation? Magnus and Tretkoff [9] have conjectured that there is no such representation over any field. In the case of free groups of rank ≥ 3, the automorphism group is not linear [6].

The above conjecture is closely connected with the old problem of linearity of the braid groups (see [1, 4]). It was proved in [4] that if B_4, the braid group on four strings, has a faithful representation of degree m, then Φ_2 has a faithful representation of degree $2m$. For a very recent account of representations of braid groups see [2].

We consider a more general problem of describing all representations of Φ_2 of degree n for small n. Very little is known about this problem : we know only the paper [3] where it is proved that Φ_2 has no faithful 3-dimensional representations over any field of characteristic 0.

We shall now recall some facts about the structure of Φ_2. For $a \in F_2$ let f_a be the inner automorphism of F_2 defined by a, i.e., $(z)f_a = a^{-1}za$ for all $z \in F_2$. (In order to conform with the usage in [8], we write f_a on the right hand side of the element to which it is applied.) Since F_2 has trivial center, the homomorphism $a \mapsto f_a$ is injective, and we use it to identify F_2 with its image in Φ_2.

It is well known [8, p. 169] that Φ_2 is generated by the following three elements :

$$\begin{aligned} P &: x \mapsto y, \quad y \mapsto x; \\ U &: x \mapsto xy, \quad y \mapsto y; \\ \sigma &: x \mapsto x^{-1}, \quad y \mapsto y; \end{aligned}$$

and has a presentation consisting of the following relations :

$$P^2 = \sigma^2 = (\sigma P)^4 = (P\sigma PU)^2 = (UP\sigma)^3 = 1, \quad (U\sigma)^2 = (\sigma U)^2. \tag{1}$$

Let $\rho : \Phi_2 \to \mathrm{GL}(V)$ be a linear representation, where V is an n-dimensional vector space over K. We can construct new representations :

$$P \to \epsilon_1\rho(P), \quad U \to \epsilon_2\rho(U), \quad \sigma \to \epsilon_3\rho(\sigma), \tag{2}$$

where $\epsilon_i = \pm 1$ and $\epsilon_1\epsilon_2\epsilon_3 = 1$.

We say that a representation ρ' of Φ_2 is *weakly equivalent* to the representation ρ if ρ' is equivalent to one of the representations (2) or their dual representations.

Our main result can be stated as follows.

Theorem. *Consider indecomposable representations ρ of Φ_2 of degree $n \leq 4$, over an algebraically closed field K, such that $\rho(F_2) \neq 1$. There are no such representations if*

$n \leq 2$. *If $\rho(\Phi_2)$ is infinite then, up to weak equivalence, there exist for $n = 3$ only one such representation, and for $n = 4$ two if char $K \neq 2, 3$, one if char $K = 3$, and none if char $K = 2$. All the representations mentioned above are reducible, and are listed in the last section. If $\rho(\Phi_2)$ is finite, ρ factorizes through the natural homomorphism $\Phi_2 \to \Gamma_i$, where Γ_i are some finite groups of small orders defined in Lemma 2.*

Corollary. *Φ_2 has no faithful representation of degree $n \leq 4$ over any field.*

If $\rho(F_2) = 1$, then ρ factorizes through the natural homomorphism $\Phi_2 \to \Phi_2/F_2 \simeq \mathrm{GL}(2, \mathbf{Z})$. It is easy to show that there exist infinitely many nonequivalent idecomposable 4-dimensional representations of $\mathrm{GL}(2, \mathbf{Z})$.

From our theorem it follows that for $n \leq 4$ there are only finitely many nonequivalent n-dimensional representations of Φ_2 such that $\rho(F_2) \neq 1$, and in all these cases $\rho(F_2)$ is a solvable group. On the other hand, already for $n = 6$ there exists a one-parameter family of irreducible nonequivalent representations of Φ_2 such that $\rho(F_2)$ contains a free non-Abelian subgroup. Hence it is impossible to extend our theorem to dimensions $n \geq 6$. This also explains why the proof of our theorem involves a lot of computations.

We indicate briefly how to construct the family mentioned above. For that purpose we make use of the braid group B_4 and the well known 3-dimensional Bürau representation β_t depending on a parameter t. This can be modified to obtain a one-parameter family of 3-dimensional representations β_t^* of B_4/Z_4, where Z_4 is the center of B_4. We recall that there is an embedding $B_4/Z_4 \to \Phi_2$ (see [4]) such that the image of B_4/Z_4 in Φ_2 has index 2. The representations β_t^* induce 6-dimensional representations of Φ_2 having the properties stated above. The claim about the existence of free non-Abelian subgroups follows from [10].

For $n \geq 6$ it would be interesting to describe the character variety of n-dimensional representations of Φ_2. For the case of braid group B_4, the character variety of 3-dimensional representations was recently described by Formanek [5].

In the last section of our paper we describe also some new 4-dimensional representations of B_4. Two of them are at the same time indecomposable and reducible. It would be interesting to find some applications of these representations.

By using our identification of F_2 with a subgroup of Φ_2, we have $y = (\sigma U)^2$ and $x = PyP$. Furthermore we have :

$$U^{-1}xU = xy, \quad Uy = yU, \quad \sigma y = y\sigma, \quad \sigma x \sigma = x^{-1}. \tag{3}$$

The elements U and y generate a free Abelian group of rank 2. We introduce the element $\omega = P\sigma P$, which satisfies :

$$\omega^2 = 1, \quad \sigma\omega = \omega\sigma, \quad \omega U \omega = U^{-1}, \quad \omega y \omega = y^{-1}. \tag{4}$$

The subgroup $D_4 = \langle P, \sigma \rangle$ of Φ_2 is a dihedral group of order 8. We shall use some elementary facts about the representations of D_4 over fields of characteristic $\neq 2$.

V. P. Platonov is grateful to the Max Planck Institute of Mathematics (Bonn) for the support and the hospitality during the preparation of this paper.

2 Some general facts and lemmas

In this section we recall some general facts about Φ_2 and its representations. We prove two lemmas concerning some particular factor groups of Φ_2. The proof of the theorem proper will begin in the next section.

In our proof we shall use the following simple fact : Any two primitive elements of F_2 are conjugate in Φ_2. Recall that $a \in F_2$ is called *primitive* if there exists $b \in F_2$ such that $\{a, b\}$ is a free basis of F_2. In order to prove the above fact, let a and b be primitive elements of F_2. Then it is clear that there exists $\phi \in \Phi_2$ such that $(a)\phi = b$. This implies that $\phi^{-1} \circ f_a \circ \phi = f_b$, and, by using our identification, we obtain $\phi^{-1} \cdot a \cdot \phi = b$. Thus our claim is proved.

In particular, the elements x and xy are conjugate in Φ_2. So $xy = z^{-1}xz$ for some $z \in \Phi_2$. This shows that y is a commutator in Φ_2, and consequently F_2 is contained in the commutator subgroup of Φ_2.

Given a linear representation $\rho : \Phi_2 \to \mathrm{GL}(V)$, for the sake of simplicity, we shall refer to the eigenvalues, trace, determinant, ... of $\rho(y)$ as the eigenvalues, trace, determinant, ... of y, and similarly for other elements of Φ_2. Since F_2 is contained in the commutator subgroup of Φ_2, we have

$$\det(y) = 1. \tag{5}$$

Now assume that $\rho(F_2) \neq 1$, or equivalently, that $\rho(y) \neq 1$. Under this hypothesis we claim that $\rho(y)$ is not scalar operator. Indeed, if $\rho(y)$ were a scalar, then we would have $\rho(x) = \rho(y)$ and $\rho(xy^{-1}) = 1$. This is impossible since y and xy^{-1} are conjugate in Φ_2 and $\rho(y) \neq 1$.

Lemma 1. *Denote by Γ the quotient group of Φ_2 obtained by adding the new defining relation $[U, (P\sigma)^2] = 1$ to the presentation* (1). *Then the image of F_2 in Γ is trivial.*

Proof. Since $(P\sigma)^2 = \sigma\omega = \omega\sigma$ and $\omega U \omega = U^{-1}$, we have $\sigma\omega U\omega\sigma U^{-1} = y^{-1}$. Hence, in Γ we have $y = 1$, and consequently also $x = 1$. ■

In the next lemma and its proof we denote by C_k a cyclic group of order k, by Q the quaternion group of order 8, by S_k the symmetric group of degree k, and by $E(2^k)$ an elementary Abelian group of order 2^k.

