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BY

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Dedicated to Professor Shoshichi Kobayashi on the occasion of his sixtieth birthday

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Abstract. A characterization of the Prym varieties associated with the morphisms of algebraic curves of an arbitrary degree r with a ramification point of index $r-1$ is established in terms of a generalization of the KP system. This class of the Prym varieties contains the classical cases associated with the degree 2 ramified coverings, and even all the Jacobian varieties as a subclass. Corresponding to this fact, the results of this paper reproduce the characterization of Jacobian varieties in terms of the original KP system as its special case as well as the characterization of the classical Prym varieties using the BKP system when $r = 2$.

0. Introduction.

The purpose of this paper is to establish a characterization of the Prym varieties associated with more general class of ramified coverings of algebraic curves than what have been dealt with in the classical settings, in particular, including coverings of an arbitrary degree, using a generalization of the KP system defined on the infinite-dimensional Grassmannians.

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Classically, the Prym varieties associated with the degree 2 coverings of algebraic curves were introduced in the Schottky-Jung approach to the Schottky problem [**SJ**]. Recently, the Prym varieties of higher-degree coverings have been used by Beauville-Narasimhan-Ramanan [**BNR**] in their study of the generalized theta divisors on the moduli spaces of stable vector bundles over an algebraic curve. They have obtained a formula about the dimension of the linear system of the generalized theta divisors, which provides a mathematical proof of a special case of the mysterious formula due to Verlinde [**B**], which has an origin in conformal field theory. In the context of integrable systems, it has been discovered by Novikov and his collaborators that some Prym varieties appear in the deformation theory of two-dimensional Schrödinger operators [**N**], [**NV**].

Since an arbitrary Jacobian variety is a Prym variety of a higher degree covering, a characterization of the larger class of Prym varieties should reproduce the characterization of Jacobians as its special case. Noting that the general Prym variety is defined as a subvariety of a Jacobian variety, we propose in this paper certain subsystems of the KP equations, which we call the *r-reduced KP system*, as the most natural and the right notion of the generalization of the KP system in this context of characterization, rather than using the other series of integrable systems. It is interesting to note that the B-type KP system of [**DJKM**] happens to give the same result of our 2-reduced KP system in considering the Prym varieties associated with the degree 2 branched coverings. However, if we want to deal with higher degree coverings, then of course the BKP system has nothing to do with them.

In order to explain our ideas, let us start with reviewing the characterization of the Jacobian varieties in terms of the KP system of [**AD**], [**M1**] and [**M2**]. The KP system is an infinite set of completely integrable nonlinear partial differential equations which governs the universal family of isospectral deformations of arbitrary linear ordinary differential operators. Its deep geometric meaning is discovered by Sato [**S**] which says that the KP system is in fact a system of infinitely many commuting vector fields defined on an infinite-dimensional Grassmannian. This Grassmannian is identified with the space of all solutions of the KP system, and the infinitesimal time evolution of the KP system is interpreted as the set of vector fields on the Grassmannian.

The Grassmannian that Sato discovered is the set of all vector subspaces W of the infinite-dimensional vector space $V = k((z))$ of formal Laurent series such that the natural projection $W \to V/k[[z]]z$ is a Fredholm map of index 0. Let us denote this Grassmannian by $Gr(0)$. Certainly, every $z^{-n} \in k[z^{-1}]$ for $n = 1, 2, 3, \cdots$ acts on V by the left-multiplication, and hence it acts on $Gr(0)$ infinitesimally. Now the KP system is defined as the system of equations

$$\frac{\partial W}{\partial t_n} = z^{-n} \cdot W \tag{0.1}$$

imposed on the points of the Grassmannian $W \in Gr(0)$, where t_n is a parameter for the one-parameter subgroup of the vector field on $Gr(0)$ defined by the infinitesimal action of z^{-n}.

Before the discovery of Sato, Krichever [**K**] had shown that every algebraic curve gives rise to an exact solution of the whole KP system. In the language of the Grassmannian, one can understand the Krichever's theorem as a map of certain algebro-geometric data into the Grassmannian. Actually, this map was formulated by Segal and Wilson [**SW**], which is now well-known as the Krichever map. This map assigns a unique point of the Grassmannian $Gr(0)$ to a set of data $(Z,\ q,\ z,\ \mathcal{L},\ \phi)$ consisting of a smooth algebraic curve Z of an arbitrary genus, say g, a point q on it, a local coordinate z of Z around q, a line bundle $\mathcal{L}$ on Z of degree $g-1$, and a local isomorphism

$$\phi : \mathcal{L}|_{U_q} \xrightarrow{\sim} \mathcal{O}_{U_q}(-1)$$

of $\mathcal{L}$ defined on a neighborhood U_q of q.

The key idea of the theory of [**M1**] and [**M2**] is that a point of the Grassmannian belongs to the image of the Krichever map (the Krichever locus) *if and only if* the KP system (0.1) produces finite-dimensional orbits starting at that point. It was also proved that if $W \in Gr(0)$ corresponds to the above geometric data, then the orbit of the KP system starting from W is canonically isomorphic to the Jacobian variety $J(Z)$ of the curve Z. Since no other Abelian varieties appear as an orbit in this way, a characterization theorem of the Jacobian varieties immediately follows from this theory.

It is thus quite natural to seek for a similar type of characterization for the Prym varieties, in particular, by using some kind of integrable systems defined on some infinite-dimensional variety. The pioneering work in this direction was done by Shiota [**Sh**]. He utilized the BKP system for this purpose. Since it had been observed in the influential paper [**DJKM**] that every Riemann theta function defined on the Prym variety associated with a branched double covering of an algebraic curve gives rise to an exact solution of the BKP system, it seems to be reasonable to use the BKP system in this context. Indeed, Shiota proved in [**Sh**] that an Abelian variety is a Prym variety of this type if and only if it can be a finite-dimensional orbit of the BKP system defined on the Grassmannian. The BKP system is closely related with the KdV system. In fact, if the algebraic curve is realized as a branched double covering over the projective line $\mathbf{P}^1$, then the Prym variety associated with this covering is nothing but the hyperelliptic Jacobian. From this point of view, Shiota's result is a natural generalization of the characterization of the hyperelliptic Jacobians using the KdV system due to Mumford.

But the Prym varieties are more general notion than what are appearing in the BKP theory. For example, one can define a Prym variety associated with an *arbitrary* covering of an algebraic curve. Let $\pi : Z \to Y$ be an r-sheeted covering of smooth algebraic curves. One can push-down every degree 0 divisor of Z to a degree 0 divisor of Y. It defines a homomorphism

$$Nm : \mathrm{Pic}^0(Z) \longrightarrow \mathrm{Pic}^0(Y)$$

known as the *norm homomorphism*. The *Prym variety* associated with the covering π is by definition the connected component of the identity of the kernel of this homomorphism. If π is a 2-sheeted covering, then the component of $\mathrm{Ker}(Nm)$ is the classical Prym variety. One can also observe here that every Jacobian variety is a Prym variety in this sense, because every algebraic curve has a nontrivial morphism onto $\mathbf{P}^1$, and the norm homomorphism corresponding to this covering is the trivial map.

