Sphere Packings in 3 Dimensions

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The Kepler conjecture asserts that no packing of equal balls in three dimensions can have density exceeding that of the face-centered cubic packing. The density of this packing is $\pi/\sqrt{18}$, or about 0.74048.

This conjecture was proved in 1998 [H]. In superficial terms, the proof takes over 270 pages of mathematical text, extensive computer resources, including 3 gigabytes of data, well over 40 thousand lines of code, about 10^5 linear programs each involving perhaps 200 variables and 1000 constraints. The project spanned nearly a decade of research, including substantial contributions from S. Ferguson [F]. This is not a *proof from the book*, in the sense of Erdös.

This lecture proposes a second-generation proof of the Kepler conjecture, and describes the current status of that second-generation proof. Much of the inspiration for the revised proof comes from the subject of *generative programming*, and the lecture gives a brief description of that topic as well.

There are various motivations for a second-generation proof. One is that a simple proof is more likely to be used as a model for other proofs in discrete geometry. One might try to build a proof of the kissing number problem in 4 dimensions, or a proof of the Kelvin problem (of finding the optimal partition of Euclidean 3-space into equal volumes) by similar methods. Such an approach is appealing only if the methods can be further simplified.

This lecture is primarily concerned with packings in three dimensions. But it should be pointed out that excellent progress in higher dimensions has been made in recent work of H. Cohn and N. Elkies. They have produced the best known bounds on densities of packings in dimensions 4-36, and they conjecture that their methods will lead to sharp bounds in dimensions 8 and 24 (corresponding to the E_8 and Leech lattices respectively) [CE].

Two common decompositions of space are the Voronoi decomposition and the Delaunay decomposition. These two decompositions lead to bounds of 0.755 and 0.740873, respectively (the second bound is not rigorous, being based merely on numerical evidence) [M]. In the 1998 proof of the Kepler conjecture, Euclidean space was partitioned according to a hybrid of these two decompositions, and this led to the exact bound of $\pi/\sqrt{18}$.

In the second-generation proof, a function is associated with each center λ of each ball. Assume that all balls in the packing have radius 1. The function f_λ at λ is the sum of the volume of the Voronoi cell around λ and a correction term ϵ_λ. The correction term has the form

$$\epsilon_\lambda = \sum_T \delta(T) + M \sum_S \mu_{S,\lambda}(\sqrt{8} - w(S)),$$

where T runs over all triangles with vertices at centers of balls, and one vertex at λ, such that the edges have length at most some fixed length $r \in [2, \sqrt{8}]$. By definition,

$$\delta(T) = 2L(c) - L(a) - L(b)$$

where L is a quadratic polynomial correction term. Also, a, b, and c are the lengths of the edges of T, and c is the edge opposite λ.

The term proportional to M is a bit more difficult to describe. M is a constant. The sum is over certain triangles S with vertices at the centers of balls with one vertex at λ. (We omit the definition of these triangles.) Each of these triangles S has a distinguished edge of length $w(S)$. The multiplicity μ is $\mu_{S,\lambda} = -1$ if an endpoint of the distinguished edge of S is λ, and $\mu_{S,\lambda} = 2$ otherwise.

Conjecture 1. For a suitable quadratic polynomial L, constant M, and parameter r, we have $f_\lambda \geq \nu_0$, where ν_0 is the volume of the rhombic dodecahedron (the Voronoi cell of the face-centered cubic packing).

This conjecture implies the Kepler conjecture, because the correction terms ϵ_λ cancel on average, when we sum over all centers of balls in a packing. The negative terms are designed to precisely cancel the positive terms. Thus, Conjecture 1 implies that Voronoi cells are, on average in some large region, at least as large as those of the face-centered cubic packing (up to a small error term depending on the size of the region), and this implies that packings in general can be no denser than the face-centered cubic packing.

According to a classical result of Tarski (ca. 1930), for any given L, M, and r (with algebraic constants and coefficients), the truth of the conjecture can be determined by a finite algorithm. Collins has an algorithm giving an improvement over Tarski's original algorithm, but even that is doubly exponential in the number of variables, and so it is not practical for a problem in over 100 variables [CJ].

Instead, the methods that one would use to prove this conjecture are the same as those that would be used in the original proof of the Kepler conjecture (linear programming, interval arithmetic, automatic generation of graphs). However, a solution of the Kepler conjecture obtained through Conjecture 1 would probably be much simpler than the original proof. At the very least, the minimization problem of Conjecture 1 is much simpler than the one considered in the 1998 proof.

About 200 pages of the 1998 proof can be summarized in one sentence. Break the configuration space into a bunch of cases, and then show that inequalities proved by interval arithmetic (together with a little bit of geometry) can be applied in such a way as to show that these cases are worse than the best known close-packings.

Given that 200 pages can be summarized so easily, we should try to structure the repetitive pattern into an algorithm and automate the proof. Multi-simplex optimization (to be explained in the lecture), combining linear programming and interval arithmetic, gives a general inequality-proving algorithm that can be applied to automate much of the proof in this way. Multi-simplex optimization occurs in the 1998 proof, but only in a peripheral way.

Additional simplifications of the proof would come from generative programming techniques. For example, it gives a general method for automated inequality proving based on interval arithmetic [CZ].

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[CZ] K. Czarnecki and U. Eisenecker, Generative Programming: Methods, Tools, and Applications, Addison-Wesley, 2000.

[F] S. Ferguson, Sphere Packings V, thesis, Univ. of Michigan, 1997, math.MG/9811077.

[H] T. Hales, The Kepler conjecture, math.MG/9811078.

[M] T. Hales, S. McLaughlin, A proof of the dodecahedral conjecture, math.MG/9811079.