Lemma 2. *By adding new relations to the presentation* (1), *we obtain some finite quotient groups as follows :*

(i) *relation $U^2 = 1$, quotient group $\Gamma_1 \simeq C_2 \times S_4$;*

(ii) *relation $[U, \sigma] = 1$, quotient group $\Gamma_2 \simeq C_2 \times S_4$;*

(iii) *relations $U^4 = (\sigma U)^4 = 1$, quotient group $\Gamma_3 \simeq E(64) \rtimes S_3$;*

(iv) *relations $U^4 = [P, (\sigma U)^4] = 1$, quotient group $\Gamma_4 \simeq (Q\#Q) \rtimes S_4$;*

where # denotes the central product. In particular Γ_1 and Γ_2 have order 48, *Γ_3 order* 384, *and Γ_4 order* 768.

Proof. It is straightforward to check that there exist surjective homomorphisms $f: \Gamma_1 \to \{\pm 1\} \times S_4$ and $g: \Gamma_2 \to \{\pm 1\} \times S_4$ given by :

$$f(U) = (-1, (13)), \quad f(P) = (1, (23)), \quad f(\sigma) = (-1, (12)(34));$$

and

$$g(U) = (-1, (1234)), \quad g(P) = (1, (23)), \quad g(\sigma) = (-1, (13)(24)).$$

To prove (i) and (ii) it suffices to show that $|\Gamma_1| \leq 48$ and $|\Gamma_2| \leq 48$, respectively. Let Γ be the common factor group of Γ_1 and Γ_2 obtained from the presentation of Φ_2 by adding the relations $U^2 = 1$ and $\sigma U = U\sigma$. These relations are equivalent to $U^2 = 1$, $(\sigma U)^2 = 1$, and so we have $\Gamma_1/\langle x, y\rangle \simeq \Gamma \simeq \Gamma_2/\langle x, y\rangle$.

In Γ we have $1 = (UP\sigma)^3 = UPU\sigma P\sigma UP\sigma = UPUPU\omega\sigma P\sigma = (UP)^3\omega$. Thus $(UP)^6 = 1$, and since $\omega = P\sigma P$, we have $\sigma \in \langle U, P\rangle$. It follows that $|\Gamma| \leq 12$.

In Γ_1 we have $x = U^{-2}xU^2 = U^{-1}xyU = U^{-1}xUy = xy^2$, and so $y^2 = 1$. It follows that $|\langle x, y\rangle| \leq 4$, and so $|\Gamma_1| \leq 48$. Thus (i) is proved.

In Γ_2 we have $y = (\sigma U)^2 = U^2$ and $y^{-1}xy = U^{-2}xU^2 = U^{-1}xUy = xy^2$. Hence $yxy = x$, and by conjugating by P we obtain $xyx = y$. So $x^2 = y^{-2}$. As $xyx^{-1} = y^{-1}$, by conjugating the equality $x^2 = y^{-2}$ by x, we obtain $x^2 = y^2$, and so $x^4 = 1$. If $x^2 \neq 1$ in Γ_2, then $\langle x, y\rangle = Q$ is the quaternion group. If $(P\sigma)^2 \neq 1$, as Γ has no elements of order 4, we have $(P\sigma)^2 = x^2$. It follows that $(P\sigma)^2$ is central in Γ_2, and Lemma 1 gives a contradiction. We conclude that $x^2 = 1$ in Γ_2, and so $|\Gamma_2| \leq 48$. Hence (ii) holds.

We now prove (iv). Let $G = (Q\#Q') \rtimes S_4$ where Q' is another copy of Q. We have $Q = \{\pm 1, \pm i, \pm j, \pm k\}$, where $1, i, j, k$ are the quaternionic units, and analogously $Q' = \{\pm 1, \pm i', \pm j', \pm k'\}$. We now describe the action of S_4 on $Q\#Q'$. First of all, both Q and Q' are normal in G. The normal 4-group, say V, of S_4 acts trivially on Q, while the subgroup S_3 acts as follows :

$$(12): \quad i \to j, \quad j \to i\,;$$

$$(123): \quad i \to -j, \quad j \to k.$$

The alternating subgroup A_4 acts trivially on Q' and the odd permutations interchange i' and j'. It is now straightforward to verify that there is a surjective homomorphism $h: \Gamma_4 \to G$ such that :

$$h(U) = (kj', (1432)), \quad h(P) = (1, (12)), \quad h(\sigma) = (jj', (13)(24)).$$

In order to prove (iv), it suffices to show that $|\Gamma_4| \leq 768$. In Γ_4 we have $x = U^{-4}xU^4 = xy^4$, and so $x^4 = y^4 = 1$. As $y = (\sigma U)^2$, P and y^2 commute in Γ_4, and so $x^2 = y^2$ and $|\langle x, y\rangle| \leq 8$. Let Δ be the factor group $\Gamma_4/\langle x, y\rangle$. Clearly $\Delta \simeq \mathrm{GL}_2(\mathbf{Z})/N$, where N is the normal closure in $\mathrm{GL}_2(\mathbf{Z})$ of $\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$. The image of N in the modular group $\mathrm{SL}_2(\mathbf{Z})/\{\pm 1\}$ is the unique normal subgroup of level 4, and so it has index 24. For these facts we refer the reader to [11, Chapter VIII]. Hence the index of N in $\mathrm{SL}_2(\mathbf{Z})$ is at most 48, and in $\mathrm{GL}_2(\mathbf{Z})$ at most 96. It follows that $|\Gamma_4| \leq 96 \cdot 8 = 768$ and (iv) is proved.

We have shown above that h is an isomorphism. Since $\Gamma_3 = \Gamma_4/P$ where P is the normal closure of $y^2 = (\sigma U)^4$ in Γ_4, and $h(y)^2 = (-1, 1)$, (iii) follows from (iv). ∎

This lemma was proved first by using GAP, the symbolic computation package [7]. Subsequently we have constructed the homomorphisms f, g, h and succeded to eliminate the reliance on GAP in our proof.

3 Representations of degree 2 and 3

For $n = 1$ the assertion of the theorem is obvious. In this section we prove the assertion of the theorem when $n = 2$ or 3 and char $K \neq 2$.

Let $n = 2$. Since $\rho(F_2) \neq 1$, Lemma 1 implies that $\rho(P\sigma)^2 \neq 1$, and so the restriction of ρ to D_4 is faithful. Hence we may assume that

$$\rho(\sigma) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho(P) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since $\sigma y = y\sigma$ and $\det(y) = 1$, we have

$$\rho(y) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

As $x = PyP$, we have $\rho(xy) = 1$. Since y and xy are conjugate, we obtain that $\lambda = 1$, a contradiction.

Now let $n = 3$. By Lemma 1, V is a sum of two irreducible D_4-modules : a 2-dimensional and a 1-dimensional. Up to weak equivalence, we may assume that

$$\rho(\sigma) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho(P) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6}$$

As $\sigma y = y\sigma$, we have

$$\rho(y) = \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & d & e \end{pmatrix}.$$

From $(\omega y)^2 = 1$, we obtain that $c(b-e) = d(b-e) = 0$ and $b^2 = e^2 = cd + 1$.

If $b \neq e$, then $c = d = 0$, $b = -e = \pm 1$. As $\det(y) = 1$, we have $a = -1$. From $\rho(y) = \mathrm{diag}(-1, b, -b)$ and $\rho(x) = \rho(PyP) = \mathrm{diag}(b, -1, -b)$, we obtain that $\rho(xy) = \mathrm{diag}(-b, -b, 1)$. As $\rho(xy) \neq 1$, we must have $b = 1$. By using the fact that y and U commute, we have

$$\rho(U) = \begin{pmatrix} \alpha & 0 & \beta \\ 0 & \gamma & 0 \\ \delta & 0 & \epsilon \end{pmatrix}.$$

The eqation $Uxy = xU$ implies that $\alpha = \epsilon = 0$. Since $y = (\sigma U)^2$, we must have $\beta\delta = \gamma^2 = 1$. Hence $\rho(U^2) = 1$, and so Lemma 2 applies.