Therefore, if one wants to establish a characterization of the Prym varieties in this more general context, then one should make the characterization containing all the Jacobian varieties as a special subclass. But of course, it is impossible to deal with an arbitrary covering π in the currently available framework of the integrable systems and the Krichever-type construction, because the theory of Grassmannians cannot be applied to arbitrary coverings. Then what would be a natural class of the coverings that can be handled in the context of the Grassmannian, and how does the characterization theorem of the Jacobians extend to the Prym varieties?

These are the questions the authors asked in the beginning. Since the Prym variety is defined as a subvariety of a Jacobian variety, and since the KP system produces these Jacobians, we need a *subsystem* to produce the Prym varieties as its orbit. Then what kind of subsystem should we take, and how is it related with the BKP system for the degree 2 cases? There is a more fundamental question. The Krichever map gives a point of the Grassmannian to the geometric data of an algebraic curve and a line bundle on it. Then what would be a counterpart of this map when we have a covering $\pi : Z \to Y$ of two curves, not just a single curve? And how can we characterize the image of the new map on the Grassmannian, and how can we recover the geometry of the covering morphism from the data given on the Grassmannian? We will give an answer to all of these questions in this paper.

First of all, let us consider another KP system

$$\frac{\partial W}{\partial t_j} = y^{-j} \cdot W \tag{0.2}$$

defined on the same Grassmannian, but with a different set of vector fields defined by

$$y = z^r + c_1 z^{r+1} + c_2 z^{r+2} + \cdots \in k((z)) .$$

We will show in Section 3 that a point W of the Grassmannian corresponds to an r-sheeted covering $\pi : Z \to Y$ with a ramification point $q \in Z$ of index $r-1$ *if and only if* both of the KP systems (0.1) and (0.2) produce finite-dimensional orbits starting from W. And if this is the case, then the orbit of (0.2) is canonically isomorphic to the Jacobian variety $J(Y)$ of Y, while the orbit of (0.1) gives $J(Z)$, which contains $J(Y)$ as a subvariety (Theorem 3.6). This is our generalization of the Grassmannian characterization of the Krichever locus to the case of covering morphisms of algebraic curves. In order to obtain an algebraic curve, we apply the KP flows (0.1) on the Grassmannian. If it produces a finite-dimensional orbit at $W \in Gr(0)$, then it corresponds to the Krichever data $(Z,\ q,\ z,\ \mathcal{L},\ \phi)$. Now we let the second KP system (0.2) act on $Gr(0)$ in order to detect if the curve Z has a morphism onto another curve Y. The finite-dimensionality test determines the existence of such a morphism. Therefore, by using the two different KP systems, we can see a much finer structure of the Jacobian variety through the covering morphisms. In the proof of this theorem, we use the notion of the *Krichever functor* discovered in [**M3**].

The Prym variety associated with this covering π appears in the transversal direction of $J(Y)$ in $J(Z)$. Thus the subsystem we need in order to generate the Prym variety is the complement of (0.2) in (0.1), which is our *r-reduced* KP system. Now we can state our main classification theorem (Theorem 3.10):

MAIN THEOREM. *The r-reduced KP system characterizes all the Prym varieties associated with the r-sheeted coverings having a ramification point of index $r-1$. More precisely, a complete algebraic variety is a Prym variety of the above type if and only if it can be realized as a finite-dimensional orbit of the r-reduced KP system.*

Since the BKP system becomes equivalent to our 2-reduced system associated with $y = z^2$ when it is restricted on certain points of the Grassmannian (see 4.3), the main theorem generalizes the characterization of the Prym varieties associated with the branched double coverings of curves in terms of the BKP system. Moreover, if (0.2) produces a zero-dimensional orbit, then the r-reduced system and the original KP system (0.1) determine the same orbit. This corresponds to the fact that every Jacobian is a special Prym variety. Therefore, our theorem includes all the Jacobians as a trivial example. The restriction of the KP system (0.1) on the points of the Grassmannian on which the other system (0.2) with respect to $y = z^r$ acts trivially is known as the *mod r reduction* of the KP system. For example, the well-known KdV equations are the mod 2 reduction of the KP system. But of course, our r-reduced KP system is not equivalent with the mod r reduction in general, and hence our system has a lot of non-Jacobian orbits.

Unfortunately, the coverings we can deal with in our theory is not the most general class of coverings. In particular, we cannot consider the unramified coverings. However, the Prym varieties we discuss in this paper are interesting objects by themselves and form a very natural class, because of the following reasons: Firstly, it contains all the Prym varieties appearing in the study of integrable systems [**DJKM**], [**NV**], [**Sh**]. Actually, this class is the largest one which can be handled in the Grassmannian framework. Secondly, our Prym varieties contain all the Jacobian varieties as a special case. It is a natural generalization of the fact that the classical Prym varieties contain all the hyperelliptic Jacobians as a special subclass. And thirdly, our class has abundantly many nonclassical examples.

This paper is organized as follows: we review some standard results on the Prym varieties in Section 1. We also determine the largest class of coverings of curves which can be dealt with in the Grassmannian language. In Section 2, we review the KP theory on the Grassmannian and the Krichever functor. Section 3 is devoted to the proof of our main classification theorem. In order to prove that the r-reduced KP system produces the Prym varieties, we have to show that the orbit of this system coincides with the kernel of the norm homomorphism. For this purpose, we use an alternative definition of the norm map in terms of the determinant of the direct image sheaf, and calculate the determinant. We will see in Section 4 how the special cases of Prym varieties appear in our more general context of characterization.

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1. Prym varieties and the cyclic coverings.

In this section, we start with reviewing several standard results about the Prym varieties. Then we introduce the notion of the *cyclic coverings* of algebraic curves, and prove that there are abundantly many cyclic coverings for any given algebraic curve. The Prym varieties we will discuss in this paper are the ones associated with these cyclic coverings.

1.1. DEFINITION. *Let $\pi : Y \to X$ be a morphism of degree r between smooth algebraic curves Y and X, and let $Nm : J(Y) \to J(X)$ be the norm homomorphism from the Jacobian variety $J(Y)$ of Y to the Jacobian $J(X)$ of X, which assigns to an element $\sum_q n_q \cdot q \in J(Y)$ its image $\sum_q n_q \cdot \pi(q) \in J(X)$. This is a surjective homomorphism, and hence the kernel $\mathrm{Ker}(Nm)$ is an abelian subscheme of $J(Y)$ of dimension $g(Y) - g(X)$, where $g(C)$ denotes the genus of the curve C. We call the connected component of $\mathrm{Ker}(Nm)$ containing the origin the Prym variety associated with the covering π, and denote it by P_π.*

1.2. REMARK: Any two connected components of $\mathrm{Ker}(Nm)$ are translations of each other in $J(Y)$. On the other hand, if the pull-back homomorphism $\pi^* : J(X) \to J(Y)$ is injective, then the norm homomorphism can be identified with the transpose of π^*, and hence its kernel is connected.