If $b = e$, then $\det(y) = 1$ implies that $a = 1$. Hence

$$\rho(y) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & b & c \\ 0 & d & b \end{pmatrix}, \quad \rho(x) = \begin{pmatrix} b & 0 & c \\ 0 & 1 & 0 \\ d & 0 & b \end{pmatrix}.$$

Since xy and y are conjugate, we have $\operatorname{tr}(xy) = \operatorname{tr}(y) = 1 + 2b$. This gives $b^2 = 1$, and so $cd = 0$. By replacing ρ by its dual (if necessary) we may assume that $d = 0$.

If $b = -1$, then $Uy = yU$ implies that $\rho(\sigma)$ and $\rho(U)$ commute, and Lemma 2 applies.

If $b = 1$, then $c \neq 0$ and we may assume that $c = 1$. Since $Uy = yU$, we have

$$\rho(y) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho(U) = \begin{pmatrix} \alpha & 0 & \beta \\ \gamma & \delta & \epsilon \\ 0 & 0 & \delta \end{pmatrix}.$$

The equation $(\omega U)^2 = 1$ implies that $\beta = 0$, $\delta = \alpha$, and $\alpha^2 = 1$. The equation $Uxy = xU$ implies that $\alpha = 1$ and $\gamma = -1$. Since $y = (\sigma U)^2$, we must have $\epsilon = 1/2$. Thus we obtain

$$\rho(U) = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1/2 \\ 0 & 0 & 1 \end{pmatrix}. \tag{7}$$

The equations (6) and (7) define an indecomposable representation of Φ_2. Obviously this representation is reducible.

4 Representations of degree 4

In this section we begin the proof of the theorem when $n = 4$ and char $K \neq 2$. This part of the proof will be completed in the next three sections.

We claim that the eigenvalues of y can be written as

$$\lambda, \lambda^{-1}, \mu, \mu^{-1} \tag{8}$$

for some $\lambda, \mu \in K^*$. If all eigenvalues of y are ± 1, this follows from (5). If y has an eigenvalue $\lambda \neq \pm 1$, then $\omega y \omega = y^{-1}$ implies that λ^{-1} is also an eigenvalue of y. Since $\lambda^{-1} \neq \lambda$, (5) implies that the remaining two eigenvalues of y can be written as μ, μ^{-1}. This proves our claim.

By replacing ρ with a weakly equivalent representation, if necessary, we may assume that

$$\operatorname{tr}(\sigma) = 0, 2. \tag{9}$$

We shall denote by V^+ resp. V^- the eigenspace of σ for eigenvalue $+1$ resp. -1. Since ω and y commute with σ, these subspaces are invariant under ω and y. We shall denote by $\rho(\omega)^+$ and $\rho(y)^+$ the restrictions of $\rho(\omega)$ and $\rho(y)$ to V^+, respectively.

We conclude this section with two lemmas.

Lemma 3. *Let ρ be a 4-dimensional representation of Φ_2 and assume that char $K \neq 2$. If* $\mathrm{tr}\,(\sigma) = 2$, *then all eigenvalues of y are ± 1.*

Proof. We shall assume that y has an eigenvalue $\lambda \neq \pm 1$ and obtain a contradiction. As $\mathrm{tr}\,(\sigma) = 2$, $\dim V^+ = 3$ and $\dim V^- = 1$. If $e_4 \in V^-$, $e_4 \neq 0$, then e_4 is an eigenvector of y. Say $y(e_4) = \mu e_4$. Since $\omega y \omega = y^{-1}$ and V^- is ω-invariant, we conclude that $\mu = \pm 1$.

It follows that $\rho(y)^+$ has three distinct eigenvalues λ, λ^{-1}, and μ. Let e_1 and e_3 be eigenvectors of $\rho(y)^+$ belonging to λ and μ, respectively. Set $e_2 = \omega(e_1)$. Then

$$y(e_2) = y\omega(e_1) = \omega y^{-1}(e_1) = \lambda^{-1}\omega(e_1) = \lambda^{-1}e_2,$$

and so $\{e_1, e_2, e_3, e_4\}$ is a basis of V.

Since $\rho(\omega)^+\rho(y)^+\rho(\omega)^+ = \rho(y^{-1})^+$, the subspace Ke_3 is ω-invariant. From $P\sigma P = \omega$ we deduce that $\mathrm{tr}\,(\omega) = 2$, and so

$$\omega(e_1) = e_2,\ \omega(e_2) = e_1,\ \omega(e_3) = e_3,\ \omega(e_4) = e_4\ .$$

By identifying linear operators with their matrices with respect to this basis, we have

$$\rho(\sigma) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},\ \rho(\omega) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ \rho(y) = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda^{-1} & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix}.$$

As U and y commute,

$$\rho(U) = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & u & v \\ 0 & 0 & w & z \end{pmatrix}.$$

The equality $(\omega U)^2 = 1$ implies that $\alpha\beta = 1$ and

$$u^2 = z^2 = 1 - vw, \quad v(u+z) = w(u+z) = 0. \tag{10}$$

The equality $y = (\sigma U)^2$ implies that $\alpha^2 = \lambda$ and

$$u^2 = z^2 = \mu + vw, \quad v(u-z) = w(u-z) = 0. \tag{11}$$

If $\mu = 1$, the above equations imply $v = w = 0$. Hence $\rho(\sigma)$ and $\rho(U)$ commute, and Lemma 2 implies that $\rho(y)^2 = 1$. This contradicts the assumption that $\lambda \neq \pm 1$.

If $\mu = -1$, then (10) and (11) imply that $u = z = 0$ and $vw = 1$. By conjugating by the diagonal matrix $\mathrm{diag}(1,1,1,w)$, we may assume that $v = w = 1$. Thus

$$\rho(U) = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha^{-1} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Since $P\sigma P = \omega$ and $P^2 = 1$, we must have

$$\rho(P) = \begin{pmatrix} a & a & b & c \\ a & a & b & -c \\ d & d & e & 0 \\ f & -f & 0 & 0 \end{pmatrix},$$

where

$$2cf = 1, \quad b(2a+e) = d(2a+e) = 0, \quad e^2 = 4a^2 = 1 - 2bd.$$

By conjugating by $\mathrm{diag}(1,1,f,f)$, we may assume that $c = 1/2$ and $f = 1$.

If $b = d = 0$, then the $(1,4)$ entries in $\rho(UP\sigma))^3 = 1$ give $ae(\alpha^2 - 1) = 0$. As $\alpha^2 \neq 1$, we have $ae = 0$. Since $e^2 = 4a^2$, we have $a = e = 0$. As $\rho(P)$ is nonsingular, we have a contradiction.

If $b \neq 0$ or $d \neq 0$, then $e = -2a$ and by comparing the $(4,3)$ entries in $\rho(UP\sigma))^3 = 1$, we obtain that $a(\alpha^2 - 1) = 0$, and so $a = 0$. By comparing $(4,4)$ entries, we obtain a contradiction. ■

Lemma 4. *Let ρ be a 4-dimensional representation of Φ_2 and assume that char $K \neq 2$. Then the Jordan canonical form of $\rho(y)$ contains no Jordan blocks of size 3.*

Proof. Assume that $\rho(y)$ has a Jordan block of size 3. Then $\mathrm{tr}\,(\sigma) \neq 0$, and so by (9) we have $\mathrm{tr}\,(\sigma) = 2$. We can choose a basis of V such that

$$\rho(\sigma) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix}.$$

As $\omega y \omega = y^{-1}$, we have $\lambda^2 = 1$. Since $\det(y) = 1$, we have $\lambda = \mu$.

Since $\omega\sigma = \sigma\omega$, $\rho(\omega) = A \oplus B$ with A of size 3 and $B = (\pm 1)$. Since $\omega y \omega = y^{-1}$, we have $A \neq 1$ and $\mathrm{tr}\,(\omega) = \mathrm{tr}\,(\sigma) = 2$ implies that $B = (1)$. By using $\omega y \omega = y^{-1}$ again, we conclude that $\rho(\omega)$ is upper triangular and that it has the form

$$\rho(\omega) = \begin{pmatrix} 1 & u & u(u-\lambda)/2 & 0 \\ 0 & -1 & \lambda - u & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

By conjugating with a suitable matrix which commutes with $\rho(\sigma)$ and $\rho(y)$, we may assume that $u = 0$.

Since U and y commute, we have

$$\rho(U) = \begin{pmatrix} a & b & c & d \\ 0 & a & b & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & e & f \end{pmatrix}, \quad \rho(\omega U) = \begin{pmatrix} a & b & c & d \\ 0 & -a & \lambda a - b & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & e & f \end{pmatrix}.$$

From $(\omega U)^2 = 1$ we obtain that $d(a+f) = e(a+f) = 0$, and from $y = (\sigma U)^2$ that $d(a-f) = e(a-f) = 0$. Since $a+f$ or $a-f$ is not zero, it follows that $d = e = 0$. Hence $\rho(U)$ and $\rho(\sigma)$ commute and, by Lemma 2, $\rho(\Phi_2)$ is finite. As $\rho(y)$ has infinite order, we have a contradiction. ■

We now divide the proof into three cases, which will be treated separately in the next three sections.

5 Case 1 : $\lambda \neq \mu, \mu^{-1}$

Up to weak equivalence, we may assume that $\operatorname{tr}(\sigma) = 0, 2$.

Subcase 1 : $\operatorname{tr}(\sigma) = 0$. Both V^+ and V^- have dimension 2. If $\det \rho(y)^+ = 1$, then $\rho(\sigma)$ is a central element of the centralizer of $\rho(y)$ in $\mathrm{GL}(V)$, and in particular it commutes with $\rho(U)$. By Lemma 2, ρ factors through the homomorphism $\Phi_2 \to \Gamma_2$.

Now let $\det \rho(y)^+ \neq 1$. Then the eigenvalues of $\rho(y)^+$ are, say, λ and μ, and those of $\rho(y)^-$ are λ^{-1} and μ^{-1}. Since ω leaves invariant V^+ and V^- and inverts y, it follows that $\lambda = -\mu = \pm 1$ and that $\rho(y)$ and $\rho(\omega)$ commute. By choosing a suitable basis, we may assume that

$$\rho(\sigma) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \rho(P) = \begin{pmatrix} r & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & s \end{pmatrix}$$

where $r, s = \pm 1$. Then $\rho(\omega)$ and $\rho(y)$ have the form

$$\rho(\omega) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & -a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & -b \end{pmatrix},$$

where $a, b = \pm 1$. As $\rho(x) \neq \rho(y)$, we have $b = a$. Hence $\rho(\omega y) = \pm 1$. It follows that $\rho(U) = \rho(\omega y U (\omega y)^{-1}) = \rho(U)^{-1}$. Hence ρ factors through the homomorphism $\Phi_2 \to \Gamma_1$ of Lemma 2.

Subcase 2 : $\operatorname{tr}(\sigma) = 2$. Now V^+ has dimension 3 and V^- dimension 1. By Lemma 3, all eigenvalues of y are ± 1, and so $\lambda = -\mu = \pm 1$.

Assume first that $\rho(y)$ is diagonalizable. Then $\rho(y^2) = 1$, and $\rho(\sigma)$, $\rho(\omega)$, and $\rho(y)$ commute. We can diagonalize them simultaneously. By Lemma 1, $\rho(\sigma) \neq \rho(\omega)$. Hence we may assume that

$$\rho(\sigma) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \rho(\omega) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & \epsilon_2 & 0 & 0 \\ 0 & 0 & \epsilon_3 & 0 \\ 0 & 0 & 0 & \epsilon_4 \end{pmatrix},$$

where $\epsilon_i = \pm 1$, $\det(y) = 1$, and $\operatorname{tr}(y) = 0$.

The equations $P^2 = 1$ and $P\sigma P = \omega$ imply that

$$\rho(P) = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & 0 & 1/e \\ 0 & 0 & e & 0 \end{pmatrix}.$$

We may assume that $e = 1$. Since $x = PyP$ and $\rho(xy) \neq 1$, we must have $\epsilon_2 = -\epsilon_1$ and $\epsilon_4 = -\epsilon_3$.

If $\epsilon_3 = -\epsilon_1$, then

$$\rho(U) = \begin{pmatrix} u & 0 & 0 & v \\ 0 & f & g & 0 \\ 0 & h & i & 0 \\ w & 0 & 0 & z \end{pmatrix}.$$

The equation $Uxy = xU$ implies that $i = 0$ (and so $gh \neq 0$), $v = w = 0$, and $ac = bc = bd = 0$. Consequently $b = c = 0$. This is impossible since ρ is indecomposable.

If $\epsilon_3 = \epsilon_1$, then

$$\rho(U) = \begin{pmatrix} u & 0 & v & 0 \\ 0 & f & 0 & g \\ w & 0 & z & 0 \\ 0 & h & 0 & i \end{pmatrix}.$$

The equation $Uxy = xU$ now implies that $z = 0$ (and so $vw \neq 0$) and $ad = bc = 0$. This is impossible since $ad - bc = \pm 1$.

Hence $\rho(y)$ is not diagonalizable. By choosing a suitable basis $\{e_1, e_2, e_3, e_4\}$ of V and by replacing λ with $-\lambda$, if necessary, we may assume that

$$\rho(\sigma) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{pmatrix}.$$

Since $\omega y \omega = y^{-1}$, the subspaces Ke_1, $Ke_1 + Ke_2$, and Ke_3 are ω-invariant. As $\mathrm{tr}(\omega) = \mathrm{tr}(\sigma) = 2$, $\rho(\omega)$ must have the form :

$$\begin{pmatrix} -1 & s & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & s & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By replacing ρ with its dual representation, we may assume that $\rho(\omega)$ is given by the first of these two matrices. By replacing e_2 with $e_2 + (s/2)e_1$, we may assume that $s = 0$.

As U and y commute, we have

$$\rho(U) = \begin{pmatrix} \alpha & \beta & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix}.$$

From $(\omega U)^2 = 1$, we obtain the equations $\alpha^2 = 1$, $a^2 = d^2 = 1 - bc$, and from $y = (\sigma U)^2$ the equations $\lambda = 1$, $\beta = \alpha/2$, $a^2 = d^2 = bc - 1$. It follows that $a = d = 0$ and $bc = 1$. By conjugating by $\mathrm{diag}(1, 1, 1, c)$, we may assume that $b = c = 1$. Hence

$$\rho(U) = \begin{pmatrix} \alpha & \alpha/2 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \alpha^2 = 1.$$

Since $P\sigma P = \omega$ and $P^2 = 1$, P must map the eigenspaces of σ to the corresponding eigenspaces of ω. It follows that

$$\rho(P) = \begin{pmatrix} 0 & 0 & 0 & e \\ 0 & f & g & 0 \\ 0 & h & i & 0 \\ 1/e & 0 & 0 & 0 \end{pmatrix},$$

where

$$\begin{pmatrix} f & g \\ h & i \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The equation $(UP\sigma)^3 = 1$ implies that $f = \alpha$, $i = -\alpha$, $g = 0$, and $h = \alpha/2e$. By conjugating by $\mathrm{diag}(1, 1, e, e)$, we may assume that $e = 1$. We compute $\rho(x)$ and find that

$$\rho(x) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & \alpha & 0 & 1 \end{pmatrix}.$$

We obtain indeed an indecomposable representation of Φ_2. The choices $\alpha = 1$ and $\alpha = -1$ give weakly equivalent representations.