1.3. REMARK: Let $\Delta \subset Y$ be the ramification divisor of the morphism π of (1.1) and $\mathcal{O}_Y(\Delta)$ the locally free sheaf associated with Δ. Then it can be shown that for any line bundle $\mathcal{L}$ on Y, we have $Nm(\mathcal{L}) = \det(\pi_*\mathcal{L}) \otimes \det(\pi_*\mathcal{O}_Y(\Delta))$. Thus up to a translation, the norm homomorphism can be identified with the map assigning the determinant of the direct image to the line bundle on Y. Therefore, one can talk about the Prym varieties in $\mathrm{Pic}^d(Y)$ for an arbitrary d, not just in $J(Y) = \mathrm{Pic}^0(Y)$.

1.4. REMARK: According to our definition (1.1), the Jacobian variety of an arbitrary algebraic curve X can be viewed as a Prym variety. Indeed, let us take a nontrivial morphism of X onto $\mathbf{P}^1$. Then the induced norm homomorphism is the zero-map. Obviously there are infinitely many ways to realize $J(X)$ as a Prym variety in this manner. Thus the class of Prym varieties contains Jacobians as a subclass.

Let us consider the polarization of the Prym varieties. Let Θ_Y and Θ_X be the Riemann theta divisors on $J(Y)$ and $J(X)$, respectively. Then the restriction of Θ_Y to P_π gives an ample divisor H on P_π. However, this is never a principal polarization. In fact, the eigenvalues of its Riemann form are $(1, \cdots, 1, r, \cdots, r)$, where the entry r is repeated $g(X)$-times. There is a natural homomorphism $\psi : J(X) \times P_\pi \to J(Y)$ which sends $(\mathcal{L}, \mathcal{M})$ to $\mathcal{L} \otimes \pi^*(\mathcal{M})$. This is an isogeny, and the pull-back of Θ_Y under this isogeny is given by

$$\psi^*(\Theta_Y) \cong \mathcal{O}_{J(X)}(r\Theta_X) \otimes \mathcal{O}_{P_\pi}(H)\,.$$

1.5. DEFINITION. *A degree r morphism $\pi : Y \to X$ of algebraic curves is called a cyclic covering if there is a point $p \in X$ such that $\pi^*(p) = r \cdot q$ for some $q \in Y$. We call the Prym variety associated with a cyclic covering of degree r the Prym variety of r-cyclic type.*

1.6. PROPOSITION. *Every smooth projective curve X has infinitely many cyclic coverings of an arbitrary degree.*

PROOF: We use the theory of spectral curves to prove this statement. For a detailed account of spectral curves, we refer to [**BNR**] and [**H**].

Let us take a line bundle $\mathcal{L}$ of sufficiently large degree. For such $\mathcal{L}$ we can choose sections $s_i \in H^0(X, \mathcal{L}^i)$, $i = 1, 2, \cdots, r$, satisfying the following conditions:

(1) All s_i's have a common zero point, say $p \in X$;
(2) $s_r \notin H^0(X, \mathcal{L}^r(-2p))$.

Now consider the sheaf $\mathcal{R}$ of symmetric $\mathcal{O}_X$-algebras generated by $\mathcal{L}^{-1}$. As an $\mathcal{O}_X$-module this algebra can be written as

$$\mathcal{R} = \bigoplus_{i=0}^{\infty} \mathcal{L}^{-i} .$$

In order to construct a cyclic covering curve X_s of X, we consider the ideal $\mathcal{I}$ of the algebra $\mathcal{R}$ generated by the image of the sum of the homomorphisms

$$s_i : \mathcal{L}^{-r} \longrightarrow \mathcal{L}^{-r+i} .$$

We define $X_s = \operatorname{Spec}(\mathcal{R}/\mathcal{I})$, where $s = (s_1, s_2, \cdots, s_r)$. Then X_s is a spectral curve and the natural projection gives a degree r covering of X. For sufficiently general sections s_i with properties (1) and (2), we may also assume that (see [**BNR**])

(3) The spectral curve X_s defined by the line bundle $\mathcal{L}$ and the sections s_i's is integral, i.e. reduced and irreducible.

We claim here that X_s is smooth in a neighborhood of the inverse image of p. In fact, let us take a local parameter z of X around p and a local coordinate x in the fiber direction of the total space of the line bundle $\mathcal{L}$. Then the local Jacobian criterion for smoothness in a neighborhood of $\pi^{-1}(p)$ states that the following system

$$\begin{cases} x^r + s_1(z)x^{r-1} + \cdots + s_r(z) = 0 \\ rx^{r-1} + s_1(z)(r-1)x^{r-2} + \cdots + s_{r-1}(z) = 0 \\ s_1(z)'x^{r-1} + s_2(z)'x^{r-2} + \cdots + s_r(z)' = 0 \end{cases}$$

of equations in (x, z) has no solutions. But this is clearly the case in our situation because of the conditions (1), (2) and (3). Thus we have verified the claim. It is also clear that $\pi^*(p) = r \cdot q$, where q is the point of X_s defined by $x^r = 0$ and $z = 0$. Then by taking the normalization of X_s we obtain a smooth cyclic covering of X. This completes the proof.

1.7. PROPOSITION. *Let $\pi : Y \to X$ be a cyclic covering of degree r. Then the induced homomorphism*

$$\pi^* : J(X) \longrightarrow J(Y)$$

of Jacobians is injective. In particular, the kernel Ker(Nm) of the norm homomorphism $Nm : J(Y) \to J(X)$ is connected.

PROOF: Let us suppose in contrary that $\mathcal{L} \not\cong \mathcal{O}_X$ and $\pi^*\mathcal{L} \cong \mathcal{O}_Y$ for some $\mathcal{L} \in J(X)$. Then by the projection formula we have $\pi_*\mathcal{O}_Y \cong \mathcal{L} \otimes \pi_*\mathcal{O}_Y$. This implies that $\mathcal{L}$ is an r-torsion point in $J(X)$, i.e. $\mathcal{L}^r \cong \mathcal{O}_X$. Let n be the smallest positive integer satisfying that $\mathcal{L}^n \cong \mathcal{O}_X$. Let us consider the spectral curve

$$Z = \operatorname{Spec}(\mathcal{O}_X \oplus \mathcal{L}^{-1} \oplus \cdots \oplus \mathcal{L}^{-n+1})$$

given by the line bundle $\mathcal{L}$ and sections

$$s = (s_1, s_2, \cdots, s_{n-1}, s_n) = (0, 0, \cdots, 0, 1) \in \bigoplus_{i=1}^{n} H^0(X, \mathcal{L}^i)\,.$$

Obviously, Z is an unramified covering of X of degree n.