6 Case 2 : $\lambda = \mu \neq \pm 1$

By Lemma 3, $\operatorname{tr}(\sigma) = 0$, and so both V^+ and V^- have dimension 2. Choose $e_1 \in V^+$, $e_1 \neq 0$, such that $y(e_1) = \lambda e_1$. Then the vector $e_2 = \omega(e_1)$ is in V^+ and $y(e_2) = \lambda^{-1} e_2$. We can choose similarly nonzero vectors e_3, e_4 in V^- such that $y(e_3) = \lambda e_3$, $y(e_4) = \lambda^{-1} e_4$, and $\omega(e_3) = e_4$. With respect to the basis $\{e_1, e_2, e_3, e_4\}$ of V, we have

$$\rho(\sigma) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \ \rho(\omega) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ \rho(y) = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda^{-1} & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda^{-1} \end{pmatrix}.$$

Since $P\sigma P = \omega$ and $P^2 = 1$, P must map the eigenspaces of σ to the corresponding eigenspaces of ω. It follows that $\rho(P)$ must have the form :

$$\rho(P) = \begin{pmatrix} a & c & \alpha & \gamma \\ a & c & -\alpha & -\gamma \\ b & d & \beta & \delta \\ b & d & -\beta & -\delta \end{pmatrix}.$$

From $P^2 = 1$ it follows that $a = c = \pm 1/2$, $\beta = -\delta = \pm 1/2$, $\alpha = \gamma$, $b = -d$, and $4\alpha b = 1$. By replacing ρ with a weakly equivalent representation, we may assume that $a = 1/2$. By conjugating with the diagonal matrix $\operatorname{diag}(1, 1, 2b, 2b)$, we may assume that $b = \alpha = 1/2$. Hence

$$\rho(P) = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & \epsilon & -\epsilon \\ 1 & -1 & -\epsilon & \epsilon \end{pmatrix}, \quad \epsilon = \pm 1.$$

Since U and y commute, we have

$$\rho(U) = \begin{pmatrix} u & 0 & v & 0 \\ 0 & u' & 0 & v' \\ w & 0 & z & 0 \\ 0 & w' & 0 & z' \end{pmatrix}.$$

From $y = (\sigma U)^2$ we obtain the equations:

$$v(u - z) = w(u - z) = 0, \quad u^2 = z^2 = vw + \lambda,$$

and from $(\omega U)^2 = 1$ the equality

$$\begin{pmatrix} u' & v' \\ w' & z' \end{pmatrix} = \begin{pmatrix} u & v \\ w & z \end{pmatrix}^{-1}.$$

Assume first that $u \neq z$. Then $v = w = 0$, and consequently $v' = w' = 0$. Furthermore, we have $u' = 1/u$, $z = -u$, and $z' = -1/u$. By using $x = PyP$ and the equation $Uxy = xU$, we obtain $u^2 = 1$. Hence $\lambda = 1$, which is a contradiction.

Hence, we must have $u = z$, and so $u' = z'$. It follows that

$$\rho(U) = \begin{pmatrix} u & 0 & v & 0 \\ 0 & u/\lambda & 0 & -v/\lambda \\ w & 0 & u & 0 \\ 0 & -w/\lambda & 0 & u/\lambda \end{pmatrix}, \quad \lambda = u^2 - vw.$$

If $\epsilon = 1$, by equating the (3,1)-entries of the matrices $\rho(Uxy)$ and $\rho(xU)$, we obtain the equation $\lambda^2(u+w) = u-w$. Similarly, the (4,2)-entries give the equation $\lambda^2(u-w) = u+w$. Hence $\lambda^4 = 1$. As $\lambda \neq \pm 1$, we must have $\lambda^2 = -1$. It follows that $u = 0$ and $w = -\lambda/v$. By equating the (1,1)-entries of the above mentioned matrices, we obtain that $v = 0$, which is impossible.

So we have $\epsilon = -1$. The equation $\rho(Uxy) = \rho(xU)$ now implies that $\lambda^2 = -1$ and $w = -v$. The relation $(UP\sigma)^3 = 1$ implies that

$$4u^2(u - v) = \lambda(3u - v) + \lambda - 1,$$

$$4u^2(u + v) = \lambda(3u + v).$$

By taking into account that $u^2+v^2 = \lambda$, we obtain only one solution : $u = v = -(1+\lambda)/2$. In this case we indeed obtain an indecomposable representation of Φ_2. Since $\rho(U)^4 = 1$ and $\rho(y^2) = -1$, ρ factorizes through the homomorphism $\Phi_2 \to \Gamma_4$ of Lemma 2.

7 Case 3 : $\lambda = \mu = \pm 1$

Recall that D_4 has (up to equivalence) only one 2-dimensional irreducible module and four 1-dimensional ones. Assume that V, as a D_4-module, is a direct sum of two irreducible 2-dimensional modules. On an irreducible 2-dimensional D_4-module the element $(P\sigma)^2$ acts as minus the identity operator and so $\rho(P\sigma)^2$ lies in the center of $\mathrm{GL}(V)$. By Lemma 1, $\rho(F_2) = 1$ and we have a contradiction. The same argument applies when V is a sum of four 1-dimensional D_4-modules. Thus we may assume that V is a direct sum of one 2-dimensional irreducible D_4-module and two 1-dimensional modules.

Subcase 1 : $\mathrm{tr}(\sigma) = 0$. Up to weak equivalence, we may assume that (with respect to a suitable basis of V)

$$\rho(\sigma) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \rho(P) = \begin{pmatrix} r & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $r = \pm 1$. As $\omega = P\sigma P$ and $y\sigma = \sigma y$, we have

$$\rho(\omega) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} \alpha' & \beta' & 0 & 0 \\ \gamma' & \delta' & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & \gamma & \delta \end{pmatrix}.$$

Since all eigenvalues of y are equal $\lambda = \pm 1$, we have $\alpha + \delta = 2\lambda$ and $\alpha\delta - \beta\gamma = 1$. Since $\omega y \omega = y^{-1}$, it follows that $\alpha = \delta = \lambda$ and $\beta\gamma = 0$. Similarily $\alpha' = \delta' = \lambda$ and $\beta'\gamma' = 0$.

Up to weak equivalence, we have the following four possibilities :

(i) $\beta' \neq 0$, $\gamma' = \beta = \gamma = 0$;

(ii) $\beta' \neq 0$, $\gamma' = \beta = 0$, $\gamma \neq 0$;

(iii) $\beta' \neq 0$, $\beta \neq 0$, $\gamma' = \gamma = 0$;

(iv) $\beta' = \gamma' = \gamma = 0$, $\beta \neq 0$.

In fact, by using some elementary considerations, one can show that (i) and (iv) are weakly equivalent. Furthermore, by conjugating by a suitable diagonal matrix which commutes with $\rho(P)$, we may assume that the nonzero parameters among β', β, and γ are all equal to 1. We now consider each of the first three possibilities separately.

(i) We have

$$\rho(y) = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad \rho(U) = \begin{pmatrix} a & b & c & d \\ 0 & a & 0 & 0 \\ 0 & e & g & h \\ 0 & f & i & j \end{pmatrix}.$$

The relation $Uxy = xU$ implies that $\lambda = 1$, $h = 0$, $g = a$, and $e = ra$. The relation $y = (\sigma U)^2$ implies that $a^2 = j^2 = 1$, $di = 0$, $(a+j)i = 0$, and $(a-j)f = air$. The relation $(P\sigma PU)^2 = 1$ implies that $c = 0$, $(a-j)i = 0$, $2ab = 1$, and $(a+j)f = air$. It follows that $i = f = 0$. Finally the relation $(UP\sigma)^3 = 1$ implies that $j = -1$, $a = r$, and $d = 0$. Since $d = h = f = i = 0$, ρ is decomposable, contrary to the hypothesis.

(ii) We have

$$\rho(y) = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 1 & \lambda \end{pmatrix}, \quad \rho(U) = \begin{pmatrix} a & b & e & f \\ 0 & a & f & 0 \\ g & h & c & 0 \\ h & 0 & d & c \end{pmatrix}.$$

From $\rho(Uxy) = \rho(xU)$, by equating $(4,4)$ and $(2,3)$ entries, we find that $c(1-\lambda) = 0$ and $f(1-\lambda) = 0$. As c and f cannot both be 0, we infer that $\lambda = 1$. From $(3,2)$ entries we obtain $g = 0$. The entries $(1,2)$, $(1,3)$, $(4,2)$, and $(4,3)$ provide the equations $a + f = rh$, $c - a = rf$, $a = c + h$, and $f = c + rh$, respectively. These equations imply that $c = -a$, $h = 2a$, $f = -2ar$, and $a(4r-1) = 0$. As $r = \pm 1$, we obtain $a = 0$, which is impossible since $\rho(U)$ is invertible.

(iii) We have

$$\rho(y) = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad \rho(U) = \begin{pmatrix} a & b & c & d \\ 0 & a & 0 & c \\ e & f & g & h \\ 0 & e & 0 & g \end{pmatrix}.$$

From $Uxy = xU$ we obtain $a(1-\lambda) = e$ and $e(1-\lambda) = 0$. As a and e are not both zero, we must have $\lambda = 1$. Taking this into account, the same relation implies that $e = 0$, $g = a$, $f - a$, $r = -1$, and $h = a - b - c$. The relation $y = (\sigma U)^2$ implies that $a^2 = 1$ and $a = 2b + c$. From $(P\sigma PU)^2 = 1$ we obtain that $c = 0$, and so $h = a - b$. From $(UP\sigma)^3 = 1$ we find that $a = -1$, $b = -1/2$, and $3d = 1/4$. In particular char $K \neq 3$. Thus $\rho(U)$ is uniquely determined and all the defining relations are satisfied. One can easily check that this representation of Φ_2 is indeed indecomposable.

Subcase 2 : $\mathrm{tr}\,(\sigma) = 2$. By choosing a suitable basis of V, we have

$$\rho(\sigma) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \rho(P) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & r \end{pmatrix},$$

$$\rho(\omega) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & a & b & c \\ 0 & d & e & f \\ 0 & g & h & i \end{pmatrix},$$

where $\alpha, \beta, \lambda = \pm 1$.

By Lemma 4, $\rho(y)$ has no Jordan blocks of size 3, and so $(\rho(y) - \lambda)^2 = 0$. From this equality and $\rho(\omega y)^2 = 1$ we obtain that $\rho(\omega y \omega) = 2\lambda - \rho(y)$. Hence we have $a = e = i = \lambda$ and $f = h = 0$. Now the equation $(\rho(y) - \lambda)^2 = 0$ implies that $bd = cd = bg = cg = 0$. Hence $\rho(y)$ has one of the forms :

$$\begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & b & c \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & d & \lambda & 0 \\ 0 & g & 0 & \lambda \end{pmatrix}.$$

By replacing ρ by its dual, we may assume that $\rho(y)$ has the form given by the first of these two matrices. At least one of b and c is not 0. By conjugating by a suitable diagonal matrix, which commutes with $\rho(P)$, we may assume that b and c are either 0 or 1. Hence there are three possibilities to consider :

(i) $b = 1$, $c = 0$;

(ii) $b = 0$, $c = 1$;

(iii) $b = c = 1$.

Furthermore, if $r = 1$ in $\rho(P)$ then, without any loss of generality, it suffices to consider the possibility (i) only. This can be achieved by conjugation by a matrix which commutes with $\rho(\sigma)$ and $\rho(P)$. We analyze each of these possibilities separately.

(i) Since y and U commute, we have

$$\rho(y) = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad \rho(U) = \begin{pmatrix} a & 0 & b & c \\ d & e & f & g \\ 0 & 0 & e & 0 \\ h & 0 & i & j \end{pmatrix},$$

where we are now reusing the letters a-j in a different role.

From $Uxy = xU$ we obtain first $e(1-\lambda) = 0$, and so $\lambda = 1$, and then $e = a$, $d = -a$, and $h = 0$. From $y = (\sigma U)^2$ we find that $a^2 = j^2 = 1$, $c(a-j) = 0$, $i(a+j) = 0$, $g(a+j) + ac = 0$, and $ab + 2af + gi = 1$. From $(P\sigma PU)^2 = 1$ we obtain from $(1,4)$ entries that $c(a+j) = 0$. Since $a \neq 0$, this equation when combined with $c(a-j) = 0$ gives $c = 0$. From $(2,4)$ entries we obtain $g(a-j) = 0$. When combined with $g(a+j) = 0$, we conclude that $g = 0$. From $(1,3)$ entries we obtain that $b = 0$. One of the previous equations now gives $f = 1/2a$. Next we exploit the relation $(UP\sigma)^3 = 1$. From $(1,1)$ entries we obtain $a^3 = 1$. Since $a^2 = 1$, it follows that $a = 1$. From $(4,3)$ entries we obtain $i(2r+j) = 0$. As $j^2 = r^2 = 1$, it follows that $i = 0$. Since $c = g = h = i = 0$, ρ is decomposable, and so we have a contradiction.

(ii) We have $r = -1$ and

$$\rho(y) = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 1 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad \rho(U) = \begin{pmatrix} a & 0 & b & c \\ d & e & f & g \\ h & 0 & i & j \\ 0 & 0 & 0 & e \end{pmatrix},$$

From $Uxy = xU$ we obtain first from $(2,2)$ entries the equation $e(1-\lambda) = 0$, and so $\lambda = 1$. Next from $(3,4)$ entries we obtain $h = 0$, from $(1,4)$ entries $e = a$, and from $(2,4)$ entries $d = a$. From $(\sigma U)^2 = (U\sigma)^2$ by comparing $(1,3)$ entries we obtain $b(a-i) = 0$. Next we use the relation $(P\sigma PU)^2 = 1$. From diagonal entries we find that $a^2 = i^2 = 1$. From $(1,3)$ entries we obtain $b(a+i) = 0$. By combining this equation with $b(a-i) = 0$, we conclude that $b = 0$. From $(1,4)$ entries we find that $c = 0$. Finally we use the relation $(UP\sigma)^3 = 1$. From diagonal entries we find that $a^3 = -1$ and $i^3 = 1$. As $a^2 = i^2 = 1$, we have $a = -1$ and $i = 1$. Now from (1,4) entries we find that $f = 0$, and from (3,4) entries $j = 0$. Since $b = f = h = j = 0$, ρ is decomposable and so we have a contradiction.

(iii) We have $r = -1$ and

$$\rho(y) = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 1 & 1 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad \rho(U) = \begin{pmatrix} a & 0 & b & c \\ d & e & f & g \\ h & 0 & i & j \\ -h & 0 & e-i & e-j \end{pmatrix},$$

From $Uxy = xU$ we obtain first from (2,2) entries the equation $e(1-\lambda) = 0$, and so $\lambda = 1$. Now the (2,3) entries give $e = -d$, while (2,4) entries give $e = d$. We infer that $e = 0$, which is a contradiction.

8 Characteristic 2 case

Let $n = 2$ and assume only that ρ is nontrivial. Since $(\omega U)^2 = 1$ and $\omega^2 = 1$, it follows that $\det(U) = 1$. Let λ and λ^{-1} be the eigenvalues of U. Since $(P\sigma)^4 = 1$, $\rho(P\sigma)$ is unipotent. As $n = 2$, we have $\rho(P\sigma)^2 = 1$. Hence $\rho(P)$ and $\rho(\sigma)$ commute, and so $\rho(\sigma) = \rho(\omega)$.

Assume first that $\lambda \neq 1$. Since $\omega U \omega = U^{-1}$, we can choose a basis of V such that

$$\rho(U) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \rho(\sigma) = \rho(\omega) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since $P^2 = 1$, and $\rho(P)$ commutes with $\rho(\sigma)$, we must have

$$\rho(P) = \begin{pmatrix} a & a+1 \\ a+1 & a \end{pmatrix}$$

for some $a \in K$. By examining the equation $\rho(UP\sigma)^3 = 1$, one can show that $a = 0$ and $\lambda^2 + \lambda + 1 = 0$, i.e., λ is a primitive cube root of 1. Hence we have an indecomposable representations of Φ_2 such that $\rho(\Phi_2) \simeq S_3$.

Assume now that $\lambda = 1$. If $\rho(U) = 1$, then also $\rho(P) = \rho(\sigma)$ and $\rho(\Phi_2) \simeq C_2$. Thus we may assume that

$$\rho(\sigma) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Now let $\rho(U) \neq 1$. If $\rho(\sigma) \neq 1$, we can choose a basis of V such that

$$\rho(\sigma) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho(P) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad \rho(U) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad b \neq 0,$$

because both $\rho(P)$ and $\rho(U)$ commute with $\rho(\sigma)$. From $(UP\sigma)^3 = 1$ we conclude that $a + b = 1$. Hence we obtain a 1-parameter family of non-equivalent indecomposable representation of Φ_2 with $\rho(\Phi_2) \simeq C_2 \times C_2$. If $\rho(\sigma) = 1$, then $\rho(UP)^3 = 1$ implies that either, say,

$$\rho(U) = \rho(P) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

or $\rho(UP)$ has order 3, in which case we may assume that

$$\rho(U) = \begin{pmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{pmatrix}, \quad \rho(P) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where λ is a primitive cube root of 1. Hence we obtain another indecomposable representation of Φ_2 with $\rho(\Phi_2) \simeq S_3$, which is not equivalent to the previous one.