Now we claim that the morphism $\pi : Y \to X$ factors through Z, but this leads to a contradiction to our assumption. The construction of such a morphism amounts to defining an $\mathcal{O}_X$-algebra homomorphism

$$f : \mathcal{O}_X \oplus \mathcal{L}^{-1} \oplus \cdots \oplus \mathcal{L}^{-n+1} \longrightarrow \pi_*\mathcal{O}_Y\,. \tag{1.8}$$

In order to give (1.8), it is sufficient to define an $\mathcal{O}_X$-module homomorphism $\mathcal{L}^{-1} \to \pi_*\mathcal{O}_Y$. This is in turn equivalent to giving a nontrivial section of $\mathcal{L} \otimes \pi_*\mathcal{O}_Y$. But such a section exists because of the isomorphism $\mathcal{L} \otimes \pi_*\mathcal{O}_Y \cong \pi_*\mathcal{O}_Y$! This completes the proof.

2. The Krichever functor and the KP system.

There is a natural correspondence between algebraic geometry of curves and their vector bundles and geometry of certain infinite-dimensional Grassmannians. The KP system is a commuting set of vector fields defined on these Grassmannians, and the vector fields correspond to the deformations of vector bundles on curves. In this section, we first review the correspondence (called the *Krichever Functor*) of **[M3]**, and then define the KP system in the coordinate-free setting following **[M4]**.

Let $V = k((z))$ be the field of formal Laurent series in one variable z with coefficients in an algebraically closed field k of characteristic zero. The space V has a natural adic-topology determined by the (pole-)order of the elements. Here we define ord $z^n = -n$.

The *Grassmannian of index* μ, which is denoted by $Gr(\mu)$, is the set of all vector subspaces $W \subset V$ such that the natural map

$$\gamma_W : W \to V/k[[z]]z \tag{2.1}$$

is Fredholm of index μ. This space has a structure of a pro-algebraic variety (see for example, [**KUS**]). The tangent space of the Grassmannian at a point $W \in Gr(\mu)$ is given by the space of continuous homomorphisms of W into V/W:

$$T_W Gr(\mu) = \mathrm{Hom}_{\mathrm{cont}}(W, V/W)\,. \tag{2.2}$$

Every element v of V defines the left-multiplication map $v : V \to V$, and hence a continuous homomorphism

$$W \hookrightarrow V \xrightarrow{v\times} V \to V/W\,. \tag{2.3}$$

This homomorphism then determines a tangent vector $\Phi_W(v) \in T_W Gr(\mu)$ by (2.2). Therefore, $v \in V$ gives a vector field

$$\Phi(v) : Gr(\mu) \ni W \longmapsto \Phi_W(v) \in T_W Gr(\mu)$$

on the Grassmannian.

2.4. Definition. *A pair (A, W) of subsets of V is said to be a Schur pair of index μ and rank r if*

(1) *$W \in Gr(\mu)$;*
(2) *A is a k-subalgebra of V such that $k \subset A$ and $A \setminus k \neq \phi$;*
(3) *A stabilizes W, i.e.*

$$A \cdot W \subset W\,;$$

(4) *r is equal to the greatest common divisor of $\{\mathrm{ord}\, a \mid a \in A\}$.*

Because of (2.3), we have $\Phi_W(a) = 0$ for all $a \in A$ if (A, W) is a Schur pair. We also note that W is a torsion-free module over A of rank r.

One can define a category of Schur pairs by supplying a morphism between (A_2, W_2) and (A_1, W_1) to be a pair of inclusions $\alpha : A_2 \hookrightarrow A_1$ and $\iota : W_2 \hookrightarrow W_1$. It has been established in [**M3**] that there is a natural fully-faithful contravariant functor called the *Krichever functor* between the category of Schur pairs and a category of geometric *quintets* $(Y, p, \pi, \mathcal{F}, \phi)$. Here, Y is a reduced complete irreducible algebraic curve over k, p is a smooth point of Y, and $\mathcal{F}$ is a sheaf of torsion-free $\mathcal{O}_Y$-modules on Y of rank r. Let U_p be the formal completion of Y at the point p and U_o the formal completion of the affine line $\mathbb{A}^1(k)$ at the origin o. The above π is an r-sheeted covering map

$$\pi : U_o \longrightarrow U_p$$

of the formal schemes ramified at $p \in U_p$, and ϕ is an $\mathcal{O}_{U_p}$-module isomorphism

$$\phi : \mathcal{F}_{U_p} \xrightarrow{\sim} \pi_* \mathcal{O}_{U_o}(-1) ,$$

where $\mathcal{F}_{U_p}$ is the formal completion of $\mathcal{F}$ at p.

The inverse of the Krichever functor is essentially the cohomology functor:

$$\text{(2.5)} \quad \begin{cases} A = \pi^*(H^0(Y \setminus p, \mathcal{O}_Y)) \subset H^0(U_o \setminus o, \mathcal{O}_{U_o}) \\ W = \phi(H^0(Y \setminus p, \mathcal{F})) \subset H^0(U_p \setminus p, \pi_* \mathcal{O}_{U_o}(-1)) = H^0(U_o \setminus o, \mathcal{O}_{U_o}(-1)) , \end{cases}$$

where we identify $H^0(U_o \setminus o, \mathcal{O}_{U_o}) = k((z))$ and $H^0(U_o \setminus o, \mathcal{O}_{U_o}(-1)) = k((z))z$. When $r = 1$, the Krichever functor is known as the *Krichever construction* (or the *Krichever map*) of Segal-Wilson [**SW**]. For general r, the construction is rather complicated. Basically, Y is a one-point completion of Spec(A), p is the point added, and $\mathcal{F}$ is the extension of the torsion-free sheaf $W^\sim$ on Spec(A) to the completion Y.

For our future convenience, let us set the following notations here. First, let us choose a monic element $a_0 \in A$ of the lowest positive order. It is unique up to the constant addition $a_0 + c$. Now we define

$$\text{(2.6)} \quad \begin{aligned} A_p &= \{ b a_0^{-n} \mid b \in A,\ n \geq 0,\ \operatorname{ord}(b a_0^{-n}) \leq 0 \} \\ W_p &= \{ w a_0^{-n} \mid w \in W,\ n \geq 0,\ \operatorname{ord}(w a_0^{-n}) \leq -1 \} . \end{aligned}$$

If we choose an element $y = b a_0^{-n} \in A_p$ of order $-r$, then we can identify the completion A_p^∞ of A_p in $k((z))$ with $k[[y]]$. The completion W_p^∞ of W_p in $k((z))$ is equal to $k[[z]]z$. The curve Y is defined by patching Spec(A) and Spec(A_p) together, and the sheaf $\mathcal{F}$ is defined by gluing $W^\sim$ and $W_p^\sim$ near p. The formal scheme morphism π is determined by the inclusion $k[[y]] \hookrightarrow k[[z]]$, and the sheaf isomorphism ϕ comes from the identification $W_p^\infty = k[[z]]z$. The parameter y is a coordinate on Y defining the morphism π. Through the Krichever functor, we have the following natural isomorphisms:

$$\begin{cases} H^0(Y, \mathcal{F}) \cong \operatorname{Ker}(\gamma_W) \\ H^1(Y, \mathcal{F}) \cong \operatorname{Coker}(\gamma_W) . \end{cases}$$

The Riemann-Roch formula gives us

$$\mu = \dim H^0(Y, \mathcal{F}) - \dim H^1(Y, \mathcal{F}) = d - r(g(Y) - 1) ,$$

where d is the degree of the torsion-free sheaf $\mathcal{F}$ and $g(Y)$ is the (arithmetic) genus of Y.