In all of the representaiton mentioned above we have $\rho(y) = \rho(\sigma U)^2 = 1$, and so $\rho(F_2) = 1$. In particular the assertion of the theorem holds if $n = 2$.

Now let $n = 3$ and assume that ρ is indecomposable and $\rho(F_2) \neq 1$. Since $\omega U \omega = U^{-1}$, the eigenvalues of U are λ, λ^{-1}, and 1.

If $\rho(y)$ is diagonalizable, then $\rho(y) \neq 1$ implies that y has three distinct eigenvalues. As $y\sigma = \sigma y$, $\rho(\sigma)$ is diagonalizable. Since $\rho(\sigma)$ is also unipotent, we obtain $\rho(\sigma) = 1$, a contradiction.

Hence $\rho(y)$ is not diagonalizable, and so must be unipotent. Since $yU = Uy$, it follows that $\lambda = 1$, i.e., $\rho(U)$ is unipotent. Consequently $\rho(U)^4 = 1$. Since $y = (\sigma U)^2$ and $\rho(y)$ is unipotent, we conclude that $\rho(y)^2 = 1$. Hence ρ factorizes through the homomorphism $\Phi_2 \to \Gamma_3$.

Finally let $n = 4$. We assume, as in the statement of the theorem, that ρ is indecomposable and that $\rho(F_2) \neq 1$. The eigenvalues of y have the form $\lambda, \lambda^{-1}, \mu, \mu^{-1}$. We divide the proof into three subcases.

Subcase 1 : $\lambda = \mu = 1$. Since $\rho(y)$ is unipotent and $y = (\sigma U)^2$, $\rho(\sigma U)$ is also unipotent. As $n = 4$, we conclude that $\rho(y)^2 = 1$. Since x, y, and xy are conjugate in Φ_2, we have also $\rho(x)^2 = \rho(xy)^2 = 1$. As $\rho(F_2) \neq 1$, we conclude that $\rho(F_2)$ is a four-group. The subspace $W \subset V$ consisting of all vectors v such that $\rho(x)(v) = \rho(y)(v) = v$ has dimension $1, 2$, or 3. Since F_2 is normal in Φ_2, W is Φ_2-invariant.

We choose a basis of W and extend it to a basis of V. With respect to such a basis we have

$$\rho = \begin{pmatrix} \rho' & * \\ 0 & \rho'' \end{pmatrix}$$

where ρ' (resp. ρ'') is the representation of Φ_2 on W (resp. V/W) induced by ρ.

If $\rho(U)$ is unipotent, then $\rho(U^4) = 1$ and so ρ factorizes through the homomorphism $\Phi_2 \to \Gamma_3$. From now, untill the end of this subcase, we shall assume that $\rho(U)$ is not unipotent.

If U has an eigenvalue 1, then we may assume that

$$\rho(U) = \begin{pmatrix} 1 & \alpha & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta^{-1} \end{pmatrix}, \quad \beta \neq 1,$$

with respect to some basis $\{e_1, e_2, e_3, e_4\}$. Since $yU = Uy$, $\rho(y)^2 = 1$, and $\rho(y) \neq 1$, we have

$$\rho(y) = \begin{pmatrix} 1 & \gamma & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \gamma \neq 0.$$

Hence $\beta \cdot \sigma U \sigma(e_3) = (\sigma U)^2(e_3) = y(e_3)$, i.e., $U\sigma(e_3) = \beta^{-1}\sigma(e_3)$. This implies that $\sigma(e_3) = ae_4$ for some $a \in K^*$. As $\sigma^2 = 1$ and $\sigma y = y\sigma$, we infer that

$$\rho(\sigma) = \begin{pmatrix} 1 & \delta & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & a^{-1} \\ 0 & 0 & a & 0 \end{pmatrix}.$$

An easy computation shows that $\rho(\sigma U)^2 = 1$. As $y = (\sigma U)^2$ and $\rho(y) \neq 1$, we have a contradiction.

Now assume that U has no eigenvalue 1. This implies that $\dim(W) = 2$ and that $\rho'(\Phi_2)$ and $\rho''(\Phi_2)$ are both isomorphic to S_3. For these representations we have $\rho'(P\sigma) = \rho''(P\sigma) = 1$, and consequently $\rho(P\sigma)^2 = 1$. Now Lemma 1 gives a contradiction.

Subcase 2 : $\{\lambda, \lambda^{-1}\} \neq \{\mu, \mu^{-1}\}$. If $\lambda, \mu \neq 1$, then $y\sigma = \sigma y$ and $\sigma^2 = 1$ imply that $\rho(\sigma) = 1$, a contradiction. Now let, say, $\mu = 1$. If $\rho(y)$ is not diagonalizable, its centralizer in $\mathrm{GL}(V)$ is Abelian. Hence $\rho(\sigma)$ and $\rho(U)$ commute. By Lemma 2, ρ factorizes through the homomorphism $\Phi_2 \to \Gamma_2$. We now assume that $\rho(y)$ is diagonalizable. Since σ and y commute, σ leaves invariant the eigenspaces of y. Consequently we can choose a basis of V such that

$$\rho(y) = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \rho(\sigma) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since $\omega y \omega = y^{-1}$ and $\omega = P\sigma P$, we may also assume that

$$\rho(\omega) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since $Uy = yU$, we have

$$\rho(U) = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix}.$$

From $y = (\sigma U)^2$ we obtain

$$\alpha^2 = \lambda, \quad \beta = \alpha^{-1}, \quad c = a + d, \quad ad + bc = 1,$$

and from $(\omega U)^2 = 1$ we obtain that $a + d = 0$. Consequently $c = 0$, $d = a = 1$. Thus $\rho(\sigma)$ and $\rho(U)$ commute and we can apply Lemma 2.

Subcase 3 : $\lambda = \mu \neq 1$. Both eigenspaces of y have the same dimension. If $\rho(y)$ is not diagonalizable, then the centralizer of $\rho(y)$ in $\mathrm{GL}(V)$ is Abelian and we can use Lemma 2 once again. Now let $\rho(y)$ be diagonalizable. Then both eigenspaces of y have dimension 2, and ω interchanges these eigenspaces. It follows that $1+\omega$ has rank 2. Since $\omega = P\sigma P$, $1+\sigma$ also has rank 2. As y and σ commute, we can choose a basis of V such that

$$\rho(y) = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda^{-1} & 0 \\ 0 & 0 & 0 & \lambda^{-1} \end{pmatrix}, \quad \rho(\sigma) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{12}$$

Since ω inverts y and commutes with σ, we must have

$$\rho(\omega) = \begin{pmatrix} 0 & 0 & a' & b' \\ 0 & 0 & 0 & a' \\ c' & d' & 0 & 0 \\ 0 & c' & 0 & 0 \end{pmatrix}, \quad a'd' = b'c', \quad a'c' = 1.$$

By conjugating $\rho(\omega)$ by a suitable matrix which commutes with $\rho(y)$ and $\rho(\sigma)$, we may assume that $a' = c' = 1$ and $b' = d' = 0$, i.e.,

$$\rho(\omega) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{13}$$

Since $Uy = yU$, we have

$$\rho(U) = \begin{pmatrix} u' & v' & 0 & 0 \\ z' & w' & 0 & 0 \\ 0 & 0 & u & v \\ 0 & 0 & z & w \end{pmatrix}.$$

From $y = (\sigma U)^2$ we obtain the equations

$$u^2 + z^2 + z(v+w) = w^2 + z(v+w) = \lambda^{-1},$$

and so $z = u + w$ and

$$\lambda^{-1} = uv + vw + wu.$$

The equation $(\omega U)^2 = 1$ gives

$$\begin{pmatrix} u' & v' \\ z' & w' \end{pmatrix} = \begin{pmatrix} u & v \\ z & w \end{pmatrix}^{-1},$$

and so

$$\rho(U) = \begin{pmatrix} \lambda w & \lambda v & 0 & 0 \\ \lambda(u+w) & \lambda u & 0 & 0 \\ 0 & 0 & u & v \\ 0 & 0 & u+w & w \end{pmatrix}. \tag{14}$$