A generic point of the Grassmannian has only the trivial stabilizer, namely, the maximal commutative stabilizer

$$M = M_W = \{ v \in V \mid v \cdot W \subset W \}$$

is equal to k. Since the Krichever functor does not apply to these points, they do not correspond to any geometric objects directly. But they do have a rather different and more elaborate geometric information related with the topology of the moduli spaces of Riemann surfaces through the matrix model and the two-dimensional quantum gravity. In this paper, we deal with only those points of the Grassmanninan having nontrivial stabilizers.

Not all the vector fields in $\Phi(V)$ given by V are interesting. From the point of view of algebraic geometry, the vector field $\Phi(v)$ for $v \in k[[z]]z$ should be considered trivial because it acts only on ϕ in the quintet and leaves the other geometric data unchanged. This motivates us to define a new vector field $\Phi^+(v)$ associated with $v \in V$ by

$$\Phi^+(v) = \Phi(v) - \Phi(v^-) ,$$

where we decompose $v = v^+ + v^-$ according to the canonical direct sum decomposition

$$k((z)) = k[z^{-1}] \oplus k[[z]]z$$

with $v^+ \in k[z^{-1}]$ and $v^- \in k[[z]]z$.

2.7. DEFINITION. *We call the system*

$$\{\Phi^+(v) \mid v \in V\}$$

of commuting vector fields defined on the Grassmannian $Gr(\mu)$ the KP system.

Our main subject of this paper is to study the finer structure of the finite-dimensional orbits of the KP system.

2.8. DEFINITION. *Let V' be a vector subspace of V. A finite-dimensional smooth subvariety X of $Gr(\mu)$ is said to be a finite-dimensional orbit of the flows defined by $\{\Phi^+(v) \mid v \in V'\}$ if $T_W X$ and $\{\Phi^+_W(v) \mid v \in V'\}$ are equal as a subspace of $T_W Gr(\mu)$ for every point $W \in X$.*

Let us consider a finite-dimensional orbit of the KP system. Because of the Fredholm condition imposed on our Grassmannian, the KP orbit X containing W is of finite-dimensional if and only if (M, W) is a Schur pair of rank one, where $M = M_W$ is the maximal stabilizer of W. Let $(Z,\ q,\ id,\ \mathcal{L},\ \phi)$ be the quintet corresponding to the maximal Schur pair (M, W). In the rank-one case, we can choose the coordinate z on U_o so that the morphism π becomes the identity. Since $\Phi^+(M)$ and $\Phi^+(k[[z]]z)$ act on W trivially, we have

$$T_W X = \Phi^+(V) = \frac{k((z))}{M + k[[z]]} \cong H^1(Z, \mathcal{O}_Z) .$$

Note that every element of the above set is written as a polynomial

$$t_1 z^{-1} + t_2 z^{-2} + t_3 z^{-3} + \cdots ,$$

and the corresponding dynamical motion of the KP system at W is given by the system of linear equations

$$\frac{\partial W}{\partial t_j} = z^{-j} \cdot W \,. \tag{2.9}$$

Formal integration of the above equation is given by

$$W(t) = \exp(t_1 z^{-1} + t_2 z^{-2} + t_3 z^{-3} + \cdots) \cdot W \,,$$

but of course this is not an element of our Grassmannian. One can choose a basis for V and write down the action of the exponential function as an infinite-size matrix multiplication. Then one can realize the solution $W(t)$ as a point of the Grassmannian defined over the ring $k[[t]] = k[[t_1, t_2, t_3, \cdots]]$ (see [**S**]). For our purpose, it is easier to understand the equation (2.9) in terms of the quintet. Obviously, the solution $W(t)$ has always the same maximal stabilizer M. Thus the data $(Z,\, q,\, id)$ in the quintet do not change. If we denote by h the transition function of the line bundle $\mathcal{L}$ on $U_q \setminus q$, then the line bundle corresponding to $W(t)$ has the transition function

$$\exp(t_1 z^{-1} + t_2 z^{-2} + t_3 z^{-3} + \cdots) \cdot h \,.$$

Let $\mathcal{L}(t)$ be the degree 0 line bundle on Z defined by the transition function $\exp(t_1 z^{-1} + t_2 z^{-2} + t_3 z^{-3} + \cdots)$. Then the orbit of the full KP system starting at W coincides with

$$\{(Z, P, \pi, \mathcal{L}(t) \otimes \mathcal{L}, \phi(t)) \mid \mathcal{L}(t) \in \mathrm{Pic}^0(Z)\} \,,$$

which is isomorphic to the Jacobian variety $J(Z)$ of the curve Z (see [**M2**], [**M3**]).

3. The subsystems of the KP system and the Prym varieties.

In this section, we define the subsystems of the KP system we need, and prove the main theorem. First of all, we have to determine points of the Grassmannian which correspond to the geometric data of r-cyclic coverings. These points are characterized by the finite-dimensionality of orbits of the two different dynamical systems defined on the Grassmannian, the full KP system and the *r-KP system*. We will show that the r-KP system produces the Jacobian variety of the curve downstairs as the orbit, while the full KP system gives the Jacobian of the curve upstairs. Then the *r-reduced KP system* produces an orbit which is transversal to the smaller Jacobian inside the larger Jacobian. We will show that this transversal orbit is indeed the Prym variety associated with an r-cyclic covering by calculating the determinant of the direct image sheaf of line bundles on the top curve.

The full KP system gives the Jacobian variety as we have seen in the previous section. We can see much finer structure of this Jacobian through the r-KP system and the reduced systems.

Let us start with defining the r-KP systems. Let $V_r \subset V$ be a subfield such that V is a cyclic extension of V_r of degree r. If we choose an arbitrary element $y \in V_r$ of order $-r$, say

$$y = z^r + c_1 z^{r+1} + c_2 z^{r+2} + \cdots , \tag{3.1}$$

then we have a natural identification $k((y)) = V_r$ as a subfield of $k((z))$.