The matrix

$$P_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

satisfies the equation $\rho(\omega)P_0 = P_0\rho(\sigma)$. Since $\rho(P)$ satisfies the same equation, the matrix $P_0^{-1}\rho(P)$ commutes with σ. Consequently $\rho(P)$ has the form

$$\rho(P) = P_0 \cdot \begin{pmatrix} a & b & c & d \\ 0 & a & 0 & c \\ \alpha & \beta & \gamma & \delta \\ 0 & \alpha & 0 & \gamma \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ \alpha & \beta & \gamma & \delta \\ a & a+b & c & c+d \\ \alpha & \alpha+\beta & \gamma & \gamma+\delta \end{pmatrix}.$$

Since $P^2 = 1$, we have the equations:

$$a(a+c) + \alpha(b+d) = 1, \quad \alpha(a+c) = 1, \tag{15}$$

$$\alpha(a+\beta+\delta) = a\gamma, \quad \alpha(\alpha+\gamma) = 0, \tag{16}$$

$$d(\alpha+\gamma) + \gamma(c+\delta) + \delta(\beta+\delta) = 0, \quad \delta(\alpha+\gamma) + \gamma^2 = 1. \tag{17}$$

The second equations of (15),(16), and (17) imply that $\alpha = \gamma = 1$. The second equation of (15) and the first equations of (16) and (17) give $c = \beta = \delta = 1 + a$. From the first equation in (15) we now obtain that $d = 1 + a + b$. Thus

$$\rho(P) = \begin{pmatrix} a & b & 1+a & 1+a+b \\ 1 & 1+a & 1 & 1+a \\ a & a+b & 1+a & b \\ 1 & a & 1 & a \end{pmatrix}$$

By conjugating by the matrix

$$\begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

we may assume that

$$\rho(P) = \begin{pmatrix} 0 & t & 1 & 1+t \\ 1 & 1 & 1 & 1 \\ 0 & t & 1 & t \\ 1 & 0 & 1 & 0 \end{pmatrix},$$

where $t = a + b + a^2$. Although $\rho(U)$ will change under this conjugation, it will still have the form (14). By using this expression for $\rho(P)$, we find that

$$\rho(x) = \begin{pmatrix} \lambda^{-1} + rt & rt & rt & rt \\ r & \lambda + rt & 0 & rt \\ rt & rt & \lambda^{-1} + rt & rt \\ 0 & rt & r & \lambda + rt \end{pmatrix}$$

where

$$r = \lambda + \lambda^{-1}.$$

By equating the diagonal entries of the matrices $\rho(xU)$ and $\rho(Uxy)$, we obtain the equations

$$\begin{aligned}
&(v + wt)\lambda^3 + ut\lambda^2 + (v + w + wt)\lambda + w + ut = 0, \\
&(u + wt)\lambda^3 + (u + v + ut)\lambda^2 + wt\lambda + v + ut = 0, \\
&wt\lambda^3 + (v + ut)\lambda^2 + (u + wt)\lambda + u + v + ut = 0, \\
&(v + w + wt)\lambda^3 + (w + ut)\lambda^2 + (v + wt)\lambda + ut = 0.
\end{aligned}$$

By adding the first two equations, we obtain

$$(\lambda + 1) \cdot [v + w + (u + v)\lambda^2] = 0,$$

and by adding the last two, we obtain

$$(\lambda + 1) \cdot [u + v + (v + w)\lambda^2] = 0.$$

Since $\lambda \neq 1$, we have

$$u + v = \lambda^{-2}(v + w) = \lambda^2(v + w),$$

and so $u = v = w$. By (12) and (14), $\rho(\sigma)$ and $\rho(U)$ commute and so, by Lemma 2, $\rho(y)^2 = 1$. This gives $\lambda = 1$, a contradiction.

This completes the proof of the theorem. ∎

9 Some indecomposable representations of Φ_2 and B_4

In this section we list all, up to weak equivalence, indecomposable representations ρ of Φ_2 of degree ≤ 4 such that $\rho(F_2) \neq 1$ and $\rho(\Phi_2)$ is infinite. According to the previous section, such reperesentations do not exist if char $K = 2$. We also include an interesting example of an indecomposable representation of degree 4 with $\rho(\Phi_2)$ finite.

One can use the above mentioned representations ρ of Φ_2 in order to construct new representations of B_4. Recall that the braid group B_4 has the following presentation :

$$B_4 = \langle \sigma_1, \sigma_2, \sigma_3 \ : \ [\sigma_1, \sigma_3] = 1, \ \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \ \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3 \rangle.$$

Furthermore there is a homomorphism $h : B_4 \to \Phi_2$ given by :

$$h(\sigma_1) = PUP, \quad h(\sigma_2) = U\sigma U^{-1}P, \quad h(\sigma_3) = P\sigma U^{-1}\sigma P.$$

For readers convenience, we have also computed the images of σ_i's in each case.

Representation 1. The generators σ, P, and U of Φ_2 are represented by the matrices

$$\rho(\sigma) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho(P) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho(U) = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1/2 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is easy to verify that these matrices satisfy the defining relations (1) of Φ_2. A simple computation shows that $x = PyP$ and $y = (\sigma U)^2$ are represented by the matrices

$$\rho(x) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence $\rho(F_2)$ is a free Abelian group of rank 2.

The corresponding representation of B_4 is determined by :

$$\sigma_1 \to \begin{pmatrix} 1 & -1 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma_2 \to \begin{pmatrix} 0 & -1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma_3 \to \begin{pmatrix} 1 & -1 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Representation 2. The second representation ρ is defined by :

$$\rho(\sigma) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \rho(P) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1/2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \rho(U) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

In this case we find that

$$\rho(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Now $\rho(F_2)$ is a solvable group which is not nilpotent.

For B_4 we have :

$$\sigma_1 \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 0 & -1 & 0 \\ 1/2 & -1 & 0 & 0 \\ 1/2 & 0 & 0 & 1 \end{pmatrix}, \quad \sigma_2 \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1/2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \sigma_3 \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 0 & 1 & 0 \\ 1/2 & 1 & 0 & 0 \\ -1/2 & 0 & 0 & 1 \end{pmatrix}.$$

Representation 3. If characteristic of K is not 2 or 3, then we have a representation ρ defined by :

$$\rho(\sigma) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \rho(P) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \rho(U) = \begin{pmatrix} 1 & 1/2 & 0 & 1/12 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In this case we have

$$\rho(x) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In this case $\rho(F_2)$ is a non-Abelian unipotent group.

The corresponding representation of B_4 is given by :

$$\sigma_1 \to \begin{pmatrix} 1 & 0 & 1/2 & -1/12 \\ 0 & 1 & 1 & -1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \sigma_2 \to \begin{pmatrix} 1 & 0 & 0 & 1/16 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\sigma_3 \to \begin{pmatrix} 1 & 0 & -1/2 & -1/12 \\ 0 & 1 & 1 & 1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

All three representations above of Φ_2 and B_4 are at the same time indecomposable and reducible.

Representation 4. This representation ρ is defined by :

$$\rho(\sigma) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \rho(P) = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix},$$

$$\rho(U) = \frac{1}{2}\begin{pmatrix} -1-i & 0 & -1-i & 0 \\ 0 & -1+i & 0 & 1-i \\ 1+i & 0 & -1-i & 0 \\ 0 & -1+i & 0 & -1+i \end{pmatrix},$$

where $i^2 = -1$. One can show that $\rho(\Phi_2) \simeq (Q\#Q) \rtimes S_3$, a quotient of the group Γ_4 defined in Lemma 2. The images of x and y generate one of the two quaternion groups Q. The basic vectors are common eigenvectors of σ and y and, up to scalar multiples, there are no other common eigenvectors. Since P does not preserve these eigenspaces, ρ has no 1-dimensional invariant subspace. As $\rho(F_2) \neq 1$, ρ cannot be direct sum of two 2-dimensional representations. Hence ρ is irreducible.

In this case the representation of B_4 is given by :

$$\sigma_1 \to \frac{1}{2}\begin{pmatrix} -1 & i & 1 & -i \\ -i & -1 & -i & -1 \\ -1 & -i & -1 & -i \\ -i & 1 & i & -1 \end{pmatrix}, \quad \sigma_2 \to \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix},$$

$$\sigma_3 \to \frac{1}{2}\begin{pmatrix} -1 & -i & 1 & i \\ i & -1 & i & -1 \\ -1 & i & -1 & i \\ i & 1 & -i & -1 \end{pmatrix}.$$

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