3.2. DEFINITION. *We call the commuting system of vector fields given by* $\{\Phi^+(v) \mid v \in V_r\}$ *the r-KP system associated with* V_r.

The r-KP system is *not* a subset of the equations among the KP system. It should be viewed rather as another full KP system written in the different coordinate y. These two different systems are mutually related on the Grassmannian through (3.1).

3.3. DEFINITION. *With the above choice of the coordinate, we define the r-reduced KP system to be the system of vector fields given by* $\{\Phi^+(v) \mid v \in \Pi_r\}$, *where*

$$\Pi_r = \bigoplus_{j \not\equiv 0 \bmod r} k \cdot y^{j/r} .$$

3.4 REMARK: Our r-reduced KP system is different from the *mod r reduction* of the KP system. The latter is defined as the entire KP system restricted on the points of the Grassmannian on which the r-KP system associated with $V_r = k((z^r))$ acts trivially. Of course on these points, all the three systems, the full KP system, our r-reduced system associated with $y = z^r$, and the mod r reduction agree.

We note here that the KP system on the Grassmannian is a coordinate-free notion, but the systems we have defined in the above really depend on the choice of the coordinate y. It is natural because we are looking for dynamical systems which can detect the geometry of covering morphisms $\pi : Z \to Y$. Indeed, the coordinate y is nothing but the morphism π, and hence it should appear anyway.

Now let us study the orbits of the r-KP systems. So let $V_r = k((y))$ as above. Like the case of the full KP system, we have a natural isomorphism

$$\Phi^+_W(V_r) \cong \frac{k((y))}{(M_W \cap k((y))) + k[[y]]} .$$

Therefore, the r-KP system produces a finite-dimensional orbit starting from a point W of the Grassmannian if and only if the above set has finite dimension over k.

3.5. EXAMPLE: Let $W = k[\mathfrak{p}(z), \mathfrak{p}'(z)]z \in Gr(0)$, where $\mathfrak{p}(z)$ is the Weierstrass elliptic function. Obviously the maximal stabilizer of W is given by $M = M_W = k[\mathfrak{p}(z), \mathfrak{p}'(z)]$. Now consider the 3-KP system defined by $V_3 = k((z^3))$. Since $M \cap V_3 = k$, the orbit of the 3-KP system starting at W has an infinite dimension! Of course the full KP system applied to W produces a one-dimensional orbit which is nothing but the elliptic curve associated with $\mathfrak{p}(z)$.

The above example shows that even though the full KP system gives a finite-dimensional orbit starting at a point W of the Grassmannian, the r-KP system may produce an infinite-dimensional orbit starting at W. This is the reason why we have to think these two systems different ones which are inter-related only by (3.1). The r-KP system associated with V_r gives a finite-dimensional orbit at W if and only if (A, W) is a Schur pair of rank r, where $A = M_W \cap V_r$. We are interested in the points of the Grassmannian where both the full KP system and the r-KP system associated with $V_r = k((y))$ produce finite-dimensional orbits.

3.6 THEOREM. *A point of the Grassmannian $W \in Gr(\mu)$ corresponds to an r-cyclic covering $\pi : Z \to Y$ if and only if both the full KP system and some r-KP system starting at W generate finite-dimensional orbits. Moreover, the orbit of the r-KP system is canonically isomorphic to the Jacobian variety $J(Y)$ of the curve Y, while the orbit of the full KP system is isomorphic to $J(Z)$.*

PROOF: Let W be a point at which the full KP system and the r-KP system associated with some V_r produce finite-dimensional orbits. Let M be the maximal stabilizer of W, and set $A = M \cap V_r$. Then (A, W) is a Schur pair of rank r. Let $(Z,\ q,\ id,\ \mathcal{L},\ \phi)$ be the quintet corresponding to (M, W), and $(Y,\ p,\ \pi,\ \mathcal{F},\ \phi')$ to (A, M). Note that we have a natural inclusion $A \hookrightarrow M$, which then corresponds to a morphism (π, β) of quintets. Here,

$$\pi : Z \longrightarrow Y$$

is the globalization of the morphism $\pi : U_q \to U_p$ of the formal schemes, and β is an isomorphism

$$\beta : \mathcal{F} \xrightarrow{\sim} \pi_* \mathcal{L} .$$

The morphism $\pi : Z \to Y$ is an r-cyclic covering of (1.5) because $\pi^*(p) = rq$.

Conversely, let $\pi : Z \to Y$ be an r-cyclic covering of smooth algebraic curves such that $\pi^*(p) = rq$ for some points $p \in Y$ and $q \in Z$. We choose a local coordinate z of Z defined on a neighborhood of q so that it defines an identity map $id : U_o \xrightarrow{\sim} U_q$ between the formal schemes, where U_q is the formal completion of Z at q as before. As an abuse of notation, we use π also for the formal covering map $\pi : U_o = U_q \to U_p$ onto the formal completion U_p of Y at p. Let us choose an arbitrary line bundle $\mathcal{L}$ on Z. For example, we can choose the line bundle determined by the divisor $q \subset Z$. Let $d = \deg(\mathcal{L})$, and set $\mu = d - g(Z) + 1$. Finally, we choose an isomorphism

$$\phi : \mathcal{L}_{U_q} \xrightarrow{\sim} \mathcal{O}_{U_q}(-1) ,$$

where $\mathcal{L}_{U_q}$ is the formal completion of $\mathcal{L}$ at q. Thus we have defined a geometric quintet $(Z,\ q,\ id,\ \mathcal{L},\ \phi)$. It automatically determines another quintet $(Y,\ p,\ \pi,\ \pi_*\mathcal{L},\ \pi(\phi))$, where $\pi(\phi)$ is the isomorphism

$$\pi(\phi) : \pi_*\mathcal{L} \xrightarrow{\sim} \pi_*\mathcal{O}_{U_q}(-1)$$

given by ϕ. The cohomology functor of (2.5) gives us two Schur pairs, (M, W) of rank 1 and index μ associated with $(Z,\ q,\ id,\ \mathcal{L},\ \phi)$, and (A, W) of rank r and the same index associated with $(Y,\ p,\ \pi,\ \pi_*\mathcal{L},\ \pi(\phi))$. Since M is a normal ring of rank 1, it is automatically the maximal stabilizer M_W of W (see [M5]). Therefore, the orbit of the full KP system starting from W is of finite-dimensional and canonically isomorphic to the Jacobian variety $J(Z)$ of Z.

Let $y = \pi(z)$ be the local coordinate on Y defining the divisor p. As an element of $k((z))$, y has order $-r$. Define $V_r = k((y))$ and consider the r-KP system associated with V_r. Since $A \subset k((y))$ and $\mathrm{rank}(A) = r$, the orbit of the r-KP system starting at W has a finite dimension. The same argument we gave in Section 2 shows that the r-KP orbit is canonically isomorphic to the Jacobian variety $J(Y)$ of the curve Y. This completes the proof of the theorem.

Let us denote by $J(Z)$ the full KP orbit starting at W and by $J(Y)$ the orbit of the r-KP systems starting at W. We have a natural inclusion map

$$H^1(Y, \mathcal{O}_Y) \cong \frac{k((y))}{A + k[[y]]} \longrightarrow \frac{k((z))}{M + k[[z]]} \cong H^1(Z, \mathcal{O}_Z)\,, \tag{3.7}$$

because $A + k[[y]] = \big(M + k[[z]]\big) \cap k((y))$. This is the infinitesimal version of the fact, which was proved in Proposition 1.7, that $J(Y)$ sits inside $J(Z)$ as a subvariety. Now let us apply the r-reduced KP system to our situation.

3.8. LEMMA. *Let W be a point of the Grassmannian satisfying the condition of Theorem 3.6. Then the orbit of the r-reduced KP system starting from W is canonically isomorphic to the Prym variety associated with the covering π.*

PROOF: Let X be the orbit of the r-reduced KP system starting at W. The dimension of X at W is the dimension of

$$T_W P = \Phi_W^+(\Pi_r) = \frac{\Pi_r}{M \cap \Pi_r + k[[z]] \cap \Pi_r}\,,$$

which is finite because we have $k[[z]] = k[[y^{1/r}]]$ and $k((z)) = k((y^{1/r}))$. Moreover, it can be easily seen from (3.7) that

$$\dim\ X = \dim\ J(Z) - \dim\ J(Y)\,.$$

Since $k((z)) = k((y^{1/r}))$, one can write the full KP system as

$$\frac{\partial W}{\partial t_j} = y^{-j/r} \cdot W\,.$$

As before, the orbit of the full KP system is spanned by the deformations of the line bundle:

$$h \longmapsto \exp\Big(\sum_{j=1}^{\infty} t_j y^{-j/r}\Big) \cdot h\,.$$

Let us denote the line bundle of the right-hand side by $\mathcal{L}(t)$. Then all what we need is to show that $\det \pi_*\mathcal{L}(t)$ is a constant for all t. For this purpose, we have to determine the action of $y^{-j/r}$ on the vector bundle $\pi_*\mathcal{L}$.

We define $\overline{A}_p^\infty$ to be the field of fractions of A_p^∞ and $\overline{W}_p^\infty$ the $\overline{A}_p^\infty$-module generated by W_p^∞ (see (2.6)). They are in fact equal to $k((y))$ and $k((z))$, respectively. We also define M_p, M_p^∞ and $\overline{M}_p^\infty$ similarly. Then we have natural isomorphisms

$$H^1(Z, End(\mathcal{L})) \cong \frac{\mathrm{End}_{\overline{M}_p^\infty} \overline{W}_p^\infty}{\mathrm{End}_M W + \mathrm{End}_{M_p^\infty} W_p^\infty}$$

and

$$H^1(Y, End(\pi_*\mathcal{L})) \cong \frac{\mathrm{End}_{\overline{A}_p^\infty} \overline{W}_p^\infty}{\mathrm{End}_A W + \mathrm{End}_{A_p^\infty} W_p^\infty} .$$

Since $A \subset M$, we have

$$\begin{array}{ccc}
\mathrm{End}_{\overline{M}_p^\infty} \overline{W}_p^\infty & \xrightarrow{\rho} & \mathrm{End}_{\overline{A}_p^\infty} \overline{W}_p^\infty \\
\uparrow & & \uparrow \\
\mathrm{End}_M W + \mathrm{End}_{M_p^\infty} W_p^\infty & \longrightarrow & \mathrm{End}_A W + \mathrm{End}_{A_p^\infty} W_p^\infty ,
\end{array}$$

where all arrows are natural inclusion maps. This induces a natural homomorphism

$$\rho : H^1(Z, End(\mathcal{L})) \to H^1(Y, End(\pi_*\mathcal{L})) .$$

We have to determine the image of the infinitesimal deformation

$$\sum t_j y^{-j/r} \in H^1(Z, End(\mathcal{L}))$$

under this map ρ. Note that we can identify $\overline{W}_p^\infty = k((y^{1/r}))$. Then $y^{1/r} \in \mathrm{End}_{\overline{M}_p^\infty} \overline{W}_p^\infty$ maps to an $r \times r$ matirx

$$\rho(y^{1/r}) = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & 0 & \ddots & \\ 0 & & & & \ddots & 1 \\ y & 0 & & & & 0 \end{pmatrix} \in \mathrm{End}_{\overline{A}_p^\infty} \overline{W}_p^\infty$$

with respect to the basis $\{1,\ y^{1/r},\ y^{2/r},\ \cdots,\ y^{(r-1)/r}\}$ for the free $\overline{A}_p^\infty$-module $\overline{W}_p^\infty$. Therefore, the diagonal entries of $\rho(y^{j/r})$ is 0 for all $j \not\equiv 0 \bmod r$. Thus we have

$$\det\left(\exp \rho\left(\sum_{j \not\equiv 0 \bmod r} t_j y^{-j/r}\right)\right) = \exp\left(\mathrm{trace}\, \rho\left(\sum_{j \not\equiv 0 \bmod r} t_j y^{-j/r}\right)\right) = 1 ,$$

where j runs over positive integers. This implies that $\det(\pi_*\mathcal{L}(t)) = \det(\pi_*\mathcal{L})$ for all t. Therefore, our orbit X is a subvariety of the Prym variety P_π associated with the cyclic covering $\pi : Z \to Y$. But since $\dim X = \dim P_\pi = g(Z) - g(Y)$, we conclude that X is indeed equal to the Prym variety P_π. This completes the proof of the lemma.

In order to show the converse, namely, that every Prym variety of r-cyclic type can be obtained as an orbit of the r-reduced KP system, we have to determine a point of the Grassmannian corresponding to the given geometric setting. So let $\pi : Z \to Y$ be an r-cyclic covering. As we have done in the proof of Theorem 3.6, we can choose a maximal Schur pair (M, W) and a rank r Schur pair (A, W) out of our geometric setting. (Of course, such choice is not unique.)

In Section 1, we defined the Prym variety P_π associated with the covering morphism π to be a connected component of the subset

$$\{\mathcal{N} \in \mathrm{Pic}^d(Z) \mid \det(\pi_*\mathcal{N}) = \det(\pi_*\mathcal{L})\} \subset \mathrm{Pic}^d(Z)\,, \tag{3.9}$$

which has dimension $g(Z)-g(Y)$. In our case, since π is a cyclic covering, (3.9) is itself connected by Proposition 1.7. Now, we know that the orbit X of the r-reduced KP system starting at W lies in the same subset (3.9) and having the same dimension, as we have observed in the proof of Lemma 3.8. Therefore, X must be equal to the entire (3.9)! This proves that the Prym variety P_π is realized as an orbit of the r-reduced KP system. Thus we have established our main theorem:

3.10. THEOREM. *Let $W \in Gr(\mu)$ be a point of the Grassmannian such that both the full KP system and a certain r-KP system produce finite-dimensional orbits. Then the orbit of the r-reduced KP system starting at W is canonically isomorphic to the Prym variety associated with an r-cyclic covering of algebraic curves.*

Conversely, every Prym variety of r-cyclic type can be obtained as a finite-dimensional orbit of the r-reduced KP system.

4. Examples.

4.1. EXAMPLE: The simplest example of our theory is the hyperelliptic Jacobians. A hyperelliptic curve Z has a unique degree 2 cyclic covering map π onto $\mathbf{P}^1$. Thus $J(Z)$ is the Prym variety associated with $\pi : Z \to \mathbf{P}^1$.

Let $W \in Gr(\mu)$ be a point of the Grassmannian satisfying that

$$z^{-2} \cdot W \subset W\,.$$

If the maximal stabilizer $M = M_W$ of W has rank one, then it has to be of the form

$$M = k[z^{-2}, b(z)]$$

with a monic element $b(z) \in k((z))$ of order $2g + 1$. In this case, (M, W) corresponds to a hyperelliptic curve Z of genus g and a line bundle of degree $\mu + g - 1$. Let us define $A = k[z^{-2}]$ and $V_2 = k((z^2))$, and consider the 2-KP system associated with V_2. Since $M \cap V_2 = A = k[z^{-2}]$, the space

$$\Phi^+(V_2) = \frac{k((z^2))}{M \cap k((z^2)) + k[[z^2]]}$$

has dimension 0. Therefore, the orbit of the 2-KP system starting at W is the point W itself. In particular, the orbit of the full KP system is the same as the orbit of the 2-reduced KP system starting at W. As we have noted in (3.4), the 2-reduced KP system appearing here is *by definition* the KdV system! Therefore, our characterization theorem recovers the characterization of the hyperelliptic Jacobians by the KdV system due to Mumford.

4.2. EXAMPLE: More generally, let Z be an arbitrary smooth algebraic curve and q its (rational) point. If we choose a sufficiently large $r > 0$, then there exists a nontrivial element $a \in H^0(Z, \mathcal{O}_Z(rq))$ which defines an r-cyclic covering $\pi : Z \to \mathbf{P}^1$. Let us choose a local coordinate z of Z around q, and let $y = 1/a \in k[[z]]$, which is an order $-r$ element. As in Theorem 3.6, we have a point W of the Grassmannian corresponding to our geometric situation. Since

$$k[a] = k[y^{-1}] \subset M = H^0(Z \setminus q, \mathcal{O}_Z)\,,$$

the r-KP system associated with $V_r = k((y))$ is trivial at W. Therefore, the full KP system and the r-reduced KP system produce the same orbit starting at W, which is the Jacobian variety of Z, and simultaneously, the Prym variety associated with the covering π. Thus our main theorem contains the characterization of all the Jacobian varieties in terms of the KP system as a special case.

4.3 EXAMPLE: Let $Z \to Y$ be a branched double covering of smooth algebraic curves. Let us pick up one of the ramification points, say $q \in Z$, and let $p = \pi(q)$. We can choose a local coordinate z of Z around q and y of Y around p such that the morphism π is locally given by the equation $y = z^2$. By supplying a *generic* line bundle $\mathcal{L}$ on Z of degree $g(Z) - 1$ and its local trivialization suitably, we have a maximal Schur pair (M, W) of rank 1 and index 0 corresponding to Z, and another pair (A, W) of rank 2 and index 0 corresponding to Y, as in Section 3. Here we mean by generic that $H^0(Z, \mathcal{L}) = H^1(Z, \mathcal{L}) = 0$. Thus the point W belongs to the big-cell of $Gr(0)$, where γ_W of (2.1) is an isomorphism. Then our theorem shows that the orbit of the 2-reduced KP system starting at W is the Prym variety associated with π.

Now, let us consider the τ-function of Hirota-Sato. (A τ-function of the KP system is the canonical section of the determinant line bundle on the Grassmannian $Gr(0)$ cut along on an orbit, or equivalently, the *bosonization* of the semi-infinite wedge product corresponding to the point of the Grassmannian. For more detail, we refer to [**KNTY**].) Since the 2-reduced KP system here does not contain the variables t_{2n} for all n, the τ-function of the full KP system restricted on the orbit of the 2-reduced KP system is simply given by

$$\tau(t_1, 0, t_3, 0, t_5, 0, \cdots)\,.$$

But of course (see [**DJKM**]), the square root of the τ-function $\tau(t_1, 0, t_3, 0, t_5, 0, \cdots)^{1/2}$ satisfies the Hirota bilinear differential equations associated with the BKP system! Note here that since our W belongs to the big-cell of the index 0 Grassmannian, we have $\tau(0, 0, \cdots) = 1$. Therefore, we can define the square root of the τ-function uniquely as a formal power series in t_1, t_3, t_5, $\cdots$. In this way, our theory contains all the Prym varieties appearing in the BKP theory.

4.4. EXAMPLE: Finally, we give an example of the three dimensional Prym variety associated with a degree 3 overing over an elliptic curve. Let us consider the following three elements of $k((z))$:

$$\begin{cases} u(z) = z^{-2} \\ v(z) = u(z)^3 = z^{-6} \\ w(z) = \sqrt{v^3 + g_2 v + g_3} = z^{-9} + \frac{g_2}{2} z^3 + \frac{g_3}{2} z^9 + \cdots . \end{cases}$$

The parameters g_2 and g_3 are chosen generically so that the hyperelliptic curve Z of genus 4 defined by

$$w^2(z) = u^9(z) + g_2 u^3(z) + g_3$$

and the elliptic curve Y defined by

$$w^2(z) = v^3(z) + g_2 v^3(z) + g_3$$

are both nonsingular. We have a cyclic covering $\pi : Z \to Y$ of degree 3 defined by

$$\pi : (u, w) \longmapsto (u^3, w) ,$$

which is ramified at the point at infinity. Let

$$W = k[u, w] \cdot z^4 \in Gr(0) .$$

As far as the curve Z is nonsingular, the maximal stabilizer of W is given by $M = k[u, w]$. Let us define $A = k[u^3, w]$, which is the coordinate ring of Y. Note that $A = M \cap k((z^3))$. Now both the Jacobians of Z and Y, and the Prym variety associated with π appear as an orbit of the three different systems applied to the point W:

$$\begin{aligned} J(Z) &= \exp\big(t_1 z^{-1} + t_3 z^{-3} + t_5 z^{-5} + t_7 z^{-7}\big) \cdot W \\ J(Y) &= \exp\big(t_3 z^{-3}\big) \cdot W \\ P_\pi &= \exp\big(t_1 z^{-1} + t_5 z^{-5} + t_7 z^{-7}\big) \cdot W . \end{aligned}$$